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A NOTE ON A NONRESONANCE CONDITION AT ZERO FOR FIRST-ORDER PLANAR SYSTEMS

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Abstract. We introduce a Landesman-Lazer type nonresonance condition at zero for planar systems and discuss its rotational interpretation. We then show an application concerning multiplicity of T-periodic solutions to unforced Hamiltonian systems like

$$Ju' = \nabla H(t,u), \quad \nabla H(t,0) \equiv 0,$$

for which the nonlinearity is resonant both at zero and at infinity, refining and complementing some recent results.

1. Introduction

This note can be seen as a complement to some recent multiplicity results for periodic solutions of first order Hamiltonian systems of ODEs in the plane, wishing to fill in a little asymmetric gap in this kind of results. Our starting point is embodied by planar systems at resonance having the form

$$Ju' = F(u) + R(t,u), \quad t \in [0,T], \tag{1.1}$$

where J denotes the standard symplectic matrix. Here, resonance - always considered with respect to T-periodic boundary conditions - essentially arises when looking at a sublinear perturbation of a center-type dynamics ($Ju' = F(u)$) for which the minimal period on the orbits is *constant and equal to a submultiple of* T; nevertheless, with reference to the beginning of Section 2, the stronger assumption

$$F(u) = \nabla V(u) \quad \text{with } V \in \mathcal{P}$$

(roughly speaking, the principal part of the nonlinearity being a positively homogeneous gradient) is usually needed in order to formulate plain and readable results. Indeed, if $V \in \mathcal{P}$ the origin is an isochronous center for $Ju' = \nabla V(u)$, and the common period of all the solutions can be computed in an immediate way from the expression of V, namely

$$\tau_V = \int_0^{2\pi} \frac{d\theta}{2V(\cos\theta, \sin\theta)}.$$

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This setting, introduced in the paper [7], takes inspiration from scalar second order asymmetric equations and represents a possible extension of the concept of Dancer-Fučik spectrum to general planar systems.

We incidentally remark that similar considerations seem to be possible imposing that F is only positively 1-homogeneous, without being a gradient, provided that there exists a closed orbit for $Ju' = F(u)$. However, apart from its possible interest in the context of general homogeneous systems, this extension seems to be quite artificial, so that we prefer to set our results in the mentioned framework.

It is well known (for the precise statements, see [7]) that, under a resonance assumption, the existence of a T-periodic solution to (1.1) is not ensured. In the last years, several conditions were produced in order to overcome this problem and obtain the solvability of (1.1). Among these, we have both Landesman-Lazer [8] and Ahmad-Lazer-Paul [3] type ones, the former being particularly useful when employing topological methods, the latter being more suitable to be exploited in a variational setting. The common point between such conditions is that they are formulated (up to some few exceptions, mainly for Ahmad-Lazer-Paul type results) mostly at infinity, requiring a certain behavior of the perturbation $R(t, u)$ when $|u|$ becomes large. To the author's knowledge, much less has been said regarding existence conditions at zero, namely when $|u|$ is small.

Wishing to deal with this situation, it is clear that, since we are working with sublinear perturbations, $R(t, u)$ has to satisfy $R(t, 0) \equiv 0$, so that mere existence results will be useless in principle, as they could lead to find the trivial solution. However, the Landesman-Lazer condition has been likewise shown to be particularly effective [2, Section 5] in order to refine results of *multiplicity* of T-periodic solutions for unforced planar Hamiltonian systems like

$$Ju' = \nabla H(t, u), \quad \nabla H(t, 0) \equiv 0, \tag{1.2}$$

when $\nabla H(t, u)$ has different asymptotic expansions with positively homogeneous principal term at zero and at infinity - for some references about this kind of statements, we remind the reader, e.g., to the bibliography in [2]. In this setting, a successful way of obtaining multiple solutions is represented by the Poincaré-Birkhoff fixed point theorem, providing a number of fixed points of the Poincaré map associated with (1.2) which grows according to the gap between the integer parts of the number of revolutions made by "small" and "large" solutions around the origin, with precise nodal characterizations. As has been shown in [2], assuming a Landesman-Lazer condition at infinity turns out to be particularly useful when the principal term (at infinity) is resonant, making it possible to give finer estimates of the rotation number of large solutions, in principle "dangerously close" to an integer. One can then conclude the existence of multiple T-periodic solutions in larger number than predicted by rougher estimates (see [2, Theorem 5.2]).

The main goal of this paper is to extend this picture at zero, after having introduced a suitable Landesman-Lazer type condition. We observe that this way of proceeding in order to obtain multiplicity of T-periodic solutions to systems like (1.2), which works in the general positively homogeneous framework, turns to be fruitful in dimension two but finds more difficulty to be applied in higher dimension (even if a significant step in this direction seems to have been achieved in the recent work [10]). Alternatively, particularly in the asymptotically linear case

$$\nabla H(t, u) = B_0(t)u + R_0(t, u) \text{ at } 0, \quad \nabla H(t, u) = B_\infty(t)u + R_\infty(t, u) \text{ at } \infty,$$

where $R_0(t,u), R_\infty(t,u)$ are negligible with respect to $|u|$, Morse, Conley-Zehnder or Maslov index theory are quite successful in any dimension, possibly considered jointly with Landesman-Lazer or Ahmad-Lazer-Paul conditions (we refer, e.g., to [5, 13] and the references therein; we mention [15] as a bridge between Maslov index type results and Poincaré-Birkhoff type ones). However, one of their possible drawbacks is represented by the fact that the number of solutions found does not change significantly according to the gap between the behaviors at zero and infinity; moreover, in case the indexes at zero and infinity are the same - for instance, if the principal parts coincide, they seldom allow to infer existence.

It is looking at this second group of results that we find one of the few contributions, to the best of our knowledge, about a possible formulation of Landesman-Lazer type conditions at zero for second order problems [12] (to be compared with similar expressions at infinity, e.g., [14, 16]). The role of such conditions is therein to allow a precise computation of the critical groups of the action functional associated with the considered problem at the origin. However, also in this case the indexes at zero and infinity are required to be different, otherwise existence is not guaranteed.

In this article, adopting a qualitative approach in the plane, we thus try to make the situation for first order planar systems a little bit more symmetric, providing a nonresonance condition at zero which carries a suitable rotational characterization. Consequently, we will be able to complement some existing results of multiplicity of T-periodic solutions, taking into account resonant principal parts at zero; furthermore, we will be able to admit the same resonant expansion at zero and at infinity.

The rotational characterization will be obtained by directly estimating the winding number of small solutions, while multiplicity of solutions will follow from the Poincaré-Birkhoff fixed point theorem, having the possibility of exploiting a finer estimate of the rotations of "small" solutions. Section 2 is completed by few applications to scalar second order equations, while in Section 3 we will highlight the relationships between Landesman-Lazer type nonresonance conditions at zero and at infinity, through a suitable change of variables. Finally, we dedicate Section 4 to a final discussion about some complementary issues related to recent papers on planar systems.

2. A Landesman-Lazer condition at zero

Let us denote by J the standard 2×2 symplectic matrix, namely $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and by $\mathcal{P}$ the set of the C^1-functions $V : \mathbb{R}^2 \to \mathbb{R}$, with locally Lipschitz continuous gradient, which are positively homogeneous of degree 2 and positive, i.e.,

$$0 < V(\lambda u) = \lambda^2 V(u), \quad \lambda > 0,\ u \in \mathbb{R}^2 \setminus \{0\};$$

if we replace *positive* with *nonnegative* in this definition, we will speak about the class $\mathcal{P}^*$, as in [11]. We remind that, given $V \in \mathcal{P}$, all the solutions to $Ju' = \nabla V(u)$ are periodic with the same minimal period $\tau_V > 0$; moreover, fixed one of them, denoted by φ_V, all the other ones can be written as $u(t) = C\varphi_V(t+\omega)$, for suitable $C \geq 0$, $\omega \in [0, \tau_V[$. We choose φ_V in such a way that $V(\varphi_V(t)) \equiv 1/2$ (we recall that V is preserved along the solutions).

We are interested in the planar T-periodic boundary value problem

$$\begin{aligned} Ju' &= \nabla V(u) + R(t,u) \\ u(0) &= u(T), \end{aligned} \tag{2.1}$$

where $V \in \mathcal{P}$ and $R : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is an L^1-Carathéodory function, i.e.,

- $t \mapsto R(t,u)$ is measurable, for every $u \in \mathbb{R}^2$;
- $u \mapsto R(t,u)$ is continuous, for almost every $t \in [0,T]$;
- for every compact subset $\mathcal{K} \subset \mathbb{R}^2$, there exists $h_{\mathcal{K}} \in L^1(0,T)$ such that $|R(t,u)| \le h_{\mathcal{K}}(t)$, for almost every $t \in [0,T]$ and every $u \in \mathcal{K}$.

Wishing to study a resonant problem, we will assume that there exists a positive integer N such that

$$\tau_V = \frac{T}{N}, \tag{2.2}$$

according to the setting introduced in [7]. To guarantee the solvability of (2.1) when $R : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is sublinear at infinity - as is usual in the spirit of resonance - and satisfies some mild control from below (see condition (LL+$_\infty$) below), a planar version of the Landesman-Lazer condition was introduced in [8] and extensively discussed in [2, 8]. Its explicit expression reads as

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t, \lambda\varphi_V(t+\omega)) | \varphi_V(t+\omega)\rangle \, dt > 0, \tag{2.3}$$

for every $\theta \in [0,T]$. The effect of (2.3) is to force the angular coordinate (normalized by 2π) of "large solutions" to the differential equation in (2.1) not to display an integer gap between the values taken for $t = 0$ and $t = T$ (see also [2]). The solvability follows then from a standard application of the Poincaré-Bohl theorem, providing a fixed point of the Poincaré map associated with (2.1).

Our aim is now to introduce a counterpart of condition (2.3) at zero - namely, for $\lambda \to 0$, when considering perturbations $R(t,u)$ which are sublinear at zero. Precisely, we will always assume that, uniformly for almost every $t \in [0,T]$,

$$\lim_{|u|\to 0} \frac{R(t,u)}{|u|} = 0. \tag{2.4}$$

This immediately implies, in view of the continuity in the u-variable, that $R(t,0) = 0$ a.e., so that problem (2.1) has the trivial solution $u(t) \equiv 0$. As a consequence, in order to formulate a Landesman-Lazer condition at zero, the mere replacement of "$+\infty$" with "0" in condition (2.3) would be useless, since the left-hand side in (2.3) would vanish. For this reason and inspired also by [8, Section 8], it seems quite natural to divide the quantity appearing in (2.3) by some power of λ, in order to avoid that the integral vanishes. Precisely, we give the following definition.

Definition 2.1. Let $R(t,u)$ fulfill (2.4). We will say that $R(t,u)$ *satisfies (LL+$_0$) with respect to* $V \in \mathcal{P}$ if there exists $\alpha > 1$ such that the two following conditions are satisfied:

- there exist a neighborhood $\mathcal{U}$ of the origin and $\eta_0^+ \in L^1(0,T)$ such that

  $$\frac{\langle R(t,u)|u\rangle}{|u|^{\alpha+1}} \ge \eta_0^+(t), \tag{2.5}$$

  for almost every $t \in [0,T]$ and every $u \in \mathcal{U}$;

- for every $\theta \in [0,T]$,

$$\int_0^T \liminf_{(\lambda,\omega)\to(0^+,\theta)} \frac{\langle R(t,\lambda\varphi_V(t+\omega))|\varphi_V(t+\omega)\rangle}{\lambda^\alpha}\,dt > 0. \tag{2.6}$$

Conditions (2.5) and (2.6) have to be satisfied for the same value of α; the fact that $\alpha > 1$ comes from the sublinearity assumption (2.4). Moreover, we observe that (2.6) is weaker than a sign assumption like $\langle R(t,u)|u\rangle \geq h(t)$ with $\int_0^T h(t)\,dt > 0$, as is immediately seen. We prove the following rotational interpretation of (LL$+_0$).

Proposition 2.2. *Let $V \in \mathcal{P}$ satisfy* (2.2) *and let $R(t,u)$ be an L^1-Carathéodory function fulfilling* (2.4)*. Then, if $R(t,u)$ satisfies (LL$+_0$), there exists $0 < R_0 \ll 1$ such that every solution $u(t)$ to*

$$Ju' = \nabla V(u) + R(t,u),$$

with $|u(0)| = R_0$, fulfills

$$\mathrm{Rot}(u(t);[0,T]) > \frac{T}{\tau_V} = N.$$

In the statement, we have denoted by $\mathrm{Rot}(u(t);[0,T])$ the standard (clockwise) rotation number of the path $t \mapsto u(t)$ around the origin, in the interval $[0,T]$, namely

$$\mathrm{Rot}(u(t);[0,T]) = \frac{1}{2\pi}\int_0^T \frac{\langle Ju'(t)|u(t)\rangle}{|u(t)|^2}\,dt.$$

In the proof, we will also make use of the notion of modified rotation number (see [2, 18]) associated with $V \in \mathcal{P}$, given by

$$\mathrm{Rot}_V\,(u(t);[0,T]) = \frac{1}{\tau_V}\int_0^T \frac{\langle Ju'(t)|u(t)\rangle}{2V(u(t))}\,dt.$$

Proof of Proposition 2.2. By contradiction, assume that the thesis is not true: then there exists a sequence of functions $u_n(t)$, with $|u_n(0)| \to 0$, such that

$$Ju_n' = \nabla V(u_n) + R(t,u_n),$$

and

$$\mathrm{Rot}(u_n(t);[0,T]) \leq N. \tag{2.7}$$

First, we observe that, by Gronwall's lemma (see [8, Lemma 2.2]), one has that $\|u_n\|_{L^\infty(0,T)} \to 0$. Moreover, setting

$$v_n(t) = \frac{u_n(t)}{\|u_n\|_{L^\infty(0,T)}},$$

we have that $v_n(t)$ satisfies, for every n, the equation

$$Jv_n' = \nabla V(v_n) + \frac{R(t,u_n)}{\|u_n\|_{L^\infty(0,T)}}, \tag{2.8}$$

so that, thanks to (2.4), there exists $\zeta \in L^1(0,T)$ such that $|v_n'(t)| \leq \zeta(t)$. Hence, using Ascoli's theorem we have that there exists a nonzero $v \in C([0,T])$ such that $v_n \to v$ uniformly; more precisely, thanks to the Dunford-Pettis theorem, we can pass to the weak L^1-limit in (2.8) and infer that $v(t)$ satisfies $Jv' = \nabla V(v)$, implying that $v(t) = R\varphi_V(t+\theta)$ for suitable constants $R > 0$ and $\theta \in [0,\tau_V[$. Now we estimate the modified rotation number for $u_n(t)$: recalling that it coincides with the usual one when taking integer values (see, for instance, [2, Proposition 2.1]), and thanks to (2.7), we have

$$
\begin{aligned}
N &\geq \mathrm{Rot}_V(u_n(t);[0,T]) \\
&= \frac{1}{\tau_V}\int_0^T \frac{\langle \nabla V(u_n(t)) + R(t,u_n(t)) | u_n(t)\rangle}{2V(u_n(t))}\,dt \\
&= N + \int_0^T \frac{\langle R(t,u_n(t)) | u_n(t)\rangle}{2V(u_n(t))}\,dt.
\end{aligned}
$$

Hence,

$$
\int_0^T \frac{\langle R(t,u_n(t)) | u_n(t)\rangle}{2V(u_n(t))}\,dt \leq 0. \tag{2.9}
$$

We now aim at showing that (2.9) leads to a contradiction. To this end, we change variables setting $u_n(t) = r_n(t)\varphi_V(t+\omega_n(t))$, with $r_n(t) \geq 0$ and $\omega_n(0) \in [0,\tau_V[$, and observe that $\|u_n\|_{L^\infty(0,T)} \to 0$ implies that $r_n \to 0^+$ uniformly. On the other hand, since $v_n \to v$ uniformly, we have that

$$
\frac{r_n}{\|u_n\|_{L^\infty(0,T)}} \to R \tag{2.10}
$$

uniformly, and it can also be seen that $\omega_n \to \theta$ uniformly (cf. [8]). Now, with reference to (2.9) and using the 2-homogeneity of V, we have

$$
\int_0^T \frac{\langle R(t,r_n(t)\varphi_V(t+\omega_n(t))) | \varphi_V(t+\omega_n(t))\rangle}{r_n(t)}\,dt \leq 0
$$

for every n, and dividing by $\|u_n\|^{\alpha-1}_{L^\infty(0,T)}$ it follows that

$$
\int_0^T \frac{r_n(t)^{\alpha-1}}{\|u_n\|^{\alpha-1}_{L^\infty(0,T)}} \frac{\langle R(t,r_n(t)\varphi_V(t+\omega_n(t))) | \varphi_V(t+\omega_n(t))\rangle}{r_n(t)^\alpha}\,dt \leq 0.
$$

In view of (2.5), we can thus use Fatou's lemma to infer that

$$
\int_0^T \liminf_{n\to+\infty} \frac{\langle R(t,r_n(t)\varphi_V(t+\omega_n(t))) | \varphi_V(t+\omega_n(t))\rangle}{r_n(t)^\alpha}\,dt \leq 0,
$$

using standard properties of the inferior limit and recalling (2.10). Consequently, observing that the inferior limit grows when computed on subsequences, we obtain

$$
\int_0^T \liminf_{(\lambda,\omega)\to(0^+,\theta)} \frac{\langle R(t,\lambda\varphi_V(t+\omega)) | \varphi_V(t+\omega)\rangle}{\lambda^\alpha}\,dt \leq 0,
$$

against the assumption. □

As already remarked, it is clear that, in view of this rotational interpretation, condition (LL+$_0$) ensures the existence of a T-periodic solution via the Poincaré-Bohl theorem. However, without any other localization of the fixed point found, there could be the risk of recovering the trivial solution.

As a counterpart of assumption (LL+$_0$) with reversed signs, we give the following definition.

Definition 2.3. Let $R(t,u)$ fulfill (2.4). We will say that $R(t,u)$ *satisfies (LL-$_0$) with respect to* $V \in \mathcal{P}$ if there exists $\alpha > 1$ such that the two following conditions are satisfied:

- there exist a neighborhood $\mathcal{U}$ of the origin and $\eta_0^- \in L^1(0,T)$ such that
$$\frac{\langle R(t,u)|u\rangle}{|u|^{\alpha+1}} \le \eta_0^-(t),$$
for almost every $t \in [0,T]$ and every $u \in \mathcal{U}$;
- for every $\theta \in [0,T]$,
$$\int_0^T \limsup_{(\lambda,\omega)\to(0^+,\theta)} \frac{\langle R(t,\lambda\varphi_V(t+\omega))|\varphi_V(t+\omega)\rangle}{\lambda^\alpha}\,dt < 0.$$

The following statement can be proved analogously to Proposition 2.2.

Proposition 2.4. *Let $V \in \mathcal{P}$ satisfy* (2.2) *and let $R(t,u)$ be an L^1-Carathéodory function fulfilling* (2.4)*. Then, if $R(t,u)$ satisfies (LL-$_0$), there exists $0 < R_0 \ll 1$ such that every solution $u(t)$ to*
$$Ju' = \nabla V(u) + R(t,u),$$
with $|u(0)| = R_0$, is such that
$$\mathrm{Rot}(u(t);[0,T]) < \frac{T}{\tau_V} = N.$$

With these preliminaries, we can give a slightly more symmetric version of the multiplicity results in [2] for unforced planar Hamiltonian systems with positively homogeneous principal part, by possibly considering a resonant situation at zero. Once taken into account Propositions 2.2 and 2.4, the proof is the same as in [2], so that we will only briefly sketch the reasoning leading to the conclusion. For the sake of clarity, we recall the version of the Landesman-Lazer conditions at infinity which will be used, together with the needed controls in order for the integrals to make sense:

(LL+$_\infty$) for every $\theta \in [0,T]$,
$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t,\lambda\varphi_V(t+\omega))|\varphi_V(t+\omega)\rangle\,dt > 0,$$
assuming that there exists $\eta_\infty^+ \in L^1(0,T)$ such that for almost every $t \in [0,T]$, for every $u \in \mathbb{R}^2$ with $|u| \le 1$ and every $\lambda > 1$,
$$\langle R(t,\lambda u)|u\rangle \ge \eta_\infty^+(t);$$

(LL-$_\infty$) for every $\theta \in [0,T]$,
$$\int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\theta)} \langle R(t,\lambda\varphi_V(t+\omega))|\varphi_V(t+\omega)\rangle\,dt < 0,$$
assuming that there exists $\eta_\infty^- \in L^1(0,T)$ such that for almost every $t \in [0,T]$, for every $u \in \mathbb{R}^2$ with $|u| \le 1$ and every $\lambda > 1$,
$$\langle R(t,\lambda u)|u\rangle \le \eta_\infty^-(t).$$

We first give the statement for a common expansion at zero and infinity.

Proposition 2.5. *Let $H : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be differentiable in the second variable, with $\nabla H(t,u)$ an L^1-Carathéodory function, and assume that the uniqueness for the solutions to the Cauchy problems associated with*

$$Ju' = \nabla H(t,u) \tag{2.11}$$

is guaranteed. Moreover, let $\nabla H(t,0) \equiv 0$ and assume that there exist a positive integer k and $V \in \mathcal{P}$ in such a way that

$$\frac{T}{\tau_V} = k,$$

and, setting

$$R(t,u) = \nabla H(t,u) - \nabla V(u),$$

it holds

$$\lim_{|u|\to 0} \frac{R(t,u)}{|u|} = 0, \quad \lim_{|u|\to +\infty} \frac{R(t,u)}{|u|} = 0.$$

Finally, suppose that

$$R(t,u) \text{ satisfies } (LL+_0) \text{ and } (LL\text{-}_\infty) \text{ with respect to } V.$$

Then, there exist at least two nontrivial T-periodic solutions $u_1(t), u_2(t)$ to (2.11) such that

$$\mathrm{Rot}\,(u_1(t);[0,T]) = \mathrm{Rot}\,(u_2(t);[0,T]) = k.$$

Proof. It suffices to observe that, due to the Landesman-Lazer assumptions (cf. Proposition 2.2 and [2, Proposition 4.1]), there exist $0 < r < R$ such that

$$|u(0)| = r \quad \Rightarrow \quad \mathrm{Rot}(u(t);[0,T]) > k$$

and

$$|u(0)| = R \quad \Rightarrow \quad \mathrm{Rot}(u(t);[0,T]) < k.$$

The conclusion follows then from the Poincaré-Birkhoff fixed point theorem. □

Of course, in presence of different asymptotic expansions at zero and infinity, a larger gap between the minimal periods associated with the principal terms corresponds to a higher number of solutions. We briefly summarize this into the following proposition.

Proposition 2.6. *Let $H : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be differentiable in the second variable, with $\nabla H(t,u)$ an L^1-Carathéodory function, and assume that the uniqueness for the solutions to the Cauchy problems associated with (2.11) is guaranteed. Moreover, let $\nabla H(t,0) \equiv 0$ and assume that there exist V_0, $V_\infty \in \mathcal{P}$, with*

$$\nabla H(t,u) = \nabla V_0(u) + o(|u|), \quad |u| \to 0, \tag{2.12}$$

$$\nabla H(t,u) = \nabla V_\infty(u) + o(|u|), \quad |u| \to \infty, \tag{2.13}$$

such that it holds

$$\frac{T}{\tau_{V_0}} = k_0, \quad \frac{T}{\tau_{V_\infty}} = k_\infty,$$

for some integers $k_0 \geq k_\infty$. Then, setting

$$R_0(t,u) = \nabla H(t,u) - \nabla V_0(u) \quad \text{and} \quad R_\infty(t,u) = \nabla H(t,u) - \nabla V_\infty(u), \tag{2.14}$$

if

$$R_0(t,u) \text{ satisfies } (LL+_0) \text{ with respect to } V_0,$$

and

$$R_\infty(t,u) \text{ satisfies (LL-}_\infty\text{) with respect to } V_\infty,$$

for every integer $k \in [k_\infty, k_0]$ there exist two T-periodic solutions $u_{1,k}(t), u_{2,k}(t)$ to (2.11) such that

$$\mathrm{Rot}(u_{1,k}(t);[0,T]) = \mathrm{Rot}(u_{2,k}(t);[0,T]) = k.$$

Obviously, one can obtain the same result if $R_0(t,u)$ satisfies (LL-$_0$) and $R_\infty(t,u)$ satisfies (LL+$_\infty$), if $k_0 \le k_\infty$.

Remark 2.7. The choice of T-periodic boundary conditions is not essential in our results. In the paper [4], resonant problems with general homogeneous boundary conditions have been considered, and corresponding Landesman-Lazer type assumptions at infinity, carrying a suitable rotational characterization, were introduced. For example, in the case of a Sturm-Liouville boundary value problem like

$$\begin{gathered} Ju' = \nabla V(u) + R(t,u) \\ u(0) \in l_S, \quad u(T) \in l_A, \end{gathered}$$

where l_S, l_A are two lines through the origin, it was observed that only two different values of θ have to be considered when formulating the Landesman-Lazer conditions, namely the values $\theta_1, \theta_2 \in [0, \tau_V[$ such that $\varphi_V(\theta_1), \varphi_V(\theta_2) \in l_S$. In a similar way, one can give analogous Landesman-Lazer conditions at zero, by considering (LL+$_0$) or (LL-$_0$) only for $\theta = \theta_1, \theta = \theta_2$ (we remind the reader to [4, Remark 2.2]); we omit the details for briefness. In this way, one can easily derive multiplicity of solutions satisfying prescribed homogeneous boundary conditions when both the principal terms at zero and infinity are resonant, via a shooting argument. Right because of this technique of proof, we notice that in this case the nonlinearity is not required to be of gradient type (as instead is needed for the Poincaré-Birkhoff theorem).

We conclude this section commenting briefly on condition (2.6), with particular attention to the second order case. As a first remark, we observe that (2.6) can be satisfied, for instance, by functions with indefinite weight like

$$R(t,u) = a(t)|u|^\beta u, \quad u \approx 0,$$

where $a(t)$ changes sign at least once in $[0,T]$ and $\beta > 0$; once the form of $\varphi_V(t)$ is known, it is not difficult to assume conditions on $a(t)$ to the purpose (to make a trivial example, $\int_0^T a(t)\,dt > 0$ is sufficient if $\varphi_V(t)$ describes a circumference in the plane).

Dealing, in particular, with the second-order asymmetric equation

$$x'' + \mu x^+ - \nu x^- + r(t,x) = 0, \quad \mu, \nu > 0,$$

where $r : [0,T]\times\mathbb{R} \to \mathbb{R}$ satisfies the sublinearity assumption $\lim_{|x|\to 0} r(t,x)/x = 0$, condition (2.6) reads as

$$\int_0^T \liminf_{(\lambda,\omega)\to(0^+,\theta)} \frac{r(t,\lambda\phi(t+\omega))\phi(t+\omega)}{\lambda^\alpha}\,dt > 0, \tag{2.15}$$

where

$$\phi(t) = \begin{cases} \frac{1}{\sqrt{\mu}}\sin(\sqrt{\mu}t) & \text{if } t \in [0, \frac{\pi}{\sqrt{\mu}}] \\ \frac{1}{\sqrt{\nu}}\sin\Big(\sqrt{\nu}\Big(\frac{\pi}{\sqrt{\mu}} - t\Big)\Big) & \text{if } t \in [\frac{\pi}{\sqrt{\mu}}, \frac{T}{N}], \end{cases}$$

and

$$\varphi_V(t) = (\phi(t), \phi'(t)), \quad \tau_V = \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} = \frac{T}{N}. \tag{2.16}$$

Exploiting the properties of inferior limits, we can easily obtain that (2.15) is equivalent to

$$\begin{aligned} &\int_{\{\phi(\cdot+\theta)>0\}} \Big(\liminf_{x\to 0^+} \frac{r(t,x)}{|x|^\alpha}\Big) |\phi(t+\theta)|^{\alpha+1}\, dt \\ &- \int_{\{\phi(\cdot+\theta)<0\}} \Big(\limsup_{x\to 0^-} \frac{r(t,x)}{|x|^\alpha}\Big) |\phi(t+\theta)|^{\alpha+1}\, dt > 0, \end{aligned}$$

recovering the same condition used in [12]. Incidentally, we notice that, with reference to the form of the usual conditions at infinity, "0^+" and "0^-" replace "$+\infty$" and "$-\infty$", respectively.

As an example, if $r(t,x) \approx x^{2j+1}$ near 0, with j a positive integer, the condition is always satisfied for $\alpha = 2j+1$, since both the summands in the above inequality are positive. This is indeed intuitive, because the contribution given by $r(t,x)$ to the angular speed of the solutions in the phase plane is here sign-defined (in clockwise sense). More in general, a similar situation occurs if $r_x(t,0) > 0$, which guarantees the validity of a similar sign condition. Of course, on the other hand, condition (2.6) cannot be satisfied by a nonlinearity which goes to zero faster than any power of x, like, for instance, $r(t,x) \approx \exp(-1/|x|)$.

If a weight is present in front of the power, the situation can be of course less easy. For instance, for the equation

$$x'' + \mu x^+ - \nu x^- + a(t)x^{2j+1} = 0,$$

where j is a positive integer and (2.16) is satisfied, setting $\alpha = 2j+1$ condition (2.6) reads as

$$\begin{aligned} &\int_{\{\phi(\cdot+\theta)>0\}} a(t)|\phi(t+\theta)|^{2j+2}\, dt + \int_{\{\phi(\cdot+\theta)<0\}} a(t)|\phi(t+\theta)|^{2j+2}\, dt \\ &= \int_0^T a(t)|\phi(t+\theta)|^{2j+2}\, dt > 0, \end{aligned}$$

for every $\theta > 0$. We notice that such a condition can be fulfilled by a sign-indefinite weight $a(t)$ (up to suitably balancing the positive and the negative contribution in the integral) and, in the context of indefinite nonlinearities, it was used, e.g., in [17], to guarantee the existence of an entire solution to a Schrödinger equation at resonance.

3. A change of variables: analogies between zero and infinity

We are now going to introduce a change of variables highlighting the relationships between the Landesman-Lazer conditions (LL$+_0$) and (LL$+_\infty$) exploited in the previous section. In order to make the situation more symmetric and the computations homogeneous, we adopt slightly different notation and formulations than in Section 2, not affecting, however, the validity of the previous results. We thus focus on the following two conditions:

(LL$+_0$)$_\alpha$ there exists $\alpha > 1$ such that, for every $\theta \in [0,T]$,

$$\int_0^T \liminf_{(\mu,\omega)\to(0^+,\theta)} \frac{\langle R(t,\mu\varphi_V(t+\omega)) | \varphi_V(t+\omega)\rangle}{\mu^\alpha}\, dt > 0, \tag{3.1}$$

provided that there exist a neighborhood $\mathcal{U}$ of the origin and $\eta_0^+ \in L^1(0,T)$ such that
$$\frac{\langle R(t,u)|u\rangle}{|u|^{\alpha+1}} \geq \eta_0^+(t), \tag{3.2}$$
for almost every $t \in [0,T]$ and every $u \in \mathcal{U}$;

$(\mathrm{LL}+_\infty)_k$ there exists $k \geq 0$ such that, for every $\theta \in [0,T]$,
$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \lambda^k \langle R(t, \lambda\varphi_V(t+\omega))|\varphi_V(t+\omega)\rangle \, dt > 0, \tag{3.3}$$
provided that there exists $\eta_\infty^+ \in L^1(0,T)$ such that for almost every $t \in [0,T]$, for every $u \in \mathbb{R}^2$ with $|u| \leq 1$ and every $\lambda > 1$, it holds
$$\lambda^k \langle R(t,\lambda u)|u\rangle \geq \eta_\infty^+(t). \tag{3.4}$$

We remark that, for $k = 0$, condition $(\mathrm{LL}+_\infty)_k$ turns into the usual Landesman-Lazer condition introduced in [8]; observe also that the rotational effect recalled in the previous section is not affected by the value of k.

To analyze the link between these two conditions, let us first consider the system
$$Ju' = \nabla V(u) + R(t,u), \tag{3.5}$$
where we assume that $R(t,0) \equiv 0$, as previously. By the uniqueness of the solutions to the associated Cauchy problems, no nontrivial solutions to (3.5) can reach the origin in a finite time. Thus, for $\beta > 1$ (and $u \in \mathbb{R}^2 \setminus \{0\}$), we can perform the change of variables
$$w = \frac{u}{V(u)^{\beta/2}} \tag{3.6}$$
(observe that the denominator only vanishes in $u = 0$). Notice that (3.6) leaves the set $\{u \mid V(u) = 1\}$ invariant and does not rescale 1-homogeneously when applied to λu.

Denoting by $\odot$ one between 0^+ and $+\infty$, and by $\odot^{-1}$ its reciprocal in extended sense, i.e. $1/0^+ = +\infty$ and $1/+\infty = 0^+$, in view of the fact that $V(u)$ is positively 2-homogeneous and $\beta > 1$ we first see that
$$|w| \to \odot \iff |u| \to \odot^{-1}. \tag{3.7}$$
Moreover, we immediately derive the relations
$$V(w) = \frac{1}{V(u)^{\beta-1}}, \quad V(u) = \frac{1}{V(w)^{\frac{1}{\beta-1}}}, \quad u = \frac{w}{V(w)^{\frac{\beta}{2(\beta-1)}}}\,.$$
Recalling now that, if $u(t)$ is a nontrivial solution to (3.5), then $w(t)$ given by (3.6) is everywhere well defined and differentiable, and using that
$$\nabla(V(u)^{\beta/2}) = \frac{\beta}{2} V(u)^{\frac{\beta-2}{2}} \nabla V(u),$$
we can compute the differential system corresponding to (3.5) in the new variable w, obtaining
$$Jw' = \nabla V(w) + \tilde{R}(t,w) - \frac{\beta}{2} V(w)^{-1} \langle J\nabla V(w)|\tilde{R}(t,w)\rangle Jw,$$
where we have set
$$\tilde{R}(t,w) = R\Big(t, \frac{w}{V(w)^{\frac{\beta}{2(\beta-1)}}}\Big) V(w)^{\frac{\beta}{2(\beta-1)}} = \frac{1}{V(u)^{\beta/2}} R(t,u).$$

We thus notice that the new system has the form

$$Jw' = \nabla V(w) + \tilde{R}(t,w) + S(t,w)Jw, \tag{3.8}$$

where the last term, being parallel to Jw, only influences the radial component of the solutions.

We now aim at showing that, for suitable choices of k and α, there is a complete equivalence between condition $(\mathrm{LL}+_0)_\alpha$ for system (3.5) (resp. (3.8)) and condition $(\mathrm{LL}+_\infty)_k$ for system (3.8) (resp. (3.5)), based on the fact that such conditions only act on the angular component of the solutions.

Keeping the same notation as before for $\odot$, we preliminarily observe that

$$\frac{\tilde{R}(t,w)}{|w|} = \frac{R(t,u)}{|w|V(u)^{\beta/2}} = \frac{R(t,u)}{|u|},$$

so that, in view of (3.7),

$$\lim_{|u|\to\odot} \frac{R(t,u)}{|u|} = 0 \quad \Longleftrightarrow \quad \lim_{|w|\to\odot^{-1}} \frac{\tilde{R}(t,w)}{|w|} = 0.$$

We now check the equivalence between (3.1) and (3.3).

Assume first that $(\mathrm{LL}+_\infty)_k$ is satisfied for (3.8), namely there exists $k \geq 0$ such that, for every $\theta \in [0,T]$, $\tilde{R}(t,w)$ satisfies

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \lambda^k \langle \tilde{R}(t,\lambda\varphi_V(t+\omega)) | \varphi_V(t+\omega)\rangle \, dt > 0 \tag{3.9}$$

(since the term coming from $S(t,w)Jw$ clearly vanishes). Writing the explicit expression of $\tilde{R}(t,w)$, using the positive 2-homogeneity of $V(u)$ and the fact that $V(\varphi_V(t)) \equiv 1/2$, we have that (3.9) is equivalent to

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \lambda^k \Big\langle R\Big(t, \frac{\lambda\varphi_V(t+\omega)}{\lambda^{\frac{2\beta}{2(\beta-1)}}(1/2)^{\frac{\beta}{2(\beta-1)}}}\Big) \lambda^{\frac{\beta}{\beta-1}} (1/2)^{\frac{\beta}{2(\beta-1)}} \Big| \varphi_V(t+\omega)\Big\rangle \, dt > 0,$$

so that

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \lambda^{\frac{\beta}{\beta-1}+k} \Big\langle R\Big(t, \big(\frac{2^{\beta/2}}{\lambda}\big)^{\frac{1}{\beta-1}} \varphi_V(t+\omega)\Big) \Big| \varphi_V(t+\omega)\Big\rangle \, dt > 0.$$

Hence, setting

$$\mu = \Big(\frac{2^{\beta/2}}{\lambda}\Big)^{\frac{1}{\beta-1}},$$

we obtain

$$\int_0^T \liminf_{(\mu,\omega)\to(0^+,\theta)} \frac{\langle R\left(t,\mu\varphi_V(t+\omega)\right) | \varphi_V(t+\omega)\rangle}{\mu^{\beta+k\beta-k}} \, dt > 0,$$

after having observed that we can omit multiplicative positive constants when considering the sign of this quantity. Then, given $\alpha > 1$, condition $(\mathrm{LL}+_\infty)_k$ for system (3.8) and condition $(\mathrm{LL}+_0)_\alpha$ for system (3.5) are equivalent through the choice

$$\beta = \frac{\alpha+k}{1+k} > 1. \tag{3.10}$$

Conversely, let us assume condition $(\mathrm{LL}+_0)_\alpha$ for system (3.8), namely there exists $\alpha > 1$ such that, for every $\theta \in [0,T]$, it holds

$$\int_0^T \liminf_{(\mu,\omega)\to(0^+,\theta)} \frac{\langle \tilde{R}(t,\mu\varphi_V(t+\omega)) | \varphi_V(t+\omega)\rangle}{\mu^\alpha} \, dt > 0.$$

Performing the same computations as before, we have that this implies

$$\int_0^T \liminf_{(\mu,\omega)\to(0^+,\theta)} \mu^{\frac{\beta}{\beta-1}-\alpha}\Big\langle R\Big(t,\big(\frac{2^{\beta/2}}{\mu}\big)^{\frac{1}{\beta-1}}\varphi_V(t+\omega)\Big)\Big|\varphi_V(t+\omega)\Big\rangle\,dt>0,$$

and with the position

$$\lambda=\Big(\frac{2^{\beta/2}}{\mu}\Big)^{\frac{1}{\beta-1}}$$

we obtain, omitting again superflous multiplicative constants,

$$\int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\theta)} \frac{\langle R\left(t,\lambda\varphi_V(t+\omega)\right)|\varphi_V(t+\omega)\rangle}{\lambda^{\beta+\alpha-\alpha\beta}}\,dt>0.$$

Then, given $k>0$, condition $(\mathrm{LL}+_0)_\alpha$ for system (3.8) and condition $(\mathrm{LL}+_\infty)_k$ for system (3.5) are equivalent through the choice

$$\beta=\frac{\alpha+k}{\alpha-1}>1. \tag{3.11}$$

We finally have to check that the controls from below make possible to apply Fatou's lemma after the change of variables. However, this is quite immediate: assume for instance $(\mathrm{LL}+_\infty)_k$ for system (3.5), with the aim of proving Proposition 2.2 for system (3.8). After computations similar to the ones appearing in the proof of Proposition 2.2, we obtain the inequality

$$\int_0^T \frac{r_n(t)^{\alpha-1}}{\|u_n\|^{\alpha-1}_{L^\infty(0,T)}}\frac{\langle\tilde R(t,r_n(t)\varphi_V(t+\omega_n(t)))|\varphi_V(t+\omega_n(t))\rangle}{r_n(t)^\alpha}\,dt\le 0,$$

where $r_n\to 0^+$ uniformly and $\omega_n\to\theta$ uniformly. In view of the previous computations, this gives

$$\int_0^T \frac{r_n(t)^{\alpha-1}}{\|u_n\|^{\alpha-1}_{L^\infty(0,T)}}\rho_n(t)^{\alpha\beta-\alpha-\beta}\langle R\left(t,\rho_n(t)\varphi_V(t+\omega)\right)|\varphi_V(t+\omega)\rangle\,dt\le 0,$$

where

$$\rho_n(t)\approx\frac{1}{r_n(t)^{\frac{1}{\beta-1}}}$$

(up to multiplicative constants), so that $\rho_n\to+\infty$ uniformly. Using (3.11), this is equivalent to

$$\int_0^T \frac{r_n(t)^{\alpha-1}}{\|u_n\|^{\alpha-1}_{L^\infty(0,T)}}\rho_n(t)^{k}\langle R\left(t,\rho_n(t)\varphi_V(t+\omega)\right)|\varphi_V(t+\omega)\rangle\,dt> 0.$$

The possibility of applying Fatou's lemma now follows in a standard way, since the ratio under the integral sign converges to R in view of (2.10), and the reminder is bounded from below in view of (3.4). This also shows that the use of (3.6) may be an alternative strategy to show the validity of the rotational interpretation of $(\mathrm{LL}+_0)_\alpha$, but to make a better comparison with the existing literature we have preferred to show explicitly the computations in the proof of Proposition 2.2.

Remark 3.1. It is clear that all the discussion could be carried out, in the same way, for conditions $(\mathrm{LL}\text{-}_0)_\alpha$ and $(\mathrm{LL}\text{-}_\infty)_k$. We omit any other detail for briefness.

Remark 3.2. The change of variables (3.6) suggests that analogous versions of recent results of existence and multiplicity of harmonic and subharmonic solutions formulated at infinity, possibly with different growth than asymptotically positively homogeneous (like, for instance, [1]), could be easily stated at zero only at the expense of suitably translating the assumptions on the nonlinearity.

4. Final remarks

In this last section we collect some final observations, possibly related to little improvements of known results through the use of condition ($\mathrm{LL}+_0$).

1. *Existence of subharmonics.* In the spirit of [2], we can deal with the existence of subharmonic solutions for (2.11), assuming that $H(t,u)$ is defined on $\mathbb{R}\times\mathbb{R}^2$ and T-periodic in the first variable. As usual in this setting, we say that an mT-periodic function $u(t)$ solving (2.11) is a *subharmonic of order* m - where m is an integer greater than or equal to 2 - if mT is its least period in the class of the integer multiples of T, i.e., $u(t)$ is not lT-periodic for any integer $l < m$. Notice that, if $\mathrm{Rot}\,(u(t);[0,mT]) = k$, then $u(t)$ is a subharmonic of order m whenever m and k are relatively prime integers, namely their greatest common divisor is 1.

Proceeding very similarly as in [2], to which we refer for further details, we can prove, for instance, the validity of the following.

Proposition 4.1. *Let $H:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ be differentiable in the second variable, with $\nabla H(t,u)$ an L^1-Carathéodory function, and assume that the uniqueness for the solutions of the Cauchy problems associated with (2.11) is guaranteed. Moreover, assume* (2.12) *and* (2.13)*, and suppose that*

$$\frac{T}{\tau_{V_\infty}} = \frac{T}{\tau_{V_0}}.$$

Using the same notation as in (2.14) *for $R_0(t,u)$ and $R_\infty(t,u)$, assume finally that*

$$R_0(t,u) \text{ satisfies } (LL\text{-}_0) \text{ with respect to } V_0,$$
$$R_\infty(t,u) \text{ satisfies } (LL+_\infty) \text{ with respect to } V_\infty.$$

Then, for every positive integer r, there exists an integer $m^*(r)$ such that, for every $m \ge m^*(r)$, there exist $2r$ subharmonics of order m solving (2.11).

Of course, a symmetric statement can be obtained exchanging the considered Landesman-Lazer conditions at zero and infinity.

Proof. The proof can be obtained similarly as in [2, Theorem 5.5], observing that, by virtue of the hypotheses, there is a gap in the rotation numbers at zero and infinity:

$$\mathrm{Rot}_0(u(t);[0,T]) < \frac{T}{\tau_{V_0}} \quad \text{(at zero)},$$
$$\mathrm{Rot}_\infty(u(t);[0,T]) > \frac{T}{\tau_{V_\infty}} \quad \text{(at infinity)}.$$

One can now find subharmonics of order m making k turns around the origin whenever there exist two integers k and m, relatively prime, such that

$$\mathrm{Rot}_0\,(u(t);[0,T]) < \frac{k}{m} < \mathrm{Rot}_\infty\,(u(t);[0,T]).$$

We refer to [2, Theorem 5.5] for further details. $\square$

2. *Comparison with different unperturbed systems.* Condition (LL$+_0$) can also be formulated in the framework of more general positively homogeneous comparison systems at zero and infinity, like

$$Ju' = \gamma_0(t)\nabla V_0(u), \quad Ju' = \gamma_\infty(t)\nabla V_\infty(u),$$

where γ_0, γ_∞ are positive functions with $\frac{1}{T}\int_0^T \gamma_0(s)\,ds = 1$, $\frac{1}{T}\int_0^T \gamma_\infty(s)\,ds = 1$. For the first system, in particular, (2.6) takes then the form

$$\int_0^T \liminf_{(\lambda,\omega)\to(0^+,\theta)} \frac{\langle R(t,\lambda\varphi_V(\Gamma_0(t)+\omega))|\varphi_V(\Gamma_0(t)+\omega)\rangle}{\lambda^\alpha}\,dt > 0,$$

where $\Gamma_0(t) = \int_0^t \gamma_0(s)\,ds$. The influence of such an assumption on the rotation number of small solutions can be shown similarly as in the case when $\gamma_0 \equiv 1$.

On the other hand, one could also consider nonlinearities whose principal term belongs to the class $\mathcal{P}^*$, namely it vanishes on some lines (see [11]). In this case, fixed $V \in \mathcal{P}^*$, the usual Landesman-Lazer condition at infinity is given by:

- for every $\xi \in \mathbb{S}^1$ satisfying $V(\xi) = 0$,

$$\int_0^T \liminf_{(\lambda,\eta)\to(+\infty,\xi)} \langle R(t,\lambda\eta)|\eta\rangle\,dt > 0$$

  (actually, it could also be formulated in a slightly weaker form, with $F(t,u) = \nabla V(u)+R(t,u)$ replacing $R(t,u)$, as seen in [11]; we omit to write the analogous of (2.5) and further details for the sake of briefness).

The counterpart of such an assumption at zero would be the following:

- there exists $\alpha > 1$ such that, for every $\xi \in \mathbb{S}^1$ satisfying $V(\xi) = 0$, it holds

$$\int_0^T \liminf_{(\lambda,\eta)\to(0^+,\xi)} \frac{\langle R(t,\lambda\eta)|\eta\rangle}{\lambda^\alpha}\,dt > 0.$$

Notice that, for the way it is used in the proof, in condition (2.6) it is not necessary that α is independent of θ (in this case, the controls from below and above required to apply Fatou's lemma have to change according to the value of θ), even if in the applications this is usually fulfilled. Just to give an example, in the framework of the class $\mathcal{P}^*$, let $T = 2\pi$ and consider a Hamiltonian $V \in \mathcal{P}^*$ which vanishes in the points $(0,1)$ and $(1,0)$. If, for instance,

$$R(t,x,y) = (x^2y + |\sin t|\sqrt{x^4+y^4}, -x^3+y^3),$$

one has

$$\langle R(t,\lambda x,\lambda y)|(x,y)\rangle = \lambda^2|\sin t|x\sqrt{x^4+y^4} + \lambda^3y^4. \tag{4.1}$$

Now, if $\xi = (1,0)$, it is sufficient to choose $\alpha = 2$ to get the sign assumption (since the first summand in (4.1) has positive integral), while for $\xi = (0,1)$ a convenient choice is given by $\alpha = 3$ (since the first summand in (4.1) vanishes).

3. *Comparison with Ahmad-Lazer-Paul type conditions.* In the spirit of [3, 9], we briefly compare (2.6) with Ahmad-Lazer-Paul type conditions at zero, in the case when $R(t,u) = \nabla Q(t,u)$ (so that the problem can be dealt with in a variational framework). Our considerations will consist purely in a matter of computation, comparing the explicit expression of the two conditions, without any other restriction imposed instead by the specific problem considered (referring, in particular, to the way the range of exponents to be taken into account has to be chosen).

We thus consider an Ahmad-Lazer-Paul type condition like

$$\liminf_{\substack{\|u\|\to 0\\ u\in H_0^0}} \frac{\int_0^T Q(t,u)\,dt}{\|u\|^{2\beta}} = M > 0, \tag{4.2}$$

(see, for instance, [12]), for $\beta > 0$, where $M \in \mathbb{R}\cup\{+\infty\}$ and H_0^0 is the nullity space associated with the principal part of the nonlinearity at zero. This condition may be fulfilled, for instance, if $Q(t,u)$ satisfies a suitable pointwise control at zero, as in [12].

We now proceed in comparing conditions (2.6) and (4.2). First, using the same argument in [3, Lemma 4.2], it is possible to show that (2.6) implies the existence of $\lambda_0 > 0$, $\theta_1,\dots,\theta_j \in [0,T]$, $\delta_1,\dots,\delta_j > 0$ and $h_1,\dots,h_j \in L^1(0,T)$ ($j\in\mathbb{N}$), with $\int_0^T h_i(t)\,dt > 0$ for every $i = 1,\dots,j$, such that $[0,T] \subset \cup_{i=1}^j [\theta_i - \delta_i, \theta_i + \delta_i]$, and, for every $i = 1,\dots,j$ and almost every $t\in[0,T]$,

$$\frac{\langle \nabla Q(t,\lambda\varphi_V(t+\omega)) | \varphi_V(t+\omega)\rangle}{\lambda^\alpha} \geq h_i(t), \quad \text{if } |\omega-\theta_i| \leq \delta_i,\ 0<\lambda\leq\lambda_0.$$

For almost every $t\in[0,T]$ and every $0<\lambda\leq\lambda_0$, $\theta\in[0,T]$, we now have

$$\begin{aligned}
\frac{Q(t,\lambda\varphi_V(t+\theta))}{\lambda^{2\beta}} &= \frac{1}{\lambda^{2\beta}}\int_0^1 \frac{d}{d\kappa} Q(t,\kappa\lambda\varphi_V(t+\theta))\,d\kappa \\
&= \frac{1}{\lambda^{2\beta}}\int_0^\lambda s^\alpha \frac{\langle\nabla Q(t,s\varphi_V(t+\theta))|\varphi_V(t+\theta)\rangle}{s^\alpha}\,ds \\
&\geq \frac{\lambda^{\alpha+1-2\beta}}{\alpha+1} h_i(t),
\end{aligned}$$

being the index i such that $\theta\in[\theta_i-\delta_i,\theta_i+\delta_i]$. Integrating on $[0,T]$ we now obtain

$$\int_0^T \frac{Q(t,\lambda\varphi_V(t+\theta))}{\lambda^{2\beta}}\,dt \geq \frac{\lambda^{\alpha+1-2\beta}}{\alpha+1}\int_0^T h_i(t)\,dt,$$

so that (4.2) is satisfied (recall that $\int_0^T h_i > 0$) if (2.6) holds with $\alpha \leq 2\beta - 1$. As can be expected, the situation concerning the range of the exponents appears quite different with respect to what happens with the usual Landesman-Lazer condition at infinity (see [3, Remark 4.3] and [9]).

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