MULTI TIME PERIOD STOCHASTIC PROGRAMMING FOR MEDIUM TERM PRODUCTION PLANNING

by

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SUMMARY

Exact solutions to stochastic, capacitated, multi-commodity, multi-stage production/inventory models are in general computationally intractable. The practical application of such models is therefore inhibited. In this thesis a general stochastic, capacitated, multi-commodity, multi-stage production/inventory model with linear cost structure is proposed. Under convexity conditions it is a stochastic linear program. A good computationally efficient approximate solution technique is developed and some numerical results reported.

It is important to assess the merit of approximate techniques and this is done statistically by replicative simulation. But the accuracy of this method improves only as the square root of the number of simulation trials made, so it is important to eliminate any unnecessary variability in each trial. It is proposed that this be done by the use of control statistics. Several novel control statistics are developed, the most powerful being a martingale control statistic constructed independently for each trial from information provided by the approximate technique being tested.

Results are reported of testing the approximate solution technique developed for the general model, ordinary linear programming ignoring all the stochastic elements in the problem, and two other approximate techniques, by replicative simulation. These suggest that the penalty incurred by ignoring the stochastic nature of the problem is significant, but that first order deviations from optimal decisions may lead only to second order penalties. This is a desirable feature of the stochastic models, for it indicates that approximate solution techniques to stochastic programs may be more reliable than would be supposed from the approximations made.

PART 1

CHAPTER 1

INTRODUCTION.

1. MEDIUM TERM PRODUCTION PLANNING

This thesis develops and studies a dynamic stochastic model for use in medium term production and inventory planning. In this context 'medium term' planning is intended to mean decisions about such aspects of the production system as production levels of individual finished products and manpower levels for different categories of employee. Short term scheduling problems which involve a detailed analysis of the day to day running of the production system and which examine, for example, which components can be produced in which order on what machine, are very specific to the industry and plant being studied and are excluded. Also excluded are long term strategic problems which, for example, arise in decisions to expand or contract production facilities, to produce a new product line or to enter new markets.

The problems addressed herein are essentially of a tactical nature and typically concern the setting of monthly or quarterly production targets, workforce levels and buffer stocks over a planning horizon of a year. Some authors refer to this as production smoothing. There are two principal aspects of this problem that require further discussion.

Firstly, there is a trade off between holding large quantities of products in stock and frequent changes of production and manpower levels. Fluctuating demand might be handled by continually varying the production rate and hiring or laying off sections of the workforce, but keeping stocks of finished items low. However these changes are

often expensive so it might be more profitable to keep the production rates and manpower levels constant whilst meeting fluctuations in demand from high stock levels. In general the best decision will lie between these two extremes. Determination of precisely what the best decisions are involves quantification of the costs involved and study of the appropriate mathematical model of the system.

Secondly, the demand requirements themselves are rarely known exactly in the medium term for they depend on future decisions made by customers who are outside the control of the production system. These demand requirements may only be known probabilistically. There is an obvious trade off between producing only as much as can definitely be sold, which keeps stock levels low but takes little advantage of the likely demand, and producing so much that demand can always be satisfied which risks carrying inordinately large stocks. Determination of the best production tactics in the face of this problem involves the decision maker's attitude toward risk, quantification of the uncertainty in demand and the study of the appropriate probabilistic mathematical model.

The problem is usually further complicated by constraints on permissible production rates and items that can be held in stock. These may require the production of items to stock in order to take most advantage of the peak in cyclical or seasonal demand.

Of the two aspects discussed above the former is easier both from the point of view of acquiring sufficient cost data and in the necessary analysis of the mathematical models. The latter problem is

much more difficult from both points of view. The quantification of the uncertainty in future demand is a difficult task and stochastic models present formidable problems both in their theoretical and computational aspects. However, it is in a sense more general for models designed to handle the latter problem can easily be extended to handle the former problem but not vice-versa. The model developed in this thesis although motivated by the uncertainty problem is designed to handle both. It is presented in Chapter 5. In order to explain the structure of the work some problems associated with stochastic models need to be discussed. This is now done.

2. STOCHASTIC PROGRAMMING

Models used for the analysis of the problems outlined above fall naturally into the ambit of stochastic programming. This is the study of certain models (stochastic programs) which explicitly incorporate random variables into their formulation and which reduce to deterministic mathematical programmes as the variability in the random variables tends to zero. The formulation of such models has not only been motivated by production planning problems but also by the need to control water resources and to tackle problems arising from economic and financial planning. Each source of "real world" problems has generated different classes of stochastic programs. But there is much common ground between them and theoretical study has led to their being classified on the basis of their more abstract properties. In consequence most classes of stochastic programmes have something to offer in the modelling of production systems. A brief review of stochastic programming from this viewpoint is therefore given in Chapter 2. However, for the medium term production/inventory problems described above, one class of stochastic programs is more natural to use than any other. This is the class of active multistage programs. Each stage can be identified with time periods in the "real world" problem, typically months or quarters, and decisions which must be made at each stage are only allowed to depend on the realisations of random variables in previous (and possibly the present) stages and the distributions of the random variables in later time periods conditional on these realisations. Thus production decisions are only allowed to depend on the demand in previous time periods and not that in future ones.

Unfortunately, in general, exactly optimal solutions to multi-stage stochastic programs reduce at best to dynamic programming methods and these become computationally intractable as the number of commodities being modelled increases. This is shown in Chapter 4 which develops some dynamic programming models. Approximate solution methods are therefore of interest. This thesis contains a generalisation and development of one of the most promising approximate methods due to Beale, Forrest and Taylor [4]. Their method is described in Chapter 3 and the development of it is presented in Chapter 6.

It is important to assess the merit of approximate solution techniques. The way in which this is done is now reviewed.

3. THE EVALUATION OF APPROXIMATE SOLUTION TECHNIQUES

The optimal utility returned by the objective function of, and the optimal decision given by, an approximate solution method to a stochastic model may be in error. This can be tested on sufficiently simple examples by comparisons with those obtained by a method known to be exact. However, this comparison may be misleading if it is used to assess the suboptimality of the decisions recommended by the approximate method. Firstly, the utility gained by actually using an approximate solution technique may be very different from that returned by the model's objective function. Secondly, deviations from the optimal decisions are not in themselves important. What is important is the drop in utility consequent upon them and this may be hard to gauge.

The method suggested in this thesis for handling these problems is that of statistical simulation. The environment within which the stochastic program operates is modelled on a computer. The random variables in the problem are simulated by pseudo-random numbers. Under the influence of these, and the control of the approximate method being tested, the stochastic process then evolves from the first time period in the problem to the time horizon. This is known as a simulation trial. It is repeated a large number of times in order to assess the performance of the approximate solution method statistically.

Unfortunately statistical estimates of attributes of interest in the process made in this way are unacceptably inaccurate. This problem is overcome by the use of control statistics. These are

described and developed in Chapter 7, but the application of them requires the derivation of formulae specific to the process being simulated and the algorithm tested. These formulae are derived in Chapter 8 for the approximate solution algorithm developed in Chapter 6. However the formulae are not restricted to this algorithm. This is shown in Chapter 9 which reports the results of simulation experiments in which four approximate algorithms were tested on two simple examples. The results of these experiments suggest that first order deviations in the decisions made by approximate algorithms from their truly optimal values produce only second order deviations in the utility realised by using algorithm from its optimal value. This is a very desirable feature of the process for it indicates that the suboptimality of approximate solution methods may be very much smaller than the approximations made by it might suggest. The thesis ends with a brief summary and conclusions in Chapter 10, in which suggestions are made for further research.

4. THESIS PLAN

This thesis is devoted to the study and development of stochastic models for medium term production planning. It divides into three parts. Part I reviews established models and solution techniques. Chapter 2 surveys stochastic programming and Chapter 3 presents an exposition of the methods of Beale, Forrest and Taylor [4]. Part II deals with novel contributions to modelling production/inventory problems. Chapter 4 describes some dynamic programming techniques, suggests an efficient algorithm for the single-commodity case, and reports some computational experience with it. Chapter 5 presents a fairly general production/inventory model and describes an application of it to a production/manpower/inventory planning problem. In Chapter 6 an approximate solution technique to it is developed, and some numerical results are given. The work contained in both Chapters 5 and 6 is a generalisation and extension of that of Beale et al. [4]. It is important to assess the merit of approximate techniques and this is done in Part III. Chapter 7 describes the techniques of replicative simulation and control statistics. It develops some novel ways of constructing control statistics. Some of these are based upon the derivation of a martingale for each simulation trial from information about the process provided by the algorithm being tested. Detailed formulae for the computation of these are derived in Chapter 8. Chapter 9 describes simulation experiments which test both approximate algorithms and the efficacy of the control statistics. The results are reported and conclusions drawn from them. This thesis is concluded with a brief summary in Chapter 10.

CHAPTER 2

A REVIEW OF STOCHASTIC PROGRAMMING TECHNIQUES FOR MEDIUM TERM PRODUCTION PLANNING

1. INTRODUCTION

The principal difficulty of studying production/inventory problems is that decisions have to be made in the face of uncertainty, not just of unreliable data, but also that in future decisions made by others, for example customers, over whom the decision maker has no control. Analysis of the consequent uncertainty in the system is essential in the determination of the best production strategy and other salient aspects of the production/inventory system, particularly safety stocks. These have traditionally been studied by statistical methods in isolation from the rest of the system. See, for example, Whitin [61], Nador [42] and Chapter four of Hadley and Whitin [26]. Properly, however, they ought to be studied in the context of the whole production process by appropriate modelling.

Stochastic programs form the natural choice of models to use in this context. Much attention has been devoted to them, although it has been more directed to a study of their abstract properties than computationally effective methods of solution. This chapter presents a brief review of the principal forms of stochastic programs. The different forms that have been proposed are surveyed in Section 2. These divide into two categories : the passive and the active forms. The former, in which decisions are made after the outcome of the random variables in the problem becomes known, may be of importance in stregic planning where the decision maker may want to assess the impact of a new production facility on the probability distribution of his total revenue. Since the concern of this thesis is with tactical or medium term planning problems, these models are only of

passing interest here. However, a brief description of them has been included for the sake of completeness.

The active stochastic programs require decisions to be made before the outcomes of some or all of the random variables in the problem are known, and themselves divide into two types: the single or two-period problems and the more general multi-stage problems. The former are, of course, simpler and the theory behind them better developed than for the latter. However production/inventory problems are better modelled by multi-stage active programs, each stage representing a unit of time, say a month or quarter, so it is these that are of most interest here. A discussion of single and two stage programs is given below in order to present a clearer picture of the complexities that arise in their multi-stage generalisations, and also because some of the techniques used to handle them can be extended to the multi-period case. A review of the approaches that have been adopted for the solution of active stochastic programs is given in Section 3 and the chapter is concluded with a summary in Section 4.

2. THE MODELS STUDIED

2.1. The Basic Structure of Stochastic Models

Nearly all stochastic models whose formulation has been motivated by the need to tackle production planning problems divide naturally into a finite number of discrete time periods. Key attributes of the system being modelled are considered to be fixed during each period, but may, of course, vary between time periods. The models are then formulated in terms of these key attributes, some of which may be random variables. It is assumed that the decision maker wishes to maximise or minimise some function of these attributes subject to the constraints imposed upon them by the system.

One of the most established classes of models used for deterministic production planning is that of linear programs. These have the merit of being straightforward to formulate and solution methods for them are well-advanced. Developments of the simplex algorithm have enabled computer programs to be written which solve very large linear programs indeed. Thus linear programs have formed the natural starting point for the development of stochastic models. The concern of this chapter will be with these stochastic linear programs

$$\text{"max"}\ c^T x \text{ over } x \text{ subject to "}Ax = b\text{"} \qquad (1)$$

where b,c and x are column vectors and A is a matrix, and (A,b,c) are random variables. There are two different interpretations to this problem. In the passive approach the decision x is made after the

random variables (A,b,c) are realised and the objective function and constraints are well defined. In the active approach some or all of the x's must be chosen before all the random variables are realised and so both the objective function and the constraints have to be more carefully specified. The former approach is discussed first.

2.2. The Passive Approach

In this approach otherwise called the "wait and see" problem by Madanasky [39] or "distribution" problem by Vajda [54], the decisions x are taken after the random variables (A,b,c) are realised in the program

$$\max z = c^T x \text{ over } x \text{ subject to } Ax = b. \qquad (2)$$

So it is desired to construct an optimal map or decision rule from the outcome space of the random variables to the decision space.

It can be shown theoretically (See Dempster [17]) that the outcome space can be partitioned into a finite set of decision regions such that the optimal decision, x^0, is constant in each decision region. Furthermore, each decision region can be identified with a basis of (2) and the decision regions form a cellular structure whose faces have Lebesgue measure zero.

Having found the set of decision regions and the optimal decision rule

$$x^{o} = x^{o}(A,b,c) \qquad (3)$$

the problem is then to compute the distribution function of

$$z = c^{T} x^{o}(A,b,c).$$

The characterisation of this distribution function in terms of general random (A,b,c) has not yet been obtained, but special cases in which A and c (dually A and b) are fixed have been studied. For example, Bereanu [6] has treated the case where there is only a single random variable in the problem and later extended his work [7] to the case where A is stochastic but imposing restrictions on the random variables.

In general, the alternative active approach is a more natural one for the modelling of production planning problems and it is this to which attention is now directed.

2.3. The Active Approach

In this approach, also known as the "here and now" approach, some or all of the decision variables must be chosen before the outcome of all the random variables in the problem is known. When the process is explicitly periodic and the decision variables and random variables pertain to individual time periods, it is common to make the decision variables in each period a function of the random variables realised up to (and perhaps including) that period.

Care must be exercised over the definitions of the objective function and constraints. The objective function is designed to model the decision maker's preference about how the system should behave. There are three such models commonly treated in the literature. These are

(a) E-models, in which it is assumed that the decision maker has a neutral attitude towards risk and so wishes to maximise (minimise) his expected profit (cost),

(b) P-models, in which it is assumed that the decision maker wishes to maximise (minimise) the probability of his profit (cost) being greater (less than) some target value, and

(c) V-models, in which it is assumed that the decision maker wishes to minimise the total variability of his profit or cost.

See, for example Charnes and Cooper [9] for a further discussion of such models with reference to chance constrained programming. The majority of work published in this area deals with E-models. In what follows reference will only be made to these. However, P and V model analogues should be readily apparent.

There are two alternative interpretations of the constraints $Ax = b$. They can be regarded as holding almost surely (a.s.) i.e. with probability one, or with some prescribed high probability. The latter approach is known as chance constrained programming.

The remainder of this sub-section will be devoted to one or two stage models. Their generalisation to many stages leads to an even greater variety of interpretations and is discussed in Section 2.4 below.

The two-stage model in which the constraints hold almost surely is now addressed. Explicitly stated it is

$$\underset{x}{\text{Max}} \; E\{c^T x - \min_y d^T y\} \tag{5}$$

$$\text{subject to} \quad Rx \leq r$$

$$Ax + By = b \quad \text{a.s.} \tag{6}$$

$$\text{and} \quad x \geq 0,\ y \geq 0, \text{ a.s.}$$

A,B and R are matrices and b,c,d,r,x and y are vectors. Formally this is called the two stage stochastic linear program with recourse. The initial decision, x, must be made before the random variables (A,B,b,c,d) are realised; the realised constraint discrepancy b - Ax yields a loss by the second stage which is to choose a recourse decision y to

$$\text{Minimise } d^T y \text{ subject to } By = b - Ax,\ y \geq 0. \tag{7}$$

The problem is considerably simplified if the recourse matrix, B, is fixed and equal to (I,-I) where I is the identity matrix. Dempster [17] refers to the problem thus obtained as that with simple recourse. Beale [3] and Wets [58] refer to it as the complete problem.

The single stage chance constrained problem may be written

$$\text{Max } Ec^T x \tag{8}$$

over x and subject to

$$P\{Ax \leq b\} \geq \alpha \tag{9}$$

$$\text{and} \quad P\{x \geq 0\} \geq \beta \tag{10}$$

where A,b and c are random variables, α and β lie between 0 and 1, and the decision x must be chosen before the random variables are realised. There are a variety of ways in which the constraints may be regarded. The two principal ones are

(a) Total chance constraints, where (9) may be written

$$P\{(Ax)_i \leq b_i, \forall i\} \geq \alpha \qquad (11)$$

and

(b) Joint chance constraints where (9) may be written

$$P\{(Ax)_i \leq b_i\} \geq \alpha_i, \forall i. \qquad (12)$$

Usually β is taken to be 1, so (10) holds almost surely, but Charnes and Kirby [12] allow β to be less than one.

The study of such models for general random A,b, and c is very complex. Usually authors restrict their attention to the case where only b is random. See, for example, Miller and Wagner [41] and Charnes, Kirby and Raike [13]. However, Ishii, Shiode, Nitshida and Iguchi [34] study a model in which one row of the technology matrix is random.

Under certain conditions the two-stage stochastic linear program with recourse is equivalent to the single stage chance constrained problem. Gartska [23] reviews results dealing with this equivalence.

Just as the active approach provides a more natural setting for the modelling of production planning problems, so multi-stage versions of it are more appropriate than single or two stage programs. These are summarised below.

2.4. Multi-stage Versions of the Active Problem.

In multi-stage problems, the process being modelled divides naturally into time periods. Each decision variable and random variable can be associated with a particular time period. It will be expedient to review those models in which the constraints hold almost surely first. Four principal variants have received attention. The most general is discussed first.

(a) The General Lower Triangular Model

Explicitly stated it is

$$\text{Max } E \sum_{t=0}^{T} c_t^T x_t \tag{13}$$

$$\text{subject to} \quad \sum_{u=0}^{t} A_{tu} x_u = b_t \quad \text{a.s.} \qquad \text{for } t = 0,\ldots,T \tag{14}$$

$$\text{and} \quad x_t \geq 0 \quad \text{a.s.}$$

where the A_{tu}'s are matrices, the b_t's, c_t's and x_t's are vectors. A_{tu}, b_t and c_t are random variables for all $t \geq 1$, and are supposed realised at period t. Let all the random variables realised in period t be a function of a more general random variable ξ_t. Then the decisions x_t are restricted to be functions of $\xi_1,\ldots,\xi_t$,

$$\text{i.e. } x_t \equiv x_t(\xi_1,\ldots,\xi_t) \tag{15}$$

When all the A_{tu}'s are identically zero for $u \leq t-2$ the following problem is obtained

(b) The Staircase Model

$$\text{Max } E \sum_{t=0}^{T} c_t^T x_t \tag{16}$$

subject to

$$\begin{aligned} A_0 x_0 &= b_0 \\ -B_0 x_0 + A_1 x_1 &= b_1 \text{ a.s.} \\ &\vdots \\ -B_{t-1} x_{t-1} + A_t x_t &= b_t \text{ a.s.} \end{aligned} \tag{17}$$

and $x_t \geq 0$.

Again, the t th stage decision is restricted to be a function of the random variables realised up to and including that period. The special case when only the b's are random has received attention from Dantzig [15], Wets [57] and Birge [8].

(c) The Control Theoretic Formulation

This is a special case of the staircase problem in which only the b's are random. If the decision variables x_t are partitioned into $(y_t^T, u_t^T)^T$ where y_t is a state variable and u_t a control variable, and if the system matrices A_t and B_t can be correspondingly partitioned:

$$A_t = \begin{pmatrix} I & 0 \\ C_t & D_t \end{pmatrix}$$

and $$B_t = \begin{pmatrix} F_t & G_t \\ 0 & 0 \end{pmatrix}$$

and if the random vector b_t is partitioned correspondingly into $(0, \xi_t^T)^T$ then the constraints become

$$\left.\begin{aligned} & F_t y_t + G_t u_t = y_{t+1} \\ \text{and} \quad & C_t y_t + D_t u_t = \xi_t \end{aligned}\right\} \quad (18)$$

Gaalman [20] studies a special case of this where (18) is

$$y_{t+1} = A_t y_t + B_t u_t + C_t \xi_t \qquad (19)$$

and the process evolves over the infinite horizon. Using modern control theoretic techniques and making assumptions about the distribution of the random variables ξ_t and stability of the process, Gaalman derives the optimal decision rules u_t as a function of the state variable y_{t-1}. However, the non-negativity constraints on u_t and y_t have been dropped and his model cannot handle capacitated production/ inventory systems that are of interest here.

(d) The Multi-Stage Recourse Problem

This is a natural multi-stage generalisation of the two-stage stochastic program with recourse. Dempster [17] writes it as

$$\underset{x}{\text{Max}} \{ E \sum_{t=1}^{T} c_t^T x_t - \min_{y} d^T y \} \qquad (20)$$

subject to

$$\begin{aligned} & A_{01} x_1 && = b_o \\ & A_{11} x_1 - B_1 y_1 && = b_1 \quad \text{a.s.} \\ & && \vdots \\ & \sum_{u=1}^{t} A_{tu} x_u - B_t y_t && = b_t \quad \text{a.s.} \end{aligned} \qquad (21)$$

and $x_t \geq 0$ a.s.

In general A_{tu}, B_t, b_t and c_t are random for $t \geq 1$. If these t th stage random variables are regarded as functions of a more general random variable, ξ_t, then the decisions x_t are restricted to be functions of $\xi_1,\ldots,\xi_{t-1}$ and the recourse decisions y_t are restricted to be functions of $\xi_1,\ldots,\xi_t$.

It can be shown that the multi-stage general triangular problem (a) is equivalent to the multi-stage recourse problem (d), for each can be regarded as a special case of the other.

To see this let the superscript ^{a} or ^{d} denote an attribute pertaining to problem (a) or (d) respectively. Then to show that problem (a) is a special case of problem (d) set $y_t^d = x_{t+1}^d$ and $x_t^a = x_{t+1}^d$. Then x_t^a is a function of $\xi_1,\ldots,\xi_t$ and furthermore if A_{tt}^a is defined to be $-B_t^d$ and A_{tu}^a to be A_{tu-1}^d then

$$b_t = \sum_{u=1}^{t} A_{tu}^d x_u^d - B_t^d x_t^d$$

$$= \sum_{u=0}^{t} A_{tu}^a x_u^a.$$

Also defining c_t^a to be $c_{t+1}^d - d_t^d$ for $1 \leq t \leq T-1$, c_o^a to be c_1^d and c_1^d and c_T^a to be $-d_t^d$ it is seen that

$$\underset{x^d}{\text{Max}}\ E \sum_{t=1}^{T} c_t^{dT} x_t^d - \min_{y^d} d_t^{dT} y^d = \underset{x^a}{\text{Max}}\ E \sum_{t=0}^{T} c_t^{aT} x_t^a.$$

To show that problem (d) is a special case of problem (a), partition x_t according to

$$x_t^a = \begin{pmatrix} x_{t+1}^d \\ y_t^d \end{pmatrix} \qquad \text{for } 1 \le t \le T-1$$

$$x_0^a = \begin{pmatrix} x_1^d \\ 0 \end{pmatrix} \quad \text{and } x_T^a = \begin{pmatrix} 0 \\ y_t^d \end{pmatrix}$$

and corresponding partition c_t^a:

$$c_t^a = \begin{pmatrix} c_{t+1}^d \\ -d_t^d \end{pmatrix} \qquad \text{for} \quad 1 \le t \le T-1$$

$$c_0^a = \begin{pmatrix} c_1^d \\ 0 \end{pmatrix} \quad \text{and} \quad c_T^a = \begin{pmatrix} 0 \\ -d_T^d \end{pmatrix}$$

it is seen that x_t^d is a function of $\xi_1,\ldots,\xi_{t-1}$ and

$$\max_{x^a} \; E \sum_{t=0}^{T} c_t^{aT} x_t^a = \max_{x^d} \; E \{ \sum_{t=1}^{T} c_t^{dT} x_t^d - \min_{y^d} d_t^{dT} y_t^d \}$$

Also if A_{tu}^a is correspondingly partitioned into $(A_{tu+1}^d, 0)$ for $0 \le u \le t-1$ and $(0, -B_t^d)$ for $u = t$, then

$$b_t = \sum_{u=0}^{t} A_{tu}^a x_u^a$$

$$= \sum_{u=1}^{t} A_{tu}^d x_u^d - B_t^d y_t^d,$$

so problem (d) is a special case of problem (a) and therefore the problems are equivalent.

Hence the staircase problem (b) and the control theoretic problem (c) can be regarded as special cases of the multi-stage recourse problem (d).

(e) Multi-stage Chance Constrained Problems

The single stage chance-constrained problem generalises easily to the multi-stage case, although there is a greater variety of possible interpretations of the constraints. As in the multi-stage recourse problem the technology matrix has a lower triangular block structure. In general terms the model may be stated

$$\text{Max } E \sum_{t=1}^{T} c_t x_t^T \tag{21}$$

over x and subject to

$$\begin{aligned} A_{01}x_1 &\leq b_0 \\ P\{A_{11}x_1 &\leq b_1\} \geq \alpha_1 \\ P\{A_{21}x_1 + A_{22}x_2 &\leq b_2\} \geq \alpha_2 \\ &\vdots \\ P\{\sum_{u=1}^{t} A_{tu}x_u &\leq b_t\} \geq \alpha_t \end{aligned} \tag{22}$$

$$\text{and } \{P\, x_t \geq 0\} \geq \beta_t \tag{23}$$

the A_{tu}'s are matrices, the b_t's, c_t's and x_t's are vectors and A_{tu}, b_t and c_t are in general random variables for $t \geq 1$. If A_{tu}, b_t and c_t are regarded as being functions of a more general t th stage random variable, ξ_t, then the decisions x_t are restricted

to be functions of $\xi_1,\ldots,\xi_{t-1}$. Most authors restrict their attention to the case of fixed technology matrices A_{tu} when in certain cases the problem is equivalent to the multi-stage recourse problem (d), Gartska [23]. The further restriction that the matrices A_{tu} should all be of dimension (1×1) is also made in most of the literature. The different ways in which the probability of the constraints (22) and (23) can be interpreted needs further discussion.

If the probabilities in (22) and (23) are computed using the joint distribution of all the random variables in the problem then Charnes and Kirby [11] term the problem one of "total chance constraints". It was this approach that Charnes, Cooper and Symonds [10] used in their original formulation of a chance-constrained problem to model the production of heating oil.

If the probabilities in (22) and (23) are computed using the distribution of ξ_t conditional on the realised values of $\xi_1,\ldots,\xi_{t-1}$ then the problem is termed one of "conditional chance constraints" by Charnes and Kirby [11].

Eisner, Kaplan and Soden [18] also considered another interpretation which they called the "conditional-go" approach. At stage t the u th stage constraint (where $u > t$) is regarded as not actually being revealed until the u th stage. So the probabilities in (22) are computed with the marginal distribution of ξ_u given $\xi_1,\ldots,\xi_{t-1}$ i.e. the u th stage constraint becomes

$$P\{\sum_{v=1}^{u} A_{uv}x_v(\xi_1,\ldots,\xi_{v-1}) \leq b_u | \xi_1,\ldots,\xi_{t-1}\} \geq \alpha_u \qquad (24)$$

where $t < u$.

Other variants of the problem arise out of different choices of admissible decision rules. This is discussed in Section 3 on solution methods below.

3. METHODS OF SOLUTION

In this section some solution techniques that have been proposed for the stochastic programs presented above will be briefly discussed. Since production planning problems are more naturally modelled by the active or "here and now" stochastic programs than the "passive" or "wait and see" variety it is the solution of the former which is of interest here; lines of attack on the latter have already been briefly mentioned.

The two or single-stage active stochastic programs are easier to solve than their multi-stage counterparts. Exact computationally effective methods of solution have been devised for the former, but not for the latter when there are many constraints per stage, and these are necessary for the explicit modelling of many commodities. The only effective approximate method that has been proposed is that of Beale, Forrest and Taylor [4] which solves a simple production/inventory model.

3.1. Two Stage Stochastic Programs with Recourse

As with the other stochastic programs, solution of the general case of (5) and (6) in which A,B,b,c and d are all random is very difficult both theoretically and computationally. Attention has usually been restricted to the special case in which only b is random, although Evers[19] tackles a random A matrix by Monte Carlo methods and El-Agizy [1] has shown that if c is random then it can without loss of generality be replaced by its expected value and the correlation between its components ignored.

Dantzig [15] tackled the problem with simple recourse in which only b is random. He showed that the stochastic program is equivalent to the deterministic program

$$\max\{c^T x - d^{+T} \int_{b \le x} (x-b)dF(b) + d^{-T} \int_{b \ge x} (b-x)dF(b)\}$$

subject to

$$Rx \le r$$
$$Ax + y^+ - y^- = b$$
$$x, y^+, y^- \ge 0 .$$

Here the recourse matrix B has been partitioned into (I,-I) and the vectors d and y have been correspondingly partitioned into

$\binom{d^+}{d^-}$ and $\binom{y^+}{y^-}$ respectively, and

F is the distribution function of b. This program can be shown to be convex if $(d^+ + d^-) \ge 0$.

Solutions to the above program can be approximated by assuming a discrete distribution for the random vector b, in which case a different recourse decision, y, must be associated with each point of the discrete distribution. Dantzig and Madanasky [16] adopt this approach and use decomposition methods to exploit the program's structure. Strazicky [52] takes this approach further by using basis decomposition and reports some numerical results.

Wets [60] has investigated the derivation of deterministic equivalents for problems with general fixed recourse.

Other approaches have been proposed by, for example, Van Slyke and Wets [51] who use gradient methods and Garkska and Rutenberg [24] who use lattice points.

Of most interest in modelling stochastic production planning problems are multi-period models. It is to the solution of these that attention is now directed.

3.2. Multi-Stage Stochastic Programs with Almost Sure Constraints

These are stochastic programs in which the constraints hold almost surely (i.e. with probability one). As with the two-stage case, the solution of the general multi-stage recourse problem is very difficult, although Wets [59] shows theoretically that any solution algorithm for the two stage case can be extended to the multi-stage case.

Dantzig [15] was the first to study the special case of programs with staircase structure in which only the right hand side vector, b, is random. Again he suggested discretizing the distribution of b to derive an equivalent deterministic program. Birge [8] does the same and extends Dantzig's methods of exploiting the structure to the problem thus generated by using large scale decomposition, partitioning and basis factorization. He presents a number of ways of doing this one of which uses the interesting result due to Wets [57] that the staircase problem thus discretized is equivalent to a deterministic convex program of the form

$$\begin{aligned} &\underset{x}{\text{Max}}\ c^T x + Q(x) \\ &\text{subject to}\ \ Ax = b \\ &\qquad\qquad x \in D \end{aligned}$$

where Q is a concave function and D is a convex polyhedron.

Birge [8] goes on to show that his techniques are really methods of dynamic programming, differing only in the way in which decisions are approximated as functions of the state variables.

3.3. Chance Constrained Problems

In discussing solution methods for these, the general multi-stage chance constrained problem will be addressed. Apart from Ishii et al. [34] who only deal with a single stage model, solution methods have only been proposed for the case in which the technology matrices are fixed, i.e. just b and c are random. As in the models in which the constraints hold almost surely, solution techniques proceed by the derivation of an equivalent deterministic program. The ease with which these deterministic programs can be solved depends upon their convexity properties. The work of Prekopa [43], [44] and [45] on logarithmically concave measures has shed much light on this.

Usually authors restrict their attention to searching for optimal first order decision rules in which the decision x_t made at stage t is restricted to be a linear function of the random variables already realised, $b_1, c_1, \ldots, b_{t-1}, c_{t-1}$. Charnes, Cooper and Symonds [10] adopted this technique in modelling the production of heating oil. The model which they studied had total chance-constraints and only the b's were random. Moreover, there was only one constraint per stage in their model. They were able to calculate the optimal linear decision rules by dynamic programming starting at the time horizon and working backwards. In fact, piecewise linear decision rules are optimal for such a problem

as was shown by Charnes and Kirby [11], even when there is more than one constraint per stage. Kortonek and Soden [36] give another proof of this result and also consider the case where the cost vector c is random. Later, Charnes and Kirby [12] proved that piecewise linear rules are optimal under conditional chance constraints, although there they restricted their attention to only one constraint per stage. This enabled them to derive computationally efficient solution techniques, involving in some instances a series of simple one variable non-linear optimisation problems.

This completes the discussion of solution techniques for the stochastic models described in Section 2.

4. CONCLUSIONS

In this chapter it has been proposed that production/inventory problems be modelled by stochastic programs. A review has been provided of those most frequently studied in the literature and the lines of approach that have been suggested for their solution. The stochastic programs divide into two classes, the passive and the active. The active ones then divide into separate classes according to whether the constraints hold only with some prescribed high probability (chance constrained programs) or almost surely (i.e. with probability one), and according to whether one, two or many time periods or stages are modelled. Further details may be found in Sengupta and Tintner [48] who review stochastic linear programming and Kirby [35] who surveys chance constrained programming.

The most useful class of stochastic programs from the point of view of medium term production planning is that of multi-stage active programs in which the constraints hold almost surely. Unfortunately in general this is the hardest class to solve. Approaches to the solution generally involve the discretization of the random variables involved and the use of advanced large scale programming techniques to take advantage of the structure of the problem thus generated. These can be shown to be equivalent to dynamic programming techniques, the other candidate for handling multi-stage active stochastic programs. These methods are unsuitable for tackling multi-commodity problems because of their computational complexity. See Chapter 4 for a discussion of dynamic programming techniques, their merits and limitations.

In view of the difficulty of solving multi-stage active problems exactly even if the random variables are assumed to be discrete, approximate techniques deserve serious consideration. A promising method is that of Beale, Forrest and Taylor [4] who study a simple multi-commodity production/inventory model which has an upper bound on the total production in any period. Their approach has provided one of the foundations of the research described in this thesis, notably the development of a more general production/inventory model which is described in Chapter 5 and an approximate solution technique described in Chapter 6. Accordingly an exposition of their work is appropriate. This is given in the next chapter.

CHAPTER 3

AN EXPOSITION OF "MULTI-TIME PERIOD STOCHASTIC SCHEDULING" BY BEALE FORREST AND TAYLOR.

1. INTRODUCTION

Beale, Forrest and Taylor [4] aim to provide a suite of computer programs that would enable production planners to obtain good reliable medium term production strategies in the face of uncertainty in the demand for their products. The authors do this by studying a simple stochastic multi-product production/inventory model and proposing a computationally tractable approximate solution technique. This technique is numerically feasible in the sense that the size of problem that can be reasonably tackled (measured by the number of product lines that can be treated individually) is of the same order as the size of problem that could be handled if the demand requirements were known with certainty.

Their paper has provided much of the impetus for the research described in Part II and so an exposition of their work together with a discussion of its merits and limitations is appropriate here. In an effort to overcome the limitations inherent in their technique, a much more general production/inventory model was formulated in Chapter 5 and studied in Chapter 6.

The production/inventory model which they study is given in Section 2. They approximate it by a non-linear program and the method by which they do this is described in Section 3. Some coefficients in it are, however, still unknown. They estimate these iteratively by a process described in Section 4. The chapter ends with a brief summary and conclusions in Section 5.

2. THE PRODUCTION/INVENTORY MODEL

Production, sales and inventory levels are to be planned for each of T time periods. Demand requirements for each time period are characterised by probability distributions. Production rates are considered to be fixed during each time period but may vary between time periods. At the start of any time period production levels are decided. During that period the demand is realised and at the end of it sales are made and stock levels become apparent. All the costs are considered to be linear in the production decisions made at the start of, sales in, and stock levels realised at the end of, each time period. There is an upper bound on the total production in each period. It is assumed that the decision maker has a neutral attitude towards risk and so desires to maximise his total expected profit.

Let the column vectors p_t, a_t, s_t and d_t denote the production in, sales in, stock at the end of, and demand in time period t. Identify the i th component of each vector with the i th product.

Let C_{Pt}, P_t and C_{St} be column vectors of unit production costs, sale prices and stockholding costs. Let T_{CAPt} be the maximum total production permitted in each period and let $\mathbf{1}$ be a vector of 1's.

Then the model which Beale, Forrest and Taylor study may be explicitly stated:

$$\text{Maximise } E \sum_{t=1}^{T} P_t^T a_t - C_{Pt}^T p_t - C_{St}^T s_t \qquad (1)$$

over a_t, p_t and s_t subject to

$$\mathbf{1}^T p_t \leq T_{CAPt} \qquad (2)$$

$$a_t \leq d_t \qquad (3)$$

$$s_{t-1} + p_t - a_t - s_t = 0 \quad \text{and} \tag{4}$$

$$a_t, p_t \text{ and } s_t \geq 0 \tag{5}$$

for $t = 1,2,\ldots,T$. $t = 0$ is the initial or starting state so s_0, for example, are the initial stocks and are thus part of the model's input data.

Notice that all that is actually required from a solution to the above model is the first time period production decisions. In subsequent periods the model would be re-run with new starting stocks and more accurate data.

The authors propose that the standard deviation of each component of the demand in each time period should be directly proportional to its mean, which in turn is a linear function of the sales in the previous time period.

If v_t is a vector pertaining to the tth time period, let v_{it} be its i th component. Then explicitly stated their demand model is

$$d_{it} = d_{Mit}(1 + C_{it}\, \eta_t + R_{it}\, \varepsilon_{it}) \tag{6}$$

and

$$d_{Mit} = B_{iot} + \sum_j B_{ijt}\, a_{jt-1} \tag{7}$$

where η_t and ε_{it} are independent real Gaussian random variables, and B_{iot}, B_{ijt}, C_{it} R_{it} are known fixed constants. The term η_t is intended to model the global variability of all products in each time period and ε_{it} is intended to model the variability in demand between individual products in each time period.

Notice that apart from the initial production levels p_1, all the variables in the model are actually random variables. This is because decisions made in future time periods are allowed to be functions of the demand realised up to that period.

The authors claim that although their model is simple, it can easily be extended by the addition of extra constraints to cover the more complicated problems that are likely to be met in practice, without altering its fundamental structure and approximate solution algorithm. This is only partly true. The production constraint (2) can be replaced by a more realistic set of technological constraints without altering their solution procedure. But their model cannot accommodate bounds on storage capacities or the cost of changes in production level. Neither can a more comprehensive demand model, for example one in which the mean demand is modelled as a linear function of a moving average of past sales, be used with their solution procedure.

3. THE FORMULATION OF A NON-LINEAR PROGRAM

The authors identify the crucial quantity of interest in their model to be the excess of supply over demand, and they are particularly interested in its variability. They call the excess of supply over demand in the t th time period e_t, where it is defined by

$$e_t = s_{t-1} + p_t - d_t \tag{8}$$

and let the variance of its i th component be σ_{it}^2. The constraint (3) can then be replaced by a constraint which restricts sales to be less than both the stock available for sale and demand

i.e. $$a_t \leq \min (s_{t-1} + p_t, d_t)$$

which can be equivalently written

$$a_t \leq s_{t-1} + p_t - \max (e_t, 0). \tag{9}$$

Substitution for d_t given by (6) and (7) in (8) shows that the i th component of e_t is

$$e_{it} = s_{it-1} + p_{it} - (B_{iot} + \sum_j B_{ijt} a_{jt-1})(1+C_{it}\eta_t+R_{it}\varepsilon_{it}) \tag{10}$$

The authors now take expected values in the problem defined by rows (1), (2), (4), (9) and (10) to yield the problem

$$\text{Maximise} \sum_{t=1}^{T} P_t{}^T \bar{a}_t + C_{pt}{}^T \bar{p}_t + C_{St} \bar{s}_t \tag{12}$$

over $\bar{a}_t$, $\bar{p}_t$, and $\bar{s}_t$ subject to

$$1^T\bar{p}_t \leq T_{CAPt} \tag{13}$$

$$\bar{e}_t - \bar{s}_{t-1} - \bar{p}_t + B_t\bar{a}_{t-1} = b_t \tag{14}$$

$$\bar{a}_t - \bar{s}_{t-1} - \bar{p}_t + E\{\max(e_t,0)\} \leq 0 \tag{15}$$

$$\bar{s}_{t+1} + \bar{p}_t - \bar{a}_t - \bar{s}_t = 0 \qquad \text{and} \tag{16}$$

$$\bar{a}_t, \bar{s}_t, \bar{p}_t \geq 0 \tag{17}$$

for $t = 1,2,\ldots,T$.

The initial production decision, p_1, has been treated for convenience as a random variable equal to its expected value with probability one. $\bar{a}_t, \bar{p}_t, \bar{s}_t$ and $\bar{e}_t$ denote the expected values of a_t, p_t, s_t and e_t respectively. b_t is a vector whose i th component is B_{iot} and B_t is a matrix whose (i,j)th component is B_{ijt}.

Thus the original problem has been approximated by one which would be a deterministic linear program except for the term

$$E\{\max(e_t,0)\} \tag{18}$$

They tackle this by supposing that e_{it} can be treated as though it has a normal distribution $N(\bar{e}_{it},\sigma_{it}^2)$, whence the i th component of (18) is

$$\sigma_{it}\, f_1(\bar{e}_{it}/\sigma_{it}) \tag{19}$$

where f_1 is a function: $\mathbf{R} \to \mathbf{R}$ defined by

$$f_1(x) = \int_{-x}^{\infty} (\xi+x)\,d\Phi(\xi),$$

Φ being the Gaussian distribution function. Thus $f_1(x)$ is the mean of a random variable whose distribution is that of a Gaussian random variable

with mode x but truncated at zero such that the probability of its being non-positive is concentrated in a point mass at zero.

If the σ's were known, (19) could be substituted into (15) to yield a deterministic non-linear program. However, the σ's are not known and have to be estimated. They derive a recursive procedure for this which is described below.

Moreover, they assume that σ_{it} is directly proportional to the i th component of the mean demand so they could set

$$\sigma_{it} = \tau_{it} \bar{d}_{it}$$

for some constant τ_{it}. But this would lead to paradoxical consequences. If it is not desired to sell a particular product, say the k th, then $s_{kt-1} + p_{kt}$ must still be positive in order to satisfy (15) and (19). The cause of this paradox is the assumption that the demand is normally distributed so there is always a positive probability that it will be negative. The authors avoid this by instead setting

$$\sigma_{it} = \tau_{it} \bar{a}_{it} \quad \text{and} \tag{20}$$

$$\tau_{it} = \hat{\sigma}_{it}/d_{Mit} \tag{21}$$

where $\hat{\sigma}_{it}$ denotes an estimate of σ_{it}. Thus they enable safety stocks to be reduced considerably if it is not desired to meet demand in full. However, this changes the structure of the problem. For if it is not desired to meet demand in full and $\bar{a}_{it}/d_{Mit}$ is small, then the variability in the problem represented by σ_{it} is treated as being greatly reduced, whereas it should not be.

Substituting for $E\{\max(e_t,0)\}$ by (19) and (20) the i th component of constraint (15) can be written

$$\bar{a}_{it}-\bar{s}_{it-1}-\bar{p}_{it} + \tau_{it}\bar{a}_{it}f_1(\bar{e}_{it}/(\tau_{it}\bar{a}_{it})) \leq 0 \qquad (22)$$

The problem defined by (12), (13), (14), (22), (16) and (17) is the non-linear program which they solve by the introduction of separable variables to derive good first time period production decisions. Their procedure for estimating σ_{it} and hence τ_{it} is an iterative one. τ_{it} is initially set to its minimum value (i.e. the value obtained by ignoring the stochastic variability in everything except demand), which is

$$\sqrt{(C_{it}^2 + R_{it}^2)} \qquad (23)$$

and then re-estimated. The procedure by which they do this deserves further discussion which is given below.

4. RE-ESTIMATION OF THE STOCHASTIC VARIABILITY

In this section the method whereby the authors re-estimate τ_{it} is briefly described. They implicitly assume, but do not explicitly state that, at any stage the state of the production/inventory process which they model can be characterised by a state vector ξ_t, which they define by

$$\xi_t^T = (a_t^T, s_t^T) \tag{24}$$

So at the end of time period t, given the input data, the process is completely described by the stock levels, s_t and sales just made, a_t.

Therefore, the production decision made at the start of time period t will be a function of the previous time period state vector ξ_{t-1}. Beale et al. assume that this function can be approximated by a linear one:

$$p_t = p_t^o + A_{pt}^{(1)} a_{t-1} + A_{Pt}^{(2)} s_{t-1} \tag{25}$$

where p_t^o is a constant vector and $A_{Pt}^{(1)}$ and $A_{Pt}^{(2)}$ are constant matrices. Only $A_{Pt}^{(1)}$ and $A_{Pt}^{(2)}$ need be estimated and the way in which they do this is described below.

σ_{it}^2 is, of course, simply the i th diagonal entry of the dispersion matrix of the excess of supply over demand, e_t, and τ_{it} is σ_{it}/d_{Mit}, so the authors desire to estimate the dispersion matrix of e_t. This they do by using the above approximation to derive an expression for it in terms of the dispersion matrices of the previous time period state vector, ξ_{t-1}, and demand d_t.

They then seek to derive an expression for the dispersion matrix of ξ_t in terms of those of ξ_{t-1} and d_t. But to facilitate their analysis they make one further approximation. They approximate

$$s_{it} = \max (e_{it}, w_{it}) \tag{26}$$

where w_{it} are the slacks associated with (15) by

$$s_{it} = S_{Cit} + S_{Vit}\, e_{it} \tag{27}$$

Hence S_{Cit} and S_{Vit} are constants, the latter being defined by

$$S_{Vit} = f_2((\bar{e}_{it} - w_{it})/\sigma_{it})$$

f_2 being a function: $\mathbb{R} \to \mathbb{R}^+$ such that

$$[f_2(x)]^2 = \Phi(x) + [1-\Phi(x)]\Phi(x)x^2 - [1-2\Phi(x)]\phi(x) - [\phi(x)]^2$$

where ϕ, Φ are the Gaussian probability density function and distribution function respectively. It is not necessary to estimate S_{Cit}.

The merit of this value of S_{Vit} is that if e_t were normally distributed then s_{it} given by (26) has the same variance as if it were given by (27). Unfortunately this does not preserve the covariances between the s_{it}'s for given t. So the variance of the total number of items in stock is not preserved either.

So, having made the two approximations above, an expression for τ_{it} in terms of the dispersion matrices of the previous time period state vector, ξ_{t-1}, and demand, d_t, is derived as is an expression for the dispersion matrix of ξ_t in terms of those of ξ_{t-1} and d_t.

Regarding the dispersion matrix of the initial position on the state space to be identically zero they can then proceed forwards recursively estimating the τ_{it}'s.

However, the procedure is limited by the need to estimate the expected value of the e_t's and the matrices of linear coefficients of the production approximation (25). They can do the latter estimation only by an analysis of the final tableau to the solution of their non-linear program and then only for the first production decision in terms of the initial stocks and sales (i.e. sales in the time period immediately preceeding the first). They suggest the following way around this problem.

They first solve the non-linear program with minimal estimates of the τ_{it}'s, namely those given by (23). This generates approximate values for $\bar{a}_t$, $\bar{s}_t$, $\bar{p}_t$ and $\bar{e}_t$. Then starting at the beginning of the last time period from a position given by $\bar{s}_{T-1}$ and $\bar{a}_{T-1}$ they run their variance estimation procedure for the last time period only and re-solve the non-linear program, again for the last time period only. This generates a new value for $\bar{e}_T$ and enables the coefficient matrices $A_{PT}^{(1)}$ and $A_{PT}^{(2)}$ to be estimated. Their recursive variance estimation procedure can then be run forwards from the start of time period T-1 to the end of time period T, assuming the dispersion matrix of ξ_{T-1} to be identically zero, and the τ_{iT}'s re-evaluated. With these new values of τ_{iT} the non linear program is re-run for the last two time periods only, starting at $\bar{a}_{T-2}$ and $\bar{s}_{T-2}$. This enables $A_{PT-1}^{(1)}$ and $A_{PT-1}^{(2)}$ to be estimated and the recursive τ estimation procedure run forwards for the last three time periods, and so on, until finally the non linear program is re-solved for all the time periods in the model.

5. CONCLUSIONS

The paper by Beale, Forrest and Taylor presents a radical new way in which to handle a multi-commodity stochastic production/inventory problem by approximate techniques. Since their work provided a basis for the development of a more general stochastic model and approximate solution procedure in Chapters 5 and 6 an exposition of it has been presented here. This chapter has provided a summary and discussion of their work which explains the crucial steps that they took without detailing the technical calculations. Essentially, they derive a deterministic non-linear program from the original stochastic problem involving only the expected values of the random variables, but without ignoring their variability. This is considered to be encapsulated in a random variable representing the excess of supply over demand. Estimation of the variability of this random variable is done iteratively by solving a sequence of non-linear programs. The first is solved with minimal estimates of its variability. This provides the first approximate solution, which the authors term their "first pass", and it enables their procedure which estimates the variability of supply over demand to be run over the last time period and a non-linear program modelling the last time period only to be formulated and solved. This provides information about the process enabling better estimates of the variability of supply over demand in the last time period to be made. These revised estimates are used in the formulation of a non-linear program modelling the last two time periods, and so on until a non-linear program modelling all the time periods in the model is solved again. This the authors term their "full method".

They test their approach on four simple examples. The first two are sufficiently simple to allow a dynamic programming solution to be obtained. This they do by restricting the demand distribution to discrete values. They demonstrate that the solution thus obtained agrees well with that yielded by their approximate methods. They also used the examples to demonstrate the sensitivity of the initial production decisions to the valuation of the closing inventory.

Their method has been further tested statistically along with other algorithms by replicative simulation. To obtain sufficient accuracy by this method for a reasonable size of experiment control statistics were used. The theory behind them is developed in Chapters 7 and 8, and the results of the experiment are given in Chapter 9. These results show that their method performs well in practice for the simple examples and yields an expected revenue very close to that given by the dynamic programming solution. There was little difference in the performance of their "first pass" and "full" methods, but the examples had only four time periods in order to keep the computer time requirements of the simulation experiments reasonable. If there were more time periods in the model, their "full" method would out-perform their "first pass" method. However, from the similarity between the expected revenues accrued from using algorithms giving slightly different production decisions and the dynamic programming method it may be inferred that first order deviations from the optimal production rates produce only second order changes to the objective function. This shows that answers to stochastic problems obtained by approximate methods may be reliable in practice.

PART II

CHAPTER 4

DYNAMIC PROGRAMMING APPROACHES TO PRODUCTION PLANNING

1. INTRODUCTION

1.1. The Structure of Production Planning Models.

Production planning or smoothing problems are generally tackled by formulating a mathematical model of the production and commercial environment. This has been discussed in Chapter 2, but the essential points are worth reiterating here.

The production environment is usually modelled over some fixed time interval, perhaps a year, which is subdivided into time periods of, say, a month or quarter. Key attributes of the production/inventory/workforce system, for example, production levels, sales and stocks are then considered to change from one time period to the next and variation within each time period is ignored. The model is then manipulated and "solved" so as to provide the best initial decisions (for example, first time period production levels), which optimise some attribute of interest, for example, expected profit.

The attention of this chapter is primarily directed towards dynamic programming solutions to models in which the stochastic elements are important i.e. those models which directly involve uncertainty in some future attributes of the process. In this case attention is restricted to the maximisation of expected profit. Minimisation of expected loss or cost can be handled in exactly the same way: just consider maximising minus the expected cost.

1.2. The Basic Elements of Dynamic Programming

The form of dynamic programming applied here will be that relevant to finite time horizon models as opposed to "steady state" models where discounted expected profit or average expected profit per time period is maximised over an infinite horizon. The optimisation problem is subdivided into subproblems pertaining to each time period. Each subproblem is concerned with the maximisation of expected revenue from the time period in question to the time horizon. The optimisation must be performed for each possible state of the system at the start of the time period, and the subproblems are solved backwards in the sense that the first subproblem is that of maximising the expected last time period profit, the second subproblem is that of maximising the expected profit in the last two time periods and so on, until the last subproblem which is that of maximising the total expected profit accrued in all time periods. At each stage use is made of the results of the previous stage. The optimisation in the final subproblem is, of course, only performed for the initial state of the system. This provides the optimal solution to such models where decisions have to be implemented over time.

1.3. Advantages and Disadvantages of Dynamic Programming

Although dynamic programming techniques theoretically provide reliably optimal solutions, computationally they suffer from the "curse of dimensionality". This is now explained. The state of the system at the start of any time period is known as the state space. This is usually

characterised by an m-dimensional vector, i.e. a point in R^m (m-dimensional Euclidean space). The optimisation for each subproblem (except the last) must be performed for each possible value of the state variable at the start of the first time period of the subproblem. In practice, in dynamic programming approaches, the domain of possible state variable values is covered by a grid and optimisation performed for each vertex of the grid. If N grid points are taken for each state space dimension then the subproblem involves some N^m optimisations. Thus the computational complexity of the dynamic programming approach explodes exponentially with the number of state space dimensions. This is a severe limitation of the method which usually restricts its application to models with four or less state space dimensions. An approximate approach, which is feasible if the number of state space dimensions is small, is to assume some parameterized functional form for the maximum expected future revenue as a function of the present state variable value and to perform the optimisation over the parameters of the function.

Thus dynamic programming is unable to handle the general multi-commodity production smoothing problem. Multiple commodities must be aggregated to be amenable to this approach. However if the number of state space dimensions is small, dynamic programming may be efficient. When the state space has only one dimension it may, indeed, be more efficient than any other method. Moreover its computational complexity expands only linearly with the number of time periods in the model. This compares favourably with other approaches, for example, linear programming, where the computational complexity would expand approximately as the cube of the number of time periods.

1.4. The Contents of the Chapter

The principle of dynamic programming is treated more formally in Section 2, first in a fairly general way and then applied to a simple production/inventory model. This model is tackled in detail in Section 3, where it is restricted to one state space dimension and an efficient algorithm for its solution is derived. In Section 4 the multi-dimensional version is discussed together with the difficulties that it presents. The chapter ends with a discussion of the outcome of this investigation into dynamic programming techniques.

2. THE DYNAMIC PROGRAMMING APPROACH

2.1. A General Formulation

Suppose that the production/inventory system is modelled as a Markov process. Characterize the state of the system or state variable at the end of time period t by the random vector q_t. At the start of the t th time period suppose that controls x_t are applied and random input y_t realised. In applications the state variable might represent stock levels, the controls: production targets and the random input: demand.

Suppose that the revenue accrued in the t th time period, V_t, is some prescribed function of q_{t-1}, x_t and y_t.

i.e. $V_t \equiv V_t\,(q_{t-1}, x_t, y_t)$

Suppose also that the state space evolves from its value at the end of time period t-1 to that at the end of time period t according to some prescribed function of x_t and y_t

i.e. $q_t = f_t(q_{t-1}, x_t, y_t)$.

Then the dynamic programming approach involves the solution of the following problem, $P_t(q_{t-1})$, for each possible value of q_{t-1} and for t = T, T-1,...,1,T being the time horizon.

Problem $P_t(q_{t-1})$:

Defining $V^{T+1} = 0$,

$$\text{find } V^t(q_{t-1}) = \max_{x_t} E\{V_t(q_{t-1},x_t,y_t) + V^{t+1}(q_t)|q_{t-1}\}$$

$$\text{subject to } q_t = f_t(q_{t-1}, x_t, y_t)$$

$$\text{and} \qquad x_t \in S_t(q_{t-1}),$$

where $S_t(q_{t-1})$ is the set of permissible controls at the start of time period t and $V^t(q_{t-1})$ is the maximum expected revenue accrued in time periods t to T inclusive, given that the state space at the start of time period t is q_{t-1}. This provides the optimal solution to such problems where the decision x_t must be made at the start of time period t. See Bellman [5].

If y_t has distribution function F_t then the above problem might more usefully be written as:

Problem $P_t(q_{t-1})$

$$\text{Find } V^t(q_{t-1}) = \max_{x_t} \int\{V_t(q_{t-1},x_t,\xi) + V^{t+1}(f_t(q_{t-1},x_t,\xi))\}dF_t(\xi)$$

$$\text{subject to } x_t \in S_t(q_{t-1})$$

In practice a grid of possible values of q_{t-1}, $\{q_{t-1}^{(i)} ; i = 1,2,\ldots,I\}$ must be constructed and I problems, $P_t(q_{t-1}^{(i)})$, $i = 1,2,\ldots,I$ solved at each stage. Thus at the (t+1) th stage values $V^{t+1}(q_t^{(i)})$ are found and at the following t th stage the function $V^{t+1}(q_t)$ must be approximated by interpolation between the points $V^{t+1}(q_t^{(i)})$.

However, this approximation may be considerably refined in the case of the single state space dimension restriction of the simple model discussed next. This is done in Section 3.

2.2. A Simple Model

The simple model outlined in this section is of a type discussed by Beale et al. [4], and as far as possible their notation will be used. It is a simple production/inventory model in which demand is uncertain and in which there is a simple upper bound on the total production in any time period.

Let the vectors p_t, a_t, s_t and d_t represent production in, sales in, stock at the end of, and demand in period t respectively. Let the i th component of these vectors be that pertaining to the i th product type. e.g. let p_{it} be the i th component of p_t, the production of item i in time period t.

Let the vectors P_t, C_{Pt} and C_{St} denote the unit selling price, production cost and inventory holding cost in time period t.

The object of the analysis is, then, to maximise the expected total revenue, V, from the first time period until the time horizon at the end of the t th time period:

$$\text{Maximise } V = E \sum_{t=1}^{T} (P_t^T a_t - C_{P_t}^T p_t - C_{St}^T s_t)$$

$$\text{over } p_1, .. p_T$$

$$\text{subject to: } 1^T p_t \leq p_t^*$$

$$s_{t-1} + p_t - a_t - s_t = 0$$

$$s_t - \max(s_{t-1} + p_t - d_t, 0) = 0,$$

where all vectors are by default column vectors, T denotes the transpose of a vector, $\mathbf{1}$ is a column vector of 1's, and p_t^* is the maximum allowable total production in time period t.

The demand, d_t, is modelled in the following manner:

$$d_t = B_t(1 + S_t\varepsilon_t)$$

where, again, $\mathbf{1}$ is a vector of 1's and ε_t is a multivariate normal random vector with distribution function F_t and with independently distributed components. B_t and S_t are known, fixed system matrices.

This problem can be decomposed into smaller dynamic programming subproblems in the manner described in the next subsection.

2.3. Dynamic Programming Applied to the Simple Model

In order to apply dynamic programming to the simple model, the model is considered to be Markov. Its state variable at the end of the t th time period is the stock level vector s_t.

Denote the maximum expected revenue acquired from time periods t to T inclusive, given that the stock level at the start of the t th time period is s_{t-1} by

$$V^t(s_{t-1}).$$

Then the objective function of the t th time period subproblem is

$$V^t(s_{t-1}) = \underset{p_t}{\text{Max}} \; E\{(p_t^T a_t - C_{pt}^T p_t - C_{st}^T s_t) + V^{t+1}(s_t) | s_{t-1}\}.$$

Using the materials balance equation

$$s_{t-1} + p_t - a_t - s_t = 0$$

to eliminate the sales variable, a_t, the t th time period subproblem, $P_t(s_{t-1})$ is more usefully written as follows.

Find

$$V^t(s_{t-1}) = P_t^T s_{t-1} + \text{Max}\{(P_t^T + C_{pt})p_t - (P_t^T + C_{st})E(s_t|s_{t-1}) + E(V^{t+1}(s_t)|s_{t-1})\}$$

where the maximisation is performed over p_t for each s_{t-1} and is subject to

$$1^T p_t \leq p_t^*$$

and $$s_t - \max(s_{t-1} + p_t - d_t, 0) = 0$$

where $$d_t - B_t(1 + S_t \varepsilon_t) = 0.$$

Again, in practice a suitable grid of points for s_{t-1} on which to find $V^t(s_{t-1})$ must be devised, as must an interpolation procedure to approximate from it V^t as a function of s_{t-1}. This is easily done in the case where s_{t-1} has only one dimension and an efficient solution algorithm can be devised. This is discussed in the next section.

3. THE CASE OF A SINGLE DIMENSIONAL STATE SPACE

3.1. The Simplification Made

As has already been mentioned, a model with only one state space dimension offers considerable scope for computational tractability and efficiency. There has been some interest shown in reducing the dimension of the state space in models with a multiple dimensional state space in order to achieve computational feasibility. Dallenbach [14] ingeniously partitions the state space of a manpower/production planning model in order to achieve one effective state space dimension. Thomas [53] adopts a similar approach to a price/production planning model. In general, however, the scope for these tricks is limited, and attention should be more properly directed to the problem of product aggregation and disaggregation. Gaalman [21] provides optimal aggregation/disaggregation rules for the now classical HMMS model of Holt, Modigliani, Muth and Simon [33], but otherwise there is little attention paid to the problem in the literature.

The remainder of this section will be devoted to the derivation of an efficient algorithm for the solution of the single product version of the simple model discussed in section 2. Some attention will, however, be paid to ways in which it might be extended to cope with more general problem constraints.

3.2. Derivation of the t th Time Period Optimisation Subproblem.

In the one-product case the stock at the end of the t th time period depends on that at the end of the preceding period according to

$$s_t = \max \{(s_{t-1} + p_t - B_t) - B_t s_t \varepsilon_t, 0\}$$

where ε_t is a Gaussian random variable. For convenience set

$$r_t = s_{t-1} + p_t - B_t$$

and $\sigma_t = B_t s_t$

Then $s_t = \{\max\ r_t - \sigma_t \varepsilon_t, 0\}$

and $E(s_t | s_{t-1}) = r_t F_t(r_t/\sigma_t) - \sigma_t G_t(r_t/\sigma_t)$,

where $G_t(x) = \int_{-\infty}^{x} \xi dF_t(\xi)$.

Therefore the objective function of the t th time period sub-problem, $P_t(s_{t-1})$ may be written

$$V^t(s_{t-1}) = P_t B_t + \underset{P_t}{\text{Max}} \{g_t(r_t) - C_{pt} p_t + E(V^{t+1}(s_t)|s_{t-1})\}$$

where $r_t = s_{t-1} + p_t - B_t$

and $g_t(r_t) = P_t r_t - (P_t + C_{St})\{r_t F_t(r_t/\sigma_t) - \sigma_t G_t(r_t/\sigma_t)\}$.

It is now necessary to evaluate $E(V^{t+1}(s_t)|s_{t-1})$.

Suppose that values for $\{s_t, V^{t+1}(s_t)\}$ are known only at discrete points: $\{(s_t)^i, (V^{t+1})^i : i = 0,1,\ldots,I\}$ where $0 = (s_t)^0 < (s_t)^i < (s_t)^{i+1}$ for $i = 1,\ldots,I-1$.

Set

$$\alpha_t^i = [(V^{t+1})^{i+1} - (V^{t+1})^i]/[(s_t)^{i+1} - (s_t)^i].$$

Then $V^{t+1}(s_t)$ may be approximated by a piecewise linear function of s_t

i.e. $$V^{t+1}(s_t) = (V^{t+1})^i + \alpha_t^i[s_t - (s_t)^i]$$

for $(s_t)^i \le s_t < (s_t)^{i+1}$.

Let $\xi_t^i(r_t) = [r_t - (s_t)^i]/\sigma_t$.

Then it is seen that

$$E(V^{t+1}(s_t)|s_{t-1}) = \int_{-\infty}^{r_t/\sigma_t} V^{t+1}(r_t - \sigma_t\xi)dF_t(\xi) + \int_{r_t/\sigma_t}^{\infty} V^{t+1}(0)dF_t(\xi)$$

$$= \sum_{i=0}^{I-1} T_i(r_t)$$

where for $i \ge 1$, $T_i(r_t)$ is defined by

$$T_i(r_t) = \int_{\xi_t^{i+1}}^{\xi_t^i} \{(V^{t+1})^i + \alpha_t^i \sigma_t(\xi_t^i - \xi)\, dF_t(\xi)$$

$$= \{(V^{t+1})^i + \alpha_t^i \sigma_t \xi_t^t\}\{F_t(\xi_t^i) - F_t(\xi_t^{i+1})\}$$

$$- \alpha_t^i \sigma_t \{G_t(\xi_t^i) - G_t(\xi_t^{i+1})\}$$

and $T_0(r_t) = (V^{t+1})^0 \{1 - F_t(\xi_t^0)\}$.

It is now convenient to define the function $f_t: \mathbb{R} \to \mathbb{R}$ by

$$f_t(x) = g_t(x + B_t) + P_tB_t + \sum_{i=0}^{I-1} T_t(x + B_t)$$

whence the t th time period subproblem $P_t(s_{t-1})$ can now be written :

Find for each s_{t-1}

$$V^t(s_{t-1}) = \text{Max } \{f_t(x_t) - C_{pt}\, p_t\}$$

where the maximisation is over p_t and is subject to

$$0 \leq p_t \leq p_t^*$$

and $\quad x_t - p_t = s_{t-1}.$

3.3. The Form of the Solution to the Subproblem P_t

There are several possible approaches to the solution of the t th time period subproblem P_t given above. The most flexible, in the sense of being able to cope with variants of the production or technological constraint, is to take a piecewise linear approximation to f_t over a fixed grid and do the maximisation by parametric linear programming, varying the element s_{t-1} on the right hand side. This would then generate a new set of points $\{(s_{t-1})^i, (V^t)^i\}$, these being the values of the right hand side and objective function when the basis changes. However, before performing the parametric programming it is more elegant to directly approximate the demand distribution with a discrete one rather than approximate f_t piecewise linearly over a fixed grid.

There is, however, a more efficient method of solution to P_t if flexibility in the sense discussed above is unimportant.

Consider the problem $P_t\ (s_{t-1})$ above. Clearly the maximising p_t is a function of s_{t-1}. Denote it by $p_t^o\ (s_{t-1})$. Then if

$$0 < p_t^o(s_{t-1}) < p_t^*$$

the problem is equivalent to one of unconstrained optimisation and

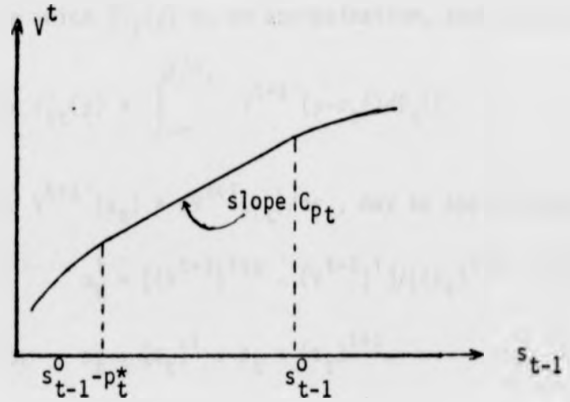
$$f_t'(s_{t-1} + p_t^o) - C_{pt} = 0, \text{ where } f'(x) = df/dx.$$

Define s_{t-1}^o to be that value of s_{t-1} at which $f_t'(s_{t-1}) - C_{pt} = 0$ or infinity if $f_t'(s_{t-1}) - C_{pt} > 0$ for all $s_{t-1} > 0$ or zero if $f_t'(s_{t-1}) - C_{pt} < 0$ for all $s_{t-1} > 0$.

It follows that

$$V^t(s_{t-1}) = \begin{cases} f_t(s_{t-1} + p_t^*) - C_{pt}p_t^* & \text{if } s_{t-1} \leq s_{t-1}^o - p_t^* \\ f_t(s_{t-1}^o) - C_{pt}(s_{t-1}^o - s_{t-1}) & \text{if } s_{t-1}^o - p_t^* < s_{t-1} \leq s_{t-1}^o \\ f_t(s_{t-1}) & \text{if } s_{t-1} > s_{t-1}^o \end{cases}.$$

So agraph of V^t against s_{t-1} looks like



It remains to find f_t' explicitly. Now

$$f_t(x) = g_t(x-B_t) + P_tB_t + \sum_{i=0}^{I-1} T_i(x-B_t), \text{ so}$$

$$f_t'(x) = g_t'(x-B_t) + \sum_{i=0}^{I-1} T_i'(x-B_t)$$

But $g_t(y) = P_ty - (P_t + C_{st})\{y\, F_t(y/\sigma_t) - \sigma_t G_t(y/\sigma_t)\}$

so $g_t'(y) = P_t - (P_t + C_{st})F_t(y/\sigma_t)$.

Instead of calculating $\sum_{i=0}^{I-1} T_i'(y)$ by differentiating each T_i and summing, it is better to differentiate

$$f_{vt}(y) = E(V^{t+1}|s_{t-1} + p_t - B_t = y)$$

$$= \int_{-\infty}^{y/\sigma_t} V^{t+1}(y-\sigma_t\xi)dF_t(\xi) + V^{t+1}(0)\int_{y/\sigma_t}^{\infty} dF_t(\xi)$$

for which $\sum T_i(y)$ is an approximation, and then approximate it.

Now $f_{vt}'(y) = \int_{-\infty}^{y/\sigma_t} V^{t+1\prime}(y-\sigma_t\xi)dF_t(\xi)$

and $V^{t+1\prime}(s_t) = dV^{t+1}(s_t)/ds_t$, may be approximated by the step function

$$\alpha_t^i = [(V^{t+1})^{i+1} - (V^{t+1})^i]/[(s_t)^{i+1} - (s_t)^i]$$

for $s_t : (s_t)^i \le s_t < (s_t)^{i+1}$.

Again, setting

$$\xi_t^i(y) = [y - (s_t)^i]/\sigma_t$$

it is seen that

$$f'_{vt}(y) = \sum_{i=0}^{I} (\alpha_t^i - \alpha_t^{i-1}) F_t(\xi_t^i(y))$$

where α_t^{-1} is taken to be zero.

Substituting into the last equation for $f'_t(x)$ the following is obtained:

$$f'_t(x) = P_t - (P_t + C_{st}) F_t((x-B_t)/\sigma_t)$$

$$+ \sum_{i=0}^{I} (\alpha_t^i - \alpha_t^{i-1}) F_t(\xi_t^i(x-B_t))$$

Therefore a value for s_{t-1}^o can be obtained by solving

$$f'_t(x) - C_{p_t} = 0.$$

3.4. An Efficient Algorithm for the Simple Model

The results of Section 3.3 can be expressed as an efficient algorithm in following manner. The following procedure is implemented for $t = T, T-1, \ldots, 1$.

STEP 0: Set $V^{T+1'}(s_t)$ equal to the unit value of the closing inventory. Set $t = T$.

STEP 1: Solve $f'_t(x) - C_{p_t} = 0$, the solution to which is s_{t-1}^o. If $t = 1$ go to step 5.

STEP 2: Calculate $V^t(s_{t-1})$ for a fixed grid of points which include $s^o_{t-1} - p^*_t$ and s^o_{t-1} as adjacent points. Call these grid points $(s_{t-1})^i$ and $(V^t)^i = V^t((s_{t-1})^i)$.

STEP 3: Estimate $V^{t'}(s_{t-1})$ by α^i_t for $(s_{t-1})^i \leq s_{t-1} < (s_{t-1})^{i+1}$ where $\alpha^i_t = [(V^t)^{i+1} - (V^t)^i]/[(s_{t-1})^{i+1} - (s_{t-1})^i]$.

STEP 4: Set t = t-1 and return to step 1.

STEP 5: Calculate $V^1(s_o)$ where s_o is the initial stock level. Stop.

This algorithm, despite the crudity of its estimates of $V^{t'}$, appears to work well. The estimation of V^t could easily be refined if desired.

3.5. A Small Numerical Example

The following small example was solved using the algorithm described. It is a four time period (season) model in which:

Unit selling price (all seasons)	=	10.0
Unit production cost (all seasons)	=	5.0
Unit storage cost (all seasons)	=	2.0
Total productive capacity (all seasons)	=	35.0 units
Mean demand in season 1	=	20.0 units
2	=	25.0 units
3	=	35.0 units
4	=	45.0 units

The stochastic element is supplied by supposing that the demand is normally distributed with standard deviation 0.2236 times its mean, suitably truncated to proscribe negative demand.

The results are given below for two cases, firstly when the closing inventory is given its "full" value i.e. 10.0/unit and secondly when given a "discounted" value of 9.5/unit.

THE RESULTS

For case (i) : full value of the closing inventory

Expected production

12.6644	35.0000	35.0000	35.0000

Expected sales

19.2405	24.9849	34.8628	42.5230

Expected stocks

3.4239	13.4390	13.5763	6.0533

Expected objective function value

611.8112	489.8358	443.2149	298.6470

For case (ii) : discounted value of the closing inventory

Expected production

12.5692	24.8830	35.0000	35.0000

Expected sales

19.2140	24.9538	34.7385	41.7114

Expected stocks

3.3553	11.2844	11.5459	4.8345

Expected objective function value

608.6293	486.0849	425.2821	278.3607

NOTE: The 'expected' values given herein are not strictly the mathematical expected values, but those values that would result from starting each time period from the expected value of the state variable.

In both cases the algorithm used about 28 seconds of C.P.U. time on a Burroughs 6700 machine. No special effort was made to produce the most efficient coding possible of the algorithm.

4. THE CASE OF A MULTI-DIMENSIONAL STATE SPACE

In the multi-dimensional state space case the state space must be discretized by a multi-dimensional grid. The t th time period subproblem $P_t(q_{t-1})$ must then be solved for every value of the state vector at the end of the t th time period, q_{t-1}, which corresponds to a vertex of the grid. As discussed in Section 1.3, this leads to horrendous computational difficulties when the number of state space dimensions is large.

This is equivalent to a discretization of the stochastic input or demand at least in the case of the production/inventory model discussed in Section 2. For discretization of demand effectively restricts the controls (production targets) to a discrete set; each discrete possible production decision being just sufficient to avoid a discrete demand point exceeding supply.

Choice of suitable points of discretization for the demand distribution is difficult. If the demand distribution is normal then suitable points and probability weights might be those of Gauss-Hermite quadrature, but even these are of doubtful value in the multi-dimensional case.

In the non-stochastic or deterministic case the problems of distribution discretization do not arise and those associated with the "curse of dimensionality" can be mitigated by starting with a very coarse grid for the state space. Dynamic programming then yields the optimal path through vertices of the grid, i.e. optimal sequence of

state space realisations, and around this path a finer grid can be constructed, and so on. However, linear programming methods, especially those associated with staircase structured technology matrices (see, for example, Madsen [40], for a review of such techniques) are more appropriate.

5. CONCLUSIONS

The basic structure of stochastic production planning models has been briefly discussed and the application of dynamic programming to such models given. It has been shown that if the production planning model is formulated as a Markov process whose state vector describes the state of the system, then dynamic programming involves the solution of a series of subproblems. Each subproblem is the optimisation of the expected revenue from the current time period until the time horizon and must be performed for each possible state of the system at the start of the current period. The subproblems are solved recursively backwards: the first involving only the last period, the second the last two periods and so on. The solution to each subproblem is necessary for the solution of the next.

It has been shown that the computational complexity of the dynamic programming approach, although expanding linearly with the number of time periods in the model, explodes exponentially with the number of state space dimensions. It is therefore unsuitable for multi-commodity models, but it may be very efficient for single commodity ones. This has been discussed and an efficient algorithm for the solution of a simple such model derived, and some computational experience reported.

So if aggregation and disaggregation schemes are readily apparent, dynamic programming may provide a useful solution technique for stochastic production planning models when all the state space

dimensions have been aggregated to form a single effective state space dimension. The difficulty of providing such schemes for capacitated models has motivated the multi-commodity model and approximate solution technique which is given in the next two chapters.

CHAPTER 5

A GENERAL MODEL FOR PRODUCTION PLANNING

1. INTRODUCTION

The formulation of multi-time period stochastic Markov models is very natural in the study of medium term production planning and workforce scheduling problems, where it is desired to account directly for the variations in some inputs, particularly demand. Medium term problems are here considered to be those with a planning horizon of, say, a year which is subdivided into decision periods of a month or quarter. Control decisions, for example production rates or workforce levels, and system variables, for example stock levels, are considered fixed during each decision period. The problem, one of dynamic stochastic theory, is well established. See, for example Dempster [17]. Solutions to it are usually approached by dynamic programming, but this is computationally restricted to cases in which the state space has only one or two dimensions.

The study and use of such models has been severely limited by a lack of computationallytractable solution techniques. The one notable exception is that of Beale et al. [4], which provides an approximate solution to a simple production-inventory model, but is capable of handling any reasonable number of products.

Simpler models have received much attention. Gaalman [20] tackles a multi-time period stochastic model with control-theoretical techniques, but it is uncapacitated and, moreover, is unable to handle the change in the system state that would arise, for example, at the point of stock-out. Other multi-time period models are approached

through the methods of chance constraints or dynamic programming. See, for example Charnes, Cooper and Symonds [10], who aggregate to restric their attention to a single product and use chance constrained programming. Dallenbach [14] and Thomas [53] ingeniously partition the state space and use dynamic programming to cope with two dimensions. Multi-period models with many dimensions are generally only treated in the deterministic case. See Gabbay [22] and Hax and Meal [30] for the treatment of such problems with extensions to a multi-echelon production process. A single period stochastic model is tackled by Hodges and Moore [32], who incorporate the randomness of the demand by a suitable marginal analysis of the linear programming solution. Two-period models reduce to the problem of stochastic programming with fixed recourse in the sense of Wets [58]. Theoretical aspects of this problem have received much attention in the literature from, for example, Walkup and Wets [56] and Vadja [54]. Application to single product inventory and manpower planning have been made by El-Agizy [1].

It is in principle desirable to have an exact solution to an O.R. model. But this is particularly difficult for multi-time period stochastic programming, and furthermore the assumption that the relevant probability distributions can be fully specified is probably quite unrealistic. So we hence concentrate on "good approximate solutions".

The model proposed in this chapter is a generalisation of that of Beale et al [4]. The generalisations enable a much more general class of production/inventory/workforce planning models to be handled

than is possible with their model. An approximate solution technique is derived in chapter 6. This is also based upon their work, but improvements are made to their method in addition to the necessary generalisations.

The model is given explicitly and discussed in section 2. An application to a specific production/inventory/workforce planning problem is given in section 3. The chapter ends with a brief summary and conclusions in section 4.

2. A DESCRIPTION OF THE MODEL

The model proposed below for the modelling of production planning problems is of a discrete time, controlled stochastic system. For a given solution algorithm, the process is Markov. Also, apart from one constraint in each time period (constraint (5) below) the model is linear. Under certain convexity conditions, which are derived and discussed in section 3.3 of chapter 6 this constraint can be replaced by two equivalent linear constraints and the model becomes a stochastic linear program.

Explicitly stated it is:

$$\text{Maximise} \quad E\left(\sum_{t=1}^{n} d_{1t}^{T} q_t + d_{2t}^{T} x_t\right) \tag{1}$$

subject to:

$$r_t = R_{1t} q_{t-1} + R_{2t} x_t \tag{2}$$

$$y_t = y_t^o + M_t q_{t-1} + e_t \tag{3}$$

$$q_t = q_t^o + A_t q_{t-1} + B_t x_t + C_t w_t + D_t z_t \tag{4}$$

$$w_t = \max(z_t, 0) \tag{5}$$

$$z_t = z_t^o + K_t q_{t-1} + L_t x_t - y_t \tag{6}$$

$$\text{and} \quad x_t \geq 0. \tag{7}$$

Constraints (2) - (7) are for $t = 1,2,\ldots,T$. State $t = 0$ is the initial or starting state. Furthermore:

The q_t are state variables realised at the end of time period t, the x_t are control decisions made at the start of time period t, the y_t are the stochastic inputs realised during time period t, z_t represents the amount by which the stochastic input is exceeded by some linear deterministic function of the control and input state variables

for the time period, as a vector with positive or negative components, whilst w_t defines the positive component of z_t.

q_t^o, y_t^o, z_t^o and r_t are fixed vectors. A_t, B_t, C_t, D_t, K_t, L_t, M_t, R_{1t} and R_{2t} are fixed system matrices. d_{1t} and d_{2t} are vectors of cost coefficients, also assumed fixed. e_t are multivariate normal random vectors with zero mean and dispersion matrix Disp (y_t), and independent in t i.e. e_t and e_u are independent if $t \neq u$.

The objective function is the simple maximisation of the expected value of a linear combination of the state and control variables from the starting state until the time horizon. Any discount factor would be incorporated in the cost coefficients.

Constraints (2) and (7) are called the control constraints. They define the set of feasible controls in the t th time period given the state space at the start of that time period, q_{t-1}, which might be used to model permissible production levels, for example. Defining this set by

$$S_t(q_{t-1}) = \{x_t : R_{1t}q_{t-1} + R_{2t}x_t = r_t,\ x_t \geq 0\}$$

it must be stipulated that $S_t(q_{t-1})$ is non-empty for all possible q_{t-1}.

Constraint (3) will be referred to as the input constraint. The input might be used to model demand for individual products. This constraint would allow the mean demand in each time period to be a linear function of the state space at the start of that time period. Thus, by choice of a suitable state space the dynamic linear model

and Bayesian forecasting techniques of Harrison and Stevens [29] can be incorporated within the model. In addition two approaches to modelling the dispersion matrix of the random input given the previous time period state space, i.e. the dispersion matrix of e_t, Disp (y_t), are considered.

(a) The 'additive' case in which Disp (y_t) is fixed, and

(b) The 'multiplicative' case in which the standard deviation of e_t is proportional to the mean of y_t.

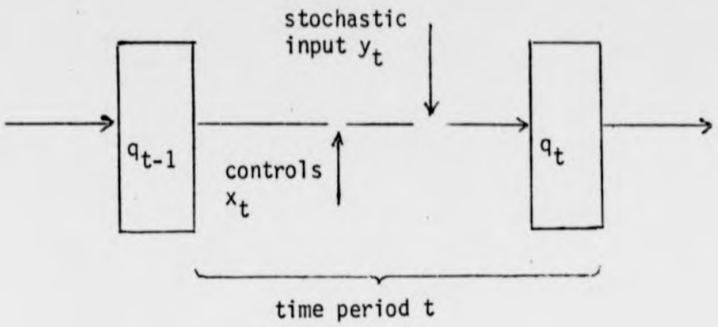
i.e. $(\text{Disp } (y_t))_{ij} = (Ey_t)_i \ (P_t^o)_{ij} \ (Ey_t)_j$

where P_t^o is a given fixed matrix.

Constraints (4), (5) and (6) will be termed the evolution equations. Given the state of the system and the controls applied at the start of time period t these equations describe how the system evolves to the start of time period t+1 according to the realised value of the t th time period stochastic input y_t.

So, at the start of time period t, the state of the system is described by q_{t-1}. Control decisions x_t are made and the system evolves through the evolution equations as the stochastic input y_t is realised to its state at the start of the next time period, q_t, and revenue $d_{1t}^T q_t + d_{2t}^T x_t$ is accrued.

Diagramatically the process may be thought of as:


stochastic
input y_t
q_{t-1}
controls
x_t
q_t
time period t

3. AN APPLICATION OF THE GENERAL MODEL TO A PRODUCTION/MANPOWER/ INVENTORY PLANNING PROBLEM

The purpose of this section is to illustrate the kind of application encompassed by the general model which has motivated its formulation. Production, workforce and inventory levels are to be planned for each of T time periods. Production, manpower and speculative stock levels are decided at the start of each period. Demand is realised during that period after which the stocks and/or backorders become apparent. Provision is made for production and workforce change costs, although these must be directly proportional to the changes made. Manpower levels can be adjusted by changing the workforce or overtime or short time (undertime) working. Speculative stock is planned by earmarking some of the available stock as being not for sale. This may be important in the production of some products, where the sale price may be greater in future time periods. Production can be increased by subcontracted work.

It is assumed that backorders are always satisfied before the current demand, and at the end of each time period stock in excess of the storage capacity must be discarded (i.e. thrown away or sold very cheaply); for the model is capacitated not only with respect to plant production facilities, but also with respect to storage facilities.

For a given planning algorithm the model is Markov. Decision and state variables are given below. It is important that the consequence of decisions made at the start of a time period which emerge only after the demand is realised be modelled as state space

realisations. After these definitions the necessary system constraints are discussed and given. These then have to be manipulated in order to reflect the form of the general model.

(a) Decisions made at the start of time period t

Description	Notation
Increase/Decrease plant production	x_{1t}/x_{2t}
Subcontracted production	x_{3t}
Increase/Decrease labour force	x_{4t}/x_{5t}
Overtime/short time worked	x_{6t}/x_{7t}
Stock withheld	x_{8t}
Backorders satisfied (Plus slacks on control constraints)	x_{9t}

(b) State variables realised at the end of time period t

Description	Notation
Plant production level	q_{1t}
Labour force employed	q_{2t}
Total stock level (including stock withheld)	q_{3t}
Total backorder level	q_{4t}
Current sales	q_{5t}
New backorders	q_{6t}
Stock discarded	q_{7t}

All the costs are linear in the variables above, which are vectors. The i th component of a variable pertaining to stocks, sales, production or backorders refers to the i th product category. The j th

component of a variable pertaining to the labour force refers to the j th category of employee. The labour force itself is not measured in numbers of employees, but in numbers of standard hours worked per time period (i.e. hours worked without over- or short-time).

(c) The stochastic input in time period t is just the demand, d_t. Again, d_t is a vector whose i th component is the demand for the i th category of product.

(d) Necessary system constraints for time period t.

It is assumed that the technological contraints on plant production can be modelled by bounds on a linear function of the total hours worked and products made per period. Now the total number of hours worked is the standard time of the labour force plus the overtime minus the short-time i.e.

$$q_{2t} + x_{6t} - x_{7t}.$$

It is assumed that the model's cost structure will proscribe simultaneous over and short time working for the same category of employee. So the technological plant production constraints are modelled by

$$R_{1t}(q_{2t} + x_{6t} - x_{7t}) + R_{2t}q_{1t} \leq r_{1t}, \qquad (1)$$

R_{1t}, R_{2t} being fixed system matrices and r_{1t} a fixed system vector.

Overtime is limited to a fraction $(1/\alpha)$ of standard time and short time must be less than the standard time. This can be specified by the single constraint

$$\alpha x_{6t} + x_{7t} \leq q_{2t}. \qquad (2)$$

The labour force at the start of the t th time period must be balanced with that at the end of the t th time period, so

$$q_{2t} = q_{2t-1} + x_{4t} - x_{5t} \quad . \tag{3}$$

Similarly, the plant production must be balanced across time periods by

$$q_{1t} = q_{1t-1} + x_{1t} - x_{2t} \quad . \tag{4}$$

The total stock level at the end of the t th time period must be equal to that at the start plus the total production minus the sum of the sales, backorders filled and stock discarded. Therefore:

$$q_{3t} = q_{3t-1} + q_{1t} + x_{3t} - q_{5t} - x_{9t} - q_{7t} \quad . \tag{5}$$

The backorders must also balance: those at the end of the time period must be equal to those at the start plus any new backorders minus the backorders filled.

$$\text{i.e. } q_{4t} = q_{4t-1} + q_{6t} - x_{9t} \, . \tag{6}$$

It will be convenient in the discussion of further system constraints to define a new vector, s_t, representing the stock available for sales and backorders:

$$s_t = q_{3t-1} + q_{1t} + x_{3t} - x_{8t} \, . \tag{7}$$

This must be constrained to be non-negative to proscribe the withholding of more stock than is physically present.

$$\text{i.e. } q_{3t-1} + q_{1t} + x_{3t} - x_{8t} \geq 0 \ . \qquad (8)$$

The stock withheld must not exceed the storage capacity, so

$$x_{8t} \leq r_{2t} \qquad (9)$$

where r_{2t} is the storage capacity. (It has been assumed that storage is specific to each product, but this assumption can easily be generalised).

Now, the backorders filled must be less than the minimum of the stock available and the total backorders at the start of the time period.

$$\text{i.e. } x_{9t} \leq \min \{s_t, q_{4t-1}\}. \qquad (10)$$

The sales must be equal to the minimum of the stock available minus the backorders filled and the demand. So

$$q_{5t} = \min \{s_t - x_{9t}, d_t\}. \qquad (11)$$

Any excess of demand over sales becomes the new backorders so

$$q_{6t} = \max \{d_t - q_{5t}, 0\} \ . \qquad (12)$$

Lastly, it is assumed that if the stock remaining at the end of the time period is in excess of the storage capacity then the difference must be discarded

$$\text{i.e. } q_{7t} = \max \{q_{3t-1} + q_{1t} + x_{3t} - q_{5t} - r_{2t}, 0\}. \qquad (13)$$

The constraints (1) - (13) have now to be manipulated in order to demonstrate that they are a special case of the general model.

A slack, x_{10t} is added to (1) and q_{1t} and q_{2t} are eliminated by substitution from (4) and (3) respectively to yield

$$\begin{aligned} r_{1t} = {} & R_{2t}q_{1t-1} + R_{1t}q_{2t-1} \\ & + R_{1t}x_{1t} - R_{1t}x_{2t} + R_{2t}x_{4t} - R_{2t}x_{5t} + R_{1t}x_{6t} \\ & + R_{1t}x_{7t} + x_{10t}. \end{aligned} \qquad (1')$$

Another slack, x_{11t} is added to (2) and q_{2t} is eliminated by substitution from (3) to yield

$$0 = q_{2t-1} + x_{4t} - x_{5t} - \alpha x_{6t} - x_{7t} - x_{11t}. \qquad (2')$$

Equations (3) and (4) are already in the appropriate form, so can be left as they are.

It is convenient to substitute for q_{1t} given by (4) in equation (7)

$$s_t = q_{1t-1} + q_{3t-1} + x_{1t} - x_{2t} + x_{3t} - x_{8t}. \qquad (7')$$

Addition of a slack, x_{12t} to constraint (8) now yields:

$$0 = q_{1t-1} + q_{3t-1} + x_{1t} - x_{2t} + x_{3t} - x_{8t} - x_{12t}. \qquad (8')$$

The addition of a slack x_{13t} to constraint (9) yields

$$r_{2t} = x_{8t} + x_{13t}. \qquad (9')$$

Constraint (10) can be replaced by the two inequalities

$$x_{9t} \leq s_t$$

and $$x_{9t} \leq q_{4t-1}.$$

Addition of slacks x_{14t} and x_{15t} respectively to the inequalities and substitution for s_t given by (8') in the first yields

$$0 = q_{1t-1} + q_{3t-1} + x_{1t} - x_{2t} + x_{3t} - x_{8t} - x_{9t} - x_{14t} \qquad (10')$$

and $$0 = q_{4t-1} - x_{9t} - x_{15t}. \qquad (10'')$$

Now equation (11) can be written:

$$q_{5t} = s_t - x_{9t} - \max\{s_t - x_{9t} - d_t, 0\}.$$

Setting $z_{1t} = s_t - x_{9t} - d_t$

$$= q_{1t-1} + q_{3t-1} + x_{1t} - x_{2t} + x_{3t} - x_{8t} - x_{9t} - d_t \qquad (14)$$

and substituting for s_t given by (8') it is seen that

$$q_{5t} = q_{1t-1} + q_{3t-1} + x_{1t} - x_{2t} + x_{3t} - x_{8t} - x_{9t} - \max(z_{1t}, 0). \qquad (11')$$

Now it is necessary to manipulate equation (12). Substitution for q_{5t} given by (11') and z_{1t} given by (14) yields

$$d_t - q_{5t} = -z_{1t} + \max(z_{1t}, 0)$$
$$= \max(-z_{1t}, 0).$$

Therefore $q_{6t} = \max(d_t - d_{5t}, 0)$
$= \max\{\max(-z_{1t}, 0), 0\}$
$= \max(-z_{1t}, 0),$

and (12) can be replaced by

$$q_{6t} = \max(z_{1t}, 0) - z_{1t}. \qquad (12')$$

Elimination of q_{5t} given by (11') in (13) yields

$$q_{7t} = \max\{\max(z_{1t}, 0) + x_{8t} - r_{2t}, 0\}.$$

But $x_{8t} - r_{2t} \leq 0$. Hence if $z_{1t} + x_{8t} - r_{2t} \leq 0$ then $q_{7t} = 0$, otherwise $q_{7t} = z_{1t} + x_{8t} - r_{2t}$, so

$$q_{7t} = \max (z_{1t}, r_{2t} - x_{8t}) + x_{8t} - r_{2t}.$$

Setting $z_{2t} = z_{1t} + x_{8t} - r_{2t}$

$$= -r_{2t} + q_{1t-1} + q_{3t-1} + x_{1t} - x_{2t} + x_{3t} - x_{9t} - d_t, \quad (15)$$

$$q_{7t} = \max (z_{2t}, 0) . \quad (13')$$

It is now possible to put (5) into the appropriate format by substitution for q_{1t}, q_{5t} and q_{7t} given by (4), (11') and (13') respectively. This yields

$$q_{3t} = x_{8t} + \max (z_{1t}, 0) - \max(z_{2t}, 0). \quad (5')$$

Lastly, substitution for q_{6t} given by (12') into (6) yields

$$q_{4t} = q_{4t-1} - x_{qt} + \max (z_{1t}, 0) - z_{1t}. \quad (6')$$

Setting

$$w_{it} = \max (z_{it}, 0) \text{ for } i = 1,2 \quad (16)$$

and $y_{1t} = y_{2t} = d_t$

the constraints are in the form demanded by the general model. They are summarised below:

$$r_{1t} = R_{2t}q_{1t-1} + R_{1t}q_{2t-1}$$

$$+ R_{1t}x_{1t} - R_{1t}x_{2t} + R_{2t}x_{4t} - R_{2t}x_{5t} + R_{1t}x_{6t} + R_{1t}x_{7t} + x_{10t} \quad (1')$$

$$0 = q_{2t-1} + x_{4t} - x_{5t} - \alpha x_{6t} - x_{7t} - x_{11t} \quad (2')$$

$$0 = q_{1t-1} + q_{3t-1} + x_{1t} - x_{2t} + x_{3t} - x_{8t} - x_{12t} \quad (8')$$

$$r_{2t} = x_{8t} + x_{13t} \quad (9')$$

$$0 = q_{1t-1} + q_{3t-1} + x_{1t} - x_{2t} + x_{3t} - x_{8t} - x_{9t} - x_{14t} \quad (10')$$

$$0 = q_{4t-1} - x_{9t} - x_{15t} \quad (10'')$$

$$q_{1t} = q_{1t-1} + x_{1t} - x_{2t} \quad (4)$$

$$q_{2t} = q_{2t-1} + x_{4t} - x_{5t} \quad (3)$$

$$q_{3t} = x_{8t} + w_{1t} - w_{2t} \quad (5')$$

$$q_{4t} = q_{4t-1} - x_{qt} + w_{1t} - z_{1t} \quad (6')$$

$$q_{5t} = q_{1t-1} + q_{3t-1} + x_{1t} - x_{2t} + x_{3t} - x_{8t} - x_{9t} - w_{1t} \quad (11')$$

$$q_{6t} = w_{1t} - z_{1t} \quad (12')$$

$$q_{7t} = w_{2t} \quad (13')$$

$$z_{1t} = q_{1t-1} + q_{3t-1} + x_{1t} - x_{2t} + x_{3t} - x_{8t} - x_{9t} - y_{1t} \quad (14)$$

$$z_{2t} = -r_{2t} + q_{1t-1} + q_{3t-1} + x_{1t} - x_{2t} + x_{3t} - x_{9t} - y_{2t} \quad (15)$$

$$w_{it} = \max (z_{it}, 0) \text{ for } i = 1,2 \quad (16)$$

and $x_{it} \geq 0$ for $i = 1,2,\ldots,15$. (17)

Constraints (1'), (2'), (8'), (9'), (10'), (10") and (17) are the control constraints, although strictly they should be augmented by two additional constraints which imply that the plant production level q_{1t}, and labour force, q_{2t} are positive

i.e. $$0 = q_{1t-1} + x_{1t} - x_{2t} - x_{16t} \qquad (18)$$

and $$0 = q_{2t-1} + x_{4t} - x_{5t} - x_{17t} \qquad (19)$$

where x_{16t} and x_{17t} are two additional slacks. For any q_{it-1}, $i = 1,2,\ldots,7$ there will always be feasible x_{jt}, $j = 1,2,\ldots,17$.

Constraints (4), (3), (5'), (6'), (11'), (12') and (13') are the evolution equations. For any q_{it-1}, $i = 1,\ldots,7$ and feasible x_{jt}, $j = 1,\ldots,17$ it can be assumed that the probability of any q_{it} not being ≥ 0 is sufficiently small to be ignored. For q_{1t} and q_{2t} will be ≥ 0 by the additional constraints given above. q_{3t} will always be ≥ 0 because $(w_{1t} - w_{2t}) \geq 0$ for all values of the stochastic input d_t. q_{4t} is always ≥ 0 because

$$w_{1t} - z_{1t} = \max(-z_{1t}, 0) \geq 0$$

and $x_{9t} \leq q_{4t-1}$ by control constraint (10"). q_{5t} will always be positive if the demand is positive for

$$q_{1t-1} + q_{3t-1} + x_{1t} - x_{2t} + x_{3t} - x_{8t} - x_{9t} \geq 0$$

by control constraint (10'), so some component of q_{5t}, $(q_{5t})_i$ say, will only be negative if the corresponding component of w_{1t}, $(w_{1t})_i$ is positive in which case

$$(q_{5t})_i = -(y_{1t})_i$$

and $(y_{1t})_i$ is simply $-(d_t)_i$, so $(q_{5t})_i$ will only be negative if the i th component of the demand is negative! Although demand is modelled along a multivariate normal distribution it may be assumed that the probability of any component of it being negative is sufficiently small

to be ignored. q_{6t} will always be ≥ 0 because $(w_{1t} - z_{1t}) \geq 0$ and (13') obviously implies that q_{7t} will always be positive.

The state vector q_t's can easily be expanded so as to enable a Bayesian forecasting technique to be incorporated into the model.

The revenue accrued during the t th time period is a linear function of q_{it}, $i = 1,2,\ldots,7$ and x_{jt}, $j = 1,2,\ldots,9$.

The model will be convex (see section 3.3 of chapter 6) if

$$n_t^T C_t \leq 0, \text{ for all } t = 1,2,\ldots,T$$

where n_t is a vector whose i th component represents the value of the i th component of q_t in time period t. Applying this to the model given above, it is seen that it will be convex if

(a) (stock value - stockholding cost) + (backorder value - backorder holding cost) - (sale price) - (cost of new backorder) is negative for each time period, and

(b) (stock value - stockholding cost) - (cost of discarding stock) is negative.

Although (b) may be assumed always to apply, otherwise stock left over at the end of each time period would be thrown away regardless of whether there was any spare storage capacity, (a) may fail to apply if it is expedient to withhold stock when the sale price in a particular time period is very low compared with the unit value of the items in stock.

4. CONCLUSIONS

The problem of medium term production planning has been addressed and a suitably general stochastic model has been proposed. An application of it to a more specific planning problem has been given in order to demonstrate its potential usefulness. The approximate solution method proposed in the following chapter is capable of handling any reasonable number of product lines and labour categories in the sense that the size of the problem that can be solved is of the same order as the size of problem that can be solved ignoring all the stochastic elements. The only major limitation of the model lies in its inability to handle set up costs e.g. fixed costs that might be incurred by the decision to produce any quantity of a particular product or to make any change to the workforce level. Nevertheless it is hoped that the model and the approximate solution technique may be of value in the field of medium term planning.

For convenience, the problem posed by the general model will be termed the full problem. It is this problem which the following chapter addresses in the derivation of an approximate solution technique.

CHAPTER 6

AN APPROXIMATE SOLUTION TECHNIQUE FOR THE GENERAL MODEL

1. INTRODUCTION

In the last chapter a general planning model was introduced and an application to a production/inventory/workforce planning problem given. (Under certain conditions, which are derived and discussed in Section 2.3 below), this model is a stochastic linear program in the sense that it reduces to an ordinary linear program as the variability in the stochastic input tends to zero.

This chapter is devoted to the derivation of an approximate solution technique to the general model. Attention has been devoted towards good approximate techniques because of the computational, intractability of exact ones. An exact solution to the model would require the full derivation of the optimal controls x_t as functions of the preceeding state variable, q_{t-1}. To appreciate the computational difficulty of this, suppose that the problem is discretized by taking a grid of points for each q_{t-1} and then solved by finding $x_t(q_{t-1})$ for each point on the grid. Some such scheme is necessary in any exact numerical approach. If the state space has n dimensions and N grid points are used for each dimension, then N^n function evaluations are required for each time period. This approach is feasible only if n is small when dynamic programming techniques could solve the discretized problem exactly. However, this method is of little practical use when n is large.

The method of solution proposed here is computationally tractable for any reasonable number of state space dimensions. Firstly notice

that in practice only the first control decision, x_1 is actually required, for at the end of the first time period better information would be available for the data on subsequent periods and the model would be re-run to determine x_2.

The technique suggested here is a development of the algorithm proposed by Beale et al. [4].It requires the formulation of a reduced problem involving only the expected values of the stochastic input, state space and control decisions but does not eliminate their variability. The state variables and control decisions are random vectors because the stochastic input is specified as a random vector. This leads to the state variable being a random vector, and thus the control decisions, which are functions of the preceeding state variable, are in general random vectors. For the purpose of this analysis the first control decision, x_1 is regarded as a random variable equal to its expected value with probability one, since x_1 is only a function of q_0 the initial state of the system, assumed to be a known item of data. Thus the expected value of x_1, returned by a solution to the reduced problem, is the exact first time period control decision.

The reduced problem is a simple non-linear program which turns out to be convex if the original or full problem is a stochastic linear program. The reduced problem can be viewed as a form of "deterministic equivalent" to the full problem although strictly the contraints on the expected values of the random variables are implied by, but do not imply,the constraints in the full problem on the random variables themselves.

However, more information is required for the full formulation of the reduced problem than is initially available. This difficulty is overcome by replacing the reduced problem by a sequence of restricted reduced problems and solving them in turn: the solution to each restricted reduced problem yields more information about the process which enables the formulation of the next restricted reduced problem. The restricted reduced problems tend to an approximate version of the reduced problem. The computational effort required to solve the sequence of restricted reduced problems is not as great as might first be imagined, for the next one in the sequence will be similar to the last and considerable advantage can be taken of this.

The matter is more fully discussed in Section 6, which is devoted to computational aspects of the procedure and gives a small numerical example.

Section 3 is devoted to the derivation and study of the reduced problem. The structure of the restricted reduced problems is given in Section 4. However, the precise formulation of the restricted reduced problems requires further study of the variability of the state variables. This is done in Section 5. A summary of the method in Section 7 concludes the chapter.

Some preliminary technicalities need to be tackled first and these are addressed in the next section where some convenient notation is introduced.

2. TECHNICAL PRELIMINARIES

2.1. Some Notational Conveniences

Throughout this chapter vectors are represented by lower case letters and matrices by upper case letters. Vectors are, as usual, always taken to be column vectors unless otherwise stated. The superscript ^{T} will always denote the transpose of a matrix or vector.

Many vectors and matrices pertain to a particular time period. That a matrix or vector pertains to the t th time period is denoted by a subscript $_{t}$. The i th component of such a vector is denoted by the subscripts $_{it}$. For example, the i th component of the t th time period stochastic input is denoted by y_{it}.

I is always used to represent the identity matrix, its dimension being obvious from the sense in which it is used.

2.2. Random Variables in the Problem

It has already been mentioned in the introduction to this chapter that the state variable, q_t, the control decisions x_t and the linking variable z_t are actually random variables as well as the stochastic input y_t. This follows from the dependence of q_t on z_t, which is a linear function of y_t, and the dependence of x_t, by the Markovian nature of the system on the preceeding state variable, q_{t-1}. Although x_1 is therefore a function of the initial system state, q_0, which is just part of the problem's input data, it is treated for convenience as a random variable equal to its expected value with probability one.

2.3. Representation of Multivariate Random Vectors

Let x be a random vector. Denote its mean, $E(x)$, by $\bar{x}$ and its dispersion matrix by Disp (x).

$$\text{i.e. Disp } (x) = E((x-\bar{x})(x-\bar{x})^T).$$

Disp (x) is, therefore, by definition positive semi-definite.

Let x and y be multivariate random vectors. Denote their covariance matrix by Cov (x,y)

$$\text{i.e. Cov } (x,y) = E((x-\bar{x})(y-\bar{y})^T).$$

Now if M is a positive semi-definite matrix there always exists a lower triangular matrix L such that $M = LL^T$. To see this let $M = (m_{ij})$; $L = (\ell_{ij})$ and given M define L by:

$$\begin{aligned}
\ell_{11} &= \surd m_{11} \\
\ell_{21} &= m_{21}/\ell_{11} \\
\ell_{22} &= \surd(m_{22} - \ell_{21}^2) \\
\ell_{31} &= m_{31}/\ell_{11} \\
\ell_{32} &= (m_{32} - \ell_{31}\ell_{21})/\ell_{22} \\
\ell_{33} &= \surd(m_{33}^2 - \ell_{31}^2 - \ell_{32}^2) \\
\ell_{41} &= m_{41}/\ell_{11} \\
\ell_{42} &= m_{42} - \ell_{41}\ell_{21})/\ell_{22} \\
\ell_{43} &= (m_{43} - \ell_{41}\ell_{31} - \ell_{42}\ell_{32})/\ell_{33} \\
\ell_{44} &= (m_{44} - \ell_{41}^2 - \ell_{42}^2 - \ell_{43}^2)
\end{aligned}$$

and so on.

Square roots all exist because M is positive semi-definite and divisions by zero do not occur for the same reason (i.e. the numerator is zero whenever the denominator is zero and the result may be taken as zero).

If x is a random vector, let the lower triangular decomposition of Disp (x) be Std(x)

$$\text{i.e. Disp } (x) = \text{Std}(x).(\text{Std}(x))^{T}.$$

Then if $\bar{x} = 0$ there always exists a vector ξ, of zero mean unit variance independent components such that x can be represented by

$$x = \text{Std}(x).\xi,$$

for even if Disp (x) and hence Std(x) are singular then one or more component of ξ will have no effect on x(and these components may be taken to have, say, a Gaussian distribution).

2.4. Representation of the Stochastic Input

The stochastic input is considered to have a multivariate normal distribution, whose mean may depend linearly on the state space at the end of the previous time period.

$$\text{i.e. } y_t \sim N(\tilde{y}_t, \text{ Disp } (y_t))$$

where $$\tilde{y}_t = y_t^o + M_t\, q_{t-1}$$

for some constant vector y_t^o and matrix M_t. Two separate cases are considered herein:

(a) The additive case, in which Disp (y_t) is fixed

(b) The multiplicative, case, in which Disp $(y_t) = \tilde{Y}_t P_t^o \tilde{Y}_t$, where P_t^o is a fixed matrix and $\tilde{Y}_t$ is a diagonal matrix whose i th diagonal entry is $\tilde{y}_{it}$.

So, in the additive case y_t may be represented by

$$y_t = \tilde{y}_t + \text{Std}(y_t).n_t$$

where Disp (y_t) = Std(y_t) [Std(y_t)]^T and n_t is a random vector distributed as N(0,I).

Whilst in the multiplicative case y_t may be represented by

$$y_t = \tilde{y}_t + \tilde{Y}_t S_t^o n_t$$

where $P_t^o = S_t^o (S_t^o)^T$ and again n_t is a random vector distributed as N(0,I). Notice that by definition P_t^o must be a positive semi-definite, symmetric matrix.

This completes the preliminary technical discussion. The formulation of the reduced model is tackled next.

3. THE REDUCED PROBLEM

3.1. Derivation of the Reduced Problem

For convenience the full problem or general model of Chapter 5 is re-stated here. It is:

$$\text{Maximise } E\{\sum_{t=1}^{T} d_{1t}^{T} q_t + d_{2t}^{T} x_t\} \tag{1}$$

subject to the following constraints for $t = 1,2,\ldots,T$

$$r_t = R_{1t}\, q_{t-1} + R_{2t}\, x_t \tag{2}$$

$$y_t = y_t^o + M_t\, q_{t-1} + e_t \tag{3}$$

$$q_t = q_t^o + A_t\, q_{t-1} + B_t x_t + C_t\, w_t + D_t\, z_t \tag{4}$$

$$w_t = \max\,(z_t, 0) \tag{5}$$

$$z_t = z_t^o + K_t\, q_{t-1} + L_t x_t - y_t \tag{6}$$

and $$x_t \geq 0 \tag{7}$$

e_t is a multivariate zero mean normal random vector whose dispersion matrix is Disp (y_t).

In the following text, (2) will be referred to as the control constraint, (3) as the input constraint, and (4), (5) and (6) are called the evolution equations.

The basic idea is to replace the random vectors in the full problem by their expected values in such a way that their randomness is neither ignored nor seriously distorted to formulate what will be called the reduced problem. The objective function is represented by simply replacing q_t and x_t by their expected values, so it becomes

$$\text{Maximise} \sum_t \{d_{1t}^T \bar{q}_t + d_{2t}^T \bar{x}_t\}. \tag{8}$$

So the full problem could be replaced by an equivalent deterministic one if necessary and sufficient constraints can be imposed upon $\bar{q}_t$ and $\bar{x}_t$. For convenience write the constraints of the full problem as

$$q_t = q_t^1 + A_t^o q_{t-1} + B_t^o x_t + C_t \max(z_t, 0) + D_t e_t \tag{9}$$

and
$$x_t \in S_t(q_{t-1}) \tag{10}$$

where $z_t = z_t^1 + K_t^o q_{t-1} + L_t x_t + e_t$

and $S_t(q_{t-1}) = \{x : r_t = R_{1t} q_{t-1} + R_{2t} x,\ x \geq 0\}$,

z_t^1 and q_t^1 being fixed system vectors equal to $(z_t^o - y_t^o)$ and $(q_t^o + D_t z_t^1)$ respetively, and K_t^o, A_t^o and B_t^o being fixed system matrices equal to $(K_t - M_t)$, $(A_t + D_t K_t^o)$ and $(B_t + D_t L_t)$ respectively.

Now the necessary and sufficient constraints on the $\bar{q}_t$'s and $\bar{x}_t$'s implied by (9) are

$$\bar{q}_t = q_t^1 + A_t^o \bar{q}_{t-1} + B_t^o \bar{x}_t + C_t E\{\max(z_t, 0)\}. \tag{11}$$

So if the control decisions were unconstrained and if the distribution of z_t were a known function of $\bar{z}_t$, then the full problem could be replaced by an equivalent deterministic one whose objective function is (8) and whose constraint set is defined by (11) and

$$\bar{z}_t = z_t^{\bar{1}} + K_t^o \bar{q}_{t-1} + L_t \bar{x}_t. \tag{12}$$

The reduced problem is constructed by approximating the control constraints (10) by

$$\bar{x}_t \in S_t(\bar{q}_{t-1}) \tag{13}$$

and assuming that $E\{\max(z_t,0)\}$ is a function of $\bar{z}_t$ if the dispersion matrix of z_t, Disp (z_t), were known. It is therefore defined by the objective function (8) and the constraints (11), (12) and (13). i.e. that problem which would be obtained from the full problem defined by (1) - (7) by replacing the random variables with their expected values in all rows except (5). Notice that the objective function value that would be returned by a solution to the reduced problem will always be greater than that returned by a solution to the full problem. The evaluation of $E\{\max(z_t,0)\} = \bar{w}_t$ requires further analysis.

$$\begin{aligned} &\text{Let } \sigma_{it} = \sqrt{(\text{Disp } (z_t))_{ii}} \\ &\text{and } \xi_{it} = (z_{it} - \bar{z}_{it})/\sigma_{it}. \end{aligned} \tag{14}$$

Then ξ_{it} is a zero mean, unit variance random variable. Now

$$E(w_{it}) = E\{\max(z_{it}, 0)\}$$

$$= \sigma_{it}\, E\{\max(\xi_{it}, -\bar{z}_{it}/\sigma_{it})\} + \bar{z}_{it}$$

$$= \sigma_{it}\, f_{it}(z_{it}/\sigma_{it}), \tag{15}$$

where f_{it} is a real function defined by

$$f_{it}(x) = \int_{-x}^{\infty} (\xi + x)\, dG_{it}(\xi), \tag{16}$$

G_{it} being the distribution function of ξ_{it}.

Defining the function $f_t : \mathbb{R}^n \to \mathbb{R}^n$, (where n is the dimension of z_t) by

$$(f_t(x))_i = f_{it}(x_i),$$

the expected value of w_t may be written as

$$E(w_t) = \Sigma_t\, f_t(\Sigma_t^{-1}\, \bar{z}_t) \tag{17}$$

where Σ_t is a diagonal matrix whose i th diagonal term is σ_{it}.

The evolution equations may now be written in terms of the expected values of q_t, x_t and y_t:

$$\begin{aligned} \bar{q}_t &= q_t^o + A_t\, \bar{q}_{t-1} + B_t\, \bar{x}_t + C_t\, \bar{w}_t + D_t \bar{z}_t \\ \bar{w}_t &= \Sigma_t\, f_t(\Sigma_t^{-1} \bar{z}_t) \\ \bar{z}_t &= z_t^o + K_t\, \bar{q}_{t-1} + L_t\, \bar{x}_t - \bar{y}_t. \end{aligned} \tag{18}$$

In the additive case where Disp (y_t) is fixed Σ_t is treated as a fixed matrix, whereas in the multiplicative case it will be treated as being a linear function of y_t.

Summarising, the reduced problem involving only the expected values of the random variables has been derived from the full problem involving the random variables themselves. The constraints of the reduced problem are implied by those of the full problem. Explicitly stated the reduced problem is:

$$\text{Maximise} \quad \sum_t (d_{1t}^T \bar{q}_t + d_{2t}^T \bar{x}_t)$$

subject to the following constraints for $t = 1,2,\ldots,T$

$$r_t = R_{1t} \bar{q}_{t-1} + R_{2t} \bar{x}_t$$

$$\bar{y}_t = y_t^o + M_t \bar{q}_{t-1}$$

$$\bar{q}_t = q_t^o + A_t \bar{q}_{t-1} + B_t \bar{x}_t + C_t \bar{w}_t + D_t \bar{z}_t$$

$$\bar{w}_t = \Sigma_t f_t (\Sigma_t^{-1} \bar{z}_t)$$

$$\bar{z}_t = z_t^o + K_t \bar{q}_{t-1} + L_t \bar{x}_t - \bar{y}_t$$

and $\quad \bar{x}_t \geq 0.$

Unfortunately both Σ_t and f_t are, in general, unknown. In the sequence of restricted reduced problems derived from the problem above, it was decided to approximate the distribution of z_t by that of a multivariate normal distribution, whence $f_{it}(x)$ is approximated by $\phi(x) + x\Phi(x)$, where ϕ,Φ are the Gaussian probability density function and distribution function respectively.

Estimation of the standard deviation of z_{it} is more difficult, and it is this quantity that the sequence of restricted reduced problems progressively estimates. Information provided by the solution to one enables a better estimate of the standard deviation of z_{it}, and hence Σ_t, to be made for the formulation of the next.

3.2. Randomised Decision Rules

Above it has been implicitly assumed that the control decisions x_t are deterministic functions of the preceeding state space realisations q_{t-1}. That this is not a restrictive assumption in the sense that optimal decision rules may always be taken to have this property is shown below. Although it is intuitively reasonable that this should be so, it is not entirely obvious. Moreover the exclusion of randomised decision rules is necessary for the reduced problem to be a realistic reflection of the full one.

Suppose that randomised decision rules are allowed in the full problem, in particular consider the effect of making x_t a randomised function of q_{t-1}. The decision x_t will not affect the process, particularly the revenue acquired, up to time period t. So for a given value of the state vector at the start of time period t, q_{t-1}, consider the maximum expected revenue acquired from time period t onwards, given that the non-randomised decision x_t is made at the start of time period t. Denote it by $V^t(x_t)$.

Now suppose that x_t^R is a randomised decision rule defined by

$$P\{x_t^R = x_t^{(i)}\} = \lambda_i, \quad \sum_i \lambda_i = 1, \ \lambda_i > 0 \quad \forall i.$$

If x_t^R is to be feasible then each point in the decision space $x_t^{(i)}$ must be feasible. It is necessary to show that there exists a feasible point in the decision space x_t^o such that

$$V^t(x_t^o) \geq E\ V^t(x_t^R).$$

But this is easy for set $x_t^o = x_t^{(j)}$, where $x_t^{(j)}$ is such that $V^t(x_t^{(j)}) \geq V^t(x_t^{(i)})$ for all i.

Then

$$\begin{aligned} E\ V^t(x_t^R) &= \sum_i \lambda_i V^t(x_t^{(i)}) \\ &\leq \sum_i \lambda_i V^t(x_t^o) \\ &= V^t(x_t^o). \end{aligned}$$

x_t^o is feasible since all the $x_t^{(i)}$ are feasible and the assertion proved for discrete randomised decision rules. The argument can easily be extended to cover general randomised decision rules.

The consequence of this result is that there is no merit in considering randomised decision rules, so without incurring any sub-optimality the decision x_t can be taken to be a deterministic function of q_{t-1}.

3.3. Convexity and Stochastic Linear Programming

In this section a condition on the reduced problem will be derived, which when satisfied ensures that its objective function is concave in the decision variables and moreover ensures that its constraints can be replaced by equivalent ones which define a convex feasible set. Moreover, this same condition when satisfied ensures that the full problem is a stochastic linear program in the sense that it reduces to a linear program as the random variation in the stochastic input tends to zero. In order to derive this condition it will be assumed that the probability density function of z_t exists.

The reduced problem may be written as:

$$\text{Maximise} \quad \sum_t (d_{1t}^T \bar{q}_t + d_{2t}^T \bar{x}_t) \tag{1}$$

$$\begin{aligned} \text{s.t.} \quad & r_t = R_{1t} \bar{q}_{t-1} + R_{2t} \bar{x}_t \\ & \bar{q}_t = q_t^1 + A_t^o \bar{q}_{t-1} + B_t^o \bar{x}_t + C_t \bar{w}_t \\ & \bar{w}_t = \Sigma_t f_t(\Sigma_t^{-1} \bar{z}_t) \\ & \bar{z}_t = z_t^1 + K_t^o \bar{q}_{t-1} + L_t \bar{x}_t \end{aligned} \tag{2}$$

$$\text{and} \quad \bar{x}_t \geq 0 \text{ for all } t.$$

$$\begin{aligned} \text{where} \quad & z_t^1 = z_t^o - y_t^o \\ & q_t^1 = q_t^o + D_t z_t^1 \\ & K_t^o = K_t - M_t \\ & A_t^o = A_t + D_t K_t^o \end{aligned}$$

$$\text{and} \quad B_t^o = B_t + D_t L_t.$$

For convenience define V_t and $V^{(t)}$ by

$$V_t = d_{1t}\,\bar{q}_t + d_{2t}\,\bar{x}_t \tag{3}$$

and $$V^{(t)} = \sum_{u \geq t} V_u. \tag{4}$$

Consider the system at the start of time period t, and the process from time period t until the time horizon. Now $\bar{q}_u$ is a fixed function of $\bar{q}_{u-1}$ and $\bar{x}_u$. Regard $V^{(t)}$ as a function of $\bar{q}_{t-1}$ and $\bar{x}_t$. It is necessary to provide conditions under which $V^{(t)}$ is a concave function of $\bar{x}_t$. This can be achieved by assuming that $V^{(t+1)}$ is a concave function of $\bar{q}_t$ for fixed $\bar{x}_u$ for $u \geq t + 1$, and providing conditions for which:

(i) $V^{(t)}$ is concave in $\bar{q}_{t-1}$ for fixed $\bar{x}_u$, $u \geq t$, and

(ii) $V^{(t)}$ is concave in $\bar{x}_t$ for fixed $\bar{q}_{t-1}$ and $\bar{x}_u$, $u \geq t+1$.

This is because $V^{(T+1)} = V_{T+1}$ can be regarded as existent but identically zero and hence concave in $\bar{q}_T$. Condition (ii) is precisely the concavity condition, whilst condition (i) is necessary for the backwards induction.

Let $$g_t(\bar{q}_t) = d_{1t}^T\,\bar{q}_t + V^{(t+1)}(\bar{q}_t)\,. \tag{5}$$

Since $V^{(t+1)}$ is a function of $\bar{q}_t$, then

$$V^{(t)} = d_{2t}^T\,\bar{x}_t + g_t(\bar{q}_t). \tag{6}$$

Condition (i) is satisfied by g_t being concave in $\bar{q}_{t-1}$ i.e. $d^2 g_t / d\bar{q}_{t-1}^2$ being negative semi-definite. (It will be shown that the second derivatives exist because, by assumption, the probability

density function of z_t exists). Condition (ii) is satisfied by g_t being concave in $\bar{x}_t$ i.e. $d^2g_t/d\bar{x}_t^2$ being negative semi-definite. The former is investigated first.

$$\text{Now } \partial g_t/\partial \bar{q}_{it-1} = \sum_k \partial \bar{q}_{kt}/\partial \bar{q}_{it-1} \cdot \partial g_t/\partial \bar{q}_{kt}, \tag{7}$$

$$\text{so } \partial^2 g_t/\partial \bar{q}_{it-1}\partial \bar{q}_{jt-1} = \sum_k \partial^2 \bar{q}_{kt}/\partial \bar{q}_{it-1}\partial \bar{q}_{jt-1} \cdot \partial g_t/\partial \bar{q}_{kt} + \sum_{k,\ell} \partial \bar{q}_{kt}/\partial \bar{q}_{it-1} \cdot \partial \bar{q}_{\ell t}/\partial \bar{q}_{jt-1} \cdot \partial^2 g_t/\partial \bar{q}_{\ell t}\partial \bar{q}_{kt}. \tag{8}$$

$$\text{For convenience let } (J)_{ij} = \partial \bar{q}_{jt}/\partial \bar{q}_{it-1} \tag{9}$$

$$\text{and } (H)_{ij} = \partial^2 g_t/\partial \bar{q}_{it}\partial \bar{q}_{jt}. \tag{10}$$

Then by assumption, $H = (H_{ij})$ is negative semi-definite. The second term on the right-hand side of (8) is

$$J^T H J \tag{11}$$

which is negative semi-definite, because for any vector y,

$$y^T J^T H J y = (Jy)^T H (Jy)$$
$$\leq 0,$$

because H is negative semi-definite.

Attention is now directed to the first term on the right-hand side of (8), namely

$$\sum_k \partial^2 \bar{q}_{kt} / \partial \bar{q}_{it-1} \partial \bar{q}_{jt-1} \cdot \partial g_t / \partial \bar{q}_{kt} . \tag{12}$$

Now $\partial \bar{q}_{kt} / \partial \bar{q}_{it-1} = (A_t^o)_{ki} + \sum_\ell (C_t)_{k\ell} \ \partial \bar{w}_{\ell t} / \partial \bar{q}_{it-1}$ (13)

and $\bar{w}_{\ell t} = \sigma_{\ell t} \ f_{\ell t}(\bar{z}_{\ell t} / \sigma_{\ell t})$,

where $\sigma_{\ell t}$ and $f_{\ell t}$ are as defined in Section 3.1.

Whence

$$\partial \bar{w}_{\ell t} / \partial \bar{q}_{it-1} = f'_{\ell t}(\bar{z}_{\ell t} / \sigma_{\ell t}) . (K_t^o)_{\ell i} + \tilde{f}_{\ell t}(\bar{z}_{\ell t} / \sigma_{\ell t}) . (\partial \sigma_{\ell t} / \partial \bar{q}_{it-1}), \tag{14}$$

where $\tilde{f}_{\ell t}(x) = \int_{-x}^{\infty} \xi \, dG_{\ell t}(\xi)$,

$G_{\ell t}(\cdot)$ being the distribution function of $(z_{\ell t} - \bar{z}_{\ell t}) / \sigma_{\ell t}$.

For convenience set $(M_t^o)_{\ell i} = \partial \sigma_{\ell t} / \partial \bar{q}_{it-1}$. For the additive model of the stochastic input this will be zero, but it need not be for the multiplicative model.

So differentiating (13) with respect to $\bar{q}_{jt-1}$ it is seen that

$$\partial^2 \bar{q}_{kt} / \partial \bar{q}_{it-1} \partial \bar{q}_{jt-1} = \sum_\ell (C_t)_{k\ell} \ \partial^2 \bar{w}_{\ell t} / \partial \bar{q}_{it-1} \partial \bar{q}_{jt-1} . \tag{15}$$

Differentiating (14) with respect to $\bar{q}_{jt-1}$ it is seen that

$$\partial^2 \bar{w}_{\ell t} / \partial \bar{q}_{it-1} \partial \bar{q}_{jt-1} = (J_t)_{\ell i} (J_t)_{\ell j} . \xi_{\ell t}, \tag{16}$$

where $(J_t)_{ij}$ is $(K_t^o)_{ij} - (\bar{z}_{it} / \sigma_{it})(M_t^o)_{ij}$ (17)

and $\xi_{\ell t} = (1/\sigma_{\ell t}) dG_{\ell t}(-\bar{z}_{it}/\sigma_{it})$. (18)

$dG_t(\cdot)$ exists as a function by assumption, since it is the probability density function of $(z_{it}-\bar{z}_{it})/\sigma_{it}$.

Writing

$$\eta_{kt}(\bar{q}_t) = \partial g_t(\bar{q}_t)/\partial \bar{q}_{kt}$$

it is seen that (12) is:

$$\sum_k \sum_\ell (C_t)_{k\ell}\, \xi_{\ell t}\, (J_t)_{\ell i}\, (J_t)_{\ell j}\, \eta_{kt} \,. \tag{19}$$

Since $\xi_{\ell t}$ is always non-negative, the matrix whose $(i,j)\underline{\text{th}}$ component is (19) will be negative semi-definite if

$$\sum_k \eta_{kt}(C_t)_{k\ell} \le 0 \text{ for all } \ell.$$

Setting η_t to be the vector whose k $\underline{\text{th}}$ component is η_{kt}, i.e. $\eta_t = dg_t/d\bar{q}_t$, it is seen that the required condition is

$$\eta_t(\bar{q}_t)^T C_t \le 0. \tag{20}$$

A similar argument reveals that condition (ii), i.e. $V^{(t)}$ being concave in $\bar{x}_t$ for fixed $\bar{q}_{t-1}$ and $\bar{x}_u$, $u \ge t+1$ is satisfied by the same condition.

So if (20) is satisfied for all t, then in the reduced problem the objective function is concave in the decision variables. The vector η_t deserves further discussion. It is equivalently defined by

$$\eta_t = dV^{(t)}/d\bar{q}_t,$$

so its i th component may be regarded as the value of the i th component of $\bar{q}_t$ in the t th time period. For example, if $\bar{q}_{it}$ represented the stock level of item i at the end of time period t, then η_{it} would be the unit value of item i at the end of that period after the t th time period stockholding costs had been subtracted.

For the above condition to be useful, it is necessary to show that the set of feasible conditions is convex. Unfortunately, in general the constraint equations (2) do not define such a convex set. However, under the convexity condition (20) it will be shown that they can be replaced by an equivalent set of constraints which do. If the problem defined by the equivalent set of constraints is called the revised reduced problem, then under the concavity condition the optimal solution to the reduced problem is the optimal solution to the revised reduced problem and vice-versa, so both problems can be regarded as being equivalent.

Although it is not possible to determine η_t before the reduced problem is solved, it is usually possible to intuitively put reliable bounds on it. An easy method of checking that the convexity condition holds in practice is given in Section 6.4.

The revised reduced problem is the same as the reduced problem except that the evolution equation

$$\bar{w}_t = \Sigma_t\, f_t(\Sigma_t^{-1}\bar{z}_t)$$

of the reduced problem is replaced by

$$\bar{w}_t \geq \Sigma_t f_t(\Sigma_t^{-1} \bar{z}_t). \tag{21}$$

This will make no difference to the problem if there is no increase in profit made by increasing $\bar{w}_t$ from $\Sigma_t f_t(\Sigma_t^{-1} \bar{z}_t)$.

$$\text{i.e. if } \partial g_t/\partial \bar{w}_{it} \leq 0 \text{ for all } i \tag{22}$$

where the decisions $\bar{x}_u$, $u \geq t$ are held constant. Now

$$\partial g_t/\partial \bar{w}_{it} = \sum_k \partial g_t/\partial \bar{q}_{kt} \cdot \partial \bar{q}_{kt}/\partial \bar{w}_{it}$$

and $\partial \bar{q}_{kt}/\partial \bar{w}_{it}$ is simply $(C_t)_{ki}$, and also $\partial g_t/\partial \bar{q}_{kt}$ is η_{kt}.

$$\text{i.e. } \partial g_t/\partial \bar{w}_{it} = \sum_k \eta_{kt}(C_t)_{ki},$$

so inequality (22) will be satisfied if $\eta_t^T C_t \leq 0$, which is precisely the convexity condition (20). So under the convexity condition the revised reduced problem is equivalent to the reduced problem. It is the former that is now discussed.

It is shown that the revised reduced problem is convex by expressing it as the limit of a sequence of linear programs P_K as $K \to \infty$. The linear program P_K is constructed as follows.

Let $k^{(0)}, k^{(1)}, k^{(2)}, \ldots, k^{(K)}$ be a sequence of real numbers between 0 and 1 such that

$$k^{(i)} = (i+1)/(K+2). \tag{23}$$

Consider the i th component of inequality (21). It is

$$\bar{w}_{it} \geq \sigma_{it}\, f_{it}(\bar{z}_{it}/\sigma_{it}). \tag{24}$$

Now $f_{it}'(x) = 1 - G_{it}(-x)$, where G_{it} is the distribution function of $(z_{it} - \bar{z}_{it})/\sigma_{it}$, which is continuous by assumption. Define the sequence of real numbers $\ell_{it}^{(0)}, \ell_{it}^{(1)}, \ell_{it}^{(2)}, \ldots, \ell_{it}^{(K)}$ by

$$1 - G_{it}(-\ell_{it}^{(j)}) = k^{(j)},\ j = 0,1,2,\ldots,K \tag{25}$$

Then $\ell_{it}^{(0)} < \ell_{it}^{(1)} < \ldots < \ell_{it}^{(K)}$.

Define the set of constants $\alpha_{it}^{(j)}, \beta_{it}^{(j)}$, $j = 1,2,\ldots,K$ by

$$\alpha_{it}^{(j)}\, x + \beta_{it}^{(j)} = f_{it}(x)$$

when $x = \ell_{it}^{(j-1)}$ and $x = \ell_{it}^{(j)}$.

The linear program P_K is constructed from the revised reduced problem by replacing (24) by the K linear constraints

$$\bar{w}_{it} \geq \alpha_{it}^{(j)}\, \bar{z}_{it} + \beta_{it}^{(j)}\, \sigma_{it}\,. \tag{26}$$

Since f_{it} is convex (because $f_{it}''(x) = dG_{it}(-x)$, which is a function by assumption, is always non-negative) the maximum error caused by this replacement will be less than

$$(\ell_{it}^{(j)} - \ell_{it}^{(j-1)})\,(k^{(j)} - k^{(j-1)})$$

which tends to zero as $K \to \infty$.

That (26) is linear follows from σ_{it} being either constant (in the additive case) or a linear function of $\bar{q}_{t-1}$ (in the multiplicative case).

The feasible set defined by the constraints of P_K is convex for all K and furthermore converges to the feasible set of the revised reduced problem as $K \to \infty$. Hence the feasible set defined by the constraints of the revised reduced problem is convex.

To see that the convexity condition (20) is sufficient to ensure that the full problem is a stochastic linear program i.e. that it reduces to a linear program as the stochastic variation of the random input tends to zero, observe that as the variability of the stochastic input tends to zero, all the random variables in the problem converge in probability to their expected values and so the reduced problem converges to the full one. Under condition (20) the constraint

$$\bar{w}_t = E\{\max(z_t, 0)\}$$

can then be replaced by the two equivalent linear constraints

$$\bar{w}_t \geq \bar{z}_t$$

and $$\bar{w}_t \geq 0,$$

whence the reduced problem and therefore the full problem become ordinary linear programs.

4. THE RESTRICTED REDUCED PROBLEM

4.1. Formulation of the Restricted Reduced Problem

The reduced problem is a non-linear program derived from the original full stochastic problem which involves only the expected values of the random variables, rather than the random variables themselves. Unfortunately its formulation requires a knowledge of the distribution of z_t, and this is, in general, not available. This problem is circumvented by making some approximations and formulating a series of restricted reduced problems, the solution to each providing information necessary for the formulation of the next. Under the approximations the solutions to the restricted reduced problems tend to the solution of the reduced problem. The apparent computational labouriousness of this approach is mitigated substantially by the advantage that can be taken of the similarity between the restricted reduced problems. In order to formulate the restricted reduced problem, a first approximation is necessary. It is:

Approximation 1: that z_t can be treated as a multivariate normal random vector.

This approximation is reasonable since exact information about even the distribution of the stochastic input y_t, is in practice, very unlikely to be available; characterisation by mean and dispersion matrix is the most that can be expected. So calculations with the precise distributions of z_t assuming that the stochastic input is precisely multivariate normal may be unhelpful. In practical applications where the probability that w_{it} is zero is small, z_{it} will be very nearly normal.

Having made this approximation, the constraint

$$\bar{w}_{it} = \sigma_{it} f_{it}(\bar{z}_{it}/\sigma_{it})$$

of the reduced model can be replaced by the constraint

$$\bar{w}_{it} = \sigma_{it} f_o(\bar{z}_{it}/\sigma_{it}) \tag{1}$$

where $\sigma_{it}^2 = \text{var}(z_{it})$ and

$$f_o(x) = \int_{-x}^{\infty} (\xi + x) d\Phi(\xi)$$

$$= \phi(x) + x\Phi(x), \tag{2}$$

ϕ, Φ being the Gaussian probability density function and distribution function respectively.

Defining f to be a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ by

$$(f(x))_i = f_o(x_i)$$

the restricted reduced problem can now be written:

$$\text{Maximise} \sum_t d_{1t}^T \bar{q}_t + d_{2t}^T \bar{x}_t \tag{3}$$

subject to:

$$r_t = R_{1t}\bar{q}_{t-1} + R_{2t} \bar{x}_t \tag{4}$$

$$\bar{y}_t = y_t^o + M_t \bar{q}_{t-1} \tag{5}$$

$$\bar{q}_t = q_t^o + A_t \bar{q}_{t-1} + B_t \bar{x}_t + C_t \bar{w}_t + D_t \bar{z}_t \tag{6}$$

$$\bar{z}_t = z_t^o + K_t \bar{q}_{t-1} + L_t \bar{x}_t - \bar{y}_t \tag{7}$$

$$\bar{w}_t = \Sigma_t f(\Sigma_t^{-1} \bar{z}_t) \tag{8}$$

$$\bar{x}_t \geq 0. \tag{9}$$

for $t = 1,2,\ldots,T$.

where Σ_t is a diagonal matrix whose i th diagonal component is σ_{it}.

4.2. The Separable Program

Separable programming versions can now be given of the restricted reduced problem. The problems corresponding to the two models of the stochastic input now have to be carefully distinguished. The additive one is treated first

(a) The additive case.

This is fairly straightforward. It will be expedient to let $v_{it}^{(k)}$ for $k = 1,2,\ldots,N$ be a grid of points for $\bar{z}_{it}/\sigma_{it}$. Let $f_{it}^{(k)} = f_o(v_{it}^{(k)})$. Introduce the separable variables $\lambda_{it}^{(k)}$ and for convenience define the vectors $\lambda_t^{(k)}$, $f_t^{(k)}$, $v_t^{(k)}$ and matrix $\Lambda_t^{(k)}$ by

$$(\lambda_t^{(k)})_i = \lambda_{it}^{(k)}$$

$$(f_t^{(k)})_i = f_{it}^{(k)}$$

$$(v_t^{(k)})_i = v_{it}^{(k)}$$

and $$(\Lambda_t^{(k)})_{ij} = \begin{cases} \lambda_{it}^{(k)} & i = j \\ 0 & i \neq j. \end{cases}$$

Then the non-linear equation (8) becomes

$$\bar{w}_t = \sum_k \Lambda_t^{(k)} f_t^{(k)} \qquad (10)$$

$$\bar{z}_t = \sum_k \Lambda_t^{(k)} v_t^{(k)} \tag{11}$$

and $$\sigma_t = \sum_k \lambda_t^{(k)} \tag{12}$$

where $(\sigma_t)_i = \sigma_{it}$.

The separable linear programming formulation of the problem thus obtained is:

$$\text{Maximise} \sum_t (d_{1t}^T \bar{q}_t + d_{2t}^T \bar{x}_t) \tag{13}$$

over $\bar{q}_t$, $\bar{x}_t$ and $\lambda_t^{(k)}$,

subject to:

$$\begin{aligned}
r_t &= R_{1t} \bar{q}_{t-1} + R_{2t} \bar{x}_t \\
\bar{y}_t &= y_t^o + M_t \bar{q}_{t-1} \\
\bar{q}_t &= q_t^o + A_t \bar{q}_{t-1} + B_t \bar{x}_t + C_t \bar{w}_t + D_t \bar{z}_t \\
\bar{z}_t &= z_t^o + K_t \bar{q}_{t-1} + L_t \bar{x}_t - \bar{y}_t \\
\bar{w}_t &= \sum_k \Lambda_t^{(k)} f_t^{(k)} \\
\bar{z}_t &= \sum_k \Lambda_t^{(k)} v_t^{(k)} \\
\sigma_t &= \sum_k \lambda_t^{(k)} \\
\bar{x}_t &\geq 0.
\end{aligned} \tag{14}$$

In practice, $\bar{y}_t$, $\bar{w}_t$ and $\bar{z}_t$ would be eliminated to give the constraints

$$
\left.
\begin{aligned}
r_t &= R_{1t}\,\bar{q}_{t-1} + R_{2t}\,\bar{x}_t \\
\bar{q}_t &= q_t^1 + A_t^o\,\bar{q}_{t-1} + B_t^o\,\bar{x}_t + \sum_k C_t\,\Lambda_t^{(k)}\,f_t^{(k)} \\
\sum_k \Lambda_t^{(k)}\,v_t^{(k)} &= z_t^1 + K_t^o\,\bar{q}_{t-1} + L_t\,\bar{x}_t \\
\sigma_t &= \sum_k \lambda_t^{(k)}, \quad \bar{x}_t \geq 0.
\end{aligned}
\right\} \qquad (15)
$$

where $z_t^1 = z_t^o - y_t^o$

$q_t^1 = q_t^o + D_t\,z_t^1$

$K_t^o = K_t - M_t$

$A_t^o = A_t + D_t K_t^o$

and $B_t^o = B_t + D_t\,L_t$.

The grid $\{v_t^{(k)};\ k = 1,2,\ldots,N\}$ would be refined under the usual interpolation procedure. If the convexity condition derived in Section 3.3 is satisfied then no additional constraint proscribing $\lambda_{it}^{(k)}$'s non-adjacent in k from being positive is required. However if it does not hold then such a constraint must be added. This is commonplace in non-linear programming and standard facilities are available in most good mathematical programming codes.

The above separable program requires values for the constraints σ_t, which are initially unknown. A procedure for estimating σ_t is derived in Sections 4.3 and 4.4 below after derivation of the separable program for the multiplicative model.

(b) The Multiplicative Case.

In the multiplicative case the standard deviation of each component of the stochastic input is directly proportional to its mean. To handle this in the formulation of a separable programming version of the restricted reduced problem the following approximation is used.

The multiplicative approximation: that the standard deviation of each component of z_t can be taken to be directly proportional to the mean of the corresponding component of y_t, i.e.

$$\text{i.e. } \sigma_{it} = \tau_{it} \bar{y}_{it} \tag{16}$$

for some constant τ_{it}.

Because of the iterative scheme of restricted reduced problems this approximation is not as restrictive as it first appears. There is a full discussion of this in Section 4.3 below.

The analysis now proceeds in a very similar way to that for the additive model. Only minor modifications are needed and these are given below.

For $q_{t-1} = \bar{q}_{t-1}$, the stochastic input y_t is distributed as

$$N(\bar{y}_t, \bar{Y}_t P_t^o \bar{Y}_t) \tag{17}$$

where $$\bar{y}_t = y_t^o + M_t \bar{q}_{t-1} \tag{18}$$

and $\bar{Y}_t$ is a diagonal matrix whose i th diagonal component is $\bar{y}_{it}$.

By the multiplicative approximation, z_{it} is distributed as:

$$N(\bar{z}_{it}, \bar{y}_{it}^{\,2}\,\tau_{it}^{\,2}). \tag{19}$$

So the analysis proceeds as before with $(\bar{y}_{it}\,\tau_{it})$ replacing (σ_{it}). In particular equation (8) becomes:

$$w_t = \bar{Y}_t\,T_t\;\; f\,(T_t^{-1}\,\bar{Y}_t^{-1}\,\bar{z}_t), \tag{20}$$

where T_t is a diagonal matrix whose i <u>th</u> diagonal entry is τ_{it}.

When introducing the separable variables, $\lambda_{it}^{(k)}$, the $v_{it}^{(k)}$ must be a grid of points for $\bar{z}_{it}/(\bar{y}_{it}\,\tau_{it})$. The equations involving the separable variables, namely equations (10) - (12) therefore become:

$$\bar{w}_t = \sum_k \Lambda_t^{(k)}\, f_t^{(k)} \tag{21}$$

$$\bar{z}_t = \sum_k \Lambda_t^{(k)}\, v_t^{(k)} \tag{22}$$

$$\text{and } \bar{Y}_t\,\tau_t = \sum_k \lambda_t^{(k)}\,. \tag{23}$$

The separable program for the restricted reduced problem in the multiplicative case can now be explicitly stated:

$$\text{Maximise} \sum_t (d_{1t}^{T}\,\bar{q}_t + d_{2t}^{T}\,\bar{x}_t)$$

over $\bar{q}_t$, $\bar{x}_t$ and $\lambda_t^{(k)}$ subject to:

$$
\begin{aligned}
r_t &= R_{1t}\,\bar{q}_{t-1} + R_{2t}\,\bar{x}_t \\
\bar{y}_t &= y_t^o + M_t\,\bar{q}_{t-1} \\
\bar{q}_t &= q_t^o + A_t\,\bar{q}_{t-1} + B_t\,\bar{x}_t + C_t\,\bar{w}_t + D_t\,\bar{z}_t \\
\bar{z}_t &= z_t^o + K_t\,\bar{q}_{t-1} + L_t\,\bar{x}_t - \bar{y}_t \\
\bar{w}_t &= \sum_k \Lambda_t^{(k)}\, f_t^{(k)} \qquad (24) \\
\bar{z}_t &= \sum_k \Lambda_t^{(k)}\, v_t^{(k)} \\
\bar{Y}_t\,\tau_t &= \sum_k \lambda_t^{(k)} \\
\bar{x}_t &\geq 0 \ .
\end{aligned}
$$

Again, in practice $\bar{y}_t$, $\bar{Y}_t$, $\bar{w}_t$ and $\bar{z}_t$ would be eliminated by substitution to give the constraints

$$
\begin{aligned}
r_t &= R_{1t}\,\bar{q}_{t-1} + R_{2t}\,\bar{x}_t \\
\bar{q}_t &= q_t^1 + A_t^o\,\bar{q}_{t-1} + B_t^o\,\bar{x}_t + \sum_k C_t\,\Lambda_t^{(k)}\, f_t^{(k)} \\
\sum_k \Lambda_t^{(k)}\, v_t^{(k)} &= z_t^1 + K_t^o\,\bar{q}_{t-1} + L_t\,\bar{x}_t \\
\sum_k \lambda_t^{(k)} &= T_t\,y_t^o + T_t\,M_t\,\bar{q}_{t-1} \\
\bar{x}_t &\geq 0
\end{aligned}
$$

where q_t^1, z_t^1, A_t^o, B_t^o and K_t^o are as for the additive case and the grid $\{v_t^{(k)},\ k = 1,\ldots,N\}$ would be successively refined.

$$
\begin{aligned}
r_t &= R_{1t}\,\bar{q}_{t-1} + R_{2t}\,\bar{x}_t \\
\bar{y}_t &= y_t^o + M_t\,\bar{q}_{t-1} \\
\bar{q}_t &= q_t^o + A_t\,\bar{q}_{t-1} + B_t\,\bar{x}_t + C_t\,\bar{w}_t + D_t\,\bar{z}_t \\
\bar{z}_t &= z_t^o + K_t\,\bar{q}_{t-1} + L_t\,\bar{x}_t - \bar{y}_t \\
\bar{w}_t &= \sum_k \Lambda_t^{(k)}\, f_t^{(k)} \qquad (24) \\
\bar{z}_t &= \sum_k \Lambda_t^{(k)}\, v_t^{(k)} \\
\bar{Y}_t\,\tau_t &= \sum_k \lambda_t^{(k)} \\
\bar{x}_t &\geq 0 \ .
\end{aligned}
$$

Again, in practice $\bar{y}_t$, $\bar{Y}_t$, $\bar{w}_t$ and $\bar{z}_t$ would be eliminated by substitution to give the constraints

$$
\begin{aligned}
r_t &= R_{1t}\,\bar{q}_{t-1} + R_{2t}\,\bar{x}_t \\
\bar{q}_t &= q_t^1 + A_t^o\,\bar{q}_{t-1} + B_t^o\,\bar{x}_t + \sum_k C_t\,\Lambda_t^{(k)}\, f_t^{(k)} \\
\sum_k \Lambda_t^{(k)}\, v_t^{(k)} &= z_t^1 + K_t^o\,\bar{q}_{t-1} + L_t\,\bar{x}_t \\
\sum_k \lambda_t^{(k)} &= T_t\,y_t^o + T_t\,M_t\,\bar{q}_{t-1} \\
\bar{x}_t &\geq 0
\end{aligned}
$$

where q_t^1, z_t^1, A_t^o, B_t^o and K_t^o are as for the additive case and the grid $\{v_t^{(k)},\ k = 1,\ldots,N\}$ would be successively refined.

The same remarks about the concavity of the objective function apply as for the additive case. The τ_t's must be estimated, and a suitable procedure enabling this to be done is given and discussed in Sections 4.3 and 4.4 below.

4.3. The Iterative Approach

In the formulation of the restricted reduced problems above the terms σ_t were still unknown. This difficulty is tackled by the provision of a procedure by which σ_t is estimated. This procedure, however, requires information about the process gained from a solution to the restricted reduced model in the first place.

So the technique is an iterative one. An initial estimate of σ_t, $\sigma_t^{(0)}$, for each t is made and the restricted reduced problem solved with these values of the σ_t's. From this solution a second estimate, $\sigma_t^{(1)}$ is made for σ_t and the restricted reduced problem re-solved with this new value of σ_t. This enables a third estimate, $\sigma_t^{(2)}$, of σ_t to be made, and so on.

This is conceptually straightforward in the case of the additive model. In the multiplicative case, however, the τ_t's have to be re-estimated and these are really functions of the $\bar{y}_t$'s, but have been treated as though they were constant by the multiplicative approximation. Thus the approximation can be seen to be one of making $\tau_t^{(n)}$ a function of $\bar{y}_t^{(n-1)}$ (where $\bar{y}_t^{(n-1)}$ is the value of $\bar{y}_t$ obtained from the solution to the n th restricted reduced problem) rather than of $\bar{y}_t^{(n)}$.

The initial estimate of σ_t, $\sigma_t^{(0)}$ is obtained by setting σ_{it} to their minimum possible values that would be consequent on q_{t-1} being fixed rather than a random variable. So in the additive case the initial value of σ_{it} is

$$\sigma_{it}^{(0)} = \sqrt{(\text{Disp } (y_t))_{ii}}$$

whilst in the multiplicative the initial estimate of τ_t is

$$\tau_{it}^{(0)} = \sqrt{(\text{Disp } (y_t))_{ii}}/\bar{y}_{it},$$

which in this case is constant in $\bar{y}_t$.

Thus the initial restricted reduced problem is that which would be obtained by replacing $\bar{q}_t$, $\bar{x}_t$ and $\bar{y}_t$ by q_t^*, x_t^* and y_t^*, where

$$q_t^* = E(q_t | q_{t-1} = q_{t-1}^*)$$

$$x_t^* = E(x_t | q_{t-1} = q_{t-1}^*)$$

and $$y_t^* = E(y_t | q_{t-1} = q_{t-1}^*).$$

The re-evaluation procedures for the σ_t's and τ_t's are given below. The additive and multiplicative models are treated separately.

5. THE RE-FORMULATION OF THE RESTRICTED REDUCED PROBLEM

5.1. The Additive Case.

In this section a procedure for the re-evaluation of σ_t is derived for the problem with the additive model of the stochastic input. Modifications necessary for the multiplicative model are given subsequently. One further approximation has been found necessary, and it is clearly stated and justified with discussion as it is introduced. The time required to numerically execute the re-evaluation procedure expands linearly with the number of time periods in the model.

Since σ_{it} is the standard deviation of z_{it}, to estimate it it is necessary to study the distribution of z_t, and indeed to study the behaviour of all the random variables in the problem. To do this it is necessary to return to the evolution equations of the full problem. For convenience they are re-stated below.

$$q_t = q_t^o + A_t\, q_{t-1} + B_t x_t + C_t w_t + D_t z_t \qquad (1)$$

$$w_t = \max\,(z_t, 0) \qquad (2)$$

$$z_t = z_t^o + K_t\, q_{t-1} + L_t x_t - y_t \,. \qquad (3)$$

Under Approximation 1 , that z_t can be treated as if it has a multivariate normal distribution, it can be expressed by

$$z_t = \bar{z}_t + \mathrm{Std}(z_t).\varepsilon_t \qquad (4)$$

where $\mathrm{Std}(z_t)$ is the lower triangular decomposition of $\mathrm{Disp}\,(z_t)$ such that $\mathrm{Disp}\,(z_t) = \mathrm{Std}(z_t).(\mathrm{Std}(z_t))^T$ as described in Section 2.3, and ε_t is a vector of mutually independent Gaussian random variables.

Also q_t can be represented by

$$q_t = \bar{q}_t + \text{Std}(q_t).\xi_t \qquad (5)$$

where ξ_t is a random vector with zero mean and dispersion matrix equal to the identity matrix.

The method of re-evaluating σ_t is a recursive one and requires the estimation of Disp (z_1), Disp (q_1), Disp (z_2), Disp (q_2),... and so on. It will be shown that Disp (z_t) can be expressed in terms of Disp (y_t) and Disp (q_{t-1}), and also that Disp (q_t) can be expressed in terms of Disp (y_t), Disp (z_t) and Disp (q_{t-1}). But this requires one last approximation.

Approximation 2 : The control applied in each time period can be considered to be a linear function of the state space at the start of that time period.

$$\text{i.e.} \quad x_t = x_t^o + N_t\, q_{t-1} \qquad (6)$$

where x_t^o is a fixed vector and N_t a fixed matrix.

By the Markovian nature of the system, the decision rule x_t must be a function of the preceeding state space, q_{t-1}. Estimation of the random variation in z_t requires that x_u be assumed to be some known function of q_{u-1} for $u \leq t$; any improvement on the linear approximation would require x_u to be derived on a grid of points for q_{u-1}. This is required in any dynamic programming approach and is precisely the technique avoided here in the interest of computational tractability.

The assumption that the decision rule is linear is not as restrictive as it might first appear, for the approximation is only used for the re-estimation of σ_t and does not alter the structure of the non-linear program whose final solution provides the actual control decisions. Thus, the approximation does not restrict the controls provided by the algorithm to be linear decision rules.

An estimate of the linear coefficient matrix, N_t, can be obtained from an analysis of the last solution to the non-linear program by using "fictitious" variables in the manner described in Section 6.2. Better estimates of the N_t's can be obtained iteratively each time the non-linear program is solved with a better estimate of the σ_t's.

Now recall the model of the stochastic input. It is

$$y_t = \bar{y}_t + \text{Std}(y_t).\eta_t \tag{7}$$

$$\bar{y}_t = y_t^o + M_t\, q_{t-1} \tag{8}$$

and η_t is a vector of mutually independent Gaussian random variates.

Introduce some new notation. Define J_t by

$$J_t = K_t + L_t N_t - M_t, \tag{9}$$

then combining (3), (5), (6), (7), (8) and (9) it is seen that

$$\begin{aligned} z_t &= \bar{z}_t + J_t\, \text{Std}(q_{t-1}).\xi_{t-1} + \text{Std}(y_t).\eta_t \qquad (10)\\ &= \bar{z}_t + \text{Std}(z_t).\varepsilon_t. \end{aligned}$$

Whence, since ξ_{t-1} and η_t are independent:

$$\text{Disp}(z_t) = J_t \, \text{Disp}(q_{t-1}) J_t^T + \text{Disp}(y_t) \tag{11}$$

and $$\sigma_{it} = \sqrt{(\text{Disp}\,(z_t))_{ii}}$$

$$= \sqrt{\{ \sum_{jk} (J_t)_{ij} (\text{Disp}(q_{t-1}))_{jk} (J_t)_{ij} + (\text{Disp}(y_t))_{ii} \}} \tag{12}$$

So, having approximated x_t by a linear function of q_{t-1} an expression for $\text{Disp}(z_t)$ in terms of $\text{Disp}\,(q_{t-1})$ and $\text{Disp}\,(y_t)$ has been derived.

It remains to estimate $\text{Disp}\,(q_t)$. As with the estimation of $\text{Disp}\,(z_t)$, this is done recursively. However, the calculations are substantially more complicated since the linear analysis that has been used so far is no longer appropriate. For convenience, some of the calculations are deferred until Section 5.3 and only the results will be quoted here.

Combining equations (1), (3), (6) and (9), q_t may be expressed as

$$q_t = q_t^1 + H_t q_{t-1} + C_t w_t - D_t e_t \tag{13}$$

where q_t^1 is a fixed vector and H_t a fixed matrix defined by

$$q_t^1 = q_t^o + B_t x_t^o + D_t (z_t^o + L_t x_t^o - y_t^o) \tag{14}$$

and $$H_t = A_t + B_t N_t + D_t J_t \; . \tag{15}$$

e_t is a zero mean, multivariate normal random vector whose dispersion matrix is $\text{Disp}\,(y_t)$, so the stochastic input, y_t is being regarded as

$$y_t = y_t^o + M_t\, q_{t-1} + e_t. \tag{16}$$

Therefore $(q_t - \bar{q}_t)$ can be expressed by

$$q_t - \bar{q}_t = H_t(q_{t-1} - \bar{q}_{t-1}) + C_t(w_t - \bar{w}_t) - D_t e_t, \tag{17}$$

so Disp (q_t) can now be evaluated:

$$\begin{aligned} \text{Disp}\ (q_t) &= E\{(q_t - \bar{q}_t)(q_t - \bar{q}_t)^T\} \\ &= H_t\ \text{Disp}\ (q_{t-1})H_t^T + C_t\ \text{Disp}(w_t)C_t^T + D_t \text{Disp}(y_t)D_t^T \\ &+ H_t\ \text{Cov}(q_{t-1}, w_t)C_t^T + C_t\ \text{Cov}(w_t, q_{t-1})C_t^T \\ &- C_t\ \text{Cov}(w_t, e_t)D_t^T - D_t\ \text{Cov}(e_t, w_t)C_t^T, \end{aligned} \tag{18}$$

where advantage has been taken of the independence of e_t and q_{t-1}.

It is shown in Section 5.3 that under Approximation 1 , that z_t can be treated as a multivariate normal random vector,

$$\text{Cov}\ (w_t, e_t) = -U_t\ \text{Disp}\ (y_t) \tag{19}$$

where U_t is a diagonal vector, whose i th diagonal entry is $\Phi(\bar{z}_{it}/\sqrt{(\text{Disp}\ (z_t))_{ii}})$, i.e. $\Phi(\bar{z}_{it}/\tau_{it})$, Φ being the Gaussian distribution function.

Also in that section it is shown that

$$\text{Cov}\ (w_t, q_{t-1}) = U_t J_t\ \text{Disp}\ (q_{t-1}) \tag{20}$$

For this calculation, again, only Approximation 1 is required.

Combining equations (18), (19) and (20) and using (11) to substitute for Disp (z_t), it is seen that

$$\text{Disp}(q_t) = F_t \text{ Disp } (q_{t-1})F_t^T + G_t \text{ Disp}(y_t)G^T + C_t E_t C_t^T \quad (21)$$

where $F_t = H_t + C_t U_t J_t$ (22)

$G_t = D_t + C_t U_t$ (23)

and $E_t = \text{Disp } (w_t) - U_t \text{ Disp } (z_t)U_t^T$ (24)

The calculation of Disp (w_t) under Approximation 1 , that z_t is normal, is in principle straightforward. It is given in Section 5.3 where an expression for it in terms of $\bar{z}_t$ and Disp (z_t) is derived.

Summarising, a method has been provided whereby the solution to one restricted reduced problem can be used to re-estimate the σ_t's and enable the formulation of the next restricted reduced problem to be made. Under the assumption that the control decisions, x_t, can be approximated by linear functions of the preceeding state space values, q_{t-1}, an expression for the dispersion matrix of z_t has been derived in terms of the dispersion matrices of q_{t-1} and y_t. An expression for the dispersion matrix of q_t has then been given in terms of the dispersion matrices of q_{t-1}, y_t and z_t. The method of re-evaluation of the σ_t's is as follows:

1. Set Disp $(q_0) = 0$ and calculate σ_1 and Disp (z_1)
2. Calculate U_1 and Disp (w_1)
3. Calculate Disp (q_1)
4. Calculate σ_2 and Disp (z_2)
5. Calculate U_2 and Disp (w_2)

and so on until the final σ has been calculated.

5.2. The multiplicative Case

Herein the multiplicative model of the stochastic input is assumed. The analysis proceeds in much the same way as that for the additive model. It is desired to use the solution to one restricted reduced problem to re-evaluate the τ_t's for the next, where

$$\tau_{it}^2 = (\text{Disp}\ (z_t))_{ii}/(\bar{y}_{it})^2. \tag{25}$$

So, as before, an estimation procedure for Disp (z_t) is derived. It is very similar to that for the additive case. The necessary modifications are given below. As in the additive case, the control decision, x_t, is approximated by a linear function of q_{t-1}.

Consider the stochastic input y_t. It can be represented by

$$y_t = \tilde{y}_t + \tilde{Y}_t S_t^o \eta_t \tag{26}$$

where $$\tilde{y}_t = y_t^o + M_t q_{t-1} \tag{27}$$

and $\tilde{Y}_t$ is a diagonal matrix, whose i th diagonal entry is $\tilde{y}_{it}$, and S_t^o is a fixed lower triangular matrix such that

$$\text{Disp}\ (y_t) = \tilde{Y}_t P_t^o \tilde{Y}_t \tag{28}$$

and $$P_t^o = S_t^o (S_t^o)^T$$

i.e. $$\text{Std}(y_t) = \tilde{Y}_t S_t^o. \tag{29}$$

η_t is a vector of mutually independent Gaussian random variables.

Recalling that q_{t-1} can be represented by

$$q_{t-1} = \bar{q}_{t-1} + \text{Std}(q_{t-1}).\xi_{t-1} \tag{30}$$

where ξ_{t-1} is another vector of mutually independent Gaussian random variables, and substituting into expression (27) for the random variable $\tilde{y}_t$ and noting that

$$\bar{y}_t = y_t^o + M_t \bar{q}_{t-1}$$

It is seen that $\tilde{y}_t$ can be represented by

$$\tilde{y}_t = \bar{y}_t + M_t \text{Std}(q_{t-1})\xi_{t-1}. \tag{31}$$

Consider now z_t. It can be represented by

$$z_t = \bar{z}_t + J_t \text{ Std}(q_{t-1})\xi_{t-1} + \text{Std}(y_t)\eta_t, \tag{32}$$

where $\text{Std}(y_t)$ is stochastic and therefore needs further examination. Combining (29) and (31) it is seen that

$$(\text{Std}(y_t)\eta_t)_i = (\bar{y}_t + M_t \text{Std}(q_{t-1})\xi_{t-1})_i (S_t^o \eta_t)_i \,. \tag{33}$$

Eliminating $\text{Std}(y_t)\eta_t$ from equations (32) and (33) it is seen that

$$\begin{aligned} z_{it} &= \bar{z}_{it} + (J_t \text{ Std}(q_{t-1})\xi_{t-1})_i \\ &\quad + (\bar{y}_t + M_t \text{Std}(q_{t-1})\xi_{t-1})_i . (S_t^o \eta_t)_i \end{aligned} \tag{34}$$

$$= \bar{z}_{it} + \alpha_{it} + \beta_{it}\,\gamma_{it}, \text{ say,} \tag{35}$$

where $\alpha_{it} = (J_t \text{Std}(q_{t-1})\xi_{t-1})_i$

$$\beta_{it} = (\bar{y}_t + M_t \text{Std}(q_{t-1})\xi_{t-1})_i \qquad (36)$$

and $\gamma_{it} = (S_t^o \eta_t)_i$.

But α_{it} is independent of γ_{jt} for all (i,j) and β_{it} is independent of γ_{jt} for all (i,j). Also $E(\alpha_{it}) = 0$ and $E(\gamma_{it}) = 0$. Therefore

$$(\text{Disp}(z_t))_{ij} = E\{(\alpha_i + \beta_i\gamma_i)(\alpha_j + \beta_j\gamma_j)\}$$

$$= E(\alpha_i\alpha_j) + E(\beta_i\beta_j)E(\gamma_i\gamma_j) .$$

But $E(\alpha_{it}\alpha_{jt}) = (J_t \text{ Disp}(q_{t-1})J_t^T)_{ij}$

and $E(\beta_{it}\beta_{jt}) = (\bar{y}_t \bar{y}_t^T + M_t \text{ Disp } (q_{t-1})M_t^T)_{ij}$.

Also $E(\gamma_{it}\gamma_{jt}) = (P_t^o)_{ij}$.

Thus, defining the matrix operation $\otimes$ by

$$(A \otimes B)_{ij} = (A)_{ij}\cdot(B)_{ij}, \qquad (37)$$

Disp (z_t) may be written as:

$$\text{Disp } (z_t) = J_t \text{ Disp } (q_{t-1})J_t^T + (\bar{y}_t \bar{y}_t^T + M_t \text{ Disp}(q_{t-1})M_t^T) \otimes P_t^o . \qquad (38)$$

This is the estimation equation for Disp (z_t) and replaces equation (11) of the additive model. Hence τ_{it} is now estimated by

$$\tau_{it} = \sqrt{\{ \sum_{jk} [(J_t)_{ij}(J_t)_{ik} + (M_t)_{ij}(M_t)_{ik}(P_t^o)_{ii}](\text{Disp}(q_{t-1}))_{jk} + \bar{y}_{it}^2 (P_t^o)_{ii}\}}/\bar{y}_{it} . \quad (39)$$

The estimation of Disp (q_t) is the same as for the additive model except that $\sqrt{(\text{Disp } (z_t))_{ii}} = \tau_{it}\bar{y}_{it}$, so the i th diagonal entry of U_t is $\Phi(\bar{z}_{it}/\bar{y}_{it}\tau_{it})$, and in the recursive expression for Disp (q_t), Disp (y_t) is replaced by $\tilde{Y}_t P_t^o \tilde{Y}_t$ which has been shown to be equivalent to

$$(\bar{y}_t \bar{y}_t^T + M_t \text{ Disp } (q_{t-1})M_t^T) \circledast P_t^o. \quad (40)$$

Therefore the recursion equation (21) becomes:

$$\text{Disp } (q_t) = F_t \text{ Disp } (q_{t-1})F_t^T + C_t E_t C_t^T + G_t[(\bar{y}_t\bar{y}_t^T + M_t \text{ Disp}(q_{t-1})M_t^T) \circledast P_t^o]G_t^T \quad (41)$$

where $F_t = H_t + C_t U_t J_t$

$G_t = -D_t + C_t U_t$

and $E_t = \text{Disp } (w_t) - U_t \text{ Disp } (z_t)U_t^T$.

The calculation of Disp (w_t) is the same as before, being done under Approximation 1, that z_t can be treated as a multivariate normal random vector, and it can be expressed in terms of $\bar{z}_t$ and Disp (z_t). The calculations are in Section 5.3.

The same procedure can now be used to re-evaluate τ_t with equations (38), (39) and (41) replacing equations (11), (12) and (21) respectively.

5.3. Some Necessary Calculations

In sections 5.1 and 5.2 reference has been made to formulae from which $\text{Cov}(w_t, e_t)$, $\text{Cov}(w_t, q_{t-1})$ and $\text{Disp}(w_t)$ can be calculated. These are now derived. It is convenient to restate the evolution equations in the form in which they have been used:

$$q_t = q_t^1 + H_t\, q_{t-1} + C_t\, w_t - D_t\, e_t$$

$$z_t = z_t^1 + J_t\, q_{t-1} - e_t$$

$$w_t = \max\,(z_t, 0)$$

where q_t^1 and z_t^1 are fixed vectors and H_t and J_t are fixed matrices defined by

$$J_t = K_t + L_t\, N_t - M_t$$

$$H_t = A_t + B_t N_t + D_t J_t.$$

and e_t is a zero mean multivariate normal random vector with dispersion matrix $\text{Disp}\,(y_t)$. Approximation 2 , that the control decisions can be treated as linear functions of the preceeding state space has been used in the above representation of the evolution equations. Approximation 1 that z_t can be treated as a multivariate normal random vector is also necessary for all the calculations of this section.

(a) The calculation of $\text{Cov}\,(w_t, e_t)$

Since e_t has zero mean it is seen that

$$\text{Cov}(w_t, e_t) = E(w_t\, e_t^T)$$

$$= E(\max\,(z_t, 0) e_t^T).$$

Under Approximation 1 , z_t and e_t are treated as being jointly normally distributed. Their covariance matrix is

$$\begin{aligned}\text{Cov}\,(z_t,e_t) &= E(z_t\, e_t^T)\\ &= E((z_t^1 + J_t q_{t-1})e_t^T - e_t e_t^T).\\ &= -E(e_t\, e_t^T),\end{aligned}$$

because e_t and q_{t-1} are mutually independent.

i.e. $\text{Cov}\,(z_t,e_t) = -\text{Disp}\,(y_t)$.

For convenience let $\sigma_{it}^2 = (\text{Disp}\,(z_t))_{ii}$, $s_{it}^2 = (\text{Disp}\,(y_t))_{ii}$ and $\sigma_{it}\, s_{jt}\, \rho_{ijt} = -(\text{Disp}\,(y_t))_{ij}$, so ρ_{ijt} is the correlation coefficient between z_{it} and e_{jt}.

Now, the (i,j)th coefficient of Cov (w_t,e_t) is

$$E(\max(z_{it},0)e_{jt}) = E(\sigma_{it} s_{jt} \max(\xi, -\bar{z}_{it}/\sigma_{it})\eta)$$

where ξ,η are bivariate normal random variables with distribution $BN(0,0,1,1,\rho_{ijt})$. For convenience drop the subscripts on ρ. Then

$$\begin{aligned}&E(\max(\xi,-\bar{z}_{it}/\sigma_{it})\eta)\\ &= \int_{-\bar{z}_{it}/\sigma_{it}}^{\infty} \int_R \xi\eta\phi(\xi,\eta;\rho)d\eta d\xi - (\bar{z}_{it}/\sigma_{it}) \int_{-\infty}^{-\bar{z}_{it}/\sigma_{it}} \int_R \eta\phi(\xi,\eta;\rho)d\eta d\xi\end{aligned}$$

where $\phi(\xi,\eta;\rho)$ is the standardised bivariate normal probability density function

$$= \int_{-\bar{z}_{it}/\sigma_{it}}^{\infty} \xi\phi(\xi) \int_{R} \eta d\Phi\left(\frac{\eta - \rho\xi}{\sqrt{(1-\rho^2)}}\right)d\xi - \frac{\bar{z}_{it}}{\sigma_{it}} \int_{-\infty}^{-z_{it}/\sigma_{it}} \phi(\xi) \int_{R} \eta d\Phi\left(\frac{\eta - \rho\xi}{\sqrt{(1-\rho^2)}}\right)d\xi$$

where ϕ and Φ are the Gaussian probability density function and distribution function respectively

$$= \rho\{(-\bar{z}_{it}/\sigma_{it})\phi(\bar{z}_{it}/\sigma_{it}) + \Phi(\bar{z}_{it}/\sigma_{it}) + (\bar{z}_{it}/\sigma_{it})\phi(\bar{z}_{it}/\sigma_{it})\}$$

$$= \rho\,\Phi(\bar{z}_{it}/\sigma_{it}).$$

Therefore

$$E(\max(z_{it},0)e_{jt}) = \sigma_{it}\, s_{jt}\rho_{ijt}\, \Phi(\bar{z}_{it}/\sigma_{it})$$

$$= -(\text{Disp}\ (y_t))_{ij}\phi(\bar{z}_{it}/\sqrt{(\text{Disp}(y_t))_{ii}}).$$

Thus setting U_t to be a diagonal matrix whose i th diagonal entry is $\Phi(\bar{z}_{it}/\sqrt{(\text{Disp}(y_t))_{ii}})$, it is seen that

$$\text{Cov}\ (w_t,e_t) = -U_t\ \text{Disp}\ (y_t).$$

(b) Calculation of Cov (w_t,q_{t-1})

The calculations are similar to those for Cov (w_t,e_t) except that q_{t-1} is not assumed to be normally distributed. Now

$$\text{Cov}\ (w_t,q_{t-1}) = E(w_t(q_{t-1}-\bar{q}_{t-1})^T)$$

$$= E(\max(z_t,0).(q_{t-1}-\bar{q}_{t-1})^T).$$

$$\text{Also Cov}\ (z_t,q_{t-1}) = E((z_t-\bar{z}_t)(q_{t-1}-\bar{q}_{t-1})^T)$$

$$= E([J_t(q_{t-1}-\bar{q}_{t-1}) - e_t](q_{t-1}-\bar{q}_{t-1})^T)$$

$$= J_t\ \text{Disp}(q_{t-1})$$

because q_{t-1} and e_t are mutually independent.

Set $\sigma_{it}^2 = (\text{Disp}(z_t))_{ii}$, $s_{it-1}^2 = (\text{Disp}\,(q_{t-1}))_{ii}$

and $\sigma_{it}\, s_{jt-1}\rho_{ijt} = (J_t\, \text{Disp}(q_{t-1}))_{ij}$.

Then ρ_{ijt} is the correlation coefficient between z_{it} and q_{jt-1}. So

$$(\text{Cov}(w_t, q_{t-1}))_{ij} = \sigma_{it}\, s_{it-1} E\{\max(\xi, -\bar{z}_{it}/\sigma_{it})\eta\}$$

where ξ is a Gaussian random variable and η is a zero mean, unit variance random variable defined by

$$\eta = (q_{jt-1} - \bar{q}_{jt-1})/s_{jt-1}$$

and therefore the correlation between ξ and η is ρ_{ijt}. Again, for convenience drop the subscripts on ρ. If $f(\xi,\eta)$ is the joint probability density function of ξ and η, the (i,j)th coefficient of $\text{Cov}(w_t, q_{t-1})$ is

$$\sigma_{it}s_{it-1}\left\{\int_{-\bar{z}_{it}/\sigma_{it}}^{\infty} \xi \int_R \eta\, f(\xi,\eta)d\eta d\xi - (\bar{z}_{it}/\sigma_{it}) \int_{-\infty}^{-\bar{z}_{it}/\sigma_{it}} \int_R \eta f(\xi,\eta)d\eta d\xi\right\}$$

$$= \sigma_{it}s_{it-1}\{I_1 - I_2\}, \text{ say.}$$

Now $I_1 = \int_{-\bar{z}_{it}/\sigma_{it}}^{\infty} \xi\; E(\eta|\xi)\phi(\xi)d\xi,$

where ϕ is the probability density function of ξ, which is by assumption the Gaussian one. Furthermore $E(\eta|\xi)$ can be taken to be a linear function of ξ because z_t is linear in q_{t-1}. So let $E(\eta|\xi) = \lambda\xi$. However,

$$\rho = E(\xi\eta)$$

$$= \int_R \xi\; E(\eta|\xi)\phi(\xi)d\xi$$

$$= \int_R \lambda\xi^2\phi(\xi)d\xi = \lambda.$$

So λ can be taken to be ρ and $E(\eta|\xi) = \rho\xi$. Therefore

$$I_1 = \rho \int_{-\bar{z}_{it}/\sigma_{it}}^{\infty} \xi^2 \phi(\xi) d\xi$$

$$= \rho\{-(\bar{z}_{it}/\sigma_{it})\phi(\bar{z}_{it}/\sigma_{it}) + \Phi(\bar{z}_{it}/\sigma_{it})\} .$$

Likewise

$$I_2 = -\rho(\bar{z}_{it}/\sigma_{it})\phi(\bar{z}_{it}/\sigma_{it}).$$

Therefore, substituting for I_1 and I_2 it is seen that

$$(\text{Cov}(w_t, q_{t-1}))_{ij} = \sigma_{it} s_{jt-1} \rho_{ijt} \Phi(\bar{z}_{it}/\sigma_{it})$$

i.e. $\quad \text{Cov}(w_t, q_{t-1}) = U_t J_t \text{ Disp } (q_{t-1}) .$

(c) Calculation of Disp (w_t)

$$\text{Disp } (w_t) = E(w_t w_t^T) - \bar{w}_t \bar{w}_t^T.$$

The calculation proceeds by evaluating the terms as the right-hand side separately. The former is tackled first. Again, it is assumed that z_t is normally distributed.

Introduce some convenient notation.

Let $\sigma_i^2 = (\text{Disp } (z_t))_{ii}$ and $\sigma_i \sigma_j \rho_{ij} = (\text{Disp } (z_t))_{ij}$. Also let $\phi(\cdot)$ and $\Phi(\cdot)$ be the Gaussian probability density function and distribution function respectively. Let $\phi(\cdot,\cdot;\rho)$ denote the probability density function of the standardised bivariate normal distribution. Define p_i by

$$p_i = -\bar{z}_{it}/\sigma_i$$

Now $E(w_{it}w_{jt}) = \sigma_i\sigma_j \int_{x\geq p_i} \int_{y\geq p_j} (x-p_i)(y-p_j)\phi(x,y;\rho_{ij})dxdy$

for $|\rho_{ij}| < 1$. Expanding the integrand it is seen that

$$E(w_{it}w_{jt}) = \sigma_i\sigma_j(I_{ij}^{(1)} - p_i I_{ij}^{(2)} - p_j I_{ji}^{(2)} + p_i p_j I_{ij}^{(3)})$$

where

$$I_{ij}^{(1)} = \int_{y\geq p_j} \int_{x\geq p_i} xy\phi(x,y;\rho_{ij})dx\,dy$$

$$= \rho_{ij}(p_i J(p_i,p_j;\rho_{ij}) + p_j J(p_j,p_i;\rho_{ij}))$$

$$+ (1-\rho_{ij}^2)\phi(p_i,p_j;\rho_{ij}) + \rho_{ij}\Phi^c(p_i,p_j;\rho_{ij})$$

$$J(x,y;\rho) = \phi(x)\Phi((\rho x-y)/\sqrt{(1-\rho^2)})$$

$$\Phi^c(x,y;\rho) = \int_{\xi\geq x} \int_{\eta\geq y} \phi(\xi,\eta;\rho)d\eta d\xi,$$

$$I_{ij}^{(2)} = \int_{y\geq p_j} \int_{x\geq p_i} y\phi(x,y;\rho_{ij})dx\,dy$$

$$= J(p_j,p_i;\rho_{ij}) + \rho_{ij}J(p_i,p_j;\rho_{ij}),$$

and $I_{ij}^{(3)} = \int_{y\geq p_j} \int_{x\geq p_i} \phi(x,y;\rho_{ij})dx\,dy$

$$= \Phi^c(p_i,p_j;\rho_{ij}).$$

It is convenient to define a new function ψ by

$$\psi(x,y,\rho) = yJ(x,y;\rho) + xJ(y,x;\rho)$$

then $E(w_{it}\, w_{jt})$ can be expressed by

$$\sigma_i\sigma_j\{(1-\rho_{ij}^2)\phi(p_i,p_j;\rho_{ij}) + (\rho_{ij}+p_ip_j)\Phi^c(p_i,p_j;\rho_{ij}) - \psi(p_i,p_j;\rho_{ij})$$

for $-1 < \rho_{ij} < 1$.

If $\rho_{ij} = 1$, then to obtain an expression for $E(w_{it}w_{jt})$, the limit of the above expression can be taken as $\rho_{ij} \to 1$. Notice, however, that

$$\lim_{\rho\to 1} \psi(x,y;\rho) = (x+y-u)\phi(u), \quad u = \max(x,y)$$

$$\lim_{\rho\to 1} \Phi^c(x,y;\rho) = \Phi(-u), \text{ and}$$

$$\lim_{\rho\to 1} (1-\rho^2)\phi(x,y;\rho) = 0.$$

So, for $\rho_{ij} = 1$, $E(w_{it}w_{jt})$ is

$$\sigma_i\sigma_j\{(1+p_ip_j)\Phi(-p_o) - (p_i+p_j-p_o)\phi(p_o)\}$$

where $p_o = \max(p_i,p_j)$.

If $\rho_{ij} = -1$ then the limit can be taken as $\rho_{ij} \to -1$. This time

$$\lim_{\rho\to -1} \psi(x,y;\rho) = \begin{cases} 0 & \text{if } x+y \geq 0 \\ y\phi(x)+x\phi(y) & \text{if } x+y < 0 \end{cases}$$

$$\lim_{\rho\to -1} \Phi^c(x,y;\rho) = \begin{cases} 0 & \text{if } x+y \geq 0 \\ \Phi(-y)+\Phi(-x)-1 & \text{if } x+y < 0 \end{cases}$$

and $\lim_{\rho\to -1} (1-\rho^2)\phi(x,y;\rho) = 0.$

Now $\bar{w}_{it} = \sigma_i(\phi(p_i) - p_i\Phi(-p_i))$

So $\bar{w}_{it}\bar{w}_{jt} = \sigma_i\sigma_j(\phi(p_i)-p_i\Phi(-p_i))(\phi(p_j)-p_j\Phi(p_j))$.

Summarising, the dispersion matrix of w_t has been calculated in terms of the mean and dispersion matrix of z_t under the assumption that z_t is normally distributed.

Writing $\phi_k = \phi(p_k)$ and $\Phi^c{}_k = \Phi(-p_k)$ for k = 0, i,j,

$$\phi_{ij} = \phi(p_i,p_j;\rho_{ij}),$$

$$\Phi^c_{ij} = \Phi^c(p_i,p_j;\rho_{ij}),$$

and $$\psi_{ij} = \psi(p_i,p_j;\rho_{ij}),$$

the (i,j)th component of Disp (w_t) is:

$$\sigma_i\sigma_j\{(1-\rho_{ij}^2)\phi_{ij} + (\rho_{ij} + p_ip_j)\Phi^c_{ij} - \psi_{ij} - (\phi_i-p_i\Phi^c_i)(\phi_j-p_j\Phi^c_j)\} \quad -1<\rho_{ij}<1$$

$$\sigma_i\sigma_j\{(1+p_ip_j)\Phi^c_o - (p_i+p_j-p_o)\phi_o - (\phi_i-p_i\Phi^c_i)(\phi_j-p_j\Phi^c_j) \quad \rho_{ij} = 1$$

$$\sigma_i\sigma_j\{[(p_ip_j-1)(\Phi^c_i+\Phi^c_j-1)-p_i\phi_j-p_j\phi_i]\delta - (\phi_i-p_i\Phi^c_i)(\phi_j-p_j\Phi^c_j) \quad \rho_{ij} = -1$$

where $\delta = \begin{cases} 0 & \text{if } p_i + p_j \geq 0 \\ 1 & \text{if } p_i + p_j < 0. \end{cases}$

6. COMPUTATIONAL ASPECTS

6.1. The Iterative Approach

It is necessary to emphasise the iterative approach to the approximate solution to the reduced problem, which provides an approximate solution to the full problem. At each iteration the solution to the restricted reduced problem provides information enabling a better estimate of the σ_t's or τ_t's to be made. This improved estimate is used in the formulation of the next restricted reduced problem for the next iteration. In general, the process may not always converge, because the matrix of coefficients, N_t, in Approximation 2 (that x_t depends linearly on q_{t-1}) associated with optimal tableau T_1 of the restricted reduced problem may produce an optimal tableau T_2 in the next restricted reduced problem, and the N_t associated with tableau T_2 may produce an optimal tableau T_1 in the following restricted reduced problem. However, in numerical tests on a four time period, two commodity, model, the process converged rapidly, there being little change in the returned objective value or first time period controls after three iterations.

Computational advantage can be taken of the similarity of the restricted reduced problems, these being identical except for the values of the σ_t's or τ_t's. Any standard revise procedure can be used to update the right hand sides and technology matrix of the restricted reduced problem and use the previous solution as a starting basis for the next. Hence successive iterations may be very fast.

6.2. Estimation of the N_t

Recall that N_t is used to describe the dependence of the control variables x_t on the state space at the end of the previous time period q_{t-1} according to the equation

$$x_t = x_t^o + N_t\, q_{t-1}.$$

It has already been remarked that N_t can be obtained from an analysis of the final tableau of the solution to the last restricted reduced problem, but the details deserve further discussion.

The most straightforward way to obtain N_t is to introduce 'fictitious' variables q_t^*, replacing every occurrence of q_t in the constraints by $(q_t - q_t^*)$. Introduce q_t^* into the objective function with a sufficiently large penalty cost, say d_t^*, to ensure that q_t^* is always non basic in the final solution.

$$\text{Now } (N_t)_{ij} = \partial \bar{x}_{it} / \partial \bar{q}_{jt-1}$$

$$= -\partial \bar{x}_{it} / \partial \bar{q}^*_{jt-1}.$$

However $\partial \bar{x}_{it} / \partial \bar{q}^*_{jt-1}$ is simply the coefficient of $\bar{q}^*_{jt-1}$ in the $\bar{x}_{it}$ row of the final tableau if $\bar{x}_{it}$ is basic.

If $\bar{x}_{it}$ is non-basic it will remain so for sufficiently small changes in $\bar{q}_{jt-1}$ and so $\partial \bar{x}_{it} / \partial \bar{q}_{jt-1}$ may then be taken to be zero.

6.3. Speed of operation

The time taken to solve a linear program is roughly proportional to the cube of the number of constraints. So the time taken by the approximate solution method of the reduced problem will expand as the cube of the dimension of q_t. This is a considerable improvement on dynamic programming methods wherein the solution time expands exponentially with the dimension of q_t.

6.4. A Practical Check that the Convexity Condition Holds

If it is desired that the problem be convex, then the convexity condition can easily be checked by replacing the non-linear constraint

$$\bar{w}_t = \Sigma_t \, f_t(\Sigma_t^{-1} \, \bar{z}_t)$$

of the reduced problem by the constraint

$$\bar{w}_t = \Sigma_t \, f_t(\Sigma_t^{-1} \, \bar{z}_t) + \bar{s}_t, \; \bar{s}_t \geq 0$$

where $\bar{s}_t$ is a slack variable. The non-linear constraints of the restricted reduced problems can be similarly modified by the explicit introduction of slack variables. If these are non-basic in their solutions then the convexity condition is satisfied.

6.5. Numerical Results

The approximate solution algorithm described in this chapter was applied to a special case of the multiplicative version of the general model, namely the simple production/inventory model of Beale et al. [4].

Their model is described fully in Chapter 3 and is summarised in Chapter 9, Section 2. The algorithm was tested on two simple examples which they provided. These are the "no dependence" and "dependence" cases of the example detailed in Chapter 9, Section 3. Briefly, it is of a two product production/inventory system wherein the total production of both products is bounded in each time period. There are four time periods. Expected demand in the last two outstrips the production capacity, so there is some need to produce to stock in the first two. In the "no dependence" case the mean demand, and therefore the demand's dispersion matrix, is fixed whereas in the dependence case half of the mean demand for the first product is directly proportional to the sales of that product in the previous time period. Thus the "no dependence" case is a special case of the additive version of the general model.

In Table A below the results for the "no dependence" case are summarised. Values of the objective function, first time period production decisions and the estimates of σ_{it} (the standard deviation of the z_{it}) are given for each of five iterations (i.e. solutions to the first five restricted reduced problems). The results of the solution algorithm of Beale et al. and the objective function and first time period decisions of a dynamic programming solution are included for comparison. For the dynamic programming the demand distribution was restricted to integer values, the corresponding probability weights being proportional to the ordinates of the probability density function.

Table B below summarises the results for the "dependence" case. Again the results of the solution algorithm of Beale et al. are included for comparison.

Estimates of the σ_{it}'s were also obtained statistically by replicative simulation. The results are summarised in Table C below.

The technique is discussed in detail in Chapter 7 and its application to the particular examples is described in Chapter 9, where further results of the simulation experiments can be found.

TABLE A : RESULTS OF THE ALGORITHM ON THE "NO DEPENDENCE" CASE

Iteration no.	Objective function value	Product no.	1st time period production decision	$\sigma_{.1}$	$\sigma_{.2}$	$\sigma_{.3}$	$\sigma_{.4}$
1	863.65	1	12.41	4.472	5.590	7.826	10.062
		2	6.39	3.354	3.354	3.354	3.354
2	857.53	1	12.87	4.472	6.418	9.871	13.260
		2	6.67	3.354	4.021	4.983	5.211
3	857.25	1	12.97	4.472	6.495	9.927	13.297
		2	6.67	3.354	4.076	5.040	5.350
4	857.23	1	12.97	4.472	6.511	9.943	13.313
		2	6.67	3.354	4.076	5.035	5.352
5	857.22	1	12.97	4.472	6.511	9.935	13.312
		2	6.67	3.354	4.076	5.046	5.353
B*	859.54	1	12.13	4.472	5.590	9.842	10.062
		2	6.05	3.354	3.354	4.980	36.464
D.P.*	859.1	1	13				
		2	7				

* B denotes the algorithm of Beale et al.,

D.P. denotes dynamic programming.

TABLE B : RESULTS OF THE ALGORITHM ON THE "DEPENDENCE" CASE

Iteration no.	Objective function value	Product no.	1st time period production decision	$\sigma_{\cdot 1}$	$\sigma_{\cdot 2}$	$\sigma_{\cdot 3}$	$\sigma_{\cdot 4}$
1	860.50	1	14.45	4.472	5.590	7.826	10.062
		2	6.39	3.354	3.354	3.354	3.354
2	850.58	1	14.45	4.472	8.033	12.594	16.471
		2	6.67	3.354	4.020	4.915	5.115
3	849.90	1	14.45	4.472	8.033	12.905	17.185
		2	6.72	3.354	4.077	5.024	5.198
4	849.89	1	14.45	4.472	8.033	12.898	17.219
		2	6.74	3.354	4.086	5.038	5.182
5	849.88	1	14.45	4.472	8.033	12.898	17.220
		2	6.74	3.354	4.090	5.042	5.186
B*	845.58	1	14.74	4.472	5.670	15.786	21.012
		2	6.74	3.354	9.758	3.354	3.354

* B denotes the algorithm of Beale et al.

TABLE C : ESTIMATES OF σ_{it} OBTAINED BY SIMULATION

(a) The "No Dependence" Case

	Time period	1	2	3	4
$\hat{\sigma}_{1t}$	Upper 2½% ile	4.221	7.056	11.514	17.309
	Mean	3.640	6.085	9.930	15.790
	Lower 2½%ile	3.196	5.342	8.718	13.862
$\hat{\sigma}_{2t}$	Upper 2½%ile	3.606	4.061	5.150	6.308
	Mean	3.110	3.502	4.398	5.440
	Lower 2½%ile	2.730	3.074	3.861	4.776

(b) The "Dependence" Case

	Time period	1	2	3	4
$\hat{\sigma}_{1t}$	Upper 2½%ile	4.221	7.132	12.907	22.670
	Mean	3.640	6.151	11.131	19.551
	Lower 2½%ile	3.196	5.400	9.772	17.164
$\hat{\sigma}_{2t}$	Upper 2½%ile	3.606	4.405	6.862	8.488
	Mean	3.110	3.799	5.918	7.320
	Lower 2½%ile	2.730	3.335	5.195	6.426

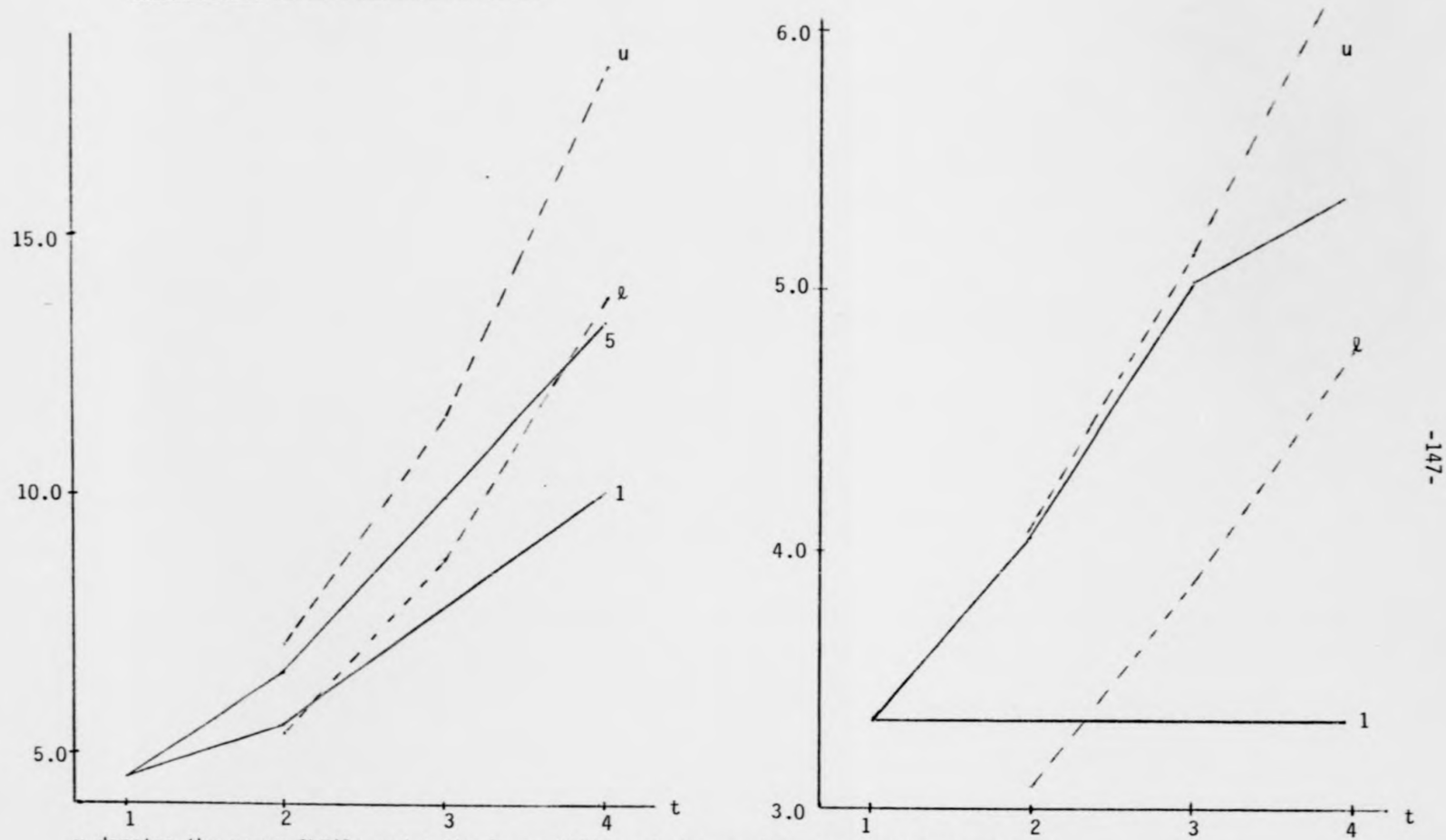
As can be seen from the above results, the solutions to the restricted reduced problems converge rapidly there being no change in the first time period production decisions and little change in the objective function value or estimates of the σ_{it}'s after three iterations. The estimates of the future uncertainty of supply over demand, represented by the σ_{it}'s converges to a pattern that is intuitively to be expected. The expected production level from iteration 2 onwards is, in fact, at capacity in the last three periods, so the system offers little scope for mitigating the future uncertainty in demand through flexibility in production levels. The uncertaintity in the excess of supply over demand might be expected to increase the further one looks into the future, and this is indeed what happens. Estimates of σ_{it} are plotted against t for each product in both the "no dependence" and "dependence" cases in graphs A and B respectively.

The pattern of the estimates of the σ_{it}'s made by the algorithm for the 5th iteration is similar to that obtained by simulation. However, the confidence intervals put upon the simulation estimates must be treated with caution, especially in the later time periods. They were computed under the assumption that the z_{it}'s are normally distributed, whence $(m-1)\hat{\sigma}_{it}^2/\sigma_{it}^2$ has a χ^2_{m-1} distribution, where $\hat{\sigma}_{it}$ is the sample estimate of σ_{it} made after m independent trials. This assumption may be unrealistic if the distribution of z_{it} is actually skew, which it is likely to be in the final two time periods.

The final returned objective function value is a fairly accurate estimate of the expected return that would be consequent on actually using the algorithm. The results of simulation experiments, detailed

in Section 4 of Chapter 9 indicate that if the algorithm were actually used the expected revenue would be 859.36 ± 0.28 and 850.32 ± 0.66 in the "no dependence" and "dependence" cases respectively; the possible errors in either direction being one estimated standard deviation from the estimate of the expected revenue.

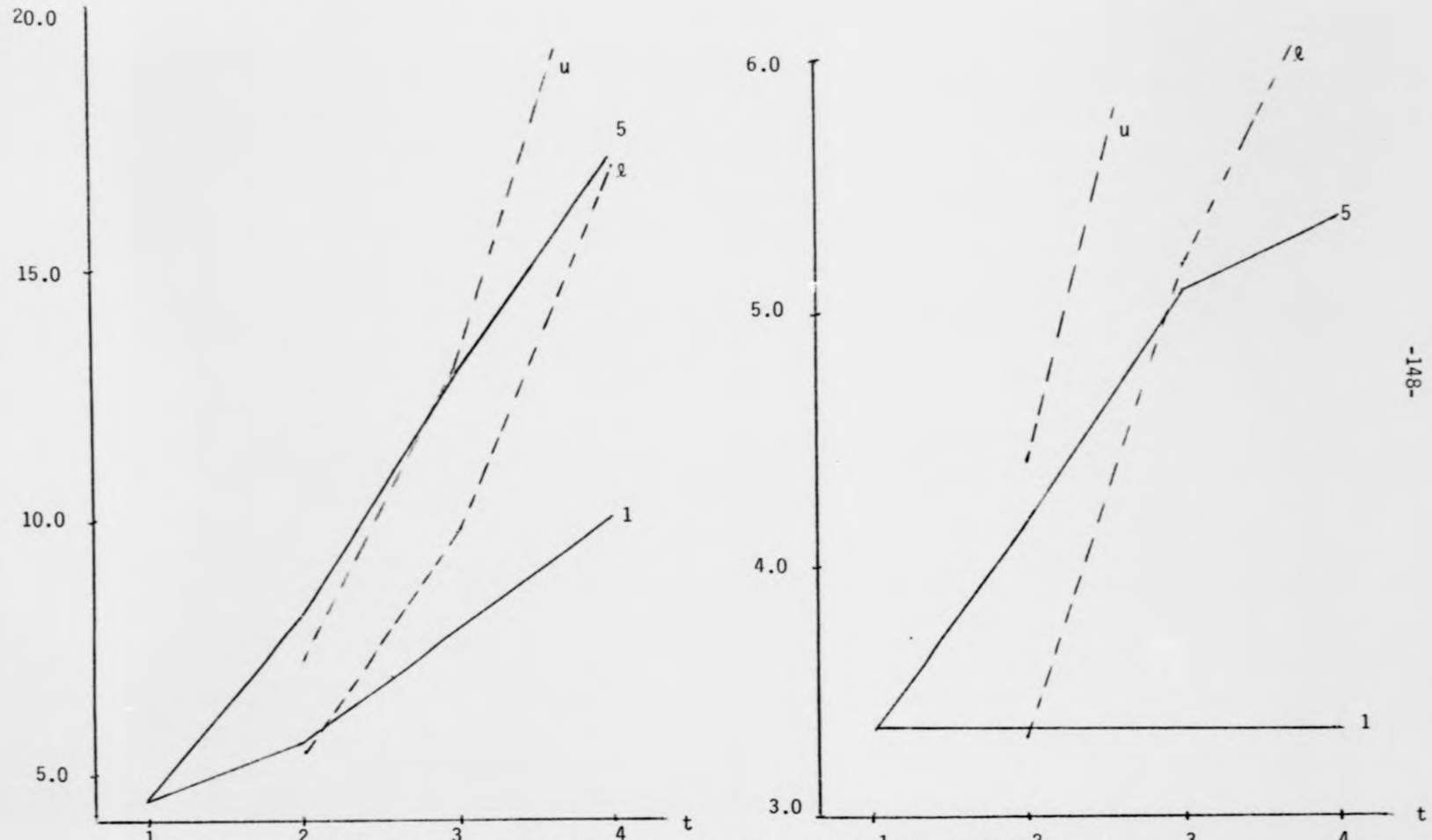
GRAPH A: The "No Dependence" Case



u denotes the upper 2½%ile and ℓ the lower 2½%ile of the results obtained by simulation

1 denotes estimates of σ_{it} made for the 1st iteration and 5 those made for the 5th iteration

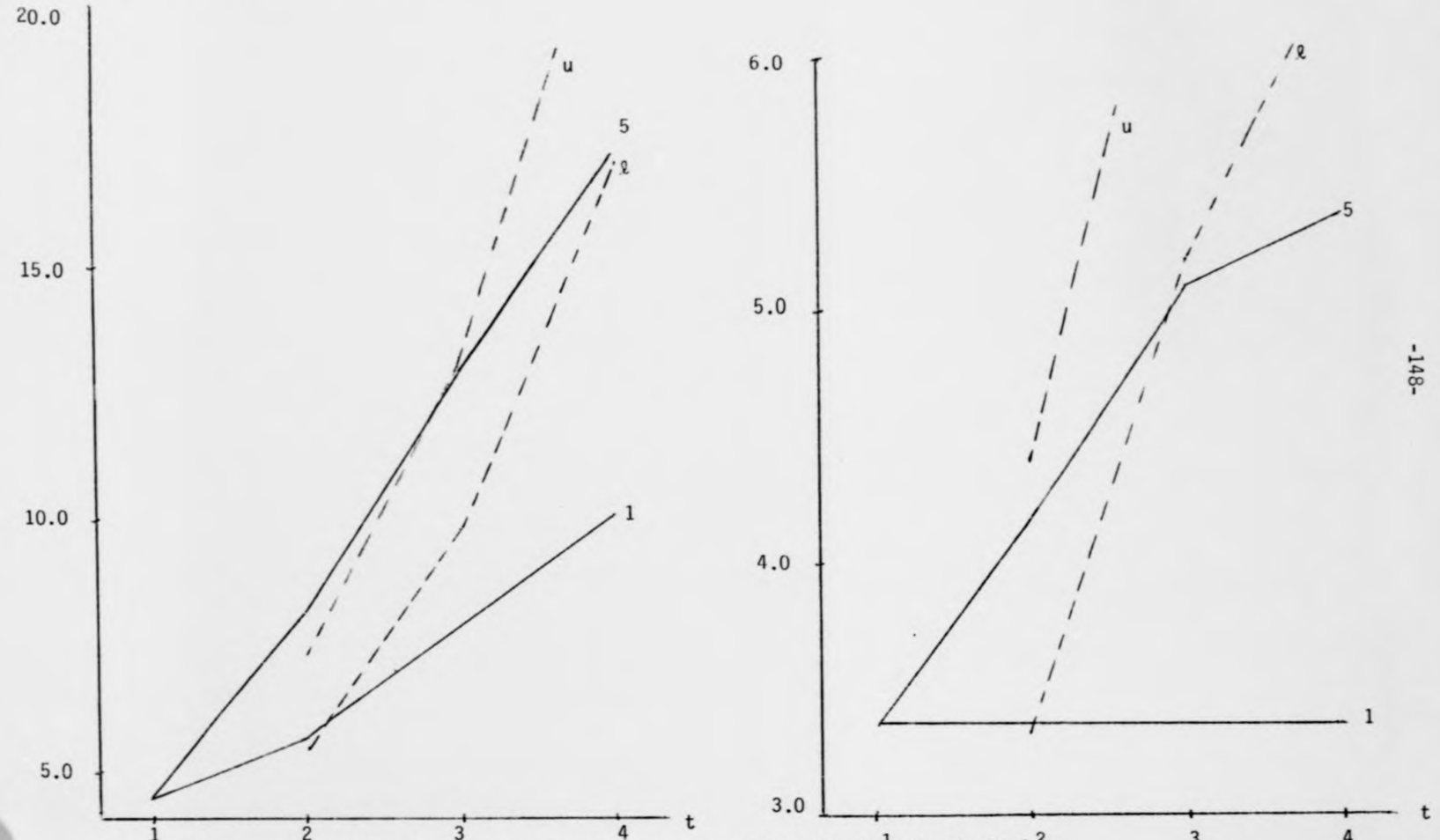
Graph B : The 'Dependence' Case



u denotes the upper 2½%ile and ℓ the lower 2½%ile of the results obtained by simulation

1 denotes estimates of σ_{i+} made for the 1st iteration and 5 those made for the 5th iteration

Graph B : The 'Dependence' Case



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7. CONCLUSIONS

7.1. The Approximate Scheme

Just as the generalised model of Chapter 5 was a generalisation of the production planning model of Beale et al. [4], so the approximate solution proposed here is based upon the one which they use. There are, however, some important differences, and these are discussed below.

Briefly, the approximate solution techniques to the full problem is as follows. Firstly the random variables in the full problem are replaced by their expected values in such a way that the constraints of the full problem directly imply those on the expected values of the random variables. This does not eliminate the variability of the random variables or distort the structure of the problem. The model thus obtained is termed the reduced problem, which is a deterministic non-linear program. Attention is then paid to the solution of the reduced problem since all that is really desired from a solution to the full problem is the best first time period controls, which if regarded as random variables,are equal to their expected values with probability one. This is because randomised decision rules are specifically excluded, it having been shown that such decision rules are no better than non-randomised ones.

However, it is still not possible to solve the reduced model,as its solution requires more information about the random variables than is initially available. The difficulty is overcome by approximating the

distribution of z_t by a multivariate normal distribution and thus formulating a special case of the reduced model, which is termed the restricted reduced problem. Explicit separable non-linear programming versions of this problem are then given. But even these require a knowledge of the standard deviation of each component of z_t, and this is not known. So initial or minimum estimates of it are made and the corresponding restricted reduced problem solved. This solution provides information which enables better estimates of the standard deviations to be made. These are used to formulate the second restricted reduced problem, the solution to which is used to re-estimate the standard deviations and formulate the next restricted reduced problem, and so on. In this way the restricted reduced problem is solved iteratively and this provides an approximate solution to the full problem.

7.2. Differences between the Proposed Solution Technique and that of Beale et al.

The problem which this chapter aims to solve is a generalisation of that of Beale et al. However, although the solution technique is based upon a generalisation of theirs, there are some important improvements.

Firstly, the additive variant, in which the dispersion matrix of the random input is fixed, is treated. This has the merit of being simpler than the version in which the random input's dispersion matrix depends upon its mean, and more robust in the sense that fewer approximations are necessary.

The model of Beale et al. can be regarded as a special case of the multiplicative variant of the general model. However, they assume that the standard deviation of z_{it} can be regarded as being directly proportional to the sales in period t, rather than the mean demands, in the formulation of the reduced problem. They then assume it to be directly proportional to the mean demand in their recursive variance estimation procedure. This has the advantage of mitigating paradoxical results that might arise from a positive probability of negative demand consequent on assuming it to be normally distributed. If it is not desired to meet demand in full then their assumption greatly reduces its variability. But the variability of demand in a particular time period should be independent of whether it is desired to meet it in that period. Consequently their approach distorts the structure of the problem and was therefore not adopted.

Secondly, the iterative scheme in which the restricted reduced problems are solved is different. In their scheme, after the solution of the initial restricted reduced problem, a problem is solved involving the last period only assuming that the state space at the start of that period is equal to its mean value given by the solution of the initial problem. The next restricted reduced problem which they solve involves only the last two time periods and so on. Their process terminates at the solution of the second restricted reduced problem to involve all time periods.

The scheme proposed here is an improvement for two reasons. Firstly the number of restricted reduced problems does not expand faster than the number of time periods in the problem, and secondly because the

similarity between the restricted reduced problems can be more easily exploited to improve computational efficiency.

Thirdly, the method by which the dispersion matrix of the state space, q_t, is recursively estimated is different. In the method proposed here there is no need to approximate $w_{it} = \max(z_{it},0)$ by a linear function of z_{it}. Their method, which does this, leads to errors even in the covariance terms.

7.3. Convexity

A condition is derived which, when satisfied, ensures that the objective function in the reduced problem is a concave function of the decision variables and also that the constraints in the reduced problem can be replaced by an equivalent set which define a convex feasible region. The same results apply to the restricted reduced problems since they are special cases of the reduced problem. Moreover, the condition is also sufficient to ensure that the full problem is a stochastic linear program.

However it is not always easy to check that the convexity condition is satisfied theoretically. In practice if it is desired that the condition holds, it can easily be checked by putting an explicit slack variable on the non-linear constraint. The condition holds if the slack is non-basic in the solution.

7.4. Computation

The approximatesolution technique here can be effectively used numerically for any reasonable dimension of the state space, q_t, for

the time required to solve each restricted reduced problem only expands as the cube of the total number of constraints, and hence as the cube of the dimension of q_t. In practical applications where the state space vector may represent stock levels it can be used to handle any reasonable number of products. This compares very favourably with dynamic programming techniques, wherein the solution time expands as the power of the number of state space dimensions and in practice could only be used on production planning problems effectively if less than four (say) products were being considered.

In all, it is hoped that the model proposed in Chapter 5 and the solution technique proposed here will be of practical use in many planning problems where uncertainties in, say, the demands for individual products needs to be considered.

PART III

CHAPTER 7

THEORETICAL APPROACHES TO THE EVALUATION OF SMOOTHING ALGORITHMS

1. INTRODUCTION

1.1. The Problem

This section is concerned with the evaluation of approximate production smoothing algorithms. The algorithms considered assume some multi-time period, finite time horizon, stochastic Markovian model of the environment. Typically, they could be scheduling the production of a number of different items over usual time periods. Stock levels at the end of each time period may then constitute the state space of the underlying Markov process. In modelling the environment, the state space is assumed to be continuous, but possibly bounded, by, for example, maximum and minimum permissible stock levels. Moreover the transition probabilities are assumed to have a continuous distribution.

It is necessary to evaluate the merit of approximate smoothing algorithms in order to assess the impact of the approximations made. It is important to determine whether the approximations are based on reliable assumptions, in which case the consequent suboptimality may be slight, or whether the approximations are less realistic in which case the consequent suboptimality may be more serious.

It is supposed that the algorithms are intended to optimise some attribute of interest, which is, by assumption, uniquely determined by the sequence ofvariable space realisations.They might, for example, be designed to maximise profit or revenue. However, since the process is stochastic, this attribute itself will be stochastic. The concern of this chapter will be the estimation of the expected or mean value of

the attribute in question. An algorithm itself may give an estimate of the expected value of this attribute; however, this cannot be assumed to be an adequate estimate of the mean value of the attribute that would be consequent on actually using the algorithm in practice. In short, it is necessary to study the actual behaviour of the process under the control of the algorithm being evaluated and hence determine a reliable estimate of the attribute of interest.

1.2. Comparison with exact algorithms

Some evaluation of the performance of an algorithm may be made by comparison of its output, for example initial production level decisions, with those from an algorithm known to be exact.

The scope of exact algorithms must, however, be limited if approximate algorithms are being considered. The obvious class of exact algorithms to use is that of those based upon dynamic programming. Except in the case where the state space is one dimensional or when the structure of the process can be exploited to reduce the state space to an equivalent one of one dimension (see, for example Dallenbach [14] and Thomas [53]), these approaches require the discretisation of the state space and probability distributions. This is, of course, an approximation, but a reliable one whose reliability can be checked, if necessary, by considering a sequence of successively finer discretisations. Furthermore, because of the familiar "curse of dimensionality", computational requirements restrict its application to simple models.

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Having obtained reliable or exactly optimal decision values from the algorithm, comparison with those obtained from an approximate algorithm may not be helpful in gauging the impact of the approximations made. This is because small errors in the decision values, say initial production decisions, will typically only make second order changes in the expected value of the attribute being optimised, say profit.

There is, however, another approach to the evaluation of algorithms available, namely that of replicative simulation. This is outlined next.

1.3. Replicative simulation

In this technique the environment within which the algorithm operates is simulated on a computer. The procedure is divided into a number of independent trials, each of which proceeds in the following manner.

Firstly the algorithm is run on a given set of data for the required number of time periods. The first time period controls, for example initial production level decisions, are then determined. Random numbers are then generated to simulate the stochastic element of the process, for example, demand for individual products. The system then evolves according to the way in which the environment is modelled until the start of the next time period. The statevector, for example stock levels, is then realised for the start of the next time period and the algorithm run again, with the new data thus generated, from the second time period until the time horizon. This procedure is repeated until the time horizon is reached, when the actual outcome of using the algorithm for the particular set of realised values of the stochastic elements is then determined.

The trial is then repated a number of times with different random numbers. From the knowledge of the outcome of each trial the mean of the attribute of interest can then be estimated.

The term replicative simulation is used to denote the replication of a number of independent trials and to distinguish it from regenerative simulation wherein some attribute of a steady state or time invariant process is to be estimated and the process is divided into a number of independent trials after the simulation run has been made. For a more detailed description of regenerative simulation see Heidelberger [31] or Lavenberg et al. [38].

The advantage of simulation is that it enables the actual effect of using a particular algorithm to be studied. The estimate of that attribute of the process on which attention is focussed can then be compared with that given by the algorithm itself. Disagreement between the two is a measure of the suboptimality of the algorithm; moreover the estimate obtained from the simulation can be compared with that obtained from an exact algorithm to provide a further assessment of the merit of the approximate algorithm being tested. The simulation technique also enables realistic comparisons to be made between different algorithms.

The principal difficulty of simulation, however, is that of accuracy. The standard error of the estimate of the mean of the attribute of interest decreases only as the reciprocal of the square root of the number of simulation trials. So straightforward simulation (naive "add up and average") may give such an inaccurate estimate from a single run that convergence is unacceptably slow. Various variance

reduction techniques are introduced to remedy this i.e. to reduce the variance with a fixed number of trials or to reduce the number of trials necessary to obtain some desired degree of accuracy. The techniques discussed here are those of control variates. The most effective one was found to be that of a martingale control variate, constructed for each simulation run.

2. VARIANCE REDUCTION TECHNIQUES

2.1. Control Variates

There has been a variety of variance reduction techniques published since Hammersley and Morton [28] and Hammersley and Mauldon [27] published in 1956. Much of it is concerned with robust estimators of location of the unknown distribution of data from a given sample. See, for example Gross [25] or Relles [46]. These are of no concern here; neither are the complex systems of antithetic variates developed in Hammersley and Morton [28] for the evaluation of multidimensional integrals occurring in atomic reactor design. The techniques of importance here are those of control and antithetic variates. The ideas behind these are described in Simon [50] and Lavenberg and Welch [37], and are outlined below.

Suppose that the attribute of interest of the stochastic process, for example, profit, is denoted by V. The concern of this chapter is the estimation of its expected value, EV. Suppose also that another random variable, U, having zero mean but highly correlated with V can be found.

Then instead of estimating E(V) it may be better to estimate E(V*) where

$$V^* = V - \alpha U, \quad \alpha \text{ being some real constant}$$

since

$$E(V^*) = EV$$

and

$$\text{var}(V^*) = \text{var}(V) - 2\alpha \, \text{cov}(V,U) + \alpha^2 \, \text{var}(U)$$

$$< \text{var}(V)$$

if

$$2\alpha \, \text{cov}(V,U) > \alpha^2 \, \text{var}(U).$$

The value of α which minimises var(V*) is

$$\text{Cov } (V,U)/\text{var } (U).$$

This is a special case of a more general result which is proved in the next subsection. The resulting minimum variance of V* is

$$(1- \rho^2) \text{ Var } (V)$$

where ρ is the correlation coefficient between V and U. Such a random variable U is said to be a control variate for V, if its correlation with V is positive and an antithetic variate for V if its correlation with V is negative.

For convenience call the random variable V* defined above the controlled random variable. If it can be constructed it is clearly expedient to estimate EV by an average of the realised values of V* rather than V.

In practice a number of different control variables say $U_1, U_2, \ldots, U_n$ may be available. The above theory easily extends to the use of multiple control variates in the following manner.

Let U be the n-vector whose ith component is the ith control variate U_i. Then define the controlled random variable V* by

$$V^* = \alpha^T U$$

where α is a fixed vector of coefficients, ^{T} denotes transpose and all vectors are by default column vectors.

The optimal control variable coefficient α is shown below to be

$$[\text{Disp } (U)]^{-1} \text{ Cov } (V,U)$$

where Disp (U) is the dispersion matrix of U and Cov (V,U) is the n-vector whose components are the covariance between V and the U_i's.

The resulting minimum variance is

$$(1 - R^2_{vu}) \text{ var } (V)$$

where

$$R^2_{vu} = \text{Cov } (V,U)^T [\text{Disp } (U)]^{-1} \text{ Cov } (V,U).$$

2.2. Optimal fixed control variate coefficients

A proof of the above result can be found in Anderson [2], but it is given briefly here.

Let $V^*(\alpha)$ be the random variable defined by

$$V^*(\alpha) = V - \alpha^T U$$

where α and U are as defined above. Then

$$\begin{aligned} \text{var } V^*(\alpha) &= E\,[(V-EV) - \alpha^T U][(V-EV)-\alpha^T U] \\ &= \text{var } V - 2\alpha^T \text{ Cov } (U,V) + \alpha^T \text{ Disp } (U)\alpha \\ &= \text{var } V - \alpha^T(2\text{Cov } (U,V) + \text{Disp } (U)\ \alpha) \end{aligned}$$

If v is a vector and M a matrix let $(v)_i$ and $(M)_{ij}$ denote the ith and (i,j)th component of v and M respectively. Then the above equation may be represented by

$$\text{var } V^*(\alpha) = \text{var } V - \sum_i \alpha_i [2(\text{Cov}(U,V))_i - \sum_j \alpha_j (\text{Disp}(U))_{ij}]$$

whence

$$\partial(\text{var } V^*(\alpha))/\partial\alpha_k = -2(\text{Cov}(U,V))_k + 2 \sum_i \alpha_i (\text{Disp}(U))_{ik}$$

which is zero if and only if

$$\sum_i (\text{Disp}(U))_{ki}\, \alpha_i = (\text{Cov}\ (U,V))_k$$

i.e. $\partial(\text{var } V^*(\alpha^0))/\partial\alpha_k = 0$ if and only if

$$\text{Disp}\ (U)\ \alpha^0 = \text{Cov}\ (U,V).$$

Furthermore

$$\partial^2(\text{var } V^*(\alpha))/\partial\alpha_i\partial\alpha_j = 2(\text{Disp}\ (U))_{ij}$$

which is positive semi-definite, therefore var $V^*(\alpha)$ is minimised by $\alpha = \alpha^0$, where

$$\alpha^0 = [\text{Disp}\ (U)]^{-1}\ \text{Cov}\ (U,V).$$

If Disp (U) is singular its inverse may be taken to be the (non-unique) psuedo-inverse and α^0 any solution of

$$\text{Disp}\ (U)\ \alpha^0 = \text{Cov}\ (U,V).$$

2.3. Estimation of the optimal control variate coefficients

In practice both Disp (U) and Cov (U,V) are usually unknown and have to be estimated in some way. The obvious estimators to use are

the sample dispersion of U and the sample covariance of U and V. If the simulation consists of m independent trials and u_i, v_i are the realised values of U and V on the ith trial, then the sample dispersion matrix of U is

$$\frac{1}{m-1} \sum_{i=1}^{m} (u_i-\bar{u})(u_i-\bar{u})^T$$

where $\bar{u} = \frac{1}{m} \sum_{i=1}^{m} u_i$, and the sample covariance vector of U and V is

$$\frac{1}{m-1} \sum_{i=1}^{m} (u_i-\bar{u})(v_i-\bar{v})$$

where $\bar{v} = \frac{1}{m} \sum_{i=1}^{m} v_i$.

For convenience denote the sample dispersion matrix of U by A and sample covariance vector of V and U by b. Then the optimal control variate coefficient, α^0, is estimated by $\hat{\alpha}$ where $\hat{\alpha}$ is a solution of

$$A\hat{\alpha} = b.$$

The controlled random variable, V* is now

$$V^* = V - \hat{\alpha}^T U.$$

There are two disadvantages to this approach. Firstly V* will not, in general, be unbiased i.e. $EV^* \neq EV$ and secondly the realisations of the controlled random variable V* for each simulation trial will not be independent. The consequence of the first remark is obvious whilst

the consequence of the second is that it may be difficult to estimate the variance of the final estimator of EV,

$$\bar{v}^* = \frac{1}{m} \sum_{i=1}^{m} v_i^*,$$

where v_i^* is the value of V^* realised on the ith simulation trial. These problems are now addressed.

Theorem 2.1.

If $\bar{V}^* = \frac{1}{m} \sum_{i=1}^{m} V_i^*$, where V_i^* is the controlled random variable of interest in the ith trial, defined by

$$V_i^* = V_i - \hat{\alpha}^T U_i,$$

$\hat{\alpha}$ being defined as above, V_i, U_i being the random variables V and U associated with the ith trial, then

$$E(\bar{V}^*) - EV \text{ is order } (\tfrac{1}{m}),$$

where m is the number of trials in the simulation.

In other words $\bar{V}^*$ is an asymptotically unbiased estimator of EV and the resultant bias is order (1/m).

Proof.

Consider the effect of adding another simulation trial to the estimate $\bar{V}^*$. Call it the (m+1)th trial. It will be shown that the expected value of $\hat{\alpha}^T U_{m+1}$ is O(1/m), where the result of the (m+1)th trial is used in the evaluation of $\hat{\alpha}$. It follows that $E(\hat{\alpha}^T U_i)$ is

is $O(1/m)$ whence $E(\frac{1}{m}\sum_{i=1}^{m}\hat{\alpha}^T U_i)$ is $O(1/m)$.

Let $A = \frac{1}{m-1}\sum_{i=1}^{m}(u_i-\bar{u})(u_i-\bar{u})^T$, $\bar{u} = \frac{1}{m}\sum_{i=1}^{m}u_i$ and

$b = \frac{1}{m-1}\sum_{i=1}^{m}(u_i-\bar{u})(v_i-\bar{v})$, $\bar{v} = \frac{1}{m}\sum_{i=1}^{m}v_i$.

A and b are the sample dispersion matrix of U and covariance vector of V and U.

Suppose that the addition of the (m+1)th trial changes the sample dispersion and covariance to A' and b' respectively.

Then $A' = (\frac{m-1}{m})A + (\frac{1}{m+1})A^0$ and

$$b' = (\frac{m-1}{m})b + (\frac{1}{m+1})b^0$$

where $A^0 = (u_{m+1} - \bar{u})(u_{m+1} - \bar{u})^T$, $\bar{u} = \frac{1}{m}\sum_{i=1}^{m}u_i$ and

$b^0 = (u_{m+1} - \bar{u})(v_{m+1} - \bar{v})$, $\bar{v} = \frac{1}{m}\sum_{i=1}^{m}v_i$.

Suppose, also, that the addition of the (m+1)th trial changes the estimate of the optimal control variate coefficient from $\hat{\alpha}$ to $\hat{\alpha}'$.

Then $\hat{\alpha}' = (A')^{-1}b'$.

Now $(A')^{-1} = (\frac{m}{m-1})\,A^{-1}(I + \frac{m}{m^2-1}A^0A^{-1})^{-1}$

$$= (\frac{m}{m-1})[A^{-1} - (\frac{m}{m^2-1})A^{-1}A^0A^{-1}] + O(1/m^2).$$

Therefore

$$\hat{\alpha}' = [A^{-1} - (\frac{m}{m^2-1})A^{-1}A^{o}A^{-1}]\,[b + (\frac{m}{m^2-1})b^{o}] + O(1/m^2)$$

$$= \hat{\alpha} + \frac{1}{m} A^{-1}[b^{o} - A^{o}\hat{\alpha}] + O(\frac{1}{m^2}).$$

Now U_{m+1} is independent of A and $\hat{\alpha}$ so

$$E(\hat{\alpha}'^{T}U_{m+1}) = \hat{\alpha}^{T}EU_{m+1} + \frac{1}{m} E\{U_{m+1}^{T}A^{-1}(b^{o} - A^{o}\hat{\alpha})\} + O(1/m^2).$$

But $EU_{m+1} = 0$, therefore

$$E(\hat{\alpha}'^{T}U_{m+1}) = \frac{1}{m} E[\text{trace}\{(b^{o} - A^{o}\hat{\alpha})U_{m+1}^{T} A^{-1}\}] + O(1/m^2)$$

which is of order (1/m) as required.

The consequence of the above theorem is that in practice it would be unwise to use $\hat{\alpha}$ as the control variate coefficient, as the resultant bias of the final estimator of EV is of the same order as its variance. However, it is shown in Lavenberg and Welch [37] that if (V,U) are jointly normally distributed there is no resultant bias.

The problem can easily be circumvented, however. For instead of defining the controlled random variable for the ith trial to be

$$V_i^{*} = V_i - \hat{\alpha}^{T}U_i$$

define it to be

$$V_i^{*} = V_i - \hat{\alpha}_i\, U_i$$

where $\hat{\alpha}_i$ is a solution to $A_i\,\hat{\alpha}_i = b_i$, A_i, b_i being the sample dispersion

matrix of U and covariance matrix of U and V calculated by excluding the result of the ith trial. Then $\hat{\alpha}_i$ and U_i are independent whence $E(\hat{\alpha}_i^T U_i) = 0$.

The above approach entails the solution of m sets of simultaneous linear equations in n unknowns (where n is the dimension of U). This may be computationally expensive, but a more efficient method of evaluating $\hat{\alpha}_i$ can be developed.

The basic idea is the exploitation of the similarity between the A_i's and b_i's for different i and hence between the $\hat{\alpha}_i$'s. As in the proof of the preceeding theorem $\hat{\alpha}_i$ is expressed as the sum of $\hat{\alpha}$ and some small quantity of order (1/m).

$$\text{Now } A_i = \frac{1}{m-2} \sum_{j \neq i} (u_j - \bar{u}_j)(u_j - \bar{u}_j)^T ; \; \bar{u}_i = \frac{1}{m-1} \sum_{j \neq i} u_j$$

$$\text{and } b_i = \frac{1}{m-2} \sum_{j \neq i} (u_j - \bar{u}_j)(v_j - \bar{v}_j) \; ; \; \bar{v}_i = \frac{1}{m-1} \sum_{j \neq i} v_j$$

$$\text{where } A_i = \left(\frac{m-1}{m-2}\right) \left[A - \frac{1}{m} A_i^o\right]$$

$$\text{and } b_i = \left(\frac{m-1}{m-2}\right) \left[b - \frac{1}{m} b_i^o\right]$$

$$\text{where } A_i^o = (u_i - \bar{u}_i)(u_i - \bar{u}_i)^T$$

$$\text{and } b_i^o = (u_i - \bar{u}_i)(v_i - \bar{v}_i).$$

Since A_i^o is the product of two vectors the Sherman-Morrison formula (See Sherman and Morrison [49]) can be used to express A_i^{-1} as

$$A_i^{-1} = A^{-1} + (m - k_i)^{-1} A^{-1} A_i^o A^{-1}$$

where $k_i = (u_i - \bar{u}_i)^T A^{-1} (u_i - \bar{u})$.

Therefore

$$\hat{\alpha}_i = A_i^{-1} b_i$$
$$= \hat{\alpha} + (m-k_i)^{-1}A^{-1}[A_i^0\hat{\alpha} - b_i^0].$$

If the number of trials, m, is large this will certainly be a more expedient way of calculating the $\hat{\alpha}_i$'s than solving $A_i \hat{\alpha}_i = b_i$ for each i. It only requires the inversion of one matrix, A, the time to do which is proportional to n^3 (where A is of dimension $n \times n$). Subsequent calculation of $\hat{\alpha}_i$ then requires matrix multiplication taking a time proportional to n^2. Whereas, solving $A_i\hat{\alpha}_i = b_i$ takes a time proportional to n^3 and would have to be repeated for each i.

However, to make use of the formula provided above it is necessary to ensure that $m-k_i \neq 0$. The following theorem provides sufficient conditions:

Theorem 2.2

If A, u_i, $\bar{u}_i$, k_i, m and n are as defined above then $(m-k_i) > 0$ if $n \leq m-2$.

Proof: Notice that

$$u_i - \bar{u}_i = \frac{m}{m-1}(u_i-\bar{u})$$

where $$\bar{u} = \frac{1}{m}\sum_{j=1}^{m} u_j.$$

So $$k_i = \frac{m^2}{(m-1)^2}(u_i-\bar{u})^T A(u_i-\bar{u}).$$

A is positive (semi)-definite by definition, so $k_i \geq 0$. It is therefore sufficient to show that if $n \leq m-2$, then $k < m$ where

$$k = \sum_{i=1}^{m} k_i.$$

Now $n = \text{trace}\{AA^{-1}\}$

$$= \text{trace}\left\{\sum_{i=1}^{m}(u_i-\bar{u})(u_i-\bar{u})^T A^{-1}\right\}$$

by the definition of A.

$$\text{So } n = \sum_{i=1}^{m} \text{trace}\{(u_i-\bar{u})(u_i-\bar{u})^T A^{-1}\}$$

$$= \sum_{i=1}^{m} (u_i-\bar{u})^T A^{-1}(u_i-\bar{u})$$

$$= \frac{(m-1)^2}{m^2} k$$

$$\text{i.e. } k = \frac{m^2}{(m-1)^2} n .$$

Therefore $k < m$ if and only if

$$n < \frac{(m-1)^2}{m} .$$

Since the left hand side is $m-2+\frac{1}{m}$, it can be seen that if $n \leq m-2$ then $k < m$. So $(m-k_i)$ is positive for each i if $n \leq m-2$ as required.

If the control variate coefficient vector is estimated from the sample dispersion matrix of the control variates and sample covariance vector of the outcome and control variates in the manner originally suggested i.e. the controlled outcome of the ith trial being

$$v_i^* = v_i - \hat{\alpha}^T u_i,$$

then the sequence of outcomes of the simulation procedure, $(v_1^*, v_2^*, \ldots, v_m^*)$ has only m-n degrees of freedom, where n is the dimension of the control vector, U. Thus, the variance of the final estimator of EV,

$$\bar{v}^* = \frac{1}{m} \sum_{i=1}^{m} v_i^*$$

will be

$$\text{var}\ (\bar{v}^*) = (\frac{1}{m-n})\ \text{var}\ v^*,$$

and var (v^*) is estimated by

$$\hat{\sigma}^2(\bar{v}^*) = (\frac{1}{m-n-1}) \sum_{i=1}^{m} (v_i^* - \bar{v}^*)^2.$$

Therefore, the appropriate estimator of the variance of $\bar{v}^*$ is

$$\hat{\sigma}^2(\bar{v}^*) = (\frac{1}{m-n})(\frac{1}{m-n-1}) \sum_{i=1}^{m} (v_i^* - \bar{v}^*)^2.$$

Lavenberg and Welch [37] assume that the vector

$$\begin{pmatrix} v \\ u \end{pmatrix}$$

has a multivariate normal distribution, and derive confidence intervals for $\bar{v}^*$ based upon Student's t distribution. They also show that the variance reduction achieved by using the estimate $\hat{\alpha}$ of the optimal coefficient vector, α^0, is the potential variance reduction, which would be achieved if α^0 were known, multiplied by the factor

$$\frac{m-2}{m-n-2}.$$

If the bias control methods outlined above are being used, then the variance estimator has one fewer degrees of freedom. The appropriate estimator of the variance of $\bar{v}^*$ becomes

$$\hat{\sigma}^2(\bar{v}^*) = (\frac{1}{m-n-1})\ (\frac{1}{m-n-2}) \sum_{i=1}^{m} (v_i^* - \bar{v}^*)^2$$

and estimation of the optimal coefficient vector degrades the potential variance reduction by a factor of

$$\frac{m-3}{m-n-3} \quad .$$

As can be seen, the realised variance reduction deteriorates as the dimension of the control variates vector, U, increases. Sometimes it is important to choose a small set of control variates from among the set of possible control variates, and this can be a problem.

In the application of the above variance reduction techniques to the study of algorithms operating within a finite time period Markov model of the environment, effective control variables were found to be the difference between the space realisations at the end of each time period and their expected values one time period before. The results are given in Chapter 9.

3. GENERAL FUNCTIONS OF CONTROL VARIATES

3.1. The Concept of General Functions of Control Variates

The form of the controlled random variate of interest that has been explored is

$$V^* = V - \alpha^T U$$

where U are the control variates and α is a fixed vector. Estimation of the optimal value for α requires the results of all the simulation trials.

In general, this approach will be inferior to one involving the use of more general functions of the control variates, U, where the controlled random variate of interest would have the form

$$V^* = V - V_C(U),$$

V_C being a function from $\mathbb{R}^n$ (n being the dimension of U) to $\mathbb{R}$.

Suppose that the algorithm being tested provides information about the process from which a suitable function V_C can be constructed. This has the merit of being potentially much more powerful than the previous approach, particularly if the algorithm being tested is not severely suboptimal. Furthermore, if a suitable function V_C can be derived from a single simulation trial, then the controlled output of each trial is independent, and so the problems in estimating the variance of the estimate of the mean of the attribute of interest do not arise.

Derivation of a suitable functional form for V_C depends on the model of the stochastic process within which the algorithms being tested operate, and on the choice of control variates, U. This is explored for the particular application being considered, namely the simulation of a finite time period Markov process, after a more detailed description of its structure in Section 3.2. Natural restrictions to the class of admissible functional forms of V_C are introduced and discussed in Section 3.3. It is shown in Section 3.4 that with this restriction, the problem of constructing V_C reduces to one of modelling the expected future contribution to the attribute of interest as a function of the next time-period state space realisation.

3.2. A Symbolic Representation of the Underlying Process.

Hereafter the concern of this chapter is with the evaluation of approximate smoothing algorithms which operate within a discrete time, finite time-period Markov model of the environment, as discussed in Section 1. The notation introduced here should be sufficiently general to cover all such applications.

The process naturally divides into T distinct time periods: 1, 2,..., T. The subscript t will be used to denote that some quantity pertains to time period t. Then pertaining to each time period is the principal attribute of interest, V_t, hereafter called revenue or profit (considered to be a scalar), the state variable at the end of the time period, Q_t (this is considered to be a vector and might represent, for example, stock levels and a weighted average of previous sales), the

controls applied, X_t (also considered to be a vector and it is an output from the algorithm being studied, perhaps production levels), and the stochastic input, Y_t. Again this is considered to be a vector and might represent demand for products or availability of raw materials.

Now suppose that a particular algorithm, assuming a particular model of the underlying process, is being studied. Then the sequence of state variables, $\{Q_0, Q_1, \ldots\}$, forms a Markov process. The revenue V_t is a random variable and the total revenue from a simulation trial is

$$V = \sum_{t=1}^{T} V_t.$$

Furthermore, it can be assumed that V_t is a function of Q_{t-1} and Q_t. For in the last resort the state space could be expanded and V_t set to zero for all $t < T$ and V_T set to V.

As far as possible random variables will be denoted by upper case letters and their outcomes by lower case letters.

Consider the progress of a simulation trial. At the start of time period t $(Y_1, \ldots, Y_{t-1})$ have been realised by $(y_1, \ldots, y_{t-1})$ and $(Q_1, \ldots, Q_{t-1})$ by $(q_1, \ldots, q_{t-1})$. The algorithm is run for time periods t, t+1,...,T. The t th time-period controls are then determined and the distribution of Q_t given that $Q_{t-1} = q_{t-1}$ is completely known.

Define the "one-step-ahead" expected revenue:

$$V_{Ht} = V_{Ht}(Q_{t-1}), \quad \text{where}$$

$$V_{Ht}(q_{t-1}) = E_{Q_t}(V_t(q_{t-1}, Q_t))$$

$$= \int V_t(q_{t-1}, \xi) dF_{Q_t | q_{t-1}}(\xi)$$

where $F_{Q_t|q_{t-1}}(\cdot)$ is the distribution function of Q_t given that $Q_{t-1} = q_{t-1}$. So $V_{Ht}(q_{t-1})$ is the expected revenue in time period t given that $Q_{t-1} = q_{t-1}$. Because of the remark in the above paragraph V_{Ht} can be calculated exactly from the output of the scheduling algorithm.

$$\text{Set } V_H = \sum_{t=1}^{T} V_{Ht}. \quad \text{Then } EV_H = EV.$$

EV might be estimated by averaging the realised values of V or V_H. When the martingle control statistic derived in Sections 3.3 and 3.4 is used, however, it will make no difference, as will be shown.

3.3. The Martingale Control Variates

A particular form of control variate function is now proposed. It will be shown that for each trial, the sequence of functions of control variates proposed forms a martingale. Although this puts a restriction on the class of functions of the control variates which will be considered, there is no consequent suboptimality. Subsequent sections deal with optimality criteria for the martingale and suggest ways in which it might be constructed from information provided by the scheduling algorithm.

Advantage is taken of the time-periodic structure of the process being simulated. Decompose the total control for each trial $V_C(U)$ into parts associated with each time period, i.e.

$$\text{let } V_C(U) = \sum_{t=1}^{T} V_{Ct}(U)$$

where $V_{Ct}(U)$ is the control pertaining to the t th time period.

It is now proposed that V_{Ct} be a function of the state variable at the start of the t th time period, Q_{t-1}, and the difference between the state variable at the end of the t th time period and its expected value one time period before, $Q_t - E(Q_t|Q_{t-1})$.

$$\text{i.e. } V_{Ct} \equiv V_{Ct}(Q_{t-1}, Q_t - E(Q_t|Q_{t-1})).$$

Since the total revenue can be decomposed into the sum of revenues accrued in each time period, which are, moreover, functions of the state variable at the start and end of that time period, this is a natural restriction. That, potentially, all the varability in the estimate of EV can be eliminated is shown at the end of Section 3.4. Also the functions V_{Ct} are restricted to be those for which

$$E(V_{Ct}|Q_{t-1}) = 0.$$

Again, nothing is lost by this restriction since EV_{Ct} must be known and may, without loss of generality, be set to the zero. Therefore $E(V_{Ct}|Q_{t-1})$ must be taken to be zero, since it cannot be assumed that the distribution of Q_{t-1} is known.

The t th time period control V_{Ct} is regarded as a control of the t th time period revenue. The total controlled attribute of interest, V*, can be expressed as:

$$V^* = \sum_{t=1}^{T} (V_t - V_{Ct}).$$

The rationale behind the use of the state random state variables rather than, say, the random input, is that in each trial the realised value of the attribute of interest (revenue) is, by assumption, a computable function of the sequence of state space realisations and is only a function of the random input through its effect on the state space realisations. Moreover, the t th time period revenue, V_t, is a directly computable function of Q_{t-1} and Q_t.

Intuitively the construction of V_{Ct} may be regarded as follows. For a particular trial, suppose that the start of time period t has been reached, $(Q_1,\ldots,Q_{t-1})$ are realised by $(q_1,\ldots,q_{t-1})$, q_{t-1} contains all the information necessary to describe the state of the process at the start of time period t. Given this information, $V_{Ct}(q_{t-1},Q_t-E(Q_t|q_{t-1}))$ is a measure of the "luck" associated with the next position on the state space, Q_t.

Now, a martingale is defined to be a sequence of random variables, $M_o,M_1,M_2,\ldots$, say with the property that

$$E(M_t|M_o,M_1,\ldots,M_{t-1}) = M_{t-1}, \text{ for all } t \geq 1.$$

The term M_t-M_{t-1} is called the t th martingale difference. The martingale property is therefore equivalent to the property that the expected value of each martingale difference given all the previous martingale differences is zero.

i.e. $E(M_{Dt}|M_{D1},M_{D2},\ldots,M_{Dt-1}) = 0$ for all $t \geq 1$

where $M_{Dt} = M_t - M_{t-1}$.

Theorem 3.1.

(a) $EV_{Ct} = 0$ for all t and

(b) The V_{Ct}'s are martingale differences

i.e. $(V_{C1}, V_{C1} + V_{C2}, V_{C1} + V_{C2} + V_{C3}, \ldots)$ is a martingale.

Proof

(a) Let $F_{Q_{t-1}}(\cdot)$, $F_{Q_t | q_{t-1}}(\cdot)$ be the distribution functions of Q_{t-1} and Q_t given that $Q_{t-1} = q_{t-1}$, respectively. Now V_{Ct} is a function of Q_{t-1} and Q_t.

$$\text{Therefore } EV_{Ct} = \int E(V_{Ct} | Q_{t-1} = \xi) dF_{Q_{t-1}}(\xi)$$

$$= 0,$$

because $E(V_{Ct} | Q_{t-1} = \xi) = 0$ for all ξ.

(b) To prove the second part of the theorem it is sufficient to show that

$$E(V_{Ct} | V_{C1}, V_{C2}, \ldots, V_{Ct-1}) = 0.$$

$$\text{i.e. } E(V_{Ct} | V_{Cu} = v_{Cu},\ u = 1,2,\ldots,t-1) = 0$$

Now let $F_{Vt-1}(\cdot; v_{C1}, v_{C2}, \ldots, v_{Ct-1})$ be the distribution function of Q_{t-1} given that $V_{Cu} = v_{Cu}$ for $u = 1,2,\ldots,t-1$. Also let S be the set of possible realisations of Q_{t-1} given $V_{Cu} = v_{Cu}$, $u = 1,2,\ldots,t-1$.

$$\text{Now } E[V_{Ct} | Q_{t-1} = \xi;\ V_{Cu} = v_{Cu}; u = 1,\ldots,t-1]$$

$$= E[V_{Ct} | Q_{t-1} = \xi], \text{ for all } \xi \in S,$$

because $\{Q_0, Q_1, \ldots\}$ is a Markov process. It follows that

$$E[V_{Ct} | V_{Cu} = v_{Cu},\ u = 1,\ldots,t-1]$$

$$= \int_{\xi \epsilon S} E[V_{Ct} | Q_{t-1} = \xi] dF_{Q_{t-1}}(\xi)$$

$$= 0,$$

because the integrand is zero for all $\xi \epsilon S$.

Therefore the V_{Ct}'s are martingale differences and will hereafter be referred to as such.

3.4. The Optimality Criterion for the Martingale Difference Functions

It is desired to choose the martingale differences to minimise the variance of the controlled revenue

$$V^* = \sum_{t=1}^{T} (V_t - V_{Ct})$$

Now $$\text{var } V^* = \sum_{t=1}^{T} \sum_{u=1}^{T} \text{Cov}\ (V_t - V_{Ct}, V_u - V_{Cu})$$

$$= \sum_t \sum_u E\ \{(V_t - V_{Ct})(V_u - V_{Cu})\} - (EV)^2$$

since $EV_{Ct} = EV_{Cu} = 0$.

Because the summand is symmetric in t and u, var V* can be expressed as

$$E\ \{\sum_t (V_{Ct}^2 - 2V_{Ct}V_t) + 2 \sum_t \sum_{u>t} (V_{Ct}V_{Cu} - V_{Ct}V_u - V_tV_{Cu})\} + \text{var } V,$$

where $V = \sum_t V_t$.

Now var V is independent of the V_{Ct}'s, so it is desired to minimise the sum of

$$S_1 = E\{\sum_t (V_{Ct}^2 - 2V_{Ct}V_t)\} \quad \text{and}$$

$$S_2 = E\{\sum_t \sum_{u>t} (V_{Ct}V_{Cu} - V_{Ct}V_u - V_tV_{Cu})\}.$$

Conditioning with respect to Q_{t-1}, S_1 can be expressed as

$$S_1 = E\{\sum_t E(V_{Ct}^2 - 2V_{Ct}V_t|Q_{t-1})\}$$

$$= E\{\sum_t \text{var}(V_{Ct}|Q_{t-1}) - 2\,\text{cov}(V_{Ct},V_t|Q_{t-1})\}.$$

Conditioning with respect to Q_{u-1}, S_2 can be expressed as

$$S_2 = 2E\{\sum_t \sum_{u>t} E(V_{Ct}V_{Cu} - V_{Ct}V_u - V_tV_{Cu}|Q_{u-1})\}$$

$$= E\{2\sum_t E[V_{Ct} \sum_{u>t} E(V_{Cu} - V_u|Q_{u-1})$$

$$- V_t \sum_{u>t} E(V_{Cu}|Q_{u-1})|Q_{t-1}]\},$$

conditioning with respect to Q_{t-1} also. Now the expectation of V_{Cu} given Q_{u-1} is zero, therefore

$$S_2 = E\{-2\sum_t E[V_{Ct} \sum_{u>t} E(V_u|Q_{u-1})|Q_{t-1}]\}$$

$$= E\{-2\sum_t E[V_{Ct} \sum_{u>t} V_u|Q_{t-1}]\}.$$

Combining the above expressions for S_1 and S_2 it is seen that var V* - var V is

$$E\{ \sum_{t} V_{Ct}^{2} - 2V_{Ct} \sum_{u \geq t} V_u | Q_{t-1} \},$$

and as var V is fixed, this is the expression to be minimised by appropriate choice of functions V_{Ct}.

Now the t th term in the summand is not dependent on V_{Cu} for $u \neq t$. Minimisation is therefore achieved by minimising, for each t,

$$E\{V_{Ct}^{2} - 2V_{Ct} V^{t} | Q_{t-1}\}$$

where V^{t} is a function of Q_t and defined to be

$$\sum_{u \geq t} E(V_u | Q_t).$$

Suppose that a simulation trial is in progress and the start of time period t has been reached; Q_{t-1} has been realised by q_{t-1}. Given this particular realisation of Q_{t-1}, the problem is then one of choosing the function $V_{Ct}(Q_t)$ to minimise

$$E(V_{Ct}^{2} - 2V_{Ct} V^{t}),$$

such that $EV_{Ct} = 0$.

The minimisation can be regarded as being over $\lambda_1, \lambda_2, \ldots, \lambda_N$; $\lambda_i \in \mathbf{R}$, where V_{Ct} has been parameterised:

$$V_{Ct} = V_{Ct}(\lambda_1, \lambda_2, \ldots, \lambda_N; Q_t),$$

V_{Ct} being twice differentiable with respect to the λ's and

$$\partial^2 V_{Ct} / \partial \lambda_i \, \partial \lambda_j = 0.$$

This is because such a parameterised function (e.g. a polynomial) can approximate any measurable function arbitrarily closely (for sufficiently large N) over any bounded subset of the range of Q_t (except for sub-subsets of null probability) and such a subset can be chosen such that the expectation over it is arbitrarily close to the expectation over the full range of Q_t.

Define the Lagrangian to be

$$L(\lambda_1,\ldots,\lambda_N,\mu) = E(V_{Ct}^2 - 2V_{Ct}V^t + \mu V_{Ct}).$$

Then $\partial L/\partial\mu = E(V_{Ct})$

and $\partial L/\partial\lambda_i = E(\partial V_{Ct}/\partial\lambda_i . (2V_{Ct} - 2V^t + \mu))$

Now $\partial^2 L/\partial\mu^2 = 0$ and moreover

$$\partial^2 L/\partial\lambda_i\,\partial\lambda_j = E(\partial V_{Ct}/\partial\lambda_i . \partial V_{Ct}/\partial\lambda_j)$$

so the matrix whose components are $\partial^2 L/\partial\lambda_i\,\partial\lambda_j$ is positive semi-definite, whence

$$E(V_{Ct}^2 - 2V_{Ct}V^t)$$

will be minimised subject to $EV_{Ct} = 0$ by $\mu, \lambda_1,\ldots,\lambda_N$ satisfying

$$\partial L/\partial\mu = 0,$$

and $$\partial L/\partial\lambda_i = 0 \quad \forall i$$

Now $\partial L/\partial\lambda_i = 0 \ \forall i$ is implied by

$$V_{Ct} = V^t - \tfrac{1}{2}\mu$$

whilst $\partial L/\partial \mu = 0$ if and only if

$$\tfrac{1}{2}\mu = EV^t.$$

Therefore the required minimising V_{Ct} is

$$V_{Ct}(Q_t - E(Q_t|q_{t-1})) = V^t(Q_t - E(Q_t|q_{t-1})) - EV^t.$$

The problem of constructing $V_{Ct}(Q_t - E(Q_t|q_{t-1}))$ therefore reduces to one of modelling V^t, the expected future revenue as a function of $Q_t - E(Q_t|q_{t-1})$. This problem is tackled in the next section where two models, a linear and a quadratic one, are proposed.

4. MODELLING THE EXPECTED FUTURE REVENUE

4.1. General Considerations

Consider one particular simulation run. At the start of time-period t suppose that the state vector Q_{t-1} has been realised by q_{t-1}. Then the process for time periods t, t+1,...,T has to be considered, and in particular the t th time period controls and the t th time period martingale difference function, V_{Ct}, have to be derived. The former is a problem for the particular algorithm being studied; it is the construction of the latter that is of concern here. From an analytical viewpoint this is equivalent to the first such function of a (T-t+1) time period process. So without loss of generality the construction of the first such martingale difference function only, V_{C1}, can be treated. This considerably simplifies the notation required.

Consider, then, the process at the start of time period 1. The first time period controls have been determined and the distribution of Q_1, given the present position on the initial state space, q_o, is completely known. It is desired to model the expected future revenue as a function of the realisation, q_1, of the state vector Q_1 at the end of the first time period without re-running the algorithm. This estimate is the martingale difference function, $V_{C1}(q_o, Q_1 - E(Q_1|q_o))$, which when applied to the actual realisation of Q_1, q_1, yields the first martingale difference $V_{C1}(q_o, q_1 - E(Q_1|q_o))$ of the martingale which will become the control statistic for the simulation run.

The martingale difference function is constructed about $\bar{q}_1 = E(Q_1|q_o)$. This is for two reasons. Firstly, the control statistic only approximates the behaviour of $E(V|Q_1)$, so it is natural to construct it about the mean of Q_1, and secondly, because its construction requires certain information about the stochastic process which may only be available in the neighbourhood of $\bar{q}_1$.

Now, the expected future revenue can be split into that accrued in the first time period and that accrued in subsequent time periods:

$$E(V|Q_1) = E(V_1|Q_1) + E(V^{(2)}|Q_1),$$

where

$$V^{(2)} = \sum_{t=2}^{T} V_t.$$

Since V_1 is, by assumption, a known function of q_o and Q_1 the first term on the RHS,

$$V_1(q_o,Q_1),$$

can be calculated analytically. So the problem reduces to one of modelling $E(V^{(2)}|Q_1)$. $V_{C1}(q_o,Q_1-\bar{q}_1)$ is split into the sum of

$V_{AC1}(q_o,Q_1-\bar{q}_1)$ and $V_{BC1}(q_o,Q_1-\bar{q}_1)$, where

$V_{AC1}(q_o,Q_1-\bar{q}_1) = V_1(q_o,Q_1) - E[V_1(q_o,Q_1)]$ and

$V_{BC1}(q_o,Q_1-\bar{q}_1)$ is an estimate of $E(V^{(2)}|Q_1)$.

Recall that the "one-step-ahead" expected revenue V_H is

$$V_H = \sum_{t=1}^{T} E(V_t|Q_{t-1})$$

$$= EV_1 + \sum_{t=2}^{T} E(V_t|Q_{t-1}).$$

Therefore the appropriate function to be modelled in order to derive a suitable control statistic for V_H is

$$E(V_H|Q_1) = EV_1 + E(V^2|Q_1),$$

so again the problem reduces to one of modelling $E(V^2|Q_1)$, the appropriate control statistic simply being

$$V_{BC1}(q_o, Q_1 - \bar{q}_1).$$

However, if the appropriate control statistics are used there is no advantage in forming two estimates of the expected revenue, EV, from the actual revenues accrued in each simulation trial and the "one-step-ahead" revenues, for they will both be the same.

The controlled value of the actual revenue accrued in each trial is

$$V^* = \sum_{t=1}^{T} (V_t - V_{Ct})$$

$$= \sum_{t=1}^{T} (V_t - V_{ACt} - V_{BCt})$$

$$= \sum_{t=1}^{T} \{V_t(Q_{t-1}, Q_t) - V_t(Q_{t-1}, Q_t) + E(V_t|Q_{t-1})\}$$

$$- \sum_{t=1}^{T} V_{BCt},$$

Since $V_{Ct} = V_{ACt} + V_{BCt}$ and

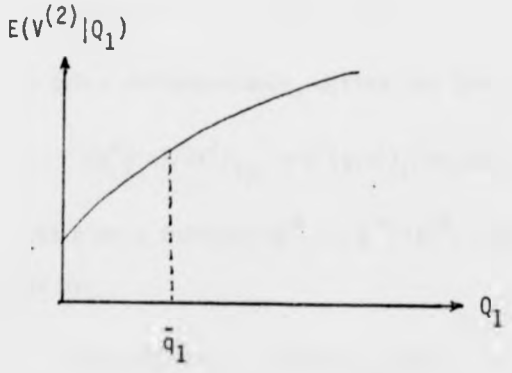
$$V_{ACt} = V_t\,(Q_{t-1}, Q_t) - E(V_t|Q_{t-1}).$$

So $$V^* = \sum_{t=1}^{T} E(V_t|Q_{t-1}) - \sum_{t=1}^{T} V_{BCt}$$

$$= \sum_{t=1}^{T} (V_{Ht} - V_{BCt}),$$

and this is the controlled value of the one-step-ahead revenue V_H.

A graph of $E(V^{(2)}|Q_1)$ against some component of Q_1 will have the following form:



and will usually be concave.

Two models of $E(V^{(2)}|Q_1)$ will be treated, namely a linear and quadratic function of $Q_1 - E(Q_1|q_0)$. These have the merit of being fairly straightforward to derive and computationally effective in practice.

4.2. Some Essential Notation

Let f be a function: $\mathbf{R}^m \dashrightarrow \mathbf{R}$. Then if f is differentiable, define the vector df(x)/dx by:

$$(df(x)/dx)_i = \partial f(x)/\partial x_i .$$

If f is twice differentiable define the matrix $d^2f(x)/dx^2$ by:

$$(d^2f(x)/dx^2)_{ij} = \partial^2 f(x)/\partial x_i \partial x_j .$$

Let g be a function: $\mathbf{R}^m \dashrightarrow \mathbf{R}^m$. Then if g is differentiable, define the matrix dg(x)/dx by:

$$(dg(x)/dx)_{ij} = \partial (g(x))_j/\partial x_i .$$

If g is twice differentiable, define the 3rd rank tensor $d^2g(x)/dx^2$ by:

$$(d^2g(x)/dx^2)_{ijk} = \partial^2 (g(x))_k/\partial x_i \partial x_j .$$

Let h be a function: $\mathbf{R}^m \dashrightarrow \mathbf{R}^m \times \mathbf{R}^m$. Define the 3rd rank tensor dh(x)/dx by:

$$(dh(x)/dx)_{ijk} = \partial (h(x))_{jk}/\partial x_i .$$

Let T be any (m × m × m) 3rd rank tensor, M an (m × m) matrix and v an m-vector. Then define the following products by:

$$TM \text{ by } (TM)_{ijk} = \sum_{\ell} (T)_{ij\ell}(M)_{\ell k}$$

$$MT \text{ by } (MT)_{ijk} = \sum_{\ell} (M)_{i\ell}\,(T)_{\ell jk}$$

$$Tv \text{ by } (Tv)_{ij} = \sum_{k} (T)_{ijk}(v)_{k}$$

and $$vT \text{ by } (vT)_{ij} = \sum_{k} (v)_{k}(T)_{kij}$$

Recall that the sequence of state space random variables $Q_0, Q_1, \ldots, Q_T$ is a Markov process. Let the distribution function of Q_t given that $Q_u = \xi$ where $u < t$ be denoted by:

$$F_t^u(\cdot|\xi)$$

4.3. The Linear Model

For the linear case a martingale difference function of the following form is proposed:

$$V_{BC1}\,(Q_1 - \bar{q}_1) = (Q_1 - q_1)^T\,\alpha(\bar{q}_1)$$

where $\bar{q}_1 = E(Q_1)$ and the tedious explicit representation of the dependence on the initial state space position, q_0, has now been dropped.

This guarantees that $EV_{BC1} = 0$ and hence that $EV_{Ct} = 0$. It is intuitively desirable that α should be set equal to the best possible estimate of the gradient of $E(V^{(2)}|Q_1)$ at $Q_1 = \bar{q}_1$. This value for α has proved computationally effective. So the problem reduces to the estimation of

$$\alpha(q_1) = dE(V^{(2)}|q_1)/dq_1 \text{ at } q_1 = \bar{q}_1$$

Decomposing $E(V^{(2)}|q_1)$ into the sum of contributions made in each time period:

$$E(V^{(2)}|q_1) = \sum_{t=2}^{T} E(V_t|q_1)$$

it is seen that

$$dE(V^{(2)}|q_1)/dq_1 = \sum_{t=2}^{T} dE(V_t|q_1)/dq_1,$$

and the problem reduces to the estimation of each term in the sum.

Now $E(V_t|q_1) = E(V_{Ht}|q_1)$ for $t \geq 2$,

where $V_{Ht} = E(V_t|Q_{t-1})$.

So $dE(V^{(2)}|q_1)/dq_1 = \sum_{t=2}^{T} dE(V_{Ht}|q_1)/dq_1$ and the concern is now with the estimation of $dE(V_{Ht}|q_1)/dq_1$. Under certain conditions to be discussed below, the following relations, which are crucial to this estimation, apply.

Relation R1: $dE(V_{Ht}|q_1)/dq_1 = dE(Q_{t-1}|q_1)dq_1\ d_t(q_1)$

Relation R2: $dE(Q_t|q_1)dq_1 = \mathcal{D}_2(q_1)\mathcal{D}_3(q_1) \ldots \mathcal{D}_t(q_1)$

The definitions of $d_t(q_1)$ and $\mathcal{D}_u(q_1)$ differ according to the conditions which apply, the following two cases only being treated:

case (i) $d_t(q_1) = dV_{Ht}(\xi)/d\xi$ evaluated at $\xi = E(Q_{t-1}|q_1)$

and $\mathcal{D}_u(q_1) = dE(Q_u|Q_{u-1} = \xi)/d\xi$ evaluated at $\xi = E(Q_{u-1}|q_1)$.

case (ii) $d_t(q_1) = E(dV_{Ht}(Q_{t-1})/dQ_{t-1}|q_1)$

and $\mathcal{D}_u(q_1) = E_{Q_{u-1}}(dE_{Q_u}(Q_u|Q_{u-1})/dQ_{u-1}|q_1)$.

So under the appropriate conditions $dE(V_{Ht}|q_1)/dq_1$ may be computed by combining relations R1 and R2 into the following formula:

Formula F1: $dE(V_{Ht}|q_1)/dq_1 = \mathcal{D}_2(q_1)\mathcal{D}_3(q_1) \ldots \mathcal{D}_{t-1}(q_1)d_t(q_1)$

The first two or the last of the following conditions are sufficient to ensure that formula F1 is applicable.

Condition C1: $V_{Ht}(q_{t-1})$ is linear in q_{t-1}.

Condition C2: $E(Q_u|q_{u-1})$ is linear in q_{u-1}, for $u = 2,3,\ldots,t-1$.

Condition C3: There exists a distribution function G_u^v such that

$$F_u^v(\xi|\eta) = G_u^v(\xi - E(Q_u|Q_v = \eta))$$

for all $v \leq u$, and for $u = 2,3,\ldots,t-1$.

Conditions C1 and C2 are self explanatory, but condition C3 deserves further explanation. It is the condition that the distribution of Q_u given knowledge of the outcome q_v of Q_v for $v < u$ depends on q_v through its location parameter only, i.e. The mean but not the shape of the distribution function of Q_u depends on q_v.

It will be shown that the case (i) interpretation of formula F1 holds if conditions C1 and C2 apply, and the case (ii) interpretation of F1 holds if condition C3 applies. Thus the 'case (i) interpretation' of the system will be used to mean that conditions C1 and C2 hold for all $t \geq 2$, and the 'case (ii) interpretation' will be used to mean that condition C3 applies for all $t \geq 2$. The following results are necessary.

Theorem 4.1.

If condition C1 holds then so does relation R1 in either case (i) or case (ii).

Proof

Under condition C1 we have:

$$E(V_{Ht}(Q_{t-1})|q_1) = V_{Ht}(E(Q_{t-1}|q_1)),$$

whence

$$dE(V_{Ht}|q_1)/dq_1 = dE(Q_{t-1}|q_1)/dq_1 \; dV_{Ht}(\xi)/d\xi|_{\xi = E(Q_{t-1}|q_1)},$$

which is R1(i). R1(ii) follows immediately since the second term on the left hand side is constant in q_1.

Theorem 4.2.

If condition C2 holds then so does relation R2 in either case (i) or case (ii).

Proof

We first show that R2(i) holds. The proof proceeds by induction on t. It is sufficient to show that:

$$dE(Q_t|q_1)/dq_1 = dE(Q_{t-1}|q_1)/dq_1 \; dE(Q_t|q_{t-1})/dq_{t-1}|_{q_{t-1}} = E(Q_{t-1}|q_1)$$

since R2(ii) trivially holds for t = 2.

Now $E(Q_t|q_1) = \int E(Q_t|Q_{t-1} = \xi) \; dF^1_{t-1}(\xi|q_1)$

$= E(Q_t|Q_{t-1} = E(Q_{t-1}|q_1))$, applying condition C2.

Therefore $dE(Q_t|q_1) = dE(Q_{t-1}|q_1)/dq_1 \; dE(Q_t|q_{t-1})/dq_{t-1}|_{q_{t-1}} = E(Q_{t-1}|q_1)$

as required.

Noting that under C2 the second term in the right hand side above is constant in q_1, it is seen that relation R2(i) is equivalent to R2(ii), which completes the proof.

Theorem 4.3.

If condition C3 holds, then so do relations R1 and R2 in case (ii).

Proof

We show that R1(ii) holds first.

Now $E(V_{Ht}(Q_{t-1})|q_1) = \int V_{Ht}(\xi)\, dF^1_{t-1}(\xi|q_1)$

$= \int V_{Ht}(\xi)\, dG^1_{t-1}(\xi - E(Q_{t-1}|q_1))$, by condition C3

$= \int V_{Ht}(\xi + E(Q_{t-1}|q_1))\, dG^1_{t-1}(\xi)$, by change of variables

Whence $dE(V_{Ht}|q_1) = dE(Q_{t-1}|q_1)/dq_1 \int dV_{Ht}(\eta)/d\eta|_{\eta=\xi+E(Q_{t-1}|q_1)} dG^1_{t-1}(\xi)$

$= dE(Q_{t-1}|q_1)/dq_1 \int dV_{Ht}(\xi)/d\xi\, dF^1_{t-1}(\xi|q_1)$, by

condition C3 and change of variables. And the above relation is R1(ii).

The second part of the proof proceeds by induction on t. It is sufficient to show that:

$$dE(Q_t|q_1)/dq_1 = dE(Q_{t-1}|q_1)/dq_1\, E_{Q_{t-1}}(dE_{Q_t}(Q_t|Q_{t-1})/dQ_{t-1}|q_1)$$

since R2(i) is trivially true for t = 2.

Now $E(Q_t|q_1) = \int E(Q_t|Q_{t-1} = \xi)\, dF^1_{t-1}(\xi|q_1)$

$= \int E(Q_t|Q_{t-1} = \xi + E(Q_{t-1}|q_1))\, dG^1_{t-1}(\xi)$,

by condition C3 and change of variables.

Whence $dE(Q_t|q_1)/dq_1 = dE(Q_{t-1}|q_1)/dq_1 \int dE(Q_t|q_{t-1})/dq_{t-1}|_{q_{t-1}=\xi+E(Q_{t-1}|q_1)} dG^1_{t-1}(\xi)$

$= dE(Q_{t-1}|q_1)/dq_1 \int dE(Q_t|Q_{t-1} = \xi)/d\xi\, dF^1_{t-1}(\xi|q_1)$

$= dE(Q_{t-1}|q_1)/dq_1\, E_{Q_{t-1}}(dE_{Q_t}(Q_t|Q_{t-1})/dQ_{t-1}|q_1)$,

as required.

If the formula F1 applies in whichever version, then it provides a ready way of decomposing $dE(V^{(2)}|q_1)/dq_1$ into more easily estimated components:

$$dE(V^{(2)}|q_1)dq_1 = \sum_{t=2}^{T} \nu_2(q_1) \ldots \nu_{t-1}(q_1)\, d_t(q_1),$$

where the first term of the sum is $d_2(q_1)$.

From a computational viewpoint the sum is efficiently calculated by the backwards recursion:

$$\beta_T = 0$$

$$\beta_{t-1} = d_t + \nu_t \beta_t$$

the β's being calculated backwards from β_T to β_1, which is the required sum.

The means of constructing the first martingale difference has now been suggested, namely

$$V_{C1}(Q_1-\bar{q}_1) = V_{AC1}(Q_1-\bar{q}_1) + V_{BC1}(Q_1-\bar{q}_1)$$

where

$$V_{AC1}(Q_1-\bar{q}_1) = V_1(Q_1) - EV_1(Q_1),$$

and

$$V_{BC1}(Q_1-\bar{q}_1) = (Q_1-\bar{q}_1)^T\alpha,$$

α being an estimate of $dE(V^{(2)}|q_1)/dq_1$, calculated at $q_1 = \bar{q}_1$.

It must be stressed that even if none of the conditions are satisfied exactly then formula F1 may still be applied if C1 and C2 or C3 hold approximately. All that is required is an estimate of $dE(V^{(2)}|q_1)/dq_1$; even a bad estimate can be better than none at all.

In general the case (i) versions of $\mathcal{D}_u(q_1)$ and $d_t(q_1)$ are easier to calculate than their case (ii) counterparts, so given the choice, it is better to use the case (i) versions.

4.4. The Quadratic Model

Herein, a quadratic form for the martingale difference function $V_{BC1}(Q_1-\bar{q}_1)$, is proposed, namely

$$V_{BC1}(Q_1-\bar{q}_1) = \kappa(\bar{q}_1) + (Q_1-\bar{q}_1)^T\alpha(\bar{q}_1) + (Q_1-\bar{q}_1)^T B(\bar{q}_1)\,(Q_1-\bar{q}_1).$$

It will be constructed to have the following two properties, the first of which is essential, and the second intuitively desirable:

(i) $E(V_{BC1}(Q_1-\bar{q}_1)) = 0$

(ii) The first and second differential of V_{BC1} at $q_1 = \bar{q}_1$ will be equal to the best estimate of those of $E(V^{(2)}|q_1)$ at $q_1 = \bar{q}_1$

Now $E(V_{BC1}(Q_1-\bar{q}_1)) = \kappa(\bar{q}_1) + E((Q_1-\bar{q}_1)^T B(\bar{q}_1)(Q_1-\bar{q}_1))$.

Let Disp (Q_1) be the dispersion matrix of Q_1, then:

$$E(V_{BC1}(Q_1-\bar{q}_1)) = \kappa(\bar{q}_1) + \text{trace}\,\{\text{Disp}\,(Q_1).B(\bar{q}_1)\}\ .$$

So, in order to satisfy the first property $\kappa(\bar{q}_1)$ is chosen to be

$$-\text{trace}\,\{\text{Disp}\,(Q_1).B(\bar{q}_1)\}\ .$$

Notice that without loss of generality, $B(q_1)$ can be taken to be symmetric, i.e. $B(q_1) = B(q_1)^T$.

Sufficient conditions are now derived to satisfy the second property.

$$\text{Now } dV_{BC1}(q_1-\bar{q}_1)/dq_1 = \alpha(\bar{q}_1) + (B(\bar{q}_1) + B(\bar{q}_1)^T)(q_1-\bar{q}_1)$$

$$= \alpha(\bar{q}_1) + 2B(\bar{q}_1)(q_1-\bar{q}_1)^T.$$

$$\text{So} \quad dV_{BC1}(q_1-\bar{q}_1)/dq_1\Big|_{q_1=\bar{q}_1} = \alpha(\bar{q}_1).$$

Hence, as in the linear case, $B_{BC1}(q_1-\bar{q}_1)$ and $E(V^{(2)}|q_1)$ will have the same differential at $q_1 = \bar{q}_1$ if:

$$\alpha(q_1) = dE(V^{(2)}|q_1)/dq_1.$$

The estimation of $dE(V^{(2)}|dq_1)/dq_1$ has already been discussed in the previous section on the linear martingale; the same methods of estimation can be used here.

$$\text{Now } d^2V_{BC1}(q_1-\bar{q}_1)/dq_1^2 = 2\,B$$

therefore, in order to satisfy equality of the second differentials, we must have

$$B = \tfrac{1}{2}d^2E(V^{(2)}|q_1)/dq_1^2\Big|_{q_1=\bar{q}_1},$$

Thus, it is sufficient to set $B(q_1)$ equal to the best estimate available of

$$\tfrac{1}{2}\, d^2E(V^{(2)}|q_1)/dq_1^2$$

and evaluate it at $q_1 = \bar{q}_1$. The remainder of this section will be devoted to its estimation.

By definition $d^2E(V^{(2)}|q_1)/dq_1^2 = d/dq_1(dE(V^{(2)}|q_1)/dq_1)$. $dE(V^{(2)}|q_1)/dq_1$ has already been estimated by $\alpha(q_1)$, so $d^2E(V^{(2)}|q_1)/dq_1^2$ can be estimated by $d\alpha(q_1)/dq_1$.

If an analytical expression for $\alpha(q_1)$, as opposed to a numerical estimate of $\alpha(\bar{q}_1)$, has been found, then $2B(q_1)$ can be obtained by simply differentiating $\alpha(q_1)$. In what follows, hoever, we assume that this is not the case and look for an explicit representation of $d\alpha(q_1)/dq_1$.

It is assumed that we can use formula F1 to estimate $dE(V^{(2)}|q_1)/dq_1$ by $\alpha(q_1)$.

i.e. $$\alpha(q_1) = \sum_{t=2}^{n} \mathcal{D}_2(q_1) \ldots \mathcal{D}_{t-1}(q_1) d_t(q_1).$$

If the case(i) interpretation of $\alpha(q_1)$ applies, i.e. $F(Q_u|q_{u-1})$ and $E(V_{Ht}|q_{t-1})$ are linear in q_{u-1} and q_{t-1} respectively, then $\alpha(q_1)$ is constant in q_1 and $d\alpha(q_1)/dq_1$ is a matrix of zeros,

i.e. $$d\alpha(q_1)/dq_1 = 0.$$

Now consider the case (ii) interpretation of $\alpha(q_1)$, when condition C3 applies and

$$\mathcal{D}_u(q_1) = E_{Q_{u-1}}(dE_{Q_u}(Q_u|Q_{u-1})/dQ_{u-1}|q_1),$$

and

$$d_t(q_1) = E(dV_{Ht}(Q_{t-1})/dQ_{t-1}|q_1).$$

Introduce some convenient notation.

Let $\mathcal{U}_t(q_1) = \mathcal{D}_2(q_1)\mathcal{D}_3(q_1) \ldots \mathcal{D}_t(q_1)$ for $t \geq 2$

$= I$, the indentity matrix, for $t = 1$

and $T_t(q_1) = \mathcal{U}_{t-1}(q_1) d_t(q_1)$ for $t \geq 2$.

Then $$\alpha(q_1) = \sum_{t=2}^{T} T_t(q_1)$$

$$\text{and} \quad dT_t(q_1)/dq_1 = \sum_{u=2}^{t} T_t^u(q_1),$$

$$\text{where } (T_t^u(q_1))_{ij} = \sum_l \sum_k (u_{u-1})_{jl}(d\mathcal{D}_u/dq_1)_{ilk}(\mathcal{D}_{u+1}\cdots\mathcal{D}_{t-1}d_t)_k \quad \text{for } u < t$$

$$= \sum_l (u_{u-1})_{jl}(dd_t/dq_1)_{il} \quad \text{for } u = t$$

$$\text{i.e.} \quad T_t^u = ((d\mathcal{D}_u/dq_1)(\mathcal{D}_{u+1}\cdots\mathcal{D}_{t-1}d_t))u_{u-1}^T \quad \text{for } u < t$$

$$= (dd_t/dq_1)u_{t-1}^T \quad \text{for } u = t$$

We need to investigate $d\mathcal{D}_u/dq_1$ and dd_t/dq_1 further. The following theorem enables them to be decomposed.

Theorem 4.4.

Under condition C3 the following relations hold

Relation R3: $dd_t/dq_1 = u_{t-1}d'_t$, where $d'_t = E(d^2V_{Ht}(Q_{t-1})/dQ_{t-1}^2|q_1)$

Relation R4: $d\mathcal{D}_u/dq_1 = u_{u-1}\mathcal{D}'_u$, where $\mathcal{D}'_u = E_{Q_{u-1}}(d^2E_{Q_u}(Q_u|Q_{u-1})/dQ_{u-1}^2|q_1)$

Proof

Let $V'_{Ht}(\xi) = dV_{Ht}(\xi)/d\xi$, and $V''_{Ht}(\xi) = d^2V_{Ht}(\xi)/d\xi^2$.

Now $d_t = E(V'_{Ht}(Q_{t-1})|q_1)$

$$= \int V'_{Ht}(\xi)\, dF_{t-1}^1(\xi|q_1)$$

$$= \int V'_{Ht}(\xi + E(Q_{t-1}\, q_1))\, dG_{t-1}^1(\xi),$$

by condition C3 and a change of variables.

Therefore $d d_t/dq_1 = dE(Q_{t-1}|q_1)/dq_1 \int V''_{Ht}(\xi + E(Q_{t-1}|q_1))\ dG^1_{t-1}(\xi)$

$= dE(Q_{t-1}|q_1)/dq_1 \int V''_{Ht}(\xi)\ \ dF^1_{t-1}(\xi|q_1)$

$= u_{t-1} d'_t,$

by Theorem 4.3 and the definition of d'_t. This is relation R5.

Let $\mu_u(\xi) = E(Q_u|Q_{u-1} = \xi)$, $\mu'_u(\xi) = d\mu_u(\xi)/d\xi$, and $\mu''_u(\xi) = d^2\mu_u(\xi)/d\xi^2$.

Then $\mathcal{D}_u = \int \mu'_u(\xi)\ dF^1_{u-1}(\xi|q_1)$

$= \int \mu'_u(\xi + E(Q_{u-1}|q_1))\ dG^1_{u-1}(\xi),$

by condition C3 and a change of variables.

i.e. $(\mathcal{D}_u)_{ij} = \int (\mu'_u(\xi + E(Q_{u-1}|q_1)_{ij}\ dG^1_{u-1}(\xi).$

Therefore:

$$\partial(\mathcal{D}_u)_{jk}/\partial q_{i1} = \sum_{\ell} \partial(E(Q_{u-1}|q_1))_1/\partial q_{i1} \int (\mu''_u(\xi+E(Q_{u-1}|q_1)))_{\ell jk}\ dG^1_{u-1}(\xi)$$

Now the integral on the right hand side is:

$\int (\mu''_u(\xi))_{\ell jk}\ dF^1_{u-1}(\xi|q_1)$

$= (E(\mu''_u(Q_{u-1})|q_1))_{\ell jk}$

$= (\mathcal{D}'_u)_{\ell jk}.$

So $d\mathcal{D}_u/dq_1 = dE(Q_{u-1}|q_1)/dq_1\ \mathcal{D}'_u$, whence relation R4 follows by applying Theorem 3.

If condition C3 and therefore Theorem 4 applies, T_u^t may be expressed by:

$$T_u^t = u_{u-1} \Delta_u^t u_{u-1}^T ,$$

where $\Delta_u^t = D_u'(D_{u+1} \cdots D_{t-1} d_t)$ for $u < t$

$= d_t'$ for $u = t$.

Now

$$2B(q_1) = d\alpha(q_1)/dq_1$$

and

$$d\alpha/dq_1 = \sum_{t=2}^{T} \sum_{u=2}^{t} T_u^t$$

$$= \sum_{u=2}^{T} \sum_{t=u}^{T} T_u^t .$$

Writing $\Delta_u = \sum_{t=2}^{T} \Delta_u^t$, we have

Formula F2: $2B(q_1) = \sum_{u=2}^{T} u_{u-1} \Delta_u u_{u-1}^T .$

Notice that $B(q_1)$ given by the above formula is indeed symmetric, since Δ_u^t is and therefore so is Δ_u.

From a computational viewpoint there is an efficient double recursion for the above sum. Define β_u and Γ_u by

$$\beta_T = 0$$

$$\beta_{u-1} = d_u + D_u \beta_u$$

and

$$\Gamma_T = 0$$

$$\Gamma_{u-1} = \Delta_u + D_u \Gamma_u D_u^T .$$

Then $\Delta_u = d_t' + D_u' \beta_u$, and the required sum is simply Γ_1.

Γ_u can be calculated by the backwards recursion from $u = T$ to 1, requiring only the temporary store of β_u, Δ_u, and Γ_u at each stage.

Again, as in the case of the linear martingale difference function, it must be emphasised that formula F2 may still be applied even if none of the requisite conditions apply exactly, for even a bad estimate of $d^2E(V^{(2)}|q_1)/dq_1^2$ may be of considerably more value than none at all.

5. CONCLUSIONS

The need to evaluate approximate production planning algorithms has been discussed and ways of doing so explored. The most promising approach is that of replicative simulation, which is the only way to gauge the actual effect of using an approximate algorithm in practice. Since the accuracy of the estimates of the parameters of interest thus obtained is only proportional to the reciprocal of the square root of the number of trials, it is important to ensure that the variability in the estimates returned by each trial is as small as possible. To this end variance reduction techniques were discussed and in particular the use of control variates investigated in detail.

For each simulation trial a number of control variates will, in general be available. So ways of combining them are of paramount importance. Fixed linear combinations are easiest, both from a conceptual and analytical viewpoint . For a given attribute of interest and a given set of control variates, the optimal fixed linear combination is derived. However, this depends on the dispersion matrix of the control variates and the covariance vector of the control variates and the attributes of interest. In general these will not be known and must be estimated. Use of the consequent estimate of the optimal linear combination leads to problems of bias and difficulty in estimating the variance of the final estimate of the parameter of interest. It is shown that the bias is of order $1/m$, where m is the number of simulation trials. A method of eliminating the bias at the expense of much computation is given and a procedure for avoiding most of the computational effort then derived. The problem of variance

estimation is approached though the work of Lavenberg and Welch [37], and it is shown that the potential variance reduction is degraded as the number of control variates increases.

The use of more general functions of control variates was then explored. These have the merit of being potentially far more powerful than fixed linear functions. If independent functions can be derived for each trial then the problems of bias control and variance estimations do not arise. The class of such functions that were considered is usefully restricted by reference to the application of algorithm evaluation, and particularly to the time periodic structure of the process. It is shown that such functions of control variates form martingale differences. Optimality criteria for these martingale difference functions are derived and the problem then reduces to one of modelling the expected future contribution to the attribute of interest as a function of the realisation of next time-period state space.

Two such models, namely a linear and a quadratic model are proposed. Their construction depends on a detailed decomposition of the process into aspects pertaining to each time period. This is only achieved under one of two assumptions. Whilst the assumptions will not, in general, hold exactly they may often hold approximately and the martingale difference function thus constructed may still be effective. Depending on the assumptions made, appropriate formulae for the models are derived.

These formulae are applied to the 'general model' of Chapter 5 in Chapter 8 and the results of using them in practice on the comparison and evaluation of the "full" algorithm of Beale et al. [4], their "first pass" method and ordinary linear programming are given in Chapter 9.

CHAPTER 8

THE CONSTRUCTION OF THE MARTINGALE CONTROL STATISTIC FOR THE GENERAL MODEL

1. INTRODUCTION

1.1. The Purpose of the Chapter

In the previous chapter it was suggested that approximate scheduling algorithms be evaluated by replicative simulation. Since the accuracy of the method is only proportional to the reciprocal of the square root of the number of simulation trails, it is desirable to remove unnecessary variability in the result of each trial. In pursuit of this object a martingale control statistic was proposed. Direct estimation of the terms of the control statistic being difficult, they were decomposed (under certain assumptions) into more easily estimated components. Estimation of these components is peculiar to the algorithm being tested and the model of the stochastic process within which it works. It is the purpose of this chapter to provide explicit formulae for the estimation of these components in the case of the "general" model developed in Chapters 5 and 6. However the formulae derived herein can, with mild procedural modifications, be applied to two other algorithms, namely the "first pass" or first step of the general model (wherein only one non-linear program is solved) and ordinary linear programming, ignoring all stochastic elements.

1.2. An Outline of the chapter

The "general" model is introduced and pertinent aspects are discussed in section 2. There are two possible interpretations of the formulae derived in the previous chapter, depending on the

assumptions under which they were derived. As in the previous chapter, these will be referenced by case (i) and case (ii). The case (i) versions are much the simpler and are treated first. Only the linear martingale difference function is appropriate in this case. The linear and quadratic martingale difference functions are then tackled under the assumptions of case (ii).

All of the approaches involve some approximations. These are confined to all or some of those needed in the solution algorithm for the general model, and so their use is natural here.

The notation used is that of Chapters 5 and 6. In particular, the convention of the previous chapter that upper case letters denote random variables and lower case letters their realisations is dropped in order to more easily distinguish between matrices and vectors.

2. THE GENERAL MODEL

2.1. A Brief Description of the 'General' Model.

The model is developed in Chapters 5 and 6. It is a generalisation of that of Beale et al. [4]. It assumes that the stochastic process can be represented as a discrete time period Markov process with a fixed, finite horizon. At the start of each time period, the algorithm provides a control vector x_t; during that time period the stochastic environment supplies a random input vector y_t and the state variable evolves from vector q_{t-1} at the start of the period to q_t at the end of the period according to certain evolution equations. The revenue accrued during the tth time period is a linear function of the state vector at the end of the time period and the control vector applied at the start of the time period. The algorithm is designed to maximise the total revenue accrued from the first time period until the time horizon. Given the initial data, the controls applied at the start of each time period are a function of the state space at the end of the preceding time period, thus preserving the Markovian structure of the process. Therefore the revenue accrued during each time period can be considered to be a function of the state vector at the start and finish of the time period only.

2.2. A Symbolic Description of the Process.

The state vector evolves from its value at the start of the t th time period, q_{t-1}, to its value at the end of that time period q_t according to the equations:

$$q_t = q_t^o + A_t q_{t-1} + B_t x_t + C_t w_t + D_t z_t$$

$$z_t = z_t^o + K_t q_{t-1} + L_t x_t - y_t$$

$$w_t = \max (z_t, 0)$$

$$y_t = y_t^o + M_t q_{t-1} + e_t$$

e_t being a multivariate normal random vector with zero mean and dispersion matrix Disp (y_t)

where

q_t^o and y_t^o are fixed vectors and A_t, B_t, C_t, D_t, K_t, L_t and M_t are fixed system matrices.

The revenue accrued during the t th time period is

$$V_t = d_{1t}^T x_t + d_{2t}^T q_t$$

where d_{1t} and d_{2t} are fixed system vectors.

2.3. The Random Input

The random input y_t is driven by the multivariate normal random vector e_t, and for fixed q_{t-1} has dispersion matrix Disp (y_t). Now there are two possible interpretations of the dispersion matrix. In the "additive" case it is fixed, whilst in the "multiplicative" case $(\text{Disp } (y_t))^{\frac{1}{2}}$ is proportional to the mean of y_t.

i.e. $\text{Disp } (y_t) = \tilde{Y}_t P_t^o \tilde{Y}_t$

where P_t^o is a fixed matrix and $\tilde{Y}_t$ is a diagonal matrix, whose i th diagonal entry is $(\tilde{y}_t)_i$ and $\tilde{y}_t = y_t^o + M_t q_{t-1}$.

For the case (ii) interpretation of the formulae derived in the previous chapter, the two models have to be treated separately, whilst for the case (i) interpretation there is no need to distinguish between the models of the stochastic input.

2.4. The Approximation of the Control Vector Provided by the Algorithm

As in the solution procedure of the general model, it is necessary to approximate the control provided by the algorithm x_t by a linear function of the previous time period state vector, q_{t-1}.

i.e. $$x_t = x_t^o + N_t q_{t-1},$$

x_t^o being a fixed vector and the coefficient matrix N_t being derived from an analysis of the final non-linear program solved by the algorithm. This approximation is an essential part of the variance estimation procedure of the general model algorithm and is discussed in more detail in Chapter 6, Section 5.1. Using it the evolution equations can be simplified and become

$$q_t = q_t^1 + H_t q_{t-1} + C_t w_t - D_t e_t$$

$$z_t = z_t^1 + J_t q_{t-1} - e_t$$

$$w_t = \max (z_t, 0)$$

where

$$z_t^1 = z_t^o + L_t x_t^o - y_t^o$$

$$q_t^1 = q_t^o + B_t x_t^o + D_t z_t^1$$

$$J_t = K_t + L_t N_t - M_t$$

$$H_t = A_t + B_t N_t + D_t J_t$$

This is the form of the evolution equations which is amenable to analysis and will therefore be the one hereafter used.

2.5. Further Approximations

Other approximations are used to split the martingale difference function into more readily estimated components. There are two possible approaches. If it can be supposed that the mean of the distribution of q_t depends linearly on the outcome of q_{t-1}, then the case (i) interpretation of the terms of the formulae derived in Chapter 7 is appropriate. Alternatively, if it may be supposed that only the mean and not the shape of the distribution of q_t depends on the outcome of q_{t-1}, then the case (ii) interpretation of the terms of those formulae is appropriate. Neither supposition will in general be exactly true but either or both may hold approximately. The latter approximation can intuitively be expected to be good in the case where the additive model of the stochastic input is being used; then the shape of the distribution of q_t only varies with the outcome of q_{t-1} through the point of truncation of w_t. Its usefulness in the case where the multiplicative model of the stochastic input is being used is, however, more doubtful.

The case (i) interpretation is treated in the next section. The case (ii) interpretation is more complex and the succeeding three sections are devoted to it.

3. THE CASE (i) INTERPRETATION OF THE MARTINGALE DIFFERENCE FUNCTION

In this section the mean of q_t will be supposed to be a linear function of the outcome of q_{t-1}. The merits and demerits of this assumption have already been discussed; this section is devoted to obtaining specific formulae.

As has already been mentioned in Chapter 7, Section 4.4, under the above assumption all the quadratic terms in the quadratic martingale difference function are identically zero, so here it is only necessary to deal with the linear case.

It is therefore sufficient to provide formulae for the calculation of d_t and D_t where these are defined by:

$$d_t = dE(V_t|q_{t-1})/dq_{t-1}$$

and $$D_t = dE(q_t|q_{t-1})/dq_{t-1}$$

both being evaluated at $q_{t-1} = E(q_{t-1}|q_1)$.

Approximating the control decisions by a linear function of the state space at the end of the preceding time period and using the notation of Section 2.4 it is seen that

$$D_t = (H_t)^T + dE(w_t|q_{t-1})/dq_{t-1}.C_t^T$$

where the differential is evaluated at $q_{t-1} = E(q_{t-1}|q_1)$.

Also

$$\begin{aligned} V_t(q_{t-1},q_t) &= d_{1t}^T q_t + d_{2t}^T x_t \\ &= d_{2t}^T x_t^o + d_{2t}^T N_t q_{t-1} + d_{1t}^T q_t, \end{aligned}$$

where the control decision x_t is being approximated by $x_t = x_t^o + N_t q_{t-1}$.

Differentiating with respect to q_{t-1} it is seen that

$$dE(V_t|q_{t-1})/dq_{t-1} = N_t^T d_{2t} + dE(q_t|q_{t-1})/dq_{t-1} \cdot d_{1t}$$

so $d_t = N_t^T d_{2t} + D_t d_{1t}$.

Therefore, the approximation necessary for the case (i) interpretation of the martingale difference function, namely the approximation of $E(q_t|q_{t-1})$ as a linear function of q_{t-1} is one of approximating $E(V_t|q_{t-1})$ as a linear function of q_{t-1}.

The approximation suggested here is one of approximating each component of w_t by a linear function of the corresponding component of z_t in such a way that the variance of each component of w_t is unchanged

i.e. $w_{it} = \psi_{it} z_{it} + (\text{constant})$

where $\psi_{it} = f(z_{it}^*/\sigma_{it})$, σ_{it} and z_{it}^* being the standard deviation and mean of $(z_{it}|q_1)$ and $f(x)$ being defined to be the positive square root of

$$x\phi(x)[1 + 2\Phi(x)] + \{1 + x^2[1 - \Phi(x)]\}\Phi(x) - [\phi(x)]^2.$$

ϕ and Φ are the Gaussian probability density function and distribution function respectively. In this way $\psi_{it} z_{it}$ and $\max(z_{it}, 0)$ have the same variance. Defining Ψ_t to be the diagonal matrix whose i th diagonal component is ψ_{it}, w_t is approximated by

$$w_t = \Psi_t z_t + (\text{constant}).$$

Although the covariance terms of w_t are not, in general, preserved by this approximation, it appears to work well in practice as the results of Chapter 9 show.

Using this approximation it is seen that

$$E(w_t|q_{t-1}) = \Psi_t J_t q_{t-1} + (\text{constant}) .$$

Hence

$$dE(w_t|q_{t-1})/dq_{t-1} = (\Psi_t J_t)^T$$

which, when substituted into the expression for D_t yields

$$D_t = H_t^T + J_t^T \Psi_t C_t^T$$

and we still have

$$d_t = N_t^T d_{2t} + D_t d_{1t}.$$

Notice that this derivation applies to both the additive and multiplicative models, for the only relevant difference between the two is in the estimation of σ_{it}. A procedure for the estimation of the entire dispersion matrix of z_t is given in Chapter 6 for both the additive and multiplicative models, although it should be run forward recursively from time period 2 instead of time period 1, starting at the expected value of the first time period state variable, $\bar{q}_1$. Here z_t^* is approximated by $E(z_t)$ given by the model's solution from which N_t and hence J_t (= $K_t + L_t N_t - M_t$) is also derived. Attention is now directed to the case (ii) interpretation of the martingale difference function where it is necessary to carefully distinguish between the additive and multiplicative models.

4. THE CASE (ii) INTERPRETATION OF THE MARTINGALE DIFFERENCE FUNCTION: THE LINEAR MODEL

4.1. The Terms Requiring Calculation

In this section it is supposed that only the mean and not the shape of the distribution of the state space at the end of each time period depends on the previous statevector realisation. With this assumption the components of the linear model of the expected future revenue which require calculations are d_t and $\mathcal{D}_t$, which are defined by

$$d_t = E_{q_{t-1}}(dE_{q_t}(V_t|q_{t-1})/dq_{t-1}|q_1)$$

and $$\mathcal{D}_t = E_{q_{t-1}}(dE_{q_t}(q_t|q_{t-1})/dq_{t-1}|q_1).$$

As in Section 3, the key calculation in the estimation of the above terms is

$$E_{q_{t-1}}[dE_{q_t}(w_t|q_{t-1})/dq_{t-1}|q_1]$$

for it was shown there that

$$dE_{q_t}(q_t|q_{t-1})/dq_{t-1} = (H_t)^T + dE(w_t|q_{t-1})/dq_{t-1}(C_t)^T$$

and $$dE_{q_t}(V_t|q_{t-1})/dq_{t-1} = (N_t)^T + [dE(q_t|q_{t-1})/dq_{t-1}]d_{1t}.$$

Taking expectations over q_{t-1} for given q_1 and setting

$$\mathcal{D}_{wt} = E_{q_{t-1}}[dE_{q_t}(w_t|q_{t-1})/d_{q_{t-1}}|q_1]$$

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As in Section 3, the key calculation in the estimation of the above terms is

$$E_{q_{t-1}}[dE_{q_t}(w_t|q_{t-1})/dq_{t-1}|q_1]$$

for it was shown there that

$$dE_{q_t}(q_t|q_{t-1})/dq_{t-1} = (H_t)^T + dE(w_t|q_{t-1})/dq_{t-1}(C_t)^T$$

and $$dE_{q_t}(V_t|q_{t-1})/dq_{t-1} = (N_t)^T + [dE(q_t|q_{t-1})/dq_{t-1}]d_{1t}.$$

Taking expectations over q_{t-1} for given q_1 and setting

$$\mathcal{D}_{wt} = E_{q_{t-1}}[dE_{q_t}(w_t|q_{t-1})/d_{q_{t-1}}|q_1]$$

it follows that

$$D_t = (H_t)^T + D_{wt}(C_t)^T$$

and $$d_t = (N_t)^T d_{2t} + D_t d_{1t}.$$

4.2. The Distributions of z_t and $\tilde{z}_t$

Before further progress can be made, it is necessary to study the distribution of z_t and $\tilde{z}_t$, where

$$z_t = \tilde{z}_t + e_t$$

and $$\tilde{z}_t = z_t^1 + J_t q_{t-1}.$$

Here e_t is a zero mean normal random vector with dispersion matrix Disp (y_t). For fixed q_{t-1}, $\tilde{z}_t$ is, of course, fixed, but consider the distribution of $\tilde{z}_t$ when q_{t-1} is considered to be a random variable conditional on q_1. For this purpose it is expedient to approximate the distribution of z_t given q_1 by a multivariate normal distribution with mean

$$z_t^* = E(\tilde{z}_t|q_1) = E(z_t|q_1)$$

and dispersion matrix Disp $(\tilde{z}_t|q_1)$.

Introduce some convenient notation. Let

$$r_{it}^2 = (\text{Disp } (\tilde{z}_t|q_1))_{ii}$$

$$s_{it}^2 = (\text{Disp } (y_t))_{ii}$$

and $$\sigma_{it}^2 = r_{it}^2 + s_{it}^2$$

then σ_{it}^2 is the i th diagonal entry of the dispersion matrix of z_t given q_1. For convenience also define the diagonal matrices S_t, Σ_t to be those whose i th diagonal entry is s_{it} and σ_{it} respectively.

Since w_t is defined by

$$w_t = \max (z_t, 0)$$

it is easy to show that

$$E(w_t|q_{t-1}) = F(S_t^{-1} \tilde{z}_t)s_t + G(S_t^{-1} \tilde{z}_t)\tilde{z}_t$$

where s_t is a vector whose i th entry is s_{it} and G and F are functions from $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ (n being the dimension of z_t), defined by

$$(F(x))_{ij} = \begin{cases} \phi(x_i) & i = j \\ 0 & i \neq j \end{cases}$$

and

$$(G(x))_{ij} = \begin{cases} \Phi(x_i) & i = j \\ 0 & i \neq j \end{cases}$$

ϕ being the Gaussian probability density function and $\Phi(x) = \int_{-\infty}^{x} \phi$.

Differentiating with respect to q_{t-1} one obtains:

$$dE(w_t|q_{t-1})/dq_{t-1} = d(F(S_t^{-1} \tilde{z}_t)s_t)/dq_{t-1} + d(G(S_t^{-1} \tilde{z}_t)\tilde{z}_t)/dq_{t-1}.$$

Calculation of the terms on the right hand side depends on the model of the random input used. In the "multiplicative" case s_{it} depends on q_{t-1}, whereas in the additive case it is fixed. The latter version is simpler and is treated first.

4.3. The Additive Model of the Random Input

In this model, the dispersion matrix of y_t, the random input is fixed, i.e. independent of the previous time period state vector q_{t-1}. Thus s_{it} is independent of q_{t-1} and

$$(d(F(S_t^{-1}\tilde{z}_t)s_t)/dq_{t-1})_{ij}$$

$$= \partial(\phi(\tilde{z}_{jt}/s_{jt}))/\partial q_{it-1}\cdot s_{jt}$$

$$= -(\tilde{z}_{jt}/s_{jt})\phi(\tilde{z}_{jt}/s_{jt})\ \partial\tilde{z}_{jt}/\partial q_{it-1}.$$

Here $\tilde{z}_{jt}$ and q_{it-1} are the j th and i th components of $\tilde{z}_t$ and q_{t-1} respectively.

Also $(d(G(S_t^{-1}\tilde{z}_t))/dq_{t-1})_{ij}$

$$= \partial(G(S_t^{-1}\tilde{z}_t)\tilde{z}_t)/\partial q_{it-1}$$

$$= (\partial\tilde{z}_{jt}/\partial q_{it-1})\{\phi(\tilde{z}_{jt}/s_{jt}) + (\tilde{z}_{jt}/s_{jt})\phi(\tilde{z}_{jt}/s_{jt})\}$$

whence, adding the two expressions above, the rate of change of $E(w_t|q_{t-1})$ with respect to q_{t-1} is seen to be:

$$(dE(w_t|q_{t-1})/dq_{t-1})_{ij} = (\partial\tilde{z}_{jt}/\partial q_{it-1})\phi(\tilde{z}_{jt}/s_{jt}).$$

But $\tilde{z}_t = z_t^1 + J_t\, q_{t-1}$, therefore

$$\partial\tilde{z}_{jt}/\partial q_{it-1} = (J_t)_{ji}, \text{ and}$$

$$dE(w_t|q_{t-1})/dq_{t-1} = J_t^T\, G(S_t^{-1}\tilde{z}_t).$$

Regarding $\tilde{z}_t$ as a random variable in the manner discussed in Section 4.2, it is seen that

$$E(G(S_t^{-1} \underset{\sim}{z}_t)) = G(\Sigma_t^{-1} z_t^*).$$

Details of the calculation are given in Section 6.1.

Therefore

$$\mathcal{D}_{wt} = E_{q_{t-1}}(dE_{q_t}(w_t|q_{t-1})/dq_{t-1}|q_1)$$

$$= J_t^T G(\Sigma_t^{-1} z_t^*),$$

where the expectation over q_{t-1} is taken conditional on q_1. This expression for $\mathcal{D}_{wt}$ may be substituted into those already derived for $\mathcal{D}_t$ and d_t to yield:

$$\mathcal{D}_t = H_t^T + J_t^T G(\Sigma_t^{-1} z_t^*) C_t^T$$

and $$d_t = N_t^T d_{2t} + \mathcal{D}_t d_{1t}.$$

A suitable procedure for the estimation of Σ_t and z_t^* is discussed in Section 4.5. Having thus estimated $\mathcal{D}_t$ and d_t for $t = 2,3,\ldots,T$ formula F1 derived in the previous chapter may then be used to calculate the first linear martingale difference function of the martingale control statistic. Notice the marked similarity between the formulae for $\mathcal{D}_t$ and d_t derived in Section 3 for the case (i) interpretation and those derived in this section. Those of Section 3 are those of this section with

$$\Psi_t: (\Psi_t)_{ij} = \begin{cases} f(z_{it}^*/\sigma_{it}) & i = j \\ 0 & i \neq j \end{cases}$$

replaced by

$$G(\Sigma_t^{-1} z_t^*): (G(\Sigma_t^{-1} z_t^*))_{ij} = \begin{cases} \Phi(z_{it}^*/\sigma_{it}), & i = j \\ 0 & i \neq j \end{cases}$$

Φ being the Gaussian distribution function and f being defined in Section 3. They both have similar properties in that for fixed σ_{it} both $(\Psi_t)_{ii}$ and $(G(\Sigma_t^{-1} z_t)_{ii}$ tend to 1 as z_{it}^* tends to infinity and to zero as z_{it}^* tends to minus infinity.

It is now necessary to compute D_t and d_t for the "multiplicative" model of the stochastic input.

4.4. The Multiplicative Model of the Stochastic Input

Here, the dispersion matrix of y_t and hence s_{it} is not fixed but is linearly dependent on q_{t-1}.

i.e. $$S_t = S_t^o \tilde{y}_t$$

and $$\tilde{y}_t = y_t^o + M_t q_{t-1}.$$

S_t^o is a fixed diagonal matrix whose i th diagonal entry is s_{it}^o and $\tilde{Y}_t$ is a diagonal matrix whose i th diagonal entry is the i th component of $\tilde{y}_t$.

Therefore

$$\partial s_{jt}/\partial q_{it-1} = (S_t^o M_t)_{ji}$$

Differentiating $E(w_t|q_{t-1})$ with respect to q_{t-1} the following is obtained:

$$dE(w_t|q_{t-1})/dq_{t-1} = (S_t^o M_t)^T F(S_t^{-1} \tilde{z}_t) + J_t^T G(S_t^{-1} \tilde{z}_t).$$

Regarding q_{t-1} and hence $\tilde{z}_t$ as random variables, it is seen that

$$E(F(S_t^{-1} \tilde{z}_t)) = S_t \Sigma_t^{-1} F(\Sigma_t^{-1} z_t^*)$$

where the expectation is conditional on q_1. Details of the calculation are given in Section 6.2. It has already been shown that the expectation of $G(S_t^{-1} \tilde{z}_t)$ conditional on q_1 is $G(\Sigma_t^{-1} z_t^*)$ so

$$\mathcal{D}_{wt} = E_{q_{t-1}}(dE_{q_t}(w_t|q_{t-1})/dq_{t-1}|q_1)$$

$$= (S_t^o M_t)^T S_t \Sigma_t^{-1} F(\Sigma_t^{-1} z_t^*) + J_t^T G(\Sigma_t^{-1} z_t^*).$$

When substituted into the expressions for $\mathcal{D}_t$, d_t in terms of $\mathcal{D}_{wt}$, this yields

$$\mathcal{D}_t = H_t^T + \{(S_t^o M_t)^T S \Sigma_t^{-1} F(\Sigma_t^{-1} z_t^*) + J_t^T G(\Sigma_t^{-1} z_t^*)\} C_t^T$$

and $d_t = N_t^T d_{2t} + \mathcal{D}_t d_{1t}$.

Again, once σ_{it} and z_t^* have been estimated the above expressions provide a ready way to estimate $\mathcal{D}_t$ and d_t and hence the first martingale difference function via formula F1 and its associated recursion given in the previous chapter.

4.5. Estimation of the Terms in the Formulae for $\mathcal{D}_t$ and d_t.

Once N_t has been estimated, J_t and H_t can be calculated from

$$J_t = K_t + L_t N_t - M_t,$$

$$H_t = A_t + B_t N_t + D_t J_t,$$

since A_t, B_t, D_t, K_t, L_t and M_t are system matrices. S_t and Σ_t are diagonal matrices whose i th diagonal term is s_{it} and σ_{it}, where s_{it} is the standard deviation of the i th component of the t th time period random input whilst σ_{it} is the standard deviation of the i th component of $(z_t|q_1)$.

z_t^* is approximated by $E(z_t)$ given in the model's solution from which N_t is also derived. σ_{it} is found by the recursive procedure in the model's solution algorithm for the estimation of the dispersion matrix of z_t and just taking the square root of its i th diagonal term. However, the recursive procedure must be run forwards from time period 2 instead of time period 1, starting at the expected position on the state space at the end of the first time period.

5. The Case (ii) Interpretation of the Martingale Control Statistic: The Quadratic Model.

5.1. The Terms Requiring Calculation

In this section concern is with the construction of a quadratic martingale difference function of the form

$$V_{C1} = V_{AC1} + V_{BC1},$$

$$V_{BC1} = V_{BC1}(q_1 - \bar{q}_1),$$

$$= \kappa(\bar{q}_1) + (q_1 - \bar{q}_1)^T \alpha(\bar{q}_1) + (q_1-\bar{q}_1)B(\bar{q}_1)(q_1-\bar{q}_1)$$

where $\bar{q}_1 = E(q_1)$.

Construction of the linear coefficient, $\alpha(\bar{q}_1)$ is identical to that of the linear martingale difference function, treated in Section 4. Construction of the quadratic matrix coefficient, $B(\bar{q}_1)$, is achieved via recursions given in the previous chapter involving d_t, $\mathcal{D}_t$, d'_t and $\mathcal{D}'_t$, there being defined by

$$d_t = E_{q_{t-1}}(dE_{q_t}(V_t|q_{t-1})/dq_{t-1}|q_1)$$

$$\mathcal{D}_t = E_{q_{t-1}}(dE_{q_t}(q_t|q_{t-1})/dq_{t-1}|q_1)$$

$$d'_t = E_{q_{t-1}}(d^2E_{q_t}(V_t|q_1)/dq^2_{t-1}|q_1)$$

and $$\mathcal{D}'_t = E_{q_{t-1}}(d^2E_{q_t}(q_t|q_{t-1})/dq^2_{t-1}|q_1).$$

Formulae enabling d_t and $\mathcal{D}_t$ to be calculated have already been derived in Section 4, so it remains to calculate $\kappa(\bar{q}_1)$, d'_t and $\mathcal{D}'_t$.

The constant term $\kappa(\bar{q}_1)$ must be calculated exactly as any errors would lead to a bias in the control statistic i.e. $E(V_{BC1}) \neq 0$, which would destroy its usefulness. It was shown in the previous chapter that the necessary value of $\kappa(\bar{q}_1)$ that ensures unbiasedness of the control statistic is

$$-\text{trace } \{\text{Disp } (q_1).B(\bar{q}_1)\},$$

where Disp (q_1) is the dispersion matrix of the first time period state variable, q_1. Fortunately the formulae given in Sections 5.1 and 5.2 of Chapter 6 for the estimation of Disp (q_t) in terms of Disp (q_{t-1}), Disp (y_t) and Disp (z_t) gives an exact value for Disp (q_1), for Disp (q_0) can be taken to be identically zero, so z_1 is normally distributed with dispersion matrix Disp (y_1), which is known exactly and Disp (q_1) is

$$(D_1 + C_1U_1)\text{Disp } (y_1)(D_1 + C_1U_1)^T + C_1[\text{Disp}(w_t - \text{Disp } (y_1)]C_1^T$$

where U_1 is a diagonal matrix whose i th diagonal entry is $\Phi(\bar{z}_{i1}/\sqrt{(\text{Disp}(y_i))_{ii}})$, Φ being the Gaussian distribution function and Disp (w_1) is calculated exactly from the formulae derived in Section 5.3 of Chapter 6.

5.2. The Additive Model of the Random Input

The form of the evolution equations derived in Section 2.4 is

$$q_t = q_t^1 + H_t q_{t-1} + C_t w_t - D_t e_t$$

$$z_t = z_t^1 + J_t q_{t-1} - e_t$$

and $$w_t = \max (z_t, 0),$$

where q_t^1, z_t^1 are fixed vectors and H_t, J_t are considered to be fixed matrices, e_t is a zero mean multivariate normal random vector with dispersion matrix Disp (y_t).

Define

$$\mathcal{D}'_{wt} = E(d^2E(w_t|q_{t-1})/dq_{t-1}^2|q_1)$$

then

$$\mathcal{D}'_t = E(d^2E(q_t|q_{t-1})/dq_{t-1}^2|q_1)$$

$$= \mathcal{D}'_{wt}\, C_t^T$$

i.e. $E(\partial^2E(q_{kt}|q_{t-1})/\partial q_{it-1}\partial q_{jt-1}|q_1) = \sum_\ell (\mathcal{D}'_{wt})_{ij\ell}(C_t)_{k\ell}$

Also in Section 3 it was shown that

$$V_t = d_{2t}^T\, x_t^o + d_{2t}^T\, N_t q_{t-1} + q_{1t}^T\, q_t,$$

x_t^o being a fixed vector, and so

$$d'_t = E(d^2E(V_t|q_{t-1})/dq_{t-1}^2) = \mathcal{D}'_{wt}\, d_{1t}$$

i.e. $$E(\partial^2E(V_t|q_{t-1})/\partial q_{it-1}\partial q_{jt-1}|q_1) = \sum_k (\mathcal{D}'_{wt})_{ijk}(d_{1t})_k.$$

Therefore the key term in the estimation of $\mathcal{D}'_t$ and d'_t is $\mathcal{D}'_{wt}$. It is to the estimation of $\mathcal{D}'_{wt}$ that attention is now directed.

In Section 4.3 it was shown that

$$\partial E(w_{kt}|q_{t-1})/\partial q_{it-1} = (J_t)_{ki}\, \Phi(\tilde{z}_{kt}/s_{kt}),$$

Φ being the Gaussian distribution function, $\tilde{z}_{kt}$, s_{kt} being the mean and standard deviation of z_{kt} for fixed q_{t-1}.

Differentiating the above expression with respect to q_{jt-1} it is seen that

$$\partial^2 E(w_k|q_{t-1})/\partial q_{it-1}\partial q_{jt-1} = (J_t)_{ki}\,\phi(\tilde{z}_{kt}/s_{kt})(\partial\tilde{z}_{kt}/\partial q_{jt-1})1/s_{kt}$$

$$= (J_t)_{ki}\,(J_t)_{kj}\phi(\tilde{z}_{kt}/S_{kt}).1/s_{kt} \quad ,$$

since $\partial\tilde{z}_{kt}/\partial q_{jt-1} = (J_t)_{kj}$.

Regarding q_{t-1} and hence $\tilde{z}_t$ as random variables it can be shown that

$$E(\phi(\tilde{z}_{kt}/s_{kt})|q_1) = (s_{kt}/\sigma_{kt})\phi(z^*_{kt}/\sigma_{kt}),$$

where $\tilde{z}_{kt}$ is assumed to be normally distributed and z^*_{kt}, σ_{kt} are the mean and standard deviation of z_{kt} for fixed q_1. Details of the calculation can be found in Section 6.2.

Therefore

$$(D'_{wt})_{ijk} = (J_k)_{ki}(J_t)_{kj}\,\phi(z^*_{kt}/\sigma_{kt})\;1/\sigma_{kt}.$$

Substituting this result into the equations for D'_t, d'_t it is seen that

$$(D'_t)_{ijk} = \sum_{\ell} (J_t)_{\ell i}\,(J_t)_{\ell j}\phi(z^*_{\ell t}/\sigma_{\ell t}).1/\sigma_{\ell t}.(C_t)_{k\ell}$$

and $$(d'_t)_{ij} = \sum_k (D'_t)_{ijk}(d_{1t})_k.$$

Once σ_{it} and z^*_{it} have been estimated the above equations provide a convenient method of estimating D'_t and d'_t. Calculation of the quadratic coefficient $B(\bar{q}_1)$ is then achieved via the recursion formulae F2 derived in the previous chapter. It is now necessary to study the multiplicative model of the random input.

5.4. The Multiplicative Model of the Random Input

In the multiplicative case, s_{it}, the standard deviation of z_{it} for fixed q_{t-1} is not fixed but a linear function of q_{t-1},

i.e. $S_t = S_t^o(y_t^o + M_t\, q_{t-1})$,

s_t, s_t^o being diagonal matrices whose i th entry is s_{it}, s_{it}^o. s_{it}^o is fixed as is the vector y_t^o. As in the additive case discussed in Section 5.3 the key term in the estimation of $\mathcal{D}_t'$ and d_t' is $\mathcal{D}_{wt}'$:

$$(\mathcal{D}_{wt}')_{ijk} = E(\partial^2 E(w_{kt}|q_{t-1})/\partial q_{it-1}\partial q_{jt-1}|q_1).$$

Now, it has already been shown in Section 4.4 that for the multiplicative model

$$\partial E(w_{kt}|q_{t-1})/\partial q_{it-1} = (R_t)_{ki}\,\phi(z_{kt}/s_{kt}) + (J_t)_{ki}\phi(z_{kt}/s_{kt})$$

where $R_t = S_t^o M_t$.

Differentiating the above expression with respect to q_{jt-1} it is seen that

$$\partial^2 E(w_{kt}|q_{t-1})/\partial q_{it-1}\partial q_{jt-1} =$$

$$[(J_t)_{ki} - (\tilde{z}_{kt}/s_{kt})(R_t)_{ki}][(J_t)_{kj} - (\tilde{z}_{kt}/s_{kt})(R_t)_{kj}]\frac{1}{s_{kt}}\phi(\tilde{z}_{kt}/s_{kt}).$$

Regarding q_{t-1} and hence z_t as random variables for given q_1 it is seen that

$$e_{0kt} = E(\phi(\tilde{z}_{kt}/s_{kt})|q_1) = (s_{kt}/\sigma_{kt})\phi(z^*_{kt}/\sigma_{kt})$$

$$e_{1kt} = E((\tilde{z}_{kt}/s_{kt})\phi(\tilde{z}_{kt}/s_{kt})|q_1) = (s^2_{kt}/\sigma^2_{kt})(z^*_{kt}/\sigma_{kt})\phi(\tilde{z}^*_{kt}/\sigma_{kt})$$

$$e_{2kt} = E((\tilde{z}_{kt}/s_{kt})^2\phi(\tilde{z}_{kt}/s_{kt})|q_1) = (r^2_{kt}/\sigma^2_{kt})e_{0kt} + (s_{kt}/\sigma_{kt})e_{1kt}.$$

Again, z^*_{kt} and σ_{kt} are the mean and standard deviation of z_{kt} given q_1, $r^2_{kt} = \sigma^2_{kt} - s^2_{kt}$, and details of the calculation which assume $\tilde{z}_t$ to be normally distributed are given in Section 6.2.

Therefore, taking expectations it is seen that $(D'_{wt})_{ijk}$ is

$$\{(J_t)_{ki}(J_t)_{kj}e_{0kt} - [(R_t)_{ki}(J_t)_{kj} + (R_t)_{kj}(J_t)_{ki}]e_{1kt} + (R_t)_{ki}(R_t)_{kj}e_{2kt}\}(1/s_{kt}).$$

Therefore $(D'_t)_{ijk}$ may be expressed by

$$\sum_{\ell} \{(J_t)_{\ell i}(J_t)_{\ell j}e_{0\ell t} - [(R_i)_{\ell i}(J_t)_{\ell j} + (R_t)_{\ell j}(J_t)_{\ell i}]e_{1\ell t} + (R_t)_{\ell i}(R_t)_{\ell j}e_{2\ell t}\}(1/S_{\ell t})C_{k\ell}$$

and $$(d'_t)_{ij} = \sum_k (D'_t)_{ijk}(d_{1t})_k.$$

As in the case of the formulae for the terms in the linear martingale difference function, R_t is a fixed system matrix; having found N_t from the model's solution, J_t is determined from a combination of N_t and given system matrices. z^*_t is approximated by the value of $E(z_t)$ given by the model's solution and σ_{it} is found by running the recursive Disp (z_t) evaluation procedure of the model's solution technique forwards starting at time period 2 instead of period 1 and then setting $\sigma^2_{it} = (\text{Disp } (z_t))_{ii}$.

6. SOME NECESSARY CALCULATIONS

6.1. The Calculation of $E(\Phi(z/s))$.

Let Φ be the Gaussian distribution function and z a (univariate) random variable, normally distributed with mean z^* and variance r^2. Let s be a real positive constant. Then it is required to calculate

$$E(\Phi(z/s)) = \int_{\xi \in R} \phi(\xi/s) d\Phi((\xi - z^*)/r). \tag{1}$$

Let $\phi(x,y;\rho)$ be the standardized bivariate normal distribution

$$\text{i.e.} \phi(x,y;\rho) = \frac{1}{2\pi\sqrt{(1-\rho^2)}} \exp - \frac{1}{2(1-\rho^2)} \{x^2 - 2\rho xy + y^2\}.$$

Consider the double integral I, defined by

$$I = \int_{\xi \in R} \int_{\eta \geq -x} \phi(\xi,\eta;\rho) d\eta d\xi.$$

Changing the order of integration and integrating over $\xi \in R$ it is seen that

$$I = \Phi(x). \tag{2}$$

However integrating over $\eta \geq -x$ first one obtains:

$$I = \int_{\xi \in R} \phi(\xi) \int_{\eta \geq -x} d\Phi((\eta - \rho\xi)/\sqrt{(1-\rho^2)}) d\xi$$

$$= \int_{\xi \in R} \phi(\xi) \Phi((\rho\xi + x)/\sqrt{(1-\rho^2)}) d\xi$$

$$= \int_{\xi \in R} \phi(\xi) d\Phi\left(\frac{\xi - x/\sqrt{(1-\rho^2)}}{\rho/\sqrt{(1-\rho^2)}}\right) \tag{3}.$$

Return to the original integral (1). Changing the variable of integration it is seen that

$$E(\phi(z/s)) = \int_{\xi \epsilon R} \phi(\xi) d\phi\left(\frac{\xi - z^*/s}{r/s}\right). \qquad (4)$$

Compare (3) with (4). The integrals are the same if

$$x/\sqrt{(1-\rho^2)} = z^*/s \text{ and } \rho/\sqrt{(1-\rho^2)} = r/s.$$

Solving the latter for ρ it is seen that

$$\rho^2 = (r/s)^2/(1 + (r/s)^2).$$

Whence $x = z^*/s\sqrt{(1+(r/s)^2)} = z^*/\sqrt{(r^2 + s^2)}$.

Therefore, comparing (4) with (2) it is seen that

$$E(\phi(z/s)) = \phi(z^*/\sqrt{r^2 + s^2})).$$

6.2. <u>The Calculation of $E((z/s)^n \phi(z/s))$ for $n = 0,1,2$.</u>

Again, z is a random variable distributed as $N(z^*, r^2)$ and s is a real positive constant. ϕ is the Gaussian probability density function.

Set $I_n = E((z/s)^n \phi(z/s))$

$$= \int_{\xi \epsilon R} (\xi/s)^n \phi(\xi/s) d\phi((\xi - z^*)/r)$$

$$= \int ((r/s)\eta + z^*/s)^n \phi(r/s)\eta + (z^*/s))\phi(\eta)d\eta,$$

by change of variables. Rearranging the exponents of the two ϕ's in the integrand it is seen that

$$I_n = \phi\left(\frac{z^*}{\sqrt{(r^2+s^2)}}\right) \int_{\eta\epsilon R} \left(\frac{r\eta+z^*}{s}\right)^n \phi\left(\frac{\eta + \frac{rz^*}{r^2+s^2}}{s/\sqrt{(r^2+s^2)}}\right) d\eta$$

$$= \phi\left(\frac{z^*}{\sqrt{(r^2+s^2)}}\right) \left(\frac{s}{\sqrt{(r^2+s^2)}}\right) \left(\frac{r}{\sqrt{(r^2+s^2)}}\right)^n J_n$$

where $J_n = \int_{\xi\epsilon R} (\xi+a)^n \phi(\xi) d\xi$

and $a = sz^*/r\sqrt{(r^2+s^2)}$

Integrating J_n by parts it is seen that

$$J_n = \int_{\xi\epsilon R} -(\xi+a)^{n-1}\, d\phi(\xi) + a\, J_{n-1} \quad \text{for } n \geq 2.$$

$$= (n-1)J_{n-2} + a\, J_{n-1}, \quad \text{for } n \geq 2.$$

Now $J_0 = 1$ and $J_1 = a$, therefore

$$J_2 = 1 + a^2.$$

Substituting for J_n the following results for I_0, I_1 and I_2 are immediate:

$$I_0 = \frac{s}{\sqrt{(r^2+s^2)}} \quad \phi\left(\frac{z^*}{\sqrt{(r^2+s^2)}}\right)$$

$$I_1 = \frac{s^2}{r^2+s^2} \quad \frac{z^*}{\sqrt{(r^2+s^2)}} \quad \phi\left(\frac{z^*}{\sqrt{(r^2+s^2)}}\right)$$

and $$I_2 = \frac{r^2}{r^2+s^2} \left[\frac{s}{\sqrt{(r^2+s^2)}} + \frac{s^3}{r^2\sqrt{(r^2+s^2)}} \quad \frac{z^{*2}}{\sqrt{(r^2+s^2)}}\right] \phi\left(\frac{z^*}{\sqrt{(r^2+s^2)}}\right).$$

7. CONCLUSIONS

In the previous chapter a martingale control statistic was proposed to improve the evaluation of approximate stochastic scheduling algorithms by simulation. Although the control statistic was decomposed under certain assumptions into simpler components, the calculation of these components is specific to the process being modelled and the algorithm studied. Formulae enabling these components to be estimated for the model of Chapter 5 and solution algorithm of Chapter 6 have been derived in this chapter for both sets of assumptions necessary for the control statistic's decomposition. Numerically, the formulae enable the martingale control statistic to be very rapidly estimated by comparison to the time required to run the solution algorithm.

Although it has been implicitly assumed that the solution algorithm described in Chapter 6 is being used, the formulae derived herein can be applied to any solution technique of the general model of Chapter 5, provided that the rate of change of the control decisions with respect to changes in the previous time period state variable can be estimated. Estimates of the dispersion matrices necessary here can then be made by running the recursive dispersion matrix estimation procedure described in Section 5 of Chapter 6 forward from the second time period.

In the next chapter a special case of the general model is considered, namely the production/inventory model of Beale et al. [4]. The formulae derived here are then applied to the construction of the martingale control statistics for four different solution algorithms. Their performance on a simple four period problem is then evaluated by simulation and the effectiveness of the control statistics can be seen.

CHAPTER 9

THE EVALUATION OF FOUR APPROXIMATE ALGORITHMS BY SIMULATION

1. INTRODUCTION

In this chapter the theory developed in chapters 7 and 8 is put into practice. Four production planning algorithms were tested by replicative simulation on a simple two product, four period example. The example is that provided by Beale et al. [4]. The four algorithms comprise the algorithm developed for the general model in chapter 6, the "full" method, of Beale et al., their "first pass" method and ordinary linear programming ignoring all stochastic elements.

The production/inventory model which the algorithms of Beale et al. are designed to solve is a special case of the general model described in chapter 5. Although their solution procedures differ from that developed for the general model in chapter 6, the formulae derived in chapter 8 for the construction of martingale control statistics can still be applied. With care they can even be used with the ordinary linear programming method.

Thus it is possible not only to compare and evaluate the algorithms, but also to investigate the practical efficiacy of the control statistics developed in chapter 7. Four separate control statistics were tried, these being that using the unbiased estimation of the optimal fixed control variate coefficients (developed in chapter 7, section 2.3), the case (i) martingale control statistic (developed in chapter 7, section 4.3), and the case (ii) martingale control statistic in the linear and quadratic versions (developed in chapter 7 sections 4.3 and 4.4). The first mentioned control statistic will be referred to as the "fixed coefficient" control statistic.

The production/inventory model of Beale et al. is described, briefly, in section 2 where it is shown to be a special case of the general model. The simple example of it on which the four algorithms are tested is given in section 3. The results of the simulation involving 100 independent trials are given in section 4 and conclusions drawn from them in section 5.

2. THE MODEL ON WHICH THE ALGORITHMS WERE TESTED AND THE CONTROL STATISTICS USED

2.1. The Production/Inventory Model of Beale et al.

The algorithms were tested on the model of Beale et al. [4]. It is necessary to explicitly show that this model is, indeed a special case of the general model and this is now done. Using their notation, let a_t, p_t, s_t and d_t be the sales in, production in, stock level at the end of, and demand in period t, respectively. Identify the ith product with the ith component of these vectors. Let e_t be the excess of supply over demand in period t. Let P_t, C_{Pt} and C_{St} be vectors of unit selling prices, production costs and inventory holding costs in time period t. Then their model may be stated:

$$\text{Maximise } E\{\sum_{t=1}^{T} P_t^T a_t - C_{Pt}^T P_t - C_{St}^T s_t\} \text{ over } p_t, a_t, s_t$$

subject to the production constraint

$$\mathbf{1}^T p_t \leq T_{CAP(t)},$$

where $\mathbf{1}$ is a vector of 1's and $T_{CAP(t)}$ is the maximum permissible total production in time period t, and subject to the evolution equations

$$a_t = s_{t-1} + p_t - f_t$$
$$s_t = f_t$$
$$e_t = s_{t-1} + p_t - d_t$$
$$f_t = \max(e_t, 0)$$

and $d_{Mt} = B_{st} - B_t a_{t-1}$, d_{Mt} being the mean demand in time period t,

given sales in the previous period. The state space is thus represented by (a_t, s_t).

In representing the model of Beale et al. in this way advantage has been taken of the fact that in most practical applications, particularly the simple example given in section 3, it is always more profitable to meet as much demand as possible than to withhold stock that could be sold.

The model used for the demand is a version of the multiplicative model for the stochastic input of the general model. They propose that

$$d_{it} = d_{Mit}\,(1 + C_{it}\eta_t + R_{it}\varepsilon_{it}),$$

where d_{it} and d_{Mit} are the ith components of d_t and d_{Mt} respectively. ε_{it} and η_t are independent real Gaussian random variables, and C_{it}, R_{it} are known constants. Thus if the dispersion matrix of d_t, Disp (d_t) has ith diagonal entry s_{it} then

$$s_{it} = s_{it}^{o}\, d_{Mit}$$

and $s_{it}^{o} = \sqrt{(C_{it}^{2} + R_{it}^{2})}$.

Indeed Disp (d_t) may be expressed by

$$(\text{Disp } (d_t))_{ij} = d_{Mit}(P_t^{o})_{ij}\, d_{Mjt}$$

where

$$(P_t^{o})_{ij} = \begin{cases} (C_{it}^{2} + R_{it}^{2}) & i = j \\ C_{it}\, C_{jt} & i \neq j \end{cases}$$

They approximate the production decision as a linear function of the last time period sales and the stock level at the end of the last time period.

$$p_t = p_t^o + A_{Pt1}\, a_{t-1} + A_{Pt2}\, s_{t-1},$$

for some fixed vector p_t^o. Using this notation, the matrices J_t and H_t needed in the estimation of the martingale control statistic of chapter 8 are

$$(A_{Pt1} - B_t \qquad A_{Pt2} + I)$$

and $\begin{pmatrix} A_{Pt1} & A_{Pt2} + I \\ 0 & 0 \end{pmatrix}$ respectively.

The system matrix C_t of the general model is

$$\begin{pmatrix} -I \\ I \end{pmatrix}$$

and the cost coefficient vectors d_{1t}, d_{2t} of the general model are

$$\begin{pmatrix} P_t \\ -C_{St} \end{pmatrix}$$

and $-C_{Pt}$ respectively.

The vector represented by z_t in the general model is simply e_t, the excess of supply over demand.

The "full" method or solution techniques of Beale et al. is similar to the appropriate special case of that developed for the general model in chapter 6. The differences are discussed in section 7.2 of that chapter.

The "first pass" method of Beale et al. is simply the first stage of the "full" method. It corresponds to solving the first non-linear program of the full method, which has initial minimal estimates of the state variable variances.

The ordinary linear programming approach is one of considering all the random variables in the model to be equal to their mean values and ignoring their stochastic variation.

2.2. Construction of the Control Statistics

The "fixed coefficient" method is the simplest for it only requires the identification of suitable control variates. Since, for a given algorithm, the process is Markov and the revenue a function of the state variable realisations it is natural to use these as control variates. But the raw state variable realisations cannot be used directly as their mean is not, in general, known. This difficulty is circumvented by noticing that at the start of any time period, t say, given the present state vector q_{t-1}, and controls x_t, the distribution of q_t and in particular its mean is known. So instead of using the raw state vector values $\{q_t, t=1,\ldots,T\}$ as control variates, the set

$$\{(q_t - E(q_t|q_{t-1})),\ t = 1,\ldots,T\}$$

is used. This is straightforward and, as can be seen from the results of section 4, effective in practice. Bias control was achieved in the manner of chapter 7, section 2.3 by not using the data from a trial in the estimation of the dispersion matrix (of the control variates) and covariance vector (between the control variates and the revenue) used to determine the control variate's coefficients for that trial. Advantage was also taken of the efficient computational procedure derived in that section.

The case (i) martingale control statistic, which assumes a linear relation between the state space realisation at the end of one time period and the mean of that at the end of the next, is the simplest of the martingale control statistics. Substitution of the system vectors and matrices given in the preceeding subsection into the equations derived in section 3 of chapter 8 yields the necessary components of the recursions given in chapter 7 section 4.3 which compute the control statistic. However, attention must be paid to the calculation of

$$E(e_{it}|a_1 = E(a_1),\ s = E(s_1)),$$

the mean of the excess of supply over demand for the ith product in the t th time period, given that the state space at the end of the first time period is equal to its mean value, its variance

$$Var(e_{it}|a_1 = E(a_1), s_1 = E(s_1))$$

and the coefficient matrices of the production control approximation

$$(A_{Pt1} \quad A_{Pt2}).$$

This is discussed below.

The case (ii) martingale control statistic, which assumes that only the mean and not the shape of the distribution of the state vectorat the end of one time period depends on the value of its realisation at the end of the preceeding time period, is computed by the recursions given in sections 4.3 and 4.4 of chapter 7, the components of which are calculated by substitution of the system vectors and matrices into the equations given in chapter 8, sections 4.4 and 5.4. Again attention must be paid to the estimation of those quantities necessary for the construction of the case (i) martingale control statistic, and in addition, to $\text{var}(d_{it} | a_1 = E(a_1),\ s_1 = E(s_1))$.

$E(e_{it} | a = E(a_1),\ s_1 = E(s_1))$ is approximated by the mean value of e_{it} returned by the algorithm, except in the ordinary linear programming method, where the returned expected value was too crude for the control statistics to realise their full potential. There it was approximated by

$$s^{*}_{it-1} + \bar{p}_{it} - \bar{d}_{it}$$

where $\bar{d}_{it}$ is the mean demand and $\bar{p}_{it}$ the mean production of item i in time period t returned by the algorithm and s^{*}_{it-1} is defined by

$$s^{*}_{it-1} = \hat{\sigma}_{it-1}\, f(e_{it-1}/\hat{\sigma}_{it-1})$$

and $f(x) = \phi(x) + x\Phi(x)$,

ϕ ,Φ being the Gaussian probability density function and distribution function respectively, and $\hat{\sigma}^2_{it-1}$ beingan estimate of $\text{var}(e_{it-1} | a_1 = E(a_1),\ s_1 = E(s_1))$.

Since an estimate of var (d_{it}) is returned by the algorithm (it being a known proportion of $E(d_{it})$), attention is directed to the estimation of var$(e_{it}|a_1 = E(a_1), s_1 = E(s_1))$.

The procedure for doing this is the same for all three algorithms being tested. The last non-linear program (or linear program) of the algorithm is run with the addition of the "fictitious" variables described in chapter 6 section 6.2. This enables the matrices A_{Pt1} and A_{Pt2} to be extracted from the program's solution. The state variable at the end of the first time period is then set equal to its mean value (so its dispersion matrix can be taken to be identically zero) and the recursive variance estimation procedure run forward from the second to the final time period. This yields the dispersion matrices of the e_t's, but they are conditional on the state space at the end of the first time period being equal to its mean value

i.e. Disp $(e_t|a_1 = E(a_1), s_1 = E(s_1))$.

The martingale control statistics can thus be applied to all four algorithms under consideration. The results of their use on the simple example below are given in section 4.

3. A SIMPLE NUMERICAL EXAMPLE

The example on which the three algorithms were tried is that of Beale et al. which is of a simple two product/inventory system. The two products are produced over four time periods. There is an upper bound on the total production in each period. Demand for the two products is uncertain; only its probability distribution is known. The stock available for sale in each time period comprises the stock at the start of that time period and the production during that period. Stockholding costs are proportional to the stock remaining at the end of each time period after that period's sales have been made. The data is as follows:

	Product I	Product II
Unit production cost (all periods)	5.0	2.0
Unit storage cost (all periods)	2.0	2.0
Unit sales price (all periods)	10.0	6.0
Mean demand in period 1	20.0	15.0
2	25.0	15.0
3	35.0	15.0
4	45.0	15.0
Unit value of the closing inventory	10.0	6.0
Initial stocks	10.0	10.0

Total productive capacity is 50.0 units per period

The demand is assumed to be normally distributed with dispersion matrix proportional to the square of the mean. The coefficient of variation was assumed to be 0.224 for each product in each period

and the correlation between products was assumed to be 0.8 in each time period. More precisely this corresponds to setting $C_{it} = 0.2$ and $R_{it} = 0.1$ for all i,t in the demand model given in section 2.1.

The above example may be considered to be a "no dependence" case. In addition, a variant called the "dependence" case in which half the demand for product I is directly proportional to the sales in the preceeding period was considered. For the demand model given in section 3.1, this means that B_{11t} has values 0.2222, 0.6250, 0.7000 and 0.6429 for t = 1,2,3 and 4 respectively. All other B_{ijt} are zero for $j \neq 0$. It has been assumed that the initial sales of product I (i.e. sales in the period immediately preceeding the first) were 45.0 units.

4. THE RESULTS

The results of testing the four algorithms each on 100 replicative simulation trials are summarised below. Those for the "no dependence" case are given in Table A and those for the "dependence" case are given in Table B. The same random number seed was used for each set of 100 trials, so the results for each algorithm are highly correlated. This is an advantage in making comparisons between them. The notation used in the tables is as follows.

Algorithms A,B,C and D are the solution method developed for the general model in chapter 6, the "full" method of Beale et al., their "first pass" method and ordinary linear programming ignoring all stochastic elements, respectively. A-B denotes the extra revenue that would be accrued from using algorithm A rather than algorithm B; similarly for A-C, B-C, and so forth.

Control statistics 1,2,3 and 4 are those of the fixed coefficients, the linear martingale assuming the case (i) conditions and the linear martingale and quadratic martingale assuming the case (ii) conditions respectively.

$\hat{\mu}$ is the estimate of the expected revenue made after 100 trials and $\hat{\sigma}$ is its estimated standard deviation. The "returned objective function value" column shows the estimate of the expected revenue given by the algorithm itself.

TABLE A : THE RESULTS FOR THE "NO DEPENDENCE" CASE

Algorithm	Returned Objective Function Value	with no control statistics		With control statistics (1)		With control statistics (2)		With control statistics (3)		With control statistics (4)	
		$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$
A	857.22	860.71	7.292	859.30	0.403	859.22	0.344	859.36	0.283	858.97	1.019
B	859.54	860.83	7.275	859.07	0.438	858.97	0.320	859.11	0.294	859.03	0.936
C	865.06	860.87	7.157	859.27	0.417	859.05	0.352	859.25	0.293	859.15	1.141
D	895.00	841.93	5.550	840.72	0.516	840.78	0.506	840.94	0.658	840.21	1.205
A-B	-2.32	-0.12	0.166	0.23	0.054	0.25	0.104	0.25	0.102	-0.06	0.408
A-C	-7.84	-0.16	0.234	0.03	0.062	0.17	0.047	0.11	0.056	-0.18	0.261
A-D	-37.78	18.78	2.333	18.58	0.532	18.44	0.589	18.42	0.609	18.76	0.845
B-C	-5.52	-0.04	0.257	-0.20	0.079	-0.12	0.341	-0.14	0.343	-0.12	0.417
B-D	-35.46	18.90	2.309	18.35	0.533	18.19	0.572	18.17	0.597	18.82	0.806
C-D	-29.94	18.94	2.187	18.55	0.527	18.27	0.583	18.31	0.593	18.94	0.823

TABLE B : THE RESULTS FOR THE "DEPENDENCE" CASE

Algorithm	Returned Objective Function Value	With no control statistics		With control statistics (1)		With control statistics (2)		With control statistics (3)		With control statistics (4)	
		$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\mu}$	$\hat{\sigma}$
A	849.88	851.04	9.455	849.09	1.229	850.24	0.783	850.32	0.656	850.36	1.788
B	845.58	850.75	9.564	850.06	0.994	850.21	0.731	850.26	0.627	849.91	1.900
C	861.70	851.51	9.217	850.29	1.074	850.03	0.670	850.17	0.633	850.07	1.880
D	895.00	828.05	6.925	824.99	0.619	826.36	0.985	826.61	1.222	828.29	1.829
A-B	4.30	0.29	0.227	-0.97	0.742	0.03	0.136	0.06	0.132	0.45	0.312
A-C	-21.82	-0.47	0.397	-0.20	0.964	0.21	0.311	0.15	0.344	0.29	0.445
A-D	-44.12	22.99	3.408	24.10	1.082	23.88	1.275	23.71	1.314	22.07	1.394
B-C	-16.12	-0.77	0.563	-0.24	0.692	0.18	0.251	0.09	0.298	-0.17	0.505
B-D	-49.42	22.70	3.530	25.07	0.841	23.85	1.206	23.65	1.258	21.61	1.394
C-D	-33.30	23.46	3.145	25.30	0.998	23.67	1.049	23.56	1.059	21.78	1.312

The results enable some conclusions to be drawn about the efficiacy of the control statistics and of the algorithms themselves.

There is little difference in the performance of the three algorithms that take some account of the stochastic variability of the problem, and they all appreciably out-perform the ordinary linear programming method which does not. However, the algorithm,A, that was developed for the general model does significantly out-perform the "full" method of Beale et al., B, in the "no dependence" case. The difference in the expected revenues made by the two was estimated at 0.23 by the use of the "fixed coefficients" control statistic with a sample standard deviation of 0.054. But this is only a very small difference in the expected return of approximately 859 consequent from using either algorithm.

The penalties incurred by ignoring the stochastic variation in the problem, as comparison with the results for the ordinary linear programming method shows, can be considerable. Moreover, the discrepancy between the expected revenue returned by the algorithm and that which was consequent from actually using it is very large. This shows that the optimal objective function value given by a deterministic model of a stochastic problem may be seriously misleading.

The control statistics were surprisingly effective, yielding reductions in the sample standard deviations of up to a factor of 25 in the "no-dependence" case and 15 in the "dependence" case. Their relative efficiacy varies in a way that might be intuitively expected. With the exception of the quadratic one, those which involve the construction of a martingale from information about the process provided by the algorithm itself perform better, the better the algorithm. In

particular they out-perform the control statistics based upon the "fixed coefficients" of the control variables for the three algorithms which take account of the stochastic nature of the system. If the revenue accrued from each trial is assumed to be normally distributed, then the F-test can be used to provide bounds in the comparison of the sample variances, if they are assumed to be independent or positively correlated. Such comparison reveals that the case (ii) martingale control statistic (that based on the assumption that only the mean and not the shape of the distribution of the state space in each time period depends on its realisation one time period before) is significantly better than the case (i) martingale control statistic (that based on the assumption that the mean of the state space at the end of one time period is a linear function of its realisation one time period before) when it is applied to the solution algorithm of the general model (A) and significantly worse when applied to the ordinary linear programming method (D). Significance in both instances is at the 95% level. This indicates that under a good planning algorithm the case (ii) assumptions about the process may be more realistic than the case (i) assumptions.

The performance of the martingale control statistics based upon a quadratic model of the process is disappointing. There may be two reasons for this. In the approximate model of the process made in the construction of the martingale control statistics, the first derivatives of the expected future revenue with respect to changes in the next time period state space realisations may change discontinuously when the basis of the approximate model changes. Also the quadratic coefficient matrix is derived from the backwards recursion, it being Γ_1 where

$$\Gamma_T = 0$$

$$\Gamma_{t-1} = \Delta_t + D_t \Gamma_t D_t^T.$$

See section 4.4 of chapter 7 for a detailed explanation of the terms and its derivation. Essentially, the matrices Δ_t and D_t have to be estimated and in practice all the Δ's and therefore the Γ's are negative semi-definite. Thus errors in the estimation of the D's build up to yield a final Γ_1 more strongly negative semi-definite than it should be. In other words any errors in the estimation of the quadratic coefficient matrix lead to a strong negative semi-definite bias.

5. CONCLUSIONS

Four production planning algorithms have been tested by replicative simulation on two variants of a simple two product, four time period example of a production/inventory model. It has been shown that this model is a special case of the general model of chapter 5, and that the two examples can be regarded as special cases of the "additive" and "multiplicative" variants of the general model. The theory developed in chapters 7 and 8 on the use of control variates has been put into practice. The simulation experiments therefore also provided a test of the control variates.

The results show that the control variates are indeed effective. No particular set of control statistics consistently out performs the others, although the quadratic martingale control statistic is uniformly worse. As might be expected, the control variates perform better on the simpler "no dependence" or "additive" example in which the dispersion matrix of the demand is fixed than on the more complicated "dependence" or "multiplicative" example in which it is directly proportional to the square of the mean. The standard deviation of the estimate of the expected revenue was reduced by up to a factor of 25 in the former case and 15 in the latter.

Although there was little difference in the performance of the algorithms that took some account of the stochastic variability in the problem, they appreciably out-performed the deterministic linear programming method. This indicates that there may be severe penalties in

ignoring the stochastic variability of demand, but that first order departures from the optimal first time period decision values may only make second order changes in the actual expected revenue consequent on using them in practice. This is a desirable feature of the system and indicates that approximate planning algorithms may be reliable.

However, on an example with more time periods, say a monthly one, the more sophisticated algorithms which re-estimate the dispersion matrices of the random variables would out-perform the simple stochastic algorithm which does not. Unfortunately, simulation experiments on a monthly model would be expensive.

CHAPTER 10

CONCLUSIONS

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1. PRODUCTION/INVENTORY MODELLING

This thesis is concerned with the potential use of stochastic programming models for medium term production planning. This involves tactical decisions, typically the determination of monthly production targets over a planning horizon of a year. Strategic planning problems such as those associated with the construction of new production facilities have been specifically excluded as have short term planning problems associated with day to day factory management.

There are two principal problems associated with medium term production planning. The first is the balancing of frequent changes in production rate against high inventory levels in order to cope with a fluctuating, but known, demand. The second is the determination of the optimal buffer or safety stocks in order to cope with uncertainties in future demand requirements. Both of these problems can be approached by the study of the appropriate mathematical model of the production/inventory system. Those associated with the latter problems must be probabilistic in the sense that they incorporate random variables into their formulation, and more general in the sense that they can be easily extended to handle the former problem, whereas the reverse is not the case.

Probabilistic planning models fall naturally into the ambit of stochastic programming. Although this field has received much attention in recent years, practical applications of stochastic programming models to production planning problems are few. This is because exact solutions to stochastic models of general multi-commodity, multi-time period planning problems are computationally intractable. Accordingly this thesis develops a good approximate solution method to a general production/inventory model.

2. THE GENERAL MODEL AND ITS APPROXIMATE SOLUTION TECHNIQUE

The model proposed in this thesis for medium term production planning is a multi-period, multi-commodity model and is an extension of that of Beale, Forrest and Taylor [4]. Under certain conditions, which would normally be met in practice. it is convex and is a stochastic linear program. The approximate method of solution developed for it is also based upon the work of Beale et al., but it is more flexible, contains fewer approximations, and is procedurally improved. A problem, termed the reduced problem, is derived from the full stochastic problem. The reduced problem involves the expected values of the random variables in the full problem. If the control decisions (for example production targets) are unrestricted then it is equivalent to the full stochastic problem. If the control decisions are restricted, then the constraints on the reduced problem are implied by but do not imply the constraints in the full problem, and the reduced problem is an approximation of the full problem. The reduced problem is tackled by making normality assumptions about the distributions of some of the random variables in the problem. It is then necessary to estimate the variability of these random variables and this is done iteratively by solving a sequence of such problems. The solution to each providing better information about the stochastic process being modelled enabling a better estimate of the variability to be made for the formulation of the next. Computational advantage can be taken of the similarity between each problem in the sequence and numerical experience suggests that convergence is rapid, only three iterations being sufficient for a four period example.

3. THE EVALUATION OF APPROXIMATE SOLUTION TECHNIQUES

It is important to assess the relative impact of the approximations made in approximate solution algorithms of stochastic models. In this thesis three approximate algorithms were compared with each other and deterministic linear programming. The method developed for the evaluation of approximate techniques is that of replicative simulation. It divides into a number of independent trials. In each trial the environment within which the stochastic models operate is simulated on a computer. Pseudo-random numbers are used to provide realisations of the random variables. The production/inventory system then evolves subject to these realisations and under the control of the approximate algorithms from the first time-period in the model until the time horizon, when the utility or revenue gained from using the algorithm is apparent. The trial is repeated a large number of times so that attributes of interest within the process can be estimated statistically. However, the accuracy of this method only improves as the reciprocal of the square root of the number of trials so that each trial may provide such an inaccurate estimate of the attribute of interest that convergence may be unacceptably slow.

To improve the accuracy of each trial control statistics have been developed. These fall into two distinct categories. The first in which data from all the trials is used in the estimation of the best coefficients of the control variables is fairly standard, although a novel way has been suggested for eliminating the resultant bias. To be effective this method requires many more trials than there are control statistics. Since the control statistics suggested here are the deviations of the state variables from their expected values one time period previously, the use of this method may involve a prohibitively large number of trials if the number of time periods and/or commodities in the problem is large.

The second type of control statistic developed involves the construction of a martingale control statistic for each trial from information about the stochastic process provided by the algorithm being tested. The construction of these control statistics does not require data from the other trials and so realises its potential for the improvement of the accuracy of the statistical estimates of the attributes of interest whatever the number of trials made. It is therefore possible to use them if it is desired only to make a small number of trials compared with the number of commodities and/or time periods in the problem. So it is possible to use them to make statistical estimates of attributes of interest in very complex examples from a small number of trials.

As might be intuitively expected, the efficacy of the latter set of control statistics depends on the merit of the algorithm being tested. For the solution technique to the general model described in Section 2, in practice they perform better than the former class of control statistics, even when the number of simulation trials is large.

The results of the simulation trials indicate that there was little difference between the three approximate algorithms that took some account of the stochastic nature of the process being modelled, but they all out performed the deterministic linear programming method which did not. However, they were tested on four period examples. If there were more time periods in the examples then the approximate algoritnm which took no account of the variability of the state variables would be inferior to those which did. Also, the difference between the utility returned by

the algorithm's objective function was often different from that accrued from actually using it. This discrepancy may be regarded as a measure of the sub-optimality of the algorithm and was greater the more approximate the algorithm. It was particularly large in the case of the deterministic linear programming method. This indicates that the utility returned by a determinsitic model of a stochastic problem may be seriously misleading.

4. SUGGESTIONS FOR FURTHER RESEARCH

In this thesis a general model for medium term production planning has been proposed and a computationally tractable approximate solution technique has been developed. It is hoped that this method will become part of a commerically available suite of computer programs, perhaps as an extension of the mathematical programming system SCICONIC [47]. The approximate solution technique to the general model has been tested statistically along with three other methods, one of which was simple linear programming ignoring all stochastic elements in the problem, on two four-period examples by replicative simulation. The results suggest that there may be significant penalities in ignoring the stochastic variation in demand, but that first order deviations from the optimal production decisions may only lead to second order penalities in practice.

This latter conjecture deserves further consideration. For if under certain conditions it is true, then approximate solution methods to stochastic models may be more reliable than previously supposed. To study the conjecture further, two separate issues require further investigation. Firstly, it may be possible to quantify the effect of small deviations from the optimal production decisions more precisely than has been possible here. Secondly,it may be possible to more accurately quantify the deviations in the decisions provided by approximate algorithms from their truly optimal values.

These problems are formidable. They might be approached by a very comprehensive series of tests on a great variety of examples by statistical simulation. If the martingale control statistics are used then it may be possible to achieve sufficient accuracy by a small number of trials (say four or five) on each example. Alternatively it may be possible,if not to solve them analytically,at least reduce them to a

series of simpler sub-proboems by suitable theoretical analysis. Each sub-problem could then be investigated statistically.

The areas of research suggested above are very difficult and beyond the scope of this thesis. It may be that the questions raised can never be fully answered theoretically, but any answers to them would advance the art of stochastic modelling.

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