THE RESIDUAL FINITENESS OF (HYPERBOLIC) AUTOMORPHISM-INDUCED HNN-EXTENSIONS

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Abstract. We classify finitely generated, residually finite automorphism-induced HNN-extensions in terms of the residual separability of a single associated subgroup. This classification provides a method to construct automorphism-induced HNN-extensions which are not residually finite. We prove that this method can never yield a "new" counter-example to Gromov's conjecture on the residual finiteness of hyperbolic groups.

1. Introduction

A group $H*_{(K,\phi)}$ is called an *automorphism-induced HNN-extension* if it has a relative presentation of the form

$$H*_{(K,\phi)} = \langle H, t; tkt^{-1} = \phi(k), k \in K\rangle$$

where $\phi \in \mathrm{Aut}(H)$ and $K \lneq H$.

The main result of this note is a classification of finitely generated, residually finite automorphism-induced HNN-extensions. A subgroup K of H is *residually separable* in H if for all $x \in H \setminus K$ there exists a finite index, normal subgroup N of H, written $N \lhd_f H$, such that $x \notin KN$ (hence if $\varphi_x : H \to H/N$ is the natural map then $\varphi_x(x) \notin \varphi_x(K)$).

Theorem A. *Suppose that H is finitely generated. Then $G = H*_{(K,\phi)}$ is residually finite if and only if H is residually finite and K is residually separable in H.*

We prove two corollaries of Theorem A. These corollaries can be easily applied to construct automorphism-induced HNN-extensions which are not residually finite. Both corollaries relate to the subgroup-quotient $N_H(K)/K$. This subgroup-quotient plays a central role in a framework for the construction of groups possessing certain properties and with specified outer automorphism group [Log15] (see also [Log16] [Log17]).

Corollary 1.1. *Suppose that H is finitely generated. If $N_H(K)/K$ is not residually finite then $G = H*_{(K,\phi)}$ is not residually finite.*

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Corollary 1.2. *Suppose that H is finitely generated and that $N_H(K)$ has finite index in H. Then $G = H*_{(K,\phi)}$ is residually finite if and only if both H and $N_H(K)/K$ are residually finite.*

Hyperbolicity. It is a famous conjecture of Gromov that all hyperbolic groups are residually finite [nib93] [KW00] [Ol'00]. One might hope to apply Corollary 1.1 to obtain a counter-example to this conjecture. However, Theorem B proves that Corollary 1.1 can produce no "new" counter-examples to Gromov's conjecture, in the sense that if $G = H*_{(K,\phi)}$ is a counter-example where the subgroup-quotient $N_H(K)/K$ is used to force G to be non-residually finite then the conditions of Theorem B hold, and so H is also a counter-example.

Theorem B. *Suppose that $G = H*_{(K,\phi)}$ is hyperbolic and non-residually finite, and that $K \lneq N_H(K)$. Then K is finite, and H is hyperbolic and non-residually finite.*

Theorem B leaves the following question:

Question 1.3. *Suppose that $G = H*_{(K,\phi)}$ is hyperbolic and non-residually finite. Then is H is hyperbolic and non-residually finite?*

We also have the following result:

Theorem C. *Suppose that $K \lneq N_H(K)$ and that K contains an element of infinite order. Then $\mathbb{Z} \times \mathbb{Z}$ embeds into $G = H*_{(K,\phi)}$.*

Automorphism-induced HNN-extensions can be thought of as "partial" mapping tori $H \rtimes_\phi \mathbb{Z}$. Theorem C proves that automorphism-induced HNN-extensions of free groups $F_n*_{(K,\phi)}$ are not hyperbolic if $K \lneq N_H(K)$, even if the "full" mapping torus $F_n \rtimes_\phi \mathbb{Z}$ is hyperbolic.

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2. Residual finiteness

We first prove Theorem A. Note that for G some group, if $P \lhd_f G$ and $H \leq G$ then $P \cap H \lhd_f H$. Also note that if H is a finitely generated group, $Q \lhd_f H$ and $\phi \in \mathrm{Aut}(H)$ then $\cap_{i\in\mathbb{Z}}\phi^i(Q) \lhd_f H$.

Proof of Theorem A. Suppose H is residually finite and K is residually separable in H. Then $H*_{(K,\phi)}$ is residually finite [BT78, Lemma 4.4].

Suppose $H*_{(K,\phi)}$ is residually finite. Then H is residually finite, as subgroups of residually finite groups are residually finite. Now, suppose that K is not residually separable in H, and let $x \in H \setminus K$ be such that

$x \in KN$ for all finite index subgroups N of H. Let $\overline{N} \lhd_f H*_{(K,\phi)}$ be arbitrary. It is sufficient to prove that $txt^{-1}\phi(x)^{-1} \in \overline{N}$. To see this inclusion, first note that $\overline{N} \cap H \lhd_f H$. Consider $L := \cap_{i\in\mathbb{Z}}\phi^i\left(\overline{N} \cap H\right)$, and note that $L \lhd_f H$. Then there exists $k \in K$ such that $xk^{-1}, \phi(xk^{-1}) \in L$. Thus, $xk^{-1}, \phi(xk^{-1}) \in \overline{N}$, and so $x\overline{N} = k\overline{N}$ and $\phi(x)\overline{N} = \phi(k)\overline{N}$. Then:

$$txt^{-1}\phi(x)^{-1}\overline{N} = tkt^{-1}\phi(k)^{-1}\overline{N} = \overline{N}$$

Hence, $txt^{-1}\phi(x)^{-1} \in \overline{N}$ as required. □

We now prove Corollary 1.1.

Proof of Corollary 1.1. Suppose that $N_H(K)/K$ is not residually finite. Then there exists some $x \in N_H(K)$ such that $x \in NK$ for all $N \lhd_f N_H(K)$. Hence, for all $\overline{N} \lhd_f H$ we have that $x \in \left(\overline{N} \cap N_H(K)\right)K$, and so $x \in \overline{N}K$. Therefore, K is not residually separable in H, and so $H*_{(K,\phi)}$ is not residually finite by Theorem A. □

We now prove Corollary 1.2. We previously proved the analogous result for the groups $H*_{(K,1)}$, so where the inducing automorphism ϕ is trivial [Log16, Proposition 2.2].

Proof of Corollary 1.2. By Theorem A and Corollary 1.1, it is sufficient to prove that if H and $N_H(K)/K$ are residually finite then K is residually separable. So, suppose that H and $N_H(K)/K$ are residually finite.

Additionally, suppose that $x \notin N_H(K)$. Clearly $x \notin N_H(K)K$ as $K \leq N_H(K)$. Then the subgroup $N := \cap_{h\in H}h^{-1}N_H(K)h$ is a finite index, normal subgroup of H such that $x \notin NK$, as required.

Suppose that $x \in N_H(K) \setminus K$. Now, as $N_H(K)/K$ is residually finite, there exists a map $\varphi_x : N_H(K)/K \to A_x$ with A_x finite and $xK \notin \ker(\varphi_x)$. Therefore, there exists a map $\widetilde{\varphi_x} : N_H(K) \to A_x$ which factors as $N_H(K) \to N_H(K)/K \xrightarrow{\varphi_x} A_x$ such that $x \notin \ker(\widetilde{\varphi_x})$. Then $K \leq \ker(\widetilde{\varphi_x})$ so $x \notin \ker(\widetilde{\varphi_x})K$. As $\ker(\widetilde{\varphi_x}) \lhd_f N_H(K) \lhd_f H$, there exists $N \lhd_f H$ such that $N \leq \ker(\widetilde{\varphi_x})$. As $x \notin \ker(\widetilde{\varphi_x})K$ and $NK \leq \ker(\widetilde{\varphi_x})K$ we have that $x \notin NK$ as required. □

3. Hyperbolicity

We first prove Theorem C, as it is applied in the proof of Theorem B. Recall that Theorem C gives a necessary condition for $\mathbb{Z} \times \mathbb{Z}$ to embed into $G = H*_{(K,\phi)}$. As $\mathbb{Z} \times \mathbb{Z}$ does not embed into any hyperbolic group, Theorem C gives a necessary condition for the hyperbolicity of automorphism-induced HNN-extensions.

Proof of Theorem C. Consider an element $k \in K$ of infinite order, and consider $a \in N_H(K) \setminus K$. Then the word $W = a^{-1}t^{-1}\phi(a)t$ has infinite order in G, and indeed no power of W is contained in K. Now, as $aka^{-1} \in K$ we have that $t^{-1}\phi(aka^{-1}) = aka^{-1}t^{-1}$. Then W and k commute as follows:

$$\begin{aligned} a^{-1}t^{-1}\phi(a)t \cdot k &= a^{-1}t^{-1}\phi(ak)t \\ &= a^{-1}t^{-1}\phi(aka^{-1})\phi(a)t \\ &= k \cdot a^{-1}t^{-1}\phi(a)t \end{aligned}$$

Therefore, $\langle W, k\rangle \cong \mathbb{Z} \times \mathbb{Z}$ as required. □

We now prove Theorem B.

Proof of Theorem B. By assumption, $G = H*_{(K,\phi)}$ is hyperbolic and non-residually finite, and $K \lneq N_H(K)$. Suppose that K is infinite. Then K is an infinite torsion group by Theorem C. Now, as $K \leq G$ with G hyperbolic, this is a contradiction [Gro87]. Hence, K is finite.

Suppose that H is residually finite. As K is finite we have that G is residually finite [BT78, Theorem 3.1], a contradiction. Hence, H is non-residually finite.

Finally, note that H is a quasi-convex subgroup of G as K and $\phi(K)$ are finite. Hence, H is hyperbolic [BH99, Proposition III.Γ.3.7]. □

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