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Stabilizing quorum-sensing networks via noise

Marianna Rotoli*, Giovanni Russo†, Mario di Bernardo*
*University of Naples Federico II, Department of Electrical Engineering and ICT
†IBM Research Ireland, Control and Optimization Group
Email: grusso@ie.ibm.com

***Abstract*—We investigate how a single node injecting noise in a quorum-sensing network can dramatically affect its convergence towards synchronization. We show that adding some white noise of sufficiently high intensity on a single node can stabilize the collective behavior of all nodes in the network towards the origin. A sketch of the proof of the convergence result is given, together with the outcome of some numerical experiments on the challenging problem of stabilizing a network of bistable systems onto a common unstable equilibrium point.**

I. Introduction

Over the past few years, much research effort has been devoted to the study of sychronization and synchronizability in complex networks, see e.g., [1], [2], [3] and references therein. This has been primarily motivated by the importance of synchronization in many application fields, with biochemical systems [4] and power networks [5] being two notable, practical, examples. Often, when studying synchronization phenomena, it is assumed that: (i) nodes communicate via exchanging information on a dedicated link connecting themselves and not others; (ii) the network is free from noise and uncertainties. Unfortunately, in some applications, these assumptions are not satisfied. For example, in biology, it is often the case that agents in the network (e.g., bacteria) communicate via a *quorum-sensing* mechanism [6], [7], i.e., via secreting an inducer molecule that diffuses through the population. Quorum-sensing networks are characterized by the fact that nodes communicate via a shared, environmental, quantity. Also, in e.g., biochemical applications, nodes are often subject to some noise [1], [8], which is due to model and/or communication uncertainties. Motivated by this, we study synchronization in quorum-sensing networks affected by noise. In particular, we show, via a stochastic Lyapunov stability argument, that the injection of noise into a single network node can drive all the agents towards a common asymptotic solution, where the state variables of all of its nodes become equal to 0. A numerical investigation complements the theoretical results.

A. Related Work

Research effort on quorum-sensing network is somehow limited, see e.g., [9], [10] for a Literature review. This is surprising as such networks, besides their pervasiveness in Nature, could also be used in designing engineered network systems with the goal of minimizing the number of node-to-node links, while achieving some desired connectivity level [11]. Recently, a large body of Literature is emerging on the study of synchronization effects in the presence of noise, see e.g., [12], [1]. In this context, an interesting phenomenon is the so-called common-noise-induced synchronization, see e.g., [13], [14], where some *environmental* noise synchronizes a network of interest. Recent results on this synchronization phenomenon include [15], which is specific to the Pikovsky-Rabinovich and Hindmarsh-Rose models, and [15], which is based on the use of Lyapunov exponents.

II. Mathematical Tools

A. Notation

We denote by: (i) I_n the $n \times n$ identity matrix; (ii) $1_{n\times m}$ the $n \times m$ matrix with unitary elements; (iii) $\|\cdot\|_F$ $(|\cdot|)$ the vector/matrix Frobenius (Euclidean) norm; (iv) $tr\left\{A\right\}$ the trace of the square matrix A; (v) $\otimes$ the Kronecker product; (vi) $\mathcal{C}$ the space of continuous functions and $\mathcal{C}^2$ the space of continuously differentiable functions (for definitions see e.g., [16]).

B. Stochastic Differential Equations

Consider an n-dimensional stochastic differential equation of the form

$$dx = f(t,x)dt + g(t,x)dB, \tag{1}$$

where: (i) $x \in \mathbb{R}^n$ is the state variable; (ii) $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ belongs to $\mathcal{C}^2$; (iii) $g : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ belongs to $\mathcal{C}$; (iv) $B = [B_1, \dots, B_d]^T$ is a d-dimensional Brownian motion. We assume that: (i) a solution of (1) exists for any initial condition $x_0 := x(0)$ [17]; (ii) $f(t,0) = g(t,0) = 0$, i.e., the *trivial solution*, $x(t) = 0$, is a solution of (1). We make use of the following definition from e.g., [18], [19], [16]

Definition 1. *The trivial solution of (1) is said to be almost surely exponentially stable if for all* $x \in \mathbb{R}^n$, $\lim_{t\to+\infty} \sup \frac{1}{t} \log\left(|x(t)|\right) < 0$, $a.s.$ *(almost surely).*

Consider the function $V(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, $V(t,x) \in \mathcal{C}^{1\times 2}$, i.e., $V(t,x)$ is twice differentiable with respect to x and differentiable with respect to t. As in [16], we let: (i) $LV(t,x) = V_t(t,x) + V_x(t,x)f(t,x) + \frac{1}{2}tr\left\{g(t,x)^T V_{xx}(t,x)g(t,x)\right\}$; (ii) $V_x = [V_{x_1}, \dots, V_{x_n}]$; (iii) V_{xx} is the $n \times n$ dimensional matrix having as ij-th element $V_{x_i x_j}$ (where $V_{x_i} := \partial V(t,x)/\partial x_i$ and $V_{x_i x_j} := \partial^2 V(t,x)/\partial x_j \partial x_i$). The following result from [16] will be used to devise our main result.

Theorem 1. *Assume that there exists a non-negative function* $V(t,x) \in \mathcal{C}^{1\times 2}$ *and constants* $p > 0$, $c_1 > 0$, $c_2 \in \mathbb{R}$, $c_3 \geq 0$, *such that* $\forall x \neq 0$ *and* $\forall t \in \mathbb{R}^+$*:* **(H1)** $c_1 |x|^p \leq V(t,x)$*;* **(H2)** $LV(t,x) \leq c_2 V(t,x)$*;* **(H3)** $|V_x(t,x)g(t,x)|^2 \geq c_3 V(t,x)^2$*. Then:* $\lim_{t\to+\infty} \sup \frac{1}{t}\log(|x(t)|) \leq -\frac{c_3 - 2c_2}{2p}$*, a.s. In particular, if* $c_3 > 2c_2$*, then the trivial solution of (1) is almost surely exponentially stable.*

III. Problem Set-up

We consider networks of $N > 1$ nodes coupled via a quorum-sensing mechanism. For the sake of brevity, we consider the case where the dynamics of the shared variable is sufficiently faster than the dynamics of the nodes in the network. Then, as remarked in [7], [10], the quorum-sensing network dynamics is given by

$$\dot{x}_i = f(t, x_i) - \bar{\sigma} x_i + \frac{\sigma_i(t)}{N}\sum_{j=1}^{N} x_j, \tag{2}$$

$x_i(0) = x_{i,0}$, $i = 1, \ldots, N$, $\bar{\sigma} > 0$ and where: (i) $x_i \in \mathbb{R}^n$, $f(t, x_i) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are the state variable and the intrinsic dynamics for the i-th network node: (ii) $\sigma_i(t) : \mathbb{R} \to \mathbb{R}$. Note that each node perceives the *aggregate* states of all the network nodes (via the term $\sum_{j=1}^N x_j$). We assume here that $f_i(t, 0) = 0$, $\forall i = 1, \ldots, N$ and we consider the case where the coupling strength for one network node, say node 1, is equal to $\sigma_1(t) = \bar{\sigma} + \sigma^* w(t)$, with $w(t)$ being 1-dimensonal white noise and σ^* being a constant that models its amplitude. For the sake of brevity, in this note we only consider the case where $\sigma_i(t) = \bar{\sigma}$, $\forall i = 2, \ldots, N$. However, our results can be generalized to a setting with heterogeneous coupling strengths, heterogeneous nodes and with the quorum-variable being characterized by its own dynamics as will be discussed elsewhere. Dynamics similar to (2) arise in opinion formation networks with uncertain parameters and in networks with noisy communication channels, [20], [21]. The dynamics (2) can be recast as the following stochastic differential equation

$$\begin{aligned} dx_1 &= \left[f(t, x_1) + \bar{\sigma}\left(\tfrac{1}{N}\textstyle\sum_{j=1}^N x_j - x_1\right)\right]dt + \tfrac{\sigma^*}{N}\textstyle\sum_{j=1}^N x_j db, \\ dx_i &= \left[f(t, x_i) + \bar{\sigma}\left(\tfrac{1}{N}\textstyle\sum_{j=1}^N x_j - x_i\right)\right]dt, \end{aligned} \tag{3}$$

$i = 2, \ldots, N$, where $b(t) \in \mathbb{R}$ is the standard Brownian process. We are interested in characterizing the behavior of (3) in terms of the behavior of its *noise-free* version ($\sigma^* = 0$):

$$dy_i = \left[f(t, y_i) + \bar{\sigma}\left(\frac{1}{N}\textstyle\sum_{j=1}^N y_j - y_i\right)\right]dt, \tag{4}$$

$i = 1, \ldots, N$ and $y_i(0) = x_i(0)$. In particular, our goal is to study how a single node injecting noise in the network can affect synchronization of (3). In order to do so, we first formalize the notion of synchronization:

Definition 2. *Let* $\bar{x}(t) = \frac{1}{N}\sum_{j=1}^N x_j(t)$ *and* $\bar{y}(t) = \frac{1}{N}\sum_{j=1}^N y_j(t)$*. We say that:*

- *the noise free network achieves synchronization if (i)* $s_n(t) := 1_N \otimes \bar{y}(t)$ *is a solution of (4); (ii)* $\lim_{t\to+\infty} \sup \frac{1}{t}\log(|y_i(t) - \bar{y}(t)|) < 0$, $\forall i = 1, \ldots, N$*;*
- *the noisy network (3) achieves stochastic synchronization if: (i)* $s(t) := 1_N \otimes \bar{x}(t)$ *is a solution of (3); (ii)* $\lim_{t\to+\infty} \sup \frac{1}{t}\log(|x_i(t) - \bar{x}(t)|) < 0$ *a.s.,* $\forall i = 1, \ldots, N$.

With our main result presented below we show that the noise injected by a single node can drive a noise-free synchronized network to a state where all of its nodes are equal to 0. That is, it happens that $\lim_{t\to+\infty} \sup \frac{1}{t}\log(|x(t)|) < 0$ a.s., i.e., $\bar{x}(t) = 0$ in Definition 2.

IV. Main Result

We first introduce the following Lemma.

Lemma 1. *Consider network (4) and assume that: (i)* $\forall x, y \in \mathbb{R}^n$ *and* $\forall t \in \mathbb{R}_+$*, there exists some* K_f *such that* $(x - y)^T [f(t,x) - f(t,y)] \leq K_f (x-y)^T (x-y)$*; (ii)* $s_n(t)$ *is a solution of the network. Then, the noise-free network achieves synchronization if* $\bar{\sigma} > K_f$*.*

Proof. The result can be proved by: (i) defining the error between the state of the network and the synchronous solution; (ii) proving, via the Lyapunov function $V(e) = \frac{1}{2}e^T e$, that $e = 0$ is exponentially stable for the error dynamics. The complete proof is omitted here for the sake of brevity. □

We are now ready to introduce our main result. For the sake of brevity, in this brief we report the main technical steps of the proof. The complete proof, which requires formalizing the notion of stability in terms of two measures [22], will be presented elsewhere.

Theorem 2. *Consider network (3) and assume that:*

- **H1** $\forall x, y \in \mathbb{R}^n$ *and* $\forall t \in \mathbb{R}_+$*, there exists some* K_f *such that* $(x-y)^T [f(t,x) - f(t,y)] \leq K_f (x-y)^T (x-y)$*;*
- **H2** $s(t) := 1_N \otimes \bar{x}(t)$ *is a solution of (3);*
- **H3** $\bar{\sigma} > K_f$*.*

Then, $\lim_{t\to+\infty} \sup \frac{1}{t}\log(|x_i(t)|) < 0$ *a.s.,* $\forall i = 1, \ldots, N$ *if:*

$$\left(\frac{\sigma^*}{\sqrt{2}N}\right)^2 > K_f. \tag{5}$$

Sketch of the proof. In order to prove the result, we first note that the dynamics for $\bar{x}$ is given by

$$d\bar{x} = \frac{1}{N}\sum_{j=1}^N f(t, x_j)dt + \frac{\sigma^*}{N}\bar{x}db. \tag{6}$$

Let $X := [x_1^T, \ldots, x_N^T]^T$. From **H2**, $s(t)$ is a solution of the network and hence we define $e := X - s = [(x_1 - \bar{x})^T, \ldots, (x_N - \bar{x})^T]^T$ as the error between the state of network (3) and the synchronous solution $\bar{x}(t)$. We then have $de = dX - ds$ and hence, from (3) and (6) we get:

$$\begin{aligned} de_1 &= \left[f(t, e_1 + \bar{x}) - \tfrac{1}{N}\textstyle\sum_{j=1}^N f(t, e_j + \bar{x}) - \bar{\sigma}e_1\right]dt + \tfrac{N-1}{N}\sigma^*\bar{x}db \\ de_i &= \left[f(t, e_i + \bar{x}) - \tfrac{1}{N}\textstyle\sum_{j=1}^N f(t, e_j + \bar{x}) - \bar{\sigma}e_i\right]dt - \tfrac{\sigma^*}{N}\bar{x}db. \end{aligned} \tag{7}$$

Now, we define the vector $\tilde{e} := [e^T, \bar{x}^T]^T$. Then, by means of (7) and (6) we have

$$d\tilde{e} = \tilde{F}(t,\tilde{e})dt + \frac{\sigma^*}{N}\tilde{G}\tilde{e}db, \qquad (8)$$

where:

- $\tilde{G} := G \otimes I_n$, where G is the $(N+1)\times(N+1)$ matrix
$$G = \begin{bmatrix} 0 & \dots & 0 & N-1 \\ 0 & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix};$$
- $\tilde{F}(t,\tilde{e}) := [F^{(e)}(t,\tilde{e})^T, \bar{F}(t,\tilde{e})^T]^T$, with $F^{(e)}(t,\tilde{e}) = [f^{(e_1)^T}(t,\tilde{e}), \dots, f^{(e_N)^T}(t,\tilde{e})]^T$, where $f^{(e_i)}(t,\tilde{e}) := f(t, e_i + \bar{x}) - \frac{1}{N}\sum_{j=1}^N f(t, e_j + \bar{x}) - \bar{\sigma} e_i$ and $\bar{F}(t,\tilde{e}) := \frac{1}{N}\sum_{j=1}^N f(t, e_j + \bar{x})$.

Note that the trivial solution $\tilde{e}(t) = 0$ is a solution of (8). In fact: (i) $\tilde{G}\tilde{e}$ is clearly equal to 0 if $\tilde{e} = 0$; (ii) $\tilde{F}(t,0)$ is also equal to 0, $\forall t \geq 0$ from the hypotheses. We now apply Theorem 1 to study stability of the trivial solution $\tilde{e}(t) = 0$ of (8). We also remark here that stability of $\tilde{e}(t) = 0$ corresponds to the state of the network where $\bar{x}(t) = x_1(t) = \dots = x_N(t) = 0$. We prove our statement by first proving stability of (8) with respect to the components $[e_1^T, \dots, e_N^T]^T$. As shown in [22] this can be done by considering the stochastic Lyapunov function $V(t,\tilde{e}) = V(\tilde{e}) = V^{(e)}(e) + V^{(\bar{x})}(\bar{x}) := \frac{1}{2}\tilde{e}^T\tilde{e} = \frac{1}{2}e^Te + \frac{1}{2}\bar{x}^T\bar{x}$. In particular, the key idea behind the proof consists in showing that: (i) independently on $\bar{x}(t)$, there exists some $c_2^{(e)} < 0$ such that $LV^{(e)}(e) \leq c_2^{(e)} V^{(e)}(e)$. Following Theorem 1, this implies that $\lim_{t\to+\infty}\sup \frac{1}{t}\log|e(t)| < 0$, a.s., thus implying that, $\forall i = 1, \dots, N$, $|e_i(t)| \to 0$ a.s. as $t \to +\infty$; (ii) for the dynamics (6) when $|e| = 0$ the solution $\bar{x}(t) = 0$ is almost surely exponentially stable. This is done again by means of Theorem 1, this time by using $V^{(\bar{x})}(\bar{x})$ as stochastic Lyapunov function.

It can be shown that $LV^{(e)}(e) = V_e^{(e)} F^{(e)}(t,\tilde{e}) = e^T F^{(e)}(t,\tilde{e})$. Now, let $F_{avg}(t,\bar{x}) := [f(t,\bar{x})^T, \dots, f(t,\bar{x})^T]^T$, we have $e^T F^{(e)}(t,\tilde{e}) = e^T F^{(e)}(t,\tilde{e}) - e^T F_{avg}(t,\bar{x}) + e^T F_{avg}(t,\bar{x})$ and, since $e^T \cdot \left(1_N \otimes \left(f(t,\bar{x}) - \frac{1}{N}\sum_{j=1}^N f(t, e_j + \bar{x})\right)\right) = 0$, we get $e^T F^{(e)}(t,\tilde{e}) = e^T[f^T(t, e_1 + \bar{x}) - f^T(t,\bar{x}) - \bar{\sigma} e_1^T, \dots, f^T(t, e_N + \bar{x}) - f^T(t,\bar{x}) - \bar{\sigma} e_N^T]^T$. Hence, from **H1** we get $e^T F^{(e)}(t,\tilde{e}) \leq (K_f - \bar{\sigma})\, e^T e$, which is indeed independent on $\bar{x}(t)$. Hence, since by hypothesis **H3** $K_f - \bar{\sigma} < 0$, then $|e_i(t)| \to 0$, a.s. as $t \to +\infty$, for all $i = 1, \dots, N$ and independently on $\bar{x}(t)$. Now, we prove that, when $|e(t)| = 0$, the solution $\bar{x}(t) = 0$ is almost surely exponentially stable. To this aim, note that, from (6), when $|e(t)| = 0$, we get:

- $LV^{(\bar{x})}(\bar{x}) = \bar{x}^T\left(\frac{1}{N}\sum_{j=1}^N f(t,\bar{x})\right) + \frac{1}{2}\left(\frac{\sigma^*}{N}\right)^2 \bar{x}^T\bar{x}$. This, by means of **H1**, leads to $LV^{(\bar{x})}(\bar{x}) \leq \left(2K_f + \left(\frac{\sigma^*}{N}\right)^2\right) V^{(\bar{x})}(\bar{x})$;
- $\left|V^{(\bar{x})} \frac{\sigma^*}{N}\bar{x}\right|^2 = 4\left(\frac{\sigma^*}{N}\right)^2 V^{(\bar{x})^2}(\bar{x})$.

Thus, $\bar{x}(t) = 0$ is almost surely exponentially stable if $4\left(\frac{\sigma^*}{N}\right)^2 > 4K_f + 2\left(\frac{\sigma^*}{N}\right)^2$, which is true by means of (5), completing the proof. □

V. Discussion

Theorem 2 implies that the presence of a single noisy node in a quorum sensing network can have dramatic effects on synchronization. In our result, nodes can be non-linear given that they fulfil **H1**, which is better known as the vector fields being QUAD in the Literature [23]. We note that, if Lemma 1 is fulfilled, then the network synchronizes onto the synchronous solution $\bar{y}(t)$ which, in general, can be different from 0. Now, if one network node becomes noisy as in (3), with the noise intensity (σ^*) being sufficiently high, then Theorem 2 implies that the states of all network nodes are driven by noise to approach 0, which is the only solution $s(t)$ common to all nodes in the noisy case. This is particularly important for certain complex networks from both Nature and Technology. For example, in certain applications arising from power networks [24] one would like to synchronize all the network nodes onto a desired periodic orbit and deviations from such a synchronous solution would result in electrical losses. In this context, following Theorem 2, one may want to mitigate the diffusion of noise through the network in order to avoid deviations from the desired synchronous solution. In other applications, instead, one may want to desynchronize a network of interest [25], [26]. To this aim, Theorem 2 could be used to *properly* inject noise in order to somehow *reset* the network to a state where all nodes are stabilized at the origin. This is the case, for example, of important applications such as neural networks, where pathological synchronization among bursting neurons might be related to the tremors observed in patients affected by the Parkinson's disease [27]. Interestingly, the key idea behind Deep Brain Stimulation techniques is indeed that of perturbing the synchronization of neurons via noise, see e.g., [28]. We also remark here that Theorem 2 offers an insight on how the size of the network has an impact on effects of noise diffusion. In fact, from (5) it is straightforward to see that the higher the number of nodes ism the higher the noise intensity σ^* needs to be in order to drive the state of all the network nodes to 0. In this sense, nodes are cooperating to protect the network from noise and this is similar in spirit to a recent result presented in [11]. Finally, we discuss here some possible generalizations of the result presented in this paper, which will be presented elsewhere: (i) the quorum sensing dynamics studied in this paper has been obtained under the assumption that the dynamics of the shared variable is sufficiently fast. This assumption is not realistic in certain applications. Our result, however, can be extended to cover the case where the shared variable has its own dynamics; (ii) the main result can be extended to consider more general network topologies. Using techniques similar to those presented in this work, it is possible to give a result analogous to Theorem 2 for complex directed networks with heterogeneous coupling strengths; (iii) Theorem 2 can be also extended to consider heterogeneous systems. In particular, let $f_i(t, x_i)$ be the intrinsic dynamics of the i-th node and assume that all the functions $f_i(\cdot,\cdot)$ fulfill condition **H1** of Theorem 2 with $f_i(t, 0) = 0$, $\forall i$ and $\forall t > 0$. Then, a result similar to Theorem 2 can be proved.

VI. Numerical Validation

We now provide a numerical validation for Theorem 2 by considering collective decision processes [29], [30]. In particular, we consider networks of the form (3) where each node/individual needs to decide between two mutually excluding alternatives. In this case, the intrinsic dynamics of the i-th node can be modeled as $\dot{x}_i = f(x_i) = rx_i - x_i^3$ where, as in [31], [32], $x_i \in \mathbb{R}$ is the attitude of the individual towards one of the two alternatives, which are represented by the two stable fixed points of the function $f(\cdot)$, i.e., $\pm r$. Note, also, that $f(0) = 0$ and, in particular, the fixed point 0 is an unstable fixed point, representing a neutral (or absence of) opinion of the node. As shown in [33], [20] the intrinsic dynamics fulfills **H1** of Theorem 2 with $K_f = r$. Hence, following Lemma 1, the network is synchronized if $\bar{\sigma} > r$. In Figure 1 the network behavior is shown when $N = 5$, $r = 1$ and $\bar{\sigma} = 2$. All the simulations presented here have been obtained via the Euler-Maruyama method [34] and the initial conditions were taken from the normal distribution. In Figure 2 the behavior of the same network is instead shown when $\sigma^* = 10$ (note that, with this value for σ^*, all the hypotheses of Theorem 2 are fulfilled). In such a figure, it is clearly shown that the injection of noise causes all the network nodes to converge to 0, which is an unstable equilibrium point for each of the nodes when they are uncoupled. That is, the network transitions from the synchronous state of Figure 1 to a state where all the nodes have neutral/no opinion. Finally, in order to provide a further numerical characterization of Theorem 2 we plotted, in Figure 3 and Figure 4, the regime value of $|\bar{x}(t)|$, say x_f, as a function of the number of nodes, N, and of the noise intensity, σ^*, respectively. In such figures, $x_f := |\bar{x}(t)| = 0$ means that all the nodes converge towards 0. That is, noise propagation drives all the network trajectories to 0.

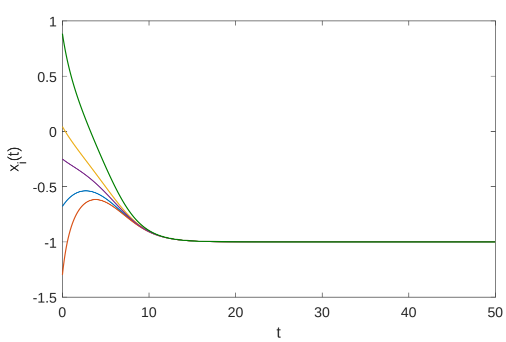

Fig. 1. Synchronization of the quorum sensing network when there is no noise, i.e. $\sigma^* = 0$. The nodes attain a common decision onto $\bar{y}(t) = 1$.

VII. Conclusions

We investigated how a single node injecting noise in a quorum-sensing network can drive a synchronized network towards a state of inactivity, where the states of all its nodes become equal to 0. In doing so, we presented a sufficient condition for the stability of the trivial solution of the network, which is based on the use of a stochastic Lyapunov argument. This allowed us to consider networks with nonlinear nodes and with the noise diffusion depending on the nodes' state. After discussing some implications of our result, we showed the effectiveness of the sufficient condition by considering a network of bistable nodes arising in the context of collective decision dynamics.

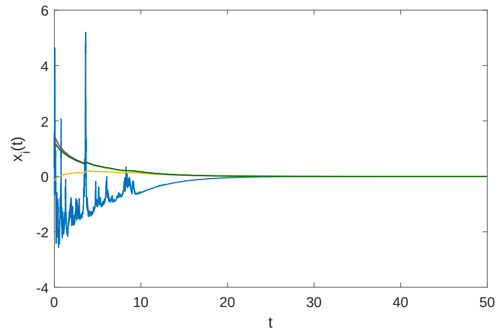

Fig. 2. Time behavior of the quorum sensing network when $\sigma^* = 10$. In this case, all the nodes converge to 0.

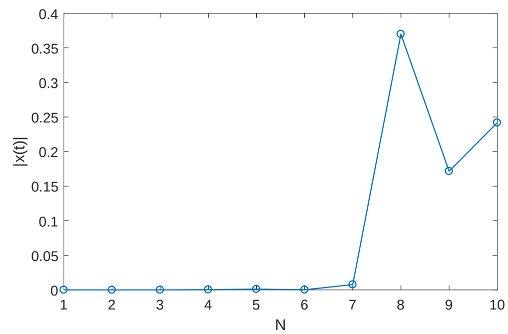

Fig. 3. The steady-state value of $|\bar{x}(t)|$, i.e. x_f, as a function of N. In the simulations, $\sigma^* = 10$.

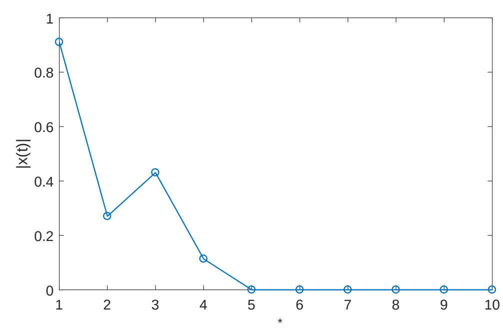

Fig. 4. The steady-state value of $|\bar{x}(t)|$, i.e. x_f, as a function of σ^*. In the simulations, $N = 5$. In accordance to the theoretical predictions of Theorem 2 values of σ^* larger than ≈ 7 drive the state variables of all the nodes to 0.

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