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George Baravdish, Ihor Borachok, Roman Chapko,
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Graphical Abstract

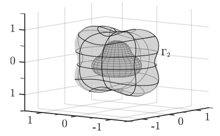
Let u be a solution to

$$Lu = 0 \quad \text{in } D$$

satisfying Cauchy data

$$u = f \quad \text{and} \quad Nu = g \quad \text{on } \Gamma_2.$$

Iterative method to find u.



Highlights

- Iterative method for Cauchy problems for elliptic equations
- Proof of convergence in Sobolev trace spaces
- Integral equations for numerical implementation
- Relation to other iterative methods investigated

An iterative method for the Cauchy problem for second-order elliptic equations

George Baravdish[a], Ihor Borachok[b], Roman Chapko[b], B. Tomas Johansson[c], Marián Slodička[d]

[a]ITN, Campus Norrköping, Linköping University, Sweden
[b]Faculty of Applied Mathematics and Informatics, Ivan Franko National University of Lviv 79000 Lviv, Ukraine
[c]School of Mathematics, Aston University, B4 7ET, Birmingham, UK
[d]Department of Mathematical Analysis, Ghent University, Galglaan 2, 9000 Gent, Belgium

Abstract

The problem of reconstructing the solution to a second-order elliptic equation in a doubly-connected domain from knowledge of the solution and its normal derivative on the outer part of the boundary of the solution domain, that is from Cauchy data, is considered. An iterative method is given to generate a stable numerical approximation to this inverse ill-posed problem. The procedure is physically feasible in that boundary data is updated with data of the same type in the iterations, meaning that Dirichlet values is updated with Dirichlet values from the previous step and Neumann values by Neumann data. Proof of convergence and stability are given by showing that the proposed method is an extension of the Landweber method for an operator equation reformulation of the Cauchy problem. Connection with the alternating method is discussed. Numerical examples are included confirming the feasibility of the suggested approach.

Keywords:
2000 MSC: 35R25, 35J05, 65R20

1. Introduction

The alternating iterative method was introduced in 1989 by Kozlov and Maz'ya [16] for solving some inverse ill-posed problems notably the Cauchy problem for self-adjoint strongly elliptic operators. For models not being self-adjoint, other iterative methods have been developed, early works are [3, 10]. In those latter procedures, the adjoint of the governing partial differential equation is involved in the iterations.

Email addresses: george.baravdish@liu.se> (George Baravdish), ihorborachok@ukr.net (Ihor Borachok), chapko@lnu.edu.ua (Roman Chapko), b.t.johansson@fastem.com (B. Tomas Johansson), marian.slodicka@ugent.be (Marián Slodička)

From a physical point of view, the alternating method is natural in that it updates function values on the boundary with function values from the previous iteration step, and the normal derivative is updated with the normal derivative from the previous step. In the Landweber type procedures that have been proposed, see for example [3], typically the function values are updated via the normal derivative of the solution of the previous iteration step.

We focus on the case of a second-order strongly elliptic and self-adjoint operator and present an iterative method of Landweber type building on [3]. In the method we propose, function values on the boundary are updated by function values from the previous step, and the normal derivative by the normal derivative of the previous step. For the proposed method, we outline convergence and stability. Compared with [3], we do not work with very weak solutions but in the classical weak sense. Numerical experiments are included both in two and three dimensions. This altogether counts as the novelty of the present work. The method given is presented in [25] for equations related to eddy-current modelling, and is also mentioned in [22]; in both these works the domain is simply connected. We also discuss connections with the alternating method. Rather surprisingly, using the results in [22], it turns out that putting the regularizing parameter to unity in the proposed method, generates the alternating method.

To formulate the problem to be studied, let $D \subset \mathbb{R}^n$, $n \geq 2$, be a doubly-connected domain being the region between the two boundary surfaces Γ_1 and Γ_2. Here, each boundary surface is simple (no self-intersections) closed (the surface has no boundary and is connected) and is at least Lipschitz smooth. Moreover, Γ_1 lies in the bounded interior of Γ_2. In the case $n = 2$, the region D is the domain between two simple closed non-intersecting curves; note that each time the word "surface" appears the reader has to keep in mind that the present work also covers the planar case.

Let u be a solution to

$$Lu = 0 \quad \text{in } D \tag{1.1}$$

and suppose additionally that u satisfies the following boundary conditions (Cauchy data) on the outer surface Γ_2,

$$u = f \quad \text{on } \Gamma_2 \qquad \text{and} \qquad Nu = g \quad \text{on } \Gamma_2. \tag{1.2}$$

The operator L is a second-order elliptic operator with N being the corresponding co-normal derivative,

$$\begin{aligned} Lu = L(x, \partial_x)u &= \sum_{i,j=1}^{n} \partial_{x_i}(a_{i,j}(x)\partial_{x_j}u) + c(x)u, \\ Nu = N(x, \partial_x)u &= \sum_{i,j=1}^{n} \nu_i a_{i,j}(x)\partial_{x_j}u, \end{aligned}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the outward unit normal to the boundary. The coefficients $a_{i,j}$ and c are assumed to be sufficiently smooth, with $a_{i,j} = a_{j,i}$; we are not after the most general setting. Moreover, L is assumed to be strongly elliptic in D meaning that for every $\xi = (\xi_1, \ldots, \xi_n)$,

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq \alpha|\xi|^2, \quad x \in D,$$

and $\alpha > 0$. Then it is known that the element $c(x)$ in L can be chosen such that $a(u,u) \geq C\|u\|_{H^1(D)}$, with $a(\cdot,\cdot)$ the standard bilinear form corresponding to the operator L. We therefore assume that the coefficients of L are such that this inequality holds. It is further assumed that data are compatible such that there exists a solution $u \in C^2(D) \cap C^1(\bar{D})$; uniqueness is clear for smooth coefficients by the Holmgren theorem. Although existence is assumed the solution will in general not depend continuously on the data, thus stability cannot be guaranteed.

The Cauchy problem for elliptic equations is classical, and we would commit to the near impossible trying to give adequate overview and references. Even narrowing down to iterative methods would be too lengthy. To at least guide the reader to some works on the alternating method, we point to the introduction in [2]; for references to a selection of other methods for Cauchy problems both direct and iterative together with applications, see the introduction in [8] and for properties of Cauchy problems [12, Chapt. 3],[1, 7].

We ask the reader to bear in mind that we are not after an optimal procedure competing with all other methods proposed. We are solely interested in how [3] can be modified to satisfy the requirement that boundary data is updated throughout the iterations with data of the same type (Dirichlet or Neumann), and how this new iterative procedure that we propose behave, and will of course relate this to the iterative methods mentioned above.

For the outline of the work, in Section 2 we state a method and a stopping rule (discrepancy principle) with notes on properties of the problems involved in the iterations, in particular well-posedness. Section 3 is devoted to convergence and stability. The proposed method is rewritten as iterations for an operator equation, and this route generates proof of convergence using [25, Theorem 2], see Theorem 3.1. Compared with [3], we do not work with very weak solutions but in the classical weak sense making the analysis different. Connections with the alternating method is given at the end of Section 3. In Section 4, it is outlined how to numerically implement the proposed method in the case of the Laplace equation, using boundary integral techniques. Section 5 presents some numerical results, both in two and three dimensions, showing the feasibility of the proposed approach.

2. An iterative method for (1.1)–(1.2)

The iterative method for the stable reconstruction of the solution to the Cauchy problem (1.1)–(1.2) involves mixed boundary value problems, and the procedure runs as follows:

- Choose an arbitrary initial approximation η_0 on the boundary part Γ_1.
- The first approximation u_0 of the solution u is obtained by solving (1.1) supplied with the boundary conditions

$$u_0 = \eta_0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nu_0 = g \quad \text{on } \Gamma_2.$$

- Next, v_0 is constructed by solving (1.1) with the boundary conditions changed to

$$Nv_0 = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad v_0 = f - u_0 \quad \text{on } \Gamma_2.$$

- Given that u_{k-1} and v_{k-1} are known, the approximation u_k is determined from (1.1) with

$$u_k = \eta_k \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nu_k = g \quad \text{on } \Gamma_2$$

  and

$$\eta_k = \eta_{k-1} + \gamma v_{k-1}|_{\Gamma_1},$$

  where $\gamma > 0$ is a relaxation parameter.
- Then v_k is determined from (1.1) with boundary conditions

$$Nv_k = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad v_k = f - u_k \quad \text{on } \Gamma_2.$$

The iterations continues with the last two steps until a suitable stopping rule has been satisfied. We make precise such a rule in the next section.

Comparing the above scheme with the alternating method [16], it is similar in the sense that the type of boundary condition alternates during the iterations, however, in the alternating method only data on Γ_1 is updated. The boundary conditions for the two problems used in that method are $u_k = v_{k-1}$ on Γ_1, $Nu_k = g$ on Γ_2, $Nv_k = Nu_k$ on Γ_1, $v_k = f$ on Γ_2, u_0 an initial guess. The proposed method is also different from [3, 13] in that η is updated by Dirichlet data; in [3, 13] the element η is updated by a normal derivative of a solution to an adjoint problem (with boundary conditions $v_k = 0$ on Γ_1 and $Nv_k = u_k - f$ on Γ_2). However, as we show at the end of Section 3, with the choice $\gamma = 1$, the sequence generated is similar to the one obtained from the alternating method.

A method of the above form is used in [25] for equations related to eddy-current modelling. Moreover, the above method is mentioned in [22]. In both those two works, the domain is simply connected making the analysis more involved due to adjustment of the classical Sobolev trace spaces for mixed problems in such domains.

The existence, uniqueness and continuous dependence of a weak solution in the Sobolev space $H^1(D)$ for boundary data in corresponding Sobolev trace spaces is standard for the problems used in the procedure, see for example [23, Chapt. 4].

3. Convergence of the proposed procedure

Define an operator K acting on the standard Sobolev trace space $H^{1/2}(\Gamma_1)$ by $K : H^{1/2}(\Gamma_1) \to H^{1/2}(\Gamma_2)$, where

$$K\eta = u|_{\Gamma_2} \quad \text{for} \quad \eta \in H^{1/2}(\Gamma_1), \tag{3.1}$$

and u satisfies (1.1) with boundary conditions

$$u = \eta \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nu = 0 \quad \text{on } \Gamma_2.$$

We also define $G : H^{-1/2}(\Gamma_2) \to H^{1/2}(\Gamma_2)$, where

$$Gg = w|_{\Gamma_2} \quad \text{for} \quad g \in H^{-1/2}(\Gamma_2), \tag{3.2}$$

and w satisfies (1.1) with

$$w = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nw = g \quad \text{on } \Gamma_2.$$

Finding a solution to the Cauchy problem (1.1)–(1.2) is then equivalent to solving for an element $\eta \in H^{1/2}(\Gamma_1)$ such that

$$K\eta = f - Gg, \tag{3.3}$$

where K and G are defined by (3.1) and (3.2), respectively.

Both the operators K and G are well-defined and bounded, due to the well-posedness (quoted at the end of the previous section) of the boundary value problems involved in their respectively definition. Moreover, the kernel of K consists of the zero element only as explained next.

To see that the kernel of K is trivial, assume that $K\eta = 0$. Then, from the definition of the operator K, there is a solution u that satisfies (1.1) together with zero Cauchy data on Γ_2. Due to uniqueness of a solution to the Cauchy problem, it follows that u is identically zero in $\overline{D}$. Hence, $\eta = 0$ and the kernel of K is trivial.

We then define an auxiliary operator T mapping in the opposite direction compared with the operator K. Let $T : H^{1/2}(\Gamma_2) \to H^{1/2}(\Gamma_1)$, where

$$Th = v|_{\Gamma_1} \quad \text{for} \quad h \in H^{1/2}(\Gamma_2), \tag{3.4}$$

and v satisfies (1.1) with boundary conditions

$$Nv = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad v = h \quad \text{on } \Gamma_2.$$

The operator T is also well-defined and bounded. The kernel of T is trivial, this follows along the similar lines as shown for the operator K.

Let u_k be the iterates obtained from the proposed algorithm. We then have

$$\begin{aligned} \eta_k &= u_{k-1}|_{\Gamma_1} + \gamma\, v_{k-1}|_{\Gamma_1} = \eta_{k-1} + \gamma\, T\,(f - u_{k-1}|_{\Gamma_2}) \\ &= \eta_{k-1} + \gamma\, T\,(f - Gg - K\eta_{k-1}). \end{aligned} \tag{3.5}$$

This is the extension of the Landweber method for solving equation (3.3) given [25]; in the classical Landweber method the operator T is equal to the adjoint of K, that is K^*. Given the above properties of K and T with TK being positive, convergence follows from [25, Theorem 2] (the assumption that TK is positive is needed but not explicitly mentioned in that work). We then have to show that TK is indeed positive, or alternatively to show that T is equal to K^*. For future reference, we show both, that is that TK is positive (shown of course without using that T equals K^*) and then we find the adjoint of K.

Note: One can obtain convergence directly using [22]. However, that analysis was carried out for a simply connected domain, making the analysis more involved (more complicated trace spaces is then needed). We believe it is of value to write out the details for the case of an annular domain and to relate to the work [25].

3.1. Positiveness of the operator TK

Let K and T be given by (3.1) and (3.2), respectively. We show that the composition TK is indeed a positive operator, with respect to a certain inner product $(\cdot,\cdot)$. Let V_1 be a closed subset of $H^{1/2}(\Gamma_1)$, which does not contain any constant functions apart from the zero element. The space $\mathscr{H}_1$ is a subset of $H^1(D)$ obtained by solving (1.1) with boundary conditions $u = \eta$ on Γ_1 and $Nu = 0$ on Γ_2, with η in V_1. One can check that $\mathscr{H}_1$ is closed.

From the assumptions on the operator L we can, without loss of generality, for the ease of presentation, assume that $Lu = \Delta u$. Following [16, 17], the required inner product on V_1 is defined by

$$(\eta, \zeta) = \int_D \nabla u \cdot \nabla v \, dx, \tag{3.6}$$

where u satisfies (1.1) with boundary conditions

$$u = \eta \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nu = 0 \quad \text{on } \Gamma_2,$$

and similarly v satisfies (1.1) with boundary conditions

$$v = \zeta \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nv = 0 \quad \text{on } \Gamma_2.$$

Here, η and ζ belong to V_1, and thus the solutions belong to $\mathscr{H}_1$. The reader can check that the above is a well-defined inner product on V_1. In particular, if the right-hand side in (3.6) is zero for $\zeta = \eta$, then u is a constant throughout the domain D. In particular, η is a constant, but the only constant element in $\mathscr{H}_1$ is the zero element. Hence, η is zero. For further details on this type of inner product, see [19].

We recall the identity

$$\int_D \nabla u \cdot \nabla v \, dx = \int_\Gamma v N u \, ds, \tag{3.7}$$

valid for $u, v \in H^1(D)$ with u being weak solution of (1.1), see for example [23, Theorem 4.4].

Let u_0 and u_1 be generated from the iterative procedure with $f = g = 0$ and initial guess $\eta \in V_1$. The element u_2 then satisfies the same type of problem as u_0 but with $u_2 = u_1$ on Γ_1. Using the inner product (3.6) together with (3.7) and employing the boundary conditions for u_0, u_1 and u_2,

$$\begin{aligned}(TK\eta, \eta) &= \int_D \nabla u_2 \cdot \nabla u_0 \, dx \\ &= \int_{\Gamma_1} u_2 N u_0 \, ds \\ &= \int_{\Gamma_1} u_1 N u_0 \, ds \\ &= \int_D \nabla u_1 \cdot \nabla u_0 \, dx.\end{aligned} \tag{3.8}$$

Continuing, we obtain

$$\begin{aligned}\int_D \nabla u_1 \cdot \nabla u_0 \, dx &= \int_{\Gamma_2} u_0 N u_1 \, ds \\ &= \int_{\Gamma_2} u_1 N u_1 \, ds \\ &= \int_D |\nabla u_1|^2 \, dx.\end{aligned} \tag{3.9}$$

Hence, from (3.8) and (3.9)

$$(TK\eta, \eta) = \int_D |\nabla u_1|^2 \, dx, \tag{3.10}$$

and TK is thereby a positive operator on V_1.

In fact, for a non-zero η in V_1, TK is a strictly positive operator. To see this, let the right-hand side in (3.10) be zero. Then the solution u_1 is a constant throughout the domain D. Since $u_0 = u_1$ on Γ_2, u_0 is therefore constant along Γ_2. Using that the normal derivative of u_0 is zero on Γ_2, we can conclude, from the uniqueness of the Cauchy problem, that u_0 is also constant in D. In particular, u_0 is constant along Γ_1. Since, by assumption, data for u_0 on Γ_1 is taken from the space V_1, and this space does only contain zero as the constant element, we have that η is identically zero. Hence, for a non-zero element η in V_1, the right-hand side in (3.10) is non-zero. Thus, (3.10) implies that TK is a strictly positive operator on V_1.

3.2. The adjoint of the operator T

Let us then show that the adjoint of K is equal to T, with K and T given by (3.1) and (3.2), respectively. Similar to the space V_1, let V_2 be a closed subset of $H^{1/2}(\Gamma_2)$, which does not contain any constant function apart from the zero element. The space $\mathscr{H}_2$ is a subset of $H^1(D)$ obtained by solving (1.1) with boundary conditions $Nu = 0$ on Γ_1 and $u = \varphi$ on Γ_2, with φ in V_2. One can check that $\mathscr{H}_2$ is closed.

An inner product on V_2 is defined by

$$(\varphi, \psi) = \int_D \nabla u \cdot \nabla v \, dx, \tag{3.11}$$

where u satisfies (1.1) with boundary conditions

$$Nu = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad u = \varphi \quad \text{on } \Gamma_2,$$

and similarly v satisfies (1.1) with boundary conditions

$$Nv = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad v = \psi \quad \text{on } \Gamma_2.$$

Here, φ and ψ belong to V_2, and the solutions therefore belong to $\mathscr{H}_2$. One can check that (3.11) is a well-defined inner product on V_2.

Let u_0, u_1 and u_2 be as above. The element w_1 satisfies (1.1) with boundary conditions

$$Nw_1 = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad w_1 = \psi \quad \text{on } \Gamma_2.$$

We also need w_2 being a solution to (1.1) with

$$w_2 = w_1 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nw_2 = 0 \quad \text{on } \Gamma_2.$$

Then

$$\begin{aligned}(K\eta, \psi) &= \int_D \nabla u_1 \cdot \nabla w_1 \, dx \\ &= \int_{\Gamma_2} u_1 N w_1 \, ds \\ &= \int_{\Gamma_2} u_0 N w_1 \, ds \\ &= \int_D \nabla u_0 \cdot \nabla w_1 \, dx.\end{aligned} \tag{3.12}$$

Furthermore,

$$\begin{aligned}(\eta, T\psi) &= \int_D \nabla u_0 \cdot \nabla w_2 \, dx \\ &= \int_{\Gamma_1} w_2 N u_0 \, ds \\ &= \int_{\Gamma_1} w_1 N u_0 \, ds \\ &= \int_D \nabla u_0 \cdot \nabla w_1 \, dx.\end{aligned} \tag{3.13}$$

Comparing (3.12) and (3.13), we see that

$$(K\eta, \psi) = (\eta, T\psi),$$

with $\eta \in \mathscr{H}_1$ and $\psi \in \mathscr{H}_2$ arbitrary. Hence, $T = K^*$.

3.3. Convergence of the iterative procedure

As has been shown above in (3.5) the proposed method can be re-written as a Landweber type iteration for an operator reformulation, (3.3), of the Cauchy problem (1.1)–(1.2). The operators K and T have been shown to satisfy the convergence criteria both for the generalized and classical Landweber method, and therefore we have the following convergence of the proposed iterative procedure.

Theorem 3.1. *Let $f \in H^{1/2}(\Gamma_2)$ and $g \in H^{-1/2}(\Gamma_2)$. Assume that the Cauchy problem (1.1)–(1.2) has a solution $u \in H^1(D)$ and that γ satisfies $0 < \gamma < 2/(\|T\|\|K\|)$. Let u_k be the k–th approximation in the given algorithm. Then*

$$\lim_{k\to\infty} \|u - u_k\|_{H^1(D)} = 0$$

for any initial function $\eta_0 \in V_1$.

Convergence of higher derivatives can also be achieved in the interior of D. Let D' be a domain such that $\overline{D'} \subset D$. Since $u_k - u$ satisfies (1.1), we can use local estimates for elliptic equations, see [23, Theorem 4.16], which gives

$$\|u_k - u\|_{H^{\ell+1}(D')} \leq C\|u_k - u\|_{H^1(D)},$$

with the choice of the non-negative integer ℓ depending on the smoothness of the coefficients in the operator L. This estimate and Theorem 3.1 show convergence of higher derivatives in D. Since $u - u_k$ has zero co-normal derivative on Γ_2, one can even allow for D' to have a non-empty intersection with Γ_2.

To formulate a stopping rule, the discrepancy principle [24], assume that we have noisy Cauchy data φ^δ and ψ^δ such that

$$\|f - f^\delta\|_{H^{1/2}(\Gamma_2)} + \|G(g - g^\delta)\|_{H^{1/2}(\Gamma_2)} \leq \delta.$$

Let u_k^δ be the iterates with f^δ and g^δ as data. For the generalized Landweber type method the discrepancy principle can be applied. This means that one should terminate the iterations when

$$\|f^\delta - u_k^\delta\|_{H^{1/2}(\Gamma_2)} \leq \tau\delta,$$

where $\tau > 1$. Stopping rules for the alternating method are given in ([16, 17, 19, 11]).

The regularizing parameter γ can be chosen as $\gamma = 1$. To see this, let u_0, u_1 and u_2 be as in the previous section. Then

$$\begin{aligned}\int_D \nabla u_2 \cdot \nabla u_2 \, dx &= \int_{\Gamma_1} u_2 N u_2 \, ds \\ &= \int_{\Gamma_1} u_1 N u_2 \, ds \\ &= \int_D \nabla u_1 \cdot \nabla u_2 \, dx.\end{aligned}$$

This implies

$$\int_D \nabla(u_1 - u_2) \cdot \nabla(u_1 - u_2) \, dx = \int_D |\nabla u_1|^2 \, dx - \int_D |\nabla u_2|^2 \, dx. \tag{3.14}$$

Similarly,

$$\begin{aligned}\int_D \nabla u_1 \cdot \nabla u_1 \, dx &= \int_{\Gamma_2} u_1 N u_1 \, ds \\ &= \int_{\Gamma_2} u_0 N u_1 \, ds \\ &= \int_D \nabla u_0 \cdot \nabla u_1 \, dx,\end{aligned}$$

which implies

$$\int_D \nabla(u_1 - u_0) \cdot \nabla(u_1 - u_0) \, dx = \int_D |\nabla u_0|^2 \, dx - \int_D |\nabla u_1|^2 \, dx. \tag{3.15}$$

Using that

$$(TK\eta, TK\eta) = \int_D \nabla u_2 \cdot \nabla u_2 \, dx$$

together with (3.14) and (3.15), give

$$(TK\eta, TK\eta) \le \int_D |\nabla u_0|^2 = \|\eta\|^2, \tag{3.16}$$

with $\|\cdot\|$ the norm induced by the inner product (3.6). The norm of the operator TK is therefore according to (3.16) less than or equal to one. Thus, from Theorem 3.1, the regularizing parameter can then be chosen as $\gamma = 1$. We point out that one is not an eigenvalue of TK.

Note: Other iterative methods, where data are updated with data of the similar type, can be proposed as well. For example, one can start by guessing the Neumann data instead of the Dirichlet data. The procedure is then:

- Choose an arbitrary initial approximation ξ_0 on the boundary part Γ_1.

- The first approximation u_0 of the solution u is obtained by solving (1.1) supplied with the boundary conditions

$$Nu_0 = \xi_0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad u_0 = f \quad \text{on } \Gamma_2.$$

- Next, v_0 is constructed by solving (1.1) with the boundary conditions changed to

$$v_0 = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nv_0 = g - Nu_0 \quad \text{on } \Gamma_2.$$

- Given that u_{k-1} and v_{k-1} are known, the approximation u_k is determined from (1.1) with

$$Nu_k = \xi_k \quad \text{on } \Gamma_1 \qquad \text{and} \qquad u_k = f \quad \text{on } \Gamma_2$$

  and

$$\xi_k = \xi_{k-1} + \gamma N v_{k-1}|_{\Gamma_1},$$

  where $\gamma > 0$ is a relaxation parameter.

- Then v_k is determined from (1.1) with boundary conditions

$$v_k = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nv_k = g - Nu_k \quad \text{on } \Gamma_2.$$

The similar analysis can be carried out, with the spaces V_1 and V_2 replaced by $H^{-1/2}(\Gamma_1)$ and $H^{-1/2}(\Gamma_2)$, respectively, to show convergence.

Moreover, in [5], another variant using Dirichlet and Robin boundary values problems is numerically investigated with indication of convergence.

3.4. Connection with other iterative methods

As it turns out, the proposed method has a connection to the classical alternating method [16, 17], although the iterations seem at first different. Relations between the alternating method and the Landweber method is investigated in [22]. In [22], the above method is mentioned (see also [25]), and a different expression is given for the adjoint operator to K in (3.1) compared with what we obtained above. We recall the expression for the adjoint from [22]. Let w_0 solve (1.1) with

$$w_0 = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad w_0 = h \quad \text{on } \Gamma_2.$$

Moreover, let w_1 satisfy (1.1) with

$$Nw_1 = -Nw_0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad w_1 = 0 \quad \text{on } \Gamma_2.$$

Then in [22] it is shown that the adjoint of K in (3.1) is $K^*h = w_1|_{\Gamma_1}$. This is then the same as what we have obtained. To see this, note that $w = w_0 + w_1$ is a solution to (1.1) supplied with

$$Nw = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad w = h \quad \text{on } \Gamma_2,$$

and $w|_{\Gamma_1} = w_1|_{\Gamma_1}$. The problem for w and the restriction of the solution to Γ_1 is precisely as in the definition of the operator T, see (3.4). This operator is, as have been shown above, equal to the adjoint of K.

With this alternative form of the adjoint operator, one can establish that

$$TK\eta = \eta - B\eta,$$

with B the operator used in the alternating method, for details see [22]. This equality in turn implies that although the proposed iterative scheme and the alternating method look different (see description on p. 2), the proposed method will generate the similar sequence as the alternating method for the choice $\gamma = 1$.

We show this in the case $f = g = 0$; then f and Gg in the right-hand side of (3.3) is zero. Let now w_0 solve (1.1) with

$$w_0 = \eta_k \quad \text{on } \Gamma_1 \qquad \text{and} \qquad Nw_0 = 0 \quad \text{on } \Gamma_2.$$

Moreover, let w_1 satisfy (1.1) with

$$Nw_1 = Nw_0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad w_1 = 0 \quad \text{on } \Gamma_2.$$

Then the alternating method (described on p. 2), updates as

$$\eta_{k+1} = w_1|_{\Gamma_1}. \tag{3.17}$$

The difference $w = w_0 - w_1$ satisfies (1.1) and

$$Nw = 0 \quad \text{on } \Gamma_1 \qquad \text{and} \qquad w = w_0 \quad \text{on } \Gamma_2.$$

Moreover, using the definition of the operator T, see (3.4), together with $w = w_0 - w_1$, it follows that

$$Tw_0 = w|_{\Gamma_1} = \eta_k - w_1|_{\Gamma_1}. \tag{3.18}$$

In the proposed procedure, see (3.5), putting $\gamma = 1$ and using (3.18), we have

$$\eta_{k+1} = \eta_k + T(0 - w_0|_{\Gamma_1}) = \eta_k - \eta_k + w_1|_{\Gamma_1} = w_1|_{\Gamma_1}. \tag{3.19}$$

From (3.17) and (3.19), it follow that the alternating method and the proposed procedure generate the same sequence when $f = g = 0$ and $\gamma = 1$.

To summarize, we have the following. To solve (3.3) in a stable way, one can apply the Landweber iteration. The generalized form in [25] forms the composition TK at each iteration step. With K from (3.1) and $T = K^*$, we obtain the proposed method, which, as a special case, when the regularizing parameter $\gamma = 1$, is similar to the alternating method [16, 17]. For acceleration of the alternating method, see [15] and for extension to Helmholtz equation, see [4]. Working in L^2 instead of the classical Sobolev trace spaces, and choosing $T = K^*_{L^2}$, the method in [3] is obtained for annular domains. In that method, Dirichlet data is updated by Neumann data and vice versa (see description on p. 2). Additionally, that method works for general elliptic equations as well as parabolic ones. The method [3] was generalised to simply connected domains in [13, 14] using weighted Sobolev spaces.

Whether there is a particular choice of T and accompanying spaces making the proposed method work for more general elliptic equations remains to be seen.

4. Numerical solution of the problems used in the iterative procedure

As mentioned in the introduction, we shall make numerical experiments with the proposed procedure in the case when the second-order operator L is equal to the Laplace operator. In the present section, we therefore briefly outline how to solve the boundary value problems occurring in the iterative procedure for this choice of L.

We start with the method for the mixed Dirichlet-Neumann boundary value problem consisting of

$$\Delta u = 0 \quad \text{in} \quad D, \tag{4.1}$$

with

$$u = h \quad \text{on } \Gamma_1 \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma_2.$$

We search for the solution as a combination of a single- and double-layer potential

$$u(x) = \int_{\Gamma_1} \mu_1(y)\Phi(x,y)\, ds(y) + \int_{\Gamma_2} \mu_2(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\, ds(y), \qquad x \in D, \tag{4.2}$$

with Φ the fundamental solution to the Laplace equation in $\mathbb{R}^n$, and the densities $\mu_1 \in C(\Gamma_1)$ and $\mu_2 \in C(\Gamma_2)$ are to be determined.

To determine these densities, using properties of single- and double-layer potentials [18, Chapt. 6.3–4], it follows that the densities satisfy the following system of integral equations

$$\begin{aligned} &\int_{\Gamma_1} \mu_1(y)\Phi(x,y)\, ds(y) + \int_{\Gamma_2} \mu_2(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\, ds(y) = h(x), \quad x \in \Gamma_1,\\ &\int_{\Gamma_1} \mu_1(y)\frac{\partial \Phi(x,y)}{\partial \nu(x)}\, ds(y) + \frac{\partial}{\partial \nu(x)}\int_{\Gamma_2} \mu_2(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\, ds(y) = g(x), \quad x \in \Gamma_2. \end{aligned} \tag{4.3}$$

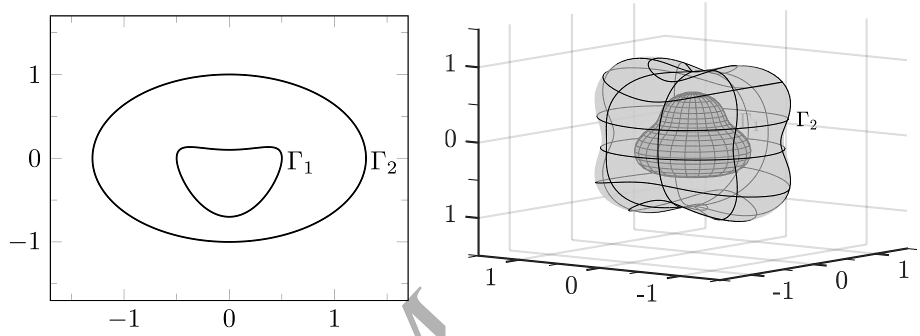
It is then advantageous, both for theoretical analysis and numerical discretisation, to parameterise these equations, that is making a specific parameterisation of the boundary parts, and then make the singularities in the kernels explicit. For two-dimensional regions, this is undertaken in [9, Sect. 3.3] together with showing existence and uniqueness of solutions to the obtained system. In three-dimensions, one can parameterise via the unit sphere as suggested in [26]. This is realised in [6, Sect. 3].

For the numerical discretisation, in two-dimensions, this is performed via a Nyström scheme, see [9, Sect. 2.2]. In three-dimensions, the approach [26] is followed, see [6, Sect. 4].

The similar representation and discretisation are employed for the other boundary value problem used in the given procedure.

5. Numerical examples

We present numerical examples with the proposed algorithm, both in two and three dimensions. The examples show that the method can be turned into a practically functioning procedure for the reconstruction of functions from Cauchy data. As pointed out in the introduction, we do not strive for full generality and do not develop a fully optimized code applicable for complicated data and domains competing with all other methods for Cauchy problems. We are only interested to see how the proposed method performs. The examples chosen are such that if the reader implements similar models in terms of boundary data and distance between the boundary surfaces, results of the similar form is to be expected.



a) The domain in Ex. 1 b) The domain in Ex. 2

Figure 1: The two solution domains used in the numerical examples

Example 1. Let the bounded domain D be the region between the following two curves,

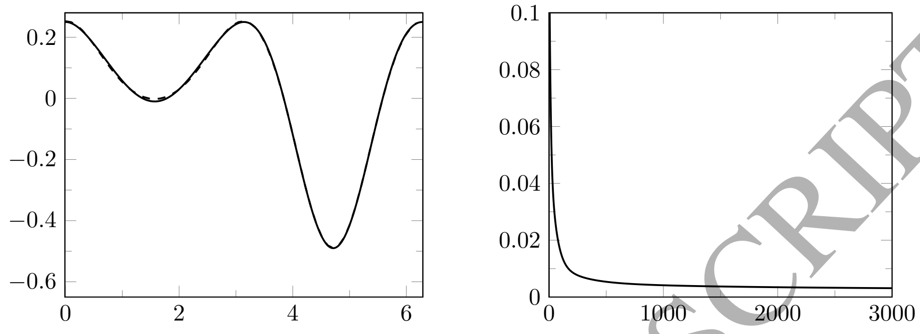
$$\Gamma_1 = \{x_1(t) = (0.5\cos t, 0.4\sin t - 0.3\sin^2 t) : t \in [0, 2\pi]\}$$

and

$$\Gamma_2 = \{x_2(t) = (1.3\cos t, \sin t) : t \in [0, 2\pi]\}.$$

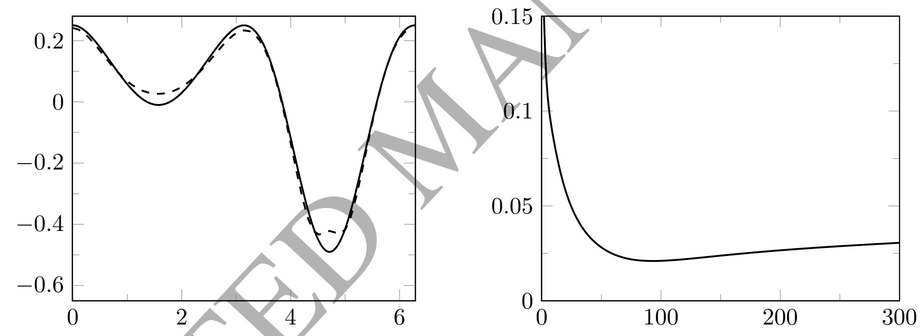
We consider the harmonic function $u_{ex}(x) = x_1^2 - x_2^2$, $x \in D$, and the necessary data for the Cauchy problem are generated as the restriction of u_{ex} and its normal derivative on the boundary Γ_2.

Results of the numerical reconstruction of the function u_{ex} on the boundary part Γ_1 of the domain D, using the given procedure for the case of exact and noisy data, are presented in Figs. 2–3. The regularizing parameter was set to $\gamma = 0.5$. The numerical solution of the corresponding mixed problems is realised by the indirect integral equation approach presented in the previous section. The Nyström method with trigonometric quadrature is then applied. The number of quadrature points is chosen as $2n = 64$. Note that for noisy data, random pointwise errors are added to the corresponding boundary function, with the percentage of error given in terms of the L^2-norm.



a) Function values b) L^2-error

Figure 2: Reconstruction on Γ_1 in Ex. 1 (exact data)



a) Function values b) L^2-error

Figure 3: Reconstruction on Γ_1 in Ex. 1 (3% noisy data)

The reconstructions behave as expected. For example, for exact data, the decrease of the error is to good order of the form $\mathcal{O}(k^{-1/2})$ and this is known decrease in the error for the Landweber method with noise free data. As the noise decreases the approximation improves. Increasing the noise makes the iterations deteriorate due to the ill-posedness of the Cauchy problem. However, the reconstructions still try to mimic the shape of the original function, thus the method is in this sense stable with respect to (moderate) noise in the data. It is also possible to reconstruct the missing Neumann data on Γ_1. This is more challenging and reconstructions will be less good. Rather then overloading with figures, we can report the similar behaviour as in [6]. For examples with the alternating method in conjunction with the boundary element method, see, for example [20, 21].

Example 2. Let the bounded domain D be the region bounded by the two surfaces

$$\Gamma_1 = \{\xi_1(\theta,\varphi) = r_1(\theta,\varphi)(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) : \theta \in [0,\pi], \varphi \in [0,2\pi]\}$$

and

$$\Gamma_2 = \{\xi_2(\theta,\varphi) = r_2(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) : \theta \in [0,\pi], \varphi \in [0,2\pi]\}$$

with radial functions

$$r_1(\theta,\varphi) = 0.2\left(0.6 + \sqrt{4.25 + 2\cos 3\theta}\right)$$

and

$$r_2(\theta,\varphi) = \sqrt{0.8 + 0.2(\cos(2\varphi) - 1)(\cos(4\theta) - 1)}.$$

The domain D is the region between an acorn shaped surface inside a cushion type surface, see Fig. 1.

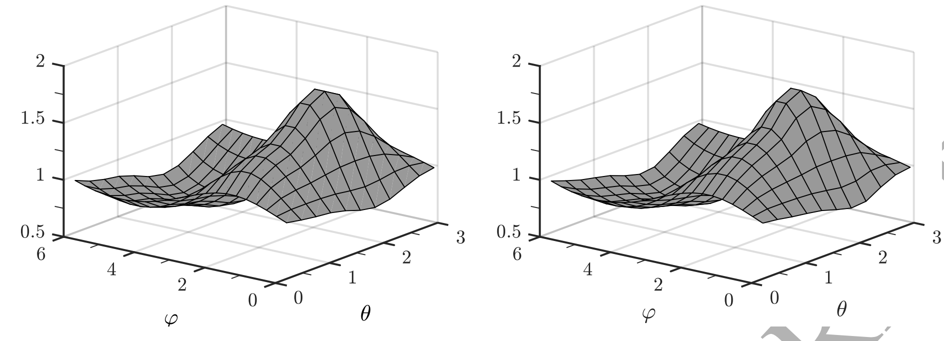
We consider the harmonic function $u_{ex}(x) = e^{x_2}\cos(x_1)$, $x \in D$. The necessary data for the Cauchy problem is generated from the exact solution on the boundary surface Γ_2 (as in the previous planar case).

The numerical solution of the corresponding mixed problems is realised by the indirect integral equation approach from the previous section.

For discretization of the system of integral equations, Wienert's method [26] is applied. This means that a Galerkin discrete projection method is used, where the unknown densities are represented as a linear combination of spherical harmonics; boundary integrals are rewritten over the unit sphere employing Gauss-trapezoid cubatures, having super-algebraic convergence.

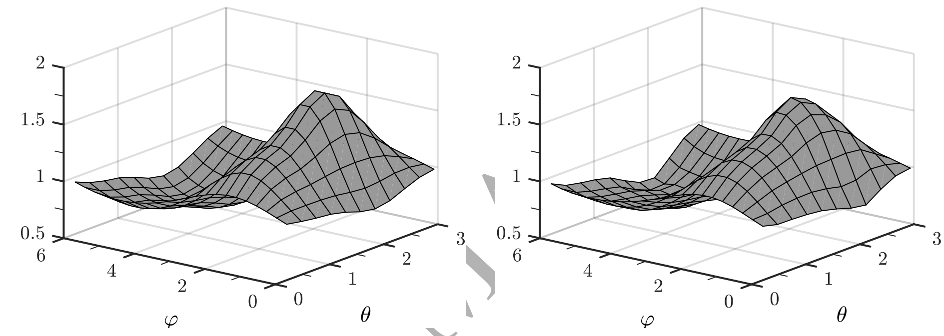
We have taken $2(n+1)^2$ basis functions and $2(n'+1)^2$ cubature points (n and n' do not necessarily have to be equal). Wienert's method [26] generates a system of linear algebraic equations of order $2(n+1)^2 \times 2(n'+1)^2$. Calculation of each coefficient of the system requires in total some computational time. Additional optimisation by using certain temporary matrices have been applied. As a result, for the construction of the matrix corresponding to the linear system obtained from discretisation, requires $\mathcal{O}(n^5)$ operations.

The results of the numerical reconstructions of the function u_{ex} on the boundary Γ_1 of the domain D, with the given algorithm for the case of exact and noisy data, are presented in Figs. 4–5. The iteration parameter γ is selected as $\gamma = 0.5$. Values of the relative error at each iteration are presented in Fig. 6. Also in this example, we leave out figures for the reconstructions of the normal derivative of Γ_1 but report that results similar to [6] are obtained.



a) Exact solution b) Approximation

Figure 4: Reconstruction of u on Γ_1 in Ex. 2 (exact data)

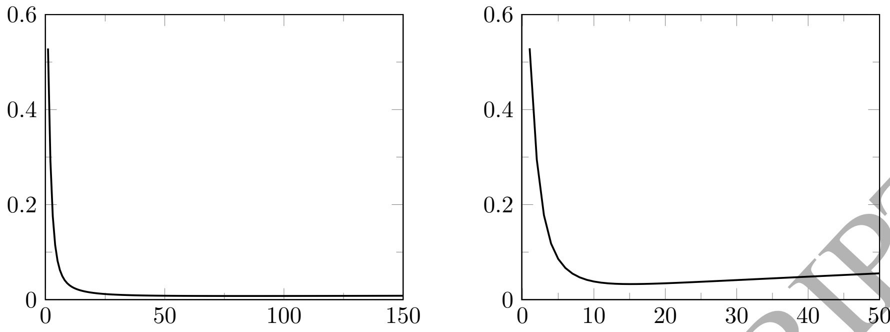


a) Exact values b) Approximation (noisy data)

Figure 5: Reconstruction of u on Γ_1 in Ex. 2 (3% noisy data)

6. Conclusion

A stable iterative procedure for the Cauchy problem for second-order strongly elliptic equations has been proposed and investigated in annular domains. This method builds on [3] but updates data with data of the same type throughout the iterations, that is Dirichlet data is updated by Dirichlet data from the previous step, and Neumann data is similarly updated by Neumann data. Boundary conditions of the problems used in the iterations are of mixed type, and the method is started by an initial guess of the Dirichlet data on the boundary part where data is missing. Convergence was established by rewriting the Cauchy problem as an operator equation on the boundary, for which the method can be written as a Landweber type procedure. Numerical experiments included show that the algorithm performed well with convergence rates as expected for a Landweber method. The similar procedure is used in [25, 22] for simply connected domains. It was shown that choosing the



a) Exact data   b) 3% noisy data

Figure 6: L^2-error for the reconstruction of u on Γ_1 in Ex. 2

regularizing parameter to unity the proposed procedure generates iterations as in the alternating method [16, 17]. As was pointed out, other variants of the iterative method can be proposed as well and one such method was given (starting with guessing the Neumann data instead of the Dirichlet data).

[1] Alessandrini, G., Rondi, L., Rosset, E., and Vessella, S., The stability for the Cauchy problem for elliptic equations, *Inverse Problems* **25** (2009), 123004.

[2] Baranger, T. N., Johansson, B. T. and Rischette, R., On the alternating method for Cauchy problems and its finite element discretisation, Springer Proceedings in Mathematics & Statistics, (Ed. L. Beilina), 183–197, (2013).

[3] Bastay, G., Kozlov, V. A. and Turesson, B. O., Iterative methods for an inverse heat conduction problem, *J. Inverse Ill-posed Probl.* **9** (2001), 375–388.

[4] Berntsson, F., Kozlov, V. A., Mpinganzima, L., and Turesson, B. O., An alternating iterative procedure for the Cauchy problem for the Helmholtz equation, *Inverse Probl. Sci. Eng.* **22** (2014), 45–62.

[5] Borachok, I. V., An iterative method for the Cauchy problem for the Laplace equation in three-dimensional domains, *J. Numer. Appl. Math.* (submitted).

[6] Borachok, I., Chapko, R. and Johansson, B. T., Numerical solution of an elliptic 3-dimensional Cauchy problem by the alternating method and boundary integral equations, *J. Inverse Ill-Posed Probl.* **24** (2016), 711–725.

[7] Cao, H., Klibanov, M. V. and Pereverzev, S. V., A Carleman estimate and the balancing principle in the quasi-reversibility method for solving the Cauchy problem for the Laplace equation, *Inverse Problems* **25** (2009), 1–21.

[8] Chapko, R. and Johansson, B. T., A direct integral equation method for a Cauchy problem for the Laplace equation in 3-dimensional semi-infinite domains, *CMES Comput. Model. Eng. Sci.* **85** (2012), 105–128.

[9] Chapko, R., Johansson, B. T. and Savka, Y., On the use of an integral equation approach for the numerical solution of a Cauchy problem for Laplace equation in a doubly connected planar domain, *Inverse Probl. Sci. Eng.* **22** (2014), 130–149.

[10] Dinh Nho Hào and Lesnic, D., The Cauchy problem for Laplace's equation via the conjugate gradient method, *IMA J. Appl. Math.* **65** (2000), 199–217.

[11] Engl, H. W. and Leitão, A., A Mann iterative regularization method for elliptic Cauchy problems, *Numer. Funct. Anal. Optim.* **22** (2001), 861–884.

[12] Isakov, V., Inverse Problems for Partial Differential Equations, ed. 3, Springer-Verlag, Cham, 2017.

[13] Johansson, T., An iterative procedure for solving a Cauchy problem for second order elliptic equations, *Math. Nachr.* **272** (2004), 46–54.

[14] Johansson, T., An iterative method for a Cauchy problem for the heat equation, *IMA J. Appl. Math.* **71** (2006), 262–286.

[15] Jourhmane, M. and Nachaoui, A., Convergence of an alternating method to solve the Cauchy problem for Poission's equation, *Appl. Anal.* **81** (2002), 1065–1083.

[16] Kozlov, V. A. and Maz'ya, V. G., On iterative procedures for solving ill-posed boundary value problems that preserve differential equations, *Algebra i Analiz* **1**, 144–170, (1989). English transl.: *Leningrad Math. J.* **1**, 1207–1228, (1990).

[17] Kozlov, V. A., Maz'ya, V. G. and Fomin, A. V., An iterative method for solving the Cauchy problem for elliptic equations, *Zh. Vychisl. Mat. i Mat. Fiz.* **31** (1991), 64–74. English transl.: *U.S.S.R. Comput. Math. and Math. Phys.* **31** (1991), 45–52.

[18] Kress, R., *Linear Integral Equations,* 3rd. ed., Springer-Verlag, New York, 2014.

[19] Leitão, A., An iterative method for solving elliptic Cauchy problems, *Numer. Funct. Anal. Optim.* **21** (2000), 715–742.

[20] Lesnic, D., Elliott, L. and Ingham, D. B., An iterative boundary element method for solving numerically the Cauchy problem for the Laplace equation, *Eng. Anal. Boundary Elements* **20** (1997), 123–133.

[21] Marin, L., Elliott, L., Heggs, P. J., Ingham, D. B., Lesnic, D., and Wen, X., An alternating iterative algorithm for the Cauchy problem associated to the Helmholtz equation, *Comput. Methods in Appl. Mech. Eng.* **192** (2003), 709–722.

[22] Maxwell, D., Kozlov-Maz'ya iteration as a form of Landweber iteration, *Inverse Probl. Imaging* **8** (2014), 537–560.

[23] McLean, W., Strongly Elliptic Systems and Boundary Integral Operators, Cambridge, Cambridge University Press, 2000.

[24] Morozov, V. A., On the solution of functional equations by the method of regularization, Dokl. Akad. Nauk SSSR **167** (1966), 510–512. English Transl.: *Soviet Math. Dokl.* **7** (1966), 414–417.

[25] Slodička, M. and Melicher, V., An iterative algorithm for a Cauchy problem in eddy-current modelling, *Appl. Math. Comput.* **217** (2010), 237–346.

[26] Wienert, L., *Die Numerische Approximation von Randintegraloperatoren für die Helmholtzgleichung im* $\mathbf{R}^3$, Ph.D. Thesis, University of Göttingen, 1990.