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Constant Modulus Algorithms via Low-Rank Approximation

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Abstract

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Keywords: Constant modulus, convex optimization, trace norm.

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Constant Modulus Algorithms via Low-Rank Approximation

Amir Adler, *Member, IEEE,* and Mati Wax, *Fellow, IEEE*

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***Index Terms*—Constant modulus, constrained constant modulus, modified constant modulus, constrained modified constant modulus. convex optimization, trace norm.**

I. Introduction

Constant modulus algorithms are based on exploiting the constant modulus of the desired signal. They are used in a variety of areas in signal processing ranging from blind equalization and blind beamforming to blind multiuser detection.

The constant modulus (CM) algorithm was first introduced in the works of Godard [1] and Triechler and Agee [2] on blind equalization. In these works, the linear equalizer weight vector was computed by exploiting the constant modulus of the desired signal, without any explicit learning of the channel impulse response, and was therefore referred as "blind". Subsequently, Triechler and Larimore [3] introduced a CM algorithm for extracting a desired constant modulus signal in the presence of non-constant-modulus interfering signals. It is referred to as blind beamforming, since the only information exploited is the constancy of the modulus of the desired signal. The extension of the CM algorithm to allow additional linear constraints to be imposed on the weight vector, referred to as linear constrained constant modulus (LCCM) algorithm, was introduced by Rude and Griffith [4]. Miguez and Castro [5] then proposed a LCCM algorithm for multiuser CDMA communications, aimed at detecting the data of one user in the presence of the data of the other users, acting as interference. It is referred to as blind multiuser detection (MUD) since the only information exploited is the constant modulus of the CDMA signals and the spreading code of the desired user. Note that since all the signals are constant-modulus, and hence indistinguishable by the CM algorithm, the LCCM algorithm was necessary in order to single out the desired signal. Another extension of the CM algorithm was proposed in [6], referred to as the modified constant modulus (MCM) algorithm. This modification was motivated by the insensitiveness of the CM algorithm to the *phase* of the signal carrier, and was aimed at enabling improved performance for high order QAM constellation and for carrier phase synchronization.

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In all these algorithms, as well as in more recent developments [7], [8], the computation of the weight vector is based on a multidimensional non-convex cost function with multiple local minima [9], making global minimization very challenging.

In this paper we present a novel framework based on convex formulations of the CM and CCM cost functions, as well as of the MCCM and the linearly constrained MCM (LCMCM) cost functions . The solutions assure global optimality and are parameter free, i.e, they do not contain any tuneable parameter and do not require any a-priori parameter setting. The performance of these solutions is better than the existing CCM based solutions, reaching the theoretical performance limit with a much lower number of samples.

The rest of the paper is organized as follows. The problem formulation is presented in section II. Section III describes the convex CM solution. Section IV describes the convex CCM solution. Sections V and VII present the convex solutions for the MCM and for the LCMCM, respectively. Section VII discusses the blind beamforming problem, while section VIII discusses the blind multiuser detection problem. The performance analysis is presented in section IX. Finally, section X presents the conclusions.

II. Problem Formulation

Let the received signal $\mathbf{x}(t)$ be a $P \times 1$ complex vextor given by:

$$\mathbf{x}(t) = \mathbf{a}_1 s_1(t) + \sum_{k=2}^{K} \mathbf{a}_k s_k(t) + \mathbf{n}(\mathbf{t}), \qquad (1)$$

where $s_1(t)$ is a *desired* constant modulus (CM) signal, $\{s_k(t)\}_{k=2}^{K}$ are *non-constant-modulus* interfering signals, $\{\mathbf{a}_k\}_{k=1}^{K}$ are *unknown* vectors, and $\mathbf{n}(\mathbf{t})$ is the $P \times 1$ noise vector. We further assume that the number of signals obeys $K \leq P$ and that the vectors $\{\mathbf{a}_k\}_{k=1}^{K}$ are *linearly independent.* The constant modulus problem can be formulated as follows: Given the received vectors $\{\mathbf{x}(t_n)\}_{n=1,}^{N}$, find a $P \times 1$ weight vector $\mathbf{w}$ such that the linear combiner output $y(t) = \mathbf{w}^H \mathbf{x}(t)$, where H denotes the conjugate transpose, *provides a good estimate of the CM signal* $s_1(t)$.

Assuming, without loss of generality, that the square of the modulus of the desired signal $s_1(t)$ is R, the common CM cost function for estimating the linear combiner weight $\mathbf{w}$ is given by minimization of the sample-average of the deviation of the linear combiner power output from R:

$$\hat{\mathbf{w}} = \underset{\mathbf{w}}{\text{argmin}} \frac{1}{N} \sum_{n=1}^{N} (|\mathbf{w}^H \mathbf{x}(t_n)|^2 - R)^2. \quad (2)$$

This is a fourth order minimization problem in the vector $\mathbf{w}$, and as such does not admit a closed form solution. Moreover, as shown in [13]-[14], it is a non-convex problem (i.e. it has multiple local minima), making global minimization very challenging. We next show how to reformulate the CMA as a convex optimization problem, which assures global optimality.

III. Convex Constant Modulus Algorithm

First, we rewrite the linear combiner power output, denoted by z(t), as[1]:

$$\begin{aligned} z(t) &= |\mathbf{w}^H \mathbf{x}(t)|^2 = \mathbf{w}^H \mathbf{x}(t)\mathbf{x}(t)^H \mathbf{w} \quad (3)\\ &= \mathrm{tr}(\mathbf{w}\mathbf{w}^H \mathbf{x}(t)\mathbf{x}(t)^H) = \mathrm{tr}(\mathbf{W}\mathbf{x}(t)\mathbf{x}(t)^H), \end{aligned}$$

where $\mathrm{tr}()$ denotes the trace of the bracketed matrix and $\boldsymbol{W}$ denotes the $P \times P$ positive semidefinite (PSD) rank-1 matrix:

$$\mathbf{W} = \mathbf{w}\mathbf{w}^H. \quad (4)$$

We can now rewrite (2) as:

$$\hat{\mathbf{W}} = \underset{\mathbf{W}}{\text{argmin}} \frac{1}{N} \sum_{n=1}^{N} |z(t_n) - R|^2, \quad (5a)$$

subject to:

$$z(t_n) = \mathrm{tr}(\mathbf{W}\mathbf{x}(t_n)\mathbf{x}(t_n)^H), \quad n = 1, ..., N, \quad (5b)$$

$$\mathbf{W} \succcurlyeq 0, \quad (5c)$$

$$\mathrm{rank}\, \mathbf{W} = 1, \quad (5d)$$

where $\mathbf{W} \succcurlyeq 0$ denotes the PSD constraint. Note however, that since the rank constraint (5d) is not convex, the minimization problem is not convex. A commonly-used convex relaxation surrogate to the rank-1 constraint is to minimize the trace norm (nuclear norm), defined as the sum of the singular values of the matrix. Recalling that $\mathbf{W}$ is a PSD matrix, it follows that its trace norm is given by $\mathrm{tr}(\mathbf{W})$. This implies that we can reformulate the CM problem as the following *convex optimization* problem:

$$\hat{\mathbf{W}} = \underset{\mathbf{W}}{\text{argmin}} \{ (\frac{1}{N} \sum_{n=1}^{N} |z(t_n) - R|^2) + \mathrm{tr}(\mathbf{W}) \}, \quad (6a)$$

subject to:

$$\mathrm{tr}(\mathbf{W}\mathbf{x}(t_n)\mathbf{x}(t_n)^H) = z(t_n), \quad n = 1, ..., N, \quad (6b)$$

$$\mathbf{W} \succcurlyeq 0. \quad (6c)$$

Since (6) is a convex optimization problem, we can use any of the convex optimization solvers [10], [11] to solve for $\hat{\mathbf{W}}$.

[1]We use the following properties of the trace operator $\mathrm{tr}()$: (i) cyclic shift: $\mathrm{tr}(ABCD) = \mathrm{tr}(BCDA) = \mathrm{tr}(CDAB) = \mathrm{tr}(DABC)$; and (ii) $\mathrm{tr}(a) = a$ for any scalar a.

With $\hat{\mathbf{W}}$ at hand, a straightforward way to estimate the weight vector $\mathbf{w}$ is by the rank-1 approximation of $\hat{\mathbf{W}}$:

$$\hat{\mathbf{W}} \simeq \lambda_1 \mathbf{v}_1 \mathbf{v}_1^H, \quad (7)$$

where λ_1 denotes the largest eigenvalue of $\hat{\mathbf{W}}$, and $\mathbf{v}_1$ denotes the eigenvector of $\hat{\mathbf{W}}$ corresponding to λ_1. Using this rank-1 approximation, we estimate the weight vector $\mathbf{w}$ as:

$$\hat{\mathbf{w}} = \mathbf{v}_1. \quad (8)$$

IV. Convex Linearly Constrained CMA

In many scenarios involving CM signals, it may be desired to impose additional constraints on the weight vector $\mathbf{w}$ in the form of the following linear constraint:

$$\mathbf{w}^H \mathbf{C} = \mathbf{v}^H, \quad (9)$$

where $\mathbf{C}$ and $\mathbf{v}$ are $P \times J$ matrix and $J \times 1$ vector, respectively, assumed to be *known*. This problem is referred to as LCCM.

To incorporate the linear constraint (9) into our convex CMA formulation, we first rewrite it as

$$\mathbf{w}^H \mathbf{c}_j = v_j, \quad j = 1, ..., J, \quad (10)$$

where $\mathbf{c}_j$ denotes the j-th column of $\mathbf{C}$ and v_j denotes the j-th element of $\mathbf{v}$. Now, using the properties of the trace operator and (9), we have

$$\mathrm{tr}(\mathbf{w}\mathbf{w}^H \mathbf{c}_j \mathbf{c}_j^H) = \mathrm{tr}(\mathbf{c}_j^H \mathbf{w}\mathbf{w}^H \mathbf{c}_j) = \mathrm{tr}(v_j v_j^H) = |v_j|^2, \quad (11)$$

which implies that we can rewrite the linear constraint (10) as,

$$\mathrm{tr}(\mathbf{W}\mathbf{c}_j \mathbf{c}_j^H) = |v_j|^2, \quad j = 1, ..., J, . \quad (12)$$

The convex LCCM cost function can now be formulated as:

$$\hat{\mathbf{W}} = \underset{\mathbf{W}}{\text{argmin}} \{ (\frac{1}{N} \sum_{n=1}^{N} |z(t_n) - R|^2) + \mathrm{tr}(\mathbf{W}) \}, \quad (13a)$$

subject to:

$$\mathrm{tr}(\mathbf{W}\mathbf{x}(t_n)\mathbf{x}(t_n))^H) = z(t_n) \quad n = 1, ..., N, \quad (13b)$$

$$\mathrm{tr}(\mathbf{W}\mathbf{c}_j \mathbf{c}_j^H) = |v_j|^2 \quad j = 1, ..., J, \quad (13c)$$

$$\mathbf{W} \succcurlyeq 0. \quad (13d)$$

V. Convex Modified Linearly Constrained CMA

The CM cost function is insensitive to the *phase* of the signal carrier. Therefore, in the presence of an unknown phase rotation the resulting estimated signal will also be rotated. A phase-sensitive modification of the CM cost function, referred to as the MCM, was introduced in [6], aimed at enabling improved performance for high order QAM and carrier phase synchronization. To introduce the MCM cost function, let $\mathbf{w}_R$, $\mathbf{w}_I$ and $y_R(t)$, $y_I(t)$ denote, respectively, the real and imaginary parts of $\mathbf{w}$ and $y(t)$. Using this notation, the MCM cost function can be written as:

$$\hat{\mathbf{w}} = \underset{\mathbf{w}}{\text{argmin}} \frac{1}{N} \sum_{n=1}^{N} [(|y_R(t_n)|^2 - R_R)^2 + (|y_I(t_n)|^2 - R_I)^2], \quad (14a)$$

where

$$R_R = \frac{E|s_R(t)|^4}{E|s_R(t)|^2}, \quad (14b)$$

and

$$R_I = \frac{E|s_I(t)|^4}{E|s_I(t)|^2}, \quad (14c)$$

where $s_R(t)$ and $s_I(t)$ denote the real and imaginary parts of $s(t)$.

To reformulate the MCM cost function as a convex optimization problem, we first rewrite $y_R(t)$ and $y_I(t)$ as

$$y_R(t) = \widetilde{\mathbf{w}}^T \mathbf{x_1}(t), \quad (15a)$$

and

$$y_I(t) = \widetilde{\mathbf{w}}^T \mathbf{x}_2(t). \quad (15b)$$

where

$$\widetilde{\mathbf{w}} = \begin{bmatrix} \mathbf{w}_R \\ \mathbf{w}_I \end{bmatrix}, \quad (15c)$$

$$\mathbf{x}_1(t) = \begin{bmatrix} \mathbf{x}_R(t) \\ \mathbf{x}_I(t) \end{bmatrix}, \quad (15d)$$

and

$$\mathbf{x}_2(t) = \begin{bmatrix} \mathbf{x}_I(t) \\ -\mathbf{x}_R(t) \end{bmatrix}. \quad (15e)$$

Using the properties of the trace operator, we have

$$z_1(t) = |y_R(t)|^2 = \widetilde{\mathbf{w}}^T \mathbf{x}_1(t)\mathbf{x}_1^T(t)\widetilde{\mathbf{w}} = \mathrm{tr}(\widetilde{\mathbf{W}}\mathbf{x}_1(t)\mathbf{x}_2^T(t)), \quad (16a)$$

and

$$z_2(t) = |y_I(t)|^2 = \widetilde{\mathbf{w}}^T \mathbf{x}_2(t)\mathbf{x}_2^T(t)\widetilde{\mathbf{w}} = \mathrm{tr}(\widetilde{\mathbf{W}}\mathbf{x}_2(t)\mathbf{x}_2^T(t)), \quad (16b)$$

where $\widetilde{\mathbf{W}}$ denotes the $2P \times 2P$ rank-1 matrix

$$\widetilde{\mathbf{W}} = \widetilde{\mathbf{w}}\widetilde{\mathbf{w}}^T. \quad (17)$$

With this notation we can rewrite the MCM cost function as

$$\hat{\widetilde{\mathbf{w}}} = \underset{\mathbf{w}}{\mathrm{argmin}} \frac{1}{N} \sum_{n=1}^{N} [(z_1(t_n)) - R_R)^2 + (z_2(t_n) - R_I)^2], \quad (18a)$$

subject to

$$z_1(t_n) = \mathrm{tr}(\widetilde{\mathbf{W}}\mathbf{x}_1(t_n)\mathbf{x}_1^T(t_n)), \quad n = 1, ..., N, \quad (18b)$$

$$z_2(t_n) = \mathrm{tr}(\widetilde{\mathbf{W}}\mathbf{x}_2(t_n)\mathbf{x}_2^T(t_n)), \quad n = 1, ..., N, \quad (18c)$$

$$\widetilde{\mathbf{W}} \succeq \mathbf{0}, \quad (18d)$$

$$\mathrm{rank}\, \widetilde{\mathbf{W}} = 1. \quad (18e)$$

Using the trace norm as a surrogate for the non-convex rank-1 constraint, we can reformulate the MCM cost function as the following convex optimization problem:

$$\hat{\widetilde{\mathbf{W}}} = \underset{\widetilde{\mathbf{W}}}{\mathrm{argmin}} \frac{1}{N} \sum_{n=1}^{N} [(z_1(t_n) - R_R)^2 + (z_2(t_n) - R_I)^2] + \mathrm{tr}(\widetilde{\mathbf{W}}), \quad (19a)$$

subject to:

$$z_1(t_n) = \mathrm{tr}(\widetilde{\mathbf{W}}\mathbf{x}_1(t_n)\mathbf{x}_1^T(t_n)), \quad n = 1, ..., N, \quad (19b)$$

$$z_2(t_n) = \mathrm{tr}(\widetilde{\mathbf{W}}\mathbf{x}_2(t_n)\mathbf{x}_2^T(t_n)), \quad n = 1, ..., N, \quad (19c)$$

$$\widetilde{\mathbf{W}} \succeq \mathbf{0}, \quad (19d)$$

We next show that for a *symmetric constellation*, the solution $\widetilde{\mathbf{W}}$ of this optimization problem has *two different rank-1 solutions* corresponding to two linearly independent vectors.

To this end, note that from (15d) and (15e) we have

$$\mathbf{x}_2(t) = \mathbf{G}\mathbf{x}_1(t), \quad (20)$$

where $\mathbf{G}$ is the $2(P + L - 1) \times 2(P + L - 1)$ block matrix

$$\mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}. \quad (21)$$

This enables us to rewrite $z_2(t)$ as

$$z_2(t) = \mathrm{tr}(\widetilde{\mathbf{W}}\mathbf{x}_2(t)\mathbf{x}_2^T(t)) = \mathrm{tr}(\widetilde{\mathbf{W}}\mathbf{G}\mathbf{x}_1(t)\mathbf{x}_1^T(t)\mathbf{G}^T), \quad (22)$$

or alternatively, using the properties of the trace operator, as

$$z_2(t) = \mathrm{tr}(\widetilde{\widetilde{\mathbf{W}}}\mathbf{x}_1(t)\mathbf{x}_1^T(t)), \quad (23)$$

where $\widetilde{\widetilde{\mathbf{W}}}$ is given by

$$\widetilde{\widetilde{\mathbf{W}}} = \mathbf{G}^T \widetilde{\mathbf{W}} \mathbf{G}, \quad (24)$$

which, by using (40), can be rewritten as

$$\widetilde{\widetilde{\mathbf{W}}} = \widetilde{\widetilde{\mathbf{w}}}\widetilde{\widetilde{\mathbf{w}}}^T, \quad (25)$$

where $\widetilde{\widetilde{\mathbf{w}}}$ is given by

$$\widetilde{\widetilde{\mathbf{w}}} = \mathbf{G}^T \widetilde{\mathbf{w}}, \quad (26)$$

which implies that $\widetilde{\widetilde{\mathbf{w}}}$ and $\widetilde{\mathbf{w}}$ are *linearly independent*.

Now, since the optimization process yields

$$z_1(t) \approx R_R, \quad (27a)$$

and

$$z_2(t) \approx R_I, \quad (27b)$$

and since in a *symmetrical constellation* we have

$$R_R = R_I \quad (28)$$

it follows that

$$z_1(t) \approx z_2(t). \quad (29)$$

This implies, as can be easily verified, that

$$\mathrm{tr}(\widetilde{\mathbf{W}}\mathbf{x}_1(t)\mathbf{x}_1^T(t) \approx \mathrm{tr}(\widetilde{\widetilde{\mathbf{W}}}\mathbf{x}_1(t)\mathbf{x}_1^T(t), \quad (30a)$$

and

$$\mathrm{tr}(\widetilde{\mathbf{W}}\mathbf{x}_2(t)\mathbf{x}_2^T(t) \approx \mathrm{tr}(\widetilde{\widetilde{\mathbf{W}}}\mathbf{x}_2(t)\mathbf{x}_2^T(t). \quad (30b)$$

which shows that both $\widetilde{\mathbf{W}}$ and $\widetilde{\widetilde{\mathbf{W}}}$ are *feasible solutions,* corresponding to two rank-1 solutions given by $\widetilde{\mathbf{w}}$ and $\widetilde{\widetilde{\mathbf{w}}}$.

Thus, given the solution $\hat{\widetilde{\mathbf{W}}}$ of (19), we can estimate vectors $\widetilde{\mathbf{w}}$ and $\widetilde{\widetilde{\mathbf{w}}}$, from the rank-2 approximation of $\hat{\widetilde{\mathbf{W}}}$, as:

$$\hat{\widetilde{\mathbf{w}}} = \mathbf{v}_1 \quad (31a)$$

and

$$\hat{\widetilde{\widetilde{\mathbf{w}}}} = \mathbf{v}_2 \quad (31b)$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ denote the two largest eigenvectors of $\hat{\widetilde{\mathbf{W}}}$.

As in the CM formulation, we can incorporate additional linear constraints on the vector $\mathbf{w}$. This problem is referred to as the Constrained Modified Constant Modulus (CMCM). To incorporate the constraints (10) into our convex MCM formulation, let $\mathbf{c}_{R_j}$, $\mathbf{c}_{I_j}$ and v_{R_j} , v_{I_j} denote, respectively, the real and imaginary parts of $\mathbf{c}_j$ and v_j. Using this notation we can rewrite (10) as

$$\widetilde{\mathbf{w}}^T \mathbf{c}_{1_j} = v_{R_j}, \quad j = 1, ..., J, \tag{32a}$$

and

$$\widetilde{\mathbf{w}}^T \mathbf{c}_{2_j} = -v_{I_j}, \quad j = 1, ..., J, \tag{32b}$$

where

$$\mathbf{c}_{1_j} = \begin{bmatrix} \mathbf{c}_{R_j} \\ \mathbf{c}_{I_j} \end{bmatrix}, \tag{32c}$$

and

$$\mathbf{c}_{2_j} = \begin{bmatrix} \mathbf{c}_{I_j} \\ -\mathbf{c}_{R_j} \end{bmatrix}. \tag{32d}$$

This implies that

$$\widetilde{\mathbf{w}}^T \mathbf{c}_{1_j} \mathbf{c}_{1_j}^T \widetilde{\mathbf{w}} = v_{R_j}^2, \quad j = 1, ..., J, \tag{33a}$$

and

$$\widetilde{\mathbf{w}}^T \mathbf{c}_{2_j} \mathbf{c}_{2_j}^T \widetilde{\mathbf{w}} = v_{I_j}^2, \quad j = 1, ..., J, \tag{33b}$$

which, using the properties of the trace operator, can be rewritten

$$\mathrm{tr}(\widetilde{\mathbf{W}}^T \mathbf{c}_{1_j} \mathbf{c}_{1_j}^T) = v_{R_j}^2, \quad j = 1, ..., J, \tag{34a}$$

and

$$\mathrm{tr}(\widetilde{\mathbf{W}}^T \mathbf{c}_{2_j} \mathbf{c}_{2_j}^T) = v_{I_j}^2, \quad j = 1, ..., J, . \tag{34b}$$

Thus, combining these two equation, we can rewite (10) as

$$\mathrm{tr}(\widetilde{\mathbf{W}}^T \mathbf{c}_{1_j} \mathbf{c}_{1_j}^T) + \mathrm{tr}(\widetilde{\mathbf{W}}^T \mathbf{c}_{2_j} \mathbf{c}_{2_j}^T) = |v_j|^2, \quad j = 1, ..., J, \tag{34c}$$

The convex formulation of CMCM is therefore given by

$$\hat{\widetilde{\mathbf{W}}} = \underset{\widetilde{\mathbf{W}}}{\mathrm{argmin}} \{ \frac{1}{N} \sum_{n=1}^{N} [(z_1(t_n) - R_R)^2 + (z_2(t_n) - R_I)^2] + \mathrm{tr}(\widetilde{\mathbf{W}}), \tag{35a}$$

subject to

$$z_1(t_n) = \mathrm{tr}(\widetilde{\mathbf{W}} \mathbf{x}_1(t_n) \mathbf{x}_1^T(t_n)), \quad n = 1, ..., N, \tag{35b}$$

$$z_2(t_n) = \mathrm{tr}(\widetilde{\mathbf{W}} \mathbf{x}_2(t_n) \mathbf{x}_2^T(t_n)), \quad n = 1, ..., N, \tag{35c}$$

$$\mathrm{tr}(\widetilde{\mathbf{W}}^T \mathbf{c}_{1_j} \mathbf{c}_{1_j}^T) + \mathrm{tr}(\widetilde{\mathbf{W}}^T \mathbf{c}_{2_j} \mathbf{c}_{2_j}^T) = |v_j|^2, \quad j = 1, ..., J, \tag{35d}$$

$$\widetilde{\mathbf{W}} \succeq \mathbf{0}, \tag{35e}$$

VI. CONVEX CMA FOR BLIND BEAMFORMING

Consider an antenna array composed of P antennas with arbitrary locations and arbitrary directional characteristics. Assume that a desired signal $s_1(t)$ is impinging on the array from an unknown direction-of-arrival θ_1 and that $K-1$ other interfering signals $s_k(t), k = 2, \ldots, K$, are impinging on the array from unknown directions-of-arrival $\theta_2, \ldots, \theta_K$. All the signals are assumed to be narrow-band, namely that the array aperture, denoted by d, obeys $d << c/B$, where c is the speed of light, and B is the signals bandwidth. Under these assumptions, the $P \times 1$ array vector $\mathbf{x}(t)$ of the complex envelopes of the received signals can be written as:

$$\mathbf{x}(\mathbf{t}) = \mathbf{a}(\theta_1) s_1(t) + \sum_{k=2}^{K} \mathbf{a}(\theta_k) s_q(t) + \mathbf{n}(\mathbf{t}), \tag{36}$$

where $\boldsymbol{a}(\theta_1)$ is the $P \times 1$ steering vector of the array toward the desired CM signal $s_1(t)$, $\mathbf{a}(\theta_q)$ is the $P \times 1$ steering vector of the array toward the interfering signal $s_q(t)$, and $\mathbf{n}(\mathbf{t})$ is the $P \times 1$ noise vector.

Since (36) is in the form of (1), the estimation of the desired CM signal $s_1(t)$ from the received array vectors $\{\mathbf{x}(t_n)\}_{n=1,}^N$ can be readily done using the unconstrained convex CMA (6), summarized in Algorithm 1, or the convex MCM (19), summarized in Algorithm 2.

Algorithm 1 Convex CMA Blind Beamforming

1: **Input:** Received array vectors $\{\mathbf{x}(t_n)\}_{n=1,}^N$
2: **Solve:**

$$\hat{\mathbf{W}} = \underset{\mathbf{W}}{\mathrm{argmin}} \{ \frac{1}{N} \sum_{n=1}^{N} |z(t_n) - R|^2 + \mathrm{tr}(\mathbf{W}) \},$$

subject to:

$$\mathrm{tr}(\mathbf{W}\mathbf{x}(t_n)\mathbf{x}(t_n)^H) = z(t_n), \quad n = 1, ..., N,$$

$$\mathbf{W} \succcurlyeq 0.$$

3: **Compute:** $\mathbf{v}_1$ = Largest Eigenvector of $\hat{\mathbf{W}}$
4: **Output:** $\hat{\mathbf{w}} = \mathbf{v}_1$.

Regarding the constrained algorithms (16) and (35), it is worth while to point out some useful linear constraints, which are special cases of (9). An example for such a constraint is the well-known "look direction" constraint:

$$\mathbf{w}^H \mathbf{a}(\theta) = 1, \tag{37}$$

constraining $\mathbf{w}$ to have a unity gain in the direction θ. Another example is the constraint,

$$\mathbf{w}^H \mathbf{B} = \mathbf{0}, \tag{38}$$

constraining $\mathbf{w}$ to be orthogonal to the columns of $\mathbf{B}$. One example for such a $\mathbf{B}$ is

$$\mathbf{B} = \mathbf{a}(\theta), \tag{39}$$

assuring deep "nulls" in the direction θ. This may be desired, for example, in case a strong interference is known to be impinging from direction θ and the desire is to put a deep null in this direction. Another example is

$$\mathbf{B} = [\mathbf{v}_{K+1}, ..., \mathbf{v}_P], \tag{40}$$

where $\mathbf{v}_i$ is the eigenvector of the array covariance matrix $\hat{\mathbf{R}} = \sum_{n=1}^{N} x(t_n) x(t_n)^H$ corresponding to the i-th eigenvalue. This constraints $\mathbf{w}$ to be orthogonal to the noise subspace, i.e., to be confined to the K-dimensional signal subspace [12]. This low-dimensional confinement reduces the number of degrees-of-freedom of $\mathbf{w}$, thereby improving the solution performance, especially in challenging conditions such as small number of samples and low signal-to-noise ratio.

Algorithm 2 Convex Modified CMA Blind Beamforming

1: **Input:** Received array vectors $\{\mathbf{x}(t_n)\}_{n=1,}^N$
2: **Set:**
$$\mathbf{x}_1(t) = \begin{bmatrix} \mathbf{x}_R(t) \\ \mathbf{x}_I(t) \end{bmatrix},$$
and
$$\mathbf{x}_2(t) = \begin{bmatrix} \mathbf{x}_I(t) \\ -\mathbf{x}_R(t) \end{bmatrix}.$$
3: **Solve:**
$$\hat{\widetilde{\mathbf{W}}} = \underset{\widetilde{\mathbf{W}}}{\operatorname{argmin}} \frac{1}{N} \sum_{n=1}^{N} [(z_1(t_n)-R_R)^2 + (z_2(t_n)-R_I)^2] + \operatorname{tr}(\widetilde{\mathbf{W}}),$$
subject to:
$$z_1(t_n) = \operatorname{tr}(\widetilde{\mathbf{W}}\mathbf{x}_1(t_n)\mathbf{x}_1^T(t_n)), \quad n = 1, ..., N,$$
$$z_2(t_n) = \operatorname{tr}(\widetilde{\mathbf{W}}\mathbf{x}_2(t_n)\mathbf{x}_2^T(t_n)), \quad n = 1, ..., N,$$
$$\widetilde{\mathbf{W}} \succeq \mathbf{0},$$
4: **Compute:** $\mathbf{v}_1$ = Largest Eigenvector of $\hat{\widetilde{\mathbf{W}}}$
5: **Set:**
$$\hat{\tilde{\mathbf{w}}} = \begin{bmatrix} \mathbf{w}_R \\ \mathbf{w}_I \end{bmatrix} = \mathbf{v}_1.$$
6: **Output:** $\hat{\mathbf{w}} = \mathbf{w}_R + \mathbf{j}\mathbf{w}_I$.

VII. Convex Linearly Constrained CMA For Blind Multiuser Detection

Consider a symbol-synchronous un-coded CDMA system with K simultaneous users. Let $\mathbf{s}_k$ denote the $P \times 1$ spreading code of user k, assuming that the spreading codes are normalized, i.e, $||\mathbf{s}_k|| = 1$. Let $b_k(t)$ denote the the transmitted QAM symbols to user k, and let B denote constellation alphabet from which $b_k(t)$ take its value. Assuming *multipath free propagation*, the $P \times 1$ vector $\mathbf{x}(t)$ of the complex envelopes of the signals received by a *single user*, say user 1, after filtering by a chip-pulse matched filter and sampled at chip rate, can be written as [5]

$$\mathbf{x}(t) = A_1 b_1(t)\mathbf{s}_1 + \sum_{k=2}^{K} A_k b_k(t)\mathbf{s}_k + \mathbf{n}(t), \tag{41}$$

where A_k and $\mathbf{s}_k$ are the received amplitude and the spreading code, respectively, of the k-th user. To insure identifiability, we further assume that the number of users obeys $K \leq L$, and that the signals' spreading codes $\{\mathbf{s}_k)\}_{k=1}^K$ are *linearly independent.*

The estimation of user 1 symbols $\{b_1(t_n)\}_{n=1,}^N$ from the received vectors $\{\mathbf{x}(t_n)\}_{n=1,}^N$, is referred to as blind multiuser detection.

Note that (41) is in the form of (1), with the difference that here all the other interfering signals $\{b_k(t)\}_{k=2}^K$ are also *constant modulus*. In this case the unconstrained algorithms (6) and (13) will capture the desired signal only if it is stronger than the interfering signals. To assure capturing of the desired signal in this case it necessary to incorporate the following desired user unit gain constraint:

$$\mathbf{w}^H \mathbf{s}_1 = \mathbf{1}, \tag{42}$$

Incorporating this constraint in the convex CCM (13), as summarized in Algorithm 3, or in the convex CMCM (35) readily solves the blind multiuser detection problem.

Algorithm 3 Convex LCCM Blind Multiuser Detection

1: **Input:** Chip-matched filter vectors $\{\mathbf{x}(t_n)\}_{n=1,}^N$, desired user signature $\mathbf{s}_1$.
2: **Solve:**
$$\hat{\mathbf{W}} = \underset{\mathbf{W}}{\operatorname{argmin}} \{ \frac{1}{N} \sum_{n=1}^{N} |z(t_n) - R|^2 + \operatorname{tr}(\mathbf{W}) \},$$
subject to:
$$\operatorname{tr}(\mathbf{W}\mathbf{x}(t_n)\mathbf{x}(t_n)^H) = z(t_n), \quad n = 1, ..., N,$$
$$\operatorname{tr}(\mathbf{W}\mathbf{s}_1\mathbf{s}_1^H) = 1,$$
$$\mathbf{W} \succcurlyeq 0.$$
3: **Compute:** $\mathbf{v}_1$ = Largest Eigenvector of $\hat{\mathbf{W}}$
4: **Output:** $\hat{\mathbf{w}} = \mathbf{v}_1$.

In case of *multipath propagation* the situation is slightly more complicated. Assuming a maximum delay of LT_c, where T_c is the chip duration, the $(P+L-1) \times 1$ vector $\mathbf{x}(t)$ of the complex envelopes of the signals received by a *single user*, say user 1, after filtering by a chip-pulse matched filter and sampled at chip rate, can be written as [5],

$$\mathbf{x}(t) = A_1 b_1(t)\mathbf{S}_1\mathbf{h}_1 + \mathbf{u}_1(t) + \sum_{k=2}^{K} (A_k b_k(t)\mathbf{S}_k\mathbf{h}_k + \mathbf{u}_k(t)) + \mathbf{n}(t), \tag{43}$$

where $\mathbf{S}_k$ is the $(P + L - 1) \times L$ matrix whose columns are shifted versions of the spreading code $\mathbf{s}_k$,

$$\mathbf{S}_k = \begin{bmatrix} \mathbf{s}_k & 0 & \cdots & 0 \\ 0 & \mathbf{s}_k & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \mathbf{s}_k \end{bmatrix}, \tag{44}$$

$\mathbf{h}_k$ is the $L \times 1$ vector of the channel response of user k, $\mathbf{u}_k(t)$ is the inter-symbol interference (ISI) for user k from the adjacent symbols, and $\mathbf{n}(t)$ is the $(L + P - 1) \times 1$ noise vector. To insure identifiability in this case, we further assume that the number of users obeys $2K \leq (L + P - 1)$.

The blind multiuser detection in this case can be similarly solved using the constrained algorithms described above, by using the following linear constraints:

$$\mathbf{w}^H \mathbf{c}_1 = 1, \tag{45}$$

where $\mathbf{c}_1$ is given by

$$\mathbf{c}_1 = \mathbf{S}_1\mathbf{h}_1, \tag{46}$$

with $\mathbf{h}_1$ and $\mathbf{S}_1$ assumed to be *known*. Since $\mathbf{h}_1$ is typically unknown in practice, it is *estimated* from the data, as discussed in [4]-[5].

Another useful constraint is given by,

$$\mathbf{w}^H \mathbf{C} = \mathbf{0}, \tag{47}$$

constraining $\mathbf{w}$ to be be orthogonal to the columns of $\mathbf{C}$. One example for such a $\mathbf{C}$, presented here for simplicity for the *multipath free* case is given by

$$\mathbf{C} = \mathbf{s}_k, \tag{48}$$

assuring "deep nulls" towards spreading code $\mathbf{s}_k$. This may be desired, for example, in case a strong interference is known to have a spreading code $\mathbf{s}_k$, and it desired to mitigate it by a "deep null". Another example, again presented for simplicity for the *multipath free* case is

$$\mathbf{C} = [\mathbf{v}_{K+1}, ..., \mathbf{v}_P] \tag{49}$$

where $\mathbf{v}_i$ is the eigenvector corresponding to the i-th eigenvalue of the covariance matrix $\hat{\mathbf{R}} = \sum_{n=1}^{N} \mathbf{x}(t_n)\mathbf{x}(t_n)^H$. This constraints $\mathbf{w}$ to be orthogonal to the noise subspace [29], i.e., to be confined to the K-dimensional signal subspace spanned by the vectors $\{\mathbf{x}(t_n)\}_{n=1}^{N}$. This confinement to a low-dimensional subspace reduces the number of degrees-of-freedom of $\mathbf{w}$, thereby improves the solution performance, especially in challenging conditions such as small number of samples and low signal-to-noise ratio.

VIII. Performance Analysis

A. Blind Beamforming Performance Evaluation

In this section we present blind beamforming simulation results illustrating the performance of the proposed solution, referred to as Trace Norm. The performance is compared to the Recursive Least Squares (RLS) [13] the Unscented Kalman Filter (UKF) [8] and the Constrained CM-RLS (CCM-RLS) [14] solutions.

The desired signal was simulated as a unit power QPSK signal. The interfering signals were simulated as complex Gaussian with zero mean and unit variance. The noise was simulated as a complex Gaussian with zero mean and covariance $\sigma_n^2\mathbf{I}$. The performance measure employed is the signal-to-interference-plus-noise ratio (SINR) at the beamformer output:

$$SINR = \frac{\mathbf{w}^H\mathbf{R}_{ss}\mathbf{w}}{\mathbf{w}^H\mathbf{R}_{nn}\mathbf{w} + \mathbf{w}^H\mathbf{R}_{ii}\mathbf{w}}, \tag{50}$$

where $\mathbf{R}_{ss} = \mathbf{a}(\theta_0)\mathbf{a}(\theta_0)^H$, $\mathbf{R}_{nn} = \sigma_n^2\mathbf{I}$, and $\mathbf{R}_{ii} = \sum_{j=1}^{q} \mathbf{a}(\theta_j)\mathbf{a}(\theta_j)^H$ are the CM signal, noise and interference covariance matrices, respectively. All presented results are averaged over 100 experiments, and employ a uniform linear array (ULA) with $P = 16$, unless specified differently.

Before discussing the results of the simulated experiments, we would like to discuss the computation time of the Trace Norm solution and its suitability to real time communication systems. To this end, we evaluated the computation time of the simulated experiments using the MATLAB CVX [15] toolbox[2]. The computation time for a typical scenario with $N = 100$ samples, consisting of a CM signal impinging from $20°$ on a Uniform Linear Array (ULA) with $P = 16$ and 3 interferers impinging from $-45°$, $-15°$ and $40°$, all having SNR of 10 dB, is 1 second. Since the average speed-up factor between CVX-based implementation and a real-time implementation, as analyzed in [16], is $\times 1000$ (single processor), this implies a 1 ms in real-time implementation, which is highly suitable to packet-based communications, especially in low mobility.

Experiment 1 evaluates the ratio between the largest (λ_1) and the second largest (λ_2) eigenvalues of $\hat{\mathbf{W}}$, which is a good measure for the goodness of the rank-1 approximation of the trace norm solution of $\hat{\mathbf{W}}$. We evaluated this ratio by simulating 500 times[3] each of the following scenarios: a CM signal in the presence of 0, 1, or 2 interferers, all signals having equal power, at SNR of 10dB or 20dB ($\sigma_n^2 = 0.1$ or 0.01, respectively). For the case of no interference, the ratio $\frac{\lambda_1}{\lambda_2}$ exceeded 10^6 with probability 1, implying a perfect rank-1 result. Fig. 1(a) presents the results for the cases of 1 and 2 interferers, and reveals that with probability 1, $\frac{\lambda_1}{\lambda_2} \geq 10$ for SNR = 10dB, and $\frac{\lambda_1}{\lambda_2} \geq 50$ for SNR = 20dB. These results demonstrate the goodness of the rank-1 approximation of $\hat{\mathbf{W}}$.

Experiment 2 evaluates the performance of the Trace-Norm solution in the presence of two CM signals: The first from $20°$ with unit power, and the second from $50°$, attenuated in each trial by a random attenuation, uniformly distributed between 0dB to -5dB. Fig. 1(b) presents the averaged array pattern, over 500 experiments, and demonstrates the "capture" effect of the Trace-Norm solution: the algorithm captures always the strongest CM signal, and cancels the weaker

Experiment 3 compares the SINR of the Trace-Norm, UKF and RLS, in the presence of interferers. Note that the UKF and the RLS are sequential algorithms, i.e., operating on the samples sequentially, from the first to the last, say n, while the Trace-Norm is a batch algorithm operating on all the n samples simultaneously. In the first scenario we simulated a CM signal impinging from $20°$, with 3 interfering signals impinging from $-45°$, $-15°$, $40°$, and noise variance $\sigma_n^2 = 0.1$. The results are presented in Fig. 2(a) and demonstrate that the Trace-Norm solution yields better SINR and converges after 100 samples, as compared to the UKF and RLS, which require 500 and $2,700$ samples, respectively, to converge. Fig. 2(b) presents the performance with an additional interferer from $60°$. In this case convergence of the UKF and RLS is slower ($1,500$ and $3,500$ samples, respectively), whereas the Trace-Norm is essentially invariant to the additional interferer, and surpasses UKF and RLS with only 100 samples. The array pattern of the Trace Norm with 200 samples (averaged over 1,000 experiments), is depicted in Fig. 2(c). The rejection of all 4 interferers is clearly visible.

Experiment 4 evaluates the robustness of the proposed linearly-constrained solution for the case of a look direction error. In this scenario, the constraint is $\mathbf{w}^H\tilde{\mathbf{a}}(\theta) = 1$ where $\tilde{\mathbf{a}(\theta)} = \mathbf{a}(\theta) + \Delta(\theta_E)$, with Δ being the steering vector error

[2]Using an Intel Core i7-5930K, 32GB RAM, desktop computer.

[3]Each solution treated different transmitted symbols, different noise realization, and different interfering signals waveforms.

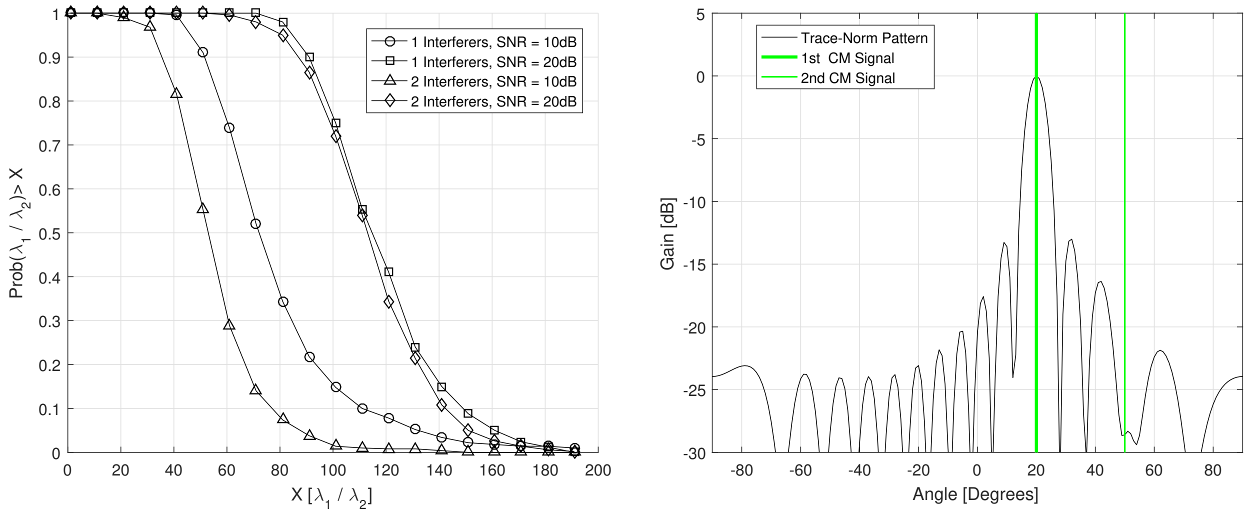

Fig. 1. (a) The ratio between the first and second largest eigenvalues of $\hat{\mathbf{W}}$. (b) Averaged array pattern of the Trace-Norm solution, over 1000 experiments, with two CM signals: unit power from $20°$, and attenuated by random attenuation ($0dB$ to $-5dB$) from $50°$.

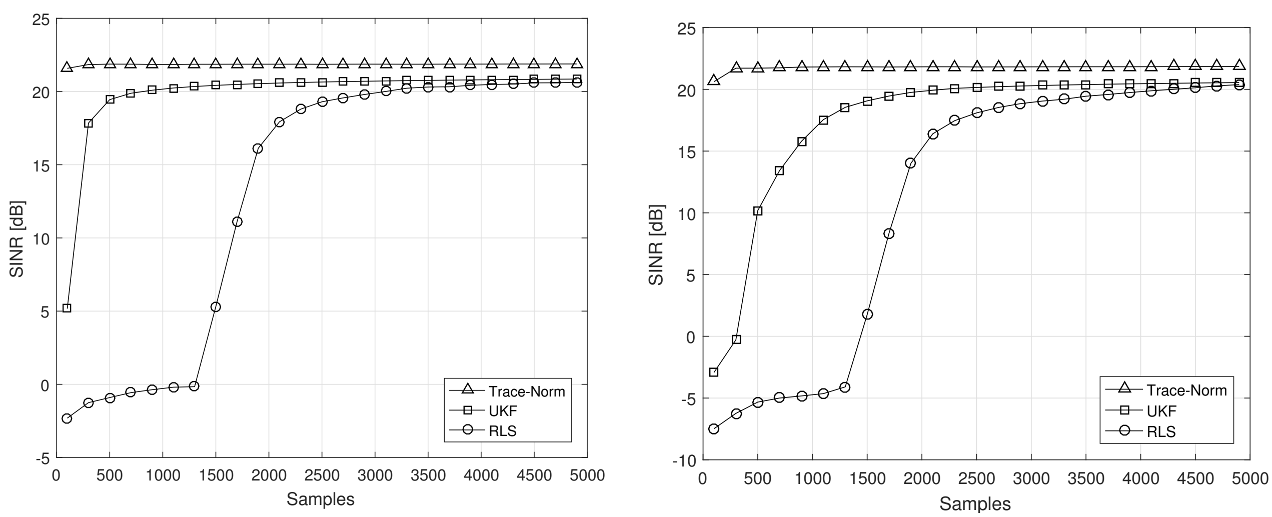

Fig. 2. (a) Output SINR of the Trace-Norm, UKF and RLS ($p = 1$, $\lambda = 0.985$, $\delta = 0.001$) vs. number of samples (N), with noise variance $\sigma_n^2 = 0.1$, CM signal at $20°$; and 3 interferers at $-45°$,$-15°$ and $40°$; (b) with a 4-th interferer at $60°$.

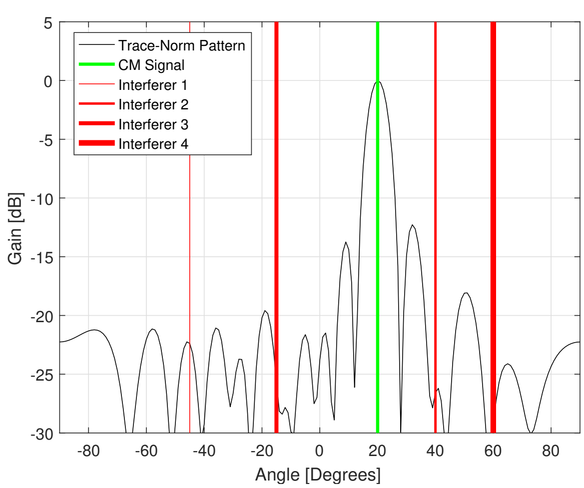

Fig. 3. Array pattern of the Trace-Norm solution, with the 4 interferers.

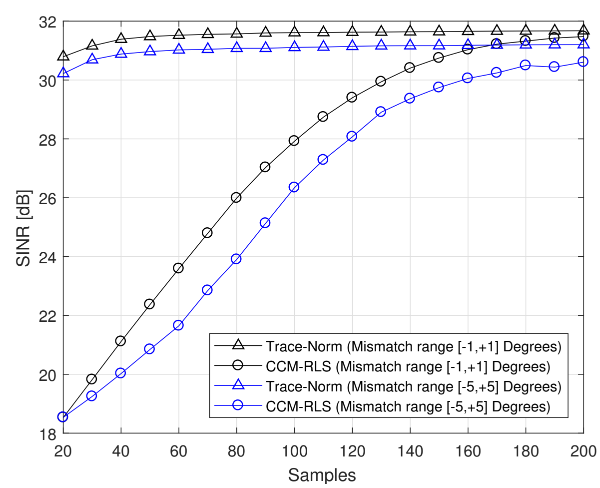

Fig. 4. SINR vs. steering angle error of the LCCMA Trace-Norm and CCM-RLS [11].

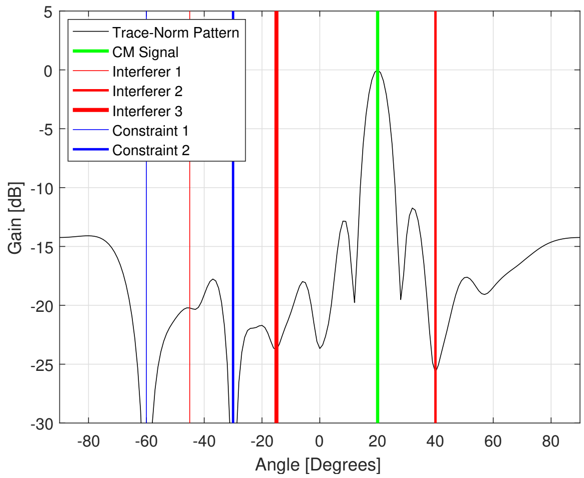

Fig. 5. LCCMA Trace-Norm array pattern, with null constraints at $-30°, -60°$ and 3 interferers ($N = 200$ samples, SNR = 10dB).

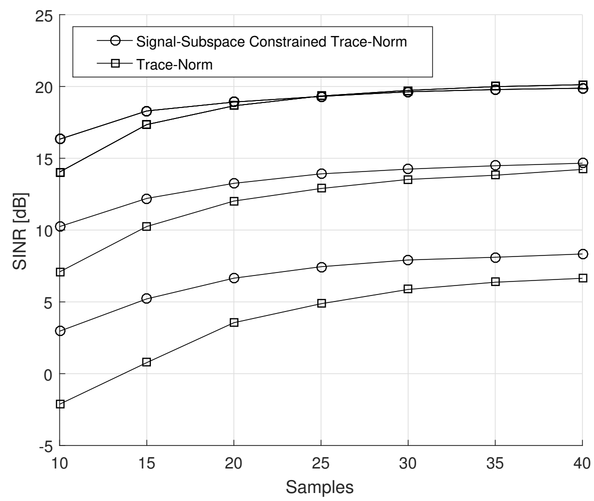

Fig. 6. SINR of the Trace-Norm vs. the Signal-Subspace Constrained Trace-Norm ($P = 32$ elements) in the presence of a CM signal at $20°$, and 2 interferers at $-45°$,$-20°$: SNR = -5dB (lower curves); SNR = 0dB (middle); and SNR = 5dB (upper).

component, resulting from a steering angle error θ_E. Fig. 3(a) compares output SINR performance of the Trace-Norm LCCMA to the CCM-RLS [14] for the following scenario: the CM signal of interest is impinging from $20°$ and two CM interferers are impinging from $40°$ and $60°$, all QPSK modulated with SNR of 20dB. Performance was evaluated for a steering angle mismatch error uniformly distributed in the range of $[-1°, +1°]$ and $[-5°, +5°]$. The robustness of the Trace Norm algorithm is clearly visible, in addition to the faster convergence rate, as compared to CCM-RLS.

Experiment 5 demonstrates the ability of the Trace-Norm LCCMA to generate deep nulls in the array pattern in predefined directions, using the constraint (11),(12). The simulated scenario includes a CM signal at $20°$, and 3 interferers from $-45°$,$-15°$ and $40°$ ($\sigma_n^2 = 0.1$). The nulls are constrained to directions $-30°$ and $-60°$. The resulting array pattern, averaged over 1,000 experiments, is depicted in Fig. 3(b). Clearly visible is the rejection of all interferers, as well as the deep nulls in the specified directions.

Experiment 6 demonstrates the performance advantage of the Trace Norm LCCMA over the Trace-Norm CMA when the constraint (11),(13) is imposed. The simulated scenario includes a CM signal impinging from $20°$ on a $P = 32$ elements ULA, with 2 interferers impinging from $-45°$ and $-20°$. The SNR per array element is varied between -5dB to 5dB. The constraint (11),(13) forces the beamforming vector to be confined to the *3-dimensional signal subspace*. Fig. 3(c) shows SINR results vs. the number of samples (N). The results demonstrate the advantage of the Trace Norm LCCMA over the Trace-Norm CMA for all signal-to-noise ratios (excluding a minor disadvantage for SNR=5dB and $N > 30$ samples).

B. Blind MUD Performance Evaluation

In this section we present blind MUD simulation results illustrating the performance of the proposed Trace Norm solution, as compared to the Linearly Constrained CMA RLS (LCCMA-RLS) [14], The Minimum Output Energy (MOE-MUD) [17], the Subspace-based blind multiuser detector (SUB-MUD) [17], and the Minimum Mean Squared Error with Tikhonov Reguralization (MMSE-Tikhonov) [18].

The MOE-MUD detector under the constraint $\mathbf{w}^H\mathbf{s}_1 = 1$ is given by [17]:

$$\hat{\mathbf{w}} = (\mathbf{s}_1^H\hat{\mathbf{R}}^{-1}\mathbf{s}_1)^{-1}\hat{\mathbf{R}}^{-1}\mathbf{s}_1, \tag{51}$$

where $\hat{\mathbf{R}}$ is the estimated covariance matrix. The blind subspace-based MUD requires explicit knowledge of the number of users K, and is given by [17]:

$$\hat{\mathbf{w}} = (\mathbf{s}_1^H\mathbf{U}_s\mathbf{\Lambda}_s^{-1}\mathbf{U}_s^H\mathbf{s}_1)^{-1}\mathbf{U}_s\Lambda_s^{-1}\mathbf{U}_s^H\mathbf{s}_1, \tag{52}$$

where $\mathbf{U}_s$ and $\mathbf{\Lambda}_s$ are computed from the eigenvalue decomposition of the estimated covariance $\hat{\mathbf{R}}\mathbf{U} = \mathbf{U}\Lambda$. $\mathbf{U}_s$ includes the K leading eigenvectors of $\mathbf{U}$, and spans the signal subspace. Similarly, Λ_s includes the K leading eigenvalues of Λ.
The Minimum Mean Squared Error with Tikhonov Regularization is given by [18]:

$$\hat{\mathbf{w}} = [\mathbf{s}_1^H(\hat{\mathbf{R}} + \alpha\mathbf{I})^{-1}\mathbf{s}_1]^{-1}(\hat{\mathbf{R}} + \alpha\mathbf{I})^{-1}\mathbf{s}_1, \tag{53}$$

where $\alpha = m \times tr(\hat{\mathbf{R}})$ ($m = 0.01$ in our experiments).
All users were simulated as QPSK signals, spread by Gold sequences with $P = 31$ chips. The desired user was simulated with unit power, whereas the interfering users were simulated with amplitudes $A_k > 1$. The noise was simulated as a complex Gaussian with zero mean and covariance $\sigma_n^2\mathbf{I}$. The performance measure employed is the SINR at the MUD detector output:

$$SINR = \frac{\mathbf{w^H}R_{ss}\mathbf{w}}{\mathbf{w}^H\mathbf{R}_{nn}\mathbf{w} + \mathbf{w}^H\mathbf{R}_{ii}\mathbf{w}}, \tag{54}$$

where $\mathbf{R}_{ss} = \mathbf{s}_1\mathbf{s}_1^H$, $\mathbf{R}_{nn} = \sigma_n^2\mathbf{I}$, and $\mathbf{R}_{ii} = \sum_{k=2}^K A_k^2\mathbf{s}_k\mathbf{s}_k^H$, are the desired signal, noise and multiple-access interference (MAI) covariance matrices, respectively. All presented results

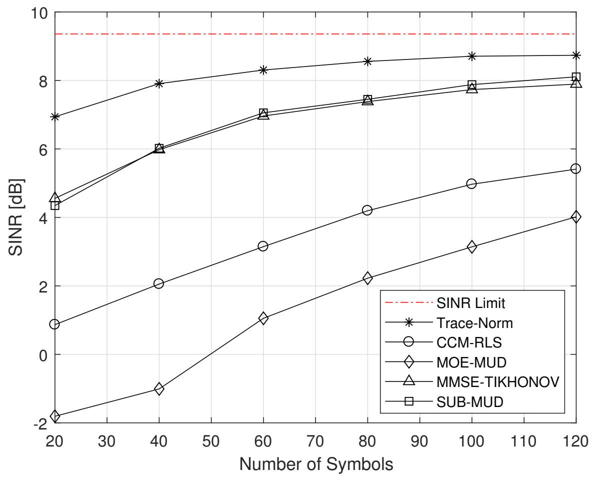

Fig. 7. MUD output SINR [dB] of the Trace-Norm solution vs. CCM-RLS, MOE, MMSE-TIKHONOV and Subspace-based MUD. P=31 chips, K=5 users, QPSK modulation, SNR=10dB, Interference-to-Signal-Ratio=10dB.

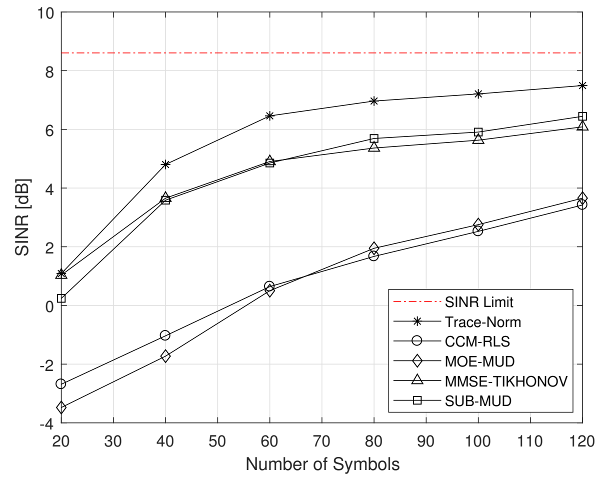

Fig. 8. MUD output SINR [dB] of the Trace-Norm solution vs. CCM-RLS, MOE, MMSE-TIKHONOV and Subspace-based MUD. P=31 chips, K=10 users, QPSK modulation, SNR=10dB, Interference-to-Signal-Ratio=10dB.

were averaged over 100 experiments.

Experiment 1 evaluates MUD SINR output as a function of the number of symbols (20 to 120). Figure 7, presents the results with $K = 5$ users, interference-to-signal ratio of 10dB, and SNR of 10dB, demonstrating the superior performance of the Trace-Norm. Figure 8, presents the results with $K = 10$ users, further demonstrating the superior performance of the Trace-Norm, in this challenging scenario. The SINR limit was computed using the ground truth covariance matrix $R = \sum_{k=1}^{K} A_k^2 \mathbf{s}_k \mathbf{s}_k^T + \sigma_n^2 \mathbf{I}$, and by computing $\mathbf{w}$ using the MMSE detector with $\hat{R} = R$.

Experiment 2 evaluates MUD SINR output as a function of the number of symbols (20 to 120), in the presence of spreading code mismatch due to multi-path propagation: the code of the desired user was distorted by the channel $\mathbf{h} = [0.925, 0, -0.1, 0, 0.2]$, resulting in an average correlation of 0.975 between the correct and distorted codes. Figure 9, presents the results with $K = 5$ users, interference-to-signal ratio of 10dB, and SNR of 10dB, demonstrating the superior performance of the Trace-Norm. Figure 10, presents the results with $K = 10$ users, further demonstrating the superior performance of the Trace-Norm, in this challenging scenario.

Experiment 3 evaluates SINR performance of the Trace-Norm solution with the multi-path constraints (45)-(46), in the presence of the channel $\mathbf{h} = [0.925, 0, -0.1, 0, 0.2]$, as depicted in Fig. 11, with $K = 10$ users, interference-to-signal ratio of 10dB, and SNR of 15dB. Fig. 12 demonstrates the results with $K = 5$ users and a more challenging channel $\mathbf{h} = [0.85, -0.125, 0.15, 0.45, -0.1, 0.05]$ (of longer delay spread). Fig. 13, presents the constellation at the detector output.

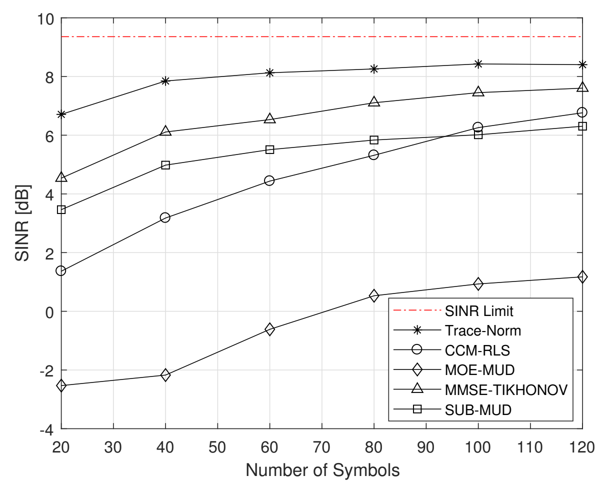

Fig. 9. MUD output SINR [dB] in the presence of spreading code mismatch due to multi-path propagation with the channel $\mathbf{h} = [0.925, 0, -0.1, 0, 0.2]$. Spreading codes of P=31 chips, K=5 users, QPSK modulation, SNR=10dB, Interference-to-Signal-Ratio=10dB.

IX. Conclusions

We have presented a new convex-optimization-based approach to the constant modulus problem, and to the related problems of linearly constrained constant modulus and modified constant modulus. This approach is based on casting these problem as rank-1 matrix minimization problems, and then transforming them to convex optimization problems by replacing the rank-1 constraint by its convex surrogate - the minimization of the trace norm. As solutions to convex optimization problems, the proposed solutions are free from the local minima problem hindering the existing solutions.

We have demonstrated the effectiveness of the proposed solutions in simulated experiments of typical scenarios in blind beamforming and blind multiuser detection. In all these experiments the proposed solution have shown superior per-

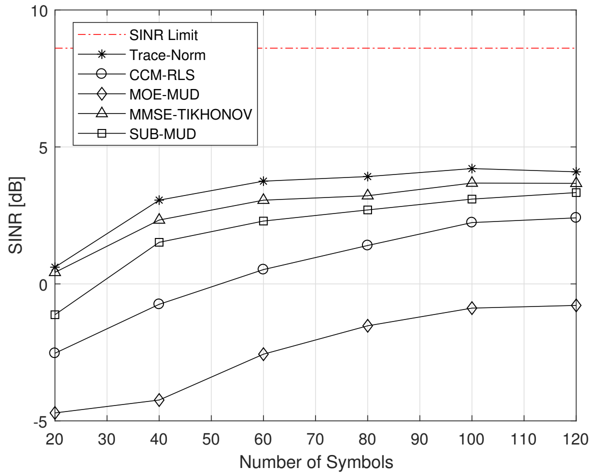

Fig. 10. MUD output SINR [dB] in the presence of spreading code mismatch due to multi-path propagation with the channel $\mathbf{h} = [0.925, 0, -0.1, 0, 0.2]$. Spreading codes of P=31 chips, K=10 users, QPSK modulation, SNR=10dB, Interference-to-Signal-Ratio=10dB.

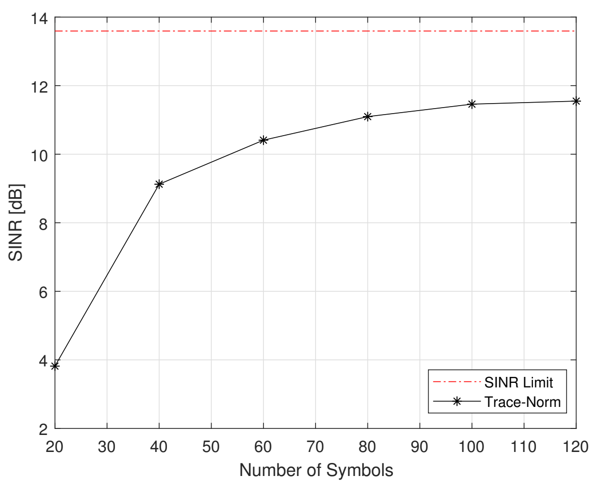

Fig. 11. Trace-Norm SINR performance with multi-path constraints (45)-(46), and the channel $\mathbf{h} = [0.925, 0, -0.1, 0, 0.2]$, spreading codes of P=31 chips, K=10 users, QPSK modulation, SNR=15dB, Interference-to-Signal-Ratio=10dB.

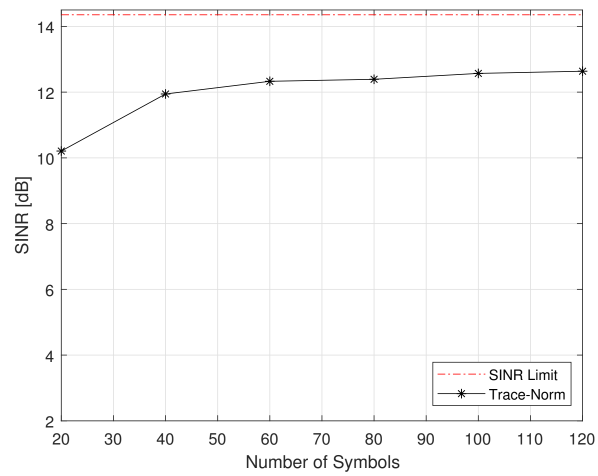

Fig. 12. Trace-Norm SINR performance with multi-path constraints (45)-(46), and the channel $\mathbf{h} = [0.85, -0.125, 0.15, 0.45, -0.1, 0.05]$, spreading codes of P=31 chips, K=5 users, QPSK modulation, SNR=15dB, Interference-to-Signal-Ratio=10dB.

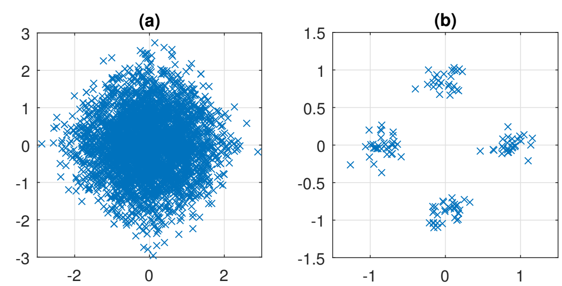

Fig. 13. Trace-Norm performance with multi-path constraints (45)-(46), and the channel $\mathbf{h} = [0.85, -0.125, 0.15, 0.45, -0.1, 0.05]$, spreading codes of P=31 chips, K=5 users, QPSK modulation, SNR=15dB, Interference-to-Signal-Ratio=10dB: (a) Detector input; (b) Trace-Norm detector output (120 symbols, SINR = 12.6dB).

formance over the existing solutions, especially in challenging conditions of low number of samples/symbols.

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