ANALYSIS OF A HYBRIDIZED/INTERFACE STABILIZED FINITE ELEMENT METHOD FOR THE STOKES EQUATIONS

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Abstract. Stability and error analysis of a hybridized discontinuous Galerkin finite element method for Stokes equations is presented. The method is locally conservative, and for particular choices of spaces the velocity field is point-wise solenoidal. It is shown that the method is inf-sup stable for both equal-order and locally Taylor–Hood type spaces, and *a priori* error estimates are developed. The considered method can be constructed to have the same global algebraic structure as a conforming Galerkin method, unlike standard discontinuous Galerkin methods that have greater number of degrees of freedom than conforming Galerkin methods on a given mesh. We assert that this method is amongst the simplest and most flexible finite element approaches for Stokes flow that provide local mass conservation. With this contribution the mathematical basis is established, and this supports the performance of the method that has been observed experimentally in other works.

Key words. Stokes equations, hybridized, discontinuous Galerkin, finite element methods.

AMS subject classifications. 65N12, 65M15, 65N30, 76D07.

1. Introduction. We present analysis of a type of hybridized discontinuous Galerkin method for the Stokes equations. These methods can be constructed to have properties usually associated with discontinuous Galerkin finite element methods, while retaining the attractive features of continuous finite element methods, such as reduced discrete problem size. The analysis includes the method known as the Interface Stabilized Finite Element Method (IS-FEM) [11, 12, 15] or Embedded Discontinuous Galerkin (EDG) method [7] in the literature. The interface stabilized formulation for the incompressible Navier–Stokes equations with continuous pressure fields was presented in [11], and generalised in [12] for discontinuous pressure fields. The formulation is closely related to that of the hybridizable discontinuous Galerkin method using interior penalty numerical traces, the so-called IP-H methods [5].

The formulation we consider has been shown previously to have a number of appealing properties when applied to the Stokes equations [11, 12]. These include local mass conservation (point-wise in cases), experimentally observed optimal rates of convergence, they can be constructed to have an algorithmic structure that is identical to a conforming finite element method, and natural incorporation of stabilizing numerical fluxes when including advective transport.

Other hybridized discontinuous Galerkin methods for the Stokes equations have been developed, for example in [6, 8]. The method discussed in this work differs from those in the aforementioned references in the following two respects. Firstly, the methods considered in [6, 8] solve the Stokes equations in the gradient-velocity-pressure format, resulting in so-called LDG-H type methods, whereas we consider here the Stokes equations in the velocity-pressure format. The velocity field resulting from the formulations in [6, 8] is not point-wise solenoidal, but it is possible to devise post-processing operators for LDG-H methods that generate approximate velocity fields that are exactly divergence-free and $H(\text{div})$-conforming. Furthermore, it can be shown that the post-processed velocity fields converge with order $k+2$ when using polynomial approximations of order k. The second difference is that in [6, 8] the velocity traces are the only global unknowns. Both velocity and pressure traces are

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the global unknowns in our formulation (although pressure traces can be eliminated in some cases).

Analysis of the framework considered in this work, but applied to the scalar advection-diffusion equation, was presented in [15]. Here, we address the Stokes problem and underpin previous numerical investigations [12] by proving stability of the formulation for the Stokes problem and providing *a priori* error estimates. A particular motivation for this work is an observation that block preconditioners, with multigrid preconditioners applied to the blocks, can be very effective for the IS-FEM/EDG type formulation.

The remainder of this work is structured as follows. In section 2 we present the Stokes equations, followed in section 3 by the definition of the method we analyse with a summary of its key features. Stability and boundedness of the method are shown in section 4, and error estimates are presented in section 5. Conclusions are drawn in section 6.

2. The Stokes problem. Let $\Omega \subset \mathbb{R}^d$ be a polygonal ($d = 2$) or polyhedral ($d = 3$) domain, with the boundary of Ω denoted by Γ. We consider the Stokes problem of finding the velocity field $u : \Omega \to \mathbb{R}^d$ and the pressure $p : \Omega \to \mathbb{R}$ such that

$$-\Delta u + \nabla p = f \qquad \text{in } \Omega, \tag{1a}$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega, \tag{1b}$$

$$u = 0 \qquad \text{on } \Gamma, \tag{1c}$$

$$\int_\Omega p \,\mathrm{d}x = 0, \tag{1d}$$

where $f : \Omega \to \mathbb{R}^d$ is the prescribed body force.

For a body force $f \in [L^2(\Omega)]^d$, the weak formulation of the Stokes problem is given by: find $(u, p) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$ such that

$$a(u, v) + b(v, p) = \int_\Omega f \cdot v \,\mathrm{d}x \qquad \forall v \in [H_0^1(\Omega)]^d, \tag{2a}$$

$$-b(u, q) = 0 \qquad \forall q \in L_0^2(\Omega), \tag{2b}$$

where $L_0^2(\Omega)$ is the space of L^2 functions on Ω with zero mean, and the forms a and b are defined as

$$a(u, v) := \int_\Omega \nabla u : \nabla v \,\mathrm{d}x \tag{3}$$

$$b(p, v) := -\int_\Omega p \nabla \cdot v \,\mathrm{d}x. \tag{4}$$

Given that the form a is coercive on $[H_0^1(\Omega)]^d$, the inf-sup condition on b for the Stokes problem (2) to be well-posed is [2, Section 4.2.2]:

$$\beta_c \|q\|_{0,\Omega} \le \sup_{w \in [H_0^1(\Omega)]^2} \frac{b(q, w)}{\|w\|_{1,\Omega}} \qquad \forall q \in L_0^2(\Omega), \tag{5}$$

where $\beta_c > 0$ is a constant depending only on Ω.

3. Hybridized discontinuous Galerkin method. The method that will be analysed is presented in this section, along with some of its key conservation properties.

3.1. Preliminaries. Let $\mathcal{T} := \{K\}$ be a triangulation of the domain Ω into non-overlapping cells K. The characteristic length of a cell K is denoted by h_K. On the boundary of a cell, ∂K, we denote the outward unit normal vector by n. An interior facet F is shared by two adjacent cells K^+ and K^-, $F := \overline{\partial K^+} \cap \overline{\partial K^-}$ and a boundary facet is a facet of $\overline{\partial K}$ that lies on Γ. The set of all facets is denoted by $\mathcal{F} = \{F\}$, and the union of all facets is denoted by Γ^0.

We will work with the following finite element function spaces on Ω:

$$
\begin{aligned}
V_h &= \left\{ v_h \in \left[L^2(\Omega)\right]^d : \ v_h \in \left[P_k(K)\right]^d, \ \forall\, K \in \mathcal{T} \right\}, \\
Q_h &= \left\{ q_h \in L^2(\Omega) : \ q_h \in P_m(K), \ \forall\, K \in \mathcal{T} \right\},
\end{aligned}
\tag{6}
$$

and the following finite element spaces on the facets of the triangulation of Ω:

$$
\begin{aligned}
\bar{V}_h &= \left\{ \bar{v}_h \in \left[L^2(\Gamma^0)\right]^d : \ \bar{v}_h \in \left[P_k(F)\right]^d \ \forall\, F \in \mathcal{F}, \ \bar{v}_h = 0 \text{ on } \Gamma \right\}, \\
\bar{Q}_h &= \left\{ \bar{q}_h \in L^2(\Gamma^0) : \ \bar{q}_h \in P_k(F) \ \forall\, F \in \mathcal{F} \right\},
\end{aligned}
\tag{7}
$$

where $P_k(D)$ denotes the space of polynomials of degree k on domain D, with $k \geq 1$ and $m \leq k$.

For notational purposes, we introduce the spaces $V_h^\star = V_h \times \bar{V}_h$, $Q_h^\star = Q_h \times \bar{Q}_h$, and $X_h^\star = V_h^\star \times Q_h^\star$. Function pairs in $V_h^\star$ and $Q_h^\star$ will be denoted by boldface, e.g., $\mathbf{v}_h = (v_h, \bar{v}_h) \in V_h^\star$ and $\mathbf{q}_h = (q_h, \bar{q}_h) \in Q_h^\star$.

3.2. Weak formulation. We consider the weak formulation as presented in [12]. For the Stokes problem, it seeks $(\mathbf{u}_h, \mathbf{p}_h) \in X_h^\star$ such that

$$
\begin{aligned}
\sum_{K\in\mathcal{T}} \int_K \nabla u_h : \nabla v_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} (\bar{u}_h - u_h) \cdot \frac{\partial v_h}{\partial n} \,\mathrm{d}s - \sum_{K\in\mathcal{T}} \int_K p_h \nabla \cdot v_h \,\mathrm{d}x \\
+ \sum_{K\in\mathcal{T}} \int_{\partial K} \hat{\sigma}_h n \cdot (v_h - \bar{v}_h) \,\mathrm{d}s = \sum_{K\in\mathcal{T}} \int_K f \cdot v_h \,\mathrm{d}x \qquad \forall \mathbf{v}_h \in V_h^\star,
\end{aligned}
\tag{8}
$$

and

$$
\sum_{K\in\mathcal{T}} \int_K u_h \cdot \nabla q_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} \hat{u}_h \cdot n (\bar{q}_h - q_h) \,\mathrm{d}s - \int_\Gamma \bar{u}_h \cdot n \bar{q}_h \,\mathrm{d}s = 0 \qquad \forall \mathbf{q}_h \in Q_h^\star,
\tag{9}
$$

with the numerical fluxes $\hat{\sigma}_h$ and $\hat{u}_h$ given by

$$
\hat{\sigma}_h := -\nabla u_h + \bar{p}_h I - \frac{\alpha_v}{h_K} (\bar{u}_h - u_h) \otimes n, \qquad \hat{u}_h := u_h - \alpha_p h_K (\bar{p}_h - p_h) n,
\tag{10}
$$

and where $\alpha_v > 0$ and $\alpha_p \geq 0$ are penalty parameters. Note that the numerical fluxes can take on different values on opposite sides of a facet. We will prove that the formulation is stable for sufficiently large α_v, akin to the standard interior penalty method [1]. For a mixed-order formulation with $m = k - 1$ we will show that α_p can be set to zero, and for the equal-order case ($k = m$) we will show that α_p must be positive.

It will be convenient to express the method in a compact form, therefore we introduce the bilinear forms:

$$a_h(\mathbf{u},\mathbf{v}) := \sum_{K\in\mathcal{T}} \int_K \nabla u : \nabla v \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} \frac{\alpha_v}{h_K}(u-\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s - \sum_{K\in\mathcal{T}} \int_{\partial K} \left[(u-\bar{u})\cdot\frac{\partial v}{\partial n} + \frac{\partial u}{\partial n}\cdot(v-\bar{v})\right]\mathrm{d}s, \tag{11a}$$

$$b_h(\mathbf{p},\mathbf{v}) := -\sum_{K\in\mathcal{T}} \int_K p\nabla\cdot v\,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} (v-\bar{v})\cdot n\bar{p}\,\mathrm{d}s, \tag{11b}$$

$$c_h(\mathbf{p},\mathbf{q}) := \sum_{K\in\mathcal{T}} \int_{\partial K} \alpha_p h_K (p-\bar{p})(q-\bar{q})\,\mathrm{d}s. \tag{11c}$$

Solutions $(\mathbf{u}_h,\mathbf{p}_h)\in X_h^\star$ satisfy

$$B_h((\mathbf{u}_h,\mathbf{p}_h),(\mathbf{v}_h,\mathbf{q}_h)) = \sum_{K\in\mathcal{T}} \int_K f\cdot v_h\,\mathrm{d}x \qquad \forall(\mathbf{v}_h,\mathbf{q}_h)\in X_h^\star, \tag{12}$$

where

$$B_h((\mathbf{u}_h,\mathbf{p}_h),(\mathbf{v}_h,\mathbf{q}_h)) := a_h(\mathbf{u}_h,\mathbf{v}_h) + b_h(\mathbf{p}_h,\mathbf{v}_h) - b_h(\mathbf{q}_h,\mathbf{u}_h) + c_h(\mathbf{p}_h,\mathbf{q}_h). \tag{13}$$

To provide some insights into the method, setting $\bar{v}_h = 0$ we note that (8) is a cell-wise statement of the momentum balance, subject to weak satisfaction of the boundary condition provided by $\bar{u}_h$ (using Nitsche's method). Setting $v_h = 0$, we note that (8) imposes weak continuity of the numerical flux $\hat{\sigma}_h$ across facets. Equation (9) can be interpreted similarly, with it enforcing the continuity equation locally (in terms of the numerical flux $\hat{u}_h$) and weak continuity of $\hat{u}_h$ across facets. Different from conventional discontinuous Galerkin methods, functions on cells are not directly coupled to their neighbours via the numerical flux. Rather, functions on cells are coupled indirectly via the 'bar' functions that live only on facets. This has the important implementation consequence that degrees-of-freedom associated with u_h and p_h can be eliminated cell-wise in favour of degrees-of-freedom associated with $\bar{u}_h$ and $\bar{p}_h$. This process is commonly known as *static condensation*. This avoids the greater number of global degrees-of-freedom associated with standard discontinuous Galerkin methods compared to conforming methods on the same mesh.

3.3. Mass and momentum conservation. It is straightforward to show that the method conserves mass locally (cell-wise) in terms of the numerical flux $\hat{u}_h$. Setting $v_h = \bar{v}_h = 0$ and $\bar{q}_h = 0$, and $q_h = 1$ on a cell K and $q_h = 0$ on $\mathcal{T}\backslash K$ in (9),

$$\int_{\partial K} \hat{u}_h\cdot n\,\mathrm{d}s = 0 \qquad \forall K\in\mathcal{T}. \tag{14}$$

In the case that $\alpha_p = 0$, $\hat{u}_h$ and u_h coincide. Setting $v_h = \bar{v}_h = 0$ and $q_h = \bar{q}_h = 1$ in (9),

$$\int_{\Gamma} \bar{u}_h\cdot n\,\mathrm{d}s = 0. \tag{15}$$

Noteworthy is that for simplices with $\alpha_p = 0$ and $m = k-1$, i.e. the divergence of a function in the velocity space V_h is contained in the pressure space Q_h, the velocity field u_h is point-wise solenoidal within a cell.

Momentum conservation is addressed in [12] for the incompressible Navier–Stokes equations, where local momentum conservation in terms of the numerical flux $\hat{\sigma}_h$ was shown. To see this, set $v_h = e_j$ on K, where e_j is a canonical unit basis vector, and $v_h = 0$ on $\mathcal{T}\backslash K$, $\bar{v}_h = 0$, $q_h = 0$ and $\bar{q}_h = 0$ in (8). This yields:

$$\int_{\partial K} \hat{\sigma}_h n \, \mathrm{d}x = \int_K f \, \mathrm{d}x \qquad \forall K \in \mathcal{T}. \tag{16}$$

3.4. Relationship to a $H(\mathrm{div})$-conforming formulation. For a mixed-order formulation with $m = k-1$, the 'bar' function $\bar{p}_h$ acts as a Lagrange multiplier enforcing continuity of the normal component of $u_h \in \bar{V}_h$ across inter-element boundaries. It is easy to see that by setting $\mathbf{v}_h = \mathbf{0}$ and $q_h = 0$ in (12) that the normal component of the velocity is continuous across facets, i.e. $u_h \in V_h^{\mathrm{BDM}}$, where V_h^{BDM} is a Brezzi–Douglas–Marini (BDM) finite element space [2]:

$$\begin{aligned} V_h^{\mathrm{BDM}}(K) &= \left\{ v_h \in \left[P_k(K)\right]^d : v_h \cdot n \in L^2(\partial K),\ v_h \cdot n|_F \in P_k(F) \right\}, \\ V_h^{\mathrm{BDM}} &= \left\{ v_h \in H(\mathrm{div};\Omega) :\ v_h|_K \in V_h^{\mathrm{BDM}}(K),\ \forall K \in \mathcal{T} \right\}. \end{aligned} \tag{17}$$

Defining $V_h^{\star\mathrm{BDM}} = V_h^{\mathrm{BDM}} \times \bar{V}_h$ and $X_h^{\star\mathrm{BDM}} = V_h^{\star\mathrm{BDM}} \times Q_h$, the formulation in (12) is a hybridized [4] form of: find the $(\mathbf{u}_h, p_h) \in V_h^{\star\mathrm{BDM}} \times Q_h$ such that

$$B_h^{\mathrm{BDM}}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \sum_{K\in\mathcal{T}} \int_K f \cdot v_h \, \mathrm{d}x \qquad \forall (\mathbf{v}_h, q_h) \in X_h^{\star\mathrm{BDM}}, \tag{18}$$

where

$$B_h^{\mathrm{BDM}}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) := a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h^{\mathrm{BDM}}(p_h, \mathbf{v}_h) - b_h^{\mathrm{BDM}}(q_h, \mathbf{u}_h), \tag{19}$$

and where

$$b_h^{\mathrm{BDM}}(p, \mathbf{v}) := -\sum_{K\in\mathcal{T}} \int_K p \nabla \cdot v \, \mathrm{d}x. \tag{20}$$

If (12) and (18) have unique solutions, then then solution pair $(\mathbf{u}_h, p_h)$ to (12) is the solution of (18).

The $H(\mathrm{div})$-conforming formulation will be convenient for subsequent analysis as we will be able to neglect the Lagrange multiplier terms. For implementation we recommend (12).

4. Consistency, stability and boundedness. In this section we demonstrate consistency, stability, boundedness and well-posedness. To do this, we introduce extended function spaces on Ω:

$$V(h) := V_h + \left[H_0^1(\Omega)\right]^d \cap \left[H^2(\Omega)\right]^d, \tag{21}$$

$$Q(h) := Q_h + L_0^2(\Omega) \cap H^1(\Omega), \tag{22}$$

and extended function spaces on Γ^0 (facets):

$$\bar{V}(h) := \bar{V}_h + [H_0^{3/2}(\Gamma^0)]^d, \tag{23}$$

$$\bar{Q}(h) := \bar{Q}_h + H_0^{1/2}(\Gamma^0), \tag{24}$$

where $[H_0^{3/2}(\Gamma^0)]^d$ and $H_0^{1/2}(\Gamma^0)$ are, respectively, the trace spaces of $\left[H_0^1(\Omega)\right]^d \cap \left[H^2(\Omega)\right]^d$ and $L_0^2(\Omega) \cap H^1(\Omega)$ on facets Γ^0. We introduce the trace operator $\gamma : H^s(\Omega) \to H^{s-1/2}(\Gamma^0)$ to restrict functions in $H^s(\Omega)$ to Γ^0. For functions in $[H^s(\Omega)]^d$ the trace operator is applied component-wise. Even when not strictly necessary, we will use the trace operator to make clear when a function, usually the exact solution, is being restricted to facets. For notational purposes we also introduce $V^\star(h) := V(h) \times \bar{V}(h)$, $Q^\star(h) := Q(h) \times \bar{Q}(h)$ and $X^\star(h) := V^\star(h) \times Q^\star(h)$.

We define two norms on $V^\star(h)$, namely,

$$|||\mathbf{v}|||_v^2 := \sum_{K\in\mathcal{T}} \|\nabla v\|_{0,K}^2 + \sum_{K\in\mathcal{T}} \frac{\alpha_v}{h_K} \|\bar{v} - v\|_{0,\partial K}^2 , \tag{25}$$

which will be used to prove stability of a_h, and

$$|||\mathbf{v}|||_{v'}^2 := |||\mathbf{v}|||_v^2 + \sum_{K\in\mathcal{T}} \frac{h_K}{\alpha_v} \left\| \frac{\partial v}{\partial n} \right\|_{0,\partial K}^2 , \tag{26}$$

which will be used to prove boundedness of a_h. From the discrete trace inequality [9, Remark 1.47],

$$h_K^{1/2} \|v_h\|_{0,\partial K} \le C_t \|v_h\|_{0,K} \qquad \forall v_h \in P_k(K), \tag{27}$$

where C_t depends on k, spatial dimension and cell shape, it follows that the norms $|||\cdot|||_v$ and $|||\cdot|||_{v'}$ are equivalent on $V_h^\star$:

$$|||\mathbf{v}_h|||_v \le |||\mathbf{v}_h|||_{v'} \le c(1 + \alpha_v^{-1}) |||\mathbf{v}_h|||_v, \tag{28}$$

with $c > 0$ a constant independent of h, see [15, Eq. (5.5)].

We introduce a 'pressure semi-norm':

$$|\mathbf{q}|_p^2 := \sum_{K\in\mathcal{T}} \alpha_p h_K \|\bar{q} - q\|_{0,\partial K}^2 , \tag{29}$$

and define a norm on $X_h^\star$ by:

$$|||(\mathbf{v}_h, \mathbf{q}_h)|||_{v,p}^2 := |||\mathbf{v}_h|||_v^2 + \|q_h\|_{0,\Omega}^2 + |\mathbf{q}_h|_p^2 , \tag{30}$$

and on $X^\star(h)$ we define

$$\begin{aligned} |||(\mathbf{v}, \mathbf{q})|||_{v',p'}^2 &:= |||(\mathbf{v}, \mathbf{q})|||_{v,p}^2 + \sum_{K\in\mathcal{T}} \frac{h_K}{\alpha_v} \left\| \frac{\partial v}{\partial n} \right\|_{0,\partial K}^2 + \sum_{K\in\mathcal{T}} h_K \|\bar{q}\|_{0,\partial K}^2 \\ &= |||\mathbf{v}|||_v^2 + \|q\|_{0,\Omega}^2 + |\mathbf{q}|_p^2 + \sum_{K\in\mathcal{T}} \frac{h_K}{\alpha_v} \left\| \frac{\partial v}{\partial n} \right\|_{0,\partial K}^2 + \sum_{K\in\mathcal{T}} h_K \|\bar{q}\|_{0,\partial K}^2 \\ &= |||\mathbf{v}|||_{v'}^2 + \|q\|_{0,\Omega}^2 + |\mathbf{q}|_p^2 + \sum_{K\in\mathcal{T}} h_K \|\bar{q}\|_{0,\partial K}^2 . \end{aligned} \tag{31}$$

Note that (29) vanishes for the case of $\alpha_p = 0$.

4.1. Consistency. We now prove consistency of the method. It is assumed that $(u, p) \in X$ solves the Stokes problem (1), where

$$X := \left(\left[H^1_0(\Omega) \right]^d \cap \left[H^2(\Omega) \right]^d \right) \times \left(L^2_0(\Omega) \cap H^1(\Omega) \right). \tag{32}$$

LEMMA 1 (Consistency). *If $(u, p) \in X$ solves the Stokes problem (1), letting $\mathbf{u} = (u, \gamma(u))$ and $\mathbf{p} = (p, \gamma(p))$, then*

$$B_h((\mathbf{u}, \mathbf{p}), (\mathbf{v}_h, \mathbf{q}_h)) = \sum_{K \in \mathcal{T}} \int_K f \cdot v_h \,\mathrm{d}x \qquad \forall (\mathbf{v}_h, \mathbf{q}_h) \in X^\star_h. \tag{33}$$

Proof. We consider each form in the definition of B_h separately. Using that $\bar{u} = \gamma(u)$ and applying integration by parts to (11a), we find that

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}_h) &= \int_\Omega \nabla u : \nabla v_h \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} \frac{\alpha_v}{h_K} (\bar{u} - u) \cdot (\bar{v}_h - v_h) \,\mathrm{d}s \\ &\quad + \sum_{K \in \mathcal{T}} \int_{\partial K} \left[\frac{\partial u}{\partial n} \cdot (\bar{v}_h - v_h) + (\bar{u} - u) \cdot \frac{\partial v_h}{\partial n} \right] \mathrm{d}s \\ &= \int_\Omega \nabla u : \nabla v_h \,\mathrm{d}x - \sum_{K \in \mathcal{T}} \int_{\partial K} \frac{\partial u}{\partial n} \cdot (v_h - \bar{v}_h) \,\mathrm{d}s \\ &= - \int_\Omega \Delta u \cdot v_h \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} \frac{\partial u}{\partial n} \cdot \bar{v}_h \,\mathrm{d}s. \end{aligned} \tag{34}$$

Next, using that $\bar{p} = \gamma(p)$ and applying integration by parts to (11b), we find that

$$\begin{aligned} b_h(\mathbf{p}, \mathbf{v}_h) &= - \int_\Omega p \nabla \cdot v_h \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} (v_h - \bar{v}_h) \cdot n \bar{p} \,\mathrm{d}s \\ &= \int_\Omega \nabla p \cdot v_h \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} v_h \cdot n (\bar{p} - p) \,\mathrm{d}s - \sum_{K \in \mathcal{T}} \int_{\partial K} \bar{v}_h \cdot n \bar{p} \,\mathrm{d}s \\ &= \int_\Omega \nabla p \cdot v_h \,\mathrm{d}x - \sum_{K \in \mathcal{T}} \int_{\partial K} \bar{v}_h \cdot n \bar{p} \,\mathrm{d}s. \end{aligned} \tag{35}$$

Adding (34) and (35) and using that $\bar{p} = \gamma(p)$, we obtain

$$\int_\Omega (-\Delta u + \nabla p) \cdot v_h \,\mathrm{d}x - \sum_{K \in \mathcal{T}} \int_{\partial K} (-\nabla u + \bar{p} I)\, n \cdot \bar{v}_h \,\mathrm{d}s = \int_\Omega f \cdot v_h \,\mathrm{d}x. \tag{36}$$

Consider the facet integrals:

$$\sum_{K \in \mathcal{T}} \int_{\partial K} (-\nabla u + \bar{p} I)\, n \cdot \bar{v}_h \,\mathrm{d}s = \int_\Gamma (-\nabla u + \bar{p} I)\, n \cdot \bar{v}_h \,\mathrm{d}s = 0, \tag{37}$$

where the first equality is due to the single-valuedness of u, $\bar{p}$ and $\bar{v}_h$ on element boundaries, and the second equality is due to $\bar{v}_h = 0$ on Γ. We therefore conclude for the momentum equation that

$$a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{p}, \mathbf{v}_h) = \int_\Omega f \cdot v_h \,\mathrm{d}x. \tag{38}$$

We next consider the continuity equation. First note that

$$b_h(\mathbf{q}_h, \mathbf{u}) = -\int_\Omega q_h \nabla \cdot u \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} (u - \bar{u}) \cdot n\bar{q} \,\mathrm{d}s = 0, \tag{39}$$

because $\bar{u} = \gamma(u)$ and $\nabla \cdot u = 0$. Furthermore,

$$c_h(\mathbf{p}, \mathbf{q}_h) = \sum_{K\in\mathcal{T}} \int_{\partial K} \alpha_p h_K (\bar{p} - p)(\bar{q}_h - q_h) \,\mathrm{d}s = 0, \tag{40}$$

because $\bar{p} = \gamma(p)$. It follows that

$$-b_h(\mathbf{q}_h, \mathbf{u}) + c_h(\mathbf{p}, \mathbf{q}_h) = 0, \tag{41}$$

concluding the proof. □

4.2. Stability and boundedness of the vector-Laplacian term. Some results from [15] are generalised in this section to the vector-Laplacian term a_h, and are provided here for completeness.

LEMMA 2 (Stability of a_h). *There exists a $\beta_v > 0$, independent of h, and a constant $\alpha_0 > 0$ such that for $\alpha_v > \alpha_0$ and for all $\mathbf{v}_h \in V_h^\star$*

$$a_h(\mathbf{v}_h, \mathbf{v}_h) \geq \beta_v |||\mathbf{v}_h|||_v^2. \tag{42}$$

Proof. By definition of the bilinear form a_h in (11a),

$$a_h(\mathbf{v}_h, \mathbf{v}_h) = \sum_{K\in\mathcal{T}} \|\nabla v_h\|_{0,K}^2 + \sum_{K\in\mathcal{T}} \frac{\alpha_v}{h_K} \|v_h - \bar{v}_h\|_{0,\partial K}^2 + 2\sum_{K\in\mathcal{T}} \int_{\partial K} (\bar{v}_h - v_h) \cdot \frac{\partial v_h}{\partial n} \,\mathrm{d}s. \tag{43}$$

Applying the Cauchy–Schwarz inequality and a trace inequality to the third term on the right-hand side of (43),

$$\begin{aligned} \left| 2\sum_{K\in\mathcal{T}} \int_{\partial K} (\bar{v}_h - v_h) \cdot \frac{\partial v_h}{\partial n} \,\mathrm{d}s \right| &\leq 2 \frac{h_K^{1/2}}{\alpha_v^{1/2}} \left\| \frac{\partial v_h}{\partial n} \right\|_{0,\partial K} \frac{\alpha_v^{1/2}}{h_K^{1/2}} \|\bar{v}_h - v_h\|_{0,\partial K} \\ &\leq 2c\alpha_v^{-1/2} \|\nabla v_h\|_{0,K} \frac{\alpha_v^{1/2}}{h_K^{1/2}} \|\bar{v}_h - v_h\|_{0,\partial K}. \end{aligned} \tag{44}$$

Combined with (43) we obtain

$$a_h(\mathbf{v}_h, \mathbf{v}_h) \geq \sum_{K\in\mathcal{T}} \left(\|\nabla v_h\|_{0,K}^2 + 2c\alpha_v^{-1/2} \|\nabla v_h\|_{0,K} \frac{\alpha_v^{1/2}}{h_K^{1/2}} \|\bar{v}_h - v_h\|_{0,\partial K} + \frac{\alpha_v}{h_K} \|v_h - \bar{v}_h\|_{0,\partial K}^2 \right). \tag{45}$$

Note that for any $0 < \Psi < 1$ the following inequality holds for $x, y \in \mathbb{R}$: $x^2 - 2\Psi xy + y^2 \geq \frac{1}{2}(1 - \Psi^2)(x^2 + y^2)$ [9]. Taking $x = \|\nabla v_h\|_{0,K}$, $y = \alpha_v^{1/2} h_K^{-1/2} \|v_h - \bar{v}_h\|_{0,\partial K}$ and $\Psi = c\alpha_v^{-1/2}$, then if $\alpha_v > c^2 = \alpha_0$ it follows that

$$a_h(\mathbf{v}_h, \mathbf{v}_h) \geq \tfrac{1}{2}(1 - \alpha_0/\alpha_v) |||\mathbf{v}_h|||_v^2, \tag{46}$$

so that the result follows with $\beta_v = \frac{1}{2}(1 - \alpha_0/\alpha_v)$. □

LEMMA 3 (Boundedness of a_h). *There exists a $c > 0$, independent of h, such that for all $\mathbf{u} \in V^\star(h)$ and for all $\mathbf{v}_h \in V_h^\star$*

$$|a_h(\mathbf{u}, \mathbf{v}_h)| \le C_a |||\mathbf{u}|||_{v'} |||\mathbf{v}_h|||_v, \tag{47}$$

with $C_a = c(1 + \alpha_v^{-1/2})$.

Proof. From the definition of a_h in (11a),

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}_h) = &\underbrace{\sum_K \int_K \nabla u : \nabla v_h \,\mathrm{d}x}_{T_1} - \underbrace{\sum_{K\in\mathcal{T}} \int_{\partial K} \frac{\partial u}{\partial n} \cdot (v_h - \bar{v}_h) \,\mathrm{d}s}_{T_2} \\ &- \underbrace{\sum_{K\in\mathcal{T}} \int_{\partial K} (u - \bar{u}) \cdot \frac{\partial v_h}{\partial n} \,\mathrm{d}s}_{T_3} + \underbrace{\sum_{K\in\mathcal{T}} \int_{\partial K} \frac{\alpha_v}{h_K} (u - \bar{u}) \cdot (v_h - \bar{v}_h) \,\mathrm{d}s}_{T_4}. \end{aligned} \tag{48}$$

A bound for T_3 follows from

$$\begin{aligned} |T_3| &\le \left(\sum_{K\in\mathcal{T}} \frac{h_K}{\alpha_v} \left\| \frac{\partial v_h}{\partial n} \right\|_{0,\partial K}^2 \right)^{1/2} \left(\sum_{K\in\mathcal{T}} \frac{\alpha_v}{h_K} \|\bar{u} - u\|_{0,\partial K}^2 \right)^{1/2} \\ &\le c\alpha_v^{-1/2} \left(\sum_{K\in\mathcal{T}} \|\nabla v_h\|_{0,K}^2 \right)^{1/2} |||\mathbf{u}|||_v \\ &\le c\alpha_v^{-1/2} |||\mathbf{v}_h|||_v |||\mathbf{u}|||_{v'}, \end{aligned} \tag{49}$$

where $c > 0$ is a constant independent of h. For the second inequality in (49) we used the discrete trace inequality (27). Similar bounds for T_1, T_2 and T_4 follow after applying the Cauchy–Schwarz inequality. Collecting all bounds proves (47). ∎

4.3. Stability of the pressure–velocity coupling term. We now examine stability of the discrete pressure–velocity coupling term b_h. The analysis of b_h for the equal-order and mixed-order cases differs, hence we will prove stability of b_h for the two cases separately.

To prove stability of the discrete pressure–velocity coupling term, b_h, we remark that satisfaction of the inf-sup condition for the infinite-dimensional problem (5) is equivalent to there existing for all $q \in L_0^2(\Omega)$ a $v_q \in \left[H_0^1(\Omega)\right]^d$ that satisfies

$$q = \nabla \cdot v_q \quad \text{and} \quad \beta_c \|v_q\|_{1,\Omega} \le \|q\|_{0,\Omega} \tag{50}$$

(see, e.g. [9, Theorem 6.5]). We make extensive use of this result.

We state now the stability lemma for the pressure–velocity coupling.

LEMMA 4 (Stability of b_h). *There exists a constant $\beta_p > 0$, independent of h, such that for all $\mathbf{q}_h \in Q_h^\star$*

$$\beta_p \|q_h\|_{0,\Omega} \le \sup_{\mathbf{w}_h \in V_h^\star} \frac{b_h(\mathbf{q}_h, \mathbf{w}_h)}{|||\mathbf{w}_h|||_v} + |\mathbf{q}_h|_p. \tag{51}$$

We prove the above lemma for the equal-order and mixed-order cases in the following sections. Recall that $|\mathbf{q}_h|_p$ is zero for the mixed-order case since $\alpha_p = 0$.

4.3.1. Equal-order case. For the equal-order case ($k = m$ in (6)), we introduce the projections Π_h and $\bar{\Pi}_h$, where $\Pi_h : [H^1(\Omega)]^d \to V_h$ is any projection such that

$$\int_K (\Pi_h v - v) \cdot y_h \,\mathrm{d}x = 0 \qquad \forall y_h \in [P_{k-1}(K)]^d \tag{52}$$

for all $K \in \mathcal{T}_h$, and $\bar{\Pi}_h : [H^1(\Omega)]^d \to \bar{V}_h$ is the L^2-projection into $\bar{V}_h$,

$$\sum_{K \in \mathcal{T}} \int_{\partial K} (\bar{\Pi}_h v - v) \cdot \bar{y}_h \,\mathrm{d}s = 0 \qquad \forall \bar{y}_h \in \bar{V}_h. \tag{53}$$

The following two inequalities will be used below to prove stability of b_h:

$$\|w - \Pi_h w\|_{0,F} \le c h_K^{1/2} |w|_{1,K} \qquad \forall F \in \mathcal{F},\ F \subset \partial K,\ \forall K \in \mathcal{T}_h, \tag{54}$$

$$\left\|\Pi_h w - \bar{\Pi}_h w\right\|_{\partial K}^2 \le h_K \|w\|_{1,K}^2 \qquad \forall K \in \mathcal{T}_h, \tag{55}$$

where $c > 0$ is independent of h. The first inequality is due to [9, Lemma 1.59], and the second is due to [6, Proposition 3.9].

We can now prove stability of the discrete pressure–velocity coupling term, b_h, for an equal-order velocity/pressure approximation.

Proof of Lemma 4 for the equal-order case. For a $q_h \in Q_h$, from (50), there exists a $v_{q_h} \in \left[H_0^1(\Omega)\right]^d$ such that $\nabla \cdot v_{q_h} = q_h$ and $\beta_c \left\|v_{q_h}\right\|_{1,\Omega} \le \|q_h\|_{0,\Omega}$. Since $v_{q_h} \in \left[H_0^1(\Omega)\right]^d$ we see that

$$\begin{aligned} \|q_h\|_{0,\Omega}^2 = \int_\Omega q_h^2 \,\mathrm{d}x &= \int_\Omega q_h \nabla \cdot v_{q_h} \,\mathrm{d}x \\ &= -\sum_{K \in \mathcal{T}} \int_K \nabla q_h \cdot v_{q_h} \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} q_h v_{q_h} \cdot n \,\mathrm{d}s. \end{aligned} \tag{56}$$

By (52), we note that $\sum_{K \in \mathcal{T}} \int_K \nabla q_h \cdot (\Pi_h v_{q_h} - v_{q_h}) \,\mathrm{d}x = 0$, and it follows that

$$\|q_h\|_{0,\Omega}^2 = -\sum_{K \in \mathcal{T}} \int_K \nabla q_h \cdot \Pi_h v_{q_h} \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} q_h v_{q_h} \cdot n \,\mathrm{d}s. \tag{57}$$

Next, we note for $\mathbf{q}_h = (q_h, \bar{q}_h) \in Q_h^*$, from the definition of b_h in (11b) and applying integration by parts we have

(58)

$$\begin{aligned} b_h(\mathbf{q}_h, (\Pi_h v_{q_h}, \bar{\Pi}_h v_{q_h})) &= \int_\Omega \nabla q_h \cdot \Pi_h v_{q_h} \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} \left[(\Pi_h v_{q_h} - \bar{\Pi}_h v_{q_h}) \cdot n \bar{q}_h - \Pi_h v_{q_h} \cdot n q_h \right] \mathrm{d}s \\ &= \int_\Omega \nabla q_h \cdot \Pi_h v_{q_h} \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} (\bar{q}_h - q_h) \Pi_h v_{q_h} \cdot n \,\mathrm{d}s, \end{aligned}$$

where the last equality is due to the single-valuedness of $\bar{\Pi}_h v_{q_h}$ and $\bar{q}_h$ across element boundaries and because $\bar{\Pi}_h v_{q_h} = 0$ on the domain boundary Γ. We may now write (57) as

$$\begin{aligned} \|q_h\|_{0,\Omega}^2 = &- b_h(\mathbf{q}_h, (\Pi_h v_{q_h}, \bar{\Pi}_h v_{q_h})) + \sum_{K \in \mathcal{T}} \int_{\partial K} (\bar{q}_h - q_h) \Pi_h v_{q_h} \cdot n \,\mathrm{d}s \\ &- \sum_{K \in \mathcal{T}} \int_{\partial K} (\bar{q}_h - q_h) v_{q_h} \cdot n \,\mathrm{d}s, \end{aligned} \tag{59}$$

where equality is due to the single-valuedness of v_{q_h} and $\bar{q}_h$ across element boundaries and because $v_{q_h} = 0$ on the domain boundary Γ. It follows then that

$$\|q_h\|_{0,\Omega}^2 = \underbrace{-b_h(\mathbf{q}_h, (\Pi_h v_{q_h}, \bar{\Pi}_h v_{q_h}))}_{T_1} + \underbrace{\sum_{K\in\mathcal{T}} \int_{\partial K} (\bar{q}_h - q_h)(\Pi_h v_{q_h} - v_{q_h}) \cdot n \, \mathrm{d}s}_{T_2} . \tag{60}$$

We now bound T_1 and T_2 separately. Starting with T_1,

$$|T_1| \le \left(\sup_{\mathbf{w}_h \in V_h^\star} \frac{b_h(\mathbf{q}_h, \mathbf{w}_h)}{|||\mathbf{w}_h|||_v} \right) |||(\Pi_h v_{q_h}, \bar{\Pi}_h v_{q_h})|||_v. \tag{61}$$

Noting that

$$|||(\Pi_h v_{q_h}, \bar{\Pi}_h v_{q_h})|||_v^2 = \sum_{K\in\mathcal{T}} \left\|\nabla(\Pi_h v_{q_h})\right\|_{0,K}^2 + \sum_{K\in\mathcal{T}} \frac{\alpha_v}{h_K} \left\|\bar{\Pi}_h v_{q_h} - \Pi_h v_{q_h}\right\|_{0,\partial K}^2 , \tag{62}$$

it is possible to bound the two terms on the right separately. By [9, Lemma 6.11] and (55) we find, respectively,

$$\begin{aligned} \sum_{K\in\mathcal{T}} \left\|\nabla(\Pi_h v_{q_h})\right\|_{0,K}^2 &\le c\left\|v_{q_h}\right\|_{1,\Omega}^2 , \\ \sum_{K\in\mathcal{T}} \frac{\alpha_v}{h_K} \left\|\bar{\Pi}_h v_{q_h} - \Pi_h v_{q_h}\right\|_{0,\partial K}^2 &\le c\alpha_v \left\|v_{q_h}\right\|_{1,\Omega}^2 , \end{aligned} \tag{63}$$

with $c > 0$ independent of h. Using (50) it follows then that

$$|T_1| \le c(1+\alpha_v)\beta_c^{-1} \left(\sup_{\mathbf{w}_h \in V_h^\star} \frac{b_h(\mathbf{q}_h, \mathbf{w}_h)}{|||\mathbf{w}_h|||_v} \right) \|q_h\|_{0,\Omega} . \tag{64}$$

We now bound T_2. Using the Cauchy–Schwarz inequality,

$$\begin{aligned} |T_2| &\le \left(\sum_{K\in\mathcal{T}} \int_{\partial K} h_K (\bar{q}_h - q_h)^2 \, \mathrm{d}s \right)^{1/2} \left(\sum_{K\in\mathcal{T}} \int_{\partial K} h_K^{-1} \left|\Pi_h v_{q_h} - v_{q_h}\right|^2 \mathrm{d}s \right)^{1/2} \\ &\le \alpha_p^{-1} |\mathbf{q}_h|_p \left(\sum_{K\in\mathcal{T}} h_K^{-1} \left\|\Pi_h v_{q_h} - v_{q_h}\right\|_{0,\partial K}^2 \right)^{1/2} , \end{aligned} \tag{65}$$

where the last inequality follows from the definition of the pressure semi-norm in (29). Note that by (54)

$$\left\|\Pi_h v_{q_h} - v_{q_h}\right\|_{0,\partial K}^2 \le \sum_{F\in\mathcal{F},\ F\subset\partial K} \left\|\Pi_h v_{q_h} - v_{q_h}\right\|_{0,F}^2 \le c h_K \left|v_{q_h}\right|_{1,K}^2 , \tag{66}$$

where $c > 0$ is independent of h. Using again (50) we find the following bound for T_2,

$$|T_2| \le c\alpha_p^{-1} |\mathbf{q}_h|_p \left\|v_{q_h}\right\|_{1,\Omega} \le c(\alpha_p \beta_c)^{-1} |\mathbf{q}_h|_p \|q_h\|_{0,\Omega} . \tag{67}$$

Combining now (60), (64) and (67), we find

$$\beta_c \|q_h\|_{0,\Omega}^2 \le c(1+\alpha_v) \left(\sup_{\mathbf{w}_h \in V_h^\star} \frac{b_h(\mathbf{q}_h, \mathbf{w}_h)}{|||\mathbf{w}_h|||_v} \right) \|q_h\|_{0,\Omega} + c\alpha_p^{-1} |\mathbf{q}_h|_p \|q_h\|_{0,\Omega} . \tag{68}$$

Dividing both sides by $\|q_h\|_{0,\Omega}$ and rearranging terms the bound in (51) follows with

$$\beta_p = \frac{c\beta_c}{\max(1+\alpha_v, \alpha_p^{-1})}. \tag{69}$$ ∎

Note that $\beta_p \to 0$ as $\alpha_p \to 0$, illustrating the need for the penalty term on the pressure jump.

4.3.2. Mixed-order case. To prove Lemma 4 for a mixed-order velocity–pressure approximation, we follow a similar approach to [10]. For this we require the definition of the BDM interpolation operator, as given in the following lemma [10, Lemma 7]. See [13] for the analogous operators on quadrilaterals and hexahedra.

LEMMA 5. *If the mesh consists of triangles in two dimensions or tetrahedra in three dimensions there is an interpolation operator* $\Pi_{\rm BDM} : [H^1(\Omega)]^d \to V_h$ *with the following properties for all* $u \in [H^{k+1}(K)]^d$*:*

(i) $[\![n \cdot \Pi_{\rm BDM} u]\!] = 0$*, where* $[\![a]\!] = a^+ + a^-$ *and* $[\![a]\!] = a$ *on, respectively, interior and boundary faces is the usual jump operator.*
(ii) $\|u - \Pi_{\rm BDM} u\|_{m,K} \le c h_K^{l-m} \|u\|_{l,K}$ *with* $m = 0, 1, 2$ *and* $m \le l \le k+1$.
(iii) $\|\nabla \cdot (u - \Pi_{\rm BDM} u)\|_{m,K} \le c h_K^{l-m} \|\nabla \cdot u\|_{l,K}$ *with* $m = 0, 1$ *and* $m \le l \le k$.
(iv) $\int_K q(\nabla \cdot u - \nabla \cdot \Pi_{\rm BDM} u) \,{\rm d}x = 0$ *for all* $q \in P_{k-1}(K)$.
(v) $\int_F \bar{q}(n \cdot u - n \cdot \Pi_{\rm BDM} u) \,{\rm d}s = 0$ *for all* $\bar{q} \in P_k(F)$*, where* F *is a face on* ∂K.

We can now prove stability of the discrete pressure–velocity coupling term, b_h, for a mixed-order velocity/pressure approximation.

Proof of Lemma 4 for the mixed-order case. Let $q_h \in Q_h$. By (50) there exists a $v_{q_h} \in \left[H_0^1(\Omega)\right]^d$ such that $\nabla \cdot v_{q_h} = q_h$ and $\beta_c \|v_{q_h}\|_{1,\Omega} \le \|q_h\|_{0,\Omega}$. Since $v_{q_h} \in \left[H_0^1(\Omega)\right]^d$ we see that

$$\|q_h\|_{0,\Omega}^2 = \int_\Omega q_h \nabla \cdot v_{q_h} \,{\rm d}x = \int_\Omega q_h \nabla \cdot \Pi_{\rm BDM} v_{q_h} \,{\rm d}x = -b_h(\mathbf{q}_h, (\Pi_{\rm BDM} v_{q_h}, \bar{\Pi}_h v_{q_h})), \tag{70}$$

by (iv) of Lemma 5 and by definition of b_h in (11b). Next, we determine a bound for $|\!|\!|(\Pi_{\rm BDM} v_{q_h}, \bar{\Pi}_h v_{q_h})|\!|\!|_v$. We first note that

$$\|\nabla(\Pi_{\rm BDM} v_{q_h})\|_{0,K} \le \|\nabla v_{q_h} - \nabla(\Pi_{\rm BDM} v_{q_h})\|_{0,K} + \|\nabla v_{q_h}\|_{0,K} \le c\|v_{q_h}\|_{1,K}, \tag{71}$$

due to (ii) of Lemma 5. We also note that

$$\begin{aligned} h_K^{-1}\|\Pi_{\rm BDM} v_{q_h} - \bar{\Pi}_h v_{q_h}\|_{0,\partial K}^2 &\le h_K^{-1}\|\Pi_{\rm BDM} v_{q_h} - v_{q_h}\|_{0,\partial K}^2 \\ &+ h_K^{-1}\|\bar{\Pi}_h v_{q_h} - \Pi_h v_{q_h}\|_{0,\partial K}^2 + h_K^{-1}\|\Pi_h v_{q_h} - v_{q_h}\|_{0,\partial K}^2. \end{aligned} \tag{72}$$

By the trace inequality (27), item (ii) of Lemma 5, and the inequalities (54) and (55),

$$\begin{aligned} h_K^{-1}\|\Pi_{\rm BDM} v_{q_h} - v_{q_h}\|_{0,\partial K}^2 &\le c h_K^{-2}\|\Pi_{\rm BDM} v_{q_h} - v_{q_h}\|_{0,K}^2 \le c\|v_{q_h}\|_{1,K}^2, \\ h_K^{-1}\|\Pi_h v_{q_h} - v_{q_h}\|_{0,\partial K}^2 &\le c|v_{q_h}|_{1,K}^2, \\ h_K^{-1}\|\bar{\Pi}_h v_{q_h} - \Pi_h v_{q_h}\|_{0,\partial K}^2 &\le \|v_{q_h}\|_{1,K}^2. \end{aligned} \tag{73}$$

Combining these results with (72), we obtain

$$\frac{\alpha_v}{h_K}\|\Pi_{\rm BDM} v_{q_h} - \bar{\Pi}_h v_{q_h}\|_{0,\partial K}^2 \le c\alpha_v \|v_{q_h}\|_{1,K}^2. \tag{74}$$

From (71) and (74) we therefore find that

$$\|\!|(\Pi_{\mathrm{BDM}} v_{q_h}, \bar{\Pi}_h v_{q_h})\|\!|_v^2 \leq c(1+\alpha_v)\left\|v_{q_h}\right\|_{1,K}^2. \tag{75}$$

Satisfaction of the inf-sup condition follows from

$$\sup_{\mathbf{w}_h \in V_h^\star} \frac{-b_h(\mathbf{q}_h, \mathbf{w}_h)}{\|\!|\mathbf{w}_h\|\!|_v} \geq \frac{-b_h(\mathbf{q}_h, (\Pi_{\mathrm{BDM}} v_{q_h}, \bar{\Pi}_h v_{q_h}))}{\|\!|(\Pi_{\mathrm{BDM}} v_{q_h}, \bar{\Pi}_h v_{q_h})\|\!|_v} \geq \frac{c\beta_c}{1+\alpha_v} \|q_h\|_{0,\Omega}, \tag{76}$$

where we have used (70), (75) and (50) for the second inequality. The bound in (51) follows with $\beta_p = c\beta_c/(1+\alpha_v)$. ∎

Note that the analysis for the mixed order case does not depend on the pressure penalty term α_p, which can be set to zero.

By Lemmas 2 and 4, it straightforward to show that a solution $(\mathbf{u}_h, p_h)$ to the $H(\mathrm{div})$-type formulation in (18) is unique. However, for the case $\alpha_p = 0$, Lemma 4 does not involve any norms of $\bar{q}_h \in \bar{Q}_h$. The following proposition shows that Lemma 4 is sufficient for the formulation in (12).

PROPOSITION 6. *If $\alpha_v > \alpha_0$, a solution $(\mathbf{u}_h, \mathbf{p}_h) \in X_h^\star$ to (12) is unique.*

Proof. We wish to show that for zero data $\mathbf{u}_h = \mathbf{0}$ and $\mathbf{p}_h = \mathbf{0}$. Setting $\mathbf{v}_h = \mathbf{u}_h$ and $\mathbf{q}_h = \mathbf{p}_h$ in (12), coercivity of $a_h(\cdot,\cdot)$ implies that $\mathbf{u}_h = \mathbf{0}$.

Substituting $\mathbf{u}_h = \mathbf{0}$ into (12) results in

$$-\sum_{K\in\mathcal{T}} \int_K p_h \nabla\cdot v_h \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} v_h \cdot n \bar{p}_h \,\mathrm{d}s = 0 \qquad \forall v_h \in V_h. \tag{77}$$

Setting $v_h = 0$ on all elements except K, and integrating by parts,

$$\int_K \nabla p_h \cdot v_h \,\mathrm{d}x + \int_{\partial K} v_h \cdot n\,(\bar{p}_h - p_h)\,\mathrm{d}s = 0 \qquad \forall v_h \in \left[P_k(K)\right]^d. \tag{78}$$

We consider a $w_h \in \left[P_k(K)\right]^3$ that satisfies

$$\int_{\partial K} w_h \cdot n \bar{r}_h \,\mathrm{d}s = \int_{\partial K} (p_h - \bar{p}_h)\,\bar{r}_h \,\mathrm{d}s \qquad \forall \bar{r}_h \in P_k(\partial K), \tag{79a}$$

$$\int_K w_h \cdot z_h \,\mathrm{d}x = 0 \qquad \forall z_h \in \mathcal{N}_{k-2}(K), \tag{79b}$$

where $\mathcal{N}_{k-2}$ is the Nédélec space [2]. We remark that such a w_h exists and is unique [2, Proposition 2.3.2]. Using w_h as test function in (78),

$$0 = \int_K \nabla p_h \cdot w_h \,\mathrm{d}x + \int_{\partial K} w_h \cdot n\,(\bar{p}_h - p_h)\,\mathrm{d}s = \int_{\partial K} (\bar{p}_h - p_h)^2 \,\mathrm{d}s, \tag{80}$$

where we used that $\nabla p_h \in \nabla P_{k-1}(K) \subset \left[P_{k-2}(K)\right]^3 \subset \mathcal{N}_{k-2}(K)$, and set $\bar{r}_h = \bar{p}_h - p_h \in P_k(\partial K)$. Equation (80) implies that $p_h = \bar{p}_h$ on ∂K. Using $p_h = \bar{p}_h$ on facets in (78) shows that the pressure p_h is defined only up to a constant. Since $p_h = \bar{p}_h$, and considering that $\bar{p}_h$ is single-valued on facets, together with $\int_\Omega p_h \,\mathrm{d}x = 0$ in (1d) we obtain that $\mathbf{p}_h = \mathbf{0}$. ∎

We have proved existence and uniqueness of a solution to (12) for simplex elements (the degrees-of-freedom in (79) are specific to simplices). For other elements, the analogous degree-of-freedom definitions can be found in [2, Chapter 2].

4.4. Well-posedness and boundedness. We are now ready to prove inf-sup stability and boundedness of the method. The analysis for the equal- and mixed-order case is identical, building on Lemma 4. We first prove satisfaction of the discrete inf-sup condition.

LEMMA 7 (Discrete inf-sup stability). *If $\alpha_v > \alpha_0$, with α_0 defined in Lemma 2, then there exists a constant $\sigma > 0$, independent of h, such that for all $(\mathbf{v}_h, \mathbf{q}_h) \in X_h^\star$*

$$\sigma |||(\mathbf{v}_h, \mathbf{q}_h)|||_{v,p} \le \sup_{(\mathbf{w}_h,\mathbf{r}_h)\in X_h^\star} \frac{B_h((\mathbf{v}_h, \mathbf{q}_h), (\mathbf{w}_h, \mathbf{r}_h))}{|||(\mathbf{w}_h, \mathbf{r}_h)|||_{v,p}}. \tag{81}$$

Proof. We first note that

$$B_h((\mathbf{v}_h, \mathbf{q}_h), (\mathbf{v}_h, \mathbf{q}_h)) = a_h(\mathbf{v}_h, \mathbf{v}_h) + c_h(\mathbf{q}_h, \mathbf{q}_h) \ge \beta_v |||\mathbf{v}_h|||_v^2 + |\mathbf{q}_h|_p^2, \tag{82}$$

by Lemma 2 and by definition of c_h (11c) and the pressure semi-norm (29). Then,

$$\beta_v |||\mathbf{v}_h|||_v^2 + |\mathbf{q}_h|_p^2 \le \sup_{(\mathbf{w}_h,\mathbf{r}_h)\in X_h^\star} \frac{B_h((\mathbf{v}_h, \mathbf{q}_h), (\mathbf{w}_h, \mathbf{r}_h))}{|||(\mathbf{w}_h, \mathbf{r}_h)|||_{v,p}} |||(\mathbf{v}_h, \mathbf{q}_h)|||_{v,p}. \tag{83}$$

It is clear that $b_h(\mathbf{q}_h, \mathbf{w}_h) = B_h((\mathbf{v}_h, \mathbf{q}_h), (\mathbf{w}_h, \mathbf{0})) - a_h(\mathbf{v}_h, \mathbf{w}_h)$, so using Lemma 4 we find that

$$\begin{aligned} \beta_p \|q_h\|_{0,\Omega} &\le \sup_{\mathbf{w}_h\in V_h^\star} \left(\frac{-a_h(\mathbf{v}_h, \mathbf{w}_h)}{|||\mathbf{w}_h|||_v} + \frac{B_h((\mathbf{v}_h, \mathbf{q}_h), (\mathbf{w}_h, \mathbf{0}))}{|||(\mathbf{w}_h, \mathbf{0})|||_{v,p}} \right) + |\mathbf{q}_h|_p \\ &\le \sup_{\mathbf{w}_h\in V_h^\star} \frac{a_h(\mathbf{v}_h, \mathbf{w}_h)}{|||\mathbf{w}_h|||_v} + \sup_{(\mathbf{w}_h,\mathbf{r}_h)\in X_h^\star} \frac{B_h((\mathbf{v}_h, \mathbf{q}_h), (\mathbf{w}_h, \mathbf{r}_h))}{|||(\mathbf{w}_h, \mathbf{r}_h)|||_{v,p}} + |\mathbf{q}_h|_p . \end{aligned} \tag{84}$$

Using the boundedness of a_h (Lemma 3) and (28),

$$\sup_{\mathbf{w}_h\in V_h^\star} \frac{a_h(\mathbf{v}_h, \mathbf{w}_h)}{|||\mathbf{w}_h|||_v} \le c C_a (1 + \alpha_v^{-1}) |||\mathbf{v}_h|||_v, \tag{85}$$

where $c > 0$ is independent of h. Let $c_{\alpha_v} = c C_a (1 + \alpha_v^{-1})$. It follows then that

$$\beta_p \|q_h\|_{0,\Omega} \le c_{\alpha_v} |||\mathbf{v}_h|||_v + \sup_{(\mathbf{w}_h,\mathbf{r}_h)\in X_h^\star} \frac{B_h((\mathbf{v}_h, \mathbf{q}_h), (\mathbf{w}_h, \mathbf{r}_h))}{|||(\mathbf{w}_h, \mathbf{r}_h)|||_{v,p}} + |\mathbf{q}_h|_p . \tag{86}$$

Applying Young's inequality twice, it follows that

$$\begin{aligned} \beta_p^2 \|q_h\|_{0,\Omega}^2 \le 4 \left(c_{\alpha_v}^2 |||\mathbf{v}_h|||_v^2 + |\mathbf{q}_h|_p^2 \right) \\ + 2 \left(\sup_{(\mathbf{w}_h,\mathbf{r}_h)\in X_h^\star} \frac{B_h((\mathbf{v}_h, \mathbf{q}_h), (\mathbf{w}_h, \mathbf{r}_h))}{|||(\mathbf{w}_h, \mathbf{r}_h)|||_{v,p}} \right)^2 . \end{aligned} \tag{87}$$

Applying Young's inequality now to (83),

$$\begin{aligned} &\beta_v |||\mathbf{v}_h|||_v^2 + |\mathbf{q}_h|_p^2 \\ &\quad \le \frac{1}{2\psi} \left(\sup_{(\mathbf{w}_h,\mathbf{r}_h)\in X_h^\star} \frac{B_h((\mathbf{v}_h, \mathbf{q}_h), (\mathbf{w}_h, \mathbf{r}_h))}{|||(\mathbf{w}_h, \mathbf{r}_h)|||_{v,p}} \right)^2 + \frac{\psi}{2} |||(\mathbf{v}_h, \mathbf{q}_h)|||_{v,p}^2, \end{aligned} \tag{88}$$

with $\psi > 0$ a constant. Define $0 < \epsilon < \min(1, \beta_v c_{\alpha_v}^{-2})/4$. Multiplying (87) by ϵ and adding to (88), we obtain

$$\beta_v \|\|\mathbf{v}_h\|\|_v^2 + |\mathbf{q}_h|_p^2 + \epsilon\beta_p^2 \|q_h\|_{0,\Omega}^2 \le 4\epsilon c_{\alpha_v}^2 \|\|\mathbf{v}_h\|\|_v^2 + 4\epsilon|\mathbf{q}_h|_p^2 + \frac{\psi}{2}\|\|(\mathbf{v}_h, \mathbf{q}_h)\|\|_{v,p}^2 + \left(\frac{1}{2\psi} + 2\epsilon\right)\left(\sup_{(\mathbf{w}_h,\mathbf{r}_h)\in X_h^\star} \frac{B_h((\mathbf{v}_h,\mathbf{q}_h),(\mathbf{w}_h,\mathbf{r}_h))}{\|\|(\mathbf{w}_h,\mathbf{r}_h)\|\|_{v,p}}\right)^2. \tag{89}$$

Re-arranging, the result follows with

$$\sigma = \left(\frac{2\min\left\{\beta_v - 4\epsilon c_{\alpha_v}^2, 1-4\epsilon, \epsilon\beta_p^2\right\} - \psi}{\psi^{-1} + 4\epsilon}\right)^{1/2}. \tag{90}$$ □

Lemma 7 proves that the discrete problem is well-posed. We now show that the bilinear form B_h in (13) is bounded.

LEMMA 8 (Boundedness). *There exists a constant $C_B > 0$, independent of h, such that for all $(\mathbf{v},\mathbf{q}) \in X^\star(h)$ and $(\mathbf{w}_h,\mathbf{r}_h) \in X_h^\star$*

$$B_h((\mathbf{v},\mathbf{q}),(\mathbf{w}_h,\mathbf{r}_h)) \le C_B \|\|(\mathbf{v},\mathbf{q})\|\|_{v',p'} \|\|(\mathbf{w}_h,\mathbf{r}_h)\|\|_{v,p}. \tag{91}$$

Proof. Let $(\mathbf{v},\mathbf{q}) \in X^\star(h)$ and $(\mathbf{w}_h,\mathbf{r}_h) \in X_h^\star$. From the definition of B_h,

$$B_h((\mathbf{v},\mathbf{q}),(\mathbf{w}_h,\mathbf{r}_h)) = a_h(\mathbf{v},\mathbf{w}_h) + b_h(\mathbf{q},\mathbf{w}_h) - b_h(\mathbf{r}_h,\mathbf{v}) + c_h(\mathbf{q},\mathbf{r}_h). \tag{92}$$

We will bound each of the terms separately. By Lemma 3 and the Cauchy–Schwarz inequality,

$$\left|a_h(\mathbf{v},\mathbf{w}_h)\right| \le C_a \|\|(\mathbf{v},\mathbf{0})\|\|_{v',p'} \|\|(\mathbf{w}_h,\mathbf{0})\|\|_{v,p}, \tag{93}$$

$$\left|c_h(\mathbf{q},\mathbf{r}_h)\right| \le c\|\|(0,\mathbf{q})\|\|_{v,p} \|\|(0,\mathbf{r}_h)\|\|_{v,p}, \tag{94}$$

with $c > 0$ independent of h. Next we consider a bound for $b_h(\mathbf{r}_h,\mathbf{v})$. From the definition of b_h (11b), we note that

$$b_h(\mathbf{r}_h,\mathbf{v}) = -\int_\Omega r_h \nabla\cdot v \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} (v - \bar{v})\cdot n \bar{r}_h \,\mathrm{d}s. \tag{95}$$

It is clear that

$$\left|-\int_\Omega r_h \nabla\cdot v\,\mathrm{d}x\right| \le \sum_{K\in\mathcal{T}} \|\nabla v\|_{0,K} \|r_h\|_{0,\Omega} \le \|\|(\mathbf{v},\mathbf{0})\|\|_{v,p} \|\|(\mathbf{0},\mathbf{r}_h)\|\|_{v,p}, \tag{96}$$

and

$$\left|\sum_{K\in\mathcal{T}} \int_{\partial K} (v-\bar{v})\cdot n\bar{r}_h\,\mathrm{d}s\right| \le \left(\sum_{K\in\mathcal{T}} h_K^{-1}\|v-\bar{v}\|_{0,\partial K}^2\right)^{1/2} \left(\sum_{K\in\mathcal{T}} h_K \|\bar{r}_h\|_{0,\partial K}^2\right)^{1/2}. \tag{97}$$

Note, however, that

$$h_K \|\bar{r}_h\|_{0,\partial K}^2 \le 2h_K\|\bar{r}_h - r_h\|_{0,\partial K}^2 + c\|r_h\|_{0,K}^2\,, \tag{98}$$

where we have used the discrete trace inequality (27) ($c > 0$ is independent of h). We therefore find that

$$\left| \sum_{K\in\mathcal{T}} \int_{\partial K} (v - \bar{v}) \cdot n \bar{r}_h \,\mathrm{d}s \right| \le c\alpha_v^{-1/2} |||(\mathbf{v}, \mathbf{0})|||_{v,p} |||(\mathbf{0}, \mathbf{r}_h)|||_{v,p}. \tag{99}$$

It follows that

$$\left| b_h(\mathbf{r}_h, \mathbf{v}) \right| \le (1 + c\alpha_v^{-1/2}) |||(\mathbf{v}, \mathbf{0})|||_{v,p} |||(\mathbf{0}, \mathbf{r}_h)|||_{v,p}. \tag{100}$$

The final term to bound is $b_h(\mathbf{q}, \mathbf{w}_h)$. Using the Cauchy–Schwarz inequality,

$$\left| \sum_{K\in\mathcal{T}} \int_{\partial K} (w_h - \bar{w}_h) \cdot n \bar{q} \,\mathrm{d}s \right| \le c\alpha_v^{-1/2} |||(\mathbf{w}_h, \mathbf{0})|||_{v,p} |||(\mathbf{0}, \mathbf{q})|||_{v',p'}. \tag{101}$$

Furthermore, as in (96), $\left| -\int_\Omega q \nabla \cdot w_h \,\mathrm{d}x \right| \le |||(\mathbf{w}_h, \mathbf{0})|||_{v,p} |||(\mathbf{0}, \mathbf{q})|||_{v,p}$. It follows that

$$\left| b_h(\mathbf{q}, \mathbf{w}_h) \right| \le (1 + c\alpha_v^{-1/2}) |||(\mathbf{w}_h, \mathbf{0})|||_{v,p} |||(\mathbf{0}, \mathbf{q})|||_{v',p'}. \tag{102}$$

Collecting the bounds (93), (94), (100) and (102), the result follows with $C_B = c(1 + \alpha_v^{-1/2})$, where we have used that $C_a = c(1 + \alpha_v^{-1/2})$. ∎

5. Error analysis. With the stability and boundedness results from the preceding section, we can now develop convergence results for the method.

5.1. Estimates in mesh-dependent norms.

THEOREM 9 ($|||\cdot|||_{v,p}$-norm error estimate). *Let $(u, p) \in X$ be the solution of the Stokes problem (1), and $\mathbf{u} = (u, \gamma(u))$ and $\mathbf{p} = (p, \gamma(p))$. Let $(\mathbf{u}_h, \mathbf{p}_h) \in X_h^\star$ solve (12). Then there exists a constant $C_I > 0$, independent of h, such that*

$$|||(\mathbf{u} - \mathbf{u}_h, \mathbf{p} - \mathbf{p}_h)|||_{v,p} \le C_I \inf_{(\mathbf{v}_h, \mathbf{q}_h) \in X_h^\star} |||(\mathbf{u} - \mathbf{v}_h, \mathbf{p} - \mathbf{q}_h)|||_{v',p'}. \tag{103}$$

Proof. This is a direct consequence of stability (Lemma 7), consistency (Lemma 1), and boundedness of B_h (Lemma 8). The theorem follows with $C_I = 1 + \sigma^{-1} C_B$. ∎

For the analysis in the remainder of this section we introduce continuous interpolants. Let $(u, p) \in \left[H^{k+1}(\Omega)\right]^d \times H^{l+1}(\Omega)$. We denote the standard continuous interpolant [3] of $\mathbf{u} = (u, \gamma(u))$ by $\mathcal{I}_h^u \mathbf{u} = (\mathcal{I}_h^u u, \bar{\mathcal{I}}_h^u u)$, hence $\mathcal{I}_h^u u \in V_h \cap C^0(\bar{\Omega})$ and $\bar{\mathcal{I}}_h^u u = \mathcal{I}_h^u u|_{\Gamma^0} \in \bar{V}_h$. Furthermore, let $\mathcal{I}_h^p \mathbf{p} = (\mathcal{I}_h^p p, \bar{\mathcal{I}}_h^p p)$ for the pressure, where $\mathcal{I}_h^p p$ is the continuous Scott–Zhang interpolant [14], and $\bar{\mathcal{I}}_h^p p = \mathcal{I}_h^p p|_{\Gamma^0}$. Then $\mathcal{I}_h^p p \in Q_h \cap C^0(\bar{\Omega})$ and $\bar{\mathcal{I}}_h^p p = \mathcal{I}_h^p p|_{\Gamma^0} \in \bar{Q}_h$. The interpolation estimates read

$$\|u - \mathcal{I}_h^u u\|_{m,K} \le c h_K^{l+1-m} |u|_{l+1,K}, \quad \left\|p - \mathcal{I}_h^p p\right\|_{m,K} \le c h_K^{l+1-m} |p|_{l+1,K}. \tag{104}$$

LEMMA 10 (Convergence rate in the $|||\cdot|||_{v,p}$-norm). *Let $(u, p) \in \left[H^{k+1}(\Omega)\right]^d \times H^{l+1}(\Omega)$ solve the Stokes problem (1), and let $(\mathbf{u}_h, \mathbf{p}_h) \in X_h^\star$ solve the finite element problem (12). Let $\mathbf{u} = (u, \gamma(u))$ and $\mathbf{p} = (p, \gamma(p))$. There exists a constant $C_R > 0$, independent of h, such that*

$$|||(\mathbf{u} - \mathbf{u}_h, \mathbf{p} - \mathbf{p}_h)|||_{v,p} \le C_R \left(h^k \|u\|_{k+1,\Omega} + h^{l+1} \|p\|_{l+1,\Omega} \right). \tag{105}$$

Proof. The proof is similar to that in [15, Lemma 5.5]. Let $\mathcal{I}_h^u \mathbf{u} = (\mathcal{I}_h^u u, \bar{\mathcal{I}}_h^u u)$ and $\mathcal{I}_h^p \mathbf{p} = (\mathcal{I}_h^p p, \bar{\mathcal{I}}_h^p p)$ be the continuous interpolants of the velocity and pressure, respectively. For the $|||\cdot|||_{v',p'}$ norm, we have

$$
\begin{aligned}
(106)\quad |||(\mathbf{u} - \mathcal{I}_h^u \mathbf{u}, \mathbf{p} - \mathcal{I}_h^p \mathbf{p})|||_{v',p'}^2 &= \sum_{K \in \mathcal{T}} \left\| \nabla (u - \mathcal{I}_h^u u) \right\|_{0,K}^2 \\
&+ \sum_{K \in \mathcal{T}} \frac{\alpha_v}{h_K} \left\| (u - \bar{\mathcal{I}}_h^u u) - (u - \mathcal{I}_h^u u) \right\|_{0,\partial K}^2 + \sum_{K \in \mathcal{T}} \left\| p - \mathcal{I}_h^p p \right\|_{0,K}^2 \\
&+ \sum_{K \in \mathcal{T}} \alpha_p h_K \left\| (p - \bar{\mathcal{I}}_h^p p) - (p - \mathcal{I}_h^p p) \right\|_{0,\partial K}^2 + \sum_{K \in \mathcal{T}} \frac{h_K}{\alpha_v} \left\| \frac{\partial u}{\partial n} - \frac{\partial \mathcal{I}_h^u u}{\partial n} \right\|_{0,\partial K}^2 \\
&+ \sum_{K \in \mathcal{T}} h_K \left\| p - \bar{\mathcal{I}}_h^p p \right\|_{0,\partial K}^2 .
\end{aligned}
$$

Using the interpolation estimate (104),

$$
\begin{aligned}
\left\| \nabla (u - \mathcal{I}_h^u u) \right\|_{0,K}^2 &\le c h^{2k} |u|_{k+1,K}^2 \\
\frac{\alpha_v}{h_K} \left\| (u - \bar{\mathcal{I}}_h^u u) - (u - \mathcal{I}_h^u u) \right\|_{0,\partial K}^2 &= 0 \\
\frac{h_K}{\alpha_v} \left\| \frac{\partial u}{\partial n} - \frac{\partial \mathcal{I}_h^u u}{\partial n} \right\|_{0,\partial K}^2 &\le c \alpha_v^{-1} \left(|u - \mathcal{I}_h^u u|_{1,K}^2 + h_K^2 |u - \mathcal{I}_h^u u|_{2,K}^2 \right) \\
&\le c \alpha_v^{-1} h^{2k} |u|_{k+1,K}^2 \\
(107)\qquad \left\| p - \mathcal{I}_h^p p \right\|_{0,K}^2 &\le c h_K^{2(l+1)} |p|_{l+1,K}^2 \\
\alpha_p h_K \left\| (p - \bar{\mathcal{I}}_h^p p) - (p - \mathcal{I}_h^p p) \right\|_{0,\partial K}^2 &= 0 \\
h_K \left\| p - \bar{\mathcal{I}}_h^p p \right\|_{0,\partial K}^2 &= h_K^2 \left\| p - \mathcal{I}_h^p p \right\|_{0,\partial K}^2 \\
&\le c h_K \left(\left\| p - \mathcal{I}_h^p p \right\|_{0,K}^2 + h_K^2 \left| p - \mathcal{I}_h^p p \right|_{1,K}^2 \right) \\
&\le c h_K^{2l+3} |p|_{l,K}^2 .
\end{aligned}
$$

It follows that

$$
(108)\qquad |||(\mathbf{u} - \mathcal{I}_h^u \mathbf{u}, \mathbf{p} - \mathcal{I}_h^p \mathbf{p})|||_{v',p'} \le c (1 + \alpha_v^{-1})^{1/2} \left(h^k |u|_{k+1,\Omega} + h^{l+1} |p|_{l+1,\Omega} \right),
$$

and by application of Theorem 9 the result follows with $C_R = c C_I (1 + \alpha_v^{-1})^{1/2}$. ∎

5.2. Estimates in the L^2 norm. To find an error estimate in the L^2-norm for the velocity, we will rely on the following regularity assumption. If $f \in [L^2(\Omega)]^d$ for (1) and (u, p) solves Stokes problem (1), we have

$$
(109)\qquad \|u\|_{2,\Omega} + \|p\|_{1,\Omega} \le c_r \|f\|_{0,\Omega} .
$$

where c_r is a constant. Satisfaction of this regularity estimate places some restrictions on the shape of the domain Ω.

THEOREM 11 (Velocity error estimate in the L^2-norm). *Let $(u, p) \in X$ solve the Stokes problem (1), and $\mathbf{u} = (u, \gamma(u))$ and $\mathbf{p} = (p, \gamma(p))$, and let $(\mathbf{u}_h, \mathbf{p}_h) \in X_h^\star$ be the solution to (12). Subject to the regularity condition in (109), there exists a constant $C_V > 0$, independent of h, such that*

$$
(110)\qquad \|u - u_h\|_{0,\Omega} \le C_V h |||(\mathbf{u} - \mathbf{u}_h, \mathbf{p} - \mathbf{p}_h)|||_{v',p'} .
$$

Proof. Let $(\zeta, \xi) \in X$ solve the Stokes problem (1) with $f = (u - u_h) \in [L^2(\Omega)]^d$. From the regularity assumption,

$$\|\zeta\|_{2,\Omega} + \|\xi\|_{1,\Omega} \le c_r \|u - u_h\|_{0,\Omega}. \tag{111}$$

From the definition of a_h (11a),

$$\begin{aligned} a_h(\mathbf{u} - \mathbf{u}_h, (\zeta, \gamma(\zeta))) &= \int_\Omega \nabla(u - u_h) : \nabla\zeta \,\mathrm{d}x - \sum_{K\in\mathcal{T}} \int_{\partial K} (\bar{u}_h - u_h) \cdot \frac{\partial \zeta}{\partial n} \,\mathrm{d}s \\ &= -\int_\Omega (u - u_h) \cdot \Delta\zeta \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} (u - \bar{u}_h) \cdot \frac{\partial \zeta}{\partial n} \,\mathrm{d}s \\ &= -\int_\Omega (u - u_h) \cdot \Delta\zeta \,\mathrm{d}x, \end{aligned} \tag{112}$$

where the second equality is due to integration by parts and the third is due to the single valuedness of u, $\bar{u}_h$ and $\nabla\zeta n$ across element boundaries and $u = \bar{u}_h = 0$ on Γ. By definition of b_h in (11b), we also find that

$$\begin{aligned} b_h((\xi, \gamma(\xi)), \mathbf{u} - \mathbf{u}_h) &= -\int_\Omega \xi \nabla \cdot (u - u_h) \,\mathrm{d}x + \sum_{K\in\mathcal{T}} \int_{\partial K} (\bar{u}_h - u_h) \cdot n\xi \,\mathrm{d}s \\ &= \int_\Omega \nabla\xi \cdot (u - u_h) \,\mathrm{d}x - \sum_{K\in\mathcal{T}} \int_{\partial K} (u - \bar{u}_h) \cdot n\xi \,\mathrm{d}s \\ &= \int_\Omega \nabla\xi \cdot (u - u_h) \,\mathrm{d}x, \end{aligned} \tag{113}$$

where the second equality is due to integration by parts and the third is due to the single-valuedness of u, $\bar{u}_h$ and ξ across element boundaries and $u = \bar{u}_h = 0$ on Γ. Combining (112) and (113) and using that $-\Delta\zeta + \nabla\xi = (u - u_h)$,

$$\begin{aligned} \|u - u_h\|^2_{0,\Omega} = \int_\Omega (u - u_h)^2 \,\mathrm{d}x &= \int_\Omega (-\Delta\zeta + \nabla\xi) \cdot (u - u_h) \,\mathrm{d}x \\ &= a_h(\mathbf{u} - \mathbf{u}_h, (\zeta, \gamma(\zeta))) + b_h((\xi, \gamma(\xi)), \mathbf{u} - \mathbf{u}_h). \end{aligned} \tag{114}$$

Furthermore, again by definition of b_h (11b),

$$b_h(\mathbf{p} - \mathbf{p}_h, (\zeta, \zeta)) = -\int_\Omega (p - p_h) \nabla \cdot \zeta \,\mathrm{d}x = 0, \tag{115}$$

due to $\nabla \cdot \zeta = 0$. From the definition of c_h (11c), it is clear that $c_h(\mathbf{p} - \mathbf{p}_h, (\zeta, \zeta)) = 0$. It therefore follows that

$$\|u - u_h\|^2_{0,\Omega} = B_h((\mathbf{u} - \mathbf{u}_h, \mathbf{p} - \mathbf{p}_h), ((\zeta, \gamma(\zeta)), (\xi, \gamma(\xi)))). \tag{116}$$

Using consistency (Lemma 1), boundedness of B_h on $X^\star(h) \times X^\star(h)$[1], we find

$$\begin{aligned} \|u - u_h\|^2_{0,\Omega} &= B_h((\mathbf{u} - \mathbf{u}_h, \mathbf{p} - \mathbf{p}_h), ((\zeta - \mathcal{I}_h^u \zeta, \zeta - \bar{\mathcal{I}}_h^u \zeta), (\xi - \mathcal{I}_h^p \xi, \xi - \bar{\mathcal{I}}_h^p \xi))) \\ &\le C_B |||(\mathbf{u} - \mathbf{u}_h, \mathbf{p} - \mathbf{p}_h)|||_{v',p'} |||((\zeta - \mathcal{I}_h^u \zeta, \zeta - \bar{\mathcal{I}}_h^u \zeta), (\xi - \mathcal{I}_h^p \xi, \xi - \bar{\mathcal{I}}_h^p \xi))|||_{v',p'}. \end{aligned} \tag{117}$$

[1] Lemma 8 provides boundedness of B_h on $X^\star(h) \times X_h^\star$ but the proof for boundedness on $X^\star(h) \times X^\star(h)$ is similar with both norms on the right-hand side of (91) being $|||\cdot|||_{v',p'}$

Using the interpolation estimate (108), we note that

$$\|\!|((\zeta-\mathcal{I}_h^u\zeta,\zeta-\bar{\mathcal{I}}_h^u\zeta),(\xi-\mathcal{I}_h^p\xi,\xi-\bar{\mathcal{I}}_h^p\xi))|\!\|_{v',p'} \leq c(1+\alpha_v^{-1})^{1/2}h\left(|\zeta|_{2,\Omega}+|\xi|_{1,\Omega}\right). \tag{118}$$

It follows from (117) and (118),

$$\|u-u_h\|_{0,\Omega}^2 \leq c(1+\alpha_v^{-1})^{1/2}C_B h|\!|\!|(\mathbf{u}-\mathbf{u}_h,\mathbf{p}-\mathbf{p}_h)|\!|\!|_{v',p'}\left(\|\zeta\|_{2,\Omega}+\|\xi\|_{1,\Omega}\right). \tag{119}$$

Using the regularity estimate (111), we obtain

$$\|u-u_h\|_{0,\Omega}^2 \leq c(1+\alpha_v^{-1})^{1/2}C_B c_r h|\!|\!|(\mathbf{u}-\mathbf{u}_h,\mathbf{p}-\mathbf{p}_h)|\!|\!|_{v',p'}\|u-u_h\|_{0,\Omega}, \tag{120}$$

from which the theorem follows with $C_V = c(1+\alpha_v^{-1})^{1/2}C_B c_r$. ∎

We can now obtain a convergence rate for the velocity error in the L^2-norm.

LEMMA 12 (Convergence rate for the velocity in the L^2-norm). *Let $(u,p) \in \left[H^{k+1}(\Omega)\right]^d \times H^{l+1}(\Omega)$ be the solution of the Stokes problem (1), and $\mathbf{u}=(u,\gamma(u))$ and $\mathbf{p}=(p,\gamma(p))$, and let $(\mathbf{u}_h,\mathbf{p}_h)\in X_h^\star$ solve (12). Subject to the regularity condition (109), there exists a constant $C_C>0$, independent of h, such that*

$$\|u-u_h\|_{0,\Omega} \leq C_C\left(h^{k+1}\|u\|_{k+1,\Omega}+h^{l+2}\|p\|_{l+1,\Omega}\right). \tag{121}$$

Proof. We first show that $|\!|\!|(\mathbf{v}_h,\mathbf{q}_h)|\!|\!|_{v,p}$ and $|\!|\!|(\mathbf{v}_h,\mathbf{q}_h)|\!|\!|_{v',p'}$ are equivalent norms on $X_h^\star$, i.e.,

$$|\!|\!|(\mathbf{v}_h,\mathbf{q}_h)|\!|\!|_{v,p}^2 \leq |\!|\!|(\mathbf{v}_h,\mathbf{q}_h)|\!|\!|_{v',p'}^2 \leq c|\!|\!|(\mathbf{v}_h,\mathbf{q}_h)|\!|\!|_{v,p}^2. \tag{122}$$

The first inequality is trivial. The second inequality follows from

$$\sum_{K\in\mathcal{T}} h_K\|\bar{q}_h\|_{0,\partial K}^2 \leq \sum_{K\in\mathcal{T}} h_K\left(\|\bar{q}_h-q_h\|_{0,\partial K}^2+\|q_h\|_{0,\partial K}^2\right) \leq |\mathbf{q}_h|_p^2 + C_t^2\sum_{K\in\mathcal{T}}\|q_h\|_{0,K}^2 \tag{123}$$

where the last inequality above is due to the discrete trace inequality (27). From the equivalence of $|\!|\!|(\mathbf{v}_h,\mathbf{q}_h)|\!|\!|_{v,p}$ and $|\!|\!|(\mathbf{v}_h,\mathbf{q}_h)|\!|\!|_{v',p'}$ on $X_h^\star$, we find by Lemma 7 that for all $(\mathbf{v}_h,\mathbf{q}_h)\in X_h^\star$

$$c\sigma|\!|\!|(\mathbf{v}_h,\mathbf{q}_h)|\!|\!|_{v',p'} \leq \sup_{(\mathbf{w}_h,\mathbf{r}_h)\in X_h^\star}\frac{B_h((\mathbf{v}_h,\mathbf{q}_h),(\mathbf{w}_h,\mathbf{r}_h))}{|\!|\!|(\mathbf{w}_h,\mathbf{r}_h)|\!|\!|_{v',p'}}. \tag{124}$$

By Lemma 8 we have boundedness of B_h on $X^\star(h)\times X_h^\star$ with respect to the $|\!|\!|(\cdot,\cdot)|\!|\!|_{v',p'}$ and $|\!|\!|(\cdot,\cdot)|\!|\!|_{v,p}$ norms. The bilinear form B_h, however, is also bounded on $X^\star(h)\times X^\star(h)$, but with respect to only the $|\!|\!|(\cdot,\cdot)|\!|\!|_{v',p'}$ norm:

$$B_h((\mathbf{v},\mathbf{q}),(\mathbf{w},\mathbf{r})) \leq C_B|\!|\!|(\mathbf{v},\mathbf{q})|\!|\!|_{v',p'}|\!|\!|(\mathbf{w},\mathbf{r})|\!|\!|_{v',p'}. \tag{125}$$

Using (124) and consistency (Lemma 1), we find that

$$\begin{aligned} c\sigma|\!|\!|(\mathbf{u}_h-\mathbf{v}_h,\mathbf{p}_h-\mathbf{q}_h)|\!|\!|_{v',p'} &\leq \sup_{(\mathbf{w}_h,\mathbf{r}_h)\in X_h^\star}\frac{B_h((\mathbf{u}_h-\mathbf{v}_h,\mathbf{p}_h-\mathbf{q}_h),(\mathbf{w}_h,\mathbf{r}_h))}{|\!|\!|(\mathbf{w}_h,\mathbf{r}_h)|\!|\!|_{v',p'}} \\ &= \sup_{(\mathbf{w}_h,\mathbf{r}_h)\in X_h^\star}\frac{B_h((\mathbf{u}-\mathbf{v}_h,\mathbf{p}-\mathbf{q}_h),(\mathbf{w}_h,\mathbf{r}_h))}{|\!|\!|(\mathbf{w}_h,\mathbf{r}_h)|\!|\!|_{v',p'}}. \end{aligned} \tag{126}$$

Boundedness of B_h (125) results in

$$\frac{c\sigma}{C_B}|||(\mathbf{u}_h-\mathbf{v}_h,\mathbf{p}_h-\mathbf{q}_h)|||_{v',p'} \le |||(\mathbf{u}-\mathbf{v}_h,\mathbf{p}-\mathbf{q}_h)|||_{v',p'}, \tag{127}$$

and by the triangle inequality (similar to Theorem 9), we find

$$|||(\mathbf{u}-\mathbf{u}_h,\mathbf{p}-\mathbf{p}_h)|||_{v',p'} \le \left(1+\frac{cC_B}{\sigma}\right)\inf_{(\mathbf{v}_h,\mathbf{q}_h)\in X_h^\star}|||(\mathbf{u}-\mathbf{v}_h,\mathbf{p}-\mathbf{q}_h)|||_{v',p'}. \tag{128}$$

By Theorem 11 and (128) we therefore find

$$\begin{aligned}\|u-u_h\|_{0,\Omega} &\le C_V h|||(\mathbf{u}-\mathbf{u}_h,\mathbf{p}-\mathbf{p}_h)|||_{v',p'}\\ &\le C_V\left(1+\frac{cC_B}{\sigma}\right)h\inf_{(\mathbf{v}_h,\mathbf{q}_h)\in X_h^\star}|||(\mathbf{u}-\mathbf{v}_h,\mathbf{p}-\mathbf{q}_h)|||_{v',p'}.\end{aligned} \tag{129}$$

In particular

$$\|u-u_h\|_{0,\Omega} \le C_V\left(1+\frac{cC_B}{\sigma}\right)h|||(\mathbf{u}-\mathcal{I}_h^u\mathbf{u},\mathbf{p}-\mathcal{I}_h^p\mathbf{p})|||_{v',p'}. \tag{130}$$

Using the interpolation estimate (108), we obtain

$$\|u-u_h\|_{0,\Omega} \le c(1+\alpha_v^{-1})^{1/2}C_V\left(1+\frac{cC_B}{\sigma}\right)\left(h^{k+1}|u|_{k+1,\Omega}+h^{l+2}|p|_{l+1,\Omega}\right), \tag{131}$$

and the result follows with $C_C = c(1+\alpha_v^{-1})^{1/2}C_V(1+cC_B\sigma^{-1})$. ∎

We end this section with the convergence rate for the pressure in the L^2-norm.

Lemma 13 (Convergence rate for the pressure in the L^2-norm). *Let $(u,p)\in\left[H^{k+1}(\Omega)\right]^d\times H^{l+1}(\Omega)$ solve the Stokes problem (1), and $\mathbf{u}=(u,\gamma(u))$ and $\mathbf{p}=(p,\gamma(p))$, and let $(\mathbf{u}_h,\mathbf{p}_h)\in X_h^\star$ solve (12). Let C_R be the constant defined in Lemma 10. Subject to the regularity condition in (109), the following inequality holds,*

$$\|p-p_h\|_{0,\Omega} \le C_R\left(h^k\|u\|_{k+1,\Omega}+h^{l+1}\|p\|_{l+1,\Omega}\right). \tag{132}$$

Proof. Since the L^2-norm of the pressure is part of the norm $|||(\cdot,\cdot)|||_{v,p}$ we note that

$$\|p-p_h\|_{0,\Omega} \le |||(\mathbf{u}-\mathbf{u}_h,\mathbf{p}-\mathbf{p}_h)|||_{v,p} \le C_R\left(h^k\|u\|_{k+1,\Omega}+h^{l+1}\|p\|_{l+1,\Omega}\right), \tag{133}$$

where the last inequality is due to Lemma 10. ∎

Lemmas 12 and 13 show that if $[P_k]^d$–P_k or $[P_k]^d$–P_{k-1} elements are used for the velocity-pressure approximation, then

$$\|u-u_h\|_{0,\Omega}\le ch^{k+1} \quad\text{and}\quad \|p-p_h\|_{0,\Omega}\le ch^k. \tag{134}$$

For the $[P_k]^d$–P_k element we therefore find a sub-optimal error estimate, while an optimal estimate is obtained for the $[P_k]^d$–P_{k-1} element.

The *a priori* error estimates are consistent with the experimentally observed convergence rates in [12] for the case of C^0-conforming facet functions.

6. Conclusions. We have analysed a hydridized DG/interface stabilized method for the Stokes equations, and proven inf-sup stability and optimal convergence rates. The developed convergence estimates are consistent with the experimentally observed convergence rates presented in earlier publications. The method is particularly appealing as it can be constructed to have the same number of global degrees of freedom and the same global matrix operator structure as a conforming formulation, yet it is locally conservative. Moreover, on simplices the local velocity field can be point-wise divergence-free. These properties make the method an excellent candidate for coupling to transport equations. When extended to the incompressible Navier–Stokes equations, the structure of the method makes the incorporation of standard DG-type stabilization of the advective terms straightforward.

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