

Entanglement entropy for (3+1)-dimensional topological order with excitations

Xueda Wen,¹ Huan He,² Apoorv Tiwari,^{3,4} Yunqin Zheng,² and Peng Ye³

¹*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

²*Physics Department, Princeton University, Princeton, New Jersey 08544, USA*

³*Department of Physics and Institute for Condensed Matter Theory, University of Illinois, 1110 W. Green Street, Urbana, Illinois 61801, USA*

⁴*Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada N2L 2Y5*



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Excitations in (3+1)-dimensional [(3+1)D] topologically ordered phases have very rich structures. (3+1)D topological phases support both pointlike and stringlike excitations, and in particular the loop (closed string) excitations may admit knotted and linked structures. In this work, we ask the following question: How do different types of topological excitations contribute to the entanglement entropy or, alternatively, can we use the entanglement entropy to detect the structure of excitations, and further obtain the information of the underlying topological order? We are mainly interested in (3+1)D topological order that can be realized in Dijkgraaf-Witten (DW) gauge theories, which are labeled by a finite group G and its group 4-cocycle $\omega \in \mathcal{H}^4[G; \text{U}(1)]$ up to group automorphisms. We find that each topological excitation contributes a universal constant $\ln d_i$ to the entanglement entropy, where d_i is the quantum dimension that depends on both the structure of the excitation and the data (G, ω) . The entanglement entropy of the excitations of the linked/unlinked topology can capture different information of the DW theory (G, ω) . In particular, the entanglement entropy introduced by Hopf-link loop excitations can distinguish certain group 4-cocycles ω from the others.

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I. INTRODUCTION

The long-range entanglement of topologically ordered phases in (2+1) dimensions [(2+1)D] is closely related with the exotic features such as fractional quasiparticle statistics and topologically protected ground-state degeneracies (GSD) [1–24]. By studying the topological entanglement entropy [4,5] (TEE) of topological phases on nontrivial spatial manifolds such as a T^2 , we can extract the universal topological data, for instance, the modular \mathcal{S} , \mathcal{T} matrices, of the emergent topological quantum field theory (TQFT) [7,8,25–33].

Topological phases in (3+1) dimensions [(3+1)D] have attracted extensive attention recently [26,34–56]. One of the demonstrative examples of topological phases in (3+1)D is the gauged symmetry-protected topological phases (gSPT) [36,51,57–60]. The symmetry-protected topological (SPT) phases are topologically nontrivial gapped phases of matter protected by global symmetries [61–87]. The global symmetries of SPT phases can be gauged to obtain gSPT phases which support fractional excitations. The underlying TQFT describing the gSPT is the Dijkgraaf-Witten (DW) gauge theory [88], which is characterized by a pair (G, ω) up to group automorphisms, where G is a finite group and $\omega \in \mathcal{H}^4[G; \text{U}(1)]$ is its group 4-cocycle. For the trivial cocycle ω , namely an identity function from $(G)^4$ to $\text{U}(1)$, the DW theory is called “untwisted,” while for nontrivial cocycles, the DW theory is called “twisted.”

Most recently, it has been proposed that the (3+1)D topological order whose pointlike excitations are all bosons can be classified by unitary pointed fusion 2-categories [89]. In particular, all of such (3+1)D topological order can be realized by DW gauge theories. Throughout this work, we are interested in the (twisted/untwisted) DW theories. The excitations of

DW theories in three spatial dimensions include pointlike excitations and stringlike excitations with possible knot and link topologies as depicted in Fig. 1.

The ground-state entanglement entropy of an untwisted DW theory with a general discrete gauge group G on R^3 has been studied based on Kitaev’s quantum double in Refs. [90,91]. The entanglement entropy with respect to a T^2 entanglement cut is $S = V \ln |G| - \gamma$, where

$$\gamma = \ln |G| \quad (1.1)$$

is the TEE, $|G|$ is the order of gauge group G , and V is the number of vertices on the boundary of the subregion which counts the area part of the entanglement entropy.

In this paper, aside from the ground-state entanglement entropy, we study the effects of various excitations on the entanglement entropy in (3+1)D DW theories, in order to detect (G, ω) that characterizes a DW theory. To calculate the contributions of the excitations to the entanglement entropies, we utilize the correspondence between minimal entangled states (MESs) and excitations [8]. It is known that MESs can be viewed as the bases for the degenerate ground states on a d -torus and correspond to different excitations by cutting the d -torus (see Fig. 4 for the case of T^3). The MESs may also be used to construct the modular \mathcal{S} and \mathcal{T} matrices. See, e.g., Refs. [8,92] for (2+1)D and Refs. [37,39,56] for (3+1)D topologically ordered phases.

Based on this MES/excitation correspondence, we find that each topological excitation contributes to the entanglement entropy with a universal constant

$$\Delta S = \ln d_i, \quad (1.2)$$



FIG. 1. Examples of pointlike and stringlike excitations in (3+1)D topologically ordered phases. From left to right: a point excitation; a loop (or closed string) excitation; a trefoil-knot loop excitation; and a Hopf-link loop excitation.

where d_i is the quantum dimension of the topological excitation. It depends on both the topologies of the excitations and (G, ω) . In this paper, we mainly focus on three types of excitations in a DW theory, i.e., pointlike excitations, single-loop excitations, and Hopf-link loop excitations, as interpreted in the following.

(i) *Pointlike excitations.* Pointlike excitations can be created either by local operators or by nonlocal open string operators (or Wilson line operators). The former are called local pointlike excitations and the latter are called topological pointlike excitations, and in this work we are mainly interested in the latter case. Two topological pointlike excitations are considered to be topologically equivalent if they differ by adding/removing local pointlike excitations. As discussed in Ref. [89], the pointlike excitations in (3+1)D topologically ordered phases are characterized by a symmetric fusion category (SFC), which describes a collection of particles with trivial mutual statistics. Depending on whether there are quasiparticles with fermionic statistics or not, the pointlike excitations are described by $\text{sRep}(G)$ or $\text{Rep}(G)$, respectively.¹ For the DW theory, there are only bosonic pointlike excitations which are described by $\text{Rep}(G)$. Then, the quantum dimension of a pointlike excitation is

$$d_i = \dim[\text{Rep}_i(G)]. \quad (1.3)$$

(ii) *Single-loop excitations.* A single-loop excitation here refers to a loop excitation without any knotted structures. It can be created at the boundary of a membrane operator on a disk. Each single-loop excitation is characterized by one flux (conjugacy class χ) and one charge (irreducible representation of the centralizer G_χ with respect to χ) [89,93–95]. That is, one may use a pair $[\chi, \text{Rep}(G_\chi)]$ to label each single-loop excitation. For a pure loop excitation which carries no charge, it is in the trivial irreducible representation of G_χ . From the group theory point of view, it is known that the number of conjugacy class equals the number of irreducible representations of the finite group G , which indicates that the number of pure single-loop excitation types (i.e., fluxes without attaching charges) equals the number of pointlike excitation types. By including the effect of charge, one can find the quantum dimension of a single-loop excitation as follows [89]:

$$d_{\chi;i} = |\chi| \cdot \dim[\text{Rep}_i(G_\chi)], \quad (1.4)$$

¹Mathematically, $\text{sRep}(G)$ is a category formed by the representations of G , with some of the irreducible representations assigned bosonic statistics, while the others assigned Fermi statistics. $\text{Rep}(G)$ is a category formed by the representations of G , with all the irreducible representations assigned bosonic statistics. See, e.g., Ref. [105] for more details.

where $|\chi|$ denotes the order of the conjugacy class χ .

(iii) *Hopf-link loop excitations.* For a Hopf-link loop excitation (see Fig. 1), there are two loops linked with each other, with each loop carrying a flux labeled by the conjugacy class $\chi_{i=1,2}$. These two conjugacy classes should commute with each other in the following sense [38,39]. (See also Ref. [26] from the dimension reduction point of view.) First, there exist $g \in \chi_1$ and $h \in \chi_2$ such that $gh = hg$. Then, one can find that the other elements $g' := kgk^{-1} \in \chi_1$ and $h' := khk^{-1} \in \chi_2$, $\forall k \in G$ also commute, i.e., $g'h' = h'g'$. To define the charge of the excitation, first we define the centralizer with respect to g and h as $G_{g,h} := \{x \in G | xg = gx, xh = hx\}$. Then, one can find that $G_{g,h}$ is isomorphic to $G_{g',h'}$, with g', h' defined above. It is convenient to denote these isomorphic centralizers as G_{χ_1, χ_2} . Then, the charge carried by this Hopf-link loop excitation is described by the irreducible *projective* representation of the centralizer G_{χ_1, χ_2} .

We note that for a non-Abelian gauge group G , there are some subtleties in characterizing a Hopf-link loop excitation and defining its quantum dimension, which is not yet well understood to our knowledge.² In this work, for the Hopf-link loop excitations, we will mainly focus on the Abelian gauge group, e.g., $G = (\mathbb{Z}_N)^4$. Then, each group element is itself a conjugacy class, and $G_{\chi_1, \chi_2} = G$. Then, the quantum dimension can be defined as [37–39]

$$d_{\chi_1, \chi_2; i} = \dim[\widetilde{\text{Rep}}_{\chi_1, \chi_2; i}(G)]. \quad (1.5)$$

Here, we use $\widetilde{\text{Rep}}_i(G)$ to distinguish it from the linear representation $\text{Rep}_i(G)$. As will be shown later, the projective representation $\widetilde{\text{Rep}}_i(G)$ carried by the Hopf-link loop excitation contains the information of the 4-cocycles ω . If the 4-cocycle ω in the DW theory is trivial, the projective representation will automatically reduce to a linear representation.³ Here, we keep the index χ_1, χ_2 in (1.5) to remind ourselves that the projective representation depends on the conjugacy classes χ_1 and χ_2 . Remarkably, it has been understood that a higher dimensional $d_{\chi_1, \chi_2; i}$ in (1.5) is closely related with non-Abelian three-loop braiding in a DW theory [38].

In this paper, through the entanglement entropy, we will detect the quantum dimensions in (1.3), (1.4), and (1.5) for different excitations. There are more topologically complicated excitations for loop excitations. For example, the loop excitation can be a trefoil knot in Fig. 1. Nevertheless, we restrict ourselves to only discussing the three simple cases listed above since it is less clear to us how the complex knotted/linked loop excitations correspond to MESs on the T^3 .

The rest of the paper is organized as follows. In Sec. II, we give a general consideration on the entanglement entropy in (3+1)D DW theory from the partition function point of view. In Sec. III, we give a brief introduction of the Hamiltonian version of (3+1)D DW theory and then introduce the form of MESs on a T^3 . Then, we study the effect of different excitations

²We thank C. Wang for pointing out this issue to us.

³As we will see later, even for nontrivial 4-cocycles, it is still possible that the projective representation reduces to a linear representation, as long as the induced 2-cocycles are trivial.

on the entanglement entropy in Sec. IV. In particular, we study the (untwisted) DW theory with a non-Abelian dihedral gauge group $G = D_3$, and the twisted type-IV DW theory with an Abelian gauge group $G = (\mathbb{Z}_N)^4$. Then, we summarize and remark on the main results of this work in Sec. V, and point out some future problems.

II. TOPOLOGICAL ENTANGLEMENT ENTROPY IN (3 + 1)D DIJKGRAAF-WITTEN THEORIES

An elegant way to study the entanglement entropy in a (3+1)D DW theory is based on the surgery and the partition function method. The spirit is similar to the calculation of the entanglement entropy in a Chern-Simons theory in (2+1)D [7]. It allows us to express the entanglement entropy in terms of partition functions of a (3+1)D DW theory on certain simple 4-manifolds.

Now, we consider the ground state of a DW theory on a 3-sphere (in three spatial dimensions) described by the wave functional $|\Psi\rangle$. By considering a bipartition into spatial subregions A and B (see Fig. 2), we may define the reduced density matrix as $\rho_A = \text{tr}_B |\Psi\rangle\langle\Psi|$. Let the entanglement cut between subregions A and B have the topology of a 2-sphere. By the general axioms of TQFT [96,97], it is known that a path integral on D^4 produces the state $|\Psi\rangle$ in the Hilbert space \mathcal{H}_{S^3} which is one dimensional. We first notice that $\text{tr} \rho_A = \text{tr}(\rho_A^n) = \mathcal{Z}(S^4)$ [7,98]. Then, the n th Rényi entropy is

$$S_A^{(n)} := \frac{1}{1-n} \ln \frac{\text{tr} \rho_A^n}{(\text{tr} \rho_A)^n} = \ln \mathcal{Z}(S^4). \tag{2.1}$$

Since the result is independent of the Rényi index n , the entanglement entropy is the same as the n th Rényi entropy as $S_A^{\text{topo}} = \lim_{n \rightarrow 1} S_A^{(n)}$. Here, we use “topo” in S_A^{topo} indicating that the partition function method based on a TQFT only captures the TEE for the ground-state case, without including the area part of the entanglement entropy.

Let us specialize to the case of a DW theory (G, ω) . Since the cohomology of the classifying space BG is same as the group cohomology, we may write $\omega \in \mathcal{H}^4[BG, U(1)]$. The classifying space has the property that isomorphism classes of flat G bundles on a compact oriented manifold M is equivalent to homotopy classes of maps $[\gamma] : M \rightarrow BG$. Alternatively, γ can be viewed as the homomorphism of the fundamental

group $\pi_1(M)$ of the manifold M into the group G , up to conjugation [88]. Then, the partition function for a DW theory can be defined as the sum over all possible G bundles over M , weighted by $W = e^{2\pi i S}$ (here, S is the action of the DW theory) [88,99]

$$\mathcal{Z}(M) = \frac{1}{|G|} \sum_{[\gamma]} W(\gamma), \tag{2.2}$$

with

$$W(\gamma) = \langle \gamma^* \omega, [M] \rangle, \tag{2.3}$$

where $\gamma^* \omega \in \mathcal{H}^4[M, U(1)]$ is the pullback of ω onto M and $[M]$ denotes the fundamental homology class of M . In other words, $\langle \gamma^* \omega, [M] \rangle$ is a topological action that furnishes a pure $U(1)$ phase when evaluated on a specific choice of a flat bundle. The normalization factor of $1/|G|$ can be justified by the fact that it gives the correct ground-state degeneracy for a three-sphere where $\text{GSD}(S^3) = \mathcal{Z}(S^3 \times S^1)$. For $M = S^1 \times S^3$, one has $\pi_1(S^1 \times S^3) = \mathbb{Z}$. In addition, $\text{Hom}(\mathbb{Z}, G) \simeq G$. Then, one has

$$\mathcal{Z}(S^1 \times S^3) = \frac{1}{|G|} (1 \times |G|) = 1. \tag{2.4}$$

Here, we have used $W(\gamma) = 1$, i.e., a cocycle evaluated on a trivial bundle is 1.⁴

For $G = (\mathbb{Z}_n)^k$, which is a main example we will study in this work, partition functions take the explicit forms

$$\begin{aligned} \mathcal{Z}_{\text{II,III}}(M) &= \frac{1}{|G|} \sum_{[A]} e^{\frac{2\pi i p}{n^2} \int_M A' U A' U \delta A^k}, \quad p \in \mathbb{Z}_n \\ \mathcal{Z}_{\text{IV}}(M) &= \frac{1}{|G|} \sum_{[A]} e^{\frac{2\pi i p'}{n} \int_M A' U A^j U A^k U A^l}, \quad p' \in \mathbb{Z}_n. \end{aligned} \tag{2.5}$$

Here, the subscripts “II, III, IV” are meant to denote the partition functions corresponding to type-II, -III, -IV cocycles, respectively [38,100]. $A^l \in \mathcal{H}^1(M, \mathbb{Z}_n)$ and $\delta A^l \in \mathcal{H}^2(M, n\mathbb{Z})$ are obtained using a Bockstein homomorphism (for technical details of simplicial calculus and Bockstein homomorphisms, we refer the reader to Appendix A of Ref. [101])

$$\beta : \mathcal{H}^1(M, \mathbb{Z}_n) \rightarrow \mathcal{H}^2(M, n\mathbb{Z}), \tag{2.6}$$

and p, p' in Eq. (2.5) are a set of \mathbb{Z}_n valued parameters that represent different choices of cocycles $\omega \in \mathcal{H}^4[G, U(1)]$. The integral may be evaluated by picking a triangulation of M and

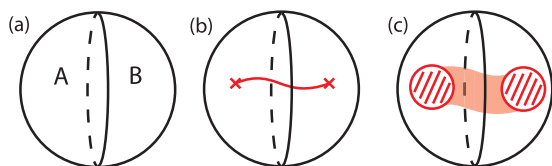


FIG. 2. Bipartition of a spatial manifold S^3 , where the interface between subsystems A and B is a S^2 . This spatial manifold S^3 can be viewed as the surface of a space-time 4-manifold D^4 , where the wave functional is defined. The system may be in (a) a ground state, (b) an excited state with a pair of pointlike excitations, and (c) an excited state with a pair of loop excitations which may admit a complicated knotted/linked structure. The two excitations are connected by a two-dimensional membrane with possible internal structure.

⁴First, since $W(\gamma)$ is evaluated on a map in a particular homotopy class $[\gamma] : M \rightarrow BG$, it must by definition be insensitive to deformations that leave it in the same homotopy class. Then, noticing that there are no nontrivial bundles on S^3 , the map γ can be deformed such that its pullback to M gives a G coloring of a triangulation of $M = S^3 \times S^1$ which only has nontrivial elements of G along S^1 and identity everywhere else. Now, we recall that in general a G coloring is an assignment of four elements of G to each 4-simplex. Clearly, all 4-simplices of the above coloring have at least one (actually three) trivial elements. Furthermore, since one builds Dijkgraaf-Witten theories from normalized cocycles, i.e., those such that for $[\omega] \in H_{\text{group}}[G, U(1)]$, we require $\omega(1, g_1, g_2, g_3) = 1$. Therefore, $W(\gamma) = 1$ for a G bundle on $S^3 \times S^1$.

then explicitly assigning $U(1)$ phases to each 4-simplex via the cochain provided above. Further different cochains are glued (i.e., multiplied) together and then we sum over all possible isomorphism classes of colorings of the triangulation. This amounts to summing over configurations $[A] \in \mathcal{H}^1(M, \mathbb{Z}_n)$.

In particular, S^4 admits a unique flat bundle which is trivial. Therefore,

$$\mathcal{Z}(S^4) = 1/|G|. \quad (2.7)$$

Based on Eqs. (2.1) and (2.7), the TEE for the ground state $|\Psi\rangle$ evaluates to

$$S_A^{\text{topo}} = -\ln |G|. \quad (2.8)$$

This generalizes previous results on an untwisted DW theory to a twisted DW theory. It is noted that the entanglement cut here is chosen as a 2-sphere S^2 . As shown in Appendix A, we also consider the type-IV DW theory with $G = (\mathbb{Z}_N)^4$ with an entanglement cut T^2 . It is found that the TEE is the same as Eq. (2.8).

Now, we give several remarks of the entanglement entropies on the case with excitations. First, the above method can be straightforwardly generalized when excitations are introduced. Shown in Figs. 2(b) and 2(c) are examples of excited states with a pair of pointlike excitations and a pair of loop excitations with possible knotted or linked structure. By repeating the above procedure, one can find the entanglement entropy

$$S_A = \ln \mathcal{Z}(S^4, i), \quad i = \circ, \# \quad (2.9)$$

where \circ represents a Wilson loop, and $\#$ represents a closed two-dimensional Wilson membrane that may potentially admit a complicate structure. Since it is difficult to evaluate the partition function $\mathcal{Z}(S^4, \circ)$ or $\mathcal{Z}(S^4, \#)$ with Wilson loop/membrane inserted in for a (3+1)D DW theory, in order to investigate the effect of excitations on the quantum entanglement, we resort to the exactly solvable model as discussed in the following sections. Based on the calculation in Sec. IV, the partition function $\mathcal{Z}(S^4, i)$ with $i = \circ, \#$ for a (3+1)D DW theory has the expression $\mathcal{Z}(S^4, i) = d_i/|G|$, where d_i are the quantum dimensions of the topological excitations, and then the entanglement entropy has the form⁵ $S_A = \ln d_i - \ln |G|$.

III. (3+1)D TWISTED GAUGE THEORY AND MES

The Hamiltonian version of a DW theory has been studied in both (2+1)D [92,102] and (3+1)D [37–39], where the canonical basis for the degenerate ground states on T^2 [in (2+1)D] and T^3 [in (3+1)D] are also constructed. By calculating the basis overlaps under the modular \mathcal{S}, \mathcal{T} transformations, one can obtain the modular matrices of the underlying TQFT [39,92] in the quasiexcitation basis. For the purpose of our work, we will give a very brief review of this model by mainly focusing on the ingredients of MESs on a T^3 . One can refer to Ref. [39] for more details.

⁵We emphasize that here the entanglement cut is a S^2 , while the entanglement cut considered in Sec. IV is a T^2 . It is our future work to investigate if the effect of topological excitations on entanglement entropy depends on the topology of entanglement cut or not and, if yes, how it depends on the entanglement cut.

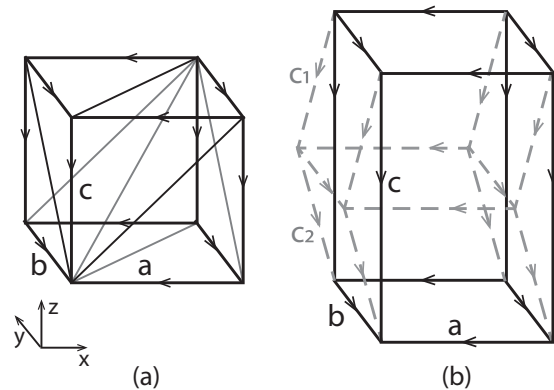


FIG. 3. (a) Triangulation of a T^3 on a single cube with six tetrahedrons. Periodic boundary conditions are imposed along each direction. (b) Evolution from a single cube to two cubes, on which a three-torus is triangulated. Not every bond in the triangulated T^3 is shown here.

The model is characterized by (H, G, ω) , where H denotes the Hamiltonian, G is a finite (Abelian or non-Abelian) gauge group, and ω is a 4-cocycle. When the 4-cocycle ω is trivial, the model reduces to an untwisted discrete gauge theory. The Hilbert space is defined as follows:

$$\mathcal{H} = \otimes_b \mathcal{H}_b, \quad \mathcal{H}_b = \text{Span}\{|g\rangle | g \in G\}, \quad (3.1)$$

where \mathcal{H}_b is the local Hilbert space on a bond b of the triangulated lattice, and the total Hilbert space is the tensor product of all local Hilbert spaces since we only consider bosonic systems in this work. In our convention, each basis $|g\rangle$ is associated with an orientation, and changing the orientation is equivalent to inverting the group element: $|g, \rightarrow\rangle = |g^{-1}, \leftarrow\rangle$. This convention is same as in Refs. [26,39,103]. The group elements associated with all local Hilbert spaces can be interpreted as gauge fields in lattice gauge theory language. The Hamiltonian is defined on a three-dimensional triangulated manifold as follows [39]:

$$H = -\sum_v A_v - \sum_p B_p, \quad (3.2)$$

where A_v acts on the vertex v , and generates a gauge transformation of the group element on each bond that connects to v , B_p imposes the zero-flux condition on each face (a triangle), and $[A_v, B_p] = 0$. The 4-cocycle ω is introduced when we act the operator $A_v = \frac{1}{|G|} \sum_{[vv'] = g \in G} A_v^g$ on vertex v , by lifting the vertex v to v' with $[vv'] = g \in G$. The amplitude of this gauge transformation can be expressed in terms of 4-cocycles ω . (See Ref. [39] for details.)

Putting many details of this model aside, for our purpose, now we mainly focus on the MESs on a triangulated T^3 (see Fig. 3). On the T^3 , we have topologically protected degenerate ground states, due to the global degrees of freedom in the flat gauge field configurations. The MES, a canonical choice of ground-state basis, is labeled by three objects [37–39]:

$$|\chi_1, \chi_2, \mu\rangle, \quad (3.3)$$

where χ_1 and χ_2 denote the “flux” and μ denotes the “charge.” Here, χ_1 and χ_2 are two mutually commuting conjugacy

classes of G , and μ corresponds to the irreducible projective representation of the centralizer G_{χ_1, χ_2} as discussed in Sec. I. Now, let us consider the MES in z direction (see Fig. 3). First, it is noted that the conjugacy class $\chi_{1,2}$ can be used to label the membrane, by considering that the loop excitation is created by a membrane operator. Then, $|\chi_1, \chi_2, \mu\rangle$ in z direction may be viewed as a state with membrane χ_1 inserted in the yz plane, and membrane χ_2 inserted in the xz plane [37]. Naively, one may consider that the ‘‘charge’’ μ corresponds to a Wilson loop inserted in z direction. However, this is not true in general, considering that this charge may be carried by a loop excitation, and has no local source (or cannot be observed locally). The charge in this case is usually called ‘‘Cheshire charge’’ [93,94,104].

The concrete form of $|\chi_1, \chi_2, \mu\rangle$ can be constructed as follows [37–39]:

$$|\chi_1, \chi_2, \mu\rangle = \frac{1}{\sqrt{|G|}} \sum_{\substack{a \in \chi_1, b \in \chi_2; \\ c \in G_{a,b}}} \text{tr}[\tilde{\rho}_\mu^{a,b}(c)]|a, b, c\rangle, \quad (3.4)$$

where a, b , and c are the group elements living on the bonds in x, y , and z directions, respectively [see Fig. 3(a)]. $\tilde{\rho}_\mu^{a,b}$ is the irreducible projective representation of the centralizer $G_{a,b}$, and is determined by

$$\tilde{\rho}_\mu^{a,b}(c)\tilde{\rho}_\mu^{a,b}(d) = \beta_{a,b}(c,d)\tilde{\rho}_\mu^{a,b}(c \cdot d), \quad (3.5)$$

where $\beta_{a,b}(c,d)$, also called a factor system, is a 2-cocycle obtained based on a slant product of the 4-cocycles ω (see Appendix B). Now, let us make a connection between $\tilde{\rho}_\mu^{a,b}(c)$ in Eq. (3.4) and $\widehat{\text{Rep}}_{\chi_1, \chi_2; i}(G)$ for an Abelian group G in Eq. (1.5). Since each group element is itself a conjugacy class, by considering $\chi_1 = \{a\}$, $\chi_2 = \{b\}$, and $c \in G_{a,b} = G$, then $\tilde{\rho}_\mu^{a,b}(G)$ is exactly the same as $\widehat{\text{Rep}}_{\chi_1, \chi_2; i}(G)$ in Eq. (1.5). For an untwisted gauge theory, $\tilde{\rho}_\mu^{a,b}$ reduces to the linear representation $\rho_\mu^{a,b}$. The basis in Eq. (3.4) gives the correct modular \mathcal{S}, \mathcal{T} matrices when we compute the wave-function overlaps under modular transformations [39].

Now, let us discuss the MESs corresponding to the three types of excitations as mentioned in Sec. I, i.e., pointlike excitations, single-loop excitations, and Hopf-link loop excitations.

For a pointlike excitation, both fluxes are trivial, i.e., $\chi_1 = \chi_2 = \{\mathbb{1}\}$, where $\mathbb{1}$ represents the identity element in group G . Then, one has

$$|\mathbb{1}, \mathbb{1}, \mu\rangle = \frac{1}{\sqrt{|G|}} \sum_{c \in G} \text{tr}[\rho_\mu(c)]|\mathbb{1}, \mathbb{1}, c\rangle. \quad (3.6)$$

Here, we use $\rho_\mu(c)$ instead of $\tilde{\rho}_\mu^{\mathbb{1}, \mathbb{1}}(c)$, indicating that $\tilde{\rho}_\mu^{\mathbb{1}, \mathbb{1}}(c)$ reduces to the linear representation of G , as discussed in Appendix B. This case corresponds to a Wilson loop in representation μ threading through z direction of the T^3 , as shown in Fig. 4(a). If we cut the T^3 , one can find a point excitation on the boundary which is a T^2 .

For a single-loop excitation, one of χ_1 and χ_2 is trivial while the other one is nontrivial. Without loss of generality, we choose $\chi_1 = \{\mathbb{1}\}$. Then, one has

$$|\mathbb{1}, \chi, \mu\rangle = \frac{1}{\sqrt{|G|}} \sum_{b \in \chi, c \in G_\chi} \text{tr}[\rho_\mu^{\mathbb{1}, b}(c)]|\mathbb{1}, b, c\rangle. \quad (3.7)$$

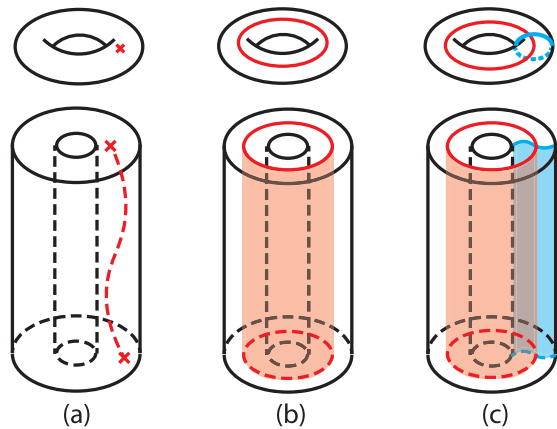


FIG. 4. Lower panel: MESs on a three-torus corresponding to different types of excitations. Upper panel: by cutting the three-torus in the xy plane, one obtains excitations on the boundaries which are T^2 , with (a) a pointlike excitation, (b) a single-loop excitation, and (c) a Hopf-link loop excitation.

Again, similar to the pointlike excitations, here the projective representation $\tilde{\rho}_\mu^{\mathbb{1}, b}(c)$ reduces to a linear representation $\rho_\mu^{\mathbb{1}, b}(c)$ of G_χ (see Appendix B). This case corresponds to the MES in Fig. 4(b). A loop excitation will appear on the boundary T^2 when we cut the T^3 .

The third type of the excitations, i.e., a Hopf-link loop excitation, corresponds to the MES in Eq. (3.4), with $\chi_1 \neq \{\mathbb{1}\}$ and $\chi_2 \neq \{\mathbb{1}\}$. In this case, both fluxes are nontrivial. As shown in Fig. 4(c), when we cut the T^3 , two linked loop excitations which carry fluxes χ_1 and χ_2 , respectively, will appear on the boundary T^2 . It is noted that the MES with nontrivial $\chi_{1,2}$ in Eq. (3.4) is characterized by the projective representation $\tilde{\rho}_\mu^{a,b}$, which is further determined by the 2-cocycle $\beta_{a,b}(c)$ in Eq. (3.5) and 4-cocycle through Eqs. (B2) and (B3). In other words, the information of 4-cocycle ω is contained in a Hopf-link loop excitation. We will illustrate how this is reflected in the entanglement entropy in Sec. IV B.

IV. EFFECT OF EXCITATIONS ON ENTANGLEMENT ENTROPY

A. Untwisted DW gauge theory with $G = D_3$

Now, we consider a discrete gauge theory with a dihedral group D_3 , which is a non-Abelian group and isomorphic to the symmetric group S_3 . Through this example, we aim to illustrate the effect of (i) pointlike excitations and (ii) single-loop excitations on the entanglement entropy.

The dihedral group D_3 is obtained by composing the six symmetries of an equilateral triangle. This group is generated by τ and θ , where τ is a flip about an axis passing through the center of the triangle and one of its vertices; θ is a rotation by $2\pi/3$ about the center of triangle. There are in total six elements, which are generated as $\mathbb{1}, a = \theta, b = \theta^2, c = \tau, d = \tau\theta, e = \tau\theta^2$, and three conjugacy classes: $\chi_{\mathbb{1}} = \{\mathbb{1}\}$, $\chi_2 = \{c, d, e\}$, and $\chi_3 = \{a, b\}$, with the corresponding centralizer subgroups $G_{\chi_1} = D_3, G_{\chi_2} = \mathbb{Z}_2$, and $G_{\chi_3} = \mathbb{Z}_3$.

As studied in Ref. [89], there are three types of pointlike excitations (denoted by p_0, p_1 , and p_2) and five types of

single-loop excitations (denoted by $s_{20}, s_{21}, s_{30}, s_{31}$, and s_{32}). For the pointlike excitations, p_0 corresponds to the trivial (one-dimensional) representation of D_3 , p_1 and p_2 correspond to the one- and two-dimensional representations of D_3 . The loop excitations are labeled by $(\chi, \text{Rep}_i(G_\chi))$ as follows: $s_{20} = (\chi_2, \text{Rep}_0(\mathbb{Z}_2))$, $s_{21} = (\chi_2, \text{Rep}_1(\mathbb{Z}_2))$, $s_{30} = (\chi_3, \text{Rep}_0(\mathbb{Z}_3))$, $s_{31} = (\chi_3, \text{Rep}_1(\mathbb{Z}_3))$, and $s_{32} = (\chi_3, \text{Rep}_2(\mathbb{Z}_3))$. The quantum dimensions of these pointlike and loop excitations are, according to Eqs. (1.3) and (1.4),

$$\begin{aligned} d_{p_0} &= d_{p_1} = 1, & d_{p_2} &= 2, \\ d_{s_{20}} &= d_{s_{21}} = 3, \\ d_{s_{30}} &= d_{s_{31}} = d_{s_{32}} = 2. \end{aligned} \quad (4.1)$$

It is interesting that each loop excitation can shrink to pointlike excitations [89]. For example, the loop excitation s_{21} can shrink to pointlike excitations: $s_{21} \rightarrow p_1 \oplus p_2$, during which the quantum dimension is conserved by considering that $d_{s_{21}} = d_{p_1} + d_{p_2}$. However, it is noted that in the configuration in Fig. 4, the loop excitation on the boundary, which is a T^2 , is along the noncontractible circle, and therefore it cannot shrink to pointlike excitations.

Here, we give a sample calculation on the entanglement entropy for an MES corresponding to the pointlike excitation p_2 , the quantum dimension of which is $d_{p_2} = 2$. Other cases can be straightforwardly calculated in the same way. To write the MES corresponding to p_2 , we first need to check the nonzero characters $\text{tr}[\rho_\theta(g)]$, $\forall g \in D_3$, where we denote the two-dimensional representation of D_3 as ρ_θ . It can be found that the nonzero characters are

$$\text{tr}[\rho_\theta(\mathbb{1})] = 2, \quad \text{tr}[\rho_\theta(a)] = \text{tr}[\rho_\theta(b)] = -1. \quad (4.2)$$

Then, based on Eq. (3.6), the MES on a single cube T^3 has the expression

$$|\mathbb{1}, \mathbb{1}, \theta\rangle = \frac{1}{\sqrt{|G|}} (|\mathbb{1}\rangle \otimes |\mathbb{1}\rangle \otimes (2|\mathbb{1}\rangle - |a\rangle - |b\rangle)), \quad (4.3)$$

where for convenience we have written $|\mathbb{1}, \mathbb{1}, c\rangle$ as $|\mathbb{1}\rangle \otimes |\mathbb{1}\rangle \otimes |c\rangle$, with $|\mathbb{1}\rangle \otimes |\mathbb{1}\rangle$ representing the trivial fluxes in both x and y directions. The next step is to write this MES in a two-cube basis, so that we can study the entanglement between two cubes. This can be done by writing

$$|c\rangle \rightarrow \frac{1}{\sqrt{|G|}} \sum_{\substack{c_1, c_2 \in G \\ c_1 \cdot c_2 = c}} |c_1\rangle \otimes |c_2\rangle, \quad (4.4)$$

where c , c_1 , and c_2 are group elements defined on the bond in z direction, as shown in Fig. 3(b). In this two-cube basis, one can obtain the reduced density matrix for cube A as $\rho_A = \text{tr}_{\bar{A}}(|\mathbb{1}, \mathbb{1}, \theta\rangle\langle\mathbb{1}, \mathbb{1}, \theta|)$. One can find the eigenvalues of ρ_A as 0 with multiplicity two, and $\frac{1}{4}$ with multiplicity four. Then, the entanglement entropy for cube A is $S_A = -\frac{1}{1-n} \ln(\text{tr}\rho_A^n) = 2 \ln 2$. By repeating this procedure for all the other pointlike and single-loop excitations, one can find

$$S_A = 2 \ln d_i, \quad (4.5)$$

where d_i are the quantum dimensions of the corresponding excitations, as given in Eq. (4.1).

There are two remarks on the result in Eq. (4.5): (i) The factor 2 in S_A arises from the fact that there are two interfaces when bipartitioning a torus. In other words, the Wilson loop/membrane operators across the entanglement cut are detected twice. (ii) One may ask why both the area-law term and the TEE term are not included in Eq. (4.5). The reason is that these two terms cancel with each other by considering that $2V \ln |G| - 2 \ln |G| = 0$ when $V = 1$ [see also Sec. I above Eq. (1.1)]. As a simple exercise, one can check the MESs on cubes with $V = 2$, and there is an extra piece $2V \ln |G| - 2 \ln |G| = 2 \ln |G|$ in Eq. (4.5). Putting the area part of the entanglement entropy back to Eq. (4.5) leads to the general formula of the entanglement entropy:

$$S_A = 2V \ln |G| + 2 \ln d_i - 2 \ln |G|. \quad (4.6)$$

B. Twisted DW theory with $G = (\mathbb{Z}_N)^4$

For a (3+1)D twisted DW gauge theory with an Abelian gauge group $G = (\mathbb{Z}_N)^4$, naively, one may expect that all the excitations are Abelian with quantum dimensions 1. What is interesting, in Ref. [38], it was found that when specific 4-cocycles ω are introduced (the so-called type-IV 4-cocycle), there exist loop excitations with a quantum dimension N (N is prime), which is related with the non-Abelian three-loop braiding. It turns out these excitations correspond to the Hopf-link loop excitations in Fig. 4(c).

Since $G = (\mathbb{Z}_N)^4$ is Abelian, each group element is itself a conjugacy class, and then the MES in Eq. (3.4) can be simplified as

$$|a, b, \mu\rangle = \frac{1}{\sqrt{|G|}} \sum_{c \in G} \text{tr}[\tilde{\rho}_\mu^{a,b}(c)] |a, b, c\rangle, \quad (4.7)$$

where $a, b \in G$ are used to label the flux in x, y directions, and μ is used to label the charge representation. In this theory, there are several kinds of 4-cocycles $\omega \in \mathcal{H}^4[G; U(1)]$, which are named type-II, -III, and -IV 4-cocycles in Ref. [38]. Then, the 2-cocycles $\beta_{a,b}(c, d)$ that enter the definition of $\tilde{\rho}_\mu^{a,b}$ in Eq. (3.5) can be obtained by doing slant product twice from 4-cocycles ω (see Appendix B for more details.).

It was found that only the type-IV 4-cocycles may lead to a higher-dimensional irreducible projective representation $\tilde{\rho}_\mu^{a,b}$. To be concrete, the type-IV 4-cocycle has the form

$$\omega_{IV}(a, b, c, d) = \exp\left(\frac{2\pi i p}{N} a_1 b_2 c_3 d_4\right). \quad (4.8)$$

For simplicity, let us choose the nontrivial parameter $p = 1$ from now on. A general $p \in \mathbb{Z}_N$ can be discussed in a similar way. From a slant product (see Appendix B), one can find the induced 2-cocycles as follows:

$$\begin{aligned} \beta_{a,b}(c, d) = \exp\left\{\frac{2\pi i}{N} [(a_4 b_3 - a_3 b_4) c_1 d_2 \right. \\ \left. + (a_2 b_4 - a_4 b_2) c_1 d_3 + (a_4 b_1 - a_1 b_4) c_2 d_3 \right. \\ \left. + (a_3 b_2 - a_2 b_3) c_1 d_4 + (a_1 b_3 - a_3 b_1) c_2 d_4 \right. \\ \left. + (a_2 b_1 - a_1 b_2) c_3 d_4\right\}. \end{aligned} \quad (4.9)$$

From Eq. (3.5), each 2-cocycle $\beta_{a,b}$ specifies a class of irreducible projective representations for G . As we will see

later, depending on the choice of the flux a and b , the dimension of the irreducible projective representation $\tilde{\rho}_\mu^{a,b}(c)$ is 1 or N .

1. Excitations with quantum dimensions 1

Based on Eq. (4.9), one can find that for an arbitrary c and d , $\beta_{a,b}(c,d)$ will become trivial, i.e., $\beta_{a,b}(c,d) = 1$, when satisfying the following condition:

$$a_i b_j = a_j b_i, \quad \forall i \neq j. \quad (4.10)$$

For these cases, the irreducible projective representations $\tilde{\rho}_\mu^{a,b}(c)$ become linear representations of $G = (\mathbb{Z}_N)^4$ that are one dimensional.

Now, let us check the quantum dimensions for pointlike and single-loop excitations in this theory. For a pointlike excitation, one has $a = b = \mathbb{1}$, where $\mathbb{1} := (0000)$ denotes the identity group element in $(\mathbb{Z}_N)^4$. For a single-loop excitation, either $a = \mathbb{1}$ or $b = \mathbb{1}$. In both cases, Eq. (4.10) is satisfied. As a result, $\beta_{a,b}(c,d)$ is trivial for both pointlike and single-loop excitations, and their quantum dimensions are 1.

The entanglement property for an MES with one-dimensional quantum dimension has been discussed in Ref. [37]. For the completeness, we give a brief review here. For one-dimensional-irreducible projective representation, one has $\text{tr}[\tilde{\rho}_\mu^{a,b}(c)] = \tilde{\rho}_\mu^{a,b}(c)$. Then, Eq. (3.5) can be rewritten as

$$\text{tr}[\tilde{\rho}_\mu^{a,b}(c)] \cdot \text{tr}[\tilde{\rho}_\mu^{a,b}(d)] = \beta_{a,b}(c,d) \cdot \text{tr}[\tilde{\rho}_\mu^{a,b}(c \cdot d)]. \quad (4.11)$$

By triangulating the T^3 with two cubes in Fig. 3(b), each one-cube basis $|a,b,c\rangle$ in Eq. (4.7) becomes [37]

$$|a,b,c\rangle \rightarrow \frac{1}{\sqrt{|G|}} \sum_{\substack{c_1, c_2 \in G; \\ c_1 \cdot c_2 = c}} \beta_{a,b}(c_1, c_2) |a,b, c_1, c_2\rangle. \quad (4.12)$$

Note that compared to the case of untwisted gauge theory in Eq. (4.4), there is an extra $U(1)$ phase $\beta_{a,b}(c_1, c_2)$ for the twisted case. Here, $\beta_{a,b}(c,d)$ is the same as the 2-cocycle that appears in Eq. (3.5), which can be understood as the phase associated with the triangulated manifold in Fig. 3(b) after an appropriate coloring of the manifold has been made [37].

Then, the MES in Eq. (4.7) can be rewritten as

$$\begin{aligned} |a,b,\mu\rangle &= \frac{1}{|G|} \sum_{c_1, c_2 \in G} \text{tr}[\tilde{\rho}_\mu^{a,b}(c_1 \cdot c_2)] \beta_{a,b}(c_1, c_2) |a,b, c_1, c_2\rangle \\ &= \frac{1}{|G|} \sum_{c_1, c_2 \in G} \text{tr}[\tilde{\rho}_\mu^{a,b}(c_1)] \text{tr}[\tilde{\rho}_\mu^{a,b}(c_2)] |a,b, c_1, c_2\rangle, \end{aligned} \quad (4.13)$$

where we have used Eq. (4.11). It is found that $|a,b,\mu\rangle$ is a direct product state for the two cubes, and the entanglement entropy for a single cube A is $S_A = 0$, i.e., one cannot detect the pointlike and single-loop excitations because of their trivial quantum dimensions. Again, as remarked below Eq. (4.5), here the area-law term cancels with the TEE term.

2. Hopf-link loop excitations with higher quantum dimensions

For Hopf-link loop excitations, both a and b correspond to nontrivial fluxes (see Fig. 4). Then, it is possible to have

nontrivial 2-cocycles $\beta_{a,b}(c,d)$, which may result in higher quantum dimensions $d_{\chi_1, \chi_2; i}$ in Eq. (1.5).⁶

For $G = (\mathbb{Z}_N)^4$, we choose the nonvanishing flux $a = (0100)$, and $b = (1000)$ to have a nontrivial 2-cocycle

$$\beta_{a,b}(c,d) = \exp\left(\frac{2\pi i}{N} c_3 d_4\right). \quad (4.14)$$

Based on Eq. (3.5), it can be easily checked that

$$\tilde{\rho}^{a,b}(ij10) \cdot \tilde{\rho}^{a,b}(ij01) = \omega \tilde{\rho}^{a,b}(ij01) \cdot \tilde{\rho}^{a,b}(ij10), \quad (4.15)$$

where $i, j \in \mathbb{Z}_N$, and we have defined $\omega := \exp(\frac{2\pi i}{N})$. That is, $\tilde{\rho}^{a,b}(ij01)$ and $\tilde{\rho}^{a,b}(ij10)$ do not commute with each other, which indicates that the projective representation $\tilde{\rho}^{a,b}$ are necessarily higher dimensional. By checking all the $\tilde{\rho}^{a,b}(c)$ that satisfy Eq. (3.5), one can find there are N^2 inequivalent irreducible projective representations labeled by (p,q) where $p, q \in \mathbb{Z}_N$. In the representation (p,q) , one has $\tilde{\rho}_{(p,q)}^{a,b}(0100) = \omega^p \mathbb{1}_N$ and $\tilde{\rho}_{(p,q)}^{a,b}(1000) = \omega^q \mathbb{1}_N$, where $\mathbb{1}_N$ is an N -dimensional identity matrix. In addition, all the $\tilde{\rho}_{(p,q)}^{a,b}(c)$ that have nonzero characters are of the form $\tilde{\rho}_{(p,q)}^{a,b}(ij00) = \omega^{j \cdot p + i \cdot q} \mathbb{1}_N$, $\forall i, j \in \mathbb{Z}_N$. For other $\tilde{\rho}_{(p,q)}^{a,b}(ijmn)$ with $m \neq 0$ or $n \neq 0$, the characters will be zero. Here, for simplicity, let us check the representation with $(p,q) = (0,0)$. Then, the corresponding MES has a simple form

$$|a,b,\mu\rangle = \frac{N}{\sqrt{|G|}} \sum_{i,j=0}^{N-1} |a,b, ij00\rangle, \quad (4.16)$$

where we have used the fact that $\text{tr}[\tilde{\rho}_{(0,0)}^{a,b}(ij00)] = N$, and here $|G| = N^4$. Based on Eq. (4.12), we may rewrite the MES in terms of two-cube basis as follows:

$$\begin{aligned} |a,b,\mu\rangle &= \frac{N}{\sqrt{|G|}} \cdot \frac{1}{\sqrt{|G|}} \underbrace{|a\rangle \otimes |b\rangle}_{\text{flux}} \otimes \\ &\times \sum_{k,l,m,n=0}^{N-1} \sum_{i,j=0}^{N-1} \omega^{m \cdot [N-n]} \underbrace{|klmn\rangle}_{\text{cube } \bar{A}} \otimes \\ &\times \underbrace{|[N-k+i][N-l+j][N-m][N-n]\rangle}_{\text{cube } A}, \end{aligned}$$

where $[i] := i \bmod N$, and the term $\omega^{m \cdot [N-n]}$ comes from the 2-cocycle in Eq. (4.14). Then, after some simple algebra, one can find that the reduced density matrix for cube A may be written as $\rho_A = \bigoplus_{m,n=0}^{N-1} \rho_{mn}$, where

$$\begin{aligned} \rho_{mn} &= \frac{1}{N^4} \underbrace{|a,b\rangle \langle a,b|}_{\text{flux}} \otimes \left(\sum_{k,l=0}^{N-1} |kl[N-m][N-n]\rangle \right) \\ &\times \left(\sum_{u,v=0}^{N-1} \langle uv[N-m][N-n] \rangle \right). \end{aligned}$$

⁶It is noted that not all Hopf-link loop excitations have higher quantum dimensions. For example, for nonvanishing flux $a = b$, Eq. (4.10) still holds. Then, the 2-cocycle $\beta_{a,b}$ is trivial, and one still has one-dimensional quantum dimension.

Each ρ_{mn} is an N^2 -dimensional matrix with all elements being $1/N^4$. The eigenvalues of each ρ_{mn} are $\lambda = \underbrace{0, 0, \dots, 0}_{N^2-1}, \frac{1}{N^2}$.

Therefore, the eigenvalues of ρ_A are

$$\lambda = \underbrace{0, \dots, 0}_{(N^4-N^2)}, \underbrace{\frac{1}{N^2}, \dots, \frac{1}{N^2}}_{N^2}. \quad (4.17)$$

Then, the n th Rényi entropy for cube A (or equivalently cube \bar{A}) has the form

$$S_A^{(n)} = \frac{1}{1-n} \ln \text{Tr}(\rho_A^n) = 2 \ln N. \quad (4.18)$$

Again, the factor 2 appears because there are two interfaces for a bipartite T^3 . As before, here the area-law term cancels with the TEE term [see also the remarks below Eq. (4.5)]. The result in Eq. (4.18) agrees with the observation in Ref. [38] that the non-Abelian stringlike excitations with nontrivial 2-cocycle $\beta_{a,b}$ have quantum dimensions N . In Ref. [38], the quantum dimensions $d_i = N$ are obtained based on the ground state counting on a three-torus T^3 . Here, we detect this quantum dimension based on the entanglement entropy.

The calculation above can be straightforwardly generalized to other irreducible projective representations $\mu = (p, q)$ with $\tilde{\rho}_\mu^{a,b}(0100) = \omega^p \mathbb{I}_N$ and $\tilde{\rho}_\mu^{a,b}(1000) = \omega^q \mathbb{I}_N$. One needs to introduce a $U(1)$ phase $\omega^{j \cdot p + i \cdot q}$ in front of the basis $|a, b, i, j00\rangle$ in Eq. (4.16). It can be found that the entanglement entropy S_A is the same as Eq. (4.18).

As a short summary for the case of type-IV twisted gauge theory with $G = (\mathbb{Z}_N)^4$, both the pointlike and single-loop excitations have quantum dimension 1, and have no contribution to the entanglement entropy. For Hopf-link excitations with nonvanishing fluxes a and b , when the 2-cocycle $\beta_{a,b}$ in Eq. (4.9) is nontrivial, these excitations will have quantum dimensions N , and contribute $\ln N$ to the entanglement entropy.

It is noted that for other types of 4-cocycles, e.g., type-II and type-III 4-cocycles, since all the corresponding 2-cocycles $\beta_{a,b}$ are trivial, then the Hopf-link loop excitations will have quantum dimensions 1 [38] and have no contribution to the entanglement entropy. From this point of view, the entanglement entropy with a Hopf-link loop excitation can be used to distinguish certain type-IV 4-cocycles from the others. To obtain further details of the 4-cocycles from the entanglement entropy, one may need to study loop excitations with more interesting structures.

V. CONCLUDING REMARKS

In summary, we study the entanglement entropy of (3+1)D topological orders in this paper which is realized in (3+1)D DW theories labeled by (G, ω) . Here, G is a finite gauge group and $\omega \in \mathcal{H}^4[G; U(1)]$ is a group 4-cocycle. We study the entanglement entropy with and without excitations. Depending on the types of excitations, it is found that the entanglement entropy can capture different information of (G, ω) as follows:

(1) For a generic DW theory on a spatial manifold S^3 , we find that the TEE in the ground state is $\gamma = -\ln |G|$, regardless of the 4-cocycle ω . In other words, the ground-state entanglement entropy captures the information of group order $|G|$.

(2) For a pointlike excitation, it contributes $\ln d_i$ to the entanglement entropy, where $d_i = \dim[\text{Rep}_i(G)]$. In other words, the entanglement entropy with pointlike excitations captures the dimension of irreducible representation of G , i.e., $|\text{Rep}(G)|$.

(3) For a single-loop excitation, its contribution to the entanglement entropy is $\ln d_{\chi,i}$ with $d_{\chi,i} = |\chi| \cdot \dim[\text{Rep}_i(G_\chi)]$. In other words, the entanglement entropy with single-loop excitations captures the information of conjugacy class size $|\chi|$ and the dimension of irreducible representation of its centralizer G_χ .

(4) For a Hopf-link loop excitation, we mainly focus on the case with Abelian gauge group G . Its contribution to the entanglement entropy is $\ln d_{\chi_1, \chi_2; i}$, where $d_{\chi_1, \chi_2; i} = \dim[\widetilde{\text{Rep}}_{\chi_1, \chi_2; i}(G)]$. That is, the dimension of irreducible *projective* representations of G can be detected. This dimension depends on the 4-cocycles ω through Eq. (3.5). In other words, the entanglement entropy with Hopf-link loop excitations captures the information of 4-cocycles ω .

Now, we give some remarks on the above results. First, the entanglement entropy with pointlike and single-loop excitations can capture different information of the finite gauge group G . However, they cannot capture the information of 4-cocycles ω . The underlying reason is that the charges of these two excitations only correspond to *linear* representation of G or G_χ . They are not “twisted” by the 4-cocycles ω . On the other hand, for the Hopf-link loop excitations, they carry irreducible *projective* representations of G , which are twisted by the 4-cocycles ω . This is why we can detect information of ω through entanglement entropy of Hopf-link loop excitations.

Second, we would like to emphasize that the Hopf-link loop excitations cannot capture *all* the information of 4-cocycles ω . The reason is simple: one can find that the projective representations carried by Hopf-link loop excitations are directly determined by the 2-cocycles $\beta_{a,b}(c, d)$ in Eq. (3.5). During the dimension reduction from 4-cocycles ω to 2-cocycles β , not *all* the information of ω is kept. This is why we can only distinguish *certain* type-IV 4-cocycles ω from the others in the case of $G = (\mathbb{Z}_N)^4$. To detect more information of the 4-cocycles, we expect that the excitations with more interesting (and complicated) structures should be considered.

Third, it is natural to ask what happens if the theory under consideration deviates from the DW theory at the fixed point. In this case, when the entanglement cut is a two-torus T^2 , the constant correction to the area-law term in entanglement entropy includes the TEE as well as the contribution from curvature of the entanglement cut, etc. The way to extract the TEE in this case has been discussed in Refs. [90,91]. It was shown that the topological contribution can be extracted via a generalized Kitaev-Preskill-Levin-Wen approach [4,5]. Moreover, it has been shown that when the theory flows to the infrared fixed point, the mean curvature contribution vanishes, and the constant part is the TEE [91].

There are many interesting problems we hope to solve in the future, and we mention some of them as follows.

(1) It is interesting to study various choices of entanglement cuts. In this work, all excitations lie inside the subsystems. It is natural to ask what happens if the entanglement cut is across the loop excitation itself. One also needs to understand

how the effect of excitations on entanglement entropy depends on the topology of the entanglement cut.

(2) The effect of excitations on the entanglement entropy is studied based on MES in this work. It would be very interesting to explore the partition function method based on a TQFT, in which one needs to evaluate the partition functions on various 4-manifolds with Wilson loops/membranes inserted.

(3) It is desirable to classify or characterize various loop excitations with interesting knotted and linked structure in (3+1)D topological order. For example, how are different excitations labeled by data based on (G, ω) ? A good understanding of these excitations may help to detect more information of the underlying theory through the entanglement entropy.

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APPENDIX A: ON THE TEE IN THE GROUND STATE OF TYPE-IV DW THEORY

In this appendix, we consider the entanglement entropy in the ground state of type-IV Dijkgraaf-Witten model. We argue that the topological part of the entropy does not depend on the genus of the entanglement surface. In particular, $S_{\text{topo}}[S^2] = S_{\text{topo}}[T^2]$, where S^2 and T^2 are entanglement surfaces.

The type-IV DW model with gauge group $\otimes_{I=1}^4 \mathbb{Z}_{n_I}$ is described by the following Lagrangian:

$$\begin{aligned} \mathcal{L} &= \sum_{I=1}^4 \frac{n_I}{2\pi} B_I dA_I + \frac{n_1 n_2 n_3 n_4 p}{(2\pi)^3 \text{gcd}(n_1, n_2, n_3, n_4)} A_1 A_2 A_3 A_4 \\ &=: \mathcal{L}_0 + \mathcal{L}_1, \end{aligned}$$

where $p \in \mathbb{Z}_{\text{gcd}(n_1, n_2, n_3, n_4)}$. We consider the space-time to be D^4 , whose boundary is $\partial D^4 = S^3$ which we identify as the spatial slice. Then, the ground state on S^3 is

$$|\psi\rangle = \mathfrak{C} \sum_{\mathcal{C}, \mathcal{C}'} \int_{\mathcal{C}'|_{S^3}} \mathcal{D}B_I \int_{\mathcal{C}|_{S^3}} \mathcal{D}A_I \exp\left(i \int_{D^4} \mathcal{L} d^4x\right) |\mathcal{C}\rangle,$$

where \mathfrak{C} is the normalization factor. \mathcal{C} and \mathcal{C}' are the configurations for the A -cochain and B -cochain, respectively. The path integral of A -cochain is over all the configurations of A over D^4 with fixed configurations on the S^3 , and similar for

B -cochain. Integrating over B_I , we get

$$|\psi\rangle = \mathfrak{C} \sum_{\mathcal{C}} \int_{\mathcal{C}|_{S^3}} \mathcal{D}A_I \delta(dA_I) \exp\left(i \int_{D^4} \mathcal{L}_1 d^4x\right) |\mathcal{C}\rangle. \quad (\text{A1})$$

The delta function in the path integral implies that the configuration of A field is a sum of closed three volumes in the dual lattice if it does not intersect with the spatial slice. On the spatial slice, the configuration of A field is a sum of closed two-surfaces in the dual lattice,

$$A_I = \frac{2\pi}{n_I} \sum_i *_4 \Sigma(V_i^I) = \frac{2\pi}{n_I} \sum_i *_3 \Sigma(S_i^I), \quad (\text{A2})$$

where $\partial V_i^I = S_i^I$, and the summation of i runs over all possible volume (surface) configurations in the dual lattice. $\Sigma(V_i^I)$ is a delta function which is one only if it is evaluated on the dual-lattice three volume V_i^I . $*_4$ is the Hodge dual operator in four dimensions, hence, $*_4 \Sigma(V_i^I)$ is a one-form. By plugging A_I into the ground state in Eq. (A1), one can find that the wave function $|\psi\rangle$ is the sum of different configurations weighted by ‘‘intersection numbers’’ of three closed dual-lattice surfaces and one volume bounded by a dual-lattice surface in S^3 :

$$\begin{aligned} |\psi\rangle &= \mathfrak{C} \sum_{\mathcal{C}} \exp\left(i \frac{2\pi p}{\text{gcd}(n_1, n_2, n_3, n_4)} \int_{S^3} \right. \\ &\quad \left. \times \sum_{i,j,k,l} \delta(S_i^1 \cap S_j^2 \cap S_k^3 \cap \partial^{-1} S_l^4)\right) |\mathcal{C}\rangle. \quad (\text{A3}) \end{aligned}$$

To compute the entanglement entropy, we first choose an entanglement cut, which bipartites the space into two parts. The ground-state wave function factorizes accordingly,

$$|\psi\rangle = \mathfrak{C} \sum_{\mathcal{C}_A \in \mathcal{H}_A, \mathcal{C}_B \in \mathcal{H}_B} p(\mathcal{C}_A, \mathcal{C}_B) |\mathcal{C}_A\rangle \otimes |\mathcal{C}_B\rangle, \quad (\text{A4})$$

where $p(\mathcal{C}_A, \mathcal{C}_B) = \exp[i \frac{2\pi p}{\text{gcd}(n_1, n_2, n_3, n_4)} \int_{S^3} \sum_{i,j,k,l} \delta(S_i^1 \cap S_j^2 \cap S_k^3 \cap \partial^{-1} S_l^4)]$ is a $U(1)$ factor. We would like to see whether the twisting parameter p can affect the entanglement entropy. One crucial observation from Ref. [91] is that unless the coefficient $p(\mathcal{C}_A, \mathcal{C}_B)$ has a term which can factorize into the product of the contribution from the region A and the contribution from the region B , i.e., $\exp[iq(\{S_i\}_A)q'(\{S_j\}_B)]$ [where $q(\{S\}_A)$ is an arbitrary quantity depending only on the configuration $\{S\}$ inside region A , and similarly for $q'(\{S\}_B)$], the $U(1)$ factor can always be absorbed by redefining the basis $|\mathcal{C}_A\rangle$ and $|\mathcal{C}_B\rangle$. However, from the form of $p(\mathcal{C}_A, \mathcal{C}_B)$ [see Eq. (A3)], to have nontrivial intersection, there must be at least one S_i in $\delta(S_i^1 \cap S_j^2 \cap S_k^3 \cap \partial^{-1} S_l^4)$ such that it intersects with the entanglement surface. Hence, the phase factor in Eq. (A3) can be absorbed by redefinition of the states $|\mathcal{C}\rangle$ and the singular values of the resulting wave function is the same as the singular values of the wave function of pure $\otimes_{I=1}^4 \mathbb{Z}_{n_I}$ gauge theory without twisting terms.

Since it is well known that for an untwisted gauge theory, the entanglement entropy across S^2 and T^2 are the same. As a corollary of the above argument, we have $S_{\text{topo}}[S^2] = S_{\text{topo}}[T^2]$ for a type-IV DW theory.

APPENDIX B: DIMENSION REDUCTION OF COCYCLES FROM A SLANT PRODUCT

In the main text, the 2-cocycles $\beta_{a,b}(c,d)$ appear in two different places: one in the definition of irreducible projective representation in Eq. (3.5), and the other in evolving from one-cube basis to two-cube basis in Eq. (4.12). In this appendix, we give a brief introduction on slant production to obtain 2-cocycles from 4-cocycles. For more details on group cohomology and cocycles, one may refer to, e.g., Refs. [37–39].

For the dimension reduction of cochains, one can use slant product to map an n -cochain c to $(n-1)$ -cochain $i_g c$ as follows:

$$i_g c(g_1, g_2, \dots, g_{n-1}) = c(g, g_1, g_2, \dots, g_{n-1})^{(-1)^{n-1}} \cdot \prod_{j=1}^{n-1} \times c(g_1, \dots, g_j, (g_1 \dots g_j)^{-1} \cdot g \cdot (g_1 \dots g_j), \dots, g_{n-1})^{(-1)^{n-1+j}}. \quad (\text{B1})$$

In this work, since we are working in $(3+1)\text{D}$, the theory will be defined based on the 4-cocycle $\omega \in \mathcal{H}^4[G; U(1)]$. We use the “canonical” 4-cocycle so that $\omega(g_1, g_2, g_3, g_4) = 1$ as long as any of g_1, g_2, g_3 , and g_4 is $\mathbb{1}$, the identity element of group G . Then, by doing slant product, we can obtain 3-cocycles from 4-cocycles:

$$\alpha_a(b, c, d) = i_a \omega(b, c, d). \quad (\text{B2})$$

From Eq. (B1), one can find that if $a = \mathbb{1}$, then the induced three-cocycle $\alpha_a(b, c, d) = 1$, which is trivial. Similarly, one

can obtain the 2-cocycle from 3-cocycle:

$$\beta_{a,b}(c, d) = i_b \alpha_a(c, d) = \frac{\alpha_a(b, c, d) \alpha_a(c, d, b)}{\alpha_a(c, b, d)}, \quad (\text{B3})$$

if any of a, b is equal to $\mathbb{1}$, then $\beta_{a,b}(c, d)$ will be trivial, i.e., $\beta_{a,b}(c, d) = 1$. It is noted that the irreducible projective representation is determined by [see Eq. (3.5)]

$$\tilde{\rho}_\mu^{a,b}(c) \tilde{\rho}_\mu^{a,b}(d) = \beta_{a,b}(c, d) \tilde{\rho}_\mu^{a,b}(c \cdot d). \quad (\text{B4})$$

Then, $\beta_{a,b}(c, d) = 1$ indicates that the projective representation $\tilde{\rho}_\mu^{a,b}$ will reduce to the linear representation that satisfies $\rho_\mu^{a,b}(c) \rho_\mu^{a,b}(d) = \rho_\mu^{a,b}(c \cdot d)$. For the pointlike excitations ($a = b = \mathbb{1}$) and single-loop excitations (one of a, b is equal to $\mathbb{1}$), one has $\beta_{a,b}(c, d) = 1$. Therefore, these two types of excitations only carry linear representations, and are insensitive to the 4-cocycles ω . On the other hand, for a Hopf-link loop excitation, one has $a \neq \mathbb{1}$ and $b \neq \mathbb{1}$, which may lead to a nontrivial $\beta_{a,b}(c, d)$ and therefore a projective representation.

For an Abelian gauge group G as studied in the main text, based on Eq. (B2), one can obtain

$$\alpha_a(b, c, d) = \frac{\omega(b, a, c, d) \omega(b, c, d, a)}{\omega(a, b, c, d) \omega(b, c, a, d)}. \quad (\text{B5})$$

Then, based on Eqs. (B3) and (B5), one can express the 2-cocycle $\beta_{a,b}(c, d)$ in terms of 4-cocycles ω .

Interestingly, there is a geometrical meaning of this 2-cocycle $\beta_{a,b}(c, d)$. For example, $\beta_{a,b}(c_1, c_2)$ can be viewed as the phase associated with the manifold in Fig. 3(b), after an appropriate coloring of the manifold. We refer the readers to Ref. [37] for more details on this geometrical picture.

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