Bisimulations for Fuzzy Transition Systems Revisited

Hengyang Wu[a], Taolue Chen[b,c,*], Tingting Han[b], Yixiang Chen[a]

[a] MoE Engineering Center for Software/Hardware Co-Design Technology and Application, East China Normal University, Shanghai, China
[b] Department of Computer Science and Information Systems, Birkbeck, University of London, London, UK
[c] State Key Laboratory for Novel Software Technology, Nanjing University, Nanjing, China

Abstract

Bisimulation is a well-known behavioral equivalence for discrete event systems, and has recently been adopted and developed in fuzzy systems. In this paper, we propose a new bisimulation, i.e., the *group-by-group fuzzy bisimulation*, for fuzzy transition systems. It relaxes the fully matching requirement of the bisimulation definition proposed by Cao et al. [*IEEE Transaction on Fuzzy Systems*, 19:540–552], and can equate more pairs of states which are deemed to be equivalent intuitively, but which cannot be equated in previous definitions. We carry out a systematic investigation on this new notion of bisimulation. In particular, a fixed point characterization of the group-by-group fuzzy bisimilarity is given, based on which, we provide a polynomial-time algorithm to check whether two states in a fuzzy transition system are group-by-group fuzzy bisimilar. Moreover, a modal logic, which is an extension of the Hennessy-Milner logic, is presented to completely characterize the group-by-group fuzzy bisimilarity.

Keywords: Bisimulation, Fuzzy transition system, Modal logic, Logical characterization

1. Introduction

Bisimulations are well-established forms of behavioural equivalences for discrete event systems, and have become a central notion in, for instance, process algebras, automata theory, etc. They are widely used in many areas of computer science, in particular, in verification where they are crucial to reduce the state space of the system under consideration.

Recently, bisimulations have been developed in fuzzy systems as well. For example, Cao et al. [2, 4] considered bisimulations for *fuzzy transition systems* (FTS) where both fuzzy transitions and nonderministic transitions co-exist.

*Corresponding author.
Email addresses: hywu@sei.ecnu.edu.cn (Hengyang Wu), taolue@dcs.bbk.ac.cn (Taolue Chen), tingting@dcs.bbk.ac.uk (Tingting Han), yxchen@sei.ecnu.edu.cn (Yixiang Chen)

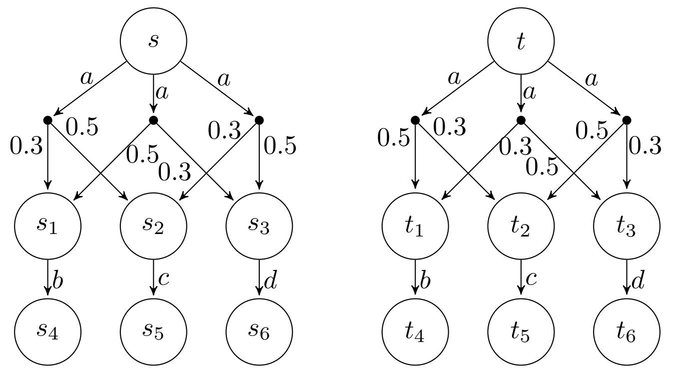

Figure 1: s and t are group-by-group fuzzy bisimilar

This model is further studied under fuzzy automata by Cao et al. [3] and Pan et al. [15]. Ćirić et al. [5] investigated bisimulations for fuzzy automata. Qiu and Deng [9], and Xing et al. [19] studied (bi)simulations for fuzzy discrete event systems. Fan [10] discussed fuzzy bisimulations for Gödel logic. Wu et al. [16, 17, 18] investigated algorithms and logical characterizations of bisimulations for FTS. For more information about fuzzy (bi)simulations, we also refer the readers to [6, 7, 11]. In addition, model checking for fuzzy systems was also studied [12, 13, 14].

A central question regarding any notion of bisimulation (or in general, any equivalence) is its distinguishing power. Namely, to which extent it will distinguish a pair of states. As a simple example, we presented two FTS in Fig. 1. Assuming that states s_i and t_i are equal for $i = 4, 5, 6$ (i.e., they cannot be distinguished in this case), and thus one can be easily convinced that s_i and t_i are also equal for $i = 1, 2, 3$, and s_i and t_j are *not* equal for $1 \leq i \neq j \leq 3$ since they have different enabled actions. We are mainly interested in whether states s and t can be related by the bisimulatiion under consideration. The bisimulation proposed by Cao et al. [4] will distinguish them. To see this, the transition $s \xrightarrow{a} \frac{0.3}{s_1} + \frac{0.5}{s_2}$ cannot be matched by either $t \xrightarrow{a} \frac{0.5}{t_1} + \frac{0.3}{t_2}$, or $t \xrightarrow{a} \frac{0.3}{t_1} + \frac{0.5}{t_3}$, or $t \xrightarrow{a} \frac{0.5}{t_2} + \frac{0.3}{t_3}$. However, arguably the two states should *not* be distinguished from the following perspective. The transition of $s \xrightarrow{a} \frac{0.3}{s_1} + \frac{0.5}{s_2}$ can be respectively matched by $t \xrightarrow{a} \frac{0.3}{t_1} + \frac{0.5}{t_3}$ (the central transition) when considering the *group of states* enabling only action b, $t \xrightarrow{a} \frac{0.5}{t_2} + \frac{0.3}{t_3}$ (the rightmost transition) when considering the *group of states* enabling only action c, and the $t \xrightarrow{a} \frac{0.5}{t_1} + \frac{0.3}{t_2}$ (the leftmost transition) when considering the *group of states* enabling action b or c. The other transitions from s can be analyzed similarly. From this point of view, s and t ought *not* be distinguished. Indeed, in [4] the bisimilar states must stepwise behave the same along two *fully matching resolutions*, which in this case unnecessary and should be relaxed.

The aim of this paper is to define a new bisimulation for an FTS, dubbed group-by-group fuzzy bisimulation. This bisimulation is different from the one proposed by Cao et al. [4] in two aspects: (1) The bisimulation in [4] considers each equivalence class of some equivalence relation R, while ours considers each subset of equivalence classes; this is why it is called a group-by-group fuzzy bisimulation, and (2) for a transition $s \xrightarrow{a} \mu$, the bisimulation in [4] requires that there exists a transition $t \xrightarrow{a} \nu$ such that μ and ν are equal at all equivalence classes of some equivalence relation R, i.e., the definition requires fully matching. In contrast, our definition requires that for each subset $\mathcal{G}$ of the equivalence classes and a transition $s \xrightarrow{a} \mu$, there exists a transition $t \xrightarrow{a} \nu$ such that μ and ν are equal for the union of $\mathcal{G}$. We note that for $\mathcal{G}_1$, ν_1 exists such that μ and ν_1 are equal for the union of $\mathcal{G}_1$, but for different $\mathcal{G}_2$, it is possible that ν_2 exists such that μ and ν_2 are equal for the union of $\mathcal{G}_2$. Loosely speaking, our new bisimulation allows *partially matching resolutions*.

We perform a systematic study on this new bisimulation by giving its characterization in different machinery, as follows.

1. We use a fixed point method to characterize group-by-group fuzzy bisimulation. This characterization shows that a group-by-group fuzzy bisimulation is a post-fixed point of some suitable monotonic function over a complete lattice, while a group-by-group fuzzy bisimilarity, the greatest group-by-group fuzzy bisimulation, is the greatest fixed point of this monotonic function.
2. We give a polynomial time algorithm to computing group-by-group fuzzy bisimilarity. Our algorithm follows the standard partition-refinement framework which is the cornerstone for the computation of almost all bisimulations in conventional labeled transition systems, and their various probabilistic and fuzzy extensions. In the current setting, while an exponential-time algorithm can be obtained almost for free, designing a polynomial-time algorithm turns out to be difficult simply because of the universal quantification over all subsets of the state space (cf. Definition 3). As a witness, for probabilistic systems considered in [1], a similar group-by-group probabilistic bisimulation is proposed, but eludes a polynomial-time algorithm[1]. In contrast, we show that in the fuzzy setting, a polynomial-time algorithm does exist, owing to that, essentially, the operations of max and min instead of addition and multiplication respectively are used.
3. We provide a logical characterization of a group-by-group fuzzy bisimilarity, which states that two states are group-by-group fuzzy bisimilar if and only if they satisfy the same logical formulae.

These characterizations suggest the robustness of our new definition of bisimulation of FTS. As mentioned, this work is inspired by the work in [1], where a group-by-group probabilistic bisimulation is investigated in probabilistic systems. However, it is probably noteworthy that our work is different from that in [1] in two aspects: (1) neither the fixed point characterization nor the algorithm

[1] We conjecture such a polynomial-time algorithm does not exist.

was discussed in [1], and (2) [1] has given a logical characterization of group-by-group probabilistic bisimilarity, but needs the condition of image-finiteness, which is not needed in our characterization. Moreover, the methods of proving logical characterization theorems are different; see Remark 2 for details.

The rest of this paper is structured as follows. We briefly review some basic concepts used in this paper in Section 2. Section 3 introduces the notion of a group-by-group fuzzy bisimulation. Some properties about it are discussed. A fixed point characterization of the group-by-group fuzzy bisimilarity is given in Section 4. In Section 5, we present a polynomial time algorithm for testing the group-by-group fuzzy bisimilarity. In the subsequent section, we provide a modal logic, which characterizes group-by-group fuzzy bisimilarity soundly and completely. Finally, this paper is concluded in Section 7 with some future work.

2. Preliminaries

In this section, we briefly recall some notions used in this paper. We write $[n]$ for $\{1, \cdots, n\}$. Given a set S and a binary relation $R \subseteq S \times S$, we write sRt if $(s,t) \in R$. An *equivalence* relation is a reflexive, symmetric, and transitive relation. An equivalence relation R partitions a set S into equivalence classes. For $s \in S$, we use $[s]_R$ to denote the (unique) equivalence class containing s. We drop the subscript R if it is clear from the context. Let $R(s)$ denote the set $\{s' \mid (s,s') \in R\}$. A set U is said to be *R-closed* if $R(s) \subseteq U$ for all $s \in U$. We write R^* for the transitive closure of R.

The following basic facts will be useful later. The proof is a fairly elementary exercise and thus is omitted. In particular, the first item follows from [16, Lemma 2.1].

Lemma 1. Let R, R_1, R_2 be equivalence relations on S. Then we have:

(i) Any R-closed set U is the union of some equivalence classes of R.
(ii) Let $R_1 \subseteq R_2$. Then U is R_2-closed implies that it is also R_1-closed.
(iii) The $(R_1 \cup R_2)^*$ is an equivalence relation.

The notions about fuzzy set are mainly borrowed from [4]. Let S be a set and μ be a *fuzzy set* in S. The *support* of μ is the set $\mathrm{supp}(\mu) = \{s \in S \mid \mu(s) > 0\}$. We denote by $\mathcal{F}(S)$ the set of all fuzzy sets in S. When $\mathrm{supp}(\mu)$ is finite, say $\{s_1, \cdots, s_n\}$, μ can be written in Zadeh's notation as follows:

$$\mu = \frac{\mu(s_1)}{s_1} + \frac{\mu(s_2)}{s_2} + \cdots + \frac{\mu(s_n)}{s_n}.$$

For any $\mu \in \mathcal{F}(S)$ and $U \subseteq S$, let $\mu(U)$ stand for $\sup_{s \in U} \mu(s)$.

Definition 1. [4] An FTS is a triple $\mathcal{M} = (S, A, \rightarrow)$, where

- S is a set of states,
- A is a set of actions, and

- $\rightarrow \subseteq S \times A \times \mathcal{F}(S)$ is the transition relation.

An FTS is *image-finite* if for each $a \in A$ and $s \in S$, the set $T_a(s) = \{\mu \mid s \xrightarrow{a} \mu\}$ is finite and for any $\mu \in T_a(s)$, the $\mathrm{supp}(\mu)$ is also finite; an FTS is *finitely branching* if for each $s \in S$, the set $\{(a, \mu) \mid s \xrightarrow{a} \mu, a \in A, \mu \in \mathcal{F}(S)\}$ is finite. Furthermore, this FTS is *finitary* if S is also finite. For readability, we usually write $s \xrightarrow{a} \mu$ for $(s, a, \mu) \in \rightarrow$.

3. New bisimulation

This section is devoted to the notions of a new bisimulation. We start with the definition proposed by Cao et al. [4], which is a straightforward adaptation of the classical bisimulation to FTSs.

Definition 2. Let $(S, A, \rightarrow)$ be an FTS. An equivalence relation $R \subseteq S \times S$ is a (fuzzy) *bisimulation* if whenever sRt, then for any transition $s \xrightarrow{a} \mu$, there exists a transition $t \xrightarrow{a} \nu$ such that $\mu([s]) = \nu([s])$ for all equivalence classes $[s] \in S/R$.

Before presenting the new bisimulation, we highlight the differences of our new definition (to be given in Definition 3) and Definition 2. The first difference is that, for an action a, the distribution over all classes of equivalent states reached by the a-transition can now be matched by *several* a-transitions, each of which may take care of a different class. Secondly, the new equivalence takes into account the possibility of reaching *groups of equivalence classes* rather than individual classes only. This is similar to the approach in [16] (cf. Definition 4). Notice that considering groups of equivalence classes in Definition 2 does *not* change the bisimulation relation, while dealing only with individual classes here would significantly weaken the discriminating power.

Definition 3. Let $(S, A, \rightarrow)$ be an FTS. An equivalence relation R over S is a *group-by-group fuzzy bisimulation* if, whenever $(s, t) \in R$, for all actions $a \in A$ and for all **groups of equivalence classes** $\mathcal{G} \in 2^{S/R}$, it holds that for each $s \xrightarrow{a} \mu$ there exists $t \xrightarrow{a} \nu$ such that $\mu(\bigcup \mathcal{G}) = \nu(\bigcup \mathcal{G})$, where $\bigcup \mathcal{G} = \bigcup_{C \in \mathcal{G}} C$. We say s and t are *group-by-group fuzzy bisimilar*, denoted by $s \sim_{FB,gbg} t$, if there exists a group-by-group fuzzy bisimulation R such that $(s, t) \in R$.

Intuitively, while in Definition 2 the quantification over $[s] \in S/R$ is *after* the transition matching, in Definition 3 the quantification over $\mathcal{G} \in 2^{S/R}$ is *before* the transition matching. This allows a transition to be matched possibly by distinct transitions depending on the target groups. Because of the difference of the quantification ordering, it is not hard to observe that the bisimulation in Definition 2 is finer than that in Definition 3.

Example 1. Consider the Fig.1. We will show that s and t are group-by-group fuzzy bisimilar. First of all, $S = \{s, t, s_i, t_i \mid i = 1, \cdots, 6\}$, we construct an equivalence relation R on S as follows:

$$R = \{s, t\} \times \{s, t\} \cup \{(s_i, s_i), (t_i, t_i), (s_i, t_i) \mid 1 \leq i \leq 6\}.$$

The equivalence class of the state space S under R is

$$S/R = \{\{s_1, t_1\}, \{s_2, t_2\} \cdots, \{s_6, t_6\}, \{s, t\}\}.$$

It suffices to prove that R is a group-by-group fuzzy bisimulation. Obviously, (s, s), (t, t), (s_i, s_i) and (t_i, t_i) $(i = 1, \cdots, 6)$ all satisfy the condition of Definition 3. Hence, we only need to verify that (s, t) and (t, s) satisfy this condition as well. We first examine (s, t). Let $\mathcal{G} \in 2^{S/R}$ be any group of equivalence classes and $s \xrightarrow{a} \mu_1 = \frac{0.3}{s_1} + \frac{0.5}{s_2}$. Now, consider the possible matching transitions from t. Since μ_1 takes none-zero values only at s_1 and s_2, and the three transition distributions from t take none-zero values only at t_1 or t_2 or t_3, it suffices to consider $\mathcal{G}$ that only contains the $\{s_1, t_1\}$ or $\{s_2, t_2\}$, or $\{s_3, t_3\}$. That is, $\{s_1, t_1\}$, $\{s_2, t_2\}$, $\{s_3, t_3\}$, $\{\{s_1, t_1\}, \{s_2, t_2\}\}$, $\{\{s_1, t_1\}, \{s_3, t_3\}\}$, $\{\{s_2, t_2\}, \{s_3, t_3\}\}$, and $\{\{s_1, t_1\}, \{s_2, t_2\}, \{s_3, t_3\}\}$. Let $\nu_1 = \frac{0.5}{t_1} + \frac{0.3}{t_2}$, $\nu_2 = \frac{0.3}{t_1} + \frac{0.5}{t_3}$ and $\nu_3 = \frac{0.5}{t_2} + \frac{0.3}{t_3}$. Then we consider the following cases according to $\bigcup \mathcal{G}$.

(1) $\bigcup \mathcal{G} = \{s_1, t_1\}$, there exists $t \xrightarrow{a} \nu_2$ such that $\mu_1(\{s_1, t_1\}) = \nu_2(\{s_1, t_1\}) = 0.3$;

(2) $\bigcup \mathcal{G} = \{s_2, t_2\}$, there exists $t \xrightarrow{a} \nu_3$ such that $\mu_1(\{s_2, t_2\}) = \nu_3(\{s_2, t_2\}) = 0.5$;

(3) $\bigcup \mathcal{G} = \{s_3, t_3\}$, there exists $t \xrightarrow{a} \nu_1$ such that $\mu_1(\{s_3, t_3\}) = \nu_1(\{s_3, t_3\}) = 0$;

(4) $\bigcup \mathcal{G} = \{s_1, s_2, t_1, t_2\}$, there exists $t \xrightarrow{a} \nu_3$ such that $\mu_1(\{s_1, s_2, t_1, t_2\}) = \nu_3(\{s_1, s_2, t_1, t_2\}) = 0.5$;

(5) $\bigcup \mathcal{G} = \{s_1, s_3, t_1, t_3\}$, there exists $t \xrightarrow{a} \nu_3$ such that $\mu_1(\{s_1, s_3, t_1, t_3\}) = \nu_3(\{s_1, s_3, t_1, t_3\}) = 0.3$;

(6) $\bigcup \mathcal{G} = \{s_2, s_3, t_2, t_3\}$, there exists $t \xrightarrow{a} \nu_2$ such that $\mu_1(\{s_2, s_3, t_2, t_3\}) = \nu_2(\{s_2, s_3, t_2, t_3\}) = 0.5$;

(7) $\bigcup \mathcal{G} = \{s_1, s_2, s_3, t_1, t_2, t_3\}$, there exists $t \xrightarrow{a} \nu_1$ such that $\mu_1(\{s_1, s_2, s_3, t_1, t_2, t_3\}) = \nu_1(\{s_1, s_2, s_3, t_1, t_2, t_3\}) = 0.5$.

That is, for the transition $s \xrightarrow{a} \mu_1$, there is corresponding transition matching coming from t. Similarly, for other transitions coming from s, there are corresponding transitions coming from t. Hence, (s, t) satisfy the condition of Definition 3. In a similar way, we can verify that (t, s) satisfy the condition of Definition 3 too. Consequently, R is a group-by-group fuzzy bisimulation as desired.

Since $\bigcup \mathcal{G}$ is the union of some equivalence classes of R, it is R-closed. Moreover, any R-closed set is the union of some equivalence classes. Hence, we have the following conclusion which effectively gives an alternative definition of the group-by-group fuzzy bisimulation. The equivalence of Definition 3 and Definition 4 is rather straightforward and thus is omitted.

Definition 4. Let $(S, A, \rightarrow)$ be an FTS. An equivalence relation R over S is a group-by-group fuzzy bisimulation iff, whenever $(s,t) \in R$, then for all actions $a \in A$ and for all R-closed sets U it holds that for each $s \xrightarrow{a} \mu$ there exists $t \xrightarrow{a} \nu$ such that $\mu(U) = \nu(U)$.

Bisimulation is preserved by the equality, inverse and the transitive closure of unions, which are summarized in the following proposition.

Proposition 1. The following statements hold.

(i) $Eq(S)$ is a group-by-group fuzzy bisimulation, where $Eq(S) = \{(s,s) \mid s \in S\}$;
(ii) If R is a group-by-group fuzzy bisimulation, then so is R^{-1} (in fact, $R^{-1} = R$ since R is an equivalence relation);
(iii) If R_i $(i = 1,2)$ is a group-by-group fuzzy bisimulation, then so is $(R_1 \cup R_2)^*$.

Proof. (i) and (ii) are trivial. Now we prove (iii).

Firstly, $(R_1 \cup R_2)^*$ is an equivalence relation by Lemma 1 (iii).

Secondly, let U be $(R_1 \cup R_2)^*$-closed. Then Lemma 1 (ii) implies that it is also R_i-closed $(i = 1,2)$ since $R_i \subseteq (R_1 \cup R_2)^*$ $(i = 1,2)$.

Finally, let $(s,t) \in (R_1 \cup R_2)^*$ and $s \xrightarrow{a} \mu$. Then there exist $s_1, s_2, \cdots, s_n$ such that $(s, s_1), (s_i, s_{i+1})(i = 1, 2, \cdots n-1), (s_n, t) \in R_1 \cup R_2$. Without loss of generality, we suppose that $(s, s_1) \in R_1$ and $(s_n, t) \in R_2$. Since R_1 is a group-by-group fuzzy bisimulation and U is R_1-closed, there exists $s_1 \xrightarrow{a} \nu_1$ such that $\mu(U) = \nu_1(U)$. Since $(s_i, s_{i+1})(i = 1, 2, \cdots n-1)$ must be in R_1 or R_2, moreover R_1, R_2 are both group-by-group fuzzy bisimulations and U is $R_i(i = 1,2)$-closed, there exists $s_i \xrightarrow{a} \nu_i(i = 2, \cdots, n)$, such that $\nu_1(U) = \nu_2(U) = \cdots = \nu_n(U)$. Hence $\mu(U) = \nu_n(U)$. Similarly, $(s_n, t) \in R_2$, R_2 is a group-by-group fuzzy bisimulation and U is R_2-closed imply that there exists $t \xrightarrow{a} \nu$ such that $\nu_n(U) = \nu(U)$. Consequently, $\mu(U) = \nu(U)$ as desired. Hence, $(R_1 \cup R_2)^*$ is a group-by-group fuzzy bisimulation by Proposition 4. The proof is completed.

□

The conclusion of Proposition 1 (iii) can be generalized into any $i \in I$.

Proposition 2. The $\sim_{FB,gbg}$ is an equivalence relation and is the greatest group-by-group fuzzy bisimulation, called group-by-group fuzzy bisimilarity.

Proof. Proposition 1 (i, ii, iii) imply that $\sim_{FB,gbg}$ is reflexive, symmetric, transitive, respectively, and hence it is an equivalence relation. By Definition 3, it is straightforward that $\sim_{FB,gbg}$ is the largest group-by-group fuzzy bisimulation. In fact, $\sim_{FB,gbg}= (\cup_{i\in I} R_i)^*$ where R_i $(i \in I)$ is a group-by-group fuzzy bisimulation. That is, $\sim_{FB,gbg}$ is the transitive closure of the union of all group-by-group fuzzy bisimulations. □

Remark 1. *Bisimulations for fuzzy systems have received much attention recently. Many different notions have been given, see, for example, [2, 5, 7, 9, 10, 11, 16, 17, 18, 19]. All these notions can be divided into two classes. In the first class, bisimulations are based on a crisp relation on the state space, and thus one state is either bisimilar to another state or not. In the second class, bisimulations are based on a fuzzy relation on the state space, which shows the degree to which one state is bisimilar to another one. We refer the readers to the related work in [16, 18] for more details. The present study belongs to the first class, but provides a coarser bisimulation than those in the existing definitions falling into the first class [2, 16]. Its advantage is that one can consider whether two systems are equivalent from the overall perspective. For example, in Fig.1, after implementing the action a and then* $\beta \in \{b, c, d\}$*, two systems reach the equivalent states with the equal possibility* 0.5*. Hence they can be regarded as equivalent.*

4. Fixed-point characterization

In this section, we will use fixed-point method to characterize group-by-group fuzzy bisimulation. We always suppose that the state space S is finite in this section.

Let $\mathcal{R}$ be the set of all equivalence relations over S. For any $R_1, R_2 \in \mathcal{R}$, define the partial order on $\mathcal{R}$ as follows:

$$R_1 \sqsubseteq R_2 \text{ if } R_1 \subseteq R_2.$$

For any $R_i(i \in I) \in \mathcal{R}$, the sup and inf on the set $\{R_i \mid i \in I\}$ are defined as follows:

$$\begin{array}{rcl} \sqcup_{i\in I} R_i & = & (\cup_{i\in I} R_i)^* \\ \sqcap_{i\in I} R_i & = & \cap_{i\in I} R_i. \end{array}$$

So, $\mathcal{R}$ is a complete lattice, its least element $\mathbf{0}$ and the greatest element $\mathbf{1}$ are $Eq(S)$ and $S \times S$, respectively.

We now review the notion of group-by-group fuzzy bisimulation in terms of suitable monotone functions over a complete lattice. We consider the function F defined as follows:

$$F : \mathcal{R} \to \mathcal{R},\ R \longmapsto \left\{ (s,t) \in S \times S \;\middle|\; \begin{array}{l} \forall\ R\text{-closed set } U \\ \forall s \xrightarrow{a} \mu\ \exists t \xrightarrow{a} \nu :\ \mu(U) = \nu(U) \\ \forall t \xrightarrow{a} \nu\ \exists s \xrightarrow{a} \mu :\ \mu(U) = \nu(U) \end{array} \right\} \quad (1)$$

Given an equivalence relation $R \in \mathcal{R}$, it is not difficult to verify that $F(R)$ is reflexive, symmetric and transitive, i.e., an equivalence relation. This shows that F is well-defined.

The following proposition establishes the relation between a group-by-group fuzzy bisimulation and a post-fixed point of F.

Proposition 3. Let R be an equivalence relation over S. Then R is a group-by-group fuzzy bisimulation if and only if R is a post-fixed point of F, i.e., $R \sqsubseteq F(R)$.

Proof. It is a straightforward result of Definition 4. □

Recall that the remarkable Tarski's theorem [8] says that each monotonic function on a complete lattice has a greatest fixed point. Therefore, to show that F has a greatest fixed point, it remains to verify that F is monotonic with respect to $\sqsubseteq$.

Recall that for a partially ordered set $(X, \leq)$, a function $f : X \to X$ is said to be monotonic if for all $x_1, x_2 \in X$, $x_1 \leq x_2$ implies that $f(x_1) \leq f(x_2)$.

Lemma 2. F is monotonic with respect to the partial order $\sqsubseteq$.

Proof. It is not difficult to verify by Lemma 1 (ii) and hence we omit its proof. □

By Tarski's fixed point theorem, we have the following theorem.

Theorem 1. F has a greatest fixed point μF. Moreover

$$\mu F = \sqcup\{R \mid R \sqsubseteq F(R)\}.$$

This theorem shows that the greatest fixed point of F is the sup of all post-fixed points of F. By the definition of sup and Propositions 2, 3, the following theorem holds.

Theorem 2. we have that $\sim_{FB,gbg} = \mu F$.

Recall that for a lattice $(L, \leq)$, a function $f : L \to L$ is said to co-continuous if for all decreasing sequence $x_0, x_1, \cdots$ (i.e., $x_{i+1} \leq x_i$ for all $i \in \mathbb{N}$) in the lattice L, we have $f(\sqcap_{i\in\mathbb{N}} x_i) = \sqcap_{i\in\mathbb{N}} f(x_i)$.

Lemma 3. F is co-continuous on the complete lattice $\mathcal{R}$.

Proof. Let $R_i \in \mathcal{R}$ ($i \in \mathbb{N}$) be a decreasing sequence. Since the state space S is finite, there are at most finite different equivalence relations. Further, $R_i \in \mathcal{R}$ ($i \in \mathbb{N}$) being a decreasing sequence implies that there exists k such that $R_k \sqsubseteq R_i$ for all $i \in \mathbb{N}$. As a result, $F(\sqcap_{i\in\mathbb{N}} R_i) = F(R_k)$. On the other hand, $F(R_k) \sqsubseteq F(R_i)$ for any $i \in \mathbb{N}$ since F is monotonic by Lemma 2. Hence $F(R_k) \sqsubseteq \sqcap_{i\in\mathbb{N}} F(R_i)$, while $\sqcap_{i\in\mathbb{N}} F(R_i) \sqsubseteq F(R_k)$ is obvious since $k \in \mathbb{N}$. That is, $F(R_k) = \sqcap_{i\in\mathbb{N}} F(R_i)$. It follows that $F(\sqcap_{i\in\mathbb{N}} R_i) = \sqcap_{i\in\mathbb{N}} F(R_i)$. The proof is completed. □

Based on this lemma, one can show that $\sqcap_{n\in\mathbb{N}} F^n(\mathbf{1})$ is the greatest fixed point of F (cf. [8]). So, the following corollary holds.

Corollary 1. we have that $\sim_{FB,gbg} = \mu F = \sqcap_{n\in\mathbb{N}} F^n(\mathbf{1})$.

This says that the group-by-group fuzzy bisimilarity is $\sqcap_{n\in\mathbb{N}} F^n(S \times S)$.

5. Algorithmic characterization

In this section, we present an algorithm for checking if two states s and t are group-by-group bisimilar in a *finitary* FTS $\mathcal{M} = (S, A, \rightarrow)$ which we fix for the rest of the section. We follow the standard partition-refinement scheme, which essentially is to compute the greatest fixed point of the function F (cf. (1)) defined in the preceding section. We start with some additional notations. Let Ξ be a partition of S, i.e., $\Xi = \{B_1, \cdots, B_m\}$ such that $\bigcup_{i=1}^m B_i = S$.

- Given $B \in \Xi$ and $C \subseteq \Xi$,

$$B_{\mu(C)} := \{t \in B \mid \exists \nu. t \xrightarrow{a} \nu \wedge \nu(C) = \mu(C)\}.$$

- For any vector $\vec{v} = (a_1, \cdots, a_n)$ and $I \subseteq [n]$, we write $\max \vec{v}[I]$ for $\max\{a_i \mid i \in I\}$.

As a routine in partition-refinement algorithms, in Algorithm 1, we start with an initial (trivial) partition Ξ and, in each iteration, we refine the current partition according to F (cf. (1)) until it is stabilized. Clearly, when the algorithm terminates, Ξ stores the greatest fixed point of F, and this is precisely the bisimulation quotient of S, i.e., $S/\sim_{FB,gbg}$, which is the output of Algorithm 1.

The challenging part, however, is to check whether the current partition Ξ of S is already a bisimulation, and in particular, to check this in *polynomial* time. According to the definition of $\sim_{FB,gbg}$, this is to check, for a pair of states (s,t) in an equivalence class of Ξ and an action a with $s \xrightarrow{a} \mu$, whether for all $C \subseteq \Xi$ there always exists some ν such that $t \xrightarrow{a} \nu$ and $\mu(C) = \nu(C)$. A naïve approach, which verifies for each $C \subseteq \Xi$ individually, is doable, but would need exponential time in the worst case. We shall show that this brute force enumeration of all $C \subseteq \Xi$ is unnecessary and the check can actually be carried out in a more efficient way, i.e., in polynomial time. This is the main purpose of the **Match** function of Algorithm 1 and Algorithm 2.

Let us fix two states s and t, an action a, and a distribution μ such that $s \xrightarrow{a} \mu$, and assume that the current partition is Ξ. Function $\mathrm{Match}(s, t, a, \mu, \Xi)$ will test whether, for all $C \subseteq \Xi$, some ν exists such that $t \xrightarrow{a} \nu$ and $\mu(C) = \nu(C)$. If this is the case, the procedure Match returns an empty set; otherwise it returns some $C \subseteq \Xi$ which one can use to refine the current partition Ξ.

Technically, assume that $\Xi = \{B_1, \cdots, B_k\}$. We let

- $\vec{v}^{\mu} := (a_1, \cdots, a_k)$ where $a_i := \mu(B_i)$. (Recall that $\mu(B) := \max\{\mu(s) \mid s \in B\}$.)
- for each ν such that $t \xrightarrow{a} \nu$, $\vec{w}^{\nu} := (b_1^{\nu}, \cdots, b_k^{\nu})$ where $b_i^{\nu} := \nu(B_i)$.

Algorithm 1: Calculate the quotient space of bisimulation $\sim_{FB,gbg}$

Data: FTS $(S, A, \to)$
Result: Bisimulation quotient space $S/\sim_{FB,gbg}$
begin
- $\Xi \longleftarrow S \times S$;
- $\Xi_{old} \longleftarrow S \times S$;
- **repeat**
  - $\Xi_{old} \longleftarrow \Xi$;
  - **for** *each* $B \in \Xi$ **do**
    - **for** *each pair of states* $s, t \in B$ *and* a, μ *with* $s \xrightarrow{a} \mu$ **do**
      - $C \longleftarrow \text{Match}(s, t, a, \mu, \Xi)$;
      - **if** $C \neq \emptyset$ **then**
        - $\Xi \longleftarrow \Pi \setminus \{B\} \cup \{B_{\mu(C)}\}$;
- **until** $\Xi_{old} = \Xi$;
- $S/\sim_{FB,gbg} \longleftarrow \Xi$

Function $\text{Match}(s, t, a, \mu, \Xi)$
Data: $s, t \in S$, $a \in A$ with $s \xrightarrow{a} \mu$, Ξ is the current partition of S
Result: true if L is fulfilled; otherwise false
begin
- $\vec{v}^\mu := (\mu(C_1), \cdots, \mu(C_k))$;
- //compute the vector $\vec{v}^\mu$.
- $h \longleftarrow 1$;
- **for** *each* ν *with* $t \xrightarrow{a} \nu$ **do**
  - $\vec{w}^\nu := (\nu(C_1), \cdots, \nu(C_i))$;
  - $W[h] \longleftarrow \vec{w}^\nu$;
  - $h \longleftarrow h + 1$;
- //compute all $\vec{w}^\nu$ and store them in the matrix W
- **for** *each* $i \in [k]$ **do**
  - $L_i \longleftarrow \{j \in [k] \mid \vec{v}^\mu[j] \leq \vec{v}^\mu[i]\}$;
  - // $\vec{v}^\mu[i]$ is the largest element of $\vec{v}^\mu$ among the indices in L_i.
  - **if** *Cover*$(i, \vec{v}^\mu, W, L_i) =$ FALSE **then**
    - $\text{Match}(s, t, a, \mu, \Xi) := L_i$;
    - Return;
- $\text{Match}(s, t, a, \mu, \Xi) := \emptyset$

Observe that $\text{Match}(s, t, a, \mu, \Xi) = \emptyset$ iff

for all subset $I \subseteq [k]$, ν exists such that $\max \vec{v}^\mu[I] = \max \vec{w}^\nu[I]$ $(\star)$

To verify $(\star)$, for each $i \in [k]$, we define an index set L_i such that a_i is the maximal element among L_i in $\vec{v}^\mu$. Formally, $L_i = \{k \mid a_k \leq a_i\}$. Furthermore,

Algorithm 2: Cover$(i, \vec{v}, W, L)$

```
Data: i ∈ L ⊆ [k], v⃗[i] = max v⃗^μ[L]
Result: TRUE iff for all i ∈ I ⊆ L, there exists some j such that
        v⃗[i] = max W[j][I]
begin
    I ⟵ L;
    while TRUE do
        S ⟵ 𝔖(I) ;
        if S = ∅ then
            Cover(i, v⃗, W, L) ⟵ FALSE;
            Return;
        else
            if i ∈ S then
                Cover(i, v⃗, W, L) ⟵ TRUE;
                Return;
            else
                I ⟵ I \ S;
```

for a given index set L with $a := \max \vec{v}^\mu[L]$, we define

$$\mathfrak{S}[L] := \{k \in L \mid \exists \nu.\, a = \vec{w}^\nu[k] = \max \vec{w}^\nu[L]\}.$$

Note that obviously $\mathfrak{S}[L] \subseteq L$.

Lemma 4. Given an index set $L \subseteq [n]$, let $\ell_M \in L$ be $\vec{v}^\mu[\ell_M] = \max \vec{v}^\mu[L]$. For all $I \subseteq L$ with $\mathfrak{S}[L] \cap I \neq \emptyset$ and $\ell_M \in I$, $\max \vec{v}^\mu[I] = \max \vec{w}^\nu[I]$.

Proof. By definition $\mathfrak{S}[L] := \{k \in L \mid \exists \nu.\, a = \vec{w}^\nu[k] = \max \vec{w}^\nu[L]\}$. As $\ell_M \in L$, $\max \vec{v}^\mu[I] = \vec{v}^\mu[\ell_M]$. It follows that there exists some ν and $k \in I \cap \mathfrak{S}[L]$ such that $\vec{v}^\mu[\ell_M] = \vec{w}^\nu[k]$, and $\vec{w}^\nu[k] = \max \vec{w}^\nu[L] \geq \max \vec{w}^\nu[I]$. Moreover, as $k \in I$, $\max \vec{w}^\nu[L] \leq \max \vec{w}^\nu[I]$. We then can conclude that $\max \vec{v}^\mu[I] = \max \vec{w}^\nu[I]$, which completes the proof. ◻

Proposition 4. [Correctness of Algorithm 2] Function Cover$(i, \vec{v}, W, L)$ returns TRUE iff for all I such $i \in I \subseteq L$, there exists some row of W indexed by ν (viz. $\vec{w}^\nu$) such that $\max \vec{v}[I] = \max \vec{w}^\nu[I]$.

Proof. Recall that, as the precondition of Cover$(i, \vec{v}, W, L)$, i satisfies that $\vec{v}[i]$ is the maximal element among $\vec{v}[I]$, i.e., for all $I \subseteq L$ with $i \in I$, $\max \vec{v}[I] = \max \vec{v}^\mu[L]$. Cover$(i, \vec{v}, W, L)$ first computes $\mathfrak{S}(I) \subseteq I$. Clearly, if $\mathfrak{S}(I) = \emptyset$, we can conclude that, for I, there is no ν such that $\max \vec{v}[I] = \max \vec{w}^\nu[I]$, hence the function returns FALSE. If, on the other hand, $\mathfrak{S}(I) \neq \emptyset$, we have the following two cases:

- $i \in \mathfrak{S}(I)$. Then clearly $\mathfrak{S}(I) \cap I \neq \emptyset$ as $i \in I$. By Lemma 4, we can conclude that some ν exists such that $\max \vec{v}[I] = \max \vec{w}^\nu[I]$.
- $i \notin \mathfrak{S}(I)$. Then by Lemma 4, we have that all I' with $\mathfrak{S}[I] \cap I' \neq \emptyset$ and $i \in I'$, $\max \vec{v}[I'] = \max \vec{w}^\nu[I']$. For I' such that $i \in I'$ but $\mathfrak{S}[I] \cap I' = \emptyset$, we have that $I' \subseteq I \setminus \mathfrak{S}[I]$, and the iteration (the while loop) will guarantee that, if the function returns TRUE, there exists some ν such that $\max \vec{v}[I'] = \max \vec{w}^\nu[I']$.

□

Proposition 5. [Correctness of Algorithm 1] The main procedure in Algorithm 1 terminates and, when it terminates, the output is $S/\sim_{FB,gbg}$.

Proof. Recall the notations introduced before, i.e., $\vec{v}^\mu$, $\vec{w}^\nu$ with respect to the fixed states s, t, action a, and partition Ξ. We only need to show that the **Match** function (from Algorithm 2) and the **Cover** (from Algorithm 2) faithfully check $(\star)$, and the rest follows standard argument for partition-refine algorithms. Note that each $I \subseteq [k]$ is verified under some $i \in I$ with L_i given in the **Match** function, and Proposition 4 shows that for such L_i, we can determine correctly whether it will invalidate the condition that $\vec{v}^\mu(L_i) = \vec{w}^\nu(L_i)$.

Termination of Algorithm 1 is straightforward. For Algorithm 2, we observe that when $S \neq \emptyset$, the size of I decreases and finally we will have $I = \{i\}$ for which the function terminates. On the other hand, when $S = \emptyset$, the function returns immediately. Hence the while loop will terminate in, at most, $|L|$ steps.

□

Complexity analysis. The partition-refinement takes at most $\mathcal{O}(|S|^2)$ steps. For each step, the computation time is dominated by Cover function in Algorithm 2. At mentioned earlier, it takes at most $\mathcal{O}(|L|) \leq \mathcal{O}(|S|)$ steps. Hence for Match function, it takes $\mathcal{O}(|S|^2 \cdot |A| \cdot |S|)$ steps. Wrapping up, the overall time is $\mathcal{O}(|S|^5|A|)$, which puts the bisimulation checking problem in (strongly) polynomial time.[2]

Example 2. We use the example FTS in Fig. 1 to illustrate some key steps of the algorithm. First of all, $S = \{s, t, s_i, t_i \mid i = 1, \cdots, 6\}$. Let's assume that the current partition $\Xi = \{\{s, t\}, \{s_i, t_i\}_{1 \leq i \leq 6}\}$. To check whether the equivalence relation induced by Ξ is a bisimulation, clearly we only need to verify whether the pair (s, t) satisfies necessary condition. To this end, as an example, we consider $s \xrightarrow{a} \frac{0.3}{s_1} + \frac{0.5}{s_2}$ and the candidate transitions from t are $t \xrightarrow{a} \frac{0.5}{t_1} + \frac{0.3}{t_2}$, $t \xrightarrow{a} \frac{0.3}{t_1} + \frac{0.5}{t_3}$, and $t \xrightarrow{a} \frac{0.5}{t_2} + \frac{0.3}{t_3}$. We then, following the **Match** function, obtain that

[2]We made no effort to optimize the algorithm, nor to provide a tightened analysis, as our main purpose is to show that the problem is in P.

- $\vec{v}^{\mu} = (0.3, 0.5, 0)$,
- $\vec{w}^1 = (0.5, 0.3, 0)$, $\vec{w}^2 = (0.3, 0, 0.5)$, $\vec{w}^3 = (0, 0.5, 0.3)$.

Here $k = 3$, and $W = \begin{bmatrix} 0.5 & 0.3 & 0 \\ 0.3 & 0 & 0.5 \\ 0 & 0.5 & 0.3 \end{bmatrix}$.

For illustration purpose, we check for $i = 1$ and thus, following the **Match** function, obtain that $L_i = \{1, 3\}$ and $\mathfrak{S}(L_i) = \{3\}$. Hence $Cover(1, \vec{v}^{\mu}, W, \{1, 3\})$ in Algorithm 2 is invoked. In this case, after one iteration, L is updated by $\{1\}$ and $\mathfrak{S}(L) = \{1\}$, and thus we have $i = 1 \in \mathfrak{S}(L)$, True is returned.

One could check further other transitions from s, i.e., $s \xrightarrow{a} \frac{0.5}{s_1} + \frac{0.3}{s_3}$ and $s \xrightarrow{a} \frac{0.3}{s_2} + \frac{0.5}{s_3}$, and readily conclude that indeed Ξ is stable, hence it is the after bisimulation quotient of the state space S.

6. Logical characterization

In this section, we embark upon the relationship between group-by-group bisimilarity and logics. More concretely, we will show that two states are group-by-group bisimilar iff they satisfy the same logical formulae.

First, we recall modal logics that are used to characterize bisimulations of Cao et al. [4]. Wu and Deng in [16] gave the following logic:

$$\varphi ::= \quad \top \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \langle a \rangle_p \varphi. \tag{2}$$

where $a \in A$ and $p \in [0, 1]$, and φ is a state formula that is interpreted on states. Under the assumption of image-finiteness, this logic can characterize the bisimularity of Cao et al. for a deterministic FTS.

For a nondeterministic FTS, Wu et al. in [17] gave the following two-sorted logic:

$$\begin{array}{lll} \varphi & ::= & \top \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \langle a \rangle \psi \\ \psi & ::= & \psi_1 \wedge \psi_2 \mid \neg\psi \mid [\varphi]_p \end{array} \tag{3}$$

where ψ is a distribution formula that is interpreted on distributions. Under the assumption of image-finiteness, this logic can characterize the bisimularity of Cao et al. for a nondeterministic FTS. For example, consider the FTS in Fig. 1, and a logical formula $\varphi = \langle a \rangle([\langle b \rangle[\top]_1]_{0.3} \wedge [\langle c \rangle[\top]_1]_{0.5})$. We have that $s \models \varphi$ but $t \not\models \varphi$. Hence, the bisimulation of Cao et al. will distinguish s and t.

In this paper, the FTS is nondeterministic. With a slight modification of the logic in (2), we obtain the following logic $\mathcal{L}$:

$$\varphi \quad ::= \quad \top \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \langle a \rangle_{[p_1, p_2]} \varphi \tag{4}$$

where $0 \leq p_1 \leq p_2 \leq 1$. This is the basic logic that we employ to establish the logical characterization of group-by-group fuzzy bisimilarity for an FTS.

Fix an FTS $(S, A, \rightarrow)$, the *satisfaction relation* is defined by

- $s \models_{\mathcal{L}} \top$, for any $s \in S$;
- $s \models_{\mathcal{L}} \varphi_1 \wedge \varphi_2$ iff $s \models \varphi_i$ for $i = 1, 2$;
- $s \models_{\mathcal{L}} \neg\varphi$ iff $s \not\models_{\mathcal{L}} \varphi$;
- $s \models_{\mathcal{L}} \langle a \rangle_{[p_1,p_2]}\varphi$ iff a transition $s \xrightarrow{a} \mu$ exists such that $p_1 \leq \mu([\![\varphi]\!]) \leq p_2$.

Let $Th_{\mathcal{L}}(s) = \{\varphi \in \mathcal{L} \mid s \models \varphi\}$ be the set of formulae that the state s satisfies.

Lemma 5. For any $\varphi \in \mathcal{L}$, $[\![\varphi]\!]$ is $\sim_{FB,gbg}$ closed.

Proof. It can be proven by structure induction on formulae, we only consider the case: $\psi = \langle a \rangle_{[p_1,p_2]}\varphi$. Let $s \sim_{FB,gbg} t$ and $s \models \langle a \rangle_{[p_1,p_2]}\varphi$. It suffices to prove that $t \models \langle a \rangle_{[p_1,p_2]}\varphi$. By induction, $[\![\varphi]\!]$ is $\sim_{FB,gbg}$ closed, which is the union of some equivalence classes of $\sim_{FB,gbg}$. Thus, there exists $\mathcal{G} \in 2^{S/\sim_{FB,gbg}}$ such that $\bigcup \mathcal{G} = [\![\varphi]\!]$. Since $s \models \langle a \rangle_{[p_1,p_2]}\varphi$, a transition $s \xrightarrow{a} \mu$ exists such that $p_1 \leq \mu([\![\varphi]\!]) \leq p_2$. That is, $p_1 \leq \mu(\bigcup \mathcal{G}) \leq p_2$. Further, by the definition of group-by-group fuzzy bisimulation, there exists $t \xrightarrow{a} \nu$ such that $\mu(\bigcup \mathcal{G}) = \nu(\bigcup \mathcal{G})$, i.e., $\mu([\![\varphi]\!]) = \nu([\![\varphi]\!])$. Hence, $p_1 \leq \nu([\![\varphi]\!]) \leq p_2$ and then $t \models \langle a \rangle_{[p_1,p_2]}\varphi$. □

Lemma 6. Let state space be finite and $R = \{(s,t) \mid Th_{\mathcal{L}}(s) = Th_{\mathcal{L}}(t)\}$. Then, for any equivalence class $[s_i]$ of R, there exists a formula $\varphi_i \in \mathcal{L}$ such that $[\![\varphi_i]\!] = [s_i]$.

Proof. Its proof can be found in the proof of Theorem 4.6 in [16]. □

Theorem 3. Let the set S be finite and $(S, A, \rightarrow)$ be an FTS. Then for any two states $s, t \in S$, $s \sim_{FB,gbg} t$ iff $Th_{\mathcal{L}}(s) = Th_{\mathcal{L}}(t)$.

Proof.

($\Longrightarrow$) First we show soundness, i.e.,

$$\forall s, t \in S.\ s \sim_{FB,gbg} t \Longrightarrow Th_{\mathcal{L}}(s) = Th_{\mathcal{L}}(t).$$

Let $s, t \in S$, $s \sim_{FB,gbg} t$ and ψ be a formula. We show that $s \models_{\mathcal{L}} \psi \Longleftrightarrow t \models_{\mathcal{L}} \psi$ by structural induction on ψ. The cases of $\top$, conjunction and negation are trivial. Now consider $\psi \equiv \langle a \rangle_{[p_1,p_2]}\varphi$.

By Lemma 5, $[\![\varphi]\!]$ is $\sim_{FB,gbg}$-closed. Further, since $\sim_{FB,gbg}$ is an equivalence relation by Proposition 2, $[\![\varphi]\!]$ is the union of some equivalence classes of $\sim_{FB,gbg}$ by Lemma 1 (i). Now let $s \models_{\mathcal{L}} \langle a \rangle_{[p_1,p_2]}\varphi$. Then a transition $s \xrightarrow{a} \mu$ exists such that $p_1 \leq \mu([\![\varphi]\!]) \leq p_2$. Since $s \sim_{FB,gbg} t$, there exists a transition $t \xrightarrow{a} \nu$ such that $\mu([\![\varphi]\!]) = \nu([\![\varphi]\!])$. As a result, $p_1 \leq \nu([\![\varphi]\!]) \leq p_2$. It follows that $t \models_{\mathcal{L}} \langle a \rangle_{[p_1,p_2]}\varphi$. Another direction holds since the symmetry of $\sim_{FB,gbg}$. Hence, $s \models_{\mathcal{L}} \psi \Longleftrightarrow t \models_{\mathcal{L}} \psi$ as desired.

($\Longleftarrow$). Let $R = \{(s,t) \mid Th_{\mathcal{L}}(s) = Th_{\mathcal{L}}(t)\}$. Obviously, R is an equivalence relation. It remains to prove that R is a group-by-group fuzzy bisimulation. Let $G = \{[s_i] \mid i \in I\}$ be the set of all equivalence classes of R. Assume by contrary that R is not a group-by-group fuzzy bisimulation. Then there exist $(s,t) \in R$, $\mathcal{G}_0 \in 2^{S/R}$ and some transition $s \xrightarrow{a} \mu$ such that for all transitions $t \xrightarrow{a} \nu$, $\mu(\bigcup \mathcal{G}_0) \neq \nu(\bigcup \mathcal{G}_0)$. For each $[s_i] \in \mathcal{G}_0$, Lemma 6 implies that a formula $\varphi_i \in \mathcal{L}$ exists such that $[\![\varphi_i]\!] = [s_i]$. Further, let $\varphi = \vee \varphi_i$ where $[\![\varphi_i]\!] = [s_i]$ and $[s_i] \in \mathcal{G}_0$. Then $\bigcup \mathcal{G}_0 = [\![\varphi]\!]$. Let $\mu([\![\varphi]\!]) = \mu(\bigcup \mathcal{G}_0) = p$. We have that $s \models \langle a \rangle_{[p,p]} \varphi$, but $t \not\models \langle a \rangle_{[p,p]} \varphi$, which contradicts $(s,t) \in R$. □

Example 3. Consider the FTS in Fig. 1 again, $s, t \models \langle a \rangle_{[0.3,0.5]} \langle \beta \rangle_{[1,1]} \top$ where $\beta \in \{b, c, d\}$. One of the benefits of a logical characterization is that we can conveniently detect two states which are *not* bisimilar. For example, if we adjust the central transition from s as $s \xrightarrow{a} \frac{0.5}{s_1} + \frac{0.4}{s_3}$, then we have that $s \models \langle a \rangle_{[0.4,0.4]} \langle d \rangle_{[1,1]} \top$ but $t \not\models \langle a \rangle_{[0.4,0.4]} \langle d \rangle_{[1,1]} \top$. Hence, s and t are not group-by-group fuzzy bisimilar.

Remark 2. *In probabilistic systems, the literature [1] (see Theorem 1, page 75) has established the logical characterization theorem of group-by-group probabilistic bisimilarity, which needs image-finite condition and the minimal probability assumption. While Theorem 3 in this paper does not need this condition and similar assumption. In addition, in order to characterize $\sim_{PB,gbg}$, the literature [1] first characterizes $\sim^i_{PB,gbg}$, $i \in \mathbb{N}$ that is a coarser equivalence relation and its limit is $\sim_{PB,gbg}$. While Theorem 3 in this paper adopts a direct approach-structural induction on formulae and the definition of group-by-group bisimulation.*

7. Conclusion and future work

In this paper, we have investigated a novel group-by-group fuzzy bisimulation for FTS, which is coarser than that in [4]. We provided both fixed point characterization and logical characterization for the group-by-group fuzzy bisimilarity, together with a polynomial-time algorithm to compute the group-by-group fuzzy bisimilarity.

As future work, two questions are worth studying. One is to study behavioral distance that can measure to what extent two states are group-by-group fuzzy bisimilar. The other is to consider coalgebraic theory of group-by-group fuzzy bisimulation.

Acknowledgment

H. Wu is supported by the National Natural Science Foundation of China (No. 61773019). T. Chen is supported by EPSRC grant (EP/P00430X/1), ARC Discovery Project (DP160101652, DP180100691), and NSFC grant (No. 61662035). T. Han is supported by EPSRC grant (EP/P015387/1). Y. Chen is supported by the National Natural Science Foundation of China (No. 61370100).

The authors would like to thank the anonymous reviewers for their invaluable suggestions and comments.

References

[1] M. Bernardo, Rocco De Nicola and M. Loreti, Revisiting bisimilarity and its modal logic for nondeterministic and probabilistic processes, Acta Informatica 52(2015) 61–106.

[2] Y. Z. Cao, G. Q. Chen and E. E. Kerre, Bisimulations for fuzzy transition systems, IEEE Transactions on Fuzzy Systems 19 (2010) 540-552.

[3] Y. Z. Cao and Y. Ezawa, Nondeterministic fuzzy automata, Information Sciences 191 (2012) 86-97.

[4] Y. Z. Cao, S. X. Sun, H. Wang and G. Q. Chen, A behavioral distance for fuzzy-transition systems, IEEE Transactions on Fuzzy Systems 21 (2013) 735-747.

[5] M. Ćirić, J. Ignjatović, N. Damljanović and M. Bašić, Bisimulations for fuzzy automata, Fuzzy Sets and Systems 186 (2012) 100-139.

[6] M. Ćirić, J. Ignjatović, I. Jančić and N. Damljanović, Computation of the greatest simulations and bisimulations between fuzzy automata, Fuzzy Sets and Systems 208 (2012) 22-42.

[7] N. Damljanović, M. Ćirić and J. Ignjatović, Bisimulations for weighted automata over an additively idempotent semiring, Theoretical Computer Science 534 (2014) 86-100.

[8] B. A. Davey and H. A. Priestley, Introduction to lattices and order, Cambridge, U. K.: Cambridge University. Press, 2002.

[9] W. L. Deng and D. W. Qiu, Supervisory control of fuzzy discrete event systems for simulation equivalence, IEEE Transactions on Fuzzy Systems 23 (2015) 178-192.

[10] T. F. Fan, Fuzzy bisimulation for Gödel logic, IEEE Transactions on Fuzzy Systems 23 (2015) 2387-2396.

[11] O. Kupferman and Y. Lustig, Latticed simulation relations and games, International Journal of Foundations of Computer Science 21 (2010) 167-189.

[12] Y. M. Li and L. J. Li, Model checking of linear-time properties based on possibility measure, IEEE Transactions on Fuzzy Systems 21 (5) (2013) 842-854.

[13] Y. M. Li and Z. Y. M, Quantitative computation tree logic model checking based on generalized possibility measures, IEEE Transactions on Fuzzy Systems 23 (6) (2015) 2034-2047.

[14] H. Y. Pan, Y. M. Li, Y. Z. Cao, M. Zhang and Z. Y. Ma, Model checking computation tree logic over finite lattice, Theoretical Computer Science 612 (2016) 45-62.

[15] H. Y. Pan, Y. M. Li, Y. Z. Cao and P. Li, Nondeterministic fuzzy automata with membership values in complete residuated lattices, International Journal of Approximate Reasoning 82 (2017) 22-38.

[16] H. Y. Wu and Y. X. Deng, Logical characterizations of simulation and bisimulation for fuzzy transition systems, Fuzzy Sets and Systems 301(2016) 19-36.

[17] H. Y. Wu, Y. X. Chen, T. M. Bu and Y. X. Deng, Algorithmic and logical characterizations of bisimulations for non-deterministic fuzzy transition systems, Fuzzy Sets and Systems, Fuzzy Sets and Systems 333(2018) 106-123.

[18] H. Y. Wu and Y. X. Deng, Distribution-based behavioural distance for nondeterministic fuzzy transition systems, IEEE Transactions on Fuzzy Systems, DOI: 10.1109/TFUZZ.2017.2670605.

[19] H. Y. Xing, Q. S. Zhang and K. S. Huang, Analysis and control of fuzzy discrete event systems using bisimulation equivalence, Theoretical Computer Science 456 (2012) 100-111.