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Wang, Zimeng (2017) Stochastic optimal controls with delay. PhD thesis, University of Nottingham.

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SCHOOL OF MATHEMATICAL SCIENCES

PHD. THESIS

STOCHASTIC OPTIMAL CONTROLS WITH DELAY

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A thesis submitted to the University of Nottingham for the degree of

DOCTOR OF PHILOSOPHY

AUGUST, 2017

To my parents Jingle Wang and Chenghong Li.

To my wife Xinming Li.

献给我的父母，王京乐和李承红

献给我的妻子，李鑫铭

ABSTRACT

This thesis investigates stochastic optimal control problems with discrete delay and those with both discrete and exponential moving average delays, using the stochastic maximum principle, together with the methods of conjugate duality and dynamic programming.

To obtain the stochastic maximum principle, we first extend the conjugate duality method presented in [2, 44] to study a stochastic convex (primal) problem with discrete delay. An expression for the corresponding dual problem, as well as the necessary and sufficient conditions for optimality of both problems, are derived. The novelty of our work is that, after reformulating a stochastic optimal control problem with delay as a particular convex problem, the conditions for optimality of convex problems lead to the stochastic maximum principle for the control problem. In particular, if the control problem involves both the types of delay and is jump-free, the stochastic maximum principle obtained in this thesis improves those obtained in [29, 30].

Adapting the technique used in [19, Chapter 3] to the stochastic context, we consider a class of stochastic optimal control problems with delay where the value functions are *separable*, i.e. can be expressed in terms of so-called *auxiliary* functions. The technique enables us to obtain second-order partial differential equations, satisfied by the auxiliary functions, which we shall call *auxiliary HJB equations*. Also, the corresponding verification theorem is obtained. If both the types of delay are involved, our auxiliary HJB equations generalize the HJB equations obtained in [22, 23] and our verification theorem improves the stochastic verification theorem there.

ACKNOWLEDGEMENTS

First of all, I would like to express my sincere thanks to my supervisors, Professor Huiling Le and Dr David J. Hodge, for their continual support, motivation, patience and inspiration in my entire PhD study and research. Without their guidance, this thesis could not have been finished. Also, I want to show my gratitude to Professors Andrew Wood and Tusheng Zhang for being my examiners and wonderful advice for future directions.

I would like to thank the University of Nottingham for awarding me the Vice-Chancellor's Scholarship for Research Excellence (International) which sponsored my entire PhD study. Also, I would like to express my gratitude to all the staff in the School of Mathematical Sciences and to my fellow students Charles, Helen and Jack. In particular, I want to thank Alban and Kevin for their wonderful suggestions about programming skills in MATLAB, C++, and LaTeX.

Finally, special thanks to my parents, Chenghong Li and Jingle Wang, and to my wife, Xinming Li. Without their support, patience, suggestions and love, I could not have pulled through the difficult times.

CONTENTS

CHAPTER 1

INTRODUCTION

There are many real-world problems providing applications for stochastic optimal control formulations. Typically, the stochastic processes involved in these problems are Markovian and are described by stochastic differential equations (SDEs) which involve controls. Such processes and equations are referred to as state processes and state systems respectively in the context of stochastic optimal control theory. Then, the so-called Markovian optimal control problem in which the aim is either minimize a cost function or maximize a performance function. Applications include the quadratic loss minimization problem in portfolio optimization and the optimal production planning problem in economics. See, for example, [26] and [45, Chapter 2].

The topic of the present thesis concerns an extension of Markovian optimal control problems to allow for time-time (or time-lag) effects. More explicitly, the state processes are no longer Markovian. In finance, for example, although the efficient-market hypothesis states that current prices of assets reveal all the necessary information from the market (see [16, Section 1.2]), investors often take the historical performance of assets into consideration and use past information in modelling the wealth processes of portfolios. There exists a way to deal with this circumstance by using stochastic differential delay equations (SDDEs), instead of the classical ones, to describe the evolution of processes. Consequently, the corresponding portfolio optimization problem becomes a so-called stochastic optimal control problem with delay (see [5]).

In the theory of stochastic optimal control, Markovian optimal control problems can be solved by using either the stochastic maximum principle or dynamic programming (see [45, Chapters 3 & 4]). Therefore, in the remainder of the present chapter, we first review these two approaches for the Markovian case and then briefly introduce some generalizations for stochastic optimal control problems with delay that motivate the results obtained in this thesis.

Markovian Optimal Control Problems

The (sufficient) stochastic maximum principle for Markovian optimal control problems involves a so-called Hamiltonian (function) and an adjoint (stochastic differential) equation together with certain convexity/concavity conditions (see [45, Section 3.3]). From the viewpoint of the characterization of SDEs, the adjoint equation is a (controlled) classical backward stochastic differential equation (BSDE) first studied by Pardoux and Peng in [32] and further developed by Karoui, Peng, and Quenez in [18]. The advantage of applying the stochastic maximum principle is that an optimal control can be verified via the maximizer of the Hamiltonian along with the corresponding solutions of the adjoint equation and the controlled SDE (see [21]). This stochastic maximum principle generalizes the original one for deterministic delay-free optimal control problems studied by Pontryagin in the 1950s. More recently, Pontryagin's work has been further extended to the cases including Lévy jumps and/or regime-switching diffusions (see [9, 15, 46]).

In another direction, Bismut in [2] demonstrates that the conjugate duality method plays an important role in the study of the stochastic maximum principle for Markovian optimal control problems. More precisely, Bismut applies the concept of conjugate convex functions, which is developed by Rockafellar in [34, 35, 37, 38] for the deterministic case, to study Markovian convex (primal) problems in the calculus of variations. The corresponding dual problems are introduced (see [2, Definition II-1]) and the necessary and sufficient conditions for optimality of both the primal and dual problems are obtained (see [2, Theorem IV-2]). Then, as presented in [2, Section 5], Bismut reformulates a Markovian optimal control problem as a particular convex problem and furthermore uses the conditions for optimality of the convex problem to obtain certain necessary conditions for optimality of the control problem, where the corresponding Hamiltonian and adjoint equation are involved (see [2, Theorem V-1]).

Unlike the stochastic maximum principle described above, the foundation for applying the dynamic programming method is to investigate a family of Markovian optimal control problems parameterized with different initial times and states. This approach solves all the control problems in that family rather than a particular one. More precisely, the Bellman principle (or dynamic programming equation), together with the Itô formula and certain smooth conditions, leads to a second-order partial differential equation (PDE) satisfied by the value function of the control problem. This PDE is called the Hamilton-Jacobi-Bellman (HJB) equation (see [45, Section 4.3]). The advantage of applying dynamic programming is that an optimal control can be constructed in terms of a solution of the HJB equation. Such a method for obtaining an optimal control is referred to as the stochastic verification theorem (or technique) in the context of dynamic programming (see [45, Section 5.5]). The original development of dynamic programming for the corresponding deterministic delay-free optimal control problem was developed by Bellman in the early 1950s. Generalizing the deterministic situation, Kushner in [20] obtains the corresponding results in the context of continuous-time diffusion. More recently, Azevedo, Pinheiro, and Weber in [1] study dynamic programming for Markovian optimal control problems including Lévy jumps and regime-switching diffusions.

Although the stochastic maximum principle and the dynamic programming described above have been developed separately and independently, there is a connection between these two approaches. More precisely, solutions of the adjoint equations can be expressed in terms of certain derivatives of sufficiently smooth solutions of the corresponding HJB equations along with the corresponding optimal controls and state processes (see [2, page 402] and [45, Chapter 5]). More recently, this connection for the Markovian optimal control problems including Lévy jumps and/or regime-switching diffusions has been studied in [9, 15, 46].

In this thesis, we shall generalize both the stochastic maximum principle and dynamic programming with the help of the conjugate duality method for stochastic optimal control problems with delay.

Stochastic Optimal Control Problems with Delay

Some progress has been made on generalizing both the stochastic maximum principle and dynamic programming for stochastic optimal control problems with delay. See, for example, [5, 6, 22, 23, 29, 30]. In particular, the types of delay considered in these papers are either just discrete delay or both discrete and exponential moving average delays.

Chen and Wu in [6] establish a stochastic maximum principle for stochastic optimal control problems with discrete delay. The corresponding Hamiltonian and adjoint equation are introduced in [6, page 1077]. Unlike the Markovian case, the adjoint equation is an anticipated (or time-advanced) BSDE first studied by Peng and Yang in [33] (see also [30, Section 5]). However, Chen and Wu in [6] do not consider further the cases with exponential moving average delay.

A stochastic maximum principle for a certain type of stochastic optimal control problems with both discrete and exponential moving average delays is obtained in [29, 30]. However, the corresponding Hamiltonian functions and adjoint equations in these two papers are different. In the former, the adjoint equation is a triple of classical BSDEs with a restriction that one of the BSDEs needs to be identically zero. On the other hand, the adjoint equation in the latter paper is a single anticipated BSDE together with a Hamiltonian differing from the one introduced in the former. Note that, from the viewpoint of characterization of stochastic optimal control problems, those with just discrete delay can obviously be regarded as a special case of those with both discrete and exponential moving average delays. However, as the argument in the proof of [29, Theorem 2.2] shows, the stochastic maximum principle presented there cannot be adapted to a model that only involves a discrete delay. Note also that, although the restriction in [29] mentioned above does not appear in [30], the stochastic maximum principle obtained in [30, Theorem 3.1] is not capable of dealing with control problems where the terminal cost depends on the terminal value of an exponential moving average delay.

Note that all the results concerning stochastic maximum principles obtained in [6, 29, 30] are proved essentially by the results and techniques of stochastic calculus. However, although the Hamiltonian and the adjoint equation together play central roles in defining the stochastic maximum principle, the results obtained in [6, 29, 30] do not offer any derivation for the corresponding expressions of Hamiltonian and adjoint equation. On the other hand, as stated in [2], the expressions for the Hamiltonian and the adjoint equation for Markovian optimal control problems can be derived by the conjugate duality method. Therefore, it is worth mentioning that such a conjugate duality method has been generalized by Tsoutsinos and Vinter to study a deterministic convex problem with discrete delay in [44]. The corresponding dual problem and the conditions for optimality are obtained (see [44, page 171 & Theorem 2.2]) which allow the authors to solve some deterministic optimal control problems with discrete delay. Nevertheless, to the best of our knowledge, the corresponding results in the context of stochastic convex problems with discrete delay and the relationship to the corresponding stochastic maximum principle, which will be studied in the present thesis, are new.

Regarding dynamic programming, Larssen in [22] obtains Bellman's principle of optimality for stochastic optimal control problems with delay, where the value functions may depend on the initial paths of the state processes in a complicated way. As noted in [23, Section 1], this causes difficulties to use the Itô formula to obtain HJB equations except for some special cases. For example, Larssen and Risebro in [23] (see also [22, page 668]) consider a class of stochastic optimal control problems with both discrete and exponential moving average delays, where the value functions depend on a certain weighted average of the initial paths. In particular, this allows them to use the Itô formula given in [22, Lemma 5.1] to obtain HJB equations. Unfortunately, using these HJB equations in the corresponding stochastic verification theorem requires that solutions of the HJB equations are independent of a secondary variable corresponding to discrete delay in the models. Such a requirement is not always fulfilled even when a model involves only an exponential moving average delay (see [22, Lemma 5.1]). Note that the result of the connection between the stochastic maximum principle in [29] and the method of dynamic programming in [23] has been obtained in [41].

As noted in [23, Lemma 3.2], the argument for deriving the HJB equation cannot be adapted to a model that only involves a discrete delay. In fact, the derivation of the HJB equation in [23] depends on the nature of the exponential moving average delay (see also [13, 22]). On the other hand, it is worth pointing out that Kolmanovskiĭ and Shaĭkhet in [19, Chapter 3] investigate a class of deterministic control problem with discrete delay by introducing an auxiliary function, which only depends on the two boundary points of the initial path, together with certain conditions to facilitate the characterization of the value function (see [19, Definition 3.1.1]). This technique allows them to obtain a first-order PDE satisfied by the auxiliary function. However, to the best of our knowledge, this technique has not been applied to the corresponding stochastic case.

Structure and Main Results of the Thesis

To address the restrictions of [6, 22, 23, 29, 30] mentioned above, our main results are presented in three chapters:

- Chapter 2 investigates the conjugate duality method for stochastic convex problems with discrete delay extending those studied in [2, 44].
- Chapter 3 uses the results for the conjugate duality method obtained in Chapter 2 to improve the stochastic maximum principle for stochastic optimal control problems with discrete delay and those with both discrete and exponential moving average delays. Our stochastic maximum principles not only recover that obtained in [6], but also extend those studied in [29, 30].
- Chapter 4 adapts the technique used in [19, Chapter 3] to obtain second-order PDEs, which are called auxiliary HJB equations, for a class of control problems studied in Chapter 3, where the value functions are separable. These results extend those studied in [22, 23].

The main results of each chapter are summarized below and we refer readers to the first section in each of these three chapters for the corresponding detailed literature reviews. Note that we only investigate minimization problems in the following chapters. With a minor modification, the arguments and results obtained in this thesis can be adapted for maximization problems. We also provide some potential future directions in Chapter 5 based on our current works.

In Chapter 2, we investigate stochastic convex (primal) problems with discrete delay by extending the approaches in [2, 44] mentioned above to obtain the corresponding dual problems and the necessary and sufficient conditions for optimality of both primal and dual problems. Unlike the corresponding deterministic case studied in [44], the 'time' cannot be reversed in the stochastic context as noted in [24, Section 1.1]. To overcome this difficulty, we apply the techniques of conditional expectations in the characterization of the stochastic processes in the corresponding dual problems and then apply the martingale representation theorem to identify those processes as solutions to certain BSDEs.

In Chapter 3, we reformulate stochastic optimal control problems with just discrete delay and those with both discrete and exponential moving average delays as specific stochastic convex problems with discrete delay investigated in Chapter 2. This allows us to use one of those conditions for optimality to derive the corresponding stochastic maximum principles. In particular, the derivation of the associated Hamiltonian functions and adjoint equations are provided. Note that, if only discrete delay is involved, our result on the stochastic maximum principle is similar to the one obtained in [6, Theorem 3.2] when the control in [6] does not depend on a discrete delay. On the other hand, if both discrete and exponential moving average delays are involved, our adjoint equations are described by a pair of BSDEs one of which is an anticipated BSDE and the other is a classical BSDE. As mentioned above, these adjoint equations are different from those in [29, 30] although our Hamiltonian is similar to the one introduced in [29]. More importantly, the restriction mentioned in [29] is eliminated and the results in [30] are improved to allow the terminal cost to depend on the terminal value of the exponential moving average delay when the model there is jump-free.

In Chapter 4, we investigate a class of the control problems studied in Chapter 3, where the value functions are separable, by adapting the technique of [19, Chapter 3] mentioned above to introduce the so-called auxiliary function. The novelty of this technique is that it allows us to apply the Itô formula to obtain second-order PDEs satisfied by the auxiliary functions. Note that, if the model only involves a discrete delay, our PDEs play a similar role to that of the classical HJB equations in the verification theorem of dynamic programming for Markovian optimal control problems. Thus, we refer these PDEs as auxiliary HJB equations. If the model involves both discrete and exponential moving average delays, our results generalize those obtained in [22, 23] and, more importantly, eliminates the restriction there noted above for some special cases. Finally, Chapter 4 provides the connection between the dynamic programming method involving the auxiliary HJB equation with the stochastic maximum principle obtained in Chapter 3.

Chapter 2

Conjugate Duality Method in Stochastic Convex Problems with Discrete Delay

2.1 Introduction

To apply the techniques and results of conjugate duality to study the stochastic maximum principles for stochastic optimal control problems with delay, this chapter extends the conjugate duality method developed in [2, 44] to investigate stochastic convex problems with discrete delay.

2.1.1 Literature Review

To access some basic results of the conjugate duality method, we first review this method reported in [37] for a delay-free deterministic convex problem in the calculus of variations. Afterward, we describe two known generalizations [2, 44] which inspire the work of the present chapter.

The Delay-Free Deterministic Convex Problem

Rockafellar in [37] investigates the deterministic convex (primal) problem: for given convex functions L and l, minimize

$$\Phi(x) = \int_0^T L(t, x(t), \dot{x}(t))\, dt + l(x(0), x(T)) \tag{2.1.1}$$

among all $\mathbb{R}^n$-valued absolutely continuous functions x on $[0, T]$ with derivative $\dot{x}$ almost everywhere, where n is a positive integer and $T \in (0, \infty)$ is the fixed time horizon. Note that, since it is absolutely continuous, x can be identified with $(x(0), \dot{x})$ in the sense that

$$x\left(t\right) = x\left(0\right) + \int_0^t \dot{x}\left(s\right) ds, \quad \forall t \in \left[0, T\right] \tag{2.1.2}$$

(see [39, Theorem 6.11]). As presented in [37, Section 5], Rockafellar defines L^* and l^* as the conjugate convex functions of L and l in (2.1.1) respectively, a concept described in his previous work [34], and then introduces the dual problem to (2.1.1): for such L^* and l^*, minimize

$$\Psi\left(p\right) = \int_0^T L^*\left(t, \dot{p}\left(t\right), p\left(t\right)\right) dt + l^*\left(p\left(0\right), -p\left(T\right)\right) \tag{2.1.3}$$

among $\mathbb{R}^n$-valued absolutely continuous functions p on $[0, T]$, where $\dot{p}$ is as defined similarly to $\dot{x}$. Note that $\bar{x}$ and $\bar{p}$ are called optimal solutions for the primal problem (2.1.1) and the dual problem (2.1.3) respectively if they achieve the corresponding minimum of the two problems.

Using certain properties of conjugate convex functions described in [34], necessary and sufficient conditions for optimality of the primal problem (2.1.1) and the dual problem (2.1.3) are obtained in [37, Theorem 5]. In particular, $\bar{x}$ and $\bar{p}$ are optimal for these two problems respectively with $\Phi(\bar{x}) + \Psi(\bar{p}) = 0$ if and only if $\bar{x}$ and $\bar{p}$ satisfy

$$L\left(t, \bar{x}\left(t\right), \dot{\bar{x}}\left(t\right)\right) + L^*\left(t, \dot{\bar{p}}\left(t\right), \bar{p}\left(t\right)\right) - \left\langle \bar{p}\left(t\right), \dot{\bar{x}}\left(t\right)\right\rangle - \left\langle \dot{\bar{p}}\left(t\right), \bar{x}\left(t\right)\right\rangle = 0 \tag{2.1.4}$$

for almost every t and

$$l\left(\bar{x}\left(0\right), \bar{x}\left(T\right)\right) + l^*\left(\bar{p}\left(0\right), -\bar{p}\left(T\right)\right) + \left\langle \bar{p}\left(T\right), \bar{x}\left(T\right)\right\rangle - \left\langle \bar{p}\left(0\right), \bar{x}\left(0\right)\right\rangle = 0, \tag{2.1.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in the Euclidean space $\mathbb{R}^n$. This method for obtaining conditions for optimality of both the primal and dual problems, involving the relationship of conjugate duality between the convex functions, is referred to as the conjugate duality method.

The Markovian Convex Problem

Bismut in [2] generalizes the above results to the stochastic context. More specifically, let m be an positive integer and write $(\Omega, \mathcal{F}, \mathbb{P})$ for a complete probability space, W for a standard m-dimensional Brownian motion (see [17, Definition 2.5.1]) and

$\mathbb{F} = \{\mathcal{F}(t)\}_{t\in[0,T]}$ for a filtration generated by W such that the usual conditions hold (see [17, Definition 1.2.25]). Then, for given convex functions L and l, Bismut in [2] minimizes

$$\Phi(X) = \mathbb{E}\left[\int_0^T L\left(t, X(t), \dot{X}(t), H_X(t)\right) dt + l\left(X(0), X(T)\right)\right] \tag{2.1.6}$$

among $\mathbb{R}^n$-valued Itô processes X which can be identified with $(X(0), \dot{X}, H_X)$ in the sense that

$$X(t) = X(0) + \int_0^t \dot{X}(s)\, ds + \int_0^t H_X(s)\, dW(s), \quad \forall t \in [0, T] \tag{2.1.7}$$

(see [43, Definition 4.4.3]), where $\dot{X}$ and H_X denote respectively the drift and diffusion coefficients. Similarly to the approach for introducing the corresponding dual problem in (2.1.3), Bismut in [2, page 389] defines L^* and l^* as the conjugate convex functions of L and l in (2.1.6) respectively and then introduces the dual problem to (2.1.6). That is, for such L^* and l^*, minimizing

$$\Psi(P) = \mathbb{E}\left[\int_0^T L^*\left(t, \dot{P}(t), P(t), H_P(t)\right) dt + l^*\left(P(0), -P(T)\right)\right] \tag{2.1.8}$$

among $\mathbb{R}^n$-valued Itô processes P, where P is identified with $(P(0), \dot{P}, H_P)$ in the same manner as X in (2.1.7) and $(\dot{P}, H_P)$ is defined similarly to $(\dot{X}, H_X)$.

Adapting the techniques applied in [37, Theorem 5], Bismut in [2, Theorem IV-2] obtains certain necessary and sufficient conditions for optimality in which, similarly to the deterministic case described by (2.1.4) and (2.1.5), $\bar{X}$ and $\bar{P}$ are optimal for the primal problem (2.1.6) and the dual problem (2.1.8) respectively with $\Phi(\bar{X}) + \Psi(\bar{P}) = 0$ if and only if $\bar{X}$ and $\bar{P}$ satisfy

$$\begin{aligned} &L\left(t, \bar{X}(t), \dot{\bar{X}}(t), H_{\bar{X}}(t)\right) + L^*\left(t, \dot{\bar{P}}(t), \bar{P}(t), H_{\bar{P}}(t)\right) - \left\langle \bar{P}(t), \dot{\bar{X}}(t)\right\rangle \\ &- \left\langle \dot{\bar{P}}(t), \bar{X}(t)\right\rangle - \left\langle H_{\bar{P}}(t), H_{\bar{X}}(t)\right\rangle = 0, \quad \mathbb{P}\otimes Leb - a.s. \end{aligned} \tag{2.1.9}$$

and

$$\begin{aligned} &l\left(\bar{X}(0), \bar{X}(T)\right) + l^*\left(\bar{P}(0), -\bar{P}(T)\right) + \left\langle \bar{P}(T), \bar{X}(T)\right\rangle \\ &- \left\langle \bar{P}(0), \bar{X}(0)\right\rangle = 0, \quad \mathbb{P} - a.s. \end{aligned} \tag{2.1.10}$$

Here $\mathbb{P}\otimes Leb$ denotes the Lebesgue measure on $\mathcal{F}\times\mathcal{B}([0,T])$, where we write $\mathcal{B}([0,T])$ for the Borel σ-algebra on $[0, T]$. Note that the primal problem (2.1.6) and the dual problem (2.1.8), together with the conditions for optimality described by (2.1.9) and (2.1.10), are similar to those offered in [37], except that the diffusion coefficients H_X and H_P have been introduced. It is worth mentioning here that the theory in [2] also allows one to consider an extra randomness in the model, related to a diffusion M, by assuming that the corresponding filtration $\mathbb{F}$ be generated jointly by M and W (see also [3, Section 3.8]). In particular, this will help us to introduce a regime-switching effect into the model.

The Deterministic Convex Problem with Discrete Delay

In a different direction, Tsoutsinos and Vinter in [44] consider the deterministic convex problem with discrete delay: for given convex functions L and l, minimize

$$\Phi(x) = \int_0^T L(t, x(t), x(t-\delta), \dot{x}(t))\, dt + l(x(T)) \tag{2.1.11}$$

among $\mathbb{R}^n$-valued absolutely continuous functions x on $[-\delta, T]$ such that $x(t) = \xi(t)$ for every $t \in [-\delta, 0]$, where $\delta \in (0, T)$ and ξ is a given continuous function on $[-\delta, 0]$. However, unlike the corresponding delay-free case, this primal problem also depends on $x(\cdot - \delta)$ which causes some difficulties. In particular, the corresponding dual problem cannot be introduced in a similar fashion to the one in (2.1.3) by using the corresponding conjugate convex functions L^* and l^* directly. However, as stated in [37, Theorem 3], the conjugate convex function of the optimal value function, a concept introduced in [38, page 2], associated with the primal problem (2.1.1) coincides with the function Ψ defined by (2.1.3). Therefore, generalizing such a relationship into the time-delay context, Tsoutsinos and Vinter in [44, Proposition 3.1] obtain the corresponding dual problem as minimizing

$$\begin{aligned} \Psi(p,q) = &\int_0^T L^*\left(t, \dot{p}(t) - \dot{q}(t+\delta) I_{[0,T-\delta]}(t), \dot{q}(t), p(t)\right) dt \\ &- \int_0^T \left\langle \dot{q}(t), \xi(t-\delta) I_{[0,\delta]}(t) \right\rangle dt - \int_0^T \langle \dot{p}(t), \xi(0) \rangle\, dt \\ &+ l^*(-p(T)) + \langle p(T), \xi(0) \rangle \end{aligned} \tag{2.1.12}$$

among $\mathbb{R}^n$-valued absolutely continuous functions p and q on $[0, T]$ such that $q(0) = 0$, where I_A denotes the indicator function of the set A.

Similarly to the conditions for optimality described by (2.1.4) and (2.1.5), Tsoutsinos and Vinter in [44, Theorem 2.1 & 2.2] obtain that $\bar{x}$ and $(\bar{p}, \bar{q})$ are optimal solution of the primal problem (2.1.11) and the dual problem (2.1.12) respectively with $\Phi(\bar{x}) + \Psi(\bar{p}, \bar{q}) = 0$ if and only if $\bar{x}$ and $(\bar{p}, \bar{q})$ satisfy

$$\begin{aligned} &L\left(t, \bar{x}(t), \bar{x}(t-\delta), \dot{\bar{x}}(t)\right) + L^*\left(t, \dot{\bar{p}}(t) - \dot{\bar{q}}(t+\delta)\, I_{[0,T-\delta]}(t), \dot{\bar{q}}(t), \bar{p}(t)\right) \\ &- \langle \dot{\bar{q}}(t), \bar{x}(t-\delta)\rangle - \left\langle \dot{\bar{p}}(t) - \dot{\bar{q}}(t+\delta)\, I_{[0,T-\delta]}(t), \bar{x}(t)\right\rangle - \langle \bar{p}(t), \dot{\bar{x}}(t)\rangle = 0 \end{aligned} \tag{2.1.13}$$

for almost every t and

$$l(\bar{x}(T)) + l^*(-\bar{p}(T)) + \langle \bar{p}(T), \bar{x}(T)\rangle = 0. \tag{2.1.14}$$

Note that the dual problem (2.1.12) and the conditions for optimality (2.1.13) and (2.1.14) involve the extra variable q with $\dot{q}(\cdot + \delta)$, where $\dot{q}$ is as defined similarly to $\dot{x}$. These time-advanced values will give rise to an issue when generalizing them to the stochastic context in the present chapter. To resolve this issue, we shall apply the techniques of conditional expectations at a certain stage in the derivation of the corresponding dual problem.

2.1.2 Main Results and Structure of the Chapter

Motivated by the results in [2, 44] described above, we explore the conjugate duality method to investigate the stochastic convex problem with discrete delay: for given convex functions L and l, minimize

$$\Phi(X) = \mathbb{E}\left[\int_0^T L\left(t, X(t), X(t-\delta), \dot{X}(t), H_X(t)\right) dt + l(X(T))\right] \tag{2.1.15}$$

among X which ranges through a certain family, to be specified in Section 2.3.1, of Itô processes satisfying $X(t) = \xi(t)$ for $t \in [-\delta, 0]$, where $\dot{X}$, H_X, ξ and δ are as defined for (2.1.6) and (2.1.11). The corresponding dual problem is derived in Theorem 2.4.4 which generalizes (2.1.12) to the stochastic context, and the necessary and sufficient conditions for optimality of the primal problem (2.1.15) and the dual problem are then obtained in Theorem 2.5.2. Note that we could equivalently maximize Φ, if L and l in (2.1.15) were concave, together with the concept of conjugate concave function (see [38, page 18]). For example, replacing L and l in (2.1.15) by $-L$ and $-l$.

The remainder of the chapter is organized as follows. Section 2.2 summarizes some basic results on convex analysis reported in [34, 35, 37, 38] and Section 2.3 gives the detailed description of the primal problem (2.1.15). Generalizing the results in [44] to the stochastic context and generalizing the results in [2] to involve a discrete delay, Section 2.4 shows the derivation of the dual problem to (2.1.15). Then, the necessary and sufficient conditions for optimality of both the primal and dual problems are obtained in Section 2.5. Finally, extending the arguments in the preceding sections of this chapter, Section 2.6 presents a stochastic convex problem in a general discrete delayed model which will be used to obtain the stochastic maximum principle for stochastic optimal control problems with both discrete and exponential moving average delays in the following chapter.

2.2 Some Results on Conjugate Convex Functions

To apply the conjugate duality method, this section recalls some basic results of conjugate convex functions taken from [34, 35, 37, 38].

The preliminary concept in the conjugate convex function is a pair of linear (or vector) spaces associated with a specified duality pairing. Let $\mathbb{X}$ and $\mathbb{Y}$ be two linear spaces. We define a bilinear map, denoted by $\ll \cdot, \cdot \gg$, on these two spaces as follows: for each $y \in \mathbb{Y}$,

$$\ll \cdot, y \gg : x \to \ll x, y \gg \tag{2.2.1}$$

is a linear function on $\mathbb{X}$ and also, for each $x \in \mathbb{X}$,

$$\ll x, \cdot \gg : y \to \ll x, y \gg \tag{2.2.2}$$

is a linear function on $\mathbb{Y}$ (see [17, page 32]). If there exist compatible topologies (specified below) with respect to $\ll \cdot, \cdot \gg$ on $\mathbb{X}$ and $\mathbb{Y}$ respectively, then the bilinear map $\ll \cdot, \cdot \gg$ is called duality pairing, or simply pairing, associated with these spaces (see [38, page 13]). Here, a topology on $\mathbb{X}$ (resp. $\mathbb{Y}$) is called compatible with respect to $\ll \cdot, \cdot \gg$ if it is a locally convex topology such that every linear function (2.2.1) (resp. (2.2.2)) is continuous and every continuous linear function on $\mathbb{X}$ (resp. $\mathbb{Y}$) can be represented in the form of (2.2.1) (resp. (2.2.2)) for some $y \in \mathbb{Y}$ (resp. $x \in \mathbb{X}$). Note that, associated with the pairing, these two spaces $\mathbb{X}$ and $\mathbb{Y}$ are referred to as duality paired linear spaces, or simply paired spaces. For example, $\mathbb{R}^n$ pairs with itself via the pairing given by the usual inner product in $\mathbb{R}^n$, where the compatible topology is induced by the Euclidean norm.

Following the convention given in [38, page 13], when we say that two linear spaces are paired spaces, then a specific pairing is implied and these two linear spaces are equipped with topologies compatible with that pairing automatically.

Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are paired spaces associated with the specified pairing $\ll \cdot, \cdot \gg$ and let F be an extended-real-valued convex function on $\mathbb{X}$. Then, the extended-real-valued function F^* on $\mathbb{Y}$, defined by

$$F^*(y) = \sup_{x \in \mathbb{X}} \{\ll x, y \gg - F(x)\}, \tag{2.2.3}$$

is called the conjugate convex function of F. Similarly, the extended-real-valued function F^{**} on $\mathbb{X}$, defined by

$$F^{**}(x) = \sup_{y \in \mathbb{Y}} \{\ll x, y \gg - F^*(y)\}, \tag{2.2.4}$$

is called the conjugate convex function of F^* and also referred to as bi-conjugate of F, where F^* and F^{**} are always convex and lower semi-continuous on $\mathbb{Y}$ and $\mathbb{X}$ respectively (see [37, page 189]). In the presence of convexity, if F is strictly greater than $-\infty$, not identically ∞, and is lower semi-continuous, then $F = F^{**}$ (see also [35, page 51]). The following gives an example of such a F.

Note that, as mentioned in Section 2.1.2, we can similarly introduce the concept of conjugate concave function if F is concave. Then the condition of lower semi-continuity and the supremum in (2.2.3) and (2.2.4) are respectively replaced by upper semi-continuity and infimum (see [38, page 18]).

Example 2.2.1. For simplicity, we set $n = 1$ and $\mathbb{X} = \mathbb{Y} = \mathbb{R}$. Suppose that the extended-real-valued convex function F is defined by

$$F(x) = \begin{cases} -\log x, & \text{if } x > 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Then, it follows from (2.2.3) that the conjugate convex function F^* of F is

$$F^*(y) = \sup_{x \in \mathbb{R}} \{xy - F(x)\} = \sup_{x \in \mathbb{R}^+} \{xy + \log x\} \tag{2.2.5}$$

for $y \in \mathbb{R}$. If $y \geq 0$, letting x tend to ∞ on the right-hand-side of the second equality of (2.2.5), we see that $F^*(y) = \infty$. Otherwise, to find an explicit expression for F^*, we take the derivative, with respect to x, of the function $xy + \log x$ to see that the corresponding derivative is zero if and only if $x = -1/y$. Moreover, since $xy + \log x$ is concave with respect to x, the supremum in (2.2.5) is attained at $x = -1/y$ so that, together with the case of $y \geq 0$,

$$F^*(y) = \begin{cases} -1 - \log(-y), & \text{if } y < 0, \\ \infty, & \text{otherwise.} \end{cases}$$

On the other hand, it follows from (2.2.4) that the corresponding bi-conjugate convex function F^{**} is

$$F^{**}(x) = \sup_{y \in \mathbb{R}} \{xy - F^*(y)\} = \sup_{y \in \mathbb{R}^-} \{xy + 1 + \log(-y)\} \tag{2.2.6}$$

for $x \in \mathbb{R}$. Similarly, if $x \leq 0$, we obtain that $F^{**}(x) = \infty$. Otherwise, taking the derivative, with respect to y, of the function within the bracket on the right-hand-side of the second equality of (2.2.6), we see that the supremum in (2.2.6) is attained at $y = -1/x$. This, together with $F^{**}(x) = \infty$ when $x \leq 0$, verifies $F^{**} = F$ as F is strictly greater than $-\infty$, not identically ∞, and is lower semi-continuous. □

2.3 The Stochastic Convex Problem with Discrete Delay

We continue to work with the fixed time horizon $T \in (0, \infty)$, complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and standard m-dimensional Brownian motion W introduced in Section 2.1.1. To include a regime-switching effect used in [8, 9, 10], we suppose that the filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{t \in [0,T]}$ here is generated jointly by W and α satisfying the usual conditions, where α is a continuous time Markov chain with the finite state $\mathbb{I} = \{1, 2, \ldots, d\}$, is independent of W, and its generator is defined by a $d \times d$ matrix $g = (g_{ij})$ and where d is an positive integer. For $i \neq j \in \mathbb{I}$, the counting process N and the intensity process λ are respectively defined by

$$N_{ij}(t) = \sum_{0 < s \leq t} I_{\{\alpha(s_-) = i\}}(t)\, I_{\{\alpha(s) = j\}}(t)$$

and

$$\lambda_{ij}(t) = g_{ij} I_{\{\alpha(t) = i\}}(t)$$

(see [9, Section 2]). Then, the set $M = \{M_{ij}, i, j \in \mathbb{I}\}$ of canonical martingales of the Markov chain α described in [9] (see also [10, 8]) is defined by

$$M_{ij}(t) = \begin{cases} N_{ij}(t) - \int_0^t \lambda_{ij}(s)\, ds, & \text{if } i \neq j, \\ 0, & \text{otherwise,} \end{cases}$$

for $t \in [0, T]$.

In addition, adapting from [2, page 386], we introduce the following four spaces which are frequently used throughout the present thesis:

$\mathbb{L}^2(\mathcal{F}(T); \mathbb{R}^n)$ denotes the space of $\mathcal{F}(T)$-measurable, $\mathbb{R}^n$-valued random variables X such that

$$\mathbb{E}\left[|X|^2\right] < \infty,$$

where $|\cdot|$ denotes the Euclidean norm and the norm on $\mathbb{L}^2(\mathcal{F}(T); \mathbb{R}^n)$ is given by

$$\|X\|_2 = \left\{\mathbb{E}\left[|X|^2\right]\right\}^{1/2};$$

$\mathbb{L}^{2\infty}_{\mathcal{F}}([0, T]; \mathbb{R}^n)$ denotes the space of $\mathcal{F}(t)$-progressively measurable (see [17, Definition 1.1.11]), $\mathbb{R}^n$-valued stochastic processes X such that

$$\mathbb{E}\left[\mathop{ess\,sup}_{0 \leq t \leq T} |X(t)|^2\right] < \infty, \tag{2.3.1}$$

where the norm on $\mathbb{L}^{2\infty}_{\mathcal{F}}([0, T]; \mathbb{R}^n)$ is given by

$$\|X\|_{2\infty} = \left\{\mathbb{E}\left[\sup_{0 \leq t \leq T} |X(t)|^2\right]\right\}^{1/2};$$

$\mathbb{L}^{21}_{\mathcal{F}}([0, T]; \mathbb{R}^n)$ denotes the space of $\mathcal{F}(t)$-progressively measurable, $\mathbb{R}^n$-valued stochastic processes X such that

$$\mathbb{E}\left[\left\{\int_0^T |X(t)| dt\right\}^2\right] < \infty, \tag{2.3.2}$$

where the norm on $\mathbb{L}^{21}_{\mathcal{F}}([0, T]; \mathbb{R}^n)$ is given by

$$\|X\|_{21} = \left\{\mathbb{E}\left[\left\{\int_0^T |X(t)| dt\right\}^2\right]\right\}^{1/2};$$

$\mathbb{L}^{22}_{\mathcal{F}}([0, T]; \mathbb{R}^{n\times m})$ denotes the space of $\mathcal{F}(t)$-progressively measurable, $\mathbb{R}^{n\times m}$-valued stochastic processes H such that

$$\mathbb{E}\left[\int_0^T |H(t)|^2 dt\right] < \infty,$$

where the elements in the Euclidean space $\mathbb{R}^{n\times m}$ are represented by $n \times m$ matrices and so that $|H(t)|^2$ is given by $\langle H(t), H(t)\rangle = \mathrm{tr}(H^\top(t)H(t))$, and the norm on $\mathbb{L}^{22}_{\mathcal{F}}([0,T];\mathbb{R}^{n\times m})$ is given by

$$\|H\|_{22} = \left\{\mathbb{E}\left[\int_0^T |H(t)|^2 dt\right]\right\}^{1/2}.$$

In what follows, we simply write the above spaces as $\mathbb{L}^2$, $\mathbb{L}^{2\infty}_{\mathcal{F}}$, $\mathbb{L}^{21}_{\mathcal{F}}$, and $\mathbb{L}^{22}_{\mathcal{F}}$ respectively if the domains and ranges of the members in those spaces are clear from the context and, as above, suppress the ω in stochastic processes for notational simplicity, unless it is necessary for clarity.

2.3.1 Identification of the Primal Variable

Fix $\delta \in (0,T)$ and let $\xi \in \mathbb{C}([-\delta,0];\mathbb{R}^n)$ be a given initial deterministic continuous function. Then, since ξ is continuous on the closed interval $[-\delta,0]$, we have

$$\max_{-\delta\leq t\leq 0} |\xi(t)|^2 < \infty \tag{2.3.3}$$

(see [40, Theorem 4.15]).

Definition 2.3.1. For the given $\xi \in \mathbb{C}([-\delta,0];\mathbb{R}^n)$, write $\mathbb{V}_1$ for the space $\mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ and identify $(\dot{X}, H_X) \in \mathbb{V}_1$ with the continuous $\mathcal{F}(t)$-adapted stochastic process $X : \Omega \times [-\delta,T] \to \mathbb{R}^n$,

$$X(t) = \begin{cases} \xi(t), & t \in [-\delta,0], \\ \xi(0) + \int_0^t \dot{X}(s)\,ds + \int_0^t H_X(s)\,dW(s), & t \in [0,T]. \end{cases} \tag{2.3.4}$$

Hereafter, we simply write $X \in \mathbb{V}_1$ to mean that X is identified with $(\dot{X}, H_X) \in \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ via Definition 2.3.1. In particular, as noted in [8, Proposition 3.2.20], the representation of $X \in \mathbb{V}_1$ is unique up to indistinguishability (see [17, Definition 1.1.3]) and also X implicitly depends on α through the filtration $\mathbb{F}$. Moreover, we define X_δ associated with X by

$$X_\delta(t) = X(t-\delta), \quad \forall t \in [0,T].$$

Proposition 2.3.2. *For $X \in \mathbb{V}_1$, we have that $X, X_\delta \in \mathbb{L}^{2\infty}_{\mathcal{F}}$ and $X(T) \in \mathbb{L}^2$.*

Proof. First, as noted in [45, Proposition 1.2.8], the continuity of X, together with being $\mathcal{F}(t)$-adapted, implies that X and X_δ are $\mathcal{F}(t)$-progressively measurable. Also, it is straightforward to see that $X(T)$ is $\mathcal{F}(T)$-measurable and

$$\mathbb{E}\left[|X(T)|^2\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T} |X(t)|^2\right]. \tag{2.3.5}$$

On the other hand, separating the interval $[-\delta, T]$ into two disjoint intervals $[-\delta, 0]$ and $[0, T]$ and noting (2.3.1), we have that

$$\begin{aligned}\mathbb{E}\left[\sup_{-\delta\leq t\leq T} |X(t)|^2\right] =&\mathbb{E}\left[\sup_{-\delta\leq t\leq T}\left\{\left|X(t)\, I_{[-\delta,0]}(t) + X(t)\, I_{[0,T]}(t)\right|^2\right\}\right] \\ \leq& 2\left\{\sup_{-\delta\leq t\leq 0} |\xi(t)|^2 + \mathbb{E}\left[\sup_{0\leq t\leq T} |X(t)|^2\right]\right\},\end{aligned} \tag{2.3.6}$$

where the '*ess*' in (2.3.1) has been relaxed since X is continuous. The first term on the right-hand-side of the inequality of (2.3.6) is due to the representation (2.3.4) associated with $t \in [-\delta, 0]$. Similarly,

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_\delta(t)|^2\right] \leq \mathbb{E}\left[\sup_{-\delta\leq t\leq T} |X(t)|^2\right]. \tag{2.3.7}$$

Thus, it can be seen from (2.3.3), (2.3.5), (2.3.6) and (2.3.7) that, to prove the required results, we only need to verify

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X(t)|^2\right] < \infty. \tag{2.3.8}$$

Indeed, it follows from (2.3.4) that, for $t \in [0, T]$,

$$|X(t)|^2 \leq 3\left\{|\xi(0)|^2 + \left\{\int_0^t \left|\dot{X}(s)\right| ds\right\}^2 + \left|\int_0^t H_X(s)\, dW(s)\right|^2\right\}. \tag{2.3.9}$$

Taking supremum over $t \in [0, T]$ and then taking expectations on the both sides of (2.3.9), we have that

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T} |X(t)|^2\right] \leq 3\Bigg\{& |\xi(0)|^2 + \mathbb{E}\left[\sup_{0\leq t\leq T}\left\{\int_0^t \left|\dot{X}(s)\right| ds\right\}^2\right] \\ &+ \mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t H_X(s)\, dW(s)\right|^2\right]\Bigg\}.\end{aligned} \tag{2.3.10}$$

In particular, the second term within the bracket on the right-hand-side of (2.3.10) can be re-expressed as

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left\{\int_0^t \left|\dot{X}(s)\right| ds\right\}^2\right] = \mathbb{E}\left[\left\{\int_0^T \left|\dot{X}(s)\right| ds\right\}^2\right] \tag{2.3.11}$$

and, by Doob's Maximal Inequality (see [17, page 14]), the last term of that equation gives

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t H_X(s)\,dW(s)\right|^2\right] \le & 4\mathbb{E}\left[\left|\int_0^t H_X(s)\,dW(s)\right|^2\right] \\ = & 4\mathbb{E}\left[\int_0^t |H_X(s)|^2\,ds\right],\end{aligned} \tag{2.3.12}$$

where the equality in (2.3.12) is due to the Itô isometry (see [43, Theorem 4.3.1]). Finally, we get (2.3.8) by substituting (2.3.11) and (2.3.12) into the right-hand-side of (2.3.10), and by noting that $(\dot{X}, H_X) \in \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$. □

Note that, although the domain for X defined via Definition 2.3.1 is $[-\delta, T]$, for simplicity, we shall in the following sections of this chapter to regard X as being in $\mathbb{L}^{2\infty}_{\mathcal{F}}$ since its path in $[-\delta, 0]$ is fixed by ξ.

2.3.2 The Primal Function and Problem

Let $L : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{\infty\}$ and $l : \Omega \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be two given functions. Define functions I_L on $\mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ and J_l on $\mathbb{L}^2$ respectively by

$$I_L(X, Y, Z, H) = \mathbb{E}\left[\int_0^T L(t, X(t), Y(t), Z(t), H(t))\,dt\right] \tag{2.3.13}$$

and

$$J_l(X_T) = \mathbb{E}[l(X_T)]. \tag{2.3.14}$$

Hereafter, as before, we suppress ω in functions for simplicity. To ensure that the measurability in (2.3.13) and (2.3.14), and that I_L and J_l are strictly greater than $-\infty$, not identically ∞, and are convex, as well as to be able to apply the conjugate duality method to obtain the corresponding dual problem in the next section, we adapt the assumptions given in [37, page 179] as follows.

Assumption I. (i) L and l are not identically ∞; L is a lower semi-continuous convex function on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times m}$, for any $(\omega, t) \in \Omega \times [0, T]$, and l is a lower semi-continuous convex function on $\mathbb{R}^n$, for any $\omega \in \Omega$.

(ii) L is $\mathcal{F}^* \times \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^{n\times m})$-measurable and l is $\mathcal{F} \times \mathcal{B}(\mathbb{R}^n)$-measurable, where we write $\mathcal{F}^*$ for the completion of $\mathcal{F} \times \mathcal{B}([0, T])$ with respect to $\mathbb{P} \otimes Leb$.

Assumption II. (i) There exist $\left(X^{*}, Y^{*}, Z^{*}, H^{*}\right) \in \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{2\infty} \times \mathbb{L}_{\mathcal{F}}^{22}$ and a $\mathbb{R}$-valued $\mathcal{F}(t)$-progressively measurable stochastic process ϖ satisfying

$$\mathbb{E}\left[\int_{0}^{T} |\varpi(t)|\, dt\right] < \infty \tag{2.3.15}$$

such that, for any $(x, y, z) \in \mathbb{R}^{n\times 3}$ and $h \in \mathbb{R}^{n\times m}$,

$$\begin{aligned} L(t, x, y, z, h) \geq & \langle x, X^{*}(t)\rangle + \langle y, Y^{*}(t)\rangle + \langle z, Z^{*}(t)\rangle \\ & + \langle h, H^{*}(t)\rangle - \varpi(t), \quad \mathbb{P}\otimes Leb - a.s. \end{aligned} \tag{2.3.16}$$

(ii) There exist $X_{T}^{*} \in \mathbb{L}^{2}$ and a $\mathbb{R}$-valued $\mathcal{F}(T)$-measurable random variable ϑ satisfying

$$\mathbb{E}\left[|\vartheta|\right] < \infty \tag{2.3.17}$$

such that, for any $x \in \mathbb{R}^{n}$,

$$l(x) \geq \langle x, X_{T}^{*}\rangle - \vartheta, \quad \mathbb{P} - a.s. \tag{2.3.18}$$

Assumption III. (i) There exist $(X, Y, Z, H) \in \mathbb{L}_{\mathcal{F}}^{2\infty} \times \mathbb{L}_{\mathcal{F}}^{2\infty} \times \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{22}$ and a $\mathbb{R}$-valued $\mathcal{F}(t)$-progressively measurable stochastic process τ satisfying

$$\mathbb{E}\left[\int_{0}^{T} |\tau(t)|\, dt\right] < \infty \tag{2.3.19}$$

such that

$$L(t, X(t), Y(t), Z(t), H(t)) \leq \tau(t), \quad \mathbb{P}\otimes Leb - a.s. \tag{2.3.20}$$

(ii) There exist $X_{T} \in \mathbb{L}^{2}$ and a $\mathbb{R}$-valued $\mathcal{F}(T)$-measurable random variable χ satisfying

$$\mathbb{E}\left[|\chi|\right] < \infty \tag{2.3.21}$$

such that

$$l(X_{T}) \leq \chi, \quad \mathbb{P} - a.s.$$

The novelty of Assumptions II & III is to ensure that the corresponding conjugate convex functions of L and l, defined in the following section, satisfy the corresponding Assumptions II & III as well (see Proposition 2.4.1).

Remark 2.3.3. As shown in [36, Corollary 5.1], Assumption I is satisfied if and only if L and l are both normal convex integrands, a concept introduced in [34, Section 2]. More precisely, a function L is called a normal convex integrand provided, in the presence of (i) of Assumption I for L, there exists a countable collection $\{(X_i, Y_i, Z_i, H_i)\}_{i\in\mathbb{N}^+}$, where X_i, Y_i, Z_i and H_i are $\mathcal{F}(t)$-progressively measurable, such that $L(\omega, t, X_i(\omega, t), Y_i(\omega, t), Z_i(\omega, t), H_i(\omega, t))$ is $\mathcal{F}^*$-measurable and the following set

$$\mathbb{D}(\omega, t) \cap \{(X_i(\omega, t), Y_i(\omega, t), Z_i(\omega, t), H_i(\omega, t))\}_{i\in\mathbb{N}^+} \tag{2.3.22}$$

is dense in the effective domain $\mathbb{D}(\omega, t)$ of L for every $(\omega, t) \in \Omega \times [0, T]$, where

$$\mathbb{D}(\omega, t) = \left\{(x, y, z, h) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \middle| L(\omega, t, x, y, z, h) < \infty\right\} \tag{2.3.23}$$

(see [37, page 180]). Similarly, with an appropriate modification, we can give the corresponding definition of normal convex integrand for l.

Note that, it follows from [34, Corollary 5] that Assumption I, together with Remark 2.3.3, guarantees that, for every $(X, Y, Z, H) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ and $X_T \in \mathbb{L}^2$, $L(\omega, t, X(\omega, t), Y(\omega, t), Z(\omega, t), H(\omega, t))$ and $l(\omega, X_T(\omega))$ are $\mathcal{F}^*$- and $\mathcal{F}$-measurable respectively as required.

Proposition 2.3.4. *Under Assumptions I, II & III for the functions L and l, we have that $I_L > -\infty$, $J_l > -\infty$, that both I_L and J_l are not identically ∞ and that both I_L and J_l are convex functions.*

Proof. The proof applies the same techniques as the proof for the deterministic and delay-free case described in [37, Proposition 1]. For the completeness, we give the details as follows. Note that, in this proof, we only give the arguments for I_L and, with an appropriate modification, the corresponding arguments for J_l can be obtained in a similar way.

First, by the definition of I_L, its convexity follows directly from the convexity of L under Assumption I. Moreover, we define a linear function $L' : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}$ by

$$L'(t, x, y, z, h) = \langle x, X^*(t)\rangle + \langle y, Y^*(t)\rangle + \langle z, Z^*(t)\rangle + \langle h, H^*(t)\rangle - \varpi(t),$$

where (X^*, Y^*, Z^*, H^*) and ϖ are those in Assumption II (i) . Similarly to I_L, we define a function $I_{L'}$ on $\mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ by

$$\begin{aligned} I_{L'}(X, Y, Z, H) =& \mathbb{E}\left[\int_0^T L'(t, X(t), Y(t), Z(t), H(t))\, dt\right] \\ =& \mathbb{E}\bigg[\int_0^T \{\langle X(t), X^*(t)\rangle + \langle Y(t), Y^*(t)\rangle + \langle Z(t), Z^*(t)\rangle \\ & + \langle H(t), H^*(t)\rangle - \varpi(t)\} dt\bigg]. \end{aligned}$$

Then, the Hölder Inequality, together with (2.3.15), gives us that

$$I_{L'}(X, Y, Z, H) > -\infty \tag{2.3.24}$$

for every $(X, Y, Z, H) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$. Furthermore, by (2.3.16), we have that $I_L(X, Y, Z, H) \geq I_{L'}(X, Y, Z, H)$ which, together with (2.3.24), implies the conclusion that $I_L > -\infty$. On the other hand, it follows from (2.3.19) and (2.3.20) that there exists $(X, Y, Z, H) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ such that

$$I_L(X, Y, Z, H) \leq \mathbb{E}\left[\int_0^T |\tau(t)|\, dt\right] < \infty,$$

where τ is as in Assumption III (i). □

Now, for the given L, l, ξ, δ described above and for $X \in \mathbb{V}_1$ defined via Definition 2.3.1, we define a function Φ of X in terms of I_L and J_l by

$$\Phi(X) = I_L\left(X, X_\delta, \dot{X}, H_X\right) + J_l(X(T)). \tag{2.3.25}$$

Since $X = (1-\lambda)X_1 + \lambda X_2$ implies that $X_\delta = (1-\lambda)X_{1\delta} + \lambda X_{2\delta}$, $\dot{X} = (1-\lambda)\dot{X}_1 + \lambda \dot{X}_2$ and $H_X = (1-\lambda)H_{X_1} + \lambda H_{X_2}$ for any $\lambda \in [0, 1]$, the facts that Φ is strictly greater than $-\infty$, that it is not identically ∞, and that it is convex with respect to X follow directly from Proposition 2.3.4. Then, building upon such a Φ, we define, in a similar fashion to the stochastic convex problem (2.1.6) studied in [2], the stochastic convex problem with discrete delay as stated by the following definition.

Definition 2.3.5. Suppose that Assumptions I, II & III hold. The function Φ on $\mathbb{V}_1$ defined by (2.3.25) is called a stochastic convex primal function with discrete delay.

The stochastic convex primal problem with discrete delay associated with Φ is to find $\bar{X} \in \mathbb{V}_1$ realizing

$$\inf_{X \in \mathbb{V}_1} \Phi(X), \tag{2.3.26}$$

where X is identified with $(\dot{X}, H_X)$ via Definition 2.3.1. We refer to the function Φ and the problem (2.3.26) as the primal function and problem respectively. Any $X \in \mathbb{V}_1$, such that $\Phi(X) < \infty$, will be called a feasible solution. Moreover, any feasible solution $\bar{X}$ that achieves the infimum in (2.3.26) will be called an optimal solution of the primal problem.

Note that this primal problem implicitly depends on α and that, if Φ is identically ∞, no $X \in \mathbb{V}_1$ will be regarded as an optimal solution. Comparing (2.3.26) with (2.1.6), our primal function and problem in Definition 2.3.5 bear a similarity to (2.1.6) studied in [2, Definition II-I]. However, the X_δ, introduced in (2.3.26), can be regarded as a function of X. As noted in Section 2.1.1, the present of X_δ makes that the techniques used in [2] for introducing the corresponding dual problem can no longer be applied directly to our problem.

2.4 The Stochastic Convex Dual Problem

To overcome the difficulty mentioned above, this section introduces an optimal value function corresponding to the problem (2.3.26) and then generalizes the techniques used in [44, Proposition 3.1] for the corresponding deterministic case to derive an explicit expression for the stochastic convex dual problem to (2.3.26).

2.4.1 Preliminaries

This subsection studies some properties of the conjugate convex functions L^*, l^*, I_{L^*} and J_{l^*} of L, l, I_L and J_l in (2.3.25), which will be used to derive the dual problem in Section 2.4.4.

For fixed $(\omega, t) \in \Omega \times [0, T]$ and $\omega \in \Omega$, let L^* and l^* be the conjugate convex functions of L and l respectively in the sense of (2.2.3), i.e. L^* and l^* are respectively given by

$$\begin{aligned} &L^*\left(t,x^*,y^*,z^*,h^*\right) \\ &= \sup_{(x,y,z,h)\in\mathbb{R}^{n\times 3}\times\mathbb{R}^{n\times m}} \left\{\left\langle (x,y,z,h),(x^*,y^*,z^*,h^*)\right\rangle - L\left(t,x,y,z,h\right)\right\} \end{aligned} \tag{2.4.1}$$

for $(x^*,y^*,z^*,h^*)\in\mathbb{R}^{n\times 3}\times\mathbb{R}^{n\times m}$ and

$$l^*\left(x^*\right) = \sup_{x\in\mathbb{R}^n}\left\{\langle x,x^*\rangle - l\left(x\right)\right\} \tag{2.4.2}$$

for $x^*\in\mathbb{R}^n$, where the associated pairings in (2.4.1) and (2.4.2) are described by the usual inner products in the corresponding Euclidean spaces.

It follows from [34, Lemma 5], noting Remark 2.3.3, that L^* and l^* also satisfy Assumption I. In fact, being a normal convex integrand is persevered under the operation of conjugation. Therefore, as noted in Section 2.3.2, $l^*(\omega, X_T^*(\omega))$ and $L^*(\omega,t,X^*(\omega,t),Y^*(\omega,t),Z^*(\omega,t),H^*(\omega,t))$ are $\mathcal{F}$- and $\mathcal{F}^*$-measurable for every $X_T^*\in\mathbb{L}^2$ and $(X^*,Y^*,Z^*,H^*)\in\mathbb{L}^{21}_{\mathcal{F}}\times\mathbb{L}^{21}_{\mathcal{F}}\times\mathbb{L}^{2\infty}_{\mathcal{F}}\times\mathbb{L}^{22}_{\mathcal{F}}$ respectively. Moreover, L^* and l^* satisfy Assumptions II & III as stated by the proposition below.

Proposition 2.4.1. *Assumptions II & III for the functions L and l described in Section 2.3.2 imply that the functions L^* and l^* also satisfy Assumptions II & III.*

Proof. The proof applies the same techniques as the proof for the deterministic and delay-free case described in [37, Theorem 2]. For the completeness, we give the details as follows. Similarly to the proof of Proposition 2.3.4, we only give the arguments for L^* as the corresponding arguments for l^* can be obtained in a similar way.

Under the given condition that L satisfies Assumption III (i) and L^* is given by (2.4.1),

$$\begin{aligned} &L^*\left(t,x^*,y^*,z^*,h^*\right) \\ &\geq \langle X(t),x^*\rangle + \langle Y(t),y^*\rangle + \langle Z(t),z^*\rangle + \langle H(t),h^*\rangle \\ &\quad - L\left(t,X(t),Y(t),Z(t),H(t)\right), \quad \mathbb{P}\otimes Leb - a.s. \end{aligned}$$

for any $(x^*,y^*,z^*,h^*)\in\mathbb{R}^{n\times 3}\times\mathbb{R}^{n\times m}$, where $X,Y\in\mathbb{L}^{2\infty}_{\mathcal{F}}, Z\in\mathbb{L}^{21}_{\mathcal{F}}$ and $H\in\mathbb{L}^{22}_{\mathcal{F}}$ are those in Assumption III (i). Then, using (2.3.20), we obtain that

$$\begin{aligned} L^*\left(t,x^*,y^*,z^*,h^*\right) \geq{}& \langle X(t),x^*\rangle + \langle Y(t),y^*\rangle + \langle Z(t),z^*\rangle \\ &+ \langle H(t),h^*\rangle - \tau(t), \quad \mathbb{P}\otimes Leb - a.s. \end{aligned}$$

where τ is given in Assumption III (i), which gives the conclusion that L^* satisfies Assumption II (i). On the other hand, under the given condition that L satisfies Assumption II (i), we obtain that

$$\begin{aligned}\varpi(t) \geq & \sup_{(x,y,z,h)\in\mathbb{R}^{n\times 3}\times\mathbb{R}^{n\times m}} \Big\{ \langle x, X^*(t)\rangle + \langle y, Y^*(t)\rangle + \langle z, Z^*(t)\rangle \\ & \qquad + \langle h, H^*(t)\rangle - L(t,x,y,z,h) \Big\} \\ = & L^*(t, X^*(t), Y^*(t), Z^*(t), H^*(t)), \quad \mathbb{P}\otimes Leb - a.s.\end{aligned}$$

where $(X^*, Y^*, Z^*, H^*) \in \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ and ϖ are given in Assumption II (i). This implies that L^* satisfies Assumption III (i). □

To obtain the conjugate convex functions of I_L and J_l, throughout this chapter, we specify the following three paired spaces associated with the pairings given in [2, Page 386] as follows. We pair $\mathbb{L}^2$ with itself via the pairing defined by

$$\ll X_T, X_T^* \gg = \mathbb{E}\left[\langle X_T, X_T^*\rangle\right]; \tag{2.4.3}$$

pair $\mathbb{L}^{22}_{\mathcal{F}}$ with itself via the pairing defined by

$$\ll H, H^* \gg = \mathbb{E}\left[\int_0^T \langle H(t), H^*(t)\rangle\, dt\right]; \tag{2.4.4}$$

pair $\mathbb{L}^{21}_{\mathcal{F}}$ with $\mathbb{L}^{2\infty}_{\mathcal{F}}$ via the pairing defined by

$$\ll X, X^* \gg = \mathbb{E}\left[\int_0^T \langle X(t), X^*(t)\rangle\, dt\right]. \tag{2.4.5}$$

Now, similarly to the functions I_L and J_l introduced in Section 2.3.2, we define functions I_{L^*} on $\mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ and J_{l^*} on $\mathbb{L}^2$ respectively by

$$I_{L^*}(X^*, Y^*, Z^*, H^*) = \mathbb{E}\left[\int_0^T L^*(t, X^*(t), Y^*(t), Z^*(t), H^*(t))\, dt\right]$$

and

$$J_{l^*}(X_T^*) = \mathbb{E}\left[l^*(X_T^*)\right].$$

As discussed above, L^* and l^* satisfy Assumptions I, II & III so that $I_{L^*} > -\infty$, $J_{l^*} > -\infty$, both I_{L^*} and J_{l^*} are not identically ∞, and both I_{L^*} and J_{l^*} are convex by Proposition 2.3.4. Consequently, it follows from [34, Theorem 2] that I_{L^*} and J_{l^*} are the conjugate convex functions of I_L and J_l respectively in the sense of (2.2.3), i.e. I_{L^*} and J_{l^*} can be respectively expressed by

$$\begin{aligned} & I_{L^*}\left(X^*, Y^*, Z^*, H^*\right) \\ = & \sup_{(X,Y,Z,H)\in\mathbb{L}^{2\infty}_{\mathcal{F}}\times\mathbb{L}^{2\infty}_{\mathcal{F}}\times\mathbb{L}^{21}_{\mathcal{F}}\times\mathbb{L}^{22}_{\mathcal{F}}} \Big\{ \ll (X,Y,Z,H), (X^*,Y^*,Z^*,H^*) \gg \\ & \qquad - I_L\left(X,Y,Z,H\right) \Big\} \end{aligned} \tag{2.4.6}$$

for $(X^*, Y^*, Z^*, H^*) \in \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ and

$$J_{l^*}\left(X_T^*\right) = \sup_{X_T\in\mathbb{L}^2} \left\{\mathbb{E}\left[\langle X_T, X_T^*\rangle\right] - J_l\left(X_T\right)\right\} \tag{2.4.7}$$

for $X_T^* \in \mathbb{L}^2$, where the pairing in (2.4.6), between the spaces $\mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ and $\mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ induced directly from the pairings (2.4.4) and (2.4.5), is described by

$$\begin{aligned} & \ll (X,Y,Z,H), (X^*,Y^*,Z^*,H^*) \gg \\ = & \mathbb{E}\left[\int_0^T \left\{\langle X(t), X^*(t)\rangle + \langle Y(t), Y^*(t)\rangle + \langle Z(t), Z^*(t)\rangle + \langle H(t), H^*(t)\rangle\right\} dt\right]. \end{aligned}$$

2.4.2 The Optimal Value Function

Similarly to that for the corresponding deterministic case studied in [44, page 172], we associate with the primal function to define a family of so-called perturbed functions F on $\mathbb{V}_1$, parameterized by $(a_T, r, k) \in \mathbb{L}^2 \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}}$, by

$$F_{a_T,r,k}(X) = I_L\left(X + r, X_\delta + k, \dot{X}, H_X\right) + J_l\left(X(T) - a_T\right). \tag{2.4.8}$$

Compared with the perturbed functions used for the delay-free deterministic convex problem in [2, Definition III-1] and for the stochastic delay-free convex problem in [37, Section 7], the perturbed function F here depends on an extra parameter k to take account of X_δ in I_L. Note that F is a composition of Φ with an affine mapping so that it is convex as the convexity is persevered under affine mappings. Note also that, by Proposition 2.3.4, F is strictly greater than $-\infty$ and is not identically ∞.

Now, building upon such a F, the family of perturbed optimization problems is to find $\bar{X} \in \mathbb{V}_1$ realizing

$$\inf_{X\in\mathbb{V}_1} F_{a_T,r,k}(X),$$

which gives an optimal value function ϕ on $\mathbb{L}^2 \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}}$ defined by

$$\phi\left(a_T, r, k\right) = \inf_{X \in \mathbb{V}_1} F_{a_T,r,k}\left(X\right). \tag{2.4.9}$$

It can be seen from the relationship between F and Φ that

$$\phi\left(0,0,0\right) = \inf_{X \in \mathbb{V}_1} F_{0,0,0}\left(X\right) = \inf_{X \in \mathbb{V}_1} \Phi\left(X\right). \tag{2.4.10}$$

Proposition 2.4.2. *The optimal value function ϕ defined by (2.4.9) is a convex function on $\mathbb{L}^2 \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}}$.*

Proof. Based on the definition of ϕ and the convexity of I_L and J_l, for any $a_T, a'_T \in \mathbb{L}^2$, $r, r' \in \mathbb{L}^{2\infty}_{\mathcal{F}}$, $k, k' \in \mathbb{L}^{2\infty}_{\mathcal{F}}$ and $X \in \mathbb{V}_1$, we obtain that

$$\begin{aligned}
&\phi\left(\lambda a_T + (1-\lambda) a'_T, \lambda r + (1-\lambda) r', \lambda k + (1-\lambda) k'\right) \\
\leq &\lambda \left\{ I_L\left(X + r, X_\delta + k, \dot{X}, H_X\right) + J_l\left(X(\cdot, T) - a_T\right)\right\} \\
&+ (1-\lambda)\left\{ I_L\left(X + r', X_\delta + k', \dot{X}, H_X\right) + J_l\left(X(\cdot, T) - a'_T\right)\right\},
\end{aligned}$$

where $\lambda \in [0,1]$. Hence, we have

$$\begin{aligned}
&\phi\left(\lambda a_T + (1-\lambda) a'_T, \lambda r + (1-\lambda) r', \lambda k + (1-\lambda) k'\right) \\
\leq &\lambda \inf_{X \in \mathbb{V}_1} F_{a_T,r,k}\left(X\right) + (1-\lambda) \inf_{X \in \mathbb{V}_1} F_{a'_T,r',k'}\left(X\right) \\
= &\lambda \phi\left(a_T, r, k\right) + (1-\lambda)\phi\left(a'_T, r', k'\right)
\end{aligned}$$

as required. ∎

Since the optimal value function ϕ is convex, let ϕ^* be the conjugate convex function of ϕ, i.e. ϕ^* is given by

$$\begin{aligned}
&\phi^*\left(a^*_T, r^*, k^*\right) \\
= &\sup_{(r,k,a_T) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^2} \left\{ \ll \left(a_T, r, k\right), \left(a^*_T, r^*, k^*\right) \gg - \phi\left(a_T, r, k\right)\right\}
\end{aligned} \tag{2.4.11}$$

for $\left(a^*_T, r^*, k^*\right) \in \mathbb{L}^2 \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}}$, where the pairing, between spaces $\mathbb{L}^2 \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}}$ and $\mathbb{L}^2 \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}}$ induced from (2.4.3) and (2.4.5), is given by

$$\begin{aligned}
&\ll \left(a_T, r, k\right), \left(a^*_T, r^*, k^*\right) \gg \\
= &\mathbb{E}\left[\int_0^T \left\langle \left(r(t), k(t)\right), \left(r^*(t), k^*(t)\right)\right\rangle dt + \left\langle a_T, a^*_T \right\rangle\right].
\end{aligned} \tag{2.4.12}$$

Using ϕ^*, we introduce the following optimization problem to find $(\bar{a}_T^*, \bar{r}^*, \bar{k}^*) \in \mathbb{L}^2 \times \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{21}$ realizing

$$\inf_{(a_T^*, r^*, k^*) \in \mathbb{L}^2 \times \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{21}} \phi^* \left(a_T^*, r^*, k^*\right). \tag{2.4.13}$$

Then, the optimality of the primal problem (2.3.26) can be related to the solution of (2.4.13). Indeed, by the relationship described by (2.4.10) between the optimal value function ϕ and the primal problem (2.3.26), we set $(a_T, r, k) = (0, 0, 0)$ on the right-hand-side of (2.4.11). Then, we see that

$$\phi^* \left(a_T^*, r^*, k^*\right) \geq -\phi\left(0, 0, 0\right) = -\inf_{X \in \mathbb{V}_1} \Phi\left(X\right)$$

for all (a_T^*, r^*, k^*), which implies

$$\inf_{\left(a_T^*, r^*, k^*\right) \in \mathbb{L}^2 \times \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{21}} \phi^* \left(a_T^*, r^*, k^*\right) + \inf_{X \in \mathbb{V}_1} \Phi\left(X\right) \geq 0. \tag{2.4.14}$$

If there exist $(\bar{a}_T^*, \bar{r}^*, \bar{k}^*) \in \mathbb{L}^2 \times \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{21}$ and $\bar{X} \in \mathbb{V}_1$ such that the equality in (2.4.14) holds, then

$$0 \leq \phi^* \left(\bar{a}_T^*, \bar{r}^*, \bar{k}^*\right) + \Phi\left(X\right) = -\Phi\left(\bar{X}\right) + \Phi\left(X\right) \tag{2.4.15}$$

for $X \in \mathbb{V}_1$. Hence, (2.4.15) implies that $\bar{X}$ is an optimal solution of the primal problem (2.3.26). Note that, by the arguments above, the optimization problem (2.4.13) can be regarded as a dual problem to (2.3.26). The reason for this claim will become clearer in Sections 2.5 & 2.6.

As noted in [2, Definition III-1] and [2, Theorem III-1], the corresponding function ϕ^* for the stochastic delay-free case is expressed in terms of the corresponding I_{L^*} and J_{l^*} in a similar manner to that for the corresponding primal function Φ given in (2.1.6) in terms of I_L and J_l. Unfortunately, the introduction of the extra parameter k^* in (2.4.11) to pair with k in (2.4.9), due to the variable X_δ introduced in the primal problem (2.3.26), makes this no longer the case. In other words, we cannot use the approach there to obtain the explicitly expression for ϕ^*.

2.4.3 Identification of the Dual Variable

Recalling that the dual variable p for the dual problem (2.1.12) to the deterministic convex problem (2.1.1) is an absolutely continuous function. As noted in [44, Proposition 3.1], p satisfies an ordinary differential equation with a given terminal value.

Hence, p can be identified with $(p(T), \dot{p})$, in a similar manner of (2.1.2) to that for x in the primal problem (2.1.11), as

$$p(t) = p(T) - \int_t^T \dot{p}(s)\, ds.$$

However, the 'time' in the stochastic context cannot be reversed in general if stochastic processes are required to be $\mathcal{F}(t)$-adapted. To illustrate this, we borrow an example from [24, Chapter 1] to consider the stochastic differential equation

$$dP(t) = 0, \quad t \in [0, T], \tag{2.4.16}$$

where the drift and diffusion coefficients are zero. If this equation equips with the initial value $P(0) = p \in \mathbb{R}^n$, then the unique solution of (2.4.16) is $P(t) \equiv p$. On the other hand, if it associates with a terminal value given by $P(T) = P_T \in \mathbb{L}^2$, then the unique solution of (2.4.16) is $P(t) = P_T$ for all $t \in [0, T]$. Unfortunately, it is not necessarily $\mathcal{F}(t)$-adapted, unless $P_T \in \mathbb{R}^n$ is fixed.

To overcome this difficulty, we define $\mathbb{V}_2 = \mathbb{L}^2 \times \mathbb{L}^{21}_{\mathcal{F}}$ and apply the technique of conditional expectation, which has been successfully used in the theory of backward stochastic differential equation (BSDE), to characterize the continuous $\mathcal{F}(t)$-adapted stochastic process P by $(P_T, \dot{P}) \in \mathbb{V}_2$ in the sense that

$$P(t) = \mathbb{E}\left[P_T - \int_t^T \dot{P}(s)\, ds \middle| \mathcal{F}(t) \right], \quad \forall t \in [0, T] \tag{2.4.17}$$

(see [24, page 2]). Clearly, $P(0)$ is a constant. On the other hand, unlike X identified with $(\dot{X}, H_X)$ via Definition 2.3.1, the identification described by (2.4.17) is implicitly. It shows that P is the solution of a BSDE. This results in that P is identified with $(P_T, \dot{P}, H_P, K_P) \in \mathbb{L}^2 \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}} \times \mathbb{K}^{22}_{\mathcal{F}}$ as stated by the following proposition, where $\mathbb{K}^{22}_{\mathcal{F}}$ denotes the space of $\mathcal{F}(t)$-progressively measurable stochastic processes $K_P(t) = (K_P^{(1)}(t), \ldots, K_P^{(n)}(t))$ with the finite norm defined by

$$\|K_P\|^{\mathbb{K}}_{22} = \left\{ \mathbb{E}\left[\sum_{r=1}^{d} \sum_{i=1}^{d} \sum_{\substack{j=1 \\ j \neq i}}^{d} \left| K_{ij}^{(r)}(t) \right|^2 \lambda_{ij}(t)\, dt \right] \right\}^{1/2}$$

and where $K_P^{(r)}(t) = \{K_{ij}^{(r)}(t)\}_{i,j=1}^{d}$ and $K_{ij}^{(r)}(t) \in \mathbb{R}$ with $K_{ii}^{(r)}(t) = 0$, $d\mathbb{P} \otimes dt$-a.s. for each $i \in \mathbb{I}$ (see [8, page 32]).

Proposition 2.4.3. *For P defined by* (2.4.17), *we have that $P \in \mathbb{L}_{\mathcal{F}}^{2\infty}$. Moreover, there exists unique $(H_P, K_P) \in \mathbb{L}_{\mathcal{F}}^{22} \times \mathbb{K}_{\mathcal{F}}^{22}$ such that, for $t \in [0, T]$,*

$$\begin{aligned} P(t) = & P_T - \int_t^T \dot{P}(s)\, ds - \int_t^T H_P(s)\, dW(s) \\ & - \int_t^T K_P(s) \bullet dM(s), \quad \mathbb{P} - a.s. \end{aligned} \tag{2.4.18}$$

where we have used the shorthand notation

$$K_P(s) \bullet dM(s) = \left\{ \sum_{i=1}^{d} \sum_{\substack{j=1 \\ j \neq i}}^{d} K_{ij}^{(1)}(s)\, dM_{ij}(s), \ldots, \sum_{i=1}^{d} \sum_{\substack{j=1 \\ j \neq i}}^{d} K_{ij}^{(n)}(s)\, dM_{ij}(s) \right\}^{\top}.$$

Proof. It follows from a similar argument to that for the proof of Proposition 2.3.2, that P is $\mathcal{F}(t)$-progressively measurable. On the other hand, since P is defined by (2.4.17), we have that

$$\begin{aligned} P(t) = & \mathbb{E}\left[P_T - \int_0^T \dot{P}(s)\, ds + \int_0^t \dot{P}(s)\, ds \Big| \mathcal{F}(t) \right] \\ = & \mathbb{E}\left[P_T - \int_0^T \dot{P}(s)\, ds \Big| \mathcal{F}(t) \right] + \int_0^t \dot{P}(s)\, ds, \end{aligned} \tag{2.4.19}$$

where the last term on the right-hand-side of the second equality of (2.4.19) is due to the fact that $\int_0^t \dot{P}(s) ds$ is $\mathcal{F}(t)$-measurable. Let

$$\mathcal{N}(t) = \mathbb{E}\left[P_T - \int_0^T \dot{P}(s)\, ds \Big| \mathcal{F}(t) \right]$$

for $t \in [0, T]$. Then,

$$P(t) = \mathcal{N}(t) + \int_0^t \dot{P}(s)\, ds, \quad \forall t \in [0, T]. \tag{2.4.20}$$

In particular, $\mathcal{N} = \{\mathcal{N}(t)\}_{t \in [0,T]}$ is a square-integrable martingale with respect to the filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{t \in [0,T]}$ given in Section 2.3. Indeed, it follows from the Conditional Jensen Inequality (see [43, Theorem 2.3.2 (v)]) and the law of total expectation (see [43, page 72]) that, for $t \in [0, T]$,

$$\begin{aligned} \mathbb{E}\left[|\mathcal{N}(t)|^2 \right] = & \mathbb{E}\left[\left\{ \mathbb{E}\left[P_T - \int_0^T \dot{P}(s)\, ds \Big| \mathcal{F}(t) \right] \right\}^2 \right] \\ \leq & \mathbb{E}\left[\mathbb{E}\left[\left\{ |P_T| + \int_0^T \left| \dot{P}(s) \right| ds \right\}^2 \Big| \mathcal{F}(t) \right] \right] \\ = & \mathbb{E}\left[\left\{ |P_T| + \int_0^T \left| \dot{P}(s) \right| ds \right\}^2 \right] \\ \leq & 2\mathbb{E}\left[|P_T|^2 + \left\{ \int_0^T \left| \dot{P}(s) \right| ds \right\}^2 \right]. \end{aligned}$$

Then, the square-integrability of $\mathcal{N}$ follows directly from the fact that $(P_T, \dot{P}) \in \mathbb{V}_2$. Moreover, for any $s \leq t$, we have

$$\begin{aligned}\mathbb{E}\left[\mathcal{N}(t)\,|\mathcal{F}(s)\right] =& \mathbb{E}\left[\mathbb{E}\left[P_T - \int_0^T \dot{P}(s)\,ds\Big|\mathcal{F}(t)\right]\Big|\mathcal{F}(s)\right] \\ =& \mathbb{E}\left[P_T - \int_0^T \dot{P}(s)\,ds\Big|\mathcal{F}(s)\right] \\ =& \mathcal{N}(s),\end{aligned}$$

where the second equality is due to the tower property of conditional expectation (see [43, Theorem 2.3.2 (iii)]). This gives the conclusion that $\mathcal{N}$ is a square-integrable martingale. Then, by the similar technique to that used in the proof of Proposition 2.3.2, we see that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|P(t)|^2\right] \leq 2\left\{\mathbb{E}\left[\sup_{0\leq t\leq T}|\mathcal{N}(t)|^2\right] + \mathbb{E}\left[\left\{\int_0^T \left|\dot{P}(t)\right| dt\right\}^2\right]\right\}$$

which gives the conclusion of $P \in \mathbb{L}_{\mathcal{F}}^{2\infty}$ noting the square-integrability of $\mathcal{N}$ and $\dot{P} \in \mathbb{L}_{\mathcal{F}}^{21}$.

Applying the martingale representation theorem (see [8, Theorem B.4.6]) to $\mathcal{N}$ on the right-hand-side of (2.4.20), we obtain that there exists unique $(H_P, K_P) \in \mathbb{L}_{\mathcal{F}}^{22} \times \mathbb{K}_{\mathcal{F}}^{22}$ such that, for $t \in [0, T]$,

$$\begin{aligned}P(t) =& \mathcal{N}(0) + \int_0^t \dot{P}(s)\,ds + \int_0^t H_P(s)\,dW(s) \\ &+ \int_0^t K_P(s) \bullet dM(s), \quad \mathbb{P}-a.s.\end{aligned} \tag{2.4.21}$$

as the filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{t\in[0,T]}$ is generated jointly by W and α. Furthermore, setting $t = T$ on the both sides of (2.4.21) and using the fact that $P(T) = P_T$, we obtain the following alternative expression for $\mathcal{N}(0)$,

$$\begin{aligned}\mathcal{N}(0) =& P_T - \int_0^T \dot{P}(s)\,ds - \int_0^T H_P(s)\,dW(s) \\ &- \int_0^T K_P(s) \bullet dM(s), \quad \mathbb{P}-a.s.\end{aligned} \tag{2.4.22}$$

Then, (2.4.18) is obtained by substituting (2.4.22) into the right-hand-side of (2.4.21). □

Hence, the identification of P in (2.4.17) is not equivalent to the one for $X \in \mathbb{V}_1$ described via Definition 2.3.1. This is different from the corresponding deterministic case described in [44].

Following the convention for $\mathbb{X}$, we write $P \in \mathbb{V}_2$ to mean that P is identified with $(P_T, \dot{P}) \in \mathbb{V}_2$ in the sense of (2.4.17).

2.4.4 Derivation of the Dual Problem

Having obtain the identification of P, this subsection generalizes the result in [44, Proposition 3.1] for the corresponding deterministic case and obtains an explicit expression for ϕ^* as stated by the following theorem.

Theorem 2.4.4. *Suppose that Assumptions I, II & III hold for the functions L and l. In addition, for any given $(a_T^*, r^*, k^*) \in \mathbb{L}^2 \times \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{21}$, let $(P_T, \dot{P}, \dot{Q}) \in \mathbb{L}^2 \times \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{21}$ be defined by*

$$\begin{cases} P_T = a_T^*, \\ \dot{P}(t) = r^*(t) + \mathbb{E}\left[k^*(t+\delta) I_{[0,T-\delta]}(t) \,|\mathcal{F}(t)\right], \\ \dot{Q}(t) = k^*(t), \end{cases} \tag{2.4.23}$$

where we identify P with $(P_T, \dot{P}) \in \mathbb{V}_2$ by (2.4.17). Then, the function Ψ on $\mathbb{V}_2 \times \mathbb{L}_{\mathcal{F}}^{21}$, defined by

$$\begin{aligned} \Psi\left(P, \dot{Q}\right) =& I_{L^*}\left(\dot{P} - \mathbb{E}\left[\dot{Q}(\cdot+\delta) I_{[0,T-\delta]}(\cdot) |\mathcal{F}(\cdot)\right], \dot{Q}, P, H_P\right) + J_{l^*}(-P_T) \\ &- \mathbb{E}\left[\int_0^T \left\langle \dot{Q}(t), \xi(t-\delta) I_{[0,\delta]}(t)\right\rangle dt\right] + \mathbb{E}\left[\langle P_T, \xi(0)\rangle\right] \\ &- \mathbb{E}\left[\int_0^T \langle \dot{P}(t), \xi(0)\rangle dt\right], \end{aligned} \tag{2.4.24}$$

satisfies $\Psi(P, \dot{Q}) = \phi^*(a_T^*, r^*, k^*)$, where H_P is specified by (2.4.18).

Proof. First, by the Conditional Jensen Inequality, the Fubini Theorem (see [39, page 416]) and the law of total expectation, we see that

$$\begin{aligned} &\mathbb{E}\left[\left\{\int_0^T \left|\mathbb{E}\left[k^*(s+\delta) I_{[0,T-\delta]}(s)\right| \mathcal{F}(s)\right]\right| dt\right\}^2\right] \\ =&\mathbb{E}\left[\left\{\mathbb{E}\left[\int_0^T \left|k^*(s+\delta) I_{[0,T-\delta]}(s)\right| ds \middle| \mathcal{F}(t)\right]\right\}^2\right] \\ \le&\mathbb{E}\left[\mathbb{E}\left[\left\{\int_0^T \left|k^*(s+\delta) I_{[0,T-\delta]}(s)\right| ds\right\}^2 \middle| \mathcal{F}(t)\right]\right] \\ =&\mathbb{E}\left[\left\{\int_0^T \left|k^*(s+\delta) I_{[0,T-\delta]}(s)\right| ds\right\}^2\right] \\ \le&\mathbb{E}\left[\left\{\int_0^T |k^*(s)| ds\right\}^2\right]. \end{aligned} \tag{2.4.25}$$

Since $k^* \in \mathbb{L}_{\mathcal{F}}^{21}$, (2.4.25) implies that $\mathbb{E}\left[k^*(\cdot+\delta) I_{[0,T-\delta]}(\cdot)|\mathcal{F}(\cdot)\right] \in \mathbb{L}_{\mathcal{F}}^{21}$. Therefore, $\dot{P}$ defined by (2.4.23) is in $\mathbb{L}_{\mathcal{F}}^{21}$ noting $r^* \in \mathbb{L}_{\mathcal{F}}^{21}$. Now, using the optimal value function

ϕ and the perturbed function F respectively defined by (2.4.9) and (2.4.8), we can re-express ϕ^* as

$$\begin{aligned}
&\phi^*\left(a_T^*, r^*, k^*\right) \\
=& \sup_{(r,k,a_T)\in\mathbb{L}_{\mathcal{F}}^{2\infty}\times\mathbb{L}_{\mathcal{F}}^{2\infty}\times\mathbb{L}^2} \left\{ \ll (a_T, r, k), (a_T^*, r^*, k^*) \gg - \inf_{(\dot{X}, H_X)\in\mathbb{V}_1} F_{a_T,r,k}(X) \right\} \\
=& \sup_{\substack{(\dot{X},H_X)\in\mathbb{V}_1 \\ (r,k,a_T)\in\mathbb{L}_{\mathcal{F}}^{2\infty}\times\mathbb{L}_{\mathcal{F}}^{2\infty}\times\mathbb{L}^2}} \left\{ \mathbb{E}\left[\int_0^T \langle (r(t), k(t)), (r^*(t), k^*(t))\rangle\, dt\right] + \mathbb{E}\left[\langle a_T^*, a_T\rangle\right] \right. \\
&\left. - I_L\left(X + r, X_\delta + k, \dot{X}, H_X\right) - J_l\left(X(T) - a_T\right) \right\}.
\end{aligned} \tag{2.4.26}$$

Then, setting $a_T' = X(T) - \alpha_T$, $r' = X + r$ and $k' = X_\delta + k$, (2.4.26) gives that

$$\begin{aligned}
&\phi^*\left(a_T^*, r^*, k^*\right) \\
=& \sup_{a_T'\in\mathbb{L}^2} \left\{ \mathbb{E}\left[\left\langle a_T', -a_T^*\right\rangle\right] - J_l\left(a_T'\right) \right\} \\
&+ \sup_{\substack{(\dot{X},H_X)\in\mathbb{V}_1 \\ (r',k')\in\mathbb{L}_{\mathcal{F}}^{2\infty}\times\mathbb{L}_{\mathcal{F}}^{2\infty}}} \left\{ \mathbb{E}\left[\int_0^T \left\{\left\langle r'(t), r^*(t)\right\rangle + \left\langle k'(t), k^*(t)\right\rangle\right\} dt\right] \right. \\
&+ \mathbb{E}\left[\langle X(T), a_T^*\rangle\right] - I_L\left(r', k', \dot{X}, H_X\right) \\
&\left. - \mathbb{E}\left[\int_0^T \left\{\langle X(t), r^*(t)\rangle + \langle X_\delta(t), k^*(t)\rangle\right\} dt\right] \right\}.
\end{aligned} \tag{2.4.27}$$

To simplify this, we use the relationship between X and X_δ and re-express the last two summands on the right-hand-side of (2.4.27) as

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left\{\langle X(t), r^*(t)\rangle + \langle X_\delta(t), k^*(t)\rangle\right\} dt\right] \\
=&\mathbb{E}\left[\int_0^T \left\{ \left\langle X(t), r^*(t) + k^*(t+\delta)\, I_{[0,T-\delta]}(t) \right\rangle \right.\right. \\
&\left.\left. + \left\langle \xi(t-\delta)\, I_{[0,\delta]}(t), k^*(t) \right\rangle \right\} dt\right] \\
=&\mathbb{E}\left[\int_0^T \left\{ \left\langle X(t), r^*(t) + \mathbb{E}\left[k^*(t+\delta)\, I_{[0,T-\delta]}(t) \middle| \mathcal{F}(t)\right] \right\rangle \right.\right. \\
&\left.\left. + \left\langle \xi(t-\delta)\, I_{[0,\delta]}(t), k^*(t) \right\rangle \right\} dt\right].
\end{aligned} \tag{2.4.28}$$

To apply the Itô formula below, noting the definition of $\dot{P}$ in (2.4.23), we have used the technique of conditional expectation in the above equation which will be clear in (2.4.31). Now, we use the expression (2.4.18) for P and then apply the Itô formula to

$\langle P(t), X(t)\rangle$ to get

$$\begin{aligned}
&\langle \xi(0), P(0)\rangle \\
=&\mathbb{E}\left[\langle X(T), P_T\rangle\right] - \mathbb{E}\left[\int_0^T \left\langle \dot{X}(t), P(t)\right\rangle dt\right] - \mathbb{E}\left[\int_0^T \left\langle X(t), \dot{P}(t)\right\rangle dt\right] \\
&- \mathbb{E}\left[\int_0^T \langle H_X(t), H_P(t)\rangle dt\right],
\end{aligned} \tag{2.4.29}$$

where, as noted in Section 2.4.3, $P(0)$ is a constant. Similarly, applying the Itô formula to $\langle P(t), \xi(0)\rangle$, we re-express $\langle \xi(0), P(0)\rangle$ in (2.4.29) as

$$\langle \xi(0), P(0)\rangle = -\mathbb{E}\left[\int_0^T \left\langle \xi(0), \dot{P}(t)\right\rangle dt\right] + \mathbb{E}\left[\langle \xi(0), P_T\rangle\right]. \tag{2.4.30}$$

Then, replacing P_T and $\dot{P}$ in (2.4.29) and in (2.4.30) by their definitions given in (2.4.23), these two equations lead to

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left\langle X(t), r^*(t) + \mathbb{E}\left[\dot{Q}(t+\delta) I_{[0,T-\delta]}(t) \middle| \mathcal{F}(t)\right]\right\rangle dt\right] \\
=&\mathbb{E}\left[\langle X(T), a_T^*\rangle - \langle \xi(0), a_T^*\rangle\right] \\
&- \mathbb{E}\Bigg[\int_0^T \Big\{ \left\langle \dot{X}(t), P(t)\right\rangle + \langle H_X(t), H_P(t)\rangle \\
&\qquad - \left\langle \xi(0), r^*(t) + \mathbb{E}\left[\dot{Q}(t+\delta) I_{[0,T-\delta]}(t) \middle| \mathcal{F}(t)\right]\right\rangle \Big\} dt\Bigg],
\end{aligned} \tag{2.4.31}$$

the left-hand-side of which is equal to the first term of the right-hand-side of the second equality in (2.4.28). Finally, we substitute (2.4.28) into (2.4.27), using (2.4.31), (2.4.6) and (2.4.7), to obtain

$$\begin{aligned}
\phi^*(a_T^*, r^*, k^*) =& \sup_{\substack{(\dot{X}, H_X) \in \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}} \\ (r', k') \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}}}} \left\{ \ll \left(r', k', \dot{X}, H_X\right), \left(r^*, \dot{Q}, P, H_P\right) \gg - I_L\left(r', k', \dot{X}, H_X\right)\right\} \\
&+ \sup_{a'_T \in \mathbb{L}^2} \left\{ \ll -a_T^*, a'_T \gg - J_l\left(a'_T\right)\right\} + \mathbb{E}\left[\langle \xi(0), a_T^*\rangle\right] \\
&- \mathbb{E}\left[\int_0^T \left\langle \xi(0), r^*(t) + \mathbb{E}\left[\dot{Q}(t+\delta) I_{[0,T-\delta]}(t) \middle| \mathcal{F}(t)\right]\right\rangle dt\right] \\
&- \mathbb{E}\left[\int_0^T \left\langle \xi(t-\delta) I_{[0,\delta]}(t), \dot{Q}(t)\right\rangle dt\right] \\
=& I_{L^*}\left(\dot{P} - \mathbb{E}\left[\dot{Q}(\cdot+\delta) I_{[0,T-\delta]}(\cdot) \middle| \mathcal{F}(\cdot)\right], \dot{Q}, P, H_P\right) + J_{l^*}(-P_T) \\
&+ \mathbb{E}\left[\langle P_T, \xi(0)\rangle\right] - \mathbb{E}\left[\int_0^T \left\langle \dot{Q}(t), \xi(t-\delta) I_{[0,\delta]}(t)\right\rangle dt\right] \\
&- \mathbb{E}\left[\int_0^T \left\langle \dot{P}(t), \xi(0)\right\rangle dt\right]
\end{aligned}$$

as required. □

Note that, although the relationship we obtained above between Ψ and ϕ^* is similar to that between the corresponding functions in the deterministic situation, our proof is different from that in [44, Proposition 3.1]. In particular, in addition to the identification of P mentioned early, we have resolved the anticipated (or time-advanced) issue for the variable $\dot{Q}(t+\delta)$ by the techniques of conditional expectation. Note also that, using (2.4.23), we can re-express the pairing $\ll (a_T, r, k), (a_T^*, r^*, k^*) \gg$ given by (2.4.12) in terms of $(P, \dot{Q}) \in \mathbb{V}_2 \times \mathbb{L}_{\mathcal{F}}^{21}$ as

$$\begin{aligned} &\ll (a_T, r, k), (P, \dot{Q}) \gg \\ =&\mathbb{E}\left[\int_0^T \left\langle \dot{P}(t) - \mathbb{E}\left[\dot{Q}(t+\delta) I_{[0,T-\delta]}(t) \,|\, \mathcal{F}(t)\right], r(t)\right\rangle dt\right] + \mathbb{E}\left[\langle P_T, \alpha_T\rangle\right] \\ &+ \mathbb{E}\left[\int_0^T \langle \dot{Q}(t), k(t)\rangle dt\right], \end{aligned} \tag{2.4.32}$$

where P is identified with $(P_T, \dot{P}) \in \mathbb{V}_2$ via (2.4.17). This generalizes the pairing for the corresponding deterministic case described in [44, page 183]. Comparing (2.4.32) with the pairing in the Markovian convex problems (see [2, page 394]), $\dot{Q}$ is introduced here to pair with k to allow X_δ in Φ. Then, using the pairing (2.4.32) and Theorem 2.4.4, we can re-express $\Psi(P, \dot{Q})$ given by (2.4.24) as follow

$$\Psi(P, \dot{Q}) = \sup_{(r,k,a_T) \in \mathbb{L}_{\mathcal{F}}^{2\infty} \times \mathbb{L}_{\mathcal{F}}^{2\infty} \times \mathbb{L}^2} \left\{ \ll (P, \dot{Q}), (a_T, r, k) \gg - \phi(a_T, r, k) \right\} \tag{2.4.33}$$

for $(P, \dot{Q}) \in \mathbb{V}_2 \times \mathbb{L}_{\mathcal{F}}^{21}$. Since $I_{L^*} > -\infty$ and $J_{l^*} > -\infty$ by Proposition 2.3.4, we see that Ψ is strictly greater than $-\infty$ and is convex.

Definition 2.4.5. Ψ defined by (2.4.24) is called a stochastic convex dual function of Φ, or simply dual function. Associated with Ψ, the stochastic convex dual problem of the primal problem (2.3.26) over $\mathbb{V}_2 \times \mathbb{L}_{\mathcal{F}}^{21}$, or simply dual problem, is to find $(\bar{P}, \dot{\bar{Q}})$ realizing

$$\inf_{(P,\dot{Q}) \in \mathbb{V}_2 \times \mathbb{L}_{\mathcal{F}}^{21}} \Psi(P, \dot{Q}). \tag{2.4.34}$$

where P is identified with $(P_T, \dot{P})$ using (2.4.17). Similarly to the primal problem described in Definition 2.3.5, any pair $(P, \dot{Q}) \in \mathbb{V}_2 \times \mathbb{L}_{\mathcal{F}}^{21}$ such that $\Psi(P, \dot{Q}) < \infty$ will be called a feasible solution. We shall call a feasible solution $(\bar{P}, \dot{\bar{Q}})$ which achieves the infimum (2.4.34) an optimal solution of the dual problem.

Note that, although we call Ψ the dual function to Φ, the space $\mathbb{V}_2 \times \mathbb{L}^{21}_{\mathcal{F}}$ on which Ψ is defined is not the paired space, with respect to the pairing introduced in Section 2.4.1, to the space $\mathbb{V}_1$ on which Φ is defined on account of the fact that the convex problems we study also depends X_δ.

Remark 2.4.6. If there is no delay in the model, corresponding to $\delta = 0$, then the X_δ is identical with X so that there exists a function $\hat{L} : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{\infty\}$ satisfying the corresponding Assumptions I, II & III such that $L(\omega, t, x, y, z, h) = \hat{L}(\omega, t, x, z, h)$ holds. Furthermore, the corresponding optimal value function ϕ depends only on (a_T, r). Consequently, Theorem 2.4.4 gives that $P = (P_T, \dot{P}) \in \mathbb{V}_2$ is identical with (a_T^*, r^*). This implies that $\Psi(P) = \phi^*(a_T^*, r^*)$, where $\Psi : \mathbb{V}_2 \to \mathbb{R} \cup \{\infty\}$ is described by

$$\begin{aligned}\Psi(P) =& I_{\hat{L}^*}\left(\dot{P}, P, H_P\right) + J_{l^*}\left(-P_T\right) + \mathbb{E}\left[\langle P_T, \xi(0)\rangle\right] \\ &- \mathbb{E}\left[\int_0^T \left\langle \dot{P}(t), \xi(0)\right\rangle dt\right].\end{aligned} \tag{2.4.35}$$

Applying the same technique as that used in (2.4.30) to the last two terms on the right-hand-side of (2.4.35), we see that

$$\Psi(P) = I_{\hat{L}^*}\left(\dot{P}, P, H_P\right) + J_{l^*}\left(-P_T\right) + \langle P(0), \xi(0)\rangle.$$

In particular, the dual function given in (2.1.8) is recovered by the above equation with the fixed initial value $\langle P(0), \xi(0)\rangle$.

2.5 Conditions for Optimality

We now study the crucial relationship between the primal problem (2.3.26) and the dual problem (2.4.34) which leads the necessary and sufficient conditions for the optimality of these two problems.

Proposition 2.5.1. *For any $X = (\dot{X}, H_X) \in \mathbb{V}_1$ and $(P, \dot{Q}) \in \mathbb{V}_2 \times \mathbb{L}^{21}_{\mathcal{F}}$,*

$$\Phi(X) + \Psi\left(P, \dot{Q}\right) \geq 0. \tag{2.5.1}$$

Note that (2.5.1) has been shown in (2.4.14) noting $\Psi(P, \dot{Q}) = \phi^*(\alpha_T^*, r^*, k^*)$ obtained by Theorem 2.4.4. In the following, we give an alternative proof for this inequality using the expression (2.4.24) directly.

Proof. Fix $X \in \mathbb{V}_1$ and $(P, \dot{Q}) \in \mathbb{V}_2 \times \mathbb{L}^{21}_{\mathcal{F}}$. Then, (2.3.25) and (2.4.24) together give that

$$\begin{aligned}
&\Phi(X) + \Psi\left(P, \dot{Q}\right) \\
=& I_{L^*}\left(\dot{P} - \mathbb{E}\left[\dot{Q}(\cdot + \delta) I_{[0,T-\delta]}(\cdot) \middle| \mathcal{F}(\cdot)\right], \dot{Q}, P, H_P\right) + I_L\left(X, X_\delta, \dot{X}, H_X\right) \\
&+ J_{l^*}(-P_T) + J_l(X(T)) + \mathbb{E}\left[\langle P_T, \xi(0)\rangle\right] - \mathbb{E}\left[\int_0^T \left\langle \dot{P}(t), \xi(0)\right\rangle dt\right] \\
&- \mathbb{E}\left[\int_0^T \left\langle \dot{Q}(t), \xi(t-\delta) I_{[0,\delta]}(t)\right\rangle dt\right].
\end{aligned} \tag{2.5.2}$$

Since I_{L^*} and J_{l^*} are the conjugate convex functions of I_L and J_l respectively, we have

$$\begin{aligned}
&I_{L^*}\left(\dot{P} - \mathbb{E}\left[\dot{Q}(\cdot + \delta) I_{[0,T-\delta]}(\cdot) \middle| \mathcal{F}(\cdot)\right], \dot{Q}, P, H_P\right) \\
\geq& \mathbb{E}\left[\int_0^T \left\langle X(t), \dot{P}(t) + \mathbb{E}\left[\dot{Q}(t+\delta) I_{[0,T-\delta]}(t) \middle| \mathcal{F}(t)\right]\right\rangle dt\right] \\
&+ \mathbb{E}\left[\int_0^T \left\langle \dot{Q}(t), X_\delta(t)\right\rangle dt\right] + \mathbb{E}\left[\int_0^T \left\langle P(t), \dot{X}(t)\right\rangle dt\right] \\
&+ \mathbb{E}\left[\int_0^T \left\langle H_P(t), H_X(t)\right\rangle dt\right] - I_L\left(X, X_\delta, \dot{X}, H_X\right)
\end{aligned} \tag{2.5.3}$$

and

$$J_{l^*}(-P_T) \geq -\mathbb{E}\left[\langle P_T, X(T)\rangle\right] - J_l(X(T)). \tag{2.5.4}$$

Then, substituting (2.5.3) and (2.5.4) into the right-hand-side of (2.5.2), we obtain that

$$\begin{aligned}
&\Phi(X) + \Psi\left(P, \dot{Q}\right) \\
\geq& \mathbb{E}\left[\int_0^T \left\langle X(t), \dot{P}(t) + \mathbb{E}\left[\dot{Q}(t+\delta) I_{[0,T-\delta]}(t) \middle| \mathcal{F}(t)\right]\right\rangle dt\right] + \mathbb{E}\left[\int_0^T \left\langle P(t), \dot{X}(t)\right\rangle dt\right] \\
&+ \mathbb{E}\left[\int_0^T \left\langle \dot{Q}(t), X_\delta(t)\right\rangle dt\right] + \mathbb{E}\left[\int_0^T \left\langle H_P(t), H_X(t)\right\rangle dt\right] - \mathbb{E}\left[\langle P_T, X(T)\rangle\right] \\
&- \mathbb{E}\left[\int_0^T \left\langle \dot{Q}(t), \xi(t-\delta) I_{[0,\delta]}(t)\right\rangle dt\right] + \langle \xi(0), P(0)\rangle,
\end{aligned}$$

where we have used (2.4.30). Moreover, substituting (2.4.29) into the right-hand-side of the above inequality, we have

$$\begin{aligned}
&\Phi(X) + \Psi\left(P, \dot{Q}\right) \\
\geq& \mathbb{E}\left[\int_0^T \left\langle X(t), \dot{Q}(t+\delta) I_{[0,T-\delta]}(t)\right\rangle dt\right] + \mathbb{E}\left[\int_0^T \left\langle \dot{Q}(t), X_\delta(t)\right\rangle dt\right] \\
&- \mathbb{E}\left[\int_0^T \left\langle \dot{Q}(t), \xi(t-\delta) I_{[0,\delta]}(t)\right\rangle dt\right],
\end{aligned}$$

which gives (2.5.1) noting the relationship between X and X_δ. □

It is straightforward to see that (2.5.1) implies

$$\inf_{(P,\dot{Q})\in\mathbb{V}_2\times\mathbb{L}^{21}_{\mathcal{F}}} \Psi\left(P,\dot{Q}\right)+\inf_{X\in\mathbb{V}_1}\Phi\left(X\right)\geq 0. \tag{2.5.5}$$

In particular, the equality in (2.5.1), as well as that in (2.5.5), is not satisfied in general. In the following theorem, we use the techniques of stochastic calculus, together with the relationship of conjugate convex functions between L and L^*, as well as that between l and l^*, obtained in Section 2.4.1 to derive the necessary and sufficient conditions for optimality of both the primal and dual problems. This generalizes those described by (2.1.9) and (2.1.10) to the context of time-delay and generalises those described by (2.1.13) and (2.1.14) to the stochastic context.

Note that Theorem 2.5.2 (iii) below plays an important role in obtaining the Hamiltonian and adjoint equation in the stochastic maximum principle for stochastic optimal control problems with discrete delay in the next chapter.

Theorem 2.5.2. *For any $\bar{X}\in\mathbb{V}_1$ and $(\bar{P},\dot{\bar{Q}})\in\mathbb{V}_2\times\mathbb{L}^{21}_{\mathcal{F}}$, the following three statements are equivalent:*

(i)

$$\Phi\left(\bar{X}\right)+\Psi\left(\bar{P},\dot{\bar{Q}}\right)=0. \tag{2.5.6}$$

(ii) $\bar{X}$ *and* $(\bar{P},\dot{\bar{Q}})$ *are optimal solutions of the primal problem* (2.3.26) *and the dual problem* (2.4.34)*, and also the equality in* (2.5.5) *is attained at* $\bar{X}$ *and* $(\bar{P},\dot{\bar{Q}})$*.*

(iii)

$$\begin{aligned}
&L^*\left(t,\dot{\bar{P}}\left(t\right)-\mathbb{E}\left[\dot{\bar{Q}}\left(t+\delta\right)I_{[0,T-\delta]}\left(t\right)\middle|\mathcal{F}\left(t\right)\right],\dot{\bar{Q}}\left(t\right),\bar{P}\left(t\right),H_{\bar{P}}\left(t\right)\right)\\
&+L\left(t,\bar{X}\left(t\right),\bar{X}_\delta\left(t\right),\dot{\bar{X}}\left(t\right),H_{\bar{X}}\left(t\right)\right)-\left\langle\dot{\bar{Q}}\left(t\right),\bar{X}_\delta\left(t\right)\right\rangle\\
&-\left\langle\dot{\bar{P}}\left(t\right)-\mathbb{E}\left[\dot{\bar{Q}}\left(t+\delta\right)I_{[0,T-\delta]}\left(t\right)\middle|\mathcal{F}\left(t\right)\right],\bar{X}\left(t\right)\right\rangle\\
&-\left\langle\bar{P}\left(t\right),\dot{\bar{X}}\left(t\right)\right\rangle-\left\langle H_{\bar{X}}\left(t\right),H_{\bar{P}}\left(t\right)\right\rangle=0,\quad\mathbb{P}\otimes Leb-a.s.
\end{aligned} \tag{2.5.7}$$

and

$$l\left(\bar{X}\left(T\right)\right)+l^*\left(-\bar{P}_T\right)+\left\langle\bar{P}_T,\bar{X}\left(T\right)\right\rangle=0,\quad\mathbb{P}-a.s. \tag{2.5.8}$$

Proof. (i)⇔(ii): Suppose that (2.5.6) holds. Then, it follows from (2.5.1) that, for any $(P,\dot{Q})\in\mathbb{V}_2\times\mathbb{L}^{21}_{\mathcal{F}}$,

$$\Phi\left(\bar{X}\right)+\Psi\left(P,\dot{Q}\right)=-\Psi\left(\bar{P},\dot{\bar{Q}}\right)+\Psi\left(P,\dot{Q}\right)\geq 0,$$

which implies that $(\bar{P}, \dot{\bar{Q}})$ is an optimal solution of the dual problem (2.4.34). The conclusion that $\bar{X}$ is an optimal solution of the primal problem (2.3.26) follows from the argument at the end of Section 2.4.2 together with Theorem 2.4.4. Then, the fact that the equality in (2.5.5) is attained at $\bar{X}$ and $(\bar{P}, \dot{\bar{Q}})$ follows from (2.5.6).

Conversely, if $\bar{X}$ and $(\bar{P}, \dot{\bar{Q}})$ are optimal to these two problems respectively, then (2.5.6) follows by combining (2.5.1) with the assumption that the equality therein is attained at $\bar{X}$ and $(\bar{P}, \dot{\bar{Q}})$.

(i)⇔(iii): Suppose that (2.5.7) and (2.5.8) hold for the given $\bar{X}$ and $(\bar{P}, \dot{\bar{Q}})$. Taking the integral of the left-hand-side of (2.5.7) over $[0, T]$, adding the left-hand-side of (2.5.8) and then taking the expectation, we have (2.5.6) using the expressions (2.3.25) for Φ and (2.4.24) for Ψ.

Conversely, it follows from the expressions (2.3.25) and (2.4.24) that (2.5.6) is equivalent to the equality

$$\mathbb{E}\left[\int_0^T A_1(t)\,dt\right] + \mathbb{E}[A_2] = 0, \tag{2.5.9}$$

where A_1 is the stochastic process defined by the left-hand-side of (2.5.7) and A_2 is the random variable defined by the left-hand-side of (2.5.8). Since, for fixed $(\omega, t) \in \Omega \times [0, T]$, L^* and l^* are the conjugate convex functions of L and l respectively described by (2.4.1) and (2.4.2), A_1 and A_2 are nonnegative. Then, the equality (2.5.9) implies that $A_1(t) = 0$, $\mathbb{P} \otimes Leb$-a.s. and $A_2 = 0$, $\mathbb{P}$-a.s., so that both (2.5.7) and (2.5.8) hold. □

2.6 A More General Model

The theory studied in the preceding sections can be extended to a more general model. In this section, we concentrate on a case which will be used in Section 3.4 to derive the maximum principle for stochastic optimal control problems with both discrete and exponential moving average delays.

The Primal Problem and Optimal Value Function

In addition to $X \in \mathbb{V}_1$, we identify $(\dot{Y}, H_Y) \in \mathbb{V}_1$ with the continuous $\mathcal{F}(t)$-adapted stochastic process $Y : \Omega \times [0, T] \to \mathbb{R}^n$ defined by

$$Y(t) = y_0 + \int_0^t \dot{Y}(s)\,ds + \int_0^t H_Y(s)\,dW(s), \quad \forall t \in [0, T],$$

in a similar fashion as the identification of X with $(\dot{X}, H_X) \in \mathbb{V}_1$ via Definition 2.3.1, where $y_0 \in \mathbb{R}^n$ is a given constant. The functions $L_a : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{\infty\}$ and $l_a : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ are modifications of L and l given in Section 2.3.2, so that the corresponding I_{L^a} and J_{l^a} also depend on $(Y, \dot{Y}, H_Y)$ and $Y(T)$ respectively. Moreover, we assume that L_a and l_a satisfy the following assumptions which are modifications of Assumptions I, II & III in Section 2.3.2 due to the introduction of Y.

Assumption* I. (i) L_a and l_a are not identically ∞; L_a is a lower semi-continuous convex function on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m}$, for any $(\omega, t) \in \Omega \times [0,T]$, and l_a is a lower semi-continuous convex function on $\mathbb{R}^n \times \mathbb{R}^n$, for any $\omega \in \Omega$.

(ii) L_a is $\mathcal{F}^* \times \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^{n\times m}) \times \mathcal{B}(\mathbb{R}^{n\times m})$-measurable and l_a is $\mathcal{F} \times \mathcal{B}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n)$-measurable.

Assumption* II. (i) There exist $(X^*, Y^*, Z^*, Z_1^*, Z_2^*, H_1^*, H_2^*) \in \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{2\infty} \times \mathbb{L}_{\mathcal{F}}^{2\infty} \times \mathbb{L}_{\mathcal{F}}^{22} \times \mathbb{L}_{\mathcal{F}}^{22}$ and a $\mathbb{R}$-valued $\mathcal{F}(t)$-progressively measurable stochastic process ϖ_a satisfying (2.3.15) such that, for any $(x, y, z, z_1, z_2) \in \mathbb{R}^{n\times 5}$ and $(h_1, h_2) \in \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m}$,

$$
\begin{aligned}
& L_a\left(t, x, y, z, z_1, z_2, h_1, h_2\right) \\
& \geq \langle x, X^*(t)\rangle + \langle y, Y^*(t)\rangle + \langle z, Z^*(t)\rangle + \langle z_1, Z_1^*(t)\rangle + \langle z_2, Z_2^*(t)\rangle \\
& \quad + \langle h_1, H_1^*(t)\rangle + \langle h_2, H_2^*(t)\rangle - \varpi_a(t), \quad \mathbb{P} \otimes Leb - a.s.
\end{aligned}
$$

(ii) There exist $(X_T^*, Y_T^*) \in \mathbb{L}^2 \times \mathbb{L}^2$ and a $\mathbb{R}$-valued $\mathcal{F}(T)$-measurable random variable ϑ_a satisfying (2.3.17) such that, for any $(x, y) \in \mathbb{R}^{n\times 2}$,

$$
l_a(x, y) \geq \langle x, X_T^*\rangle + \langle y, Y_T^*\rangle - \vartheta_a, \quad \mathbb{P} - a.s.
$$

Assumption* III. (i) There exist $(X, Y, Z, Z_1, Z_2, H_1, H_2) \in \mathbb{L}_{\mathcal{F}}^{2\infty} \times \mathbb{L}_{\mathcal{F}}^{2\infty} \times \mathbb{L}_{\mathcal{F}}^{2\infty} \times \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{21} \times \mathbb{L}_{\mathcal{F}}^{22} \times \mathbb{L}_{\mathcal{F}}^{22}$ and a $\mathbb{R}$-valued $\mathcal{F}(t)$-progressively measurable stochastic process τ_a satisfying (2.3.19) such that

$$
L_a\left(t, X(t), Y(t), Z(t), Z_1(t), Z_2(t), H_1(t), H_2(t)\right) \leq \tau_a(t), \quad \mathbb{P} \otimes Leb - a.s.
$$

(ii) There exist $(X_T, Y_T) \in \mathbb{L}^2 \times \mathbb{L}^2$ and a $\mathbb{R}$-valued $\mathcal{F}(T)$-measurable random variable χ_a satisfying (2.3.21) such that

$$l_a(X_T, Y_T) \leq \chi_a, \quad \mathbb{P} - a.s.$$

Now, under Assumptions* I, II & III for L_a and l_a, the corresponding stochastic convex (primal) problem with discrete delay is to find a pair of $(\bar{X}, \bar{Y}) \in \mathbb{V}_1 \times \mathbb{V}_1$ realizing

$$\inf_{(X,Y) \in \mathbb{V}_1 \times \mathbb{V}_1} \Phi_a(X, Y), \tag{2.6.1}$$

where Φ_a is the primal function defined by

$$\Phi_a(X, Y) = I_{L_a}\left(X, Y, X_\delta, \dot{X}, \dot{Y}, H_X, H_Y\right) + J_{l_a}(X(T), Y(T)). \tag{2.6.2}$$

Similarly to Section 2.4.2, we define the corresponding optimal value function ϕ_a on $\mathbb{L}^2 \times \mathbb{L}^2 \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}}$ by

$$\phi_a(a_1, a_2, r_1, r_2, k) = \inf_{(X,Y) \in \mathbb{V}_1 \times \mathbb{V}_1} F^a_{a_1,a_2,r_1,r_2,k}(X, Y), \tag{2.6.3}$$

where F is the perturbed function on $\mathbb{V}_1 \times \mathbb{V}_1$ expressed by

$$\begin{aligned} F^a_{a_1,a_2,r_1,r_2,k}(X, Y) =& I_{L^a}\left(X + r_1, Y + r_2, X_\delta + k, \dot{X}, \dot{Y}, H_X, H_Y\right) \\ &+ J_{l^a}(X(T) - a_1, Y(T) - a_2). \end{aligned}$$

The Dual Problem and Conditions for the Optimality

Adapting the techniques used in Section 2.4.3, in addition to $P = (P_T, \dot{P}) \in \mathbb{V}_2$, we require another continuous $\mathcal{F}(t)$-adapted stochastic process P^a to pair with $Y \in \mathbb{V}_1$, where $P^a : \Omega \times [0, T] \to \mathbb{R}^n$ is identified with $(P^a_T, \dot{P}^a) \in \mathbb{V}_2$ in the same sense to that P is identified with $(P_T, \dot{P})$ using (2.4.17).

Note that, since the primal function (2.6.2) does not involve Y_δ, the inclusion of P^a in Ψ_a does not result in the dependence of Ψ_a on an additional Q^a as was the case for the inclusion of Q in Ψ. Then, we can generalize Theorem 2.4.4 to obtain the dual function Ψ_a as follows.

Theorem 2.6.1. *Suppose that Assumptions* I, II & III hold for the functions L_a and l_a. In addition, for any given $(a_1^*, a_2^*, r_1^*, r_2^*, k^*) \in \mathbb{L}^2 \times \mathbb{L}^2 \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}}$, let $(P_T, \dot{P}, \dot{Q}) \in$*

$\mathbb{L}^2 \times \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{21}_{\mathcal{F}}$ *and* $(P^a_T, \dot{P}^a) \in \mathbb{L}^2 \times \mathbb{L}^{21}_{\mathcal{F}}$ *be defined respectively by*

$$\begin{cases} P_T = a_1^*, \\ \dot{P}(t) = r_1^*(t) + \mathbb{E}\left[k^*(t+\delta) I_{[0,T-\delta]}(t) \,\middle|\, \mathcal{F}(t)\right], \\ \dot{Q}(t) = k^*(t), \end{cases}$$

and

$$\begin{cases} P^a_T = a_2^*, \\ \dot{P}^a(t) = r_2^*(t), \end{cases}$$

where we identify P *and* P^a *respectively with* $(P_T, \dot{P}) \in \mathbb{V}_2$ *and* $(P^a_T, \dot{P}^a) \in \mathbb{V}_2$ *using* (2.4.17). *Then, the function* Ψ_a *on* $\mathbb{V}_2 \times \mathbb{V}_2 \times \mathbb{L}^{21}_{\mathcal{F}}$ *defined by*

$$\begin{aligned} \Psi_a\left(P, P^a, \dot{Q}\right) =& I_{L_a^*}\left(\dot{P} - \mathbb{E}\left[\dot{Q}(\cdot+\delta) I_{[0,T-\delta]}(\cdot) \middle| \mathcal{F}(\cdot)\right], \dot{P}^a, \dot{Q}, P, P^a, H_P, H_{P^a}\right) \\ &+ J_{l_a^*}(-P_T, -P^a_T) - \mathbb{E}\left[\int_0^T \left\langle \dot{Q}(t), \xi(t-\delta) I_{[0,\delta]}(t) \right\rangle dt\right] \\ &- \mathbb{E}\left[\int_0^T \left\{\left\langle \dot{P}(t), \xi(0)\right\rangle + \left\langle \dot{P}^a(t), y_0 \right\rangle\right\} dt\right] \\ &+ \mathbb{E}\left[\langle P_T, \xi(0)\rangle + \langle P^a_T, y_0\rangle\right] \end{aligned} \tag{2.6.4}$$

satisfies $\Psi_a(P, P^a, \dot{Q}) = \phi_a^*(a_1^*, a_2^*, r_1^*, r_2^*, k^*)$, *where* ϕ_a^* *is the conjugate convex function of* ϕ_a *defined by* (2.6.3) *and* H_P *and* H_{P^a} *are specified by* (2.4.18).

Building upon (2.6.4), the corresponding dual problem to (2.6.1) is to find $(\bar{P}, \bar{P}^a, \dot{\bar{Q}}) \in \mathbb{V}_2 \times \mathbb{V}_2 \times \mathbb{L}^{21}_{\mathcal{F}}$ realizing

$$\inf_{\left(P, P^a, \dot{Q}\right) \in \mathbb{V}_2 \times \mathbb{V}_2 \times \mathbb{L}^{21}_{\mathcal{F}}} \Psi_a\left(P, P^a, \dot{Q}\right). \tag{2.6.5}$$

Based on the arguments used at the end of Section 2.4.1, we can obtain the relationship between ϕ_a^* and Φ_a in a similar fashion to (2.4.14) which, together with Ψ_a given by (2.6.4), enables us to generalize Theorem 2.5.2 to obtain the following necessary and sufficient conditions for the optimality.

Theorem 2.6.2. *For any given* $(\bar{X}, \bar{Y}) \in \mathbb{V}_1 \times \mathbb{V}_1$ *and* $(\bar{P}, \bar{P}^a, \dot{\bar{Q}}) \in \mathbb{V}_2 \times \mathbb{V}_2 \times \mathbb{L}^{21}_{\mathcal{F}}$, *the following three statements are equivalent:*

(i)

$$\Phi_a\left(\bar{X}, \bar{Y}\right) + \Psi_a\left(\bar{P}, \bar{P}^a, \dot{\bar{Q}}\right) = 0.$$

(ii) $(\bar{X},\bar{Y})$ *and* $(\bar{P},\bar{P}^a,\dot{\bar{Q}})$ *are optimal solutions of the primal problem* (2.6.1) *and the dual problem* (2.6.5) *respectively, and*

$$\inf_{\left(P,P^a,\dot{Q}\right)\in\mathbb{V}_2\times\mathbb{V}_2\times\mathbb{L}^{21}_{\mathcal{F}}}\Psi_a\left(P,P^a,\dot{Q}\right)=-\inf_{(X,Y)\in\mathbb{V}_1\times\mathbb{V}_1}\Phi_a\left(X,Y\right).$$

(iii)

$$\begin{aligned}
&L_a^*\Big(t,\dot{\bar{P}}(t)-\mathbb{E}\left[\dot{\bar{Q}}(t+\delta)\,I_{[0,T-\delta]}(t)\,\middle|\mathcal{F}(t)\right],\dot{\bar{P}}^a(t),\dot{\bar{Q}}(t),\bar{P}(t),\bar{P}^a(t),\\
&\qquad H_{\bar{P}}(t),H_{\bar{P}^a}(t)\Big)+L_a\Big(t,\bar{X}(t),\bar{Y}(t),\bar{X}_\delta(t),\dot{\bar{X}}(t),\dot{\bar{Y}}(t),H_{\bar{X}}(t),H_{\bar{Y}}(t)\Big)\\
&-\left\langle\dot{\bar{Q}}(t),\bar{X}_\delta(t)\right\rangle-\left\langle\dot{\bar{P}}(t)-\mathbb{E}\left[\dot{\bar{Q}}(t+\delta)\,I_{[0,T-\delta]}(t)\,\middle|\mathcal{F}(t)\right],\bar{X}(t)\right\rangle\\
&-\left\langle\dot{\bar{P}}^a(t),\bar{Y}(t)\right\rangle-\left\langle\bar{P}^a(t),\dot{\bar{Y}}(t)\right\rangle-\left\langle\bar{P}(t),\dot{\bar{X}}(t)\right\rangle-\langle H_{\bar{X}}(t),H_{\bar{P}}(t)\rangle\\
&-\langle H_{\bar{P}^a}(t),H_{\bar{Y}}(t)\rangle=0,\quad\mathbb{P}\otimes Leb-a.s.
\end{aligned}\tag{2.6.6}$$

and

$$\begin{aligned}
&l_a(\bar{X}(T),\bar{Y}(T))+l_a^*(-\bar{P}_T,-\bar{P}^a_T)+\langle\bar{P}_T,\bar{X}(T)\rangle\\
&+\left\langle\bar{P}^a_T,\bar{Y}(T)\right\rangle=0,\quad\mathbb{P}-a.s.
\end{aligned}\tag{2.6.7}$$

CHAPTER **3**

STOCHASTIC MAXIMUM PRINCIPLE IN STOCHASTIC OPTIMAL CONTROL PROBLEMS WITH DELAY

3.1 Introduction

Having obtained the results for conjugate duality in the previous chapter, in particular the conditions for optimality in Theorems 2.5.2 & 2.6.2, this chapter applies these results to generalize the maximum principles studied in [6, 29, 30] for stochastic optimal control problems with discrete delay and those with both discrete and exponential moving average delays.

3.1.1 Literature Review

We first review some basic results for the stochastic maximum principle, together with some techniques of conjugate duality carried out by Bismut in [2], mainly taken from [45, Chapter 3], for Markovian optimal control problems. After that, we describe three generalizations [6, 29, 30] of this approach in solving the stochastic optimal control problems with delay.

The Markovian Optimal Control Problem

We continue to work with the fixed time horizon $T \in (0, \infty)$, complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, standard m-dimensional Brownian motion W and filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{t\in[0,T]}$ generated by W. In addition, let $\mathbb{U} \subset \mathbb{R}^r$ be a convex set throughout this chapter, where r is an positive integer. For given functions $b : [0, T] \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$

and $\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^{n\times m}$, let the continuous $\mathcal{F}(t)$-adapted state process $X : \Omega \times [0,T] \to \mathbb{R}^n$ be described by the controlled stochastic differential equation (SDE)

$$\begin{cases} dX(t) = b(t, X(t), u(t))\, dt + \sigma(t, X(t), u(t))\, dW(t), \quad t \in [0,T], \\ X(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{3.1.1}$$

where $u : \Omega \times [0,T] \to \mathbb{U}$ is an $\mathcal{F}(t)$-adapted control (process) selected from a given admissible control set $\mathcal{U}$ such that the controlled SDE (3.1.1) admits a unique strong solution (see [17, Definition 5.2.1]) for every $u \in \mathcal{U}$. For given functions $G : [0,T] \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, the cost function J is defined by

$$J(u) = \mathbb{E}\left[\int_0^T G(t, X(t), u(t))\, dt\right] + \mathbb{E}\left[g(X(T))\right], \tag{3.1.2}$$

where the first and second terms on the right-hand-side of (3.1.2) are respectively called the running and terminal costs. Then, the Markovian optimal control problem associated with the state system (3.1.1) and the cost function (3.1.2) is to find $\bar{u} \in \mathcal{U}$ realizing

$$\inf_{u \in \mathcal{U}} J(u). \tag{3.1.3}$$

We shall call $\bar{u}$ an optimal control. For notational simplicity, hereafter, we refer to this general Markovian optimal control problem through the equation describing optimality, i.e. we shall refer to the above Markovian optimal control problem and the definitions therein as (3.1.3).

Instead of minimizing the cost function among $u \in \mathcal{U}$ directly, the (sufficient) stochastic maximum principle says that $\bar{u}$ is an optimal control if, under certain concavity and convexity conditions on the Hamiltonian (function) $\mathcal{H}$ (given below) and g respectively, $\bar{u}$ maximizes $\mathcal{H}$ in the sense that, $\mathbb{P} \otimes Leb$-a.s.

$$\mathcal{H}(t, \bar{X}(t), \bar{u}(t), \bar{P}(t), \bar{H}(t)) = \max_{u \in \mathbb{U}} \mathcal{H}(t, \bar{X}(t), u, \bar{P}(t), \bar{H}(t)), \tag{3.1.4}$$

where $\mathcal{H}$ is defined by

$$\mathcal{H}(t, x, u, p, h) = \langle p, b(t,x,u)\rangle + \langle h, \sigma(t,x,u)\rangle - G(t,x,u); \tag{3.1.5}$$

$\bar{X}$ is the unique strong solution of the controlled SDE (3.1.1) with u replaced by $\bar{u}$; and $(\bar{P}, \bar{H})$ is the solution of the following controlled classical backward stochastic differential equation (BSDE) with (X, u) replaced by $(\bar{X}, \bar{u})$

$$\begin{cases} dP(t) = -\frac{\partial \mathcal{H}}{\partial x}(t, X(t), u(t), P(t), H(t)) + H(t)\, dW(t), \\ \qquad\qquad\qquad\qquad t \in [0, T], \\ P(T) = -\frac{\partial g}{\partial x}(X(T)). \end{cases} \tag{3.1.6}$$

Hereafter, the above controlled BSDE and (P, H) are respectively referred to as the adjoint equation and the adjoint process in the context of the stochastic maximum principle (see [45, Theorem 3.5.2]).

On the other hand, as noted in Chapter 1, the conjugate duality method has played an important role in the study of stochastic maximum principles. Bismut in [2, Section 5] reformulates the Markovian optimal control problem (3.1.3) with $g(x) = 0$ as a particular Markovian convex problem (2.1.6) by respectively defining the corresponding convex functions L and l as

$$L(t, x, z, h) = \begin{cases} \inf_u G(t, x, u), & \text{if } u \in \mathbb{U} \text{ such that } \begin{cases} z = b(t, x, u), \\ h = \sigma(t, x, u), \end{cases} \\ \infty, & \text{otherwise,} \end{cases}$$

and $l(x, y) = 0$. Then, the conditions for the optimality of convex problems given in [2, Theorem IV-2] provides a way to derive the Hamiltonian (3.1.5) and the adjoint equation (3.1.6) with $-\frac{\partial g}{\partial x}(X(T)) \equiv 0$ (see [2, Theorem V-1]). More precisely, if there exist $\bar{X}$ and $\bar{P}$ respectively identified with $(\bar{X}(0), \dot{\bar{X}}, H_{\bar{X}})$ and $(\bar{P}(0), \dot{\bar{P}}, H_{\bar{P}})$ in the sense of (2.1.7) satisfying the necessary and sufficient condition described by (2.1.9) and (2.1.10), then it is necessary that there exists a $\bar{u} \in \mathcal{U}$ realizing (3.1.3) with $(\bar{u}, \bar{X}, \bar{P})$ satisfying

- $\bar{X}$ is the unique strong solution of the controlled SDE (3.1.1) with u replaced by $\bar{u}$, i.e.

$$\begin{cases} \dot{\bar{X}}(t) = b(t, \bar{X}(t), \bar{u}(t)), \\ H_{\bar{X}}(t) = \sigma(t, \bar{X}(t), \bar{u}(t)); \end{cases} \qquad \mathbb{P} \otimes Leb - a.s.$$

- $(\bar{P}, H_{\bar{P}})$ is the solution of the adjoint equation (3.1.6) with (X, u) replaced by $(\bar{X}, \bar{u})$ and $P(T) \equiv 0$;

- the maximizing equation (3.1.4) holds.

However, Bismut in [2] does not investigate further the corresponding stochastic maximum principle in the context of conjugate duality.

The Stochastic Optimal Control Problem with Discrete Delay

Generalizing the Markovian optimal control problem, if the model, comprising the state system and cost function, involves a discretely delayed effect on the state process, the corresponding control problem is referred to as a stochastic optimal control problem with discrete delay. More specifically, for given functions $b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^{n\times m}$, we suppose that the continuous $\mathcal{F}(t)$-adapted state process $X : \Omega \times [-\delta, T] \to \mathbb{R}^n$ satisfies the controlled stochastic differential delay equation (SDDE)

$$\begin{cases} dX(t) = b(t, X(t), X_\delta(t), u(t))\, dt \\ \qquad\qquad + \sigma(t, X(t), X_\delta(t), u(t))\, dW(t), \quad t \in [0, T], \\ X(t) = \xi(t), \quad t \in [-\delta, 0], \end{cases} \tag{3.1.7}$$

where, as defined in the previous chapter, $\delta \in (0, T)$ is a given constant, ξ is the continuous deterministic initial path for X and $X_\delta(t) = X(t - \delta)$ for $t \in [0, T]$. The admissible control space $\mathcal{U}$ here is as defined in a similar fashion to that in (3.1.3). Moreover, the cost function J_d is defined by

$$J_d(u) = \mathbb{E}\left[\int_0^T G(t, X(t), X_\delta(t), u(t))\, dt + g(X(T))\right], \tag{3.1.8}$$

where $G : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given functions. Then, the stochastic optimal control problem with discrete delay associated with the state system (3.1.7) and the cost function (3.1.8) is to find $\bar{u} \in \mathcal{U}$ realizing

$$\inf_{u\in\mathcal{U}} J_d(u). \tag{3.1.9}$$

As before, we shall refer to this control problem and the definitions therein as (3.1.9).

Some progress has been made on the stochastic maximum principle for this type of control problems. For example, if the control problem considered in [6] is restricted to (3.1.9) i.e. the model is independent of the discrete delayed control u_δ, Chen and Wu in [6, Theorem 3.2] establish a stochastic maximum principle, where the Hamiltonian and adjoint equation are introduced. Those are,

$$\mathcal{H}_d\left(t,x,y,u,p,h\right) = \left\langle b\left(t,x,y,u\right),p\right\rangle + \left\langle \sigma\left(t,x,y,u\right),h\right\rangle - G\left(t,x,y,u\right) \tag{3.1.10}$$

and

$$\begin{cases} dP\left(t\right) = -\left\{\dfrac{\partial\mathcal{H}_d}{\partial x}\left(t\right) + \mathbb{E}\left[\dfrac{\partial\mathcal{H}_d}{\partial y}\left(t+\delta\right)I_{[0,T-\delta]}\left(t\right)\middle|\mathcal{F}\left(t\right)\right]\right\}dt \\ \qquad\qquad + H\left(t\right)dW\left(t\right), \quad t\in\left[0,T\right], \\ P\left(T\right) = -\dfrac{\partial g}{\partial x}\left(X\left(T\right)\right). \end{cases} \tag{3.1.11}$$

Here we have used the shorthand notation

$$\frac{\partial\mathcal{H}_d}{\partial x}\left(t\right) = \frac{\partial\mathcal{H}_d}{\partial x}\left(t,X\left(t\right),X_\delta\left(t\right),u\left(t\right),P\left(t\right),H\left(t\right)\right) \tag{3.1.12}$$

and similarly for the partial derivative $\frac{\partial\mathcal{H}_d}{\partial y}(t+\delta)$. Note that the above adjoint equation is an anticipated controlled BSDE which, comparing with (3.1.6), also depends on the anticipated (or time-advanced) terms $X(t+\delta), P(t+\delta)$ and $H(t+\delta)$. Nevertheless, rather than deriving them, Chen and Wu in [6] just introduce the above Hamiltonian and adjoint equation and then prove the corresponding stochastic maximum principle by techniques of stochastic calculus. However, by merely stating them, they do not consider further generalization as we do in this chapter.

Inclusion of an Exponential Moving Average Delay

Building upon the control problem (3.1.9), if the model also depends an exponential moving average delay (specified below), the corresponding control problem is referred to as having both discrete and exponential moving average delays. More precisely, for given functions $b : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{U}\to\mathbb{R}^n$ and $\sigma : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{U}\to\mathbb{R}^{n\times m}$, we suppose that the continuous $\mathcal{F}(t)$-adapted state process $X : \Omega\times[-\delta,T]\to\mathbb{R}^n$ satisfies the controlled SDDE

$$\begin{cases} dX\left(t\right) = b\left(t,X\left(t\right),X_a\left(t\right),X_\delta\left(t\right),u\left(t\right)\right)dt \\ \qquad\qquad + \sigma\left(t,X\left(t\right),X_a\left(t\right),X_\delta\left(t\right),u\left(t\right)\right)dW\left(t\right), \quad t\in\left[0,T\right], \\ X\left(t\right) = \xi\left(t\right), \quad t\in\left[-\delta,0\right], \end{cases} \tag{3.1.13}$$

where ξ, X_δ and δ are defined as before and X_a denotes the exponential moving average delay expressed by

$$X_a\left(t\right) = \int_{-\delta}^{0} e^{\lambda r}X\left(t+r\right)dr, \quad \lambda > 0.$$

For given functions $G : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the corresponding cost function J_{ad} is defined by

$$J_{ad}(u) = \mathbb{E}\left[\int_0^T G(t, X(t), X_a(t), X_\delta(t), u(t))\, dt + g(X(T), X_a(T))\right] \tag{3.1.14}$$

where G and g are given functions. Then, the stochastic optimal control problem with both discrete and exponential moving average delays associated with the state system (3.1.13) and the cost function (3.1.14) is to find $\bar{u} \in \mathcal{U}$ realizing

$$\inf_{u \in \mathcal{U}} J_{ad}(u), \tag{3.1.15}$$

where the admissible control set $\mathcal{U}$ is defined similarly as before. Similarly, this control problem and the definitions therein are referred to as (3.1.15).

As mentioned in Chapter 1, the authors of [29] and [30] use the stochastic maximum principles for solving the control problem (3.1.15). However, the Hamiltonian functions and the associated adjoint equations introduced there are very different. More explicitly, Øksendal and Sulem in [29, Section 2] introduce a Hamiltonian as

$$\begin{aligned} &\mathcal{H}_{ad}(t, x, y, z, u, p, h) \\ &= \langle b(t, x, y, z, u), p_1 \rangle + \langle \sigma(t, x, y, z, u), h_1 \rangle - G(t, x, y, z, u) \\ &\quad + \left\langle \left(x - \lambda y - e^{\lambda\delta} z\right), p_2 \right\rangle, \end{aligned} \tag{3.1.16}$$

where $p = (p_1, p_2, p_3)^\top$ and $h = (h_1, h_2)^\top$. Comparing this Hamiltonian with (3.1.5) for the Markovian optimal control problem and with (3.1.10) for the stochastic control problem with discrete delay respectively, the introduction of the last term on the right-hand-side of (3.1.16) is due to the dependence on X_a (see [29, Lemma 2.1]). Then, associated with (3.1.16), Øksendal and Sulem in [29] introduce the adjoint equations defined by a triple of classical controlled BSDEs

$$\begin{cases} dP_1(t) = -\dfrac{\partial \mathcal{H}_{ad}}{\partial x}(t, X(t), X_a(t), X_\delta(t), u(t), P(t), H(t))\, dt \\ \qquad\qquad + H_1(t)\, dW(t), \quad t \in [0,T], \\ P_1(T) = -\dfrac{\partial g}{\partial x}(X(t), X_a(t)), \end{cases} \tag{3.1.17}$$

$$\begin{cases} dP_2(t) = -\dfrac{\partial \mathcal{H}_{ad}}{\partial y}(t, X(t), X_a(t), X_\delta(t), u(t), P(t), H(t))\, dt \\ \qquad\qquad + H_2(t)\, dW(t), \quad t \in [0,T], \\ P_2(T) = -\dfrac{\partial g}{\partial y}(X(t), X_a(t)), \end{cases} \tag{3.1.18}$$

$$
\begin{cases}
dP_3(t) = -\dfrac{\partial \mathcal{H}_{ad}}{\partial z}\left(t, X(t), X_a(t), X_\delta(t), u(t), P(t), H(t)\right) dt, \\
\qquad\qquad t \in [0,T], \\
P_3(T) = 0.
\end{cases}
\tag{3.1.19}
$$

Having introduced these Hamiltonian and adjoint equations, Øksendal and Sulem in [29, Theorem 2.2] establish a stochastic maximum principle which requires P_3 to be identically zero. As noted in [29, Theorem 3.1], this imposes a restriction that the model needs to satisfy certain conditions to ensure that the requirement is satisfied. Clearly, the control problem (3.1.9) can be regarded as a special case of (3.1.15), for example, by defining b, σ, G and g in (3.1.13) and (3.1.14) to be independent of X_a and $X_a(T)$ respectively. However, the stochastic maximum principle obtained in [29, Theorem 2.2] cannot hold if the model just involves X_δ. In fact, as pointed out in [29, Lemma 2.1], introduction of the Hamiltonian (3.1.16) and the adjoint equations (3.1.17)-(3.1.19) depends on the involvement of X_a. In other words, the results obtained in [29] do not imply those studied in [6].

On the other hand, if the model studied in [30] is jump-free, Øksendal, Sulem and Zhang in [30, Theorem 3.1] provide a stochastic maximum principle for a special case of the control problem (3.1.15), where g is independent of its second component corresponding to $X_a(T)$. The Hamiltonian in [30, page 574] is defined by

$$
\begin{aligned}
&\mathcal{H}_{ad}(t,x,y,z,u,p,h) \\
&= \langle b(t,x,y,z,u), p\rangle + \langle \sigma(t,x,y,z,u), h\rangle - G(t,x,y,z,u).
\end{aligned}
\tag{3.1.20}
$$

This Hamiltonian is described in a similar manner to (3.1.10) studied in [6], but is different from (3.1.16) which, as mentioned before, has an extra term related to the exponential moving average delay. Then, instead of a triple of classical BSDEs, the authors associate the above Hamiltonian with the adjoint equation

$$
\begin{cases}
dP(t) = -\left\{\mathbb{E}\left[\dfrac{\partial \mathcal{H}_{ad}}{\partial z}(t+\delta)\, I_{[0,T-\delta]}(t) + e^{\lambda t}\displaystyle\int_t^{t+\delta} \dfrac{\partial \mathcal{H}_{ad}}{\partial y}(s)\, e^{-\lambda s} I_{[0,T]}(s) ds \,\middle|\, \mathcal{F}(t)\right]\right. \\
\qquad\qquad \left. + \dfrac{\partial \mathcal{H}_{ad}}{\partial x}(t)\right\} dt + H(t)\, dW(t), \quad t \in [0,T], \\
P(T) = -\dfrac{\partial g}{\partial x}(X(T)),
\end{cases}
\tag{3.1.21}
$$

where we have used the shorthand notation

$$\frac{\partial \mathcal{H}_{ad}}{\partial x}(t) = \frac{\partial \mathcal{H}_{ad}}{\partial x}(t, X(t), X_a(t), X_\delta(t), u(t), P(t), H(t))$$

and similarly for the partial derivatives $\frac{\partial \mathcal{H}_{ad}}{\partial z}(t+\delta)$ and $\frac{\partial \mathcal{H}_{ad}}{\partial y}(t)$. If the model just involves X_δ so that the Hamiltonian (3.1.20) is independent of y, then the term $\frac{\partial \mathcal{H}_a}{\partial y}(t)$ in (3.1.21) is identically zero. This implies that the above adjoint equation reduces to (3.1.11). In other words, the corresponding stochastic maximum principle stated in [30, Theorem 3.1] implies the one studied in [6, Theorem 3.2]. Unfortunately, although the restriction in [29] mentioned above does not appear, the results obtained in [30] cannot be applied if the terminal cost in (3.1.14) depends on $X_a(T)$.

3.1.2 Main Results and Structure of the Chapter

To resolve the restrictions in [6, 29, 30] mentioned in Section 3.1.1, this chapter studies stochastic maximum principles for both the control problems (3.1.9) & (3.1.15) using the results of conjugate duality method obtained in Chapter 2. These results, stated in Theorem 3.3.2 & 3.4.2 respectively, generalize the results of [2, Section 5]. If the model just involves a discrete delay, the Hamiltonian and the adjoint equation obtained here coincide with (3.1.10) and (3.1.11). If the model involves both discrete and exponential moving average delays, then the corresponding Hamiltonian is similar to (3.1.16) introduced in [29] but the adjoint equations are different from (3.1.17)-(3.1.19) and (3.1.21) studied in [29, 30]. In particular, those restrictions are removed by our new adjoint equations.

The remainder of the chapter is organized as follows. Section 3.2 generalizes the technique in [2] to reformulate the control problem (3.1.9) to a particular convex problem studied in the previous chapter. We also give a solvable example to describe how the conditions for optimality given in Theorem 2.5.2 can be used to obtain an optimal control. Then, under certain hypotheses, Section 3.3 applies those conditions to obtain the stochastic maximum principle for that control problem. Furthermore, modifying the arguments used in the preceding sections of this chapter, Section 3.4 applies the results obtained in Section 2.6 to establish the stochastic maximum principle for the control problem (3.1.15). Finally, Section 3.5 discusses the extension to a regime-switching model and gives a different proof for the stochastic maximum principle obtained in Section 3.4.

3.2 Reformulation to a Convex Problem

To apply the results obtained in the previous chapter, this section adapts the technique in [2, Example II-3] to reformulate the stochastic optimal control problem with discrete delay (3.1.9) as a particular stochastic convex (primal) problem with discrete delay (2.3.26) as follows. Note that, we first assume that the filtration $\mathbb{F}$ is generated only by W. The inclusion of a Markov chain α, i.e. a regime-switching model, will be considered in Section 3.5.1.

For every $t \in [0,T]$, $(x,y,z) \in \mathbb{R}^{n\times 3}$ and $h \in \mathbb{R}^{n\times m}$, we define the set $\mathcal{C} = \mathcal{C}(t,x,y,z,h)$ by

$$\mathcal{C}\left(t,x,y,z,h\right) = \left\{u \in \mathbb{U} \middle| z = b\left(t,x,y,u\right) \text{ and } h = \sigma\left(t,x,y,u\right)\right\}, \tag{3.2.1}$$

where b and σ are given in (3.1.7). Using $\mathcal{C}$, we take the functions L and l in (2.3.25) respectively to be

$$L\left(t,x,y,z,h\right) = \begin{cases} \inf\limits_{u\in\mathcal{C}} G\left(t,x,y,u\right), & \text{if } \mathcal{C} \neq \emptyset, \\ \infty, & \text{otherwise,} \end{cases} \tag{3.2.2}$$

and

$$l\left(x\right) = g\left(x\right), \tag{3.2.3}$$

where G and g are given in (3.1.8). With L and l so defined, the control problem (3.1.9) becomes a particular convex problem (2.3.26) provided Assumptions I, II & III in Section 2.3.2 are satisfied.

3.2.1 An Example

The following example demonstrates that, if b and σ in (3.1.7) are both affine functions of (x,y,u); and if G and g in (3.1.8) are convex with respect to (x,y,u) and x respectively together with appropriate assumptions on the parameters of these functions (specified below), then the stochastic convex problem with discrete delay

(2.3.26) with L and l respectively defined by (3.2.2) and (3.2.3) satisfies Assumptions I, II & III required in Definition 2.3.5.

Example 3.2.1. For simplicity, we set $n = m = r = 1$. Suppose that $\mathbb{U} = \mathbb{R}$ and $\mathcal{U} = \mathbb{L}^{22}_{\mathcal{F}}$; that b and σ in (3.1.7) are given by

$$\begin{cases} b\left(t,x,y,u\right) = a_1 x + b_1 y + c_1 u, \\ \sigma\left(t,x,y,u\right) = a_2 x + b_2 y + c_2 u, \end{cases} \tag{3.2.4}$$

where a_1, a_2, b_1, b_2, c_1 and c_2 are given constants; that G and g in (3.1.8) are given by

$$G\left(t,x,y,u\right) = \frac{1}{2} c_3 u^2 \text{ and } g\left(x\right) = \frac{1}{2} a_3 x^2, \tag{3.2.5}$$

where a_3 and c_3 are given positive constants. To simplify the following argument, we suppose further that $c_1 c_2 \neq 0$. Note that, in general, the parameters (except for a_3) in the model can be certain continuous functions on $[0, T]$.

(I) Preliminaries

First, we verify the existence of the unique strong solution of the controlled SDDE (3.1.7) and the integrability of the cost function (3.1.8) with b, σ, G and g so defined.

For every $(x, y, x', y') \in \mathbb{R}^4$,

$$\begin{aligned} &\left|b\left(t,x,y,u\right) - b\left(t,x',y',u\right)\right| + \left|\sigma\left(t,x,y,u\right) - \sigma\left(t,x',y',u\right)\right| \\ \leq &\left(|a_1| + |a_2|\right)\left|x - x'\right| + \left(|b_1| + |b_2|\right)\left|y - y'\right|, \end{aligned} \tag{3.2.6}$$

which implies that b and σ are Lipschitz continuous (see [39, Section 1.6]) with respect to (x, y) for each $u \in \mathbb{U} = \mathbb{R}$, where the Lipschitz constant is independent of (t, u). Then, by [6, Theorem 2.2], the controlled SDDE admits a unique strong solution X for every $u \in \mathbb{L}^{22}_{\mathcal{F}}$ satisfying $X \in \mathbb{L}^{22}_{\mathcal{F}}$.

Since G and g are continuous functions and $u \in \mathbb{L}^{22}_{\mathcal{F}}$, to see the integrability of the cost function, we only need to show that $X(T) \in \mathbb{L}^2$. Using the relationship between X and X_δ, we see that

$$\begin{aligned} \mathbb{E}\left[\int_0^T \left|X_\delta\left(t\right)\right|^2 dt\right] &= \mathbb{E}\left[\int_{-\delta}^{T-\delta} \left|X\left(t\right)\right|^2 dt\right] \\ &= \int_{-\delta}^0 \xi^2\left(t\right) dt + \mathbb{E}\left[\int_0^{T-\delta} \left|X\left(t\right)\right|^2 dt\right] \end{aligned}$$

$$\leq \delta \max_{t\in[-\delta,0]} \xi^2(t) + \mathbb{E}\left[\int_0^{T-\delta} |X(t)|^2 dt\right],$$

which gives that $X_\delta \in \mathbb{L}^{22}_{\mathcal{F}}$ by noting both (2.3.3) and the fact that $X \in \mathbb{L}^{22}_{\mathcal{F}}$. Since b is continuous with respect to (x, y, u), $b(t, X(t), X_\delta(t), u(t))$ is continuous $\mathcal{F}(t)$-adapted (see [39, Proposition 3.7]). On the other hand, it follows from (3.2.6) and from the triangle inequality that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T |b(t, X(t), X_\delta(t), u(t))|^2 dt\right] \\
\leq & 4\mathbb{E}\left[\int_0^T \left\{a_1^2 |X(t)|^2 + b_1^2 |X_\delta(t)|^2\right\} dt\right] + 2c_1^2 \mathbb{E}\left[\int_0^T |u(t)|^2 dt\right].
\end{aligned}$$

This implies that $b(\cdot) = b(\cdot, X(\cdot), X_\delta(\cdot), u(\cdot))$ is square-integrable by noting $(X, X_\delta, u) \in \mathbb{L}^{22}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$. Similarly, $\sigma(\cdot) = \sigma(\cdot, X(\cdot), X_\delta(\cdot), u(\cdot))$ is also square-integrable. Furthermore, by the Cauchy-Schwarz Inequality (see [39, page 142]),

$$\begin{aligned}
&\mathbb{E}\left[\left\{\int_0^T |b(t, X(t), X_\delta(t), u(t))| dt\right\}^2\right] \\
\leq & \mathbb{E}\left[\left\{\left\{\int_0^T |b(t, X(t), X_\delta(t), u(t))|^2 dt\right\}^{\frac{1}{2}} \left\{\int_0^T 1 dt\right\}^{\frac{1}{2}}\right\}^2\right] \\
= & T\mathbb{E}\left[\int_0^T |b(t, X(t), X_\delta(t), u(t))|^2 dt\right] < \infty,
\end{aligned} \tag{3.2.7}$$

which gives $b(\cdot) \in \mathbb{L}^{21}_{\mathcal{F}}$ by noting (2.3.2). Hence, the strong solution X of the controlled SDDE is identified with $(b(\cdot), \sigma(\cdot)) \in \mathbb{V}_1$ via Definition 2.3.1 and then the conclusion of $X(T) \in \mathbb{L}^2$ follows from Proposition 2.3.2.

(II) Verifying Assumptions I, II & III

It follows from (3.2.3) and (3.2.5) that l is given by

$$l(x) = \frac{1}{2} a_3 x^2. \tag{3.2.8}$$

Apparently, it is strictly greater than $-\infty$, not equal to ∞ and is a convex continuous function. Moreover, since it is continuous and is independent of $\omega \in \Omega$, l is $\mathcal{F} \times \mathcal{B}(\mathbb{R})$-measurable, so that l satisfies Assumption I (see [39, Proposition 3.3]). Similarly, it follows from (3.2.2), (3.2.4) and (3.2.5) that L is obtained by

$$L(t, x, y, z, h) = \begin{cases} \frac{c_3}{2c_1^2}(z - a_1 x - b_1 y)^2, & \text{if } (x, y, z, h) \in \mathbb{D}, \\ \infty, & \text{otherwise,} \end{cases} \tag{3.2.9}$$

where

$$\mathbb{D} = \left\{ (x, y, z, h) \in \mathbb{R}^4 \middle| c_2 (z - a_1 x - b_1 y) = c_1 (h - a_2 x - b_2 y) \right\}.$$

It is easy to see that L is not equal to ∞ identically and, when it is finite, L is continuous so that it is lower semi-continuous. Moreover, $\mathbb{D}$ is a convex set and L defined by (3.2.9) is a quadratic function for $(x, y, z, h) \in \mathbb{D}$ with a positive coefficient $c_3/(2c_1^2)$, so that L is a convex function with respect to (x, y, z, h). Furthermore, since it is independent of $(\omega, t) \in \Omega \times [0, T]$, L is a normal convex integrand using [34, Lemma 1]. Hence, L satisfies Assumption I by noting Remark 2.3.3.

Clearly, L and l defined by (3.2.9) and (3.2.8) are bounded below. Hence, they satisfy Assumption II.

Define $(\dot{X}, H_X) \in \mathbb{V}_1$ by

$$\begin{cases} \dot{X}(t) = b(t, X(t), X_\delta(t), u(t)), \\ H_X(t) = \sigma(t, X(t), X_\delta(t), u(t)), \end{cases}$$

where X is the strong solution of the controlled SDDE (3.1.7) with b and σ so defined. Moreover, as noted in (3.2.1), $\mathcal{C}\left(t, X(t), X_\delta(t), \dot{X}(t), H_X(t)\right)$ is not empty $\mathbb{P} \otimes Leb$-a.s. which implies that

$$\begin{aligned} &L\left(t, X(t), X_\delta(t), \dot{X}(t), H_X(t)\right) \\ \leq &\frac{3c_3}{2c_1^2} \left\{ \left|\dot{X}(t)\right|^2 + a_1^2 |X(t)|^2 + b_1^2 |X_\delta(t)|^2 \right\}. \end{aligned}$$

Taking τ to be $\frac{3c_3}{2c_1^2}\{|\dot{X}(t)|^2 + a_1^2 |X(t)|^2 + b_1^2 |X_\delta(t)|^2\}$, we see that τ satisfies (2.3.19) by noting that $(X, X_\delta, u) \in \mathbb{L}^{22}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ obtained in part (I) so that L satisfies Assumption III. Similarly, we take χ to be $a_3 X^2(T)/2$ which satisfies (2.3.21) by noting that $X(T) \in \mathbb{L}^2$ obtained in part (I). Thus, l satisfies Assumption III.

Since Assumptions I, II & III hold, the control problem (3.1.9), where b, σ, G and g are respectively defined by (3.2.4) and (3.2.5), can be reformulated to the convex (primal) problem (2.3.26) as

$$\inf_{X \in \mathbb{V}_1} \mathbb{E}\left[\int_0^T \frac{c_3}{2c_1^2} \left(\dot{X}(t) - a_1 X(t) - b_1 X_\delta(t)\right)^2 dt + \frac{1}{2} a_3 X^2(T)\right], \tag{3.2.10}$$

subject to, $\mathbb{P} \otimes Leb$-a.s.

$$c_2\left(\dot{X}(t) - a_1 X(t) - b_1 X_\delta(t)\right) = c_1\left(H_X(t) - a_2 X(t) - b_2 X_\delta(t)\right),$$

where X is identified with $(\dot{X}, H_X) = (b(\cdot), \sigma(\cdot)) \in \mathbb{V}_1$ via Definition 2.3.1. Then, the control u can be re-expressed in terms of $X \in \mathbb{V}_1$ as

$$u(t) = \frac{1}{c_1}\left(\dot{X}(t) - a_1 X(t) - b_1 X_\delta(t)\right). \tag{3.2.11}$$

Since $X, X_\delta \in \mathbb{L}^{2\infty}_{\mathcal{F}}$ and $\dot{X} \in \mathbb{L}^{21}_{\mathcal{F}}$, $u \in \mathcal{U} = \mathbb{L}^{22}_{\mathcal{F}}$.

(III) Deriving the Dual Problem

For P identified with $(P_T, \dot{P}) \in \mathbb{V}_2$ using (2.4.17), it follows from (2.4.2) and from the terminal term in the dual function (2.4.24) that

$$l^*(-P_T) = \sup_{x \in \mathbb{R}}\left\{-P_T x - l(x)\right\} = \sup_{x \in \mathbb{R}}\left\{-P_T x - \frac{1}{2}a_3 x^2\right\} = \frac{P_T^2}{2a_3}.$$

On the other hand, it follows from (2.4.1) and (2.4.24) that

$$\begin{aligned}
&L^*\left(t, \dot{P}(t) - \mathbb{E}\left[\dot{Q}(t+\delta) I_{[0,T-\delta]}(t) \,\middle|\, \mathcal{F}(t)\right], \dot{Q}(t), P(t), H_P(t)\right) \\
&= \sup_{(x,y,z,h) \in \mathbb{R}^4}\left\{x\left(\dot{P}(t) - \mathbb{E}\left[\dot{Q}(t+\delta) I_{[0,T-\delta]}(t) \,\middle|\, \mathcal{F}(t)\right]\right) + y\dot{Q}(t) + zP(t)\right. \\
&\qquad\qquad \left. + hH_P(t) - L(t, x, y, z, h)\right\} \\
&= \sup_{(x,y) \in \mathbb{R}^2}\left\{x\left(\dot{P}(t) - \mathbb{E}\left[\dot{Q}(t+\delta) I_{[0,T-\delta]}(t) \,\middle|\, \mathcal{F}(t)\right]\right) + y\dot{Q}(t)\right. \\
&\qquad\qquad \left. + (a_1 x + b_1 y) P(t) + (a_2 x + b_2 y) H_P(t)\right\} \\
&\quad + \sup_{u \in \mathbb{R}}\left\{u\left(c_1 P(t) + c_2 H_P(t)\right) - \frac{1}{2}c_3 u^2\right\}
\end{aligned} \tag{3.2.12}$$

for $(P, \dot{Q}) \in \mathbb{V}_2 \times \mathbb{L}^{21}_{\mathcal{F}}$. To find the explicit expression for L^*, we first take the derivatives, with respect to x and y respectively, of the function within the first bracket on the right-hand-side of the second equality of (3.2.12). Then, the corresponding derivatives are zero if and only if

$$\begin{cases}
\dot{P}(t) = \mathbb{E}\left[\dot{Q}(t+\delta) I_{[0,T-\delta]}(t) \,\middle|\, \mathcal{F}(t)\right] - a_1 P(t) - a_2 H_P(t), \\
\dot{Q}(t) = -b_1 P(t) - b_2 H_P(t).
\end{cases} \tag{3.2.13}$$

Since the function within the first bracket on the right-hand-side of the second equality of (3.2.12) is linear in (x, y), the value for the corresponding supremum is zero if (3.2.13) holds otherwise it is ∞ identically. Moreover, as $u(c_1P(t) + c_2H_P(t)) - c_3u^2/2$ is concave with respect to u, the second supremum on the right-hand-side of the second equality of (3.2.12) is attained at

$$u = \frac{1}{c_3}\left(c_1 P\left(t\right) + c_2 H_P\left(t\right)\right). \tag{3.2.14}$$

Hence,

$$\begin{aligned}
&L^*\left(t, \dot{P}\left(t\right) - \mathbb{E}\left[\dot{Q}\left(t+\delta\right) I_{[0,T-\delta]}\left(t\right) \middle| \mathcal{F}\left(t\right)\right], \dot{Q}\left(t\right), P\left(t\right), H_P\left(t\right)\right) \\
&= \begin{cases} \dfrac{1}{2c_3}\left(c_1 P\left(t\right) + c_2 H_P\left(t\right)\right)^2, & \text{if (3.2.13) holds,} \\ \infty, & \text{otherwise.} \end{cases}
\end{aligned}$$

Therefore, the dual problem to (3.2.10) is to find $(P, \dot{Q}) \in \mathbb{V}_2 \times \mathbb{L}^{21}_{\mathcal{F}}$ realizing

$$\begin{aligned}
\inf_{(P,\dot{Q})\in\mathbb{V}_2\times\mathbb{L}^{21}_{\mathcal{F}}} \quad & \left\{\mathbb{E}\left[\int_0^T \frac{1}{2c_3}\left(c_1 P\left(t\right) + c_2 H_P\left(t\right)\right)^2 dt\right] + \mathbb{E}\left[\frac{1}{2a_3}P_T^2\right]\right. \\
& + \xi\left(0\right)\mathbb{E}\left[P_T\right] - \mathbb{E}\left[\int_0^T \dot{Q}\left(t\right)\xi\left(t-\delta\right) I_{[0,\delta]}\left(t\right) dt\right] \\
& \left. - \xi(0)\mathbb{E}\left[\int_0^T \dot{P}\left(t\right) dt\right]\right\},
\end{aligned} \tag{3.2.15}$$

subject to (3.2.13), where P is identified with $(P_T, \dot{P}) \in \mathbb{V}_2$ using (2.4.17) and H_P is specified by (2.4.18). Note that, by (3.2.14), the control u can be re-expressed in terms of $P \in \mathbb{V}_2$ as

$$u\left(t\right) = \frac{1}{c_3}\left(c_1 P\left(t\right) + c_2 H_P\left(t\right)\right). \tag{3.2.16}$$

Since $(P, H_P) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$, $u \in \mathcal{U} = \mathbb{L}^{22}_{\mathcal{F}}$.

(IV) Applying Theorem 2.5.2 (iii)

To apply Theorem 2.5.2 (iii), we see that if $\bar{X} \in \mathbb{V}_1$, identified with $(\dot{\bar{X}}, H_{\bar{X}}) \in \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$, and $(\bar{P}, \dot{\bar{Q}}) \in \mathbb{V}_2 \times \mathbb{L}^{21}_{\mathcal{F}}$, where $\bar{P}$ is identified with $(\bar{P}_T, H_{\bar{P}}) \in \mathbb{L}^2 \times \mathbb{L}^{22}_{\mathcal{F}}$, satisfy, $\mathbb{P} \otimes Leb$-a.s.

$$c_2\left(\dot{\bar{X}}\left(t\right) - a_1\bar{X}\left(t\right) - b_1\bar{X}_\delta\left(t\right)\right) = c_1\left(H_{\bar{X}}\left(t\right) - a_2\bar{X}\left(t\right) - b_2\bar{X}_\delta\left(t\right)\right),$$

$$\begin{cases} \dot{\bar{P}}\left(t\right) = \mathbb{E}\left[\dot{\bar{Q}}\left(t+\delta\right) I_{[0,T-\delta]}\left(t\right) \middle| \mathcal{F}\left(t\right)\right] - a_1\bar{P}\left(t\right) - a_2 H_{\bar{P}}\left(t\right), \\ \dot{\bar{Q}}\left(t\right) = -b_1\bar{P}\left(t\right) - b_2 H_{\bar{P}}\left(t\right), \end{cases}$$

and

$$\begin{cases} \bar{X}(T) = -\frac{1}{a_3}\bar{P}_T, \\ \frac{1}{c_1}\left(\dot{\bar{X}}(t) - a_1\bar{X}(t) - b_1\bar{X}_\delta(t)\right) = \frac{1}{c_3}\left(c_1\bar{P}(t) + c_2H_{\bar{P}}(t)\right), \end{cases}$$

then the two equalities (2.5.7) and (2.5.8) are satisfied. It follows from (3.2.11) and (3.2.16) that the associated control $\bar{u}$ is expressed by

$$\bar{u}(t) = \frac{1}{c_3}\left(c_1\bar{P}(t) + c_2H_{\bar{P}}(t)\right) \text{ or } \frac{1}{c_1}\left(\dot{\bar{X}}(t) - a_1\bar{X}(t) - b_1\bar{X}_\delta(t)\right). \tag{3.2.17}$$

Therefore, $\bar{X}$ and $(\bar{P}, \dot{\bar{Q}})$ are optimal solutions of the primal problem (3.2.10) and the dual problem (3.2.15) respectively. This implies that the control $\bar{u}$ given by (3.2.17) is an optimal control of the stochastic control problem with discrete delay (3.1.9) with b, σ, G and g defined by (3.2.4) and (3.2.5). □

3.2.2 The General Case

For b, σ, G and g, to ensure that the stochastic optimal control problem with discrete delay (3.1.9) can be formulated as a stochastic convex problem with discrete delay (2.3.26), we make the following hypotheses.

Hypothesis I. $\mathbb{U}$ is a nonempty convex compact subset of $\mathbb{R}^n$. The functions b and σ are continuous with respect to $(t, u) \in [0, T] \times \mathbb{U}$; and are Lipschitz continuous with respect to $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with the Lipschitz constant independent of (t, u). Moreover, there exists a constant $C_1 > 0$ such that

$$|b(t, 0, 0, u)| + |\sigma(t, 0, 0, u)| \leq C_1, \quad \forall (t, u) \in [0, T] \times \mathbb{U}.$$

Hypothesis II. The functions G and g are continuous. Moreover, g is convex and there exist constants $C_2 \in \mathbb{R}$ and $C_3 > 0$ such that

$$\begin{cases} C_2 \leq G(t, x, y, u) \leq C_3\left(1 + |x|^2 + |y|^2\right), & \forall t \in [0, T], x, y \in \mathbb{R}^n, u \in \mathbb{U}, \\ C_2 \leq g(x) \leq C_3\left(1 + |x|^2\right), & \forall x \in \mathbb{R}^n. \end{cases}$$

These hypotheses ensure that the control SDDE (3.1.7) admits a unique strong solution and the cost function (3.1.8) is integrable for every $u \in \mathcal{U}$. More importantly, the following propositions show that, under the above hypotheses, if L is a convex function as required in Assumption I (i), then L and l defined by (3.2.2) and (3.2.3) satisfy Assumptions I, II & III.

Proposition 3.2.2. *Under Hypotheses I & II, the functions L and l defined by* (3.2.2) *and* (3.2.3) *satisfy Assumption I provided L is convex with respect to (x, y, z, h).*

Proof. It follows from Hypothesis II and (3.2.3) that l is a continuous convex function and is not equal to ∞. Similarly to the argument used in part (II) of Example 3.2.1, l is $\mathcal{F} \times \mathcal{B}(\mathbb{R}^n)$-measurable so that l satisfies Assumption I.

As noted in [2, page 393], the continuity of b, σ, G and g under Hypotheses I & II guarantees that L is lower semi-continuous. To show L is a normal convex integrand, let $\{(x_i, y_i, v_i)\}_{i\in\mathbb{N}^+}$ be a countable collection which is dense in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U}$ (see [39, Section 9.6]). Note that such a collection exists since Euclidean and compact metric spaces are separable (see [39, Theorem 9.24]). Using this collection, for every $t \in [0, T]$, we define the collections $\{z_i(t)\}_{i\in\mathbb{N}^+}$ and $\{h_i(t)\}_{i\in\mathbb{N}^+}$ respectively by

$$z_i\left(t\right) = b\left(t, x_i, y_i, v_i\right) \quad \text{and} \quad h_i\left(t\right) = \sigma\left(t, x_i, y_i, v_i\right). \tag{3.2.18}$$

Then, $\{z_i(t)\}_{i\in\mathbb{N}^+}$ and $\{h_i(t)\}_{i\in\mathbb{N}^+}$ are dense in the ranges of $b(t, \cdot)$ and $\sigma(t, \cdot)$ respectively by the continuity of b and σ under Hypothesis I so that the collection $\{(x_i, y_i, z_i(t), h_i(t))\}_{i\in\mathbb{N}^+}$ is dense in the effective domain $\mathbb{D}(\omega, t)$ defined by (2.3.23) for every $(\omega, t) \in \Omega \times [0, T]$. Let $\{(X_i, Y_i, Z_i, H_i)\}_{i\in\mathbb{N}^+}$ be a countable collection, where X_i, Y_i, Z_i and H_i are $\mathcal{F}(t)$-progressively measurable stochastic processes satisfying $X_i(\omega, t) \equiv x_i$, $Y_i(\omega, t) \equiv y_i$, $Z_i(\omega, t) \equiv z_i(t)$ and $H_i(\omega, t) \equiv h_i(t)$. Moreover, it follows from (3.2.1) and (3.2.18) that $\mathcal{C}_i = \mathcal{C}_i\left(t, X_i(\omega, t), Y_i(\omega, t), Z_i(\omega, t), H_i(\omega, t)\right)$ is not empty for each $i \in \mathbb{N}^+$ which implies that the countable collection

$$\left\{\left(X_i\left(\omega, t\right), Y_i\left(\omega, t\right), Z_i\left(\omega, t\right), H_i\left(\omega, t\right)\right)\right\}_{i\in\mathbb{N}^+} \tag{3.2.19}$$

is a subset of $\mathbb{D}(\omega, t)$ for every $(\omega, t) \in \Omega \times [0, T]$. Therefore, $\{(X_i, Y_i, Z_i, H_i)\}_{i\in\mathbb{N}^+}$ is a countable collection required in Remark 2.3.3 such that (3.2.19) is dense in (2.3.22) for each $(\omega, t) \in \Omega \times [0, T]$.

Furthermore, for each $i \in \mathbb{N}^+$, let $\{u_j^{(i)}\}_{j\in\mathbb{N}^+}$ be a countable collection of $\mathcal{F}(t)$-progressively measurable stochastic processes such that $\{u_j^{(i)}(\omega, t)\}_{j\in\mathbb{N}^+}$ is dense in

$\mathcal{C}_i$ for every $(\omega, t) \in \Omega \times [0, T]$. Such a collection exists since $\mathcal{C}_i$ is a nonempty subset of $\mathbb{U} \subset \mathbb{R}^r$, and rational numbers are countable and dense in $\mathbb{R}$. Then, as noted in [2, page 393], we have

$$\begin{aligned} & L\left(t, X_i\left(\omega, t\right), Y_i\left(\omega, t\right), Z_i\left(\omega, t\right), H_i\left(\omega, t\right)\right) \\ = & \inf_{j \in \mathbb{N}^+} G\left(t, X_i\left(\omega, t\right), Y_i\left(\omega, t\right), u_j^{(i)}\left(\omega, t\right)\right) \end{aligned}$$

by the continuity of G. Then, $L(t, X_i(\omega, t), Y_i(\omega, t), Z_i(\omega, t), H_i(\omega, t))$ is the pointwise infimum of a countable family of measurable functions so that it is $\mathcal{F}^*$-measurable (see [12, page 71]). Therefore, under the given condition that L is convex with respect to (x, y, z, h), L is a normal convex integrand. Hence, it satisfies Assumption I by noting Remark 2.3.3. □

Proposition 3.2.3. *Under Hypotheses I & II, the functions L and l defined by* (3.2.2) *and* (3.2.3) *satisfy Assumptions II & III.*

Proof. By Hypothesis II, G and g are bounded below, which implies that L and l are bounded below. Hence, L and l satisfy Assumption II.

Fix $\hat{u} \in \mathcal{U}$. Similarly to the argument used in part (I) of Example 3.2.1, Hypothesis I implies that the corresponding SDDE (3.1.7) admits the unique strong solution $\hat{X}$ with $\hat{X}, \hat{X}_\delta \in \mathbb{L}^{22}_{\mathcal{F}}$. Now, for such $(\hat{X}, \hat{u})$, we define stochastic processes $\dot{\hat{X}}$ and $H_{\hat{X}}$ respectively by

$$\begin{cases} \dot{\hat{X}}\left(t\right) = b\left(t, \hat{X}\left(t\right), \hat{X}_\delta\left(t\right), \hat{u}\left(t\right)\right), \\ H_{\hat{X}}\left(t\right) = \sigma\left(t, \hat{X}\left(t\right), \hat{X}_\delta\left(t\right), \hat{u}\left(t\right)\right), \end{cases}$$

which implies that, for any $(\omega, t) \in \Omega \times [0, T]$,

$$\hat{\mathcal{C}} = \mathcal{C}\left(t, \hat{X}\left(t\right), \hat{X}_\delta\left(t\right), \dot{\hat{X}}\left(t\right), H_{\hat{X}}\left(t\right)\right) \neq \emptyset. \tag{3.2.20}$$

On the other hand, it follows from the triangle inequality and Hypothesis I that, there is a constant $C > 0$ such that

$$\begin{aligned} \left|b\left(t, \hat{X}\left(t\right), \hat{X}_\delta\left(t\right), \hat{u}\left(t\right)\right)\right| \leq & \left|b\left(t, \hat{X}\left(t\right), \hat{X}_\delta\left(t\right), \hat{u}\left(t\right)\right) - b\left(t, 0, 0, \hat{u}\left(t\right)\right)\right| \\ & + \left|b\left(t, 0, 0, \hat{u}\left(t\right)\right)\right| \\ \leq & C\left(\left|\hat{X}\left(t\right)\right| + \left|\hat{X}_\delta\left(t\right)\right|\right) + C_1. \end{aligned}$$

This gives that

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \left|b\left(t,\hat{X}\left(t\right),\hat{X}_\delta\left(t\right),\hat{u}\left(t\right)\right)\right|^2 dt\right]\\ \leq& 4C^2\mathbb{E}\left[\int_0^T \left\{\left|\hat{X}\left(t\right)\right|^2+\left|\hat{X}_\delta\left(t\right)\right|^2\right\} dt\right]+2TC_1^2\end{aligned}$$

so that $\dot{\hat{X}}$ belongs to $\mathbb{L}^{22}_{\mathcal{F}}$. Similarly, we have $H_{\hat{X}} \in \mathbb{L}^{22}_{\mathcal{F}}$. Similarly to (3.2.7), we also have that $\dot{\hat{X}} \in \mathbb{L}^{21}_{\mathcal{F}}$ by the Cauchy-Schwarz Inequality. Hence, $\hat{X}$ is identified with $(\dot{\hat{X}}, H_{\hat{X}})$ via Definition 2.3.1. Then, by Proposition 2.3.2, we have that $\hat{X}(T) \in \mathbb{L}^2$. Since the set $\hat{\mathcal{C}}$ defined by (3.2.20) is not empty and since Hypothesis II holds, noting (3.2.2) and (3.2.3), we have that

$$\begin{aligned}&L\left(t,\hat{X}\left(t\right),\hat{X}_\delta\left(t\right),\dot{\hat{X}}\left(t\right),H_{\hat{X}}\left(t\right)\right)\\ =&\inf_{u\in\hat{\mathcal{C}}} G\left(t,\hat{X}\left(t\right),\hat{X}_\delta\left(t\right),u\right) \leq C_3\left(1+\left|\hat{X}\left(t\right)\right|^2+\left|\hat{X}_\delta\left(t\right)\right|^2\right),\quad \mathbb{P}\otimes Leb-a.s.\end{aligned}$$

and

$$l\left(\hat{X}\left(T\right)\right)=g\left(\hat{X}\left(T\right)\right)\leq C_3\left(1+\left|\hat{X}\left(T\right)\right|^2\right),\quad \mathbb{P}-a.s.$$

Then, taking τ and χ in Assumption III to be $C_3(1+|\hat{X}(t)|^2+|\hat{X}_\delta(t)|^2)$ and $C_3(1+|\hat{X}(T)|^2)$ respectively, we see that τ and χ satisfy (2.3.19) and (2.3.21) respectively since $\hat{X}, \hat{X}_\delta \in \mathbb{L}^{22}_{\mathcal{F}}$ and $\hat{X}(T) \in \mathbb{L}^2$. Therefore, L and l satisfy Assumption III. □

In the following example, we demonstrate that there exists a control problem (3.1.9) such that the corresponding b, σ, G and g satisfy Hypotheses I & II, where at least one of b and σ is not an affine function of (x, y, u).

Example 3.2.4. For simplicity, we set $n = m = r = 1$. Suppose that $\mathbb{U} = [0, 2\pi]$; that b and σ in (3.1.7) are given by

$$\begin{cases} b\left(t,x,y,u\right)=\sin\left(x+y+u\right), \\ \sigma\left(t,x,y,u\right)=y; \end{cases} \tag{3.2.21}$$

and that G and g in (3.1.8) are given by

$$G\left(t,x,y,u\right)=\left|x+\sin\left(x+y+u\right)\right| \text{ and } g\left(x\right)=x^2. \tag{3.2.22}$$

For every $x, y, x', y' \in \mathbb{R}$, it follows from (3.2.21) that

$$\begin{aligned}&\left|\sin\left(x+y+u\right)-\sin\left(x'+y'+u\right)\right|+\left|y-y'\right|\\ \leq&\left|x-x'\right|+2\left|y-y'\right|,\quad \forall u\in[0,2\pi],\end{aligned}$$

which implies that b and σ so defined are Lipschtiz continuous. It is easy to see that $b(t,0,0,u)$ and $\sigma(t,0,0,u)$ are bounded. Hence, b and σ satisfy Hypothesis I. On the other hand, similarly to Example 3.2.1, G and g so defined are continuous and are bounded below. Moreover, it follows from (3.2.22) that g is convex satisfying $g(x) = x^2 \leq c(1+x^2)$ for any $c \geq 1$ and that G satisfies

$$|x + \sin(x+y+u)| \leq |x| + 1 \leq \frac{3}{2}\left(x^2 + y^2 + 1\right).$$

Thus, Hypothesis II holds. Now, the set $\mathcal{C}$ is defined by

$$\mathcal{C}(t,x,y,z,h) = \{u \in [0,2\pi] \mid z = \sin(x+y+u) \text{ and } h = y\}$$

and $\mathcal{C}(t,x,y,z,h) \neq \emptyset$ if and only if $|z| \leqslant 1$ and $y = h$. This gives that

$$L(t,x,y,z,h) = \begin{cases} |x+z|, & \text{if } |z| \leqslant 1 \text{ and } h = y, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.2.23}$$

Similarly to part (II) of Example 3.2.1, the effective domain in (3.2.23) is convex so that it is easy to see that L is a convex function with respect to (x,y,z,h). Hence, by Propositions 3.2.2 & 3.2.3, the control problem (3.1.9) with b, σ, G and g so defined can be reformulated as the convex problem (2.3.26) with L and l respectively defined by (3.2.2) and (3.2.3). □

To end this subsection, we turn our attention to the convexity of L. The following proposition illustrates that it holds at least under certain conditions on b, σ and G in (3.1.7) and (3.1.8) respectively.

Proposition 3.2.5. *Let $\mathcal{H}_d : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}$ be defined by*

$$\mathcal{H}_d\left(t,x,y,u,p,h_p\right) = \langle b(t,x,y,u), p\rangle + \langle h_p, \sigma(t,x,y,u)\rangle - G(t,x,y,u). \tag{3.2.24}$$

If $\mathcal{H}_d$ is concave with respect to (x,y,u), then the function L defined by (3.2.2) is convex with respect to (x,y,z,h).

Proof. Let

$$\tilde{L}(t,x,y,z,h) = \inf_{u\in\mathbb{U}} \left\{ \sup_{(p,h_p)\in\mathbb{R}^n\times\mathbb{R}^{n\times m}} \left\{\langle z,p\rangle + \langle h,h_p\rangle - \mathcal{H}_d\left(t,x,y,u,p,h_p\right)\right\} \right\}. \tag{3.2.25}$$

Then, the expression (3.2.24) for $\mathcal{H}_d$ gives

$$\begin{aligned}&\tilde{L}\left(t,x,y,z,h\right)\\&=\inf_{u\in\mathbb{U}}\Big\{G\left(t,x,y,u\right)+\sup_{\left(p,h_p\right)\in\mathbb{R}^n\times\mathbb{R}^{n\times m}}\Big\{\left\langle z-b\left(t,x,y,u\right),p\right\rangle\\&\qquad\qquad+\left\langle h_p,h-\sigma\left(t,x,y,u\right)\right\rangle\Big\}\Big\}.\end{aligned}\tag{3.2.26}$$

For $\mathcal{C}=\mathcal{C}(t,x,y,z,h)$ as defined in (3.2.1), if $\mathcal{C}=\emptyset$, then

$$\left(z-b\left(t,x,y,u\right),h-\sigma\left(t,x,y,u\right)\right)\neq\left(0,0\right)$$

and so the supremum in (3.2.26) is ∞, which implies that $\tilde{L}=\infty$. Otherwise, $\tilde{L}(t,x,y,z,h)=\inf_{u\in\mathcal{C}}G\left(t,x,y,u\right)$. Hence, $\tilde{L}=L$, where L is defined by (3.2.2).

Now, since $\mathcal{H}_d$ is linear with respect to (p,h_p),

$$\langle z,p\rangle+\left\langle h,h_p\right\rangle-\mathcal{H}_d\left(t,x,y,u,p,h_p\right)$$

is convex with respect to (u,p,h_p). Then, the order of the supremum and the infimum on the right-hand-side of (3.2.25) can be exchanged (see [35, Corollary 37.2.2]) so that

$$L\left(t,x,y,z,h\right)=\sup_{\left(p,h_p\right)\in\mathbb{R}^n\times\mathbb{R}^{n\times m}}\left\{\langle z,p\rangle+\left\langle h,h_p\right\rangle-\hat{\mathcal{H}}_d\left(t,x,y,p,h_p\right)\right\},\tag{3.2.27}$$

where $\hat{\mathcal{H}}_d(t,x,y,p,h_p)=\sup_{u\in\mathbb{U}}\mathcal{H}_d\left(t,x,y,u,p,h_p\right)$. Then, given any constant $\epsilon>0$, we can find $u,u'\in\mathbb{U}$ associated with (x,y) and (x',y') respectively such that

$$\hat{\mathcal{H}}_d\left(t,x,y,p,h_p\right)-\epsilon\leq\mathcal{H}_d\left(t,x,y,u,p,h_p\right)$$

and

$$\hat{\mathcal{H}}_d\left(t,x',y',p,h_p\right)-\epsilon\leq\mathcal{H}_d\left(t,x',y',u',p,h_p\right).$$

Then, by taking weighted sum of above inequalities and noting that $\mathcal{H}_d$ is concave with respect to (x,y,u), we have

$$\begin{aligned}&\lambda\hat{\mathcal{H}}_d\left(t,x,y,p,h_p\right)+\left(1-\lambda\right)\hat{\mathcal{H}}_d\left(t,x',y',p,h_p\right)-\epsilon\\\leq&\lambda\mathcal{H}_d\left(t,x,y,u,p,h_p\right)+\left(1-\lambda\right)\mathcal{H}_d\left(t,x',y',u',p,h_p\right)\\\leq&\mathcal{H}_d\left(t,\lambda x+\left(1-\lambda\right)x',\lambda y+\left(1-\lambda\right)y',\lambda u+\left(1-\lambda\right)u',p,h_p\right)\\\leq&\hat{\mathcal{H}}_d\left(t,\lambda x+\left(1-\lambda\right)x',\lambda y+\left(1-\lambda\right)y',p,h_p\right),\end{aligned}$$

where $\lambda \in [0,T]$. This implies that $\hat{\mathcal{H}}_d$ is concave with respect to (x,y) since the above inequalities hold for every $\epsilon > 0$. Consequently, for every $x, x', y, y', z, z' \in \mathbb{R}^n$ and $h, h' \in \mathbb{R}^{n\times m}$, it follows from (3.2.27) that

$$
\begin{aligned}
& L\left(t, \lambda x + (1-\lambda) x', \lambda y + (1-\lambda) y', \lambda z + (1-\lambda) z', \lambda h + (1-\lambda) h'\right) \\
\leq & \sup_{\left(p,h_p\right)\in\mathbb{R}^n\times\mathbb{R}^{n\times m}} \Big\{ \left\langle \lambda z + (1-\lambda) z', p\right\rangle + \left\langle \lambda h + (1-\lambda) h', h_p\right\rangle \\
& \qquad\qquad - \lambda \hat{\mathcal{H}}_d\left(t,x,y,p,h_p\right) - (1-\lambda)\hat{\mathcal{H}}_d\left(t,x',y',p,h_p\right) \Big\} \\
\leq & \lambda \sup_{\left(p,h_p\right)\in\mathbb{R}^n\times\mathbb{R}^{n\times m}} \left\{ \langle z, p\rangle + \left\langle h, h_p\right\rangle - \hat{\mathcal{H}}_d\left(t,x,y,p,h_p\right)\right\} \\
& + (1-\lambda) \sup_{\left(p,h_p\right)\in\mathbb{R}^n\times\mathbb{R}^{n\times m}} \left\{ \left\langle z', p\right\rangle + \left\langle h', h_p\right\rangle - \hat{\mathcal{H}}_d\left(t,x',y',p,h_p\right)\right\} \\
= & \lambda L(t,x,y,z,h) + (1-\lambda) L\left(t,x',y',z',h'\right),
\end{aligned}
$$

as required. □

3.3 A Stochastic Maximum Principle

Since the control problem (3.1.9) can be reformulated as a particular convex problem (2.3.26), this section first generalizes [2, Theorem V-1] to derive certain necessary conditions for optimality of the control problem, where the corresponding Hamiltonian and adjoint equation are involved. Then, using these Hamiltonian and adjoint equation, we give an alternative proof for the corresponding sufficient stochastic maximum principle using the results of conjugate duality obtained in chapter 2.

3.3.1 Derivation of the Hamiltonian and Adjoint Equation

In what follows, if $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is a continuously differential function, the partial derivative $\frac{\partial F}{\partial x}$ represents the vector $(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \ldots, \frac{\partial F}{\partial x_n})^\top$. For the control problem (3.1.9), we define the processes $(P, H_P) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ by the adjoint equation

$$
\begin{cases}
dP(t) = -\left\{ \dfrac{\partial \mathcal{H}_d}{\partial x}(t) + \mathbb{E}\left[\dfrac{\partial \mathcal{H}_d}{\partial y}(t+\delta) I_{[0,T-\delta]}(t) \,\middle|\, \mathcal{F}(t)\right]\right\} dt \\
\qquad\qquad + H_P(t)\, dW(t), \quad t \in [0,T], \\
P(T) = -\dfrac{\partial g}{\partial x}(X(T)),
\end{cases}
\tag{3.3.1}
$$

where $\mathcal{H}_d$ is the Hamiltonian defined by (3.2.24) with certain necessary differentiability condition (specified below); and the shorthand notation $\frac{\partial \mathcal{H}_d}{\partial x}(t)$, as well as $\frac{\partial \mathcal{H}_d}{\partial y}(t)$, is defined as that in (3.1.12). Note that this adjoint equation is a controlled anticipated BSDE. Note also that, recalling (3.1.10) and (3.1.11), these Hamiltonian and adjoint equation coincide with those introduced in [6]. Rather than introducing them as that in [6], the following theorem uses Theorem 2.5.2 (iii) to derive these Hamiltonian and adjoint equation.

Theorem 3.3.1. *Assume that Hypotheses I & II hold and that the function L defined by* (3.2.2) *is convex with respect to (x, y, z, h). In addition, assume that b, σ, G and g in* (3.1.7) *and* (3.1.8) *are continuously differentiable with respect to (x, y) and x respectively. Suppose that $\bar{X} \in \mathbb{V}_1$ and $(\bar{P}, \dot{\bar{Q}}) \in \mathbb{V}_2 \times \mathbb{L}^{21}_{\mathcal{F}}$ satisfy* (2.5.7) *and* (2.5.8)*, where the function l is defined by* (3.2.3)*. Then, there exists a $\bar{u} \in \mathcal{U}$ realizing* (3.1.9) *with $(\bar{u}, \bar{X}, \bar{P})$ satisfying*

- *(i) $\bar{X}$ is the strong solution of the controlled SDDE* (3.1.7) *with u replaced by $\bar{u}$;*
- *(ii) $(\bar{P}, H_{\bar{P}})$ solves the adjoint equation* (3.3.1) *with (X, X_δ, u) replaced by $(\bar{X}, \bar{X}_\delta, \bar{u})$, where $H_{\bar{P}}$ is specified by $\bar{P}$ via* (2.4.18)*;*
- *(iii) $\mathbb{P} \otimes Leb$-a.s.,*

$$\begin{aligned} &\mathcal{H}_d\left(t, \bar{X}(t), \bar{X}_\delta(t), \bar{u}(t), \bar{P}(t), H_{\bar{P}}(t)\right) \\ &= \max_{u \in \mathbb{U}} \mathcal{H}_d\left(t, \bar{X}(t), \bar{X}_\delta(t), u, \bar{P}(t), H_{\bar{P}}(t)\right). \end{aligned} \tag{3.3.2}$$

Proof. First, it follows from (2.5.7) that

$$\begin{aligned} &L^*\left(t, \dot{\bar{P}}(t) - \mathbb{E}\left[\dot{\bar{Q}}(t+\delta) I_{[0,T-\delta]}(t) \,|\mathcal{F}(t)\right], \dot{\bar{Q}}(t), \bar{P}(t), H_{\bar{P}}(t)\right) \\ &= \left\langle \bar{X}(t), \dot{\bar{P}}(t) - \mathbb{E}\left[\dot{\bar{Q}}(t+\delta) I_{[0,T-\delta]}(t) \,|\mathcal{F}(t)\right]\right\rangle \\ &\quad + \left\langle \bar{X}_\delta(t), \dot{\bar{Q}}(t)\right\rangle + \left\langle \dot{\bar{X}}(t), \bar{P}(t)\right\rangle + \left\langle H_{\bar{X}}(t), H_{\bar{P}}(t)\right\rangle \\ &\quad - L\left(t, \bar{X}(t), \bar{X}_\delta(t), \dot{\bar{X}}(t), H_{\bar{X}}(t)\right), \quad \mathbb{P} \otimes Leb - a.s. \end{aligned} \tag{3.3.3}$$

where $(\dot{\bar{X}}, H_{\bar{X}}) \in \mathbb{L}^{21}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ is identified with $\bar{X}$ via Definition 2.3.1 and where $H_{\bar{P}} \in \mathbb{L}^{22}_{\mathcal{F}}$ is specified by $\bar{P}$ via (2.4.18). On the other hand, L^* is the conjugate convex function of L so that

$$L^*\left(t, \dot{\bar{P}}(t) - \mathbb{E}\left[\dot{\bar{Q}}(t+\delta) I_{[0,T-\delta]}(t) \,|\mathcal{F}(t)\right], \dot{\bar{Q}}(t), \bar{P}(t), H_{\bar{P}}(t)\right)$$

$$= \sup_{(x,y,z,h)\in\mathbb{R}^{n\times 3}\times\mathbb{R}^{n\times m}} \Big\{ \Big\langle x, \dot{\bar{P}}(t) - \mathbb{E}\Big[\dot{\bar{Q}}(t+\delta) I_{[0,T-\delta]}(t) \,|\mathcal{F}(t)\Big]\Big\rangle + \Big\langle y, \dot{\bar{Q}}(t)\Big\rangle$$
$$+ \langle z, \bar{P}(t)\rangle + \langle h, H_{\bar{P}}(t)\rangle - L(t,x,y,z,h) \Big\} \tag{3.3.4}$$

for $(\omega, t) \in \Omega \times [0, T]$. Then, using expression (3.2.2) for L, we see that the left-hand-side of (3.3.4) can be re-expressed, in terms of b, σ and G, as

$$\begin{aligned} & L^*\Big(t, \dot{\bar{P}}(t) - \mathbb{E}\Big[\dot{\bar{Q}}(t+\delta) I_{[0,T-\delta]}(t)\,|\mathcal{F}(t)\Big], \dot{\bar{Q}}(t), \bar{P}(t), H_{\bar{P}}(t)\Big) \\ = & \sup_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n} \max_{u\in\mathbb{U}} \Big\{ \Big\langle x, \dot{\bar{P}}(t) - \mathbb{E}\Big[\dot{\bar{Q}}(t+\delta) I_{[0,T-\delta]}(t)\,|\mathcal{F}(t)\Big]\Big\rangle + \Big\langle y, \dot{\bar{Q}}(t)\Big\rangle \\ & + \langle b(t,x,y,u), \bar{P}(t)\rangle + \langle \sigma(t,x,y,u), H_{\bar{P}}(t)\rangle \\ & - G(t,x,y,u) \Big\}. \end{aligned} \tag{3.3.5}$$

Now, for the given $\bar{X}$ and $(\bar{P}, \dot{\bar{Q}})$, since $\mathbb{U}$ is compact given in Hypothesis I, (3.3.3) and (3.3.5) together imply that it is necessary that there exists a $\bar{u} \in \mathcal{U}$ such that $(\dot{\bar{X}}, H_{\bar{X}})$ is expressed as

$$\begin{cases} \dot{\bar{X}}(t) = b\left(t, \bar{X}(t), \bar{X}_\delta(t), \bar{u}(t)\right), \\ H_{\bar{X}}(t) = \sigma\left(t, \bar{X}(t), \bar{X}_\delta(t), \bar{u}(t)\right), \end{cases} \quad \mathbb{P}\otimes Leb - a.s. \tag{3.3.6}$$

and that the 'sup max' on the right-hand-side of (3.3.5) is attained at $(\bar{X}(t), \bar{X}_\delta(t), \bar{u}(t))$, $\mathbb{P}\otimes Leb$-a.s. Given that Hypotheses I & II are satisfied and that L defined by (3.2.2) is convex, using Propositions 3.2.2 & 3.2.3, the control problem (3.1.9) can be reformulated as the convex problem (2.3.26) with L and l so defined. By Theorem 2.5.2 (iii), $\bar{X}$ is an optimal solution of the convex problem. Hence, (3.3.6) implies that $\bar{u}$ is an optimal control of the control problem and that $\bar{X}$ is the corresponding strong solution of the controlled SDDE (3.1.7), i.e. $\bar{u}$ realizes (3.1.9) and (i) holds.

Since the 'sup max' on the right-hand-side of (3.3.5) is attained at $(\bar{X}(t), \bar{X}_\delta(t), \bar{u}(t))$, (3.3.3) and (3.3.5) further imply that

$$\begin{aligned} & \langle b(t,\bar{X}(t),\bar{X}_\delta(t),\bar{u}(t)), \bar{P}(t)\rangle + \langle \sigma(t,\bar{X}(t),\bar{X}_\delta(t),\bar{u}(t)), H_{\bar{P}}(t)\rangle \\ & - G(t,\bar{X}(t),\bar{X}_\delta(t),\bar{u}(t)) \\ = & \max_{u\in\mathbb{U}} \Big\{ \langle b(t,\bar{X}(t),\bar{X}_\delta(t),u), \bar{P}(t)\rangle + \langle \sigma(t,\bar{X}(t),\bar{X}_\delta(t),u), H_{\bar{P}}(t)\rangle \\ & - G(t,\bar{X}(t),\bar{X}_\delta(t),u) \Big\}, \quad \mathbb{P}\otimes Leb - a.s. \end{aligned} \tag{3.3.7}$$

and

$$
\begin{aligned}
&\left\langle \bar{X}(t), \dot{\bar{P}}(t) - \mathbb{E}\left[\dot{\bar{Q}}(t+\delta) I_{[0,T-\delta]}(t) \,\middle|\, \mathcal{F}(t)\right]\right\rangle + \left\langle \bar{X}_\delta(t), \dot{\bar{Q}}(t)\right\rangle \\
&+ \langle b(t, \bar{X}(t), \bar{X}_\delta(t), \bar{u}(t)), \bar{P}(t)\rangle + \langle \sigma(t, \bar{X}(t), \bar{X}_\delta(t), \bar{u}(t)), H_{\bar{P}}(t)) \\
&- G(t, \bar{X}(t), \bar{X}_\delta(t), \bar{u}(t)) \\
= &\sup_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n} \Big\{ \left\langle x, \dot{\bar{P}}(t) - \mathbb{E}\left[\dot{\bar{Q}}(t+\delta) I_{[0,T-\delta]}(t) \,\middle|\, \mathcal{F}(t)\right]\right\rangle + \left\langle y, \dot{\bar{Q}}(t)\right\rangle \\
&\qquad + \langle b(t, x, y, \bar{u}(t)), \bar{P}(t)\rangle + \langle \sigma(t, x, y, \bar{u}(t)), H_{\bar{P}}(t)\rangle \\
&\qquad - G(t, x, y, \bar{u}(t)) \Big\}, \quad \mathbb{P}\otimes Leb - a.s.
\end{aligned}
\tag{3.3.8}
$$

In particular, using the expression (3.2.24) for $\mathcal{H}_d$, (3.3.7) implies that (iii) holds.

Under the given condition that b, σ and G are continuously differentiable with respect to (x, y), we take the derivatives, with respect to x and y, of the function within the bracket on the right-hand-side of (3.3.8). Then, the fact that the supremum is attained at $(\bar{X}(t), \bar{X}_\delta(t))$, $\mathbb{P}\otimes Leb$-a.s. implies that

$$
\begin{aligned}
\dot{\bar{P}}(t) = &-\left(\frac{\partial}{\partial x} b(t, \bar{X}(t), \bar{X}_\delta(t), \bar{u}(t))\right)^\top \bar{P}(t) \\
&- \sum_{j=1}^m \left(\frac{\partial}{\partial x}\sigma^{(j)}(t, \bar{X}(t), \bar{X}_\delta(t), \bar{u}(t))\right)^\top H_{\bar{P}}^{(j)}(t) \\
&+ \frac{\partial}{\partial x} G(t, \bar{X}(t), \bar{X}_\delta(t), \bar{u}(t)) \\
&+ \mathbb{E}\left[\dot{\bar{Q}}(t+\delta) I_{[0,T-\delta]}(t) \,\middle|\, \mathcal{F}(t)\right], \quad \mathbb{P}\otimes Leb - a.s.
\end{aligned}
\tag{3.3.9}
$$

and

$$
\begin{aligned}
\dot{\bar{Q}}(t) = &-\left(\frac{\partial}{\partial y} b(t, \bar{X}(t), \bar{X}_\delta(t), \bar{u}(t))\right)^\top \bar{P}(t) \\
&- \sum_{j=1}^m \left(\frac{\partial}{\partial y}\sigma^{(j)}(t, \bar{X}(t), \bar{X}_\delta(t), \bar{u}(t))\right)^\top H_{\bar{P}}^{(j)}(t) \\
&+ \frac{\partial}{\partial y} G(t, \bar{X}(t), \bar{X}_\delta(t), \bar{u}(t)), \quad \mathbb{P}\otimes Leb - a.s.
\end{aligned}
\tag{3.3.10}
$$

where $\sigma = (\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(m)})$ and $H_{\bar{P}} = (H_{\bar{P}}^{(1)}, H_{\bar{P}}^{(2)}, \ldots, H_{\bar{P}}^{(m)})$. Hereafter, the partial derivative $\frac{\partial b}{\partial x}$ denotes the $n \times n$-matrix $(\frac{\partial b_i}{\partial x_j})$ and similarly for the partial derivatives $\frac{\partial b}{\partial y}$, $\frac{\partial \sigma^{(j)}}{\partial x}$ and $\frac{\partial \sigma^{(j)}}{\partial y}$. Using the expression (3.2.24) for $\mathcal{H}_d$ again, (3.3.9) and (3.3.10) together give that

$$
\begin{aligned}
\dot{\bar{P}}(t) = &-\frac{\partial \mathcal{H}_d}{\partial x}(t, \bar{X}(t), \bar{X}_\delta(t), \bar{u}(t), \bar{P}(t), H_{\bar{P}}(t)) \\
&- \mathbb{E}\left[\dot{\bar{Q}}(t+\delta) I_{[0,T-\delta]}(t) \,\middle|\, \mathcal{F}(t)\right], \quad \mathbb{P}\otimes Leb - a.s.
\end{aligned}
\tag{3.3.11}
$$

and

$$\dot{Q}(t)=\frac{\partial \mathcal{H}_d}{\partial y}\left(t,\bar{X}(t),\bar{X}_\delta(t),\bar{u}(t),\bar{P}(t),H_{\bar{P}}(t)\right),\quad \mathbb{P}\otimes Leb-a.s. \tag{3.3.12}$$

Furthermore, replacing $\dot{Q}$ in (3.3.9) and using (3.3.10) give that

$$\dot{\bar{P}}(t)=-\mathbb{E}\left[\frac{\partial \bar{\mathcal{H}}_d}{\partial y}(t+\delta)\,I_{[0,T-\delta]}(t)\,\middle|\mathcal{F}(t)\right]-\frac{\partial \bar{\mathcal{H}}_d}{\partial x}(t),\quad \mathbb{P}\otimes Leb-a.s. \tag{3.3.13}$$

where we have used the shorthand notation

$$\frac{\partial \bar{\mathcal{H}}_d}{\partial x}(t)=\frac{\partial \mathcal{H}_d}{\partial x}\left(t,\bar{X}(t),\bar{X}_\delta(t),\bar{u}(t),\bar{P}(t),H_{\bar{P}}(t)\right)$$

and similarly for the partial derivative $\frac{\partial \bar{\mathcal{H}}_d}{\partial y}(t+\delta)$.

Similarly to the above argument, it follows from (2.5.8) that, for the given $\bar{X}\in\mathbb{V}_1$ and $(\bar{P},\dot{Q})\in\mathbb{V}_2\times\mathbb{L}^{21}_{\mathcal{F}}$,

$$l^*\left(-\bar{P}_T\right)=\left\langle -\bar{P}_T,\bar{X}(T)\right\rangle-l\left(\bar{X}(T)\right),\quad \mathbb{P}-a.s. \tag{3.3.14}$$

On the other hand, since l^* is the conjugate convex function of l, we have

$$\left\langle -\bar{P}_T,\bar{X}(T)\right\rangle-g\left(\bar{X}(T)\right)=\sup_{x\in\mathbb{R}^n}\left\{\left\langle x,-\bar{P}_T\right\rangle-g(x)\right\},\quad \mathbb{P}-a.s. \tag{3.3.15}$$

Thus, (3.3.14) and (3.3.15) together imply that the supremum on the right-hand-side of (3.3.15) is attained at $\bar{X}(T)$, $\mathbb{P}$-a.s. Then, under the given condition that g is continuously differentiable with respect to x, we take the derivative, with respect to x, of the function within the bracket on the right-hand-side of (3.3.15). Then, the fact that the supremum is attained at $\bar{X}(T)$, $\mathbb{P}$-a.s. implies that

$$\bar{P}_T=-\frac{\partial g}{\partial x}\left(\bar{X}(T)\right),\quad \mathbb{P}-a.s. \tag{3.3.16}$$

Now, since $\bar{P}$ is identified with $(\bar{P}_T,\dot{\bar{P}})\in\mathbb{V}_2$ via (2.4.17), (3.3.13) and (3.3.16) give that, for all $t\in[0,T]$,

$$\begin{aligned}\bar{P}(t)=&-\frac{\partial g}{\partial x}\left(\bar{X}(T)\right)+\int_t^T\left\{\mathbb{E}\left[\frac{\partial \bar{\mathcal{H}}_d}{\partial y}(s+\delta)\,I_{[0,T-\delta]}(s)\middle|\mathcal{F}(s)\right]+\frac{\partial \bar{\mathcal{H}}_d}{\partial x}(s)\right\}ds\\&-\int_t^T H_{\bar{P}}(s)\,dW(s),\quad \mathbb{P}-a.s.\end{aligned}$$

Note that we have used the result of Proposition 2.4.3, where the last term on the right-hand-side of (2.4.18) is equal to zero identically since the filtration $\mathbb{F}$ here is generated only by W. Therefore, $(\bar{P},H_{\bar{P}})$ forms a continuous $\mathcal{F}(t)$-adapted solution of the adjoint equation (3.3.1) with (X,X_δ,u) replaced by $(\bar{X},\bar{X}_\delta,\bar{u})$ so that (ii) holds. □

Note that $\dot{\bar{Q}}$, which pairs $\bar{X}_\delta$ in the corresponding dual problem, has now been replaced by (3.3.10). As a consequence, the corresponding adjoint equation (3.3.1) depends on the future value of $(\bar{P}, H_{\bar{P}})$ based on the current information $\mathcal{F}(t)$ via conditional expectation.

3.3.2 A Sufficient Condition

Recall that, in addition to Hypotheses I & II, if the Hamiltonian $\mathcal{H}_d$ satisfies the concavity condition described in Proposition 3.2.5, then the function L defined by (3.2.2) is convex. Therefore, with an appropriate modification, the proof of Theorem 3.3.1 can be reversed to give a sufficient stochastic maximum principle as stated by the following theorem.

Theorem 3.3.2. *In addition to Hypotheses I & II, we assume further that b, σ, G and g in* (3.1.7) *and* (3.1.8) *are continuously differentiable with respect to (x, y) and y respectively satisfying*

$$\mathbb{E}\left[\int_0^T \left\{\left|\frac{\partial b}{\partial x_i}(t, X(t), X_\delta(t), u(t))\right|^2 + \left|\frac{\partial \sigma}{\partial x_i}(t, X(t), X_\delta(t), u(t))\right|^2 + \left|\frac{\partial G}{\partial x_i}(t, X(t), X_\delta(t), u(t))\right|^2\right\}dt + \left|\frac{\partial g}{\partial x}(X(T))\right|^2\right] < \infty, \tag{3.3.17}$$

where $x_i = x, y$; and that the Hamiltonian $\mathcal{H}_d$ given by (3.2.24) *is concave with respect to (x, y, u). Let $\bar{u} \in \mathcal{U}$, $\bar{X}$ be the strong solution of the controlled SDDE* (3.1.7) *with u replaced by $\bar{u}$, and $(\bar{P}, H_{\bar{P}}) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ be a solution of the adjoint equation* (3.3.1) *with (u, X) replaced by $(\bar{u}, \bar{X})$. If $(\bar{u}, \bar{X}, \bar{P}, H_{\bar{P}})$ satisfies* (3.3.2), *then $\bar{u}$ is an optimal control of the stochastic optimal control problem with discrete delay* (3.1.9).

Proof. For the given $\bar{u}, \bar{X}$ and $(\bar{P}, H_{\bar{P}})$, we define $\bar{P}_T$, $\dot{\bar{P}}$ and $\dot{\bar{Q}}$ respectively by (3.3.16), (3.3.11) and (3.3.12). By the Cauchy-Schwarz Inequality, the fact that $(\bar{P}, H_{\bar{P}}) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ and (3.3.17) together imply that $\bar{P}_T \in \mathbb{L}^2$, $\dot{\bar{P}} \in \mathbb{L}^{21}_{\mathcal{F}}$ and $\dot{\bar{Q}} \in \mathbb{L}^{21}_{\mathcal{F}}$.

It follows from (2.4.18) and from the uniqueness of the martingale representation theorem that $\bar{P}$ is identified with $(\bar{P}_T, \dot{\bar{P}}) \in \mathbb{V}_2$ via (2.4.17) as required in the proof of Theorem 3.3.1. On the other hand, for the given $\bar{u}$ and $\bar{X}$, we define $\dot{\bar{X}}$ and $H_{\bar{X}}$ by (3.3.6). Then, by a similar argument to that used in the proof of Proposition 3.2.3, we have, under Hypothesis I, that $\dot{\bar{X}} \in \mathbb{L}^{21}_{\mathcal{F}}$ and $H_{\bar{X}} \in \mathbb{L}^{22}_{\mathcal{F}}$ and that $\bar{X}$ is identified with $(\dot{\bar{X}}, H_{\bar{X}})$ via Definition 2.3.1.

Under Hypothesis II that g is convex, $\bar{P}_T$ so defined attains the supremum on the right-hand-side of (3.3.15) which implies that (3.3.14) holds, where the function l is defined by (3.2.3). On the other hand, under the given conditions that $\mathcal{H}_d$ is concave with respect to (x, y) and that $(\bar{u}, \bar{X}, \bar{P}, H_{\bar{P}})$ satisfies the maximizing equation (3.3.2), we obtain that (3.3.8) and (3.3.7) hold. This implies that the 'sup max' on the right-hand-side of (3.3.5) is attained at $(\bar{X}, \bar{X}_\delta, \bar{u})$, $\mathbb{P} \otimes Leb$-a.s. Therefore, (3.3.3) holds with $(\dot{\bar{X}}, H_{\bar{X}})$ so defined, where the function L is defined by (3.2.2). Now, by Propositions 3.2.2, 3.2.3 & 3.2.5, the control problem (3.1.9) can be reformulated as the convex problem (2.3.26) with L and l so defined. Moreover, the two equalities in Theorem 2.5.2 (iii) are satisfied, which gives that $\bar{X}$ is an optimal solution of the convex problem. Hence, the control $\bar{u}$, corresponding to $\bar{X}$, is an optimal solution to the control problem. □

In the remainder of this subsection, we give an example to illustrate how to use the stochastic maximum principle described by Theorem 3.3.2 to obtain an optimal control of the control problem (3.1.9).

Example 3.3.3. Similarly to Example 3.2.1, we set $m = n = r = 1$ and suppose that $\mathbb{U} = \mathbb{R}$ and $\mathcal{U} = \mathbb{L}^{22}_{\mathcal{F}}$. Consider the control problem (3.1.9) with b, σ and G respectively defined by (3.2.4) and (3.2.5); and with g defined by $g(x) = a_3 x$ for some constant $a_3 \in \mathbb{R}$. Note that we have seen in part (I) of Example 3.2.1 that the corresponding controlled SDDE admits a unique strong solution X for every $u \in \mathcal{U} = \mathbb{L}^{22}_{\mathcal{F}}$ satisfying $X(T) \in \mathbb{L}^2$ holds. Then,

$$\mathbb{E}\left[|g(X(T))|\right] = |a_3|\, \mathbb{E}\left[|X(T)|\right] \leq \frac{1}{2}\left(a_3^2 + \mathbb{E}\left[|X(T)|^2\right]\right) < \infty, \tag{3.3.18}$$

which gives the integrability of the corresponding cost function. On the other hand, it is easy to see that (3.3.17) holds for this problem.

(I) Reformulation

We have seen in part (II) of Example 3.2.1 that the corresponding function L in this example satisfies Assumptions I, II & III. On the other hand, it follows from (3.2.3) that the corresponding function l is defined by $l(x) = g(x) = a_3 x$. It is a convex continuous function, not identically ∞ and, as discussed in part (II) of Example 3.2.1, is $\mathcal{F} \times \mathcal{B}(\mathbb{R})$-measurable. Hence, l satisfies Assumption I. Moreover,

$$l(x) = g(x) = a_3 x \geq a_3 x - c, \quad \forall c \in \mathbb{R}^+,$$

for every $x \in \mathbb{R}$. Thus, taking ϑ to be c, we see that ϑ satisfies (2.3.17) as required so that l satisfies Assumption II (ii). Furthermore, motivated by (3.3.18), we take χ to be $(a_3^2 + |X(T)|^2)/2$ which satisfies (2.3.21). Hence, although b, σ, G and g here do not fully satisfy Hypotheses I & II, L and l so defined satisfy Assumptions I, II & III which, as noted in the proof of Theorem 3.3.2, fulfils the prerequisite as did by Hypotheses I & II in Propositions 3.2.2 & 3.2.3.

(II) The Solution of the Adjoint Equation

It follows from (3.2.24) that the Hamiltonian $\mathcal{H}_d$ is given by

$$\begin{aligned} &\mathcal{H}_d(t, x, y, u, p, h_p) \\ &= (a_1 x + b_1 y + c_1 u)\, p + (a_2 x + b_2 y + c_2 u)\, h_p - \frac{1}{2} c_3 u^2, \end{aligned} \tag{3.3.19}$$

which is linear in (x, y) and is quadratic with respect to u. Since the coefficient of u^2 is negative, $\mathcal{H}_d$ is concave with respect to (x, y, u). Then, it follows from (3.3.1) that the associated adjoint equation for this control problem is

$$\begin{cases} dP(t) = -\Big\{\mathbb{E}\left[\{b_1 P(t+\delta) + b_2 H_P(t+\delta)\}\, I_{[0,T-\delta]}(t)\,|\mathcal{F}(t)\right] \\ \qquad\qquad + a_1 P(t) + a_2 H_P(t)\Big\} dt + H_P(t)\, dW(t), \quad t \in [0, T], \\ P(T) = -a_3. \end{cases} \tag{3.3.20}$$

Write

$$\begin{aligned} &F(P(t), H_P(t), P(t+\delta), H_P(t+\delta)) \\ =& a_1 P(t) + a_2 H_P(t) + \mathbb{E}\left[\{b_1 P(t+\delta) + b_2 H_P(t+\delta)\}\,|\mathcal{F}(t)\right]. \end{aligned} \tag{3.3.21}$$

Then, (3.3.20) can be re-expressed in terms of F as

$$\begin{cases} dP(t) = -F(P(t), H_P(t), P(t+\delta), H_P(t+\delta))\, dt + H_P(t)\, dW(t), \\ \qquad\qquad\qquad t \in [0, T], \\ P(T) = -a_3 \text{ and } P(t) \equiv 0, \quad t \in (T, T+\delta], \\ H_P(t) \equiv 0, \quad t \in [T, T+\delta], \end{cases} \tag{3.3.22}$$

where the indicator function in (3.3.20) has been replaced by the stated terminal values for P and H_P respectively.

Due to the property of conditional expectation with respect to $\mathcal{F}(t)$ in (3.3.21), we see that $F(P(t), H(t), P(t+\delta), H(t+\delta))$ is $\mathcal{F}(t)$-measurable for every $t \in [0, T]$ and every $(P, H) \in \mathbb{L}_{\mathcal{F}}^{22} \times \mathbb{L}_{\mathcal{F}}^{22}$. On the other hand, for every $p, p'.h, h' \in \mathbb{R}$ and $P, P', H, H' \in \mathbb{L}_{\mathcal{F}}^{22}$,

$$\begin{aligned}
&\left|F\left(p, h, P\left(t+\delta\right), H\left(t+\delta\right)\right) - F\left(p', h', P'\left(t+\delta\right), H'\left(t+\delta\right)\right)\right| \\
\leq &\left|b_1\right| \mathbb{E}\left[\left|P\left(t+\delta\right) - P'\left(t+\delta\right)\right| \left|\mathcal{F}\left(t\right)\right] + \left|b_2\right| \mathbb{E}\left[\left|H\left(t+\delta\right) - H'\left(t+\delta\right)\right| \left|\mathcal{F}\left(t\right)\right]\right.\right. \\
&+ \left|a_1\right| \left|p - p'\right| + \left|a_2\right| \left|h - h'\right|
\end{aligned}$$

and $F(0, 0, 0, 0) = 0$ so that the conditions (H1) and (H2) given in [33, page 882] are satisfied. Furthermore, by [33, Theorem 4.2] (see also [6, Theorem 2.1]) and [6, Remark 2.1], the anticipated BSDE (3.3.22), i.e. the adjoint equation (3.3.20), admits a unique solution $(\bar{P}, H_{\bar{P}}) \in \mathbb{L}_{\mathcal{F}}^{2\infty} \times \mathbb{L}_{\mathcal{F}}^{22}$ within the interval $[0, T]$.

(III) Applying Theorem 3.3.2

Taking the derivative, with respect to u, of the function on the right-hand-side of (3.3.19), we see that the corresponding derivative is zero if and only if $u = (c_1 p + c_2 h_p)/c_3$. Moreover, since the Hamiltonian $\mathcal{H}_d$ given by (3.3.19) is concave with respect to u, it implies that, if

$$\bar{u}\left(t\right) = \frac{1}{c_3}\left(c_1 \bar{P}\left(t\right) + c_2 H_{\bar{P}}\left(t\right)\right), \tag{3.3.23}$$

then $u \in \mathcal{U} = \mathbb{L}_{\mathcal{F}}^{22}$, since $(\bar{P}, H_{\bar{P}}) \in \mathbb{L}_{\mathcal{F}}^{2\infty} \times \mathbb{L}_{\mathcal{F}}^{22}$, and the corresponding maximizing equation (3.3.2) holds. Hence, by Theorem 3.3.2, $\bar{u}$ defined by (3.3.23) is an optimal control of the control problem. □

To calculate $\bar{u}$, we adopt the backward induction algorithm offered in [6, 30] to obtain $(\bar{P}, H_{\bar{P}})$ numerically as follows.

Step 1. Suppose that $t \in [T - \delta, T]$. Then, in this interval, $(\bar{P}, H_{\bar{P}})$ is the unique solution of the classic BSDE

$$\begin{cases}
d\bar{P}\left(t\right) = -\left\{a_1 \bar{P}\left(t\right) + a_2 H_{\bar{P}}\left(t\right)\right\} dt + H_{\bar{P}}\left(t\right) dW\left(t\right), \quad t \in \left[T - \delta, T\right], \\
\bar{P}\left(T\right) = -a_3.
\end{cases} \tag{3.3.24}$$

Moreover, as noted in [6, page 1079], we see that $\bar{P}$ satisfies the ordinary differential equation (ODE)

$$\begin{cases} d\bar{P}(t) = -a_1\bar{P}(t)\,dt, \quad t \in [T-\delta, T], \\ \bar{P}(T) = -a_3, \end{cases}$$

and $H_{\bar{P}}(t) \equiv 0$ for $t \in [T-\delta, T]$.

Step k. Moving backward to the interval $[T-(k+1)\delta, T-k\delta]$, where $k \in \mathbb{N}^+$ such that $T-(k+1)\delta \geq 0$. Since $H_{\bar{P}}(t+\delta) \equiv 0$ and the evolution of $\bar{P}(t+\delta)$ is known from Step $k-1$, we have

$$\begin{cases} d\bar{P}(t) = -\left\{a_1\bar{P}(t) + a_2H_{\bar{P}}(t) + \mathbb{E}\left[b_1\bar{P}(t+\delta)\,\middle|\mathcal{F}(t)\right]\right\}dt \\ \qquad\qquad + H_{\bar{P}}(t)\,dW(t), \quad t \in [T-(k+1)\delta, T-k\delta], \\ \bar{P}(T-k\delta) \text{ is known from Step } k-1, \end{cases}$$

which is also a classical BSDE. Similarly to Step 1, $\bar{P}$ satisfies the ODE

$$\begin{cases} d\bar{P}(t) = -\left\{b_1\bar{P}(t+\delta) + a_1\bar{P}(t)\right\}dt, \quad t \in [T-(k+1)\delta, T-k\delta], \\ \bar{P}(T-k\delta) \text{ is known from Step } k-1, \end{cases}$$

and $H_{\bar{P}}(t) \equiv 0$ for $t \in [T-(k+1)\delta, T-k\delta]$. Note that the above ODEs can be solved numerically using the Euler method (see [7, Chapter 10]).

Using the above backward induction algorithm, Figure 3.1 below gives an example of such a $\bar{P}$ and the corresponding optimal control $\bar{u}$. Note that, since $H_{\bar{P}}(t) \equiv 0$, it is not necessary to specify the parameters a_2, b_2 and c_2 in this case.

3.4 Inclusion of the Exponential Moving Average Delay

In addition to the discrete delay, the results and techniques obtained in the preceding sections of this chapter can be extended to include an exponential moving average delay together with the results obtained in Section 2.6.

Recalling that the stochastic optimal control problem with both discrete and exponential moving average delays is to find $\bar{u} \in \mathcal{U}$ realizing

$$\inf_{u \in \mathcal{U}} J_{ad}(u), \tag{3.4.1}$$

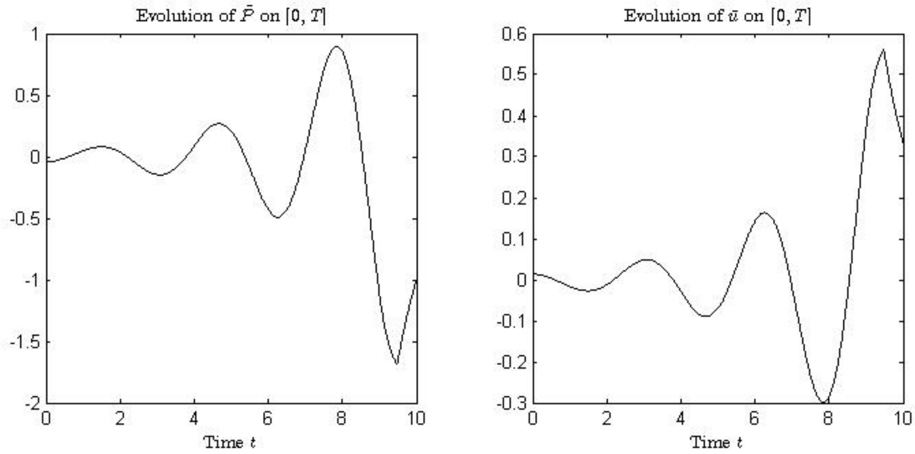

Figure 3.1: Evolution of $\bar{P}$ and $\bar{u}$ with parameters $T = 10$, $\delta = 0.5$, $\xi(t) = t + 1$ for $t \in [-\delta, 0]$, $a_1 = a_3 = 1$, $b_1 = -2$, $c_1 = -1$ and $c_3 = 3$.

where the state system is given by

$$\begin{cases} dX(t) = b(t, X(t), X_a(t), X_\delta(t), u(t))\, dt \\ \qquad\qquad + \sigma(t, X(t), X_a(t), X_\delta(t), u(t))\, dW(t), \quad t \in [0, T], \\ X_a(t) = \int_{-\delta}^{0} e^{\lambda r} X(t + r)\, dr, \quad \lambda > 0, \\ X(t) = \xi(t), \quad t \in [-\delta, 0]; \end{cases} \tag{3.4.2}$$

the admissible control set $\mathcal{U}$ is as defined in a similar fashion to that in (3.1.3) such that the above controlled SDDE admits a unique strong solution for every $u \in \mathcal{U}$; and the cost function J_{ad} is given by

$$J_{ad}(u) = \mathbb{E}\left[\int_0^T G(t, X(t), X_a(t), X_\delta(t), u(t))\, dt + g(X(T), X_a(T))\right]. \tag{3.4.3}$$

To use the results of conjugate duality obtained in Section 2.6, we introduce the continuous $\mathcal{F}(t)$-adapted state process $Y : \Omega \times [0, T] \to \mathbb{R}^n$ such that $Y(t) = X_a(t)$. Moreover, by [14, Lemma 2.1] (see also [29, Lemma 2.1]), the dependence of X_a can be removed by reformulating the controlled SDDE (3.4.2) as the higher-dimensional one with respect to $Z = (X, Y)$:

$$\begin{cases} dX(t) = b(t, X(t), Y(t), X_\delta(t), u(t))\, dt \\ \qquad\qquad + \sigma(t, X(t), Y(t), X_\delta(t), u(t))\, dW(t), \quad t \in [0, T], \\ X(t) = \xi(t), \quad t \in [-\delta, 0], \end{cases} \tag{3.4.4}$$

$$\begin{cases} dY(t) = \left\{X(t) - \lambda Y(t) - e^{-\lambda\delta} X_\delta(t)\right\} dt, & t \in [0,T], \\ Y(0) = \int_{-\delta}^{0} e^{\lambda s} \xi(s)\, ds. \end{cases} \tag{3.4.5}$$

Then, the controlled SDDE (3.4.4)-(3.4.5) for $Z = (X, Y)$ is equivalent to the original controlled SDDE (3.4.2) for X. In particular, (3.4.5) is independent of Y_δ and its diffusion coefficient is zero.

Reformulation to a Convex Problem

Adapting the technique of the reformulation used in Section 3.2, we link the control problem (3.4.1) to a particular convex problem (2.6.1) as follows. For every $t \in [0, T]$, $x, y, z, z_x \in \mathbb{R}^n$ and $h_x \in \mathbb{R}^{n\times m}$, we define the set $\mathcal{C}_{ad} = \mathcal{C}_{ad}(t, x, y, z, z_x, h_x)$ by

$$\begin{aligned} &\mathcal{C}_{ad}(t, x, y, z, z_x, h_x) \\ &= \left\{u \in \mathbb{U} \middle| z_x = b(t, x, y, z, u) \text{ and } h_x = \sigma(t, x, y, z, u)\right\}. \end{aligned}$$

Then, using $\mathcal{C}_{ad}$, we take the functions L_a and l_a in (2.6.1) respectively to be

$$\begin{aligned} &L_a\left(t, x, y, z, z_x, z_y, h_x, h_y\right) \\ &= \begin{cases} \inf_u G(t, x, y, z, u), & \text{if } \mathcal{C}_{ad} \neq \emptyset \text{ and } \begin{cases} z_y = x - \lambda y - e^{-\lambda\delta} z, \\ h_y = 0, \end{cases} \\ \infty, & \text{otherwise,} \end{cases} \end{aligned} \tag{3.4.6}$$

and

$$l_a(x, y) = g(x, y). \tag{3.4.7}$$

To ensure that the control problem can be reformulated to the convex problem with L_a and l_a so defined, so that we can apply the conditions for optimality obtained in Theorem 2.6.2, we suppose that b, σ, G and g in (3.4.2) and (3.4.3) respectively satisfy the hypotheses below. These are necessarily modified from Hypotheses I & II in Section 3.2.2 due to the inclusion of X_a.

Hypothesis* I. $\mathbb{U}$ is a nonempty convex compact subset of $\mathbb{R}^r$. The functions b and σ are continuous with respect to $(t, u) \in [0, T] \times \mathbb{U}$; and are Lipschitz continuous with respect to $(x, y, z) \in \mathbb{R}^{n\times 3}$. Moreover, there exists a constant $C_1 > 0$ such that

$$|b(t, 0, 0, 0, u)| + |\sigma(t, 0, 0, 0, u)| \le C_1, \quad \forall (t, u) \in [0, T] \times \mathbb{U}.$$

Hypothesis* II. The functions G and g are continuous. Moreover, g is convex and there exist constants $C_2 \in \mathbb{R}$ and $C_3 > 0$ such that

$$\begin{cases} C_2 \leq G(t,x,y,z,u) \leq C_3 \left(1+|x|^2+|y|^2+|z|^2\right), & \forall t \in [0,T], x,y,z \in \mathbb{R}^n, u \in \mathbb{U}, \\ C_2 \leq g(x,y) \leq C_3 \left(1+|x|^2+|y|^2\right), & \forall x,y \in \mathbb{R}^n. \end{cases}$$

Similarly to Propositions 3.2.2 & 3.2.3, we see that Hypotheses* I & II ensure that, if L_a is convex with respect to (x,y,z,z_x,z_y,h_x,h_y), L_a and l_a satisfy Assumptions* I, II & III presented in Section 2.6. Note that, following a similar argument to that in the proof of Proposition 3.2.3, we obtain that the controlled SDDE (3.4.4)-(3.4.5) admits a unique strong solution $\hat{Z} = (\hat{X},\hat{Y})$ along with $\hat{u} \in \mathcal{U}$ satisfying $(\hat{X},\hat{Y}) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{2\infty}_{\mathcal{F}}$. Moreover, we define $(\dot{\hat{X}}, H_{\hat{X}})$ and $(\dot{\hat{Y}}, H_{\hat{Y}})$ respectively by

$$\begin{cases} \dot{\hat{X}}(t) = b\left(t,\hat{X}(t),\hat{Y}(t),\hat{X}_\delta(t),\hat{u}(t)\right), \\ H_{\hat{X}}(t) = \sigma\left(t,\hat{X}(t),\hat{Y}(t),\hat{X}_\delta(t),\hat{u}(t)\right), \end{cases}$$

and

$$\begin{cases} \dot{\hat{Y}}(t) = \hat{X}(t) - \lambda\hat{Y}(t) - e^{-\lambda\delta}\hat{X}_\delta(t), \\ H_{\hat{Y}}(t) \equiv 0. \end{cases} \tag{3.4.8}$$

This gives that

$$\hat{\mathcal{C}}_{ad} = \mathcal{C}_{ad}\left(t,\hat{X}(t),\hat{Y}(t),\hat{X}_\delta(t),\dot{\hat{X}}(t),H_{\hat{X}}(t)\right) \neq \emptyset$$

so that, together with (3.4.8) and Hypothesis* II,

$$\begin{aligned} & L_a\left(t,\hat{X}(t),\hat{Y}(t),\hat{X}_\delta(t),\dot{\hat{X}}(t),\dot{\hat{Y}}(t),H_{\hat{X}}(t),H_{\hat{Y}}(t)\right) \\ = & \inf_{u\in\hat{\mathcal{C}}_{ad}} G\left(t,\hat{X}(t),\hat{Y}(t),\hat{X}_\delta(t),u\right) \\ \leq & C_3\left(1+\left|\hat{X}(t)\right|^2+\left|\hat{Y}(t)\right|^2+\left|\hat{X}_\delta(t)\right|^2\right), \quad \mathbb{P}\otimes Leb-a.s. \end{aligned}$$

Then, following a similar argument to τ in the proof of Proposition 3.2.3, we see that L_a satisfy Assumption* III (i).

The Hamiltonian and Adjoint Equation

Regarding the control problem (3.4.1), we define a Hamiltonian $\mathcal{H}_{ad} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m} \to \mathbb{R}$ in a similar fashion to (3.1.16) considered in [29] as

$$\begin{aligned}
&\mathcal{H}_{ad}\left(t,x,y,z,u,p,r,h_p,h_r\right) \\
&=\langle b\left(t,x,y,z,u\right),p\rangle+\langle\sigma\left(t,x,y,z,u\right),h_p\rangle-G\left(t,x,y,z,u\right) \\
&\quad+\left\langle\left(x-\lambda y-e^{-\lambda\delta}z\right),r\right\rangle.
\end{aligned} \tag{3.4.9}$$

Similarly to the argument used in Section 3.3.1, this Hamiltonian and the associated adjoint equations (specified below) can be derived by using Theorem 2.6.2 (iii) as stated by the following theorem.

Theorem 3.4.1. *Assume that Hypotheses* I & II hold and that the function L_a defined by* (3.4.6) *is convex with respect to (x,y,z,z_x,z_y,h_x,h_y). In addition, assume that b,σ,G and g in* (3.4.2) *and* (3.4.3) *are continuously differentiable with respect to (x,y,z) and (x,y) respectively. Suppose that $(\bar{X},\bar{Y})\in\mathbb{V}_1\times\mathbb{V}_1$ and $(\bar{P},\bar{P}^a,\dot{\bar{Q}})\in\mathbb{V}_2\times\mathbb{V}_2\times\mathbb{L}^{21}_{\mathcal{F}}$ together satisfy* (2.6.6) *and* (2.6.7)*, where l_a is defined by* (3.4.7)*. Then, there exists a $\bar{u}\in\mathcal{U}$ realizing* (3.4.1) *with $(\bar{u},\bar{X},\bar{P},\bar{P}^a)$ satisfying*

(i) $\bar{Z}=(\bar{X},\bar{Y})$ *is the unique strong solution of the controlled SDDE* (3.4.4)-(3.4.5) *with u replaced by $\bar{u}$;*

(ii) $(\bar{P},H_{\bar{P}})$ *and* $(\bar{P}^a,H_{\bar{P}^a})$ *are solutions of the adjoint equations with (X,Y,X_δ,u) replaced by $(\bar{X},\bar{Y},\bar{X}_\delta,\bar{u})$:*

$$\begin{cases}
dP\left(t\right)=-\left\{\dfrac{\partial\mathcal{H}_{ad}}{\partial x}\left(t\right)+\mathbb{E}\left[\dfrac{\partial\mathcal{H}_{ad}}{\partial z}\left(t+\delta\right)I_{[0,T-\delta]}\left(t\right)\Big|\mathcal{F}\left(t\right)\right]\right\}dt \\
\qquad\qquad+H_P\left(t\right)dW\left(t\right),\quad t\in[0,T], \\
P\left(T\right)=-\dfrac{\partial g}{\partial x}\left(X\left(T\right),Y\left(T\right)\right)
\end{cases} \tag{3.4.10}$$

$$\begin{cases}
dP^a\left(t\right)=-\dfrac{\partial\mathcal{H}_{ad}}{\partial y}\left(t\right)dt+H_{P^a}\left(t\right)dW\left(t\right),\quad t\in[0,T], \\
P^a\left(T\right)=-\dfrac{\partial g}{\partial y}\left(X\left(T\right),Y\left(T\right)\right),
\end{cases} \tag{3.4.11}$$

where $H_{\bar{P}}$ and $H_{\bar{P}^a}$ are respectively specified by $\bar{P}$ and $\bar{P}^a$ via (2.4.18)*. Here we have used the shorthand notation*

$$\frac{\partial\mathcal{H}_{ad}}{\partial x}\left(t\right)=\frac{\partial\mathcal{H}_{ad}}{\partial x}\left(t,X\left(t\right),Y\left(t\right),X_\delta\left(t\right),u\left(t\right),P\left(t\right),P^a\left(t\right),H_P\left(t\right),H_{P^a}\left(t\right)\right)$$

and similarly for the partial derivatives $\frac{\partial\mathcal{H}_{ad}}{\partial z}(t+\delta)$ and $\frac{\partial\mathcal{H}_{ad}}{\partial y}(t)$;

(iii) $\mathbb{P} \otimes$ *Leb-a.s.,*

$$\begin{aligned}&\mathcal{H}_{ad}\left(t, \bar{X}\left(t\right), \bar{Y}\left(t\right), \bar{X}_{\delta}\left(t\right), \bar{u}\left(t\right), \bar{P}\left(t\right), \bar{P}^{a}\left(t\right), H_{\bar{P}}\left(t\right), H_{\bar{P}^{a}}\left(t\right)\right)\\&=\max_{u\in\mathbb{U}}\mathcal{H}_{ad}\left(t, \bar{X}\left(t\right), \bar{Y}\left(t\right), \bar{X}_{\delta}\left(t\right), u, \bar{P}\left(t\right), \bar{P}^{a}\left(t\right), H_{\bar{P}}\left(t\right), H_{\bar{P}^{a}}\left(t\right)\right).\end{aligned} \tag{3.4.12}$$

The proof of the above theorem uses the essentially same techniques as that for the proof of Theorem 3.3.1. For example, let L_a^* be the conjugate convex function of L_a. Then, satisfying (2.6.6) implies that it is necessary that there exists a $\bar{u} \in \mathcal{U}$ such that the 'sup max' in

$$\begin{aligned}&L_a^*\left(t, \dot{\bar{P}}(t) - \mathbb{E}\left[\dot{\bar{Q}}(t+\delta) I_{[0,T-\delta]}(t) \middle| \mathcal{F}(t)\right], \dot{\bar{P}}^a(t), \dot{\bar{Q}}(t), \bar{P}(t), \bar{P}^a(t), H_{\bar{P}}(t), H_{\bar{P}^a}(t)\right)\\&= \sup_{(x,y,z)\in\mathbb{R}^{n\times 3}} \max_{u\in\mathbb{U}} \Big\{ \left\langle x, \dot{\bar{P}}\left(t\right) - \mathbb{E}\left[\dot{\bar{Q}}\left(t+\delta\right) I_{[0,T-\delta]}\left(t\right) \middle| \mathcal{F}\left(t\right)\right]\right\rangle + \left\langle y, \dot{\bar{P}}^a\left(t\right)\right\rangle\\&\qquad + \left\langle z, \dot{\bar{Q}}\left(t\right)\right\rangle + \left\langle b\left(t,x,y,z,u\right), \bar{P}\left(t\right)\right\rangle + \left\langle \sigma\left(t,x,y,z,u\right), H_{\bar{P}}\left(t\right)\right\rangle\\&\qquad + \left\langle \left(x - \lambda y - e^{-\lambda\delta} z\right), \bar{P}^a\left(t\right)\right\rangle - G\left(t,x,y,z,u\right) \Big\},\end{aligned}$$

is attained at $(\bar{X}, \bar{Y}, \bar{X}_\delta, \bar{u})$, $\mathbb{P} \otimes$ *Leb*-a.s. Similarly, satisfying (2.6.7) implies that the supremum in

$$l_a^*\left(-\bar{P}_T, -\bar{P}_T^a\right) = \sup_{(x,y)\in\mathbb{R}^{n\times 2}} \left\{\left\langle x, -\bar{P}_T\right\rangle + \left\langle y, -\bar{P}_T^a\right\rangle - g\left(x,y\right)\right\}$$

is attained at $(\bar{X}(T), \bar{Y}(T))$, $\mathbb{P}$-a.s. Then, similarly to the argument to that for the proof of Theorem 3.3.1, the above two equations enable us to obtain the adjoint equations (3.4.10)-(3.4.11) and the maximizing equation (3.4.12).

Although the Hamiltonian (3.4.9) is similar to (3.1.16), the above adjoint equations are different from (3.1.17)-(3.1.19) introduced in [29]. Instead of a triple of classical BSDEs, the adjoint equations (3.4.10)-(3.4.11) are coupled BSDEs, where (3.4.10) is an anticipated BSDE expressed in a similar fashion to (3.3.1) for the stochastic optimal control problem with discrete delay and (3.4.11) is a classical BSDE expressed in a similar fashion to (3.1.6) for Markovian optimal control problems.

A Stochastic Maximum Principle

Similarly to Theorem 3.3.2, reversing the arguments in the proof of Theorem 3.4.1, we obtain a stochastic maximum principle for the control problem (3.4.1) as stated by the following theorem.

Theorem 3.4.2. *In addition to Hypotheses* I & II, we assume further that b, σ, G and g in* (3.4.2) *and* (3.4.3) *are continuously differentiable with respect to (x, y, z) and (x, y) respectively satisfying*

$$\mathbb{E}\left[\int_0^T \left\{ \left|\frac{\partial b}{\partial x_i}(t, X(t), Y(t), X_\delta(t), u(t))\right|^2 + \left|\frac{\partial \sigma}{\partial x_i}(t, X(t), Y(t), X_\delta(t), u(t))\right|^2 + \left|\frac{\partial G}{\partial x_i}(t, X(t), Y(t), X_\delta(t), u(t))\right|^2 \right\} dt\right] + \mathbb{E}\left[\left|\frac{\partial g}{\partial x_i}(X(T), Y(T))\right|^2\right] < \infty, \tag{3.4.13}$$

where $x_i = x, y, z$; and that the Hamiltonian $\mathcal{H}_{ad}$ given by (3.4.9) *is concave with respect to (x, y, z, u). Let $\bar{u} \in \mathcal{U}$, $\bar{Z} = (\bar{X}, \bar{Y})$ be the unique strong solution of the controlled SDDE* (3.4.4)-(3.4.5) *with u replaced by $\bar{u}$, and $(\bar{P}, H_{\bar{P}}) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ and $(\bar{P}^a, H_{\bar{P}^a}) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ be solutions of the adjoint equations* (3.4.10) *and* (3.4.10) *with (u, X, Y) replaced by $(\bar{u}, \bar{X}, \bar{Y})$. If $(\bar{u}, \bar{X}, \bar{Y}, \bar{P}, H_{\bar{P}}, \bar{P}^a, H_{\bar{P}^a})$ satisfies* (3.4.12)*, then $\bar{u}$ is an optimal control of the stochastic optimal control problem with both discrete and exponential moving average delays* (3.4.1).

The proof of Theorem 3.4.2 uses the essentially same techniques as the proof of Theorem 3.3.2 for the stochastic optimal control problem with discrete delay. In particular, the concavity condition imposed on $\mathcal{H}_{ad}$ ensures that L_a is convex with respect to $(x, y, z, z_x, z_y, h_x, h_y)$ as required in the reformulation to the corresponding convex problem. More precisely, following a similar argument to that in the proof of Proposition 3.2.5, L_a can be re-expressed in term of $\mathcal{H}_{ad}$ as

$$\begin{aligned} &L_a\left(t, x, y, z, z_x, z_y, h_x, h_y\right) \\ &= \sup_{\left(p, r, h_p, h_r\right) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m}} \Big\{ \langle z_x, p\rangle + \langle h_x, h_p\rangle + \langle z_y, r\rangle + \langle h_y, h_r\rangle \\ &\qquad - \hat{\mathcal{H}}_{ad}\left(t, x, y, z, p, r, h_p, h_r\right) \Big\}, \end{aligned} \tag{3.4.14}$$

where $\hat{\mathcal{H}}_{ad}\left(t, x, y, z, p, r, h_p, h_r\right) = \sup_{u \in \mathbb{U}} \mathcal{H}_{ad}(t, x, y, z, u, p, r, h_p, h_r)$. Then, under the given condition that $\mathcal{H}_{ad}$ is concave with respect to (x, y, z, u), we see that L_a in (3.4.14) is a convex function. This, together with Hypotheses* I & II, ensures that the control problem (3.1.15) can be reformulated as the convex problem (2.6.1) with L_a and l_a so defined as required for the proof of Theorem 3.4.2.

If the controlled SDDE (3.4.2) and the cost function (3.4.3) are independent of X_a, then the corresponding Hamiltonian and adjoint equations coincide with those obtained in Section 3.3 for the control problem with discrete delay. Hence, our results in Section 3.3 become a special case of those for the control problem (3.4.1).

Comparing the above results with those obtained in [29, Theorem 2.2] and [30, Theorem 3.1], Theorem 3.4.2 does not require any adjoint process to be identically zero and also allows g in (3.1.14) to depend on y corresponding to $Y(T) = X_a(T)$. This enables us to remove the restrictions in [29, 30] mentioned in Section 3.1.1 which will become clear by the example below.

Example 3.4.3. For simplicity, we set $m = n = r = 1$. Suppose that $\mathbb{U} = \mathbb{R}$ and $\mathcal{U} = \mathbb{L}^{22}_{\mathcal{F}}$ as in Examples 3.2.1 & 3.3.3; that b and σ in (3.4.2) are given by

$$\begin{cases} b\left(t,x,y,z,u\right) = a_1 x + f_1 y + b_1 z + c_1 u, \\ \sigma\left(t,x,y,z,u\right) = a_2 x + f_2 y + b_2 z + c_2 u; \end{cases} \tag{3.4.15}$$

and that G and g in (3.4.3) are given by

$$G\left(t,x,y,z,u\right) = \frac{1}{2} c_3 u^2 \text{ and } g\left(x,y\right) = a_3 x + f_3 y, \tag{3.4.16}$$

where $a_1, a_2, a_3, b_1, b_2, c_1, c_2, c_3$ are as given in Example 3.3.3 and $f_1, f_2, f_3 \in \mathbb{R}$ are given constants. By the techniques used in part (I) of Example 3.2.1, the controlled SDDE (3.4.4)-(3.4.5) with b and σ so defined admits a unique strong solution and the cost function (3.4.3) with G and g so defined is integrable for any $u \in \mathcal{U} = \mathbb{L}^{22}_{\mathcal{F}}$.

Combining the arguments used in part (II) of Example 3.2.1 and part (I) of Example 3.3.3, we see that this control problem can be reformulated as the convex problem (2.6.1) with L_a and l_a respectively defined by (3.4.6) and (3.4.7), where Assumptions* I, II & III are satisfied.

It follows from (3.4.9) that

$$\begin{aligned} &\mathcal{H}_{ad}\left(t,x,y,z,u,p,r,h_p,h_r\right) \\ &= \left(a_1 x + f_1 y + b_1 z + c_1 u\right) p + \left(a_2 x + f_2 y + b_2 z + c_2 u\right) h_p - \frac{1}{2} c_3 u^2 \\ &\quad + \left(x - \lambda y - e^{-\lambda\delta} z\right) r, \end{aligned} \tag{3.4.17}$$

which is linear in (x, y, z) and is quadratic with u, where the coefficient of u^2 is negative, so that it satisfies the concavity condition required in Theorem 3.4.2. This particularly gives the convexity of L_a. The adjoint equations (3.4.10)-(3.4.11) for this control problem are expressed by

$$\begin{cases} dP(t) = -\Big\{ a_1 P(t) + P^a(t) + a_2 H_P(t) \\ \qquad + \mathbb{E}\Big[\Big\{ b_1 P(t+\delta) - e^{-\lambda\delta} P^a(t+\delta) \\ \qquad\qquad + b_2 H_P(t+\delta) \Big\} I_{[0,T-\delta]}(t) \Big| \mathcal{F}(t) \Big] \Big\} dt \\ \qquad + H_P(t)\, dW(t), \quad t \in [0,T], \\ P(T) = -a_3, \end{cases} \tag{3.4.18}$$

$$\begin{cases} dP^a(t) = -\{ f_1 P(t) - \lambda P^a(t) + f_2 H_P(t) \} dt + H_{P^a}(t)\, dW(t), \\ \qquad t \in [0,T] \\ P^a(T) = -f_3. \end{cases} \tag{3.4.19}$$

This can be regarded as a higher-dimensional anticipated BSDE with respect to (R, H_R), where $R = (P, P^a)$ and $H_R = (H_P, H_{P^a})$. Then, as for (3.3.20), these adjoint equations admit a unique solution $(R, H_R) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$. Taking the derivative, with respect to u, of $\mathcal{H}_{ad}$ given by (3.4.17), we obtain that, $\bar{u}$ given by

$$\bar{u}(t) = \frac{1}{c_3} \left(c_1 \bar{P}(t) + c_2 H_{\bar{P}}(t) \right), \tag{3.4.20}$$

achieves the maximum in (3.4.12), where $(\bar{P}, H_{\bar{P}})$ and $(\bar{P}^a, H_{\bar{P}^a})$ are the solutions of the adjoint equations (3.4.18)-(3.4.19) with u replaced by $\bar{u}$. We also have $\bar{u} \in \mathbb{L}^{22}_{\mathcal{F}}$ since $(\bar{P}, H_{\bar{P}}) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$. Therefore, by Theorem 3.4.2, $\bar{u}$ is an optimal control of the control problem.

Note that this control problem usually cannot be solved using the results either of [30, Theorem 3.1] or of [29, Theorem 2.2] as, for the former, g in (3.4.16) needs to be independent of y corresponding to $X_a(T)$ and, for the latter, the parameters in the model need to satisfy certain constraints. More precisely, for the latter case, the parameters in (3.4.15) and (3.4.16) need to satisfy

$$f_3 e^{-\lambda\delta} = b_1 a_3, \; b_1 \neq 0 \;\text{ and }\; \frac{e^{-\lambda\delta} f_1}{b_1} - \lambda = a_1 + b_1 e^{\lambda\delta}. \tag{3.4.21}$$

Then,

$$\bar{u}(t) = -\frac{a_3 c_1}{c_3} e^{\left(a_1 + b_1 e^{\lambda\delta}\right)(T-t)}, \quad \forall t \in [0, T]. \tag{3.4.22}$$

This is due to one of the adjoint processes in [29, Theorem 2.2] needs to be identically zero. If we set the parameters f_1 and f_3 to satisfy the conditions (3.4.21), Figure 3.2 gives an example of such $(\bar{P}, \bar{P}^a)$ and $\bar{u}$ using the backward induction algorithm described at the end of Section 3.3.2, where $H_{\bar{P}} \equiv H_{\bar{P}^a} \equiv 0$. The solid and dash lines in Figure 3.2 (c) respectively represent the corresponding results using (3.4.20) and (3.4.22). By increasing the accuracy, for example, Figure 3.2 (d) shows a segment part of this comparison. It implies that the numerical discrepancy can be narrowed down to any fixed $\varepsilon > 0$ so that these two stochastic maximum principles give numerically indistinguishable results for this control problem if the parameters in the model satisfy the conditions (3.4.21). On the other hand, Figure 3.3 gives another example of such $(\bar{P}, \bar{P}^a)$ and $\bar{u}$ where, in particular, the conditions (3.4.21) are not satisfied. □

3.5 Discussion

3.5.1 Extension to Regime-Switching Models

Using the Markov chain α and the associated canonical martingales M described in Chapter 2, the results obtained in the preceding sections of this chapter can also be extended to involve regime-switching. Note that this subsection only investigates the stochastic optimal control problem with both discrete delay and regime-switching. It can be generalized to include X_a with an appropriate modification.

Suppose that the filtration $\mathbb{F}$ is generated jointly by W and α. Let $b : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{I} \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{I} \to \mathbb{R}^{n\times m}$ be two given functions and the continuous $\mathcal{F}(t)$-adapted state process $X : \Omega \times [-\delta, T] \to \mathbb{R}^n$ be described by the controlled SDDE with regime-switching

$$\begin{cases} dX(t) = b(t, X(t), X_\delta(t), u(t), \alpha(t))\, dt \\ \qquad\qquad + \sigma(t, X(t), X_\delta(t), u(t), \alpha(t))\, dW(t), \quad t \in [0,T], \\ X(t) = \xi(t), \quad t \in [-\delta, 0], \quad \alpha(0) = i_0 \in \mathbb{I}, \end{cases} \tag{3.5.1}$$

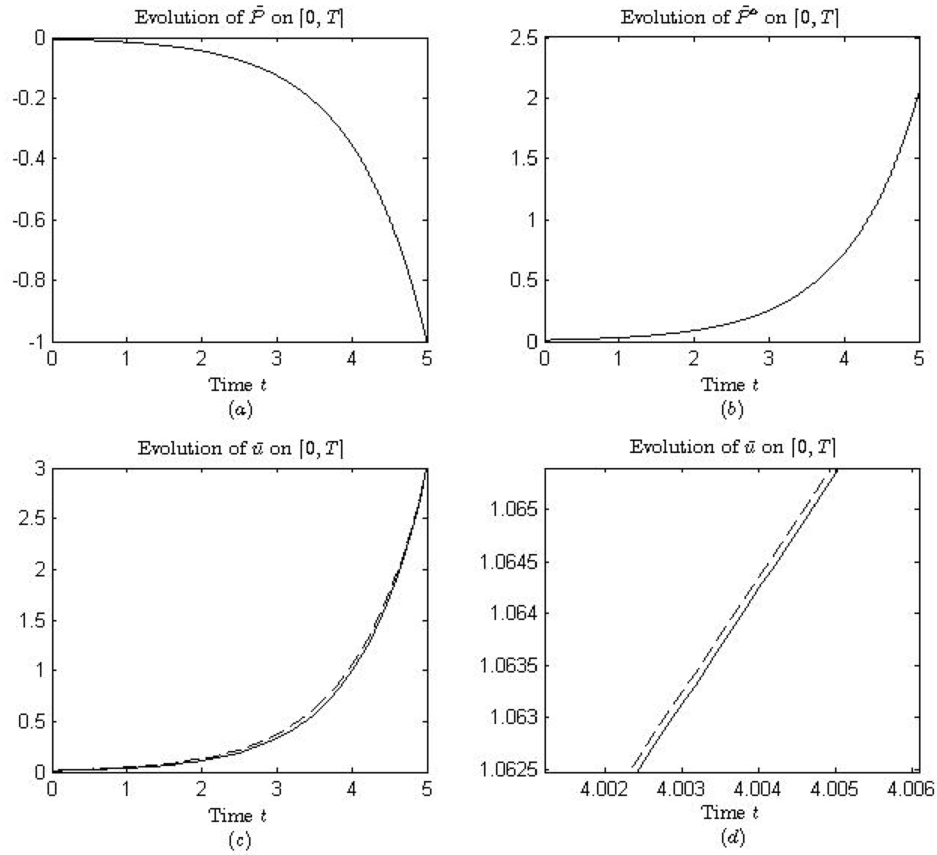

Figure 3.2: Evolution for $\bar{P}, \bar{P}^a$ and $\bar{u}$ with parameters $T = 5$, $\delta = 0.2$, $\lambda = 0.1$ $\xi(t) = t + 1$ for $t \in [-\delta, 0]$, $a_1 = a_3 = 1$, $b_1 = -2$, $c_1 = -1$ and $c_3 = 3$; f_1 and f_3 are given by using (3.4.21).

where α, i_0 are as defined in Section 2.3. For given functions $G : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{I} \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{I} \to \mathbb{R}$, the cost functional J_d^α is given by

$$J_d^\alpha(u) = \mathbb{E}\left[\int_0^T G(t, X(t), X_\delta(t), u(t), \alpha(t))\, dt + g(X(T), \alpha(T))\right]. \tag{3.5.2}$$

Then, the so-called stochastic optimal control problem with both discrete delay and regime-switching associated with the state system (3.5.1) and the cost function (3.5.2) is to find $u \in \mathcal{U}$ realizing

$$\inf_{u \in \mathcal{U}} J_d^\alpha(u). \tag{3.5.3}$$

where the admissible control set $\mathcal{U}$ is defined in a similar fashion to that in (3.1.9). It is straightforward to see that, if $\mathbb{I} = \{i_0\}$, then the above control problem reduces to the one with discrete delay.

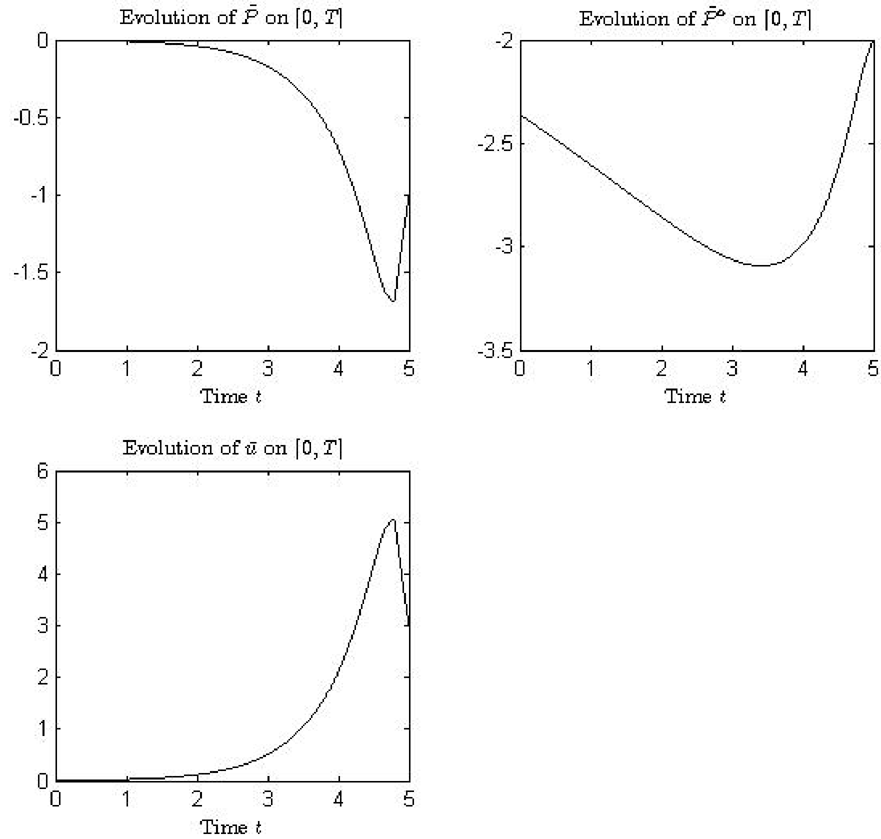

Figure 3.3: Evolution for $\bar{P}, \bar{P}^a$ and $\bar{u}$ with parameters $T = 5$, $\delta = 0.2$, $\lambda = 0.1$, $\xi(t) = t + 1$ for $t \in [-\delta, 0]$, $a_1 = a_3 = 1$, $b_1 = -2$, $c_1 = -1$ and $c_3 = 3$; $f_1 = 1$ and $f_3 = 2$.

Adapting the method used in Section 3.2, we reformulate the control problem (3.5.3) to a particular convex problem (2.3.26) as follows. For every $(\omega, t) \in \Omega \times [0, T]$, $x, y, z \in \mathbb{R}^n$ and $h \in \mathbb{R}^{n \times m}$, we define the set $\mathcal{C}^\alpha = \mathcal{C}^\alpha(t, x, y, z, h, \alpha(t))$ by

$$\begin{aligned} &\mathcal{C}^\alpha\left(t, x, y, z, h, \alpha\left(t\right)\right) \\ &= \left\{u \in \mathbb{U} \middle| z = b\left(t, x, y, u, \alpha\left(t\right)\right) \text{ and } h = \sigma\left(t, x, y, u, \alpha\left(t\right)\right)\right\}. \end{aligned}$$

Using $\mathcal{C}^\alpha$, we take L and l respectively to be

$$L\left(\omega, t, x, y, z, h\right) = \begin{cases} \inf\limits_{u \in \mathcal{C}^\alpha} G\left(t, x, y, u, \alpha\left(t\right)\right), & \text{if } \mathcal{C}^\alpha \neq \emptyset, \\ \infty, & \text{otherwise,} \end{cases}$$

and

$$l\left(\omega, x\right) = g\left(x, \alpha\left(T\right)\right).$$

Following a similar argument to that used in Section 3.2.2, L and l so defined satisfy Assumptions I, II & III given in Section 2.3.2 under certain hypotheses on b, σ, G and g which can be necessarily modified from Hypotheses I & II. Moreover, by the similar techniques as those used in Theorems 3.3.1 & 3.3.2, we can derive a stochastic maximum principle for the control problem (3.5.3) described in a similar fashion to Theorem 3.3.2, where the Hamiltonian is given by

$$\begin{aligned}&\mathcal{H}_d^{\alpha}\left(t, x, y, u, p, h_p, i\right)\\&= \langle b\left(t, x, y, u, i\right), p\rangle + \left\langle h_p, \sigma\left(t, x, y, u, i\right)\right\rangle - G\left(t, x, y, u, i\right),\end{aligned}$$

and the adjoint equation is given by

$$\begin{cases} dP\left(t\right) = -\left\{\dfrac{\partial \mathcal{H}_d^{\alpha}}{\partial x}\left(t\right) + \mathbb{E}\left[\dfrac{\partial \mathcal{H}_d^{\alpha}}{\partial y}\left(t+\delta\right) I_{[0,T-\delta]}\left(t\right)\middle|\mathcal{F}\left(t\right)\right]\right\} dt \\ \qquad\qquad + H_P\left(t\right) dW\left(t\right) + K_P\left(t\right) \bullet dM\left(t\right), \quad t \in [0, T], \\ P\left(T\right) = -\dfrac{\partial g}{\partial x}\left(X\left(T\right), \alpha\left(T\right)\right). \end{cases} \tag{3.5.4}$$

Note that the last term on the right-hand-side of the first equation of (3.5.4) is due to the results in Proposition 2.4.3. Note also that these Hamiltonian and adjoint equation and the associated stochastic maximum principle generalize those studied in [9] to the corresponding discrete delay context.

3.5.2 A Stochastic Calculus Approach

In this subsection, we use the Hamiltonian (3.4.9) and the adjoint equations (3.4.10)-(3.4.11) to offer a stochastic maximum principle as stated by the following theorem, which is in a similar fashion to those studied in [6, 9, 15, 25, 29, 30, 46], and prove it by some techniques of stochastic calculus.

Theorem 3.5.1. *Suppose that the controlled SDDE* (3.4.2) *admits a unique strong solution and the cost function* (3.4.3) *is integrable for each* $u \in \mathcal{U}$. *In addition, assume that* b, σ, G *and* g *in* (3.4.2) *and* (3.4.3) *are continuously differentiable with respect to* (x, y, z) *and* (x, y) *respectively, satisfy* (3.4.13) *and*

$$\begin{aligned}\mathbb{E}\Bigg[\int_0^T \Bigg\{ &\left|\frac{\partial b}{\partial u}\left(t, X\left(t\right), X_a\left(t\right), X_\delta\left(t\right), u\left(t\right)\right)\right|^2 + \left|\frac{\partial \sigma}{\partial u}\left(t, X\left(t\right), X_a\left(t\right), X_\delta\left(t\right), u\left(t\right)\right)\right|^2 \\ &+ \left|\frac{\partial G}{\partial u}\left(t, X\left(t\right), X_a\left(t\right), X_\delta\left(t\right), u\left(t\right)\right)\right|^2 \Bigg\} dt\Bigg] < \infty.\end{aligned}$$

Let $\bar{u} \in \mathcal{U}$, $\bar{X}$ be the unique strong solution to the controlled SDDE associated with $\bar{u}$, and $(\bar{P}, H_{\bar{P}}) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ and $(\bar{P}^a, H_{\bar{P}^a}) \in \mathbb{L}^{2\infty}_{\mathcal{F}} \times \mathbb{L}^{22}_{\mathcal{F}}$ be solutions of the adjoint equations (3.4.10)-(3.4.11) *associated with $(\bar{u}, \bar{X})$ such that, for any $u \in \mathcal{U}$ and the unique strong solution X of the controlled SDDE associated with u,*

$$\mathbb{E}\left[\int_0^T \left\{ \left|H_{\bar{P}}^\top (t)\left(\bar{X}(t) - X(t)\right)\right|^2 + \left|H_{\bar{P}^a}^\top (t)\left(\bar{X}_a(t) - X_a(t)\right)\right|^2 + \left|\left(\bar{\sigma}(t) - \sigma(t)\right)^\top \bar{P}(t)\right|^2 \right\} dt\right] < \infty, \tag{3.5.5}$$

where we have used the shorthand notation

$$\bar{\sigma}(t) = \sigma\left(t, \bar{X}(t), \bar{X}_a(t), \bar{X}_\delta(t), \bar{u}(t)\right),$$
$$\sigma(t) = \sigma\left(t, X(t), X_a(t), X_\delta(t), u(t)\right).$$

If the Hamiltonian $\mathcal{H}_{ad}$ given by (3.4.9) *satisfies the condition that, for every $t \in [0, T]$, $\mathcal{H}_{ad}(t, x, y, z, u, \bar{P}(t), \bar{P}^a(t), H_{\bar{P}}(t), H_{\bar{P}^a}(t))$ is concave with respect to (x, y, z, u), $\mathbb{P} - a.s.$ and if* (3.4.12) *holds, then $\bar{u}$ is an optimal control of the stochastic optimal control problem with both discrete and exponential moving average delays* (3.4.1).

Proof. Fix $u \in \mathcal{U}$. If X is the corresponding strong solution of the controlled SDDE (3.4.2), then

$$\begin{aligned} J_{ad}(\bar{u}) - J_{ad}(u) = &\mathbb{E}\left[\int_0^T \left\{G\left(t, \bar{X}(t), \bar{X}_a(t), \bar{X}_\delta(t), \bar{u}(t)\right) \right.\right. \\ &\left.\left. - G\left(t, X(t), X_a(t), X_\delta(t), u(t)\right)\right\} dt\right] \\ &+ \mathbb{E}\left[g\left(\bar{X}(T), \bar{X}_a(T)\right) - g\left(X(T), X_a(T)\right)\right]. \end{aligned} \tag{3.5.6}$$

Under the given condition that g is convex and is continuously differentiable with respect to (x, y), the terms within the second bracket on the right-hand-side of (3.5.6) give us that

$$\begin{aligned} &\mathbb{E}\left[g\left(\bar{X}(T), \bar{X}(T)\right) - g\left(X(T), X_a(T)\right)\right] \\ \leq &\mathbb{E}\left[\left\langle\left(\bar{X}(T) - X(T)\right), \frac{\partial g}{\partial x}\left(\bar{X}(T), \bar{X}_a(T)\right)\right\rangle \right. \\ &\left. + \left\langle\left(\bar{X}_a(T) - X_a(T)\right), \frac{\partial g}{\partial y}\left(\bar{X}(T), \bar{X}_a(T)\right)\right\rangle\right]. \end{aligned} \tag{3.5.7}$$

Now, applying the Itô formula to $\langle(\bar{X}(t) - X(t)), \bar{P}(t)\rangle$ for $t \in [0, T]$, we have

$$\langle\left(\bar{X}(T) - X(T)\right), \bar{P}(T)\rangle$$

$$
\begin{aligned}
=&-\int_0^T \Big\{ \Big\langle \left(\bar{X}(t)-X(t)\right), \frac{\partial \bar{\mathcal{H}}_{ad}}{\partial x}(t) + \mathbb{E}\left[\frac{\partial \bar{\mathcal{H}}_{ad}}{\partial z}(t+\delta)\, I_{[0,T-\delta]}(t) \,\Big|\, \mathcal{F}(t) \right] \Big\rangle \\
&\quad - \left\langle \bar{P}(t), \left(\bar{b}(t)-b(t)\right) \right\rangle - \left\langle H_{\bar{P}}(t), \left(\bar{\sigma}(t)-\sigma(t)\right)\right\rangle \Big\} dt \\
&+ \int_0^T \left\langle \bar{P}(t), \left(\bar{\sigma}(t)-\sigma(t)\right) dW(t) \right\rangle + \int_0^T \left\langle \left(\bar{X}(t)-X(t)\right), H_{\bar{P}}(t)\, dW(t) \right\rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial \bar{\mathcal{H}}_{ad}}{\partial x}(t) =& \frac{\partial \mathcal{H}_{ad}}{\partial x}\left(t, \bar{X}(t), \bar{X}_a(t), \bar{X}_\delta(t), \bar{u}(t), \bar{P}(t), \bar{P}^a(t), H_{\bar{P}}(t), H_{\bar{P}^a}(t)\right), \\
\bar{b}(t) =& b\left(t, \bar{X}(t), \bar{X}_a(t), \bar{X}_\delta(t), \bar{u}(t)\right), \\
b(t) =& b\left(t, X(t), X_a(t), X_\delta(t), u(t)\right),
\end{aligned}
$$

and similarly for the partial derivatives $\frac{\partial \bar{\mathcal{H}}_{ad}}{\partial z}(t+\delta)$ and $\frac{\partial \bar{\mathcal{H}}_{ad}}{\partial y}(t)$. Then, taking expectations on the both sides of the above equation and noting the terminal value in (3.4.10), the second term on the right-hand-side of (3.5.7) becomes

$$
\begin{aligned}
&\mathbb{E}\left[\Big\langle \left(\bar{X}(T)-X(T)\right), \frac{\partial g}{\partial x}\left(\bar{X}(T), \bar{X}_a(T)\right) \Big\rangle \right] \\
=&\mathbb{E}\Big[\int_0^T \Big\{ \Big\langle \left(\bar{X}(t)-X(t)\right), \frac{\partial \bar{\mathcal{H}}_{ad}}{\partial x}(t) \Big\rangle + \Big\langle \left(\bar{X}_\delta(t)-X_\delta(t)\right), \frac{\partial \bar{\mathcal{H}}_{ad}}{\partial z}(t) \Big\rangle \\
&\quad - \left\langle \bar{P}(t), \left(\bar{b}(t)-b(t)\right) \right\rangle - \left\langle H_{\bar{P}}(t), \left(\bar{\sigma}(t)-\sigma(t)\right) \right\rangle \Big\} dt \Big].
\end{aligned}
\tag{3.5.8}
$$

Here, we have used the fact that

$$
\mathbb{E}\left[\int_0^T \left\langle \bar{P}(t), \left(\bar{\sigma}(t)-\sigma(t)\right) dW(t) \right\rangle + \int_0^T \left\langle \left(\bar{X}(t)-X(t)\right), H_{\bar{P}}(t)\, dW(t) \right\rangle \right] = 0
$$

due to (3.5.5), and we have applied the relationship between X and X_δ to make the following simplification

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \Big\langle \left(\bar{X}(t)-X(t)\right), \mathbb{E}\left[\frac{\partial \bar{\mathcal{H}}_{ad}}{\partial z}(t+\delta)\, I_{[0,T-\delta]}(t) \,\Big|\, \mathcal{F}(t) \right] \Big\rangle dt \right] \\
=&\mathbb{E}\left[\int_0^T \Big\langle \left(\bar{X}_\delta(t)-X_\delta(t)\right), \frac{\partial \bar{\mathcal{H}}_{ad}}{\partial z}(t) \Big\rangle dt \right].
\end{aligned}
$$

Similarly, applying the Itô formula to $\langle (\bar{X}_a(t)-X_a(t)), \bar{P}^a(t) \rangle$ for $t \in [0,T]$ and then taking expectations, the third term on the right-hand-side of (3.5.7) becomes

$$
\begin{aligned}
&\mathbb{E}\left[\Big\langle \left(\bar{X}_a(T)-X_a(T)\right), \frac{\partial g}{\partial y}\left(\bar{X}(T), \bar{X}_a(T)\right) \Big\rangle \right] \\
=&\mathbb{E}\Big[\int_0^T \Big\{ \Big\langle \left(\bar{X}_a(t)-X_a(t)\right), \frac{\partial \bar{\mathcal{H}}_{ad}}{\partial y}(t) \Big\rangle \\
&\quad - \Big\langle \bar{P}^a(t), \Big\{ \left(\bar{X}(t) - \lambda \bar{X}_a(t) - e^{\lambda\delta} \bar{X}_\delta(t)\right) \\
&\qquad - \left(\bar{X}(t) - \lambda \bar{X}_a(t) - e^{\lambda\delta} \bar{X}_\delta(t)\right) \Big\} \Big\rangle \Big\} dt \Big].
\end{aligned}
\tag{3.5.9}
$$

On the other hand, using the expression (3.4.9) for $\mathcal{H}_{ad}$, the first term on the right-hand-side of (3.5.6) can be re-expressed as

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \{G(t,\bar{X}(t),\bar{X}_a(t),\bar{X}_\delta(t),\bar{u}(t)) - G(t,X(t),X_a(t),X_\delta(t),u(t))\}\,dt\right] \\
=&\mathbb{E}\Bigg[\int_0^T \Bigg\{ -\Big\{\mathcal{H}_{ad}(t,\bar{X}(t),\bar{X}_a(t),\bar{X}_\delta(t),\bar{u}(t),\bar{P}(t),\bar{P}^a(t),H_{\bar{P}}(t),H_{\bar{P}^a}(t)) \\
&\qquad - \mathcal{H}_{ad}(t,\bar{X}(t),\bar{X}_a(t),\bar{X}_\delta(t),u(t),\bar{P}(t),\bar{P}^a(t),H_{\bar{P}}(t),H_{\bar{P}^a}(t))\Big\} \\
&\quad + \langle \bar{P}(t),(\bar{b}(t)-b(t))\rangle + \langle H_{\bar{P}}(t),(\bar{\sigma}(t)-\sigma(t))\rangle \\
&\quad + \Big\langle \bar{P}^a(t), \Big\{ \Big(\bar{X}(t)-\lambda\bar{X}_a(t)-e^{\lambda\delta}\bar{X}_\delta(t)\Big) \\
&\qquad - \Big(\bar{X}(t)-\lambda\bar{X}_a(t)-e^{\lambda\delta}\bar{X}_\delta(t)\Big)\Big\}\Big\rangle\Bigg\}dt\Bigg].
\end{aligned}
\tag{3.5.10}
$$

Substituting (3.5.8), (3.5.9) and (3.5.10) into the right-hand-side of (3.5.6), we get

$$
\begin{aligned}
&J_{ad}(\bar{u}) - J_{ad}(u) \\
\le&\mathbb{E}\Bigg[\int_0^T \Bigg\{\Big\{\mathcal{H}_{ad}(t,\bar{X}(t),\bar{X}_a(t),\bar{X}_\delta(t),u(t),\bar{P}(t),\bar{P}^a(t),H_{\bar{P}}(t),H_{\bar{P}^a}(t)) \\
&\qquad - \mathcal{H}_{ad}(t,\bar{X}(t),\bar{X}_a(t),\bar{X}_\delta(t),\bar{u}(t),\bar{P}(t),\bar{P}^a(t),H_{\bar{P}}(t),H_{\bar{P}^a}(t))\Big\} \\
&\quad + \left\langle (\bar{X}(t)-X(t)), \frac{\partial\bar{\mathcal{H}}_{ad}}{\partial x}(t)\right\rangle + \left\langle (\bar{X}_a(t)-X_a(t)), \frac{\partial\bar{\mathcal{H}}_{ad}}{\partial y}(t)\right\rangle \\
&\quad + \left\langle (\bar{X}_\delta(t)-X_\delta(t)), \frac{\partial\bar{\mathcal{H}}_{ad}}{\partial z}(t)\right\rangle\Bigg\}dt\Bigg],
\end{aligned}
$$

which implies

$$
J_{ad}(\bar{u}) - J_{ad}(u) \le \mathbb{E}\left[\int_0^T \left\langle (\bar{u}(t)-u(t)), \frac{\partial\bar{\mathcal{H}}_{ad}}{\partial u}(t)\right\rangle dt\right] = 0
$$

for any $u \in \mathcal{U}$, where the last equality follows from (3.4.12). Hence, $\bar{u} \in \mathcal{U}$ is an optimal control of the control problem (3.4.1). □

Note that, to apply the conjugate duality method, some conditions in Theorem 3.4.2 are stronger than those in Theorem 3.5.1. In particular, the concavity condition imposed on the Hamiltonian in the former needs to be satisfied for each $(p, h_p) \in \mathbb{R}^n \times \mathbb{R}^{n\times m}$. However, in practically, it seems impossible to verify that the corresponding concavity condition in the latter for almost every $(\omega, t) \in \Omega \times [0, T]$. On the other hand, to use the techniques of stochastic calculus in the proof, Theorem 3.5.1 requires more integrability conditions than Theorem 3.4.2 does.

CHAPTER 4

THE AUXILIARY HJB EQUATION IN STOCHASTIC OPTIMAL CONTROL PROBLEMS WITH DELAY

4.1 Introduction

This chapter turns our attention to improving the method of dynamic programming studied in [22, 23] for stochastic optimal control problems with discrete delay and those with both discrete and exponential moving average delays.

4.1.1 Literature Review

We first summarize some basic concepts and results of dynamic programming in Markovian optimal control problems taken mainly from [45, Chapter 4]. Then, we describe some known generalizations [22, 23] for this method to stochastic optimal control problems with delay which motivate our work in this chapter.

Similarly to Chapter 3, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which $W = \{W(s)\}_{s\in[0,T]}$ is a m-dimensional standard Brownian motion where $T \in (0, \infty)$ is the given deterministic terminal time, and write $\mathbb{F} = \{\mathcal{F}(s)\}_{s\in[0,T]}$ for the filtration generated by W.

The Markovian Optimal Control Problem

Recalling that the method of dynamic programming for solving a Markovian optimal control problem is to vary its initial times and states to obtain a family of such control problems. By the Bellman principle of optimality, one can establish a relationship among these control problems via a partial differential equation (PDE), which is called the Hamilton-Jacobi-Bellman (HJB) equation. Then, rather than solving a particular control problem, one uses the HJB equation to solve all the control problems in that family (see [45, Section 4.3]).

For every $t \in [0, T]$, to put the Markovian optimal control problem (3.1.3) into a framework suitable for applying the method of dynamic programming, we let $\{\mathcal{F}_t(s)\}_{s\in[t,T]}$ denote the filtration generated by $\{W(s) - W(t) | s \in [t, T]\}$ such that the usual conditions hold, and let $\mathcal{U}[t, T]$ consist of all $u|_{[t,T]}$, where $u : \Omega \times [0, T] \to \mathbb{U} \subset \mathbb{R}^r$ belongs to the admissible control set $\mathcal{U}$ as defined in (3.1.3) and $u|_{[t,T]}$ denotes the restriction of u to $[t, T]$. Note that $\{\mathcal{F}_t(s)\}_{s\in[t,T]}$ is independent of $\{\mathcal{F}(s)\}_{s\in[0,t)}$ since $\{W(s) - W(t) | s \in [t, T]\}$ is independent of $\{W(s) | s \in [0, t)\}$. Note also that, conditioning on $\mathcal{F}(t)$, $u|_{[t,T]}$ is $\{\mathcal{F}_t(s)\}_{s\in[t,T]}$-adapted for $u \in \mathcal{U}$. Then, we associate the control problem with a family of cost functions with various starting times and initial values via conditional expectation:

$$J(u; t, x) = \mathbb{E}^{t,x}\left[\int_t^T G(s, X^u(s), u(s))\, ds + g(X^u(T))\right], \tag{4.1.1}$$

where G and g are as defined in (3.1.2); X^u denotes the unique strong solution of the controlled SDE (3.1.1); $\mathbb{E}^{t,x}$ denotes the expectation with respect to the law of X^u, conditioning on $\mathcal{F}(t)$ with $X^u(t) = x \in \mathbb{R}^n$; and $u \in \mathcal{U}[t, T]$. It is easy to see that $J(u; 0, x_0) = J(u)$ and $J(u; T, x) = g(x)$ which is independent of u, where $J(u)$ is defined by (3.1.2).

As the independence between $\{\mathcal{F}_t(s)\}_{s\in(t,T]}$ and $\{\mathcal{F}(s)\}_{s\in[0,t]}$, it follows that, conditioning $\mathcal{F}(t)$ with $X^{\tilde{u}}(t) = x$, $X^{\tilde{u}}(s)$ is equal to $X^u(s)$ in law for $s \in [t, T]$, where $\tilde{u} \in \mathcal{U}$, $u = \tilde{u}|_{[t,T]}$ and X^u satisfies

$$\begin{cases} dX^u(s) = b(s, X^u(s), u(s))\, ds + \sigma(s, X^u(s), u(s))\, dW(s), \quad s \in [t, T], \\ X^u(t) = x \in \mathbb{R}^n. \end{cases} \tag{4.1.2}$$

Here b and σ are given functions as those defined in (3.1.1). Accordingly, we have associated the Markovian optimal control problem (3.1.3) with a family of control problems described by

$$\inf_{u\in\mathcal{U}[t,T]} J(u; t, x) \tag{4.1.3}$$

with different starting times $t \in [0, T]$ and initial values $x \in \mathbb{R}$, where the state system and the cost function are respectively given by (4.1.2) and (4.1.1).

Define a function $V : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ by

$$V(t, x) = \inf_{u \in \mathcal{U}[t,T]} J(u; t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^n. \tag{4.1.4}$$

This is called the value function of the Markovian optimal control problem (3.1.3) (see [45, page 178]). In particular, $V(0, x_0)$ is the optimal value of the control problem. As noted above, to obtain the HJB equation, we use the Bellman principle of optimality. That is, for any $\hat{t} \in [t, T]$, the value function V satisfies

$$V(t, x) = \inf_{u \in \mathcal{U}[t,T]} \mathbb{E}^{t,x} \left[\int_t^{\hat{t}} G(s, X^u(s), u(s))\, ds + V\left(\hat{t}, X^u(\hat{t})\right) \right], \tag{4.1.5}$$

where X^u is the strong solution of the controlled SDE (4.1.2) (see [45, Theorem 3.3]). The above equation is also referred to as the dynamic programming equation. It can be seen from (4.1.4) that (4.1.5) tells us that if $\bar{u}$ is optimal on the interval $[t, T]$ with the initial value x, then it must be optimal on $[\hat{t}, T]$ with the initial state $X^{\bar{u}}(\hat{t})$ (see [45, page 160]). Furthermore, if V is sufficiently smooth,

$$\lim_{\hat{t} \to t+} \frac{1}{\hat{t} - t} \left\{ \mathbb{E}^{t,x} \left[V\left(\hat{t}, X^u(\hat{t})\right) \right] - V(t, x) \right\} \tag{4.1.6}$$

can be expressed in terms of V by using the Itô formula. This, together with (4.1.5), implies that V solves the second-order PDE, i.e. the HJB equation,

$$\begin{cases} \dfrac{\partial V}{\partial t}(t, x) + \inf\limits_{u \in \mathbb{U}} \mathcal{G}\left(t, x, u, \dfrac{\partial V}{\partial x}(t, x), \dfrac{\partial^2 V}{\partial x^2}(t, x)\right) = 0, \\ \qquad\qquad (t, x) \in [0, T) \times \mathbb{R}^n, \\ V(T, x) = g(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{4.1.7}$$

where $\mathcal{G}$ is given by

$$\mathcal{G}(t, x, u, p, q) = G(t, x, u) + \langle p, b(t, x, u) \rangle + \frac{1}{2} \operatorname{tr}\left(\sigma\sigma^\top(t, x, u)\, q\right). \tag{4.1.8}$$

The function $\mathcal{G}$ is called the generalized Hamiltonian (function) which is different from the Hamiltonian $\mathcal{H}$ defined by (3.1.5) in the context of the stochastic maximum principle.

The advantage of applying the dynamic programming method is that an optimal control $\bar{u} \in \mathcal{U}[t,T]$ can be constructed via a solution of the HJB equation. More precisely, suppose that the HJB equation (4.1.7) admits a solution $\bar{V}$, where the corresponding infimum is attained at a $\mathcal{B}([0,T] \times \mathbb{R}^n)$-measurable function $u_0(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$. Then, as noted in [45, Section 5.5.1], $\bar{u}$ defined by

$$\bar{u}(s) = u_0(s, \bar{X}(s)), \quad \forall s \in [t,T],$$

is an optimal control of the control problem (4.1.3) and also $\bar{V}$ coincides with the value function, where $\bar{X}$ denotes the strong solution of the controlled SDE (4.1.2) with u in $b(s,x,u)$ and $\sigma(s,x,u)$ replaced by u_0. The above procedure for obtaining such a $\bar{u}$ is called the stochastic verification technique.

As mentioned in Chapter 1, there is a connection between the stochastic maximum principle and dynamic programming for Markovian optimal control problems: the adjoint process may be expressed in terms of the derivatives of the value function together with the corresponding optimal control and the solution of the controlled SDE. More explicitly, suppose that $\bar{V}$ is a solution of the HJB equation (4.1.7) so that, by the stochastic verification technique mentioned above, one can obtain an optimal control $\bar{u} \in \mathcal{U}[t,T]$ and that $\bar{V}$ coincides with the value function defined by (4.1.4). Then, the pair $(\bar{P}, H_{\bar{P}})$ defined by

$$\begin{cases} \bar{P}(s) = -\dfrac{\partial \bar{V}}{\partial x}(s, \bar{X}(s)), \\ H_{\bar{P}}(s) = -\dfrac{\partial^2 \bar{V}}{\partial x^2}(s, \bar{X}(s))\, \sigma(s, \bar{X}(s), \bar{u}(s)), \end{cases} \quad \forall s \in [t,T], \tag{4.1.9}$$

satisfies the adjoint equation (3.1.6) with the initial time 0 replaced by t and (X,u) replaced by $(\bar{X}, \bar{u})$, where $\bar{X}$ is the strong solution of the controlled SDE (4.1.2) associated with $\bar{u}$ (see [2, page 402] and [45, Chapter 5]).

The Stochastic Optimal Control Problems with Delay

Extending the Markovian optimal control problem to the time-delay context, Larssen in [22] first considers a stochastic optimal control problem involving general delay. For every $t \in [0,T]$, write $X_{[t-\delta,t]}$ for the path segment of X from $t-\delta$ to t composed with the shift operator $s \to s+t$ for $s \in [-\delta, 0]$. That is,

$$X_{[t-\delta,t]}(s) = X(t+s), \quad s \in [-\delta, 0], \tag{4.1.10}$$

where $\delta \in (0, T)$. To use the dynamic programming, similarly to that for Markovian optimal control problems, Larssen in [22, page 661] obtains the associated family of control problems described as follows.

For $(t, \xi) \in [0, T] \times \mathbb{C}([-\delta, 0]; \mathbb{R}^n)$ and given functions $b : [0, T] \times \mathbb{C}([-\delta, 0]; \mathbb{R}^n) \times \mathbb{U} \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{C}([-\delta, 0]; \mathbb{R}^n) \times \mathbb{U} \to \mathbb{R}^{n \times m}$, let $\mathcal{F}(s)$-adapted continuous state process $X^u : \Omega \times [t - \delta, T] \to \mathbb{R}^n$ be described by the controlled stochastic differential delay equation (SDDE)

$$\begin{cases} dX^u(s) = b\left(s, X^u_{[s-\delta,s]}, u(s)\right) ds + \sigma\left(s, X^u_{[s-\delta,s]}, u(s)\right) dW(s), \\ \qquad\qquad\qquad s \in [t, T], \\ X^u(s) = \xi(s-t), \quad s \in [t-\delta, t], \end{cases} \tag{4.1.11}$$

where u is selected from a given admissible control set $\mathcal{U}[t, T]$ defined in a similar fashion to that for the Markovian optimal control problem. For given functions $G : [0, T] \times \mathbb{C}([-\delta, 0]; \mathbb{R}^n) \times \mathbb{U} \to \mathbb{R}$ and $g : \mathbb{C}([-\delta, 0]; \mathbb{R}^n) \to \mathbb{R}$, the cost function $J_{gd}(u; t, \xi)$ is defined by

$$J_{gd}(u; t, \xi) = \mathbb{E}^{t,\xi}\left[\int_t^T G\left(s, X^u_{[s-\delta,s]}, u(s)\right) ds + g\left(X^u_{[T-\delta,T]}\right)\right], \tag{4.1.12}$$

where, in a similar sense to the conditional expectation $\mathbb{E}^{t,x}$ in (4.1.1) together with the argument related to $X^{\tilde{u}}$ and X^u for Markovian optimal control problems, $\mathbb{E}^{t,\xi}$ denotes the conditional expectation with respect to the law of the strong solution X^u of the controlled SDDE (4.1.11); and $u \in \mathcal{U}[t, T]$.

Then, for given $(t, \xi) \in [0, T] \times \mathbb{C}([-\delta, 0]; \mathbb{R}^n)$, the stochastic optimal control problem with delay, associated with the state system (4.1.11) and the cost function (4.1.12), is to find $\bar{u} \in \mathcal{U}[t, T]$ realizing

$$\inf_{u \in \mathcal{U}[t,T]} J_{gd}(u; t, \xi). \tag{4.1.13}$$

Note that the stochastic optimal control problem with discrete delay and that with both discrete and exponential moving average delays studied in the previous chapter can be regarded as special cases of the above problem. As for (4.1.4), Larssen in [22, page 662] defines the corresponding value function by

$$V(t, \xi) = \inf_{u \in \mathcal{U}[t,T]} J_{gd}(u; t, \xi), \quad (t, \xi) \in [0, T] \times \mathbb{C}([-\delta, 0]; \mathbb{R}^n). \tag{4.1.14}$$

Note that $V(T,\xi) = \inf_{u \in \mathcal{U}[t,T]} J_{gd}(u;T,\xi) = g(\xi)$. Larssen in [22, Theorem 4.2] obtains the dynamic programming equation expressed by

$$V(t,\xi) = \inf_{u \in \mathcal{U}[t,T]} \mathbb{E}^{t,\xi} \left[\int_t^{\hat{t}} G\left(s, X^u_{[s-\delta,s]}, u(s)\right) ds + V\left(\hat{t}, X^u_{[\hat{t}-\delta,\hat{t}]}\right) \right], \tag{4.1.15}$$

where $\hat{t} \in [t,T]$ and X^u denotes the strong solution of the controlled SDDE (4.1.11). Unlike in the above Markovian case, the value function V here may depend on the entire initial path ξ in a complicated way which causes certain difficulty in solving this problem. More precisely, it is generally difficult to apply the Itô formula to express the corresponding limit (4.1.6) in terms of V in a similar way to that for Markovian optimal control problems except for some special cases.

For example, Larssen and Risebro in [23] (see also [5] and [22, Section 5]) study a stochastic optimal control problem with both discrete and exponential moving average delays, where they assume that the corresponding value function V depends on ξ only through $x(\xi) = \xi(0)$ and $y(\xi) = \int_{-\delta}^0 e^{\lambda r}\xi(r)dr$ satisfying

$$V(t,\xi) = V(t, x(\xi), y(\xi)) \tag{4.1.16}$$

for $(t,\xi) \in [0,T] \times \mathbb{C}([-\delta,0;\mathbb{R}^n])$. If the function V on the right-hand-side of (4.1.16) is sufficiently smooth, this hypothesis allows the authors to apply the Itô formula given in [22, Lemma 5.1] to obtain the HJB equation

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x,y) + \left\langle \dfrac{\partial V}{\partial y}(t,x,y), \left(x - \lambda y - e^{-\lambda\delta} z\right) \right\rangle \\ \qquad + \inf_{u \in \mathbb{U}} \mathcal{G}_{ad}\left(t,x,y,z,u, \dfrac{\partial V}{\partial x}(t,x,y), \dfrac{\partial^2 V}{\partial x^2}(t,x,y)\right) = 0, \\ \qquad\qquad (t,x,y,z) \in [0,T) \times \mathbb{R}^{n\times 3}, \\ V(T,x,y) = g(x,y), \quad (x,y) \in \mathbb{R}^{n\times 2}, \end{cases} \tag{4.1.17}$$

where the generalized Hamiltonian $\mathcal{G}_{ad}$ is defined by

$$\begin{aligned} &\mathcal{G}_{ad}(t,x,y,z,u,p,h) \\ &= \langle p, b(t,x,y,z,u)\rangle + \frac{1}{2}\operatorname{tr}\left(\sigma\sigma^\top(t,x,y,z,u)\,h\right) + G(t,x,y,z,u). \end{aligned}$$

Unfortunately, the stochastic verification technique associated with the HJB equation (4.1.17) has a restriction: the solution $\bar{V}$ of the HJB equation, if it exists, is required to be independent of z due to the hypothesis (4.1.16). This is not easily satisfied even when the model is independent of X_δ corresponding to z since the second term on the left-hand-side of the first equation in (4.1.17) is due to the dependence on X_a (see [22, Lemma 5.1]). It is worth mentioning that Pang and Hussain in [31] try to eliminate this restriction by increasing the time-delay in the exponential moving average delay, defining

$$\tilde{X}_a(s) = \int_{-\infty}^{0} e^{\lambda r} X(s+r)\, dr, \quad \forall s \in [t,T].$$

This type of time-delay is referred to as completed memory of exponential moving average delay in [31]. Then, Pang and Hussain impose a similar hypothesis on the corresponding value function to the one described by (4.1.16) with $y(\xi)$ replaced by $\int_{-\infty}^{0} e^{\lambda r}\xi(r)dr$. This allows them to obtain a HJB equation described in a similar manner to (4.1.17), where the corresponding second term on the left-hand-side of the first equation in (4.1.17) is $\left\langle \frac{\partial V}{\partial y}(t,x,y), (x-\lambda y)\right\rangle$ (see [31, Lemma 2.2]). Then, if the model only depends on $\tilde{X}_a$, the solution (if it exists) of the corresponding HJB equation is generally independent of z.

Note that the techniques for deriving the HJB equation (4.1.17) cannot be adapted to solve the stochastic optimal control problem with discrete delay since the hypothesis (4.1.16) is not valid if the model is independent of X_a. On the other hand, it is worth pointing out that Kolmanovskiĭ and Shaĭkhet in [19, Chapter 3] first introduce a function $\mathcal{V}$ which satisfies $\mathcal{V}(t,x,y) = \mathcal{V}_0(t,x) + \mathcal{V}_1(t,y)$ and then study the method of dynamic programming for a class of deterministic optimal control problems with discrete delay, where the value function V can be expressed in terms of $\mathcal{V}$. This allows them to obtain the following first-order PDE

$$\begin{cases} \dfrac{\partial \mathcal{V}}{\partial t}(t,x,y) + \inf\limits_{u\in\mathbb{U}}\left\{ \left\langle \dfrac{\partial \mathcal{V}_0}{\partial x}(t,x) + \dfrac{\partial \mathcal{V}_1}{\partial y}(t+\delta,x), b(t,x,y,u)\right\rangle \right. \\ \qquad\qquad\qquad \left. + G(t,x,y,u)\right\} = 0, \quad (t,x,y) \in [0,T) \times \mathbb{R}^{n\times 2}, \\ \mathcal{V}(T,x,y) = g(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{4.1.18}$$

Note that, although the above PDE is solved by $\mathcal{V}$ rather than the value function V, it plays a similar role to the classical HJB equation which allows the authors to construct an optimal control (see [19, Theorem 3.1.2]). However, to the best of our knowledge, the generalization of the PDE (4.1.18) to the corresponding stochastic case has not been investigated in the academic literature.

Having obtained the HJB equation (4.1.17), the connection with the stochastic maximum principle involving the adjoint equations (3.1.17)-(3.1.19) studied in [29] for stochastic optimal control problems with both discrete and exponential moving average delays has been carried out by Shi in [41, Theorem 3.1]. That is, if $\bar{P} = (\bar{P}_1, \bar{P}_2, \bar{P}_3)^\top$ and $\bar{H} = (\bar{H}_1, \bar{H}_2)^\top$ are defined by

$$\begin{cases} \bar{P}_1(s) = -\dfrac{\partial \bar{V}}{\partial x}\left(s, \bar{X}(s), \bar{X}_a(s)\right), \\ \bar{H}_1(s) = -\dfrac{\partial^2 \bar{V}}{\partial x^2}\left(s, \bar{X}(s), \bar{X}_a(s)\right)\sigma\left(s, \bar{X}(s), \bar{X}_a(s), \bar{X}_\delta(s), \bar{u}(s)\right), \end{cases}$$

$$\begin{cases} \bar{P}_2(s) = -\dfrac{\partial \bar{V}}{\partial y}\left(s, \bar{X}(s), \bar{X}_a(s)\right), \\ \bar{H}_2(s) = -\dfrac{\partial^2 \bar{V}}{\partial x \partial y}\left(s, \bar{X}(s), \bar{X}_a(s)\right)\sigma\left(s, \bar{X}(s), \bar{X}_a(s), \bar{X}_\delta(s), \bar{u}(s)\right), \end{cases}$$

and

$$\bar{P}_3(s) = -\frac{\partial \bar{V}}{\partial z}\left(s, \bar{X}(s), \bar{X}_a(s)\right) = 0 \tag{4.1.19}$$

for $s \in [t, T]$, where $\bar{V}$ is a solution of (4.1.17), then $(\bar{P}, \bar{H})$ solves (3.1.17)-(3.1.19) with (X, u) replaced by $(\bar{X}, \bar{u})$. Here, $(\bar{X}, \bar{u})$ is obtained in a similar way to that for the above Markovian case. In particular, it can be seen from (4.1.19) that the requirement that $\bar{P}_3(t) \equiv 0$ discussed in Section 3.1.1 coincides with the requirement of independence of z for $\bar{V}$ mentioned above. Since we have obtained the new adjoint equations (3.4.10)-(3.4.11) in Section 3.4, we wonder whether there exists an HJB equation to connect with those in a similar fashion to the above.

4.1.2 Main Results and Structure of the Chapter

This chapter investigates the method of dynamic programming for a class of stochastic optimal control problems with discrete delay and those with both discrete and exponential moving average delays, where the value functions are separable specified in Definition 4.3.1. We adapt the technique in [19, Chapter 3] mentioned above to obtain the so-called auxiliary HJB equations for the control problems in Theorems 4.3.3 & 4.6.2 and the stochastic verification theorems in Theorems 4.4.1 & 4.6.3 respectively. If the model just involves a discrete delay, the auxiliary HJB equation plays a similar role to the HJB equation (4.1.7) for Markovian optimal control problems. If the model involves both types of delays, our auxiliary HJB equation not only generalizes the HJB equation (4.1.17) studied in [22, 23] but also removes the restriction there for some special cases. The connections with the stochastic maximum principles presented in Theorems 3.3.2 & 3.4.2 are obtained in Theorems 4.5.1 & 4.6.5 respectively.

The reminder of the chapter is organized as follows. Section 4.2 puts the stochastic optimal control problem with discrete delay into a suitable framework for applying the dynamic programming method. Section 4.3 obtains the auxiliary HJB equation for the control problem, where the value function is separable. Section 4.4 states and proves the verification theorem. To demonstrate how to use the auxiliary HJB equation to find an optimal control, a solvable example is provided. The connection with the corresponding stochastic maximum principle studied in the previous chapter is obtained in Section 4.5. Section 4.6 extends the results to include the exponential moving average delay.

4.2 The Stochastic Optimal Control Problem with Discrete Delay

We continue to work with the deterministic terminal time T, probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion W and filtration $\mathbb{F} = \{\mathcal{F}(s)\}_{s\in[0,T]}$ introduced in Section 4.1.1. Recalling the stochastic optimal control problem with discrete delay studied in Chapter 3 as follows.

For given functions $b : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^{n\times m}$, let $X : \Omega \times [-\delta, T] \to \mathbb{R}^n$ be the continuous $\mathcal{F}(s)$-adapted state process satisfying the controlled SDDE

$$\begin{cases} dX(s) = b(s, X(s), X_\delta(s), u(s))\, ds \\ \qquad\qquad\qquad + \sigma(s, X(s), X_\delta(s), u(s))\, dW(s), \quad s \in [0,T], \\ X(s) = \xi_0(s), \quad s \in [-\delta, 0], \end{cases} \tag{4.2.1}$$

where $\delta \in (0,T)$, $\xi_0 \in \mathbb{C}([-\delta, 0]; \mathbb{R}^n)$ and $u : \Omega \times [0,T] \to \mathbb{U} \subset \mathbb{R}^r$ is an $\mathcal{F}(s)$-adapted control selected from a given admissible control set $\mathcal{U}$, which is defined as before, such that the controlled SDDE (4.2.1) admits a unique strong solution for every $u \in \mathcal{U}$. Moreover, for given functions $G : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, the cost function J_d is defined by

$$J_d(u) = \mathbb{E}\left[\int_0^T G(s, X(s), X_\delta(s), u(s))\, ds + g(X(T))\right]. \tag{4.2.2}$$

Then, the stochastic optimal control problem with discrete delay associated with the state system (4.2.1) and the cost function (4.2.2) is to find $\bar{u} \in \mathcal{U}$ realizing

$$\inf_{u \in \mathcal{U}} J_d(u). \tag{4.2.3}$$

To ensure that, for any $\xi_0 \in \mathbb{C}([-\delta, 0]; \mathbb{R}^n)$ and $u \in \mathcal{U}$, the controlled SDDE (4.2.1) admits a unique strong solution X^u and the cost function J_d is integrable, we consider the following two hypotheses.

Hypothesis III. The functions b and σ are continuous with respect to $(t, u) \in [0, T] \times \mathbb{U}$ and are Lipschitz continuous with respect to $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with the Lipschitz constant independent of $(t, u) \in [0, T] \times \mathbb{U}$.

Hypothesis IV. The functions G and g are continuous such that, for any $\xi_0 \in \mathbb{C}([-\delta, 0]; \mathbb{R}^n)$ and $u \in \mathcal{U}$,

$$\mathbb{E}\left[\int_0^T |G(s, X^u(s), X^u_\delta(s), u(s))|\, ds + |g(X^u(T))|\right] < \infty.$$

Note that the differences compared with Hypotheses I & II are due to that we do not need to transfer the control problem to a convex problem.

For any $(t, \xi) \in [0, T] \times \mathbb{C}([-\delta, 0]; \mathbb{R}^n)$, using a similar argument for Markovian optimal control problems described in Section 4.1.1, we associate the control problem (4.2.3) with a family of cost functions with different starting times and initial paths via conditional expectation:

$$J_d(u; t, \xi) = \mathbb{E}^{t,\xi}\left[\int_t^T G(s, X^u(s), X^u_\delta(s), u(s))\, ds + g(X^u(T))\right], \tag{4.2.4}$$

where $\mathbb{E}^{t,\xi}$ denotes the expectation with respect to the law of X^u, conditioning on $\mathcal{F}(t)$ with $X_{[t-\delta,t]} = \xi$, and $u \in \mathcal{U}[t, T]$. Then, comparing (4.2.4) with (4.2.2), we have $J_d(u; 0, \xi_0) = J_d(u)$ and

$$J_d(u; T, \xi) = g(\xi(0)),$$

which is independent of u. It follows from the independence between $\{\mathcal{F}_t(s)\}_{s\in(t,T]}$ and $\{\mathcal{F}(s)\}_{s\in[0,t]}$ that, conditioning $\mathcal{F}(t)$ with $X^{\tilde{u}}_{[t-\delta,t]} = \xi$, $X^{\tilde{u}}(s)$ is equal to $X^u(s)$ in law for $s \in [t,T]$, where $\tilde{u} \in \mathcal{U}$, $u = \tilde{u}|_{[t,T]}$ and X^u satisfies

$$\begin{cases} dX^u(s) = b\left(s, X^u(s), X^u_\delta(s), u(s)\right) ds \\ \qquad\qquad + \sigma\left(s, X^u(s), X^u_\delta(s), u(s)\right) dW(s), \quad s \in [t,T], \\ X^u(s) = \xi(s-t), \quad s \in [t-\delta, t], \end{cases} \tag{4.2.5}$$

Note that, if the controlled SDDE (4.2.1) admits a unique strong solution $X^{\tilde{u}}$, then $X^{\tilde{u}}$ is a continuous process so that $X^{\tilde{u}}_{[t-\delta,t]}$ is $\mathcal{F}(t)$-measurable and, given $\mathcal{F}(t)$, $X^{\tilde{u}}_{[t-\delta,t]}$ is determined and lies in $\mathbb{C}([-\delta,0];\mathbb{R}^n)$. Note also that, under Hypotheses III & IV, the controlled SDDE (4.2.5) has a unique strong solution and the cost function (4.2.4) is integrable for any $\xi \in \mathbb{C}([-\delta,0];\mathbb{R}^n)$ and $u \in \mathcal{U}[t,T]$. Hence, similarly to (4.1.3), we have associated the stochastic optimal control problem with discrete delay (4.2.3) with a family of control problems

$$\inf_{u\in\mathcal{U}[t,T]} J_d(u;t,\xi) \tag{4.2.6}$$

with different starting times $t \in [0,T]$ and initial paths $\xi \in \mathbb{C}([-\delta,0];\mathbb{R}^n)$, where the state system and the cost function are respectively given by (4.2.5) and (4.2.4).

Similarly to that in Markovian optimal control problems, we define the value function $V : [0,T] \times \mathbb{C}([-\delta,0];\mathbb{R}^n) \to \mathbb{R}$ by

$$V(t,\xi) = \inf_{u\in\mathcal{U}[t,T]} J_d(u;t,\xi), \quad (t,\xi) \in [0,T) \times \mathbb{C}\left([-\delta,0];\mathbb{R}^n\right). \tag{4.2.7}$$

In particular, $V(0,\xi_0)$ is the optimal value for the control problem (4.2.3). Since the control problem (4.2.6) can be regarded as a special case of (4.1.13), we can rewrite the dynamic programming equation (4.1.15) as

$$V(t,\xi) = \inf_{u\in\mathcal{U}[t,T]} \mathbb{E}^{t,\xi}\left[\int_t^{\hat{t}} G\left(s, X^u(s), X^u_\delta(s), u(s)\right) ds + V\left(\hat{t}, X^u_{[\hat{t}-\delta,\hat{t}]}\right)\right] \tag{4.2.8}$$

for any $(t,\xi) \in [0,T] \times \mathbb{C}([-\delta,0];\mathbb{R}^n)$ and any $\hat{t} \in [t,T]$. Note that the above value function V generally depends on the initial path ξ in a complicated way which makes the control problem (4.2.6) less easily solvable. As discussed in Section 4.1.1, it is usually difficult to apply the classical Itô formula to the above dynamic programming in a similar way to that for Markovian optimal control problems and the hypothesis in [23], described by (4.1.16), is not valid in this context. Thus, the results of [23] cannot be applied to our control problem.

4.3 The Auxiliary HJB Equation

To overcome the difficulty mentioned in the previous section, this section first adapts the technique used in [19, Chapter 3] to introduce a so-called auxiliary function for a class of stochastic optimal control problems with discrete delay, where the value functions are separable as stated by the definition below, and then generalizes the PDE (4.1.18) to the stochastic context.

Let $\mathbb{C}^{1,2}([0,T] \times \mathbb{R}^n)$ be the space of continuous functions $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ such that the partial derivatives $\frac{\partial F}{\partial t}$ and $\frac{\partial^2 F}{\partial x_i \partial x_j}$ for $i, j \in \{1, 2, \ldots, n\}$ exist and are continuous.

Definition 4.3.1. The value function V for the stochastic optimal control problem with discrete delay (4.2.3) is called separable if there are two functions $\mathcal{V}_0, \mathcal{V}_1 \in \mathbb{C}^{1,2}([0,T] \times \mathbb{R}^n)$, with $V_1(T, \cdot) \equiv 0$, such that, for any $(t, \xi) \in [0,T] \times \mathbb{C}([-\delta, 0]; \mathbb{R}^n)$,

$$\begin{aligned} V(t,\xi) =& \mathcal{V}_0(t, \xi(0)) + \mathcal{V}_1(\min\{t+\delta, T\}, \xi(0)) \\ &- \int_t^{\min\{t+\delta,T\}} \frac{\partial \mathcal{V}_1}{\partial s}(s, \xi(s-\delta-t))\, ds. \end{aligned} \tag{4.3.1}$$

If V is separable, we call

$$\mathcal{V}(t,x,y) = \mathcal{V}_0(t,x) + \mathcal{V}_1(t,y),$$

an auxiliary function of V.

Note that, if the value function V of the control problem (4.2.3) is separable, we can also re-express (4.3.1) as

$$V(t,\xi) = \mathcal{V}(t, \xi(0), \xi(-\delta)) + \mathcal{C}(t, \xi(\cdot - t)), \tag{4.3.2}$$

where

$$\begin{aligned} \mathcal{C}(t, \xi(\cdot - t)) =& \mathcal{V}_1(t+\delta, \xi(0)) - \mathcal{V}_1(t, \xi(-\delta)) \\ &- \int_t^{t+\delta} \frac{\partial \mathcal{V}_1}{\partial s}(s, \xi(s-\delta-t))\, ds, \end{aligned} \tag{4.3.3}$$

and where we have taken $\mathcal{V}_1(t,x) \equiv 0$ for all $t > T$ for notational simplicity. Note also that, for a given separable value function V, the functions $\mathcal{V}_0$ and $\mathcal{V}_1$ are not unique. More precisely, if $\mathcal{V}_0$ and $\mathcal{V}_1$ satisfy (4.3.1), then $\mathcal{V}_0 + F$ and $\mathcal{V}_1 - F$ also satisfy (4.3.1) for any continuously differentiable function F on $[0,T]$ with $F(T) = 0$. Nevertheless, the function $\mathcal{C}(t,\xi(\cdot - t))$ corresponding to $\mathcal{V}_1 - F$ is identical to that corresponding to $\mathcal{V}_1$, i.e. it is invariant with respect to the change from $\mathcal{V}_1$ to $\mathcal{V}_1 - F$. On the other hand, the auxiliary function $\mathcal{V}$ is uniquely defined. Indeed, if $\tilde{\mathcal{V}}$ is also an auxiliary function of V, then there is a continuously differentiable function F with $F(T) = 0$, such that $\tilde{\mathcal{V}}(t,x,y) = \tilde{\mathcal{V}}_0(t,x) + \tilde{\mathcal{V}}_1(t,y) = \mathcal{V}(t,x,y) + F(t)$. One may take $\tilde{\mathcal{V}}_0(t,x) = \mathcal{V}_0(t,x)$ and $\tilde{\mathcal{V}}_1(t,y) = \mathcal{V}_1(t,y) + F(t)$. Since $\mathcal{C}(t,\xi(\cdot - t))$ is invariant with respect to the change from $\mathcal{V}_1$ to $\tilde{\mathcal{V}}_1$, the expression (4.3.2) for V in terms of $\mathcal{V}$ and $\tilde{\mathcal{V}}$ implies that $\mathcal{V} = \tilde{\mathcal{V}}$. In the following section, we shall give certain conditions to guarantee that the control problem (4.2.6) admit an auxiliary function.

When the value function V is separable, although it generally still depends on the entire initial path ξ, we are able to express the limit (4.3.4) (see below) in terms of the auxiliary function $\mathcal{V}$. As noted in Section 4.1.1, the expression for the limit (4.1.6) in Markovian optimal control problems plays an important role in deriving the HJB equation from the dynamic programming equation.

Lemma 4.3.2. *Assume that Hypotheses III & IV hold. Suppose that the value function V for the stochastic optimal control problem with discrete delay* (4.2.3) *is separable associated with an auxiliary function $\mathcal{V}(t,x,y) = \mathcal{V}_0(t,x) + \mathcal{V}_1(t,y)$. Then, for any $(t,\xi) \in [0,T] \times \mathbb{C}([-\delta,0];\mathbb{R}^n)$ and $u \in \mathcal{U}[t,T]$,*

$$\lim_{\hat{t}\to t+} \frac{1}{\hat{t}-t} \left\{ \mathbb{E}^{t,\xi} \left[V\left(\hat{t}, X^u_{[\hat{t}-\delta,\hat{t}]}\right) \right] - V\left(t,\xi\right) \right\}, \tag{4.3.4}$$

where $\hat{t} \in [t,T]$, can be expressed as

$$\begin{aligned}
&\frac{\partial \mathcal{V}}{\partial t}\left(t,\xi\left(0\right),\xi\left(-\delta\right)\right) \\
&+ \left\langle \left(\frac{\partial \mathcal{V}_0}{\partial x}\left(t,\xi\left(0\right)\right) + \frac{\partial \mathcal{V}_1}{\partial y}\left(t+\delta,\xi\left(0\right)\right) \right), b\left(t,\xi\left(0\right),\xi\left(-\delta\right),u\left(t\right)\right) \right\rangle \\
&+ \frac{1}{2}\operatorname{tr}\left(\sigma\sigma^\top\left(t,\xi\left(0\right),\xi\left(-\delta\right),u\left(t\right)\right) \left(\frac{\partial^2 \mathcal{V}_0}{\partial x^2}\left(t,\xi\left(0\right)\right) \right.\right. \\
&\qquad\qquad \left.\left. + \frac{\partial^2 \mathcal{V}_1}{\partial y^2}\left(t+\delta,\xi\left(0\right)\right) \right) \right),
\end{aligned} \tag{4.3.5}$$

where $X^u_{[\hat{t}-\delta,\hat{t}]}$ *is defined by* (4.1.10) *with* t *replaced by* $\hat{t}$ *and* X^u *is the strong solution of the controlled SDDE* (4.2.5).

Proof. For simplicity of notation, we extend the domain of $\mathcal{V}_1$ to $[0, T+\delta] \times \mathbb{R}^n$ such that $\mathcal{V}_1(t,\cdot) \equiv 0$ for $t > T$.

Fix $u \in \mathcal{U}[t,T]$ and $\hat{t} \in [t,T]$, and write X^u for the corresponding strong solution of the controlled SDDE (4.2.5). Then, by (4.3.1), the term within the bracket in (4.3.4) can be rewritten as

$$\begin{aligned}
&\mathbb{E}^{t,\xi}\left[V\left(\hat{t}, X^u_{[\hat{t}-\delta,\hat{t}]}\right)\right] - V(t,\xi) \\
=&\left\{\mathbb{E}^{t,\xi}\left[\mathcal{V}_0\left(\hat{t}, X^u\left(\hat{t}\right)\right)\right] - \mathcal{V}_0\left(t,\xi(0)\right)\right\} \\
&+\left\{\mathbb{E}^{t,\xi}\left[\mathcal{V}_1\left(\hat{t}+\delta, X^u\left(\hat{t}\right)\right)\right] - \mathcal{V}_1\left(t+\delta,\xi(0)\right)\right\} \\
&+\left\{\int_t^{t+\delta}\frac{\partial\mathcal{V}_1}{\partial s}\left(s,\xi(s-\delta-t)\right)ds - \mathbb{E}^{t,\xi}\left[\int_{\hat{t}}^{\hat{t}+\delta}\frac{\partial\mathcal{V}_1}{\partial s}\left(s,X^u_\delta(s)\right)ds\right]\right\}.
\end{aligned} \tag{4.3.6}$$

Applying the Itô formula to $\mathcal{V}_0(s, X^u(s))$ for $s \in \left[t,\hat{t}\right]$, we obtain that

$$\begin{aligned}
&\mathcal{V}_0\left(\hat{t}, X^u\left(\hat{t}\right)\right) - \mathcal{V}_0\left(t,\xi(0)\right) \\
=&\int_t^{\hat{t}}\left\{\frac{\partial\mathcal{V}_0}{\partial s}\left(s,X^u(s)\right) + \left\langle\frac{\partial\mathcal{V}_0}{\partial x}\left(s,X^u(s)\right), b(s)\right\rangle\right. \\
&\qquad\left.+\frac{1}{2}\operatorname{tr}\left(\sigma\sigma^\top(s)\frac{\partial^2\mathcal{V}_0}{\partial x^2}\left(s,X^u(s)\right)\right)\right\}ds \\
&+\int_t^{\hat{t}}\left\langle\frac{\partial\mathcal{V}_0}{\partial x}\left(s,X^u(s)\right),\sigma(s)\,dW(s)\right\rangle,
\end{aligned} \tag{4.3.7}$$

where we have used the shorthand notation

$$\begin{aligned}
b(s) &= b\left(s, X^u(s), X^u_\delta(s), u(s)\right), \\
\sigma(s) &= \sigma\left(s, X^u(s), X^u_\delta(s), u(s)\right).
\end{aligned} \tag{4.3.8}$$

Then, taking conditional expectations $\mathbb{E}^{t,\xi}$ on the both sides of (4.3.7), we see that the terms within the first pair of brackets on the right-hand-side of (4.3.6) can be expanded in the form as

$$\begin{aligned}
&\mathbb{E}^{t,\xi}\left[\mathcal{V}_0\left(\hat{t}, X^u\left(\hat{t}\right)\right)\right] - \mathcal{V}_0\left(t,\xi(0)\right) \\
=&\mathbb{E}^{t,\xi}\left[\int_t^{\hat{t}}\left\{\frac{\partial\mathcal{V}_0}{\partial s}\left(s,X^u(s)\right) + \left\langle\frac{\partial\mathcal{V}_0}{\partial x}\left(s,X^u(s)\right), b(s)\right\rangle\right.\right. \\
&\qquad\left.\left.+\frac{1}{2}\operatorname{tr}\left(\sigma\sigma^\top(s)\frac{\partial^2\mathcal{V}_0}{\partial x^2}\left(s,X^u(s)\right)\right)\right\}ds\right].
\end{aligned} \tag{4.3.9}$$

Similarly, application of the Itô formula to $\mathcal{V}_1(s+\delta, X^u(s))$ for $s \in [t, \hat{t}]$ gives the following expansion of the terms which the second pair of brackets on the right-hand-side of (4.3.6):

$$\begin{aligned}
&\mathbb{E}^{t,\xi}\left[\mathcal{V}_1\left(\hat{t}+\delta, X^u\left(\hat{t}\right)\right)\right] - \mathcal{V}_1\left(t+\delta, \xi\left(0\right)\right) \\
=&\mathbb{E}^{t,\xi}\left[\int_t^{\hat{t}}\left\{\frac{\partial \mathcal{V}_1}{\partial s}\left(s+\delta, X^u\left(s\right)\right) + \left\langle \frac{\partial \mathcal{V}_1}{\partial y}\left(s+\delta, X^u\left(s\right)\right), b\left(s\right)\right\rangle\right.\right. \\
&\left.\left.+\frac{1}{2}\operatorname{tr}\left(\sigma\sigma^\top\left(s\right)\frac{\partial^2 \mathcal{V}_1}{\partial y^2}\left(s+\delta, X^u\left(s\right)\right)\right)\right\}ds\right].
\end{aligned} \tag{4.3.10}$$

In particular, using the relationship between X and X_δ, we re-express the first term of the right-hand-side of (4.3.10) as

$$\begin{aligned}
&\mathbb{E}^{t,\xi}\left[\int_t^{\hat{t}}\frac{\partial \mathcal{V}_1}{\partial s}\left(s+\delta, X^u\left(s\right)\right)ds\right] \\
=&\mathbb{E}^{t,\xi}\left[\int_{t+\delta}^{\hat{t}+\delta}\frac{\partial \mathcal{V}_1}{\partial s}\left(s, X^u\left(s-\delta\right)\right)ds\right] \\
=&\mathbb{E}^{t,\xi}\left[\int_t^{\hat{t}}\frac{\partial \mathcal{V}_1}{\partial s}\left(s, X_\delta^u\left(s\right)\right)ds\right] + \mathbb{E}^{t,\xi}\left[\int_{\hat{t}}^{\hat{t}+\delta}\frac{\partial \mathcal{V}_1}{\partial s}\left(s, X_\delta^u\left(s\right)\right)ds\right] \\
&-\int_t^{t+\delta}\frac{\partial \mathcal{V}_1}{\partial s}\left(s, \xi\left(s-\delta-t\right)\right)ds.
\end{aligned}$$

Thus, substituting (4.3.9) and (4.3.10), as well as using the above equality, into the right-hand-side of (4.3.6), we obtain that

$$\begin{aligned}
&\mathbb{E}^{t,\xi}\left[V\left(\hat{t}, X^u_{\left[\hat{t}-\delta,\hat{t}\right]}\right)\right] - V\left(t,\xi\right) \\
=&\mathbb{E}^{t,\xi}\left[\int_t^{\hat{t}}\left\{\left\langle\left(\frac{\partial \mathcal{V}_0}{\partial x}\left(s, X^u\left(s\right)\right) + \frac{\partial \mathcal{V}_1}{\partial y}\left(s+\delta, X^u\left(s\right)\right)\right), b\left(s\right)\right\rangle\right.\right. \\
&+\frac{1}{2}\operatorname{tr}\left(\sigma\sigma^\top\left(s\right)\left(\frac{\partial^2 \mathcal{V}_0}{\partial x^2}\left(s, X^u\left(s\right)\right) + \frac{\partial^2 \mathcal{V}_1}{\partial y^2}\left(s+\delta, X^u\left(s\right)\right)\right)\right) \\
&\left.\left.+\frac{\partial \mathcal{V}}{\partial s}\left(s, X^u\left(s\right), X_\delta^u\left(s\right)\right)\right\}ds\right].
\end{aligned} \tag{4.3.11}$$

Finally, dividing by $\hat{t}-t$ on the both sides of (4.3.11) and letting $\hat{t} \to t+$, we have the required equation (4.3.5), where the existence of the limit follows from the continuity of b and σ described in Hypothesis III. □

With the result of Lemma 4.3.2, we derive a PDE solved by the auxiliary function $\mathcal{V}$ of the value function for the control problem (4.2.3) as follows.

Theorem 4.3.3. *Assume that the conditions of Lemma 4.3.2 hold. Then, the auxiliary function $\mathcal{V}(t,x,y) = \mathcal{V}_0(t,x) + \mathcal{V}_1(t,y)$ of the value function for the stochastic optimal control problem with discrete delay* (4.2.3) *is a solution of the following terminal value problem of the second-order PDE:*

$$\begin{cases} \dfrac{\partial \mathcal{V}}{\partial t}(t,x,y) \\ \quad + \inf_{u\in\mathbb{U}} \mathcal{G}_d\Big(t,x,y,u,\dfrac{\partial \mathcal{V}_0}{\partial x}(t,x) + \dfrac{\partial \mathcal{V}_1}{\partial y}(t+\delta,x), \dfrac{\partial^2 \mathcal{V}_0}{\partial x^2}(t,x) \\ \qquad\qquad + \dfrac{\partial^2 \mathcal{V}_1}{\partial y^2}(t+\delta,x)\Big) = 0, \quad (t,x,y) \in [0,T) \times \mathbb{R}^{n\times 2}, \\ \mathcal{V}(T,x,y) = \mathcal{V}_0(T,x) = g(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{4.3.12}$$

where $\mathcal{G}_d : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ *is given by*

$$\begin{aligned} &\mathcal{G}_d(t,x,y,u,p,q) \\ &= \langle p, b(t,x,y,u)\rangle + \frac{1}{2}\operatorname{tr}\Big(\sigma\sigma^\top(t,x,y,u)\,q\Big) + G(t,x,y,u), \end{aligned} \tag{4.3.13}$$

and where $\mathbb{S}^n$ *denotes the space of* $n \times n$ *symmetric matrices.*

As (4.3.12) is a PDE solved by the auxiliary function $\mathcal{V}$, rather than the value function, and is expressed in a similar form to the HJB equation (4.1.7) for Markovian optimal control problems (see [45, Chapter 4]), we shall call it the auxiliary HJB equation for the control problem (4.2.3), where the function $\mathcal{G}_d$ defined by (4.3.13) is referred to as the corresponding generalized Hamiltonian.

Note that the proof below uses essentially the same technique as that for the Markovian case described in [45, Proposition 3.5]. For completeness, we give the details as follows.

Proof. Fix $(t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ and $u \in \mathbb{U}$ (i.e. value for a control). Let $\xi \in \mathbb{C}([-\delta,0];\mathbb{R}^n)$ be a path such that $x = \xi(0)$ and $y = \xi(-\delta)$; $v \in \mathcal{U}[t,T]$ be the control such that $v(t) \equiv u$; and X^v be the corresponding strong solution of the controlled SDDE (4.2.5) with u replaced by v.

It follows from the dynamic programming equation (4.2.8) that

$$V(t,\xi) \le \mathbb{E}^{t,\xi}\left[\int_t^{\hat{t}} G\left(s, X^v(s), X^v_\delta(s), v(s)\right) ds + V\left(\hat{t}, X^v_{[\hat{t}-\delta,\hat{t}]}\right)\right],$$

where $\hat{t} \in [t, T]$. Then, dividing by $\hat{t} - t$ on the both sides of the above inequality, we obtain that

$$\begin{aligned} 0 \leq & \frac{1}{\hat{t}-t} \mathbb{E}^{t,\xi} \left[\int_t^{\hat{t}} G\left(s, X^v(s), X_\delta^v(s), v(s)\right) ds \right] \\ & + \frac{1}{\hat{t}-t} \left\{ \mathbb{E}^{t,\xi} \left[V\left(\hat{t}, X^v_{[\hat{t}-\delta,\hat{t}]}\right) \right] - V(t,\xi) \right\}. \end{aligned} \tag{4.3.14}$$

Rewriting the value function V on the right-hand-side of the above inequality in terms of the auxiliary function $\mathcal{V}$, and then letting $\hat{t} \to t$ and using (4.3.5) for the limit, we obtain that

$$\begin{aligned} 0 \leq & \frac{\partial \mathcal{V}}{\partial t}(t,x,y) + G(t,x,y,u) \\ & + \left\langle \left(\frac{\partial \mathcal{V}_0}{\partial x}(t,x) + \frac{\partial \mathcal{V}_1}{\partial y}(t+\delta,x) \right), b(t,x,y,u) \right\rangle \\ & + \frac{1}{2} \operatorname{tr}\left(\sigma\sigma^\top(t,x,y,u) \left(\frac{\partial^2 \mathcal{V}_0}{\partial x^2}(t,x) + \frac{\partial^2 \mathcal{V}_1}{\partial y^2}(t+\delta,x) \right) \right). \end{aligned} \tag{4.3.15}$$

This gives that

$$\begin{aligned} 0 \leq & \frac{\partial \mathcal{V}}{\partial t}(t,x,y) + \inf_{u \in \mathbb{U}} \mathcal{G}_d \bigg(t,x,y,u, \frac{\partial \mathcal{V}_0}{\partial x}(t,x) + \frac{\partial \mathcal{V}_1}{\partial y}(t+\delta,x), \frac{\partial^2 \mathcal{V}_0}{\partial x^2}(t,x) \\ & \qquad + \frac{\partial^2 \mathcal{V}_1}{\partial y^2}(t+\delta,x) \bigg). \end{aligned} \tag{4.3.16}$$

On the other hand, it follows from the dynamic programming equation (4.2.8) that, for any sufficiently small $\varepsilon > 0$, we can find a $v^\varepsilon \in \mathcal{U}[t,T]$ such that

$$\begin{aligned} V(t,\xi) + \varepsilon\left(\hat{t}-t\right) \geq & \mathbb{E}^{t,\xi} \left[\int_t^{\hat{t}} G\left(s, X^{v^\varepsilon}(s), X_\delta^{v^\varepsilon}(s), v^\varepsilon(s)\right) ds \right] \\ & + \mathbb{E}^{t,\xi} \left[V\left(\hat{t}, X^{v^\varepsilon}_{[\hat{t}-\delta,\hat{t}]}\right) \right]. \end{aligned} \tag{4.3.17}$$

Then, as that for (4.3.14) and (4.3.15), we divide by $\hat{t} - t$ on the both sides of (4.3.17) to get the following inequality

$$\begin{aligned} \varepsilon \geq & \frac{1}{\hat{t}-t} \mathbb{E}^{t,\xi} \left[\int_t^{\hat{t}} G\left(s, X^{v^\varepsilon}(s), X_\delta^{v^\varepsilon}(s), v^\varepsilon(s)\right) ds \right] \\ & + \frac{1}{\hat{t}-t} \left\{ \mathbb{E}^{t,\xi} \left[V\left(\hat{t}, X^{v^\varepsilon}_{[\hat{t}-\delta,\hat{t}]}\right) \right] - V(t,\xi) \right\}, \end{aligned}$$

which, letting $\hat{t} \to t$ and using (4.3.5), implies that

$$\begin{aligned} \varepsilon \geq & \frac{\partial \mathcal{V}}{\partial t}(t,x,y) + G(t,x,y,u) \\ & + \left\langle \left(\frac{\partial \mathcal{V}_0}{\partial x}(t,x) + \frac{\partial \mathcal{V}_1}{\partial y}(t+\delta,x) \right), b(t,x,y,u) \right\rangle \end{aligned}$$

$$+\frac{1}{2}\operatorname{tr}\left(\sigma\sigma^{\top}(t,x,y,u)\left(\frac{\partial^2\mathcal{V}_0}{\partial x^2}(t,x)+\frac{\partial^2\mathcal{V}_1}{\partial y^2}(t+\delta,x)\right)\right)$$
$$\geq\frac{\partial\mathcal{V}}{\partial t}(t,x,y)+\inf_{u\in\mathbb{U}}\mathcal{G}_d\bigg(t,x,y,u,\frac{\partial\mathcal{V}_0}{\partial x}(t,x)+\frac{\partial\mathcal{V}_1}{\partial y}(t+\delta,x),\frac{\partial^2\mathcal{V}_0}{\partial x^2}(t,x)$$
$$+\frac{\partial^2\mathcal{V}_1}{\partial y^2}(t+\delta,x)\bigg).$$

Finally, combining the above inequalities with (4.3.16), we obtain (4.3.12), where the boundary condition follows immediately from the definitions of both the value and auxiliary functions given in (4.2.7) and Definition 4.3.1 respectively. □

If $\delta = 0$, X_δ is identical with X. Under the conditions given in Theorem 4.3.3 with $\delta = 0$, the corresponding $\mathcal{C}(t,\xi(\cdot - t))$ defined by (4.3.3) is identically zero. Consequently, there exists a function $\hat{\mathcal{V}} : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ such that $\hat{\mathcal{V}}(t,x) = \mathcal{V}(t,x,x)$ which coincides with the value function given by (4.1.4). Then, noting Definition 4.3.1, the corresponding auxiliary HJB equation and generalized Hamiltonian coincide with (4.1.7) and (4.1.8) respectively.

4.4 A Stochastic Verification Theorem

As mentioned in Section 4.1.1, the stochastic verification technique for Markovian optimal control problems offers one way to construct an optimal control $\bar{u} \in \mathcal{U}[t,T]$ via a solution of the HJB equation (4.1.7). For the stochastic optimal control problem with discrete delay (4.2.3), although we do not in general have an HJB equation that the value function V satisfies, it is possible to have the following stochastic verification theorem in terms of auxiliary functions when V is separable.

Theorem 4.4.1. *Assume that Hypotheses III & IV hold. Suppose that there exist $\bar{\mathcal{V}}_0, \bar{\mathcal{V}}_1 \in \mathbb{C}^{1,2}([0,T]\times\mathbb{R}^n)$ with $\bar{\mathcal{V}}_1(T,\cdot)\equiv 0$ such that*

$$\bar{\mathcal{V}}(t,x,y) = \bar{\mathcal{V}}_0(t,x) + \bar{\mathcal{V}}_1(t,y)$$

satisfies the auxiliary HJB equation (4.3.12). *Then, for any $(t,\xi)\in[0,T]\times\mathbb{C}([-\delta,0];\mathbb{R}^n)$,*

$$J_d(u;t,\xi) \geq \bar{\mathcal{V}}(t,\xi(0),\xi(-\delta)) + \bar{\mathcal{C}}(t,\xi(\cdot - t)) \tag{4.4.1}$$

for all $u\in\mathcal{U}[t,T]$, where $\bar{\mathcal{C}}(t,\xi(\cdot - t))$ is given by (4.3.3) *with $\mathcal{V}_1$ replaced by $\bar{\mathcal{V}}_1$. Moreover, if there exists a $\mathcal{B}([0,T]\times\mathbb{R}^{n\times 2})$-measurable function $u_0 : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{U}$ such that* *the infimum in* (4.3.12) *is attained at* $u_0(t,x,y)$ *for all* $(t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ *in the sense that*

$$\begin{aligned}
&\mathcal{G}_d\bigg(t,x,y,u_0\left(t,x,y\right),\frac{\partial \bar{\mathcal{V}}_0}{\partial x}\left(t,x\right) \\
&\qquad + \frac{\partial \bar{\mathcal{V}}_1}{\partial y}\left(t+\delta,x\right),\frac{\partial^2 \bar{\mathcal{V}}_0}{\partial x^2}\left(t,x\right)+\frac{\partial^2 \bar{\mathcal{V}}_1}{\partial y^2}\left(t+\delta,x\right)\bigg) \\
&= \inf_{u\in\mathbb{U}} \mathcal{G}_d\bigg(t,x,y,u,\frac{\partial \bar{\mathcal{V}}_0}{\partial x}\left(t,x\right) \\
&\qquad + \frac{\partial \bar{\mathcal{V}}_1}{\partial y}\left(t+\delta,x\right),\frac{\partial^2 \bar{\mathcal{V}}_0}{\partial x^2}\left(t,x\right)+\frac{\partial^2 \bar{\mathcal{V}}_1}{\partial y^2}\left(t+\delta,x\right)\bigg),
\end{aligned} \tag{4.4.2}$$

where $\mathcal{G}_d$ *is defined by* (4.3.13), *then the control* $\bar{u}$, *defined by*

$$\bar{u}\left(s\right) = u_0\left(s,\bar{X}\left(s\right),\bar{X}_\delta\left(s\right)\right), \quad \forall s \in [t,T], \tag{4.4.3}$$

is an optimal control of the stochastic optimal control problem with discrete delay (4.2.6), *where* $\bar{X}$ *denotes the strong solution of the controlled SDDE* (4.2.5) *with* u *in* $b(s,x,y,u)$ *and* $\sigma(s,x,y,u)$ *replaced by* u_0.

Proof. As before, for notational ease, we extend the domain of $\bar{\mathcal{V}}_1$ to $[0,T+\delta] \times \mathbb{R}^n$ such that $\mathcal{V}_1(t,\cdot) \equiv 0$ for $t > T$.

Fix $u \in \mathcal{U}[t,T]$ and $\xi \in \mathbb{C}([-\delta,0];\mathbb{R}^n)$, and write X^u for the corresponding strong solution of the controlled SDDE (4.2.5). Then, it follows from the condition that $\bar{\mathcal{V}}(t,x,y) = \bar{\mathcal{V}}_0(t,x) + \bar{\mathcal{V}}_1(t,y)$ that

$$\begin{aligned}
&\mathbb{E}^{t,\xi}\left[\bar{\mathcal{V}}\left(\hat{t},X^u\left(\hat{t}\right),X^u_\delta\left(\hat{t}\right)\right)\right] - \bar{\mathcal{V}}\left(t,\xi\left(0\right),\xi\left(-\delta\right)\right) \\
&= \left\{\mathbb{E}^{t,\xi}\left[\bar{\mathcal{V}}_0\left(\hat{t},X^u\left(\hat{t}\right)\right)\right] - \bar{\mathcal{V}}_0\left(t,\xi\left(0\right)\right)\right\} \\
&\quad + \left\{\mathbb{E}^{t,\xi}\left[\bar{\mathcal{V}}_1\left(\hat{t},X^u_\delta\left(\hat{t}\right)\right)\right] - \bar{\mathcal{V}}_1\left(t,\xi\left(-\delta\right)\right)\right\},
\end{aligned} \tag{4.4.4}$$

where $\hat{t} \in [t,T]$. The terms in the first pair of brackets on the right-hand-side of (4.4.4) can be expanded by (4.3.9) with $\mathcal{V}_0$ replaced by $\bar{\mathcal{V}}_0$. For the terms within the second pair of brackets on the right-hand-side of (4.4.4), we write

$$\begin{aligned}
&\mathbb{E}\left[\bar{\mathcal{V}}_1\left(\hat{t},X^u_\delta\left(\hat{t}\right)\right)\right] - \bar{\mathcal{V}}_1\left(t,\xi\left(-\delta\right)\right) \\
&= \left\{\mathbb{E}^{t,\xi}\left[\bar{\mathcal{V}}_1\left(\hat{t},X^u_\delta\left(\hat{t}\right)\right)\right] - \bar{\mathcal{V}}_1\left(t+\delta,\xi\left(0\right)\right)\right\} \\
&\quad + \left\{\bar{\mathcal{V}}_1\left(t+\delta,\xi\left(0\right)\right) - \bar{\mathcal{V}}_1\left(t,\xi\left(-\delta\right)\right)\right\}.
\end{aligned} \tag{4.4.5}$$

Applying the Itô formula to $\bar{\mathcal{V}}_1(s, X_\delta^u(s))$ for $s \in \left[t+\delta, \hat{t}\right]$ and using the relationship between X and X_δ, we have

$$
\begin{aligned}
& \mathcal{V}_1\left(\hat{t}, X_\delta^u\left(\hat{t}\right)\right) - \mathcal{V}_1\left(t+\delta, \xi\left(0\right)\right) \\
= & \int_t^{\hat{t}-\delta} \left\{ \frac{\partial \mathcal{V}_1}{\partial s}\left(s+\delta, X^u\left(s\right)\right) + \left\langle \frac{\partial \mathcal{V}_1}{\partial y}\left(s+\delta, X^u\left(s\right)\right), b\left(s\right) \right\rangle \right. \\
& \qquad \left. + \frac{1}{2}\operatorname{tr}\left(\sigma\sigma^\top\left(s\right) \frac{\partial^2 \mathcal{V}_1}{\partial y^2}\left(s+\delta, X^u\left(s\right)\right) \right) \right\} ds \\
& + \int_t^{\hat{t}-\delta} \left\langle \frac{\partial \mathcal{V}_1}{\partial y}\left(s+\delta, X^u\left(s\right)\right), \sigma\left(s\right) dW\left(s\right) \right\rangle \\
= & \int_t^{\hat{t}} \left\{ \frac{\partial \mathcal{V}_1}{\partial s}\left(s, X_\delta^u\left(s\right)\right) + \left\langle \frac{\partial \mathcal{V}_1}{\partial y}\left(s+\delta, X^u\left(s\right)\right), b\left(s\right) \right\rangle I_{\left[t, \hat{t}-\delta\right]}\left(s\right) \right. \\
& \qquad \left. + \frac{1}{2}\operatorname{tr}\left(\sigma\sigma^\top\left(s\right) \frac{\partial^2 \mathcal{V}_1}{\partial y^2}\left(s+\delta, X^u\left(s\right)\right) \right) I_{\left[t, \hat{t}-\delta\right]}\left(s\right) \right\} ds \\
& + \int_t^{\hat{t}-\delta} \left\langle \frac{\partial \mathcal{V}_1}{\partial y}\left(s+\delta, X^u\left(s\right)\right), \sigma\left(s\right) dW\left(s\right) \right\rangle \\
& - \int_t^{t+\delta} \frac{\partial \bar{\mathcal{V}}_1}{\partial s}\left(s, \xi\left(s-\delta-t\right)\right) ds,
\end{aligned}
\tag{4.4.6}
$$

where we have used the shorthand notation $b(s)$ and $\sigma(s)$ given by (4.3.8). Then, taking conditional expectations $\mathbb{E}^{t,\xi}$ on the both sides of (4.4.6), we have that

$$
\begin{aligned}
& \mathbb{E}^{t,\xi}\left[\bar{\mathcal{V}}_1\left(\hat{t}, X_\delta^u\left(\hat{t}\right)\right)\right] - \bar{\mathcal{V}}_1\left(t+\delta, \xi\left(0\right)\right) \\
= & \mathbb{E}^{t,\xi}\left[\int_t^{\hat{t}} \left\{ \frac{\partial \mathcal{V}_1}{\partial s}\left(s, X_\delta^u\left(s\right)\right) + \left\langle \frac{\partial \mathcal{V}_1}{\partial y}\left(s+\delta, X^u\left(s\right)\right), b\left(s\right) \right\rangle I_{\left[t, \hat{t}-\delta\right]}\left(s\right) \right.\right. \\
& \qquad \left.\left. + \frac{1}{2}\operatorname{tr}\left(\sigma\sigma^\top\left(s\right) \frac{\partial^2 \mathcal{V}_1}{\partial y^2}\left(s+\delta, X^u\left(s\right)\right) \right) I_{\left[t, \hat{t}-\delta\right]}\left(s\right) \right\} ds \right] \\
& - \int_t^{t+\delta} \frac{\partial \bar{\mathcal{V}}_1}{\partial s}\left(s, \xi\left(s-\delta-t\right)\right) ds.
\end{aligned}
$$

This allows us to rewrite the right-hand-side of (4.4.5) as

$$
\begin{aligned}
& \mathbb{E}^{t,\xi}\left[\bar{\mathcal{V}}_1\left(\hat{t}, X_\delta^u\left(\hat{t}\right)\right)\right] - \bar{\mathcal{V}}_1\left(t, \xi\left(-\delta\right)\right) \\
= & \mathbb{E}^{t,\xi}\left[\int_t^{\hat{t}} \left\{ \frac{\partial \bar{\mathcal{V}}_1}{\partial s}\left(s, X_\delta^u\left(s\right)\right) + \left\langle \frac{\partial \bar{\mathcal{V}}_1}{\partial y}\left(s+\delta, X^u\left(s\right)\right), b\left(s\right) \right\rangle I_{\left[t, \hat{t}-\delta\right]}\left(s\right) \right.\right. \\
& \qquad \left.\left. + \frac{1}{2}\operatorname{tr}\left(\sigma\sigma^\top\left(s\right) \frac{\partial^2 \bar{\mathcal{V}}_1}{\partial y^2}\left(s+\delta, X^u\left(s\right)\right) \right) I_{\left[t, \hat{t}-\delta\right]}\left(s\right) \right\} ds \right] \\
& + \left\{ \bar{\mathcal{V}}_1\left(t+\delta, \xi\left(0\right)\right) - \bar{\mathcal{V}}_1\left(t, \xi\left(-\delta\right)\right) \right\} - \int_t^{t+\delta} \frac{\partial \bar{\mathcal{V}}_1}{\partial s}\left(s, \xi\left(s-\delta-t\right)\right) ds.
\end{aligned}
\tag{4.4.7}
$$

Now, substituting (4.4.7) and the corresponding (4.3.9) into the right-hand-side of (4.4.4) and then letting $\hat{t} = T$, we obtain that

$$\begin{aligned}
&\mathbb{E}^{t,\xi}\left[\bar{\mathcal{V}}_0\left(T, X^u\left(T\right)\right)\right] - \bar{\mathcal{V}}\left(t, \xi\left(0\right), \xi\left(-\delta\right)\right) - \bar{\mathcal{C}}\left(t, \xi\left(\cdot - t\right)\right) \\
=&\mathbb{E}^{t,\xi}\Bigg[\int_t^T \Bigg\{\frac{\partial \bar{\mathcal{V}}}{\partial s}\left(s, X^u\left(s\right), X^u_\delta\left(s\right)\right) \\
&\qquad + \left\langle \left\{\frac{\partial \bar{\mathcal{V}}_0}{\partial x}\left(s, X^u\left(s\right)\right) + \frac{\partial \bar{\mathcal{V}}_1}{\partial y}\left(s+\delta, X^u\left(s\right)\right)\right\}, b\left(s\right)\right\rangle \\
&\qquad + \frac{1}{2}\operatorname{tr}\Bigg(\sigma\sigma^\top\left(s\right)\left\{\frac{\partial^2 \bar{\mathcal{V}}_0}{\partial x^2}\left(s, X^u\left(s\right)\right)\right. \\
&\qquad\qquad \left.+ \frac{\partial^2 \bar{\mathcal{V}}_1}{\partial y^2}\left(s+\delta, X^u\left(s\right)\right)\right\}\Bigg)\Bigg\} ds\Bigg],
\end{aligned} \tag{4.4.8}$$

where the indicator function in (4.4.7) is eliminated since $\bar{\mathcal{V}}_1(t, \cdot) \equiv 0$ for all $t \geq T$. Furthermore, rearranging the terms in (4.4.8) and using the expression (4.3.13) for $\mathcal{G}_d$, we have that

$$\begin{aligned}
&\bar{\mathcal{V}}\left(t, \xi\left(0\right), \xi\left(-\delta\right)\right) + \bar{\mathcal{C}}\left(t, \xi\left(\cdot - t\right)\right) \\
=&\mathbb{E}^{t,\xi}\left[\int_t^T G\left(s, X^u\left(s\right), X^u_\delta\left(s\right), u\left(s\right)\right) ds + \bar{\mathcal{V}}_0\left(T, X^u\left(T\right)\right)\right] \\
&- \mathbb{E}^{t,\xi}\Bigg[\int_t^T \Bigg\{\frac{\partial \bar{\mathcal{V}}}{\partial s}\left(s, X^u\left(s\right), X^u_\delta\left(s\right)\right) \\
&\qquad + \mathcal{G}_d\Bigg(s, X^u\left(s\right), X^u_\delta\left(s\right), u\left(s\right), \frac{\partial \bar{\mathcal{V}}_0}{\partial x}\left(s, X^u\left(s\right)\right) \\
&\qquad\qquad + \frac{\partial \bar{\mathcal{V}}_1}{\partial y}\left(s+\delta, X^u\left(s\right)\right), \frac{\partial^2 \bar{\mathcal{V}}_0}{\partial x^2}\left(s, X^u\left(s\right)\right) \\
&\qquad\qquad + \frac{\partial^2 \bar{\mathcal{V}}_1}{\partial y^2}\left(s+\delta, X^u\left(s\right)\right)\Bigg)\Bigg\} ds\Bigg].
\end{aligned}$$

If $\bar{\mathcal{V}}$ is a solution of the auxiliary HJB equation (4.3.12), then we have $\mathcal{V}_0(T, x) = g(x)$ so that, noting (4.2.4), the above equation is equivalent to

$$\begin{aligned}
&\bar{\mathcal{V}}\left(t, \xi\left(0\right), \xi\left(-\delta\right)\right) + \bar{\mathcal{C}}\left(t, \xi\left(\cdot - t\right)\right) \\
=&J_d\left(u; t, \xi\right) \\
&- \mathbb{E}^{t,\xi}\Bigg[\int_t^T \Bigg\{\frac{\partial \bar{\mathcal{V}}}{\partial s}\left(s, X^u\left(s\right), X^u_\delta\left(s\right)\right) \\
&\qquad + \mathcal{G}_d\Bigg(s, X^u\left(s\right), X^u_\delta\left(s\right), u\left(s\right), \frac{\partial \bar{\mathcal{V}}_0}{\partial x}\left(s, X^u\left(s\right)\right) \\
&\qquad\qquad + \frac{\partial \bar{\mathcal{V}}_1}{\partial y}\left(s+\delta, X^u\left(s\right)\right), \frac{\partial^2 \bar{\mathcal{V}}_0}{\partial x^2}\left(s, X^u\left(s\right)\right)
\end{aligned} \tag{4.4.9}$$

$$+ \frac{\partial^2 \bar{\mathcal{V}}_1}{\partial y^2}(s+\delta, X^u(s))\Big)\Big\}ds\Big].$$

This implies that

$$\begin{aligned}
&\bar{\mathcal{V}}(t,\xi(0),\xi(-\delta)) + \bar{\mathcal{C}}(t,\xi(\cdot - t)) \\
\leq &J_d(u;t,\xi) \\
&- \mathbb{E}^{t,\xi}\Bigg[\int_t^T \Bigg\{\frac{\partial \bar{\mathcal{V}}}{\partial s}(s, X^u(s), X^u_\delta(s)) \\
&\qquad + \inf_{u\in\mathbb{U}} \mathcal{G}_d\Bigg(s, X^u(s), X^u_\delta(s), u, \frac{\partial \bar{\mathcal{V}}_0}{\partial x}(s, X^u(s)) \\
&\qquad\qquad + \frac{\partial \bar{\mathcal{V}}_1}{\partial y}(s+\delta, X^u(s)), \frac{\partial^2 \bar{\mathcal{V}}_0}{\partial x^2}(s, X^u(s)) \\
&\qquad\qquad + \frac{\partial^2 \bar{\mathcal{V}}_1}{\partial y^2}(s+\delta, X^u(s))\Bigg)\Bigg\}ds\Bigg]
\end{aligned}$$

so that

$$J_d(u;t,\xi) \geq \bar{\mathcal{V}}(t,\xi(0),\xi(-\delta)) + \bar{\mathcal{C}}(t,\xi(\cdot - t)),$$

giving the conclusion (4.4.1).

Since u_0 is Borel-measurable with values in $\mathbb{U}$, Hypothesis III implies that the functions $b(t,x,y,u_0(t,x,y))$ and $\sigma(t,x,y,u_0(t,x,y))$ are also $\mathcal{B}([0,T]\times\mathbb{R}^{n\times 2}\times\mathbb{U})$-measurable. Then, the corresponding controlled SDDE (4.2.5) admits a unique strong solution (see [6, Theorem 2.2]). Hence, the control $\bar{u}$ given by (4.4.3) is in $\mathcal{U}[t,T]$. Furthermore, substituting $\bar{u}$ described by (4.4.3), as well as the corresponding strong solution $\bar{X}$ of (4.2.5), into the right-hand-side of (4.4.9) and noting (4.4.2), we have

$$\begin{aligned}
&\bar{\mathcal{V}}(t,\xi(0),\xi(-\delta)) + \bar{\mathcal{C}}(t,\xi(\cdot - t)) \\
= &J_d(\bar{u};t,\xi) \\
&- \mathbb{E}^{t,\xi}\Bigg[\int_t^T \Bigg\{\frac{\partial \bar{\mathcal{V}}}{\partial s}(s, \bar{X}(s), \bar{X}_\delta(s)) \\
&\qquad + \mathcal{G}_d\Bigg(s, \bar{X}(s), \bar{X}_\delta(s), \bar{u}(s), \frac{\partial \bar{\mathcal{V}}_0}{\partial x}(s, \bar{X}(s)) \\
&\qquad\qquad + \frac{\partial \bar{\mathcal{V}}_1}{\partial y}(s+\delta, \bar{X}(s)), \frac{\partial^2 \bar{\mathcal{V}}_0}{\partial x^2}(s, \bar{X}(s)) \\
&\qquad\qquad + \frac{\partial^2 \bar{\mathcal{V}}_1}{\partial y^2}(s+\delta, \bar{X}(s))\Bigg)\Bigg\}ds\Bigg]
\end{aligned}$$

$$
\begin{aligned}
=&J_d\left(\bar{u};t,\xi\right)\\
&-\mathbb{E}\Bigg[\int_t^T\Bigg\{\frac{\partial\bar{\mathcal{V}}}{\partial s}\left(s,\bar{X}\left(s\right),\bar{X}_\delta\left(s\right)\right)\\
&\qquad+\inf_{u\in\mathbb{U}}\mathcal{G}_d\Bigg(s,\bar{X}\left(s\right),\bar{X}_\delta\left(s\right),u,\frac{\partial\bar{\mathcal{V}}_0}{\partial x}\left(s,\bar{X}\left(s\right)\right)\\
&\qquad\qquad+\frac{\partial\bar{\mathcal{V}}_1}{\partial y}\left(s+\delta,\bar{X}\left(s\right)\right),\frac{\partial^2\bar{\mathcal{V}}_0}{\partial x^2}\left(s,\bar{X}\left(s\right)\right)\\
&\qquad\qquad+\frac{\partial^2\bar{\mathcal{V}}_1}{\partial y^2}\left(s+\delta,\bar{X}\left(s\right)\right)\Bigg)\Bigg\}ds\Bigg]\\
=&J_d\left(\bar{u};t,\xi\right),
\end{aligned}
$$

where the last equality is due to the fact that $\bar{\mathcal{V}}$ is a solution of the auxiliary HJB equation (4.3.12). By (4.4.1), we have that $J_d\left(\bar{u};t,\xi\right)\leq J_d\left(u;t,\xi\right)$ for every $u\in\mathcal{U}[t,T]$ which implies that $\bar{u}$ is an optimal control of the control problem (4.2.6). □

One of the immediate consequences of Theorem 4.4.1 is that, if one can find a solution $\bar{\mathcal{V}}$ of the auxiliary HJB equation (4.3.12) and the associated u_0 as required in Theorem 4.4.1, then the equality in (4.4.1) holds with respect to $\bar{u}$ given by (4.4.3). This offers the sufficient conditions for the existence of auxiliary function as stated by the following corollary.

Corollary 4.4.2. *Assume that Hypotheses III & IV hold. If there exist $\bar{\mathcal{V}}_0,\bar{\mathcal{V}}_1\in\mathbb{C}^{1,2}([0,T]\times\mathbb{R}^n)$ with $\bar{\mathcal{V}}_1(T,\cdot)\equiv 0$ such that $\bar{\mathcal{V}}\left(t,x,y\right)=\bar{\mathcal{V}}_0\left(t,x\right)+\bar{\mathcal{V}}_1\left(t,y\right)$ satisfies the auxiliary HJB equation* (4.3.12)*; and the associated infimum is attained at a $\mathcal{B}([0,T]\times\mathbb{R}^{n\times 2})$-measurable function $u_0(t,x,y)$ for all $(t,x,y)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ in the sense of* (4.4.2)*, then*

$$
V\left(t,\xi\right)=J_d\left(\bar{u};t,\xi\right)=\bar{\mathcal{V}}\left(t,\xi\left(0\right),\xi\left(-\delta\right)\right)+\bar{\mathcal{C}}\left(t,\xi\left(\cdot-t\right)\right)
$$

for any $(t,\xi)\in[0,T]\times\mathbb{C}([-\delta,0];\mathbb{R}^n)$, where $\bar{\mathcal{C}}\left(t,\xi\left(\cdot-t\right)\right)$ is given by (4.3.3) *with $\mathcal{V}_1$ replaced by $\bar{\mathcal{V}}_1$ and $\bar{u}$ is given by* (4.4.3)*. Hence, $\bar{\mathcal{V}}$ is an auxiliary function of the value function for the stochastic optimal control problem with discrete delay* (4.2.3).

4.4.1 An Example

Although we do not have conditions which guarantee the existence of a solution of the auxiliary HJB equation (4.3.12), the following provides an example to demonstrate the existence of auxiliary function.

For simplicity, we set $m = n = r = 1$. Suppose that $\mathbb{U} = \mathbb{R}$ and $\mathcal{U} = \mathbb{L}^{22}_{\mathcal{F}}$; that b and σ in (4.2.5) are given by

$$\begin{cases} b\left(s,x,y,u\right) = a_1 x + c_1 u, \\ \sigma\left(s,x,y,u\right) = \left(a_2 x^2 + b_2 y^2 + p\right)^{\frac{1}{2}}, \end{cases} \tag{4.4.10}$$

where a_1, a_2, b_2 and p are given constants such that $a_2, b_2 \geq 0$ and $p > 0$; and that G and g in (4.2.4) are given by

$$G\left(s,x,y,u\right) = \frac{1}{2}\left(c_3 u^2 + a_3 x^2 + b_3 y^2\right) \quad \text{and} \quad g\left(x\right) = \frac{1}{2} a_4 x^2, \tag{4.4.11}$$

where a_3, a_4, b_3 and c_3 are given constants such that $c_3 > 0$. Note that, since the corresponding Hamiltonian given by (3.2.24) is not concave with respect to (x, y, u) if a_3 and b_3 are negative, neither the conjugate duality method described in Theorem 2.5.2 nor the stochastic maximum principle described in Theorem 3.3.2 can be applied to this problem.

(I) Verifying Hypotheses III & IV

Clearly, the functions b, σ, G and g defined by (4.4.10) and (4.4.11) are continuous with respect to (t, x, y, u) and x respectively. Moreover, it has been seen in Examples 3.2.1 & 3.3.3 that b is Lipschitz continuous. On the other hand, for every $x, x', y, y', u \in \mathbb{R}$, we obtain that

$$\begin{aligned} & \left|\sigma\left(t,x,y,u\right) - \sigma\left(t,x',y',u\right)\right| \\ \leq & \left|\sigma\left(t,x,y,u\right) - \sigma\left(t,x',y,u\right)\right| + \left|\sigma\left(t,x',y,u\right) - \sigma\left(t,x',y',u\right)\right| \\ \leq & \frac{a_2}{\sqrt{a_2}}\left|x - x'\right| + \frac{b_2}{\sqrt{b_2}}\left|y - y'\right|, \end{aligned}$$

where the second inequality is due to the fact that the derivatives of σ, with respect to x and y, are bounded by $a_2/\sqrt{a_2}$ and $b_2/\sqrt{b_2}$ respectively. This indicates that σ is Lipschitz continuous with respect to (x, y) so that Hypothesis III is satisfied. Applying the technique used at the beginning of Example 3.2.1, the controlled SDDE (4.2.1) with b and σ so defined admits a unique strong solution X for every $\tilde{u} \in \mathcal{U} = \mathbb{L}^{22}_{\mathcal{F}}$ with $X, X_\delta \in \mathbb{L}^{22}_{\mathcal{F}}$. This implies that the corresponding controlled SDDE (4.2.5) admits a unique strong solution X^u satisfying $X^u, X^u_\delta \in \mathbb{L}^{22}_{\mathcal{F}_t}([t,T];\mathbb{R})$ for every $u \in \mathcal{U}[t,T] = \mathbb{L}^{22}_{\mathcal{F}_t}([t,T];\mathbb{R})$, where $\mathbb{L}^{22}_{\mathcal{F}_t}([t,T];\mathbb{R})$ is defined in a similar fashion to $\mathbb{L}^{22}_{\mathcal{F}}$ in Section 2.3. Also, similarly to Example 3.2.1, the cost function (4.2.2) with G and g so defined is integrable.

(II) Applying Theorem 4.4.1

It follows from (4.3.12) that the auxiliary HJB equation for the control problem (4.2.6) with b, σ, G and g so defined is

$$\begin{cases} \frac{\partial \mathcal{V}}{\partial t}(t,x,y) \\ \quad + \inf_{u\in\mathbb{U}} \left\{ \frac{1}{2}\left(a_3x^2 + b_3y^2 + c_3u^2\right) + \left(\frac{\partial \mathcal{V}_0}{\partial x}(t,x) + \frac{\partial \mathcal{V}_1}{\partial y}(t+\delta,x)\right)(a_1x + c_1u) \right. \\ \qquad\qquad \left. + \frac{1}{2}\left(\frac{\partial^2 \mathcal{V}_0}{\partial x^2}(t,x) + \frac{\partial^2 \mathcal{V}_1}{\partial y^2}(t+\delta,x)\right)\left(a_2x^2 + b_2y^2 + p\right) \right\} = 0, \\ \qquad\qquad\qquad\qquad (t,x,y) \in [0,T) \times \mathbb{R}^2, \\ \mathcal{V}(T,x,y) = \mathcal{V}_0(T,x) = \frac{1}{2}a_4x^2, \quad x \in \mathbb{R}. \end{cases} \tag{4.4.12}$$

To find a solution $\bar{\mathcal{V}}(t,x,y) = \bar{\mathcal{V}}_0(t,x) + \bar{\mathcal{V}}_1(t,y)$ of the above auxiliary HJB equation, we suppose that $\bar{\mathcal{V}}_0(t,x)$ and $\bar{\mathcal{V}}_1(t,y)$ take the form

$$\bar{\mathcal{V}}_0(t,x) = \frac{1}{2}\left(A(t)x^2 + C(t)\right) \text{ and } \bar{\mathcal{V}}_1(t,y) = \frac{1}{2}B(t)y^2, \tag{4.4.13}$$

where A, B and C need to be determined, and satisfy the condition that A, B and C are $\mathbb{R}$-valued continuously differentiable functions on $[0,T]$ with $B(T) = 0$. For simplicity of notation, as those in the proofs of Lemma 4.3.2 and Theorem 4.4.1, we extend the domain of B to $[0, T+\delta]$ such that $B(t) = 0$ for $t > T$. Substituting $\bar{\mathcal{V}}$ defined by (4.4.13) into (4.4.12) and taking the derivative, with respect to u, of the function within the bracket on the left-hand-side of the first equation of (4.4.12), we see that the corresponding derivative is zero if and only if

$$u = -\frac{c_1}{c_3}\left(A(t) + B(t+\delta)\right)x. \tag{4.4.14}$$

Then, taking the function u_0 in Theorem 4.4.1 to be $-c_1\left(A(t) + B(t+\delta)\right)x/c_3$, we see that u_0 satisfies the required conditions since A and B are continuously differentiable and the function within the bracket on the left-hand-side of the first equation of (4.4.12) is convex with respect to u.

Define a control $\bar{u}$ via u_0 as

$$\bar{u}(s) = -\frac{c_1}{c_3}(A(s) + B(s+\delta))\bar{X}(s), \quad \forall s \in [t, T], \tag{4.4.15}$$

where $\bar{X}$ is the unique strong solution of the corresponding controlled SDDE (4.2.5) with u in (4.4.10) replaced by u_0. Note that such a $\bar{u}$ is in $\mathcal{U}[t,T] = \mathbb{L}^{22}_{\mathcal{F}_t}([t,T];\mathbb{R})$ since $\bar{X} \in \mathbb{L}^{22}_{\mathcal{F}_t}([t,T];\mathbb{R})$ and A and B are continuous. By Theorem 4.4.1, $\bar{u}$ described by (4.4.15) is an optimal control of the control problem.

(III) Solving the Auxiliary HJB Equation

Substituting (4.4.13) and u_0 obtained by (4.4.14) into the auxiliary HJB equation (4.4.12), we see that A, B and C satisfy

$$\begin{aligned} &\frac{dA}{dt}(t)x^2 + \frac{dB}{dt}(t)y^2 + \frac{dC}{dt}(t) \\ &= \left\{\frac{c_1^2}{c_3}(A(t) + B(t+\delta))^2 - (2a_1 + a_2)(A(t) + B(t+\delta)) - a_3\right\}x^2 \\ &\quad - \{(A(t) + B(t+\delta))b_2 + b_3\}y^2 - p(A(t) + B(t+\delta)). \end{aligned} \tag{4.4.16}$$

Then, comparing the corresponding coefficients of x^2 and y^2 on the both sides of (4.4.16), we see that (4.4.16) is satisfied if A and B solve the system of ordinary differential equations (ODEs):

$$\begin{cases} \dfrac{dA}{dt}(t) = \dfrac{c_1^2}{c_3}(A(t) + B(t+\delta))^2 \\ \qquad\qquad - (2a_1 + a_2)(A(t) + B(t+\delta)) - a_3, \quad t \in [0,T], \\ A(T) = a_4; \end{cases} \tag{4.4.17}$$

$$\begin{cases} \dfrac{dB}{dt}(t) = -(A(t) + B(t+\delta))b_2 - b_3, \quad t \in [0,T], \\ B(t) = 0, \quad t \in [T, T+\delta]; \end{cases} \tag{4.4.18}$$

where the terminal values are derived from those in (4.4.12); and that C satisfies

$$C(t) = \int_t^T p(A(s) + B(s+\delta))\,ds, \quad \forall t \in [0,T]. \tag{4.4.19}$$

Note that, since the function $F(x,y) = c_1^2(x+y)^2/c_3 - (2a_1 + a_2)(x+y) - a_3$ corresponding to the terms on the right-hand-side of the first equation of (4.4.17) is not Lipschitz continuous with respect to (x,y), this system of ODEs only admits a local solution ([4, Theorem 2.8.1]).

We summarize the above by the following theorem.

Theorem 4.4.3. *If the system of ODEs* (4.4.17)-(4.4.18) *admits a solution* (A, B)*, then* $\bar{\mathcal{V}}(t,x,y) = \bar{\mathcal{V}}_0(t,x) + \bar{\mathcal{V}}_1(t,y)$ *described by* (4.4.13) *is the solution of the auxiliary HJB equation* (4.4.12)*, where* C *is obtained by* (4.4.19)*, and the control* $\bar{u}$ *described by* (4.4.15) *is optimal to the stochastic optimal control problem with discrete delay* (4.2.6)*, where* b, σ, G *and* g *are respectively defined by* (4.4.10) *and* (4.4.11).

(IV) The Backward Induction Algorithm

It is generally difficult to find an analytic solution of the above system of ODEs. Hence, we solve it numerically using a similar backward induction algorithm to the one given in Example 3.3.3, as follows.

Step 1. Suppose that $t \in [T-\delta, T]$. Then, $B(t+\delta) \equiv 0$ and the ODE (4.4.17) becomes

$$\begin{cases} \dfrac{dA}{dt}(t) = \dfrac{c_1^2}{c_3} A^2(t) - (2a_1 + a_2) A(t) - a_3, \quad t \in [T-\delta, T], \\ A(T) = a_4. \end{cases}$$

This allows us to obtain $A(t)$ in $[T-\delta, T]$. Then, $B(t)$ for $t \in [T-\delta, T]$ is obtained by the ODE (4.4.18) in which $A(t)$ is already obtained and $B(t+\delta) \equiv 0$.

Step k. Moving backward to the interval $[T-(k+1)\delta, T-k\delta]$, where $k \in \mathbb{N}^+$ such that $T-(k+1)\delta \geq 0$. Note that $B(t+\delta)$ is already known by Step $k-1$. Then, the corresponding numerical solutions for $A(t)$ and $B(t)$ are obtained in a similar way to Step 1.

Using the above backward induction algorithm, Figures 4.1 & 4.2 below give an example of such A, B and C and the corresponding pair $(\bar{X}, \bar{u})$ respectively, where a_3 and b_3 are negative.

4.5 Connection with the Stochastic Maximum Principle

Having studied the dynamic programming method for the stochastic optimal control problem with discrete delay (4.2.3), this section investigates its connection with the stochastic maximum principle obtained in Section 3.3.

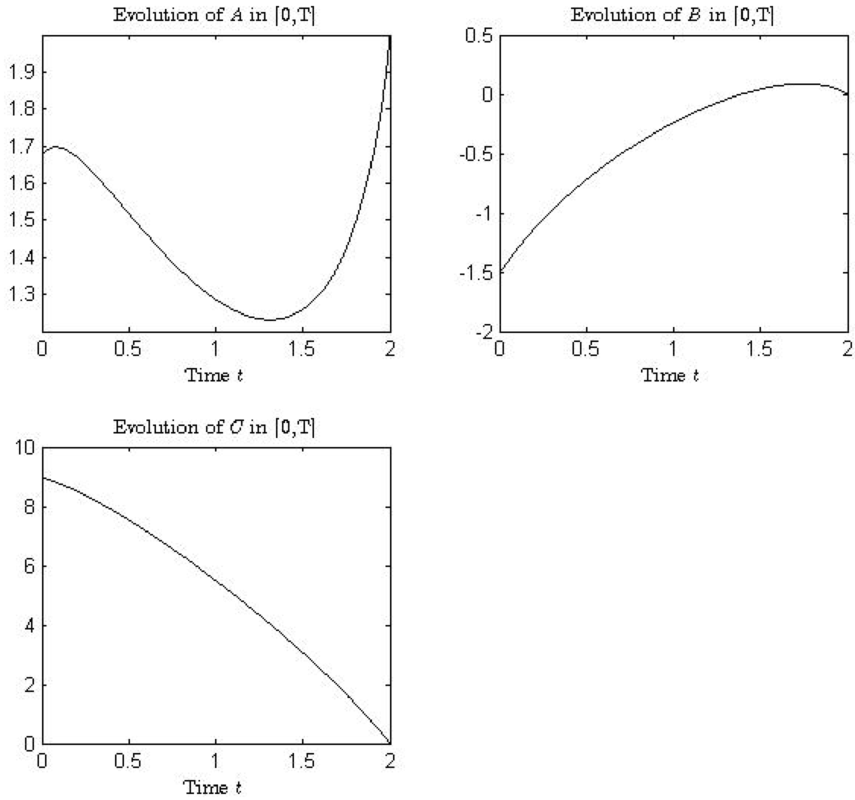

Figure 4.1: Evolution of A, B and C with parameters $T = 2$, $\delta = 0.1$, $a_1 = 3, a_2 = 2, a_3 = -3, a_4 = 2, b_2 = 2, b_3 = -3, c_1 = -3, c_3 = 2$ and $p = 4$.

Let $\mathbb{C}^{1,3}([0,T] \times \mathbb{R}^n)$ be the space of continuous functions $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ such that the partial derivatives $\frac{\partial^2 F}{\partial t \partial x_i}$ and $\frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}$, for $i, j, k \in \{1, 2, \dots, n\}$, exist and are continuous. To explore such a connection, in addition to Hypotheses III & IV, we assume further in this section the following hypothesis.

Hypothesis V. The functions b, σ and G are continuously differentiable with respect to $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$; the function g is continuously differentiable with respect to $x \in \mathbb{R}^n$; and $\mathbb{U} = \mathbb{R}^r$.

Theorem 4.5.1. *Assume that Hypotheses III, IV & V hold. Suppose that there are $\bar{\mathcal{V}}_0, \bar{\mathcal{V}}_1 \in \mathbb{C}^{1,3}([0,T] \times \mathbb{R}^n)$ with $\bar{\mathcal{V}}_1(T, \cdot) = 0$ such that $\bar{\mathcal{V}}(t, x, y) = \bar{\mathcal{V}}_0(t, x) + \bar{\mathcal{V}}_1(t, y)$ is a solution of the auxiliary HJB equation* (4.3.12). *Suppose further that the infimum of* (4.3.12) *is attained at a $\mathcal{B}([0,T] \times \mathbb{R}^{n \times 2})$-measurable function u_0 in the sense of* (4.4.2). *Let $\bar{u}$ be the optimal control defined by* (4.4.3) *and $\bar{X}$ be the strong solution of controlled SDDE* (4.2.5) *with u in*

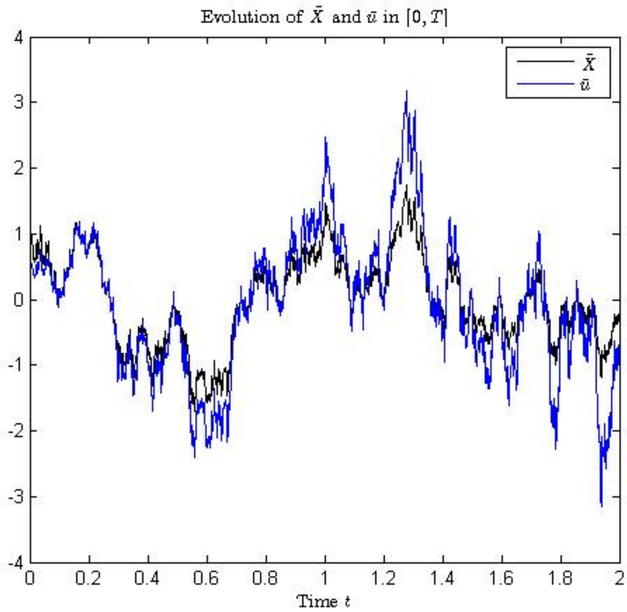

Figure 4.2: Evolution of $\bar{X}$ and $\bar{u}$ with parameters $T = 2$, $\delta = 0.1$, $\xi(t) = t + 1$ for $t \in [-0.1, 0]$, $a_1 = 3, a_2 = 2, a_3 = -3, a_4 = 2, b_2 = 2, b_3 = -3, c_1 = -3, c_3 = 2$ and $p = 4$.

$b(s,x,y,u)$ and $\sigma(s,x,y,u)$ replaced by u_0. Then, for any $(t,\xi) \in [0,T] \times \mathbb{C}([-\delta,0];\mathbb{R}^n)$, $(\bar{P}, H_{\bar{P}})$ defined by

$$\begin{cases} \bar{P}_i(s) = -\left\{ \dfrac{\partial \bar{\mathcal{V}}_0}{\partial x_i}(s, \bar{X}(s)) + \dfrac{\partial \bar{\mathcal{V}}_1}{\partial y_i}(s+\delta, \bar{X}(s)) \right\}, \\ H_{\bar{P}}^{ij}(s) = -\displaystyle\sum_{k=1}^{n} \left\{ \dfrac{\partial^2 \bar{\mathcal{V}}_0}{\partial x_i \partial x_k}(s, \bar{X}(s)) + \dfrac{\partial^2 \bar{\mathcal{V}}_1}{\partial y_i \partial y_k}(s+\delta, \bar{X}(s)) \right\} \quad \forall s \in [t,T], \\ \qquad \times \sigma_{kj}(s, \bar{X}(s), \bar{X}_\delta(s), \bar{u}(s)), \end{cases} \tag{4.5.1}$$

where $i \in \{1,2,\ldots,n\}$ and $j \in \{1,2,\ldots,m\}$, satisfies

$$\begin{cases} d\bar{P}(s) = -\left\{ \dfrac{\partial \bar{\mathcal{H}}_d}{\partial x}(s) + \mathbb{E}\left[\dfrac{\partial \bar{\mathcal{H}}_d}{\partial y}(s+\delta) I_{[t,T-\delta]}(s) \middle| \mathcal{F}(s) \right] \right\} ds \\ \qquad + H_{\bar{P}}(s) dW(s), \quad s \in [t,T], \\ \bar{P}(T) = -\dfrac{\partial g}{\partial x}(\bar{X}(T)). \end{cases} \tag{4.5.2}$$

where the Hamiltonian $\mathcal{H}_d$ is given by (3.2.24)*; and*

$$\frac{\partial \bar{\mathcal{H}}_d}{\partial x}(s) = \frac{\partial \mathcal{H}_d}{\partial x}(s, \bar{X}(s), \bar{X}_\delta(s), \bar{u}(s), \bar{P}(s), H_{\bar{P}}(s))$$

and similarly for the partial derivative $\frac{\partial \bar{\mathcal{H}}_d}{\partial y}(s+\delta)$.

Proof. Under the conditions that $\bar{\mathcal{V}}(t,x,y) = \bar{\mathcal{V}}_0(t,x) + \bar{\mathcal{V}}_1(t,y)$ solves the auxiliary HJB equation (4.3.12), we have

$$\begin{aligned}
0 = &\frac{\partial \bar{\mathcal{V}}}{\partial t}(t,x,y) + G\left(t,x,y,u_0(t,x,y)\right) \\
&+ \left\langle \left(\frac{\partial \bar{\mathcal{V}}_0}{\partial x}(t,x) + \frac{\partial \bar{\mathcal{V}}_1}{\partial y}(t+\delta,x) \right), b\left(t,x,y,u_0(t,x,y)\right) \right\rangle \\
&+ \frac{1}{2}\operatorname{tr}\left(\sigma\sigma^\top\left(t,x,y,u_0(t,x,y)\right) \left(\frac{\partial^2 \bar{\mathcal{V}}_0}{\partial x^2}(t,x) + \frac{\partial^2 \bar{\mathcal{V}}_1}{\partial y^2}(t+\delta,x) \right) \right).
\end{aligned} \tag{4.5.3}$$

In addition, since u_0 attains the infimum in (4.3.12) in the sense of (4.4.2) and $\mathbb{U} = \mathbb{R}^r$, we obtain that the derivative, with respect to u, of the function on the right-hand-side of (4.5.3) vanishes at $u = u_0(t,x,y)$. Then, for any $\hat{t} \in [t,T]$, differentiating the both sides of (4.5.3) with respect to x_i and then evaluating at $(t,x,y) = (\hat{t}, \bar{X}(\hat{t}), \bar{X}_\delta(\hat{t}))$, we have that

$$\begin{aligned}
&\frac{\partial^2 \bar{\mathcal{V}}_0}{\partial t \partial x_i}\left(\hat{t}, \bar{X}(\hat{t})\right) \\
= &-\frac{\partial G}{\partial x_i}\left(\hat{t}, \bar{X}(\hat{t}), \bar{X}_\delta(\hat{t}), \bar{u}(\hat{t})\right) \\
&- \sum_{j=1}^{n} \Bigg\{ \left(\frac{\partial^2 \bar{\mathcal{V}}_0}{\partial x_i \partial x_j}\left(\hat{t}, \bar{X}(\hat{t})\right) + \frac{\partial^2 \bar{\mathcal{V}}_1}{\partial y_i \partial y_j}\left(\hat{t}+\delta, \bar{X}(\hat{t})\right) \right) b_j\left(\hat{t}, \bar{X}(\hat{t}), \bar{X}_\delta(\hat{t}), \bar{u}(\hat{t})\right) \\
&\qquad + \left(\frac{\partial \bar{\mathcal{V}}_0}{\partial x_j}\left(\hat{t}, \bar{X}(\hat{t})\right) + \frac{\partial \bar{\mathcal{V}}_1}{\partial y_j}\left(\hat{t}+\delta, \bar{X}(\hat{t})\right) \right) \frac{\partial b_j}{\partial x_i}\left(\hat{t}, \bar{X}(\hat{t}), \bar{X}_\delta(\hat{t}), \bar{u}(\hat{t})\right) \Bigg\} \\
&- \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \left\{ \frac{\partial^3 \bar{\mathcal{V}}_0}{\partial x_i \partial x_j \partial x_k}\left(\hat{t}, \bar{X}(\hat{t})\right) + \frac{\partial^3 \bar{\mathcal{V}}_1}{\partial y_i \partial y_j \partial y_k}\left(\hat{t}+\delta, \bar{X}(\hat{t})\right) \right\} \\
&\qquad \times \left\{ \sum_{r=1}^{m} \left(\sigma_{jr}\sigma_{kr}\right)\left(\hat{t}, \bar{X}(\hat{t}), \bar{X}_\delta(\hat{t}), \bar{u}(\hat{t})\right) \right\} \\
&- \sum_{r=1}^{m}\sum_{j=1}^{n} \Bigg\{ \sum_{k=1}^{n} \Bigg\{ \left(\frac{\partial^2 \bar{\mathcal{V}}_0}{\partial x_j \partial x_k}\left(\hat{t}, \bar{X}(\hat{t})\right) + \frac{\partial^2 \bar{\mathcal{V}}_1}{\partial y_j \partial y_k}\left(\hat{t}+\delta, \bar{X}(\hat{t})\right) \right) \\
&\qquad \times \sigma_{kr}\left(\hat{t}, \bar{X}(\hat{t}), \bar{X}_\delta(\hat{t}), \bar{u}(\hat{t})\right) \Bigg\} \frac{\partial \sigma_{jr}}{\partial x_i}\left(\hat{t}, \bar{X}(\hat{t}), \bar{X}_\delta(\hat{t}), \bar{u}(\hat{t})\right) \Bigg\},
\end{aligned} \tag{4.5.4}$$

where, for the last summation on the right-hand-side of (4.5.4), we have used the fact that

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \Bigg\{ \left(\frac{\partial^2 \bar{\mathcal{V}}_0}{\partial x_j \partial x_k}\left(\hat{t}, \bar{X}(\hat{t})\right) + \frac{\partial^2 \bar{\mathcal{V}}_1}{\partial y_j \partial y_k}\left(\hat{t}+\delta, \bar{X}(\hat{t})\right) \right) \\
&\qquad \times \frac{\partial}{\partial x_i}\left(\sum_{r=1}^{m} \left(\sigma_{jr}\sigma_{kr}\right)\left(\hat{t}, \bar{X}(\hat{t}), \bar{X}_\delta(\hat{t}), \bar{u}(\hat{t})\right) \right) \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
=\sum_{r=1}^{m}\sum_{j=1}^{n}\Bigg\{\sum_{k=1}^{n}\Bigg\{&\left(\frac{\partial^2\bar{\mathcal{V}}_0}{\partial x_j\partial x_k}\left(\hat{t},\bar{X}\left(\hat{t}\right)\right)+\frac{\partial^2\bar{\mathcal{V}}_1}{\partial y_j\partial y_k}\left(\hat{t}+\delta,\bar{X}\left(\hat{t}\right)\right)\right)\\
&\times\sigma_{kr}\left(\hat{t},\bar{X}\left(\hat{t}\right),\bar{X}_\delta\left(\hat{t}\right),\bar{u}\left(\hat{t}\right)\right)\Bigg\}\\
&\times\frac{\partial\sigma_{jr}}{\partial x_i}\left(\hat{t},\bar{X}\left(\hat{t}\right),\bar{X}_\delta\left(\hat{t}\right),\bar{u}\left(\hat{t}\right)\right)\Bigg\}.
\end{aligned}
$$

Similarly, differentiating the both sides of (4.5.3) with respect to y_i, evaluating at $(t,x,y) = (\hat{t}+\delta,\bar{X}(\hat{t}+\delta),\bar{X}(\hat{t}))$ and then taking the conditional expectation with respect to $\mathcal{F}(\hat{t})$, we have that

$$
\begin{aligned}
&\frac{\partial\bar{\mathcal{V}}_1}{\partial t\partial y_i}\left(\hat{t}+\delta,\bar{X}\left(\hat{t}\right)\right)\\
&=-\mathbb{E}\Bigg[\frac{\partial G}{\partial y_i}\left(\hat{t}+\delta,\bar{X}\left(\hat{t}+\delta\right),\bar{X}\left(\hat{t}\right),\bar{u}\left(\hat{t}+\delta\right)\right)\\
&\quad+\sum_{j=1}^{n}\left\{\frac{\partial\bar{\mathcal{V}}_0}{\partial x_j}\left(\hat{t}+\delta,\bar{X}\left(\hat{t}+\delta\right)\right)+\frac{\partial\bar{\mathcal{V}}_1}{\partial y_j}\left(\hat{t}+2\delta,\bar{X}\left(\hat{t}+\delta\right)\right)\right\}\\
&\qquad\times\frac{\partial b_j}{\partial y_i}\left(\hat{t}+\delta,\bar{X}\left(\hat{t}+\delta\right),\bar{X}\left(\hat{t}\right),\bar{u}\left(\hat{t}+\delta\right)\right)\\
&\quad+\sum_{r=1}^{m}\sum_{j=1}^{n}\Bigg\{\sum_{k=1}^{n}\left\{\frac{\partial^2\bar{\mathcal{V}}_0}{\partial x_j\partial x_k}\left(\hat{t}+\delta,\bar{X}\left(\hat{t}+\delta\right)\right)+\frac{\partial^2\bar{\mathcal{V}}_1}{\partial y_j\partial y_k}\left(\hat{t}+2\delta,\bar{X}\left(\hat{t}+\delta\right)\right)\right\}\\
&\qquad\times\sigma_{kr}\left(\hat{t}+\delta,\bar{X}\left(\hat{t}+\delta\right),\bar{X}\left(\hat{t}\right),\bar{u}\left(\hat{t}+\delta\right)\right)\\
&\qquad\times\frac{\partial\sigma_{jr}}{\partial y_i}\left(\hat{t}+\delta,\bar{X}\left(\hat{t}+\delta\right),\bar{X}\left(\hat{t}\right),\bar{u}\left(\hat{t}+\delta\right)\right)\Bigg\}\Bigg|\mathcal{F}\left(\hat{t}\right)\Bigg].
\end{aligned}
\tag{4.5.5}
$$

Under the conditions that $\bar{\mathcal{V}}_0,\bar{\mathcal{V}}_1\in\mathbb{C}^{1,3}([0,T]\times\mathbb{R}^n)$, we apply the Itô formula to $\bar{P}_i$ defined by the first equation of (4.5.1) to get

$$
\begin{aligned}
d\bar{P}_i\left(s\right)=-\Bigg\{&\left\{\frac{\partial^2\bar{\mathcal{V}}_0}{\partial x_i\partial s}\left(s,\bar{X}\left(s\right)\right)+\frac{\partial^2\bar{\mathcal{V}}_1}{\partial y_i\partial s}\left(s+\delta,\bar{X}\left(s\right)\right)\right\}\\
&+\sum_{j=1}^{n}\left\{\frac{\partial^2\bar{\mathcal{V}}_0}{\partial x_i\partial x_j}\left(s,\bar{X}\left(s\right)\right)+\frac{\partial^2\bar{\mathcal{V}}_1}{\partial y_i\partial y_j}\left(s+\delta,\bar{X}\left(s\right)\right)\right\}\\
&\qquad\times b_j\left(s,\bar{X}\left(s\right),\bar{X}_\delta\left(s\right),\bar{u}\left(s\right)\right)\\
&+\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left\{\frac{\partial^3\bar{\mathcal{V}}_0}{\partial x_i\partial x_j\partial x_k}\left(s,\bar{X}\left(s\right)\right)+\frac{\partial^3\bar{\mathcal{V}}_1}{\partial y_i\partial y_j\partial y_k}\left(s+\delta,\bar{X}\left(s\right)\right)\right\}\\
&\qquad\times\left\{\sum_{r=1}^{m}\left(\sigma_{jr}\sigma_{kr}\right)\left(s,\bar{X}\left(s\right),\bar{X}_\delta\left(s\right),\bar{u}\left(s\right)\right)\right\}\Bigg\}ds\\
-\sum_{j=1}^{n}&\left\{\frac{\partial^2\bar{\mathcal{V}}_0}{\partial x_i\partial x_j}\left(s,\bar{X}\left(s\right)\right)+\frac{\partial^2\bar{\mathcal{V}}_1}{\partial y_i\partial y_j}\left(s+\delta,\bar{X}\left(s\right)\right)\right\}
\end{aligned}
\tag{4.5.6}
$$

$$\times \sum_{r=1}^{m} \sigma_{jr}\left(s, \bar{X}\left(s\right), \bar{X}_{\delta}\left(s\right), \bar{u}\left(s\right)\right) dW_r\left(s\right).$$

Then, the substitution of (4.5.4) and (4.5.5) into the first bracket on the right-hand-side of (4.5.6) leads to

$$\begin{aligned}
& d\bar{P}_i\left(s\right) \\
= & -\Bigg\{ -\frac{\partial G}{\partial x_i}\left(s, \bar{X}\left(s\right), \bar{X}_{\delta}\left(s\right), \bar{u}\left(s\right)\right) + \sum_{j=1}^{n} \bar{P}_i\left(s\right) \frac{\partial b_j}{\partial x_i}\left(s, \bar{X}\left(s\right), \bar{X}_{\delta}\left(s\right), \bar{u}\left(s\right)\right) \\
& \quad + \sum_{r=1}^{m}\sum_{j=1}^{n} H_{\bar{P}}^{ij}\left(s\right) \frac{\partial \sigma_{jr}}{\partial x_i}\left(s, \bar{X}\left(s\right), \bar{X}_{\delta}\left(s\right), \bar{u}\left(s\right)\right) \\
& \quad + \mathbb{E}\Bigg[-\frac{\partial G}{\partial y_i}\left(s+\delta, \bar{X}\left(s+\delta\right), \bar{X}\left(s\right), \bar{u}\left(s+\delta\right)\right) \\
& \qquad + \sum_{j=1}^{n} \bar{P}_i\left(s+\delta\right) \frac{\partial b_j}{\partial y_i}\left(s+\delta, \bar{X}\left(s+\delta\right), \bar{X}\left(s\right), \bar{u}\left(s+\delta\right)\right) \\
& \qquad + \sum_{r=1}^{m}\sum_{j=1}^{n} H_{\bar{P}}^{ij}\left(s+\delta\right) \frac{\partial \sigma_{jr}}{\partial y_i}\left(s+\delta, \bar{X}\left(s+\delta\right), \bar{X}\left(s\right), \bar{u}\left(s+\delta\right)\right) \Bigg| \mathcal{F}(s) \Bigg] \Bigg\} ds \\
& + \sum_{r=1}^{m} H_{\bar{P}}^{ir}\left(s\right) dW_r\left(s\right),
\end{aligned}$$

which coincides with (4.5.2) by noting the expression (3.2.24) for $\mathcal{H}_d$, where the terminal value for $\bar{P}$ follows immediately from that for the auxiliary HJB equation (4.3.12) together with (4.5.1). □

By Corollary 4.4.2, the function $\bar{\mathcal{V}}$ required in Theorem (4.5.1) is an auxiliary function of the value function V for the control problem (4.2.3). Hence, (4.5.1) generalizes (4.1.9) for Markovian optimal control problems and shows that the adjoint process (4.5.2) can be expressed in terms of the auxiliary function together with the corresponding optimal control and the solution of the controlled SDDE.

In the reminder of this section, we give an example to verify the result obtained in Theorem 4.5.1.

Example 4.5.2. We revisit the control problem studied in Example 3.3.3. Recalling that $m = n = r = 1$, $\mathbb{U} = \mathbb{R}$ and $\mathcal{U} = \mathbb{L}_{\mathcal{F}}^{22}$; that b and σ are given by

$$\begin{cases} b\left(s, x, y, u\right) = a_1 x + b_1 y + c_1 u, \\ \sigma\left(s, x, y, u\right) = a_2 x + b_2 y + c_2 u, \end{cases}$$

and that G and g are given by

$$G(s,x,y,u) = \frac{1}{2}c_3u^2 \text{ and } g(x) = a_3x,$$

where $a_1, a_2, a_3, b_1, b_2, c_1, c_2 \in \mathbb{R}$ and $c_3 > 0$ are given constants. Note that, following the arguments used in Examples 3.3.3 & 3.2.1, Hypotheses III, IV &V are satisfied for this problem.

(I) Applying Dynamic Programming

It follows from (4.3.12) that the auxiliary HJB equation for this control problem is expressed by

$$\begin{cases} \frac{\partial \mathcal{V}}{\partial t}(t,x,y) \\ \quad + \inf_{u\in\mathbb{U}} \left\{ \frac{1}{2}c_3u^2 + \left(\frac{\partial \mathcal{V}_0}{\partial x}(t,x) + \frac{\partial \mathcal{V}_1}{\partial y}(t+\delta,x) \right)(a_1x + b_1y + c_1u) \right. \\ \qquad \left. + \frac{1}{2}\left(\frac{\partial^2 \mathcal{V}_0}{\partial x^2}(t,x) + \frac{\partial^2 \mathcal{V}_1}{\partial y^2}(t+\delta,x) \right)(a_2x + b_2y + c_2u)^2 \right\} = 0, \\ \qquad\qquad (t,x,y) \in [0,T) \times \mathbb{R}^2, \\ \mathcal{V}(T,x,y) = \mathcal{V}_0(T,x) = a_3x, \quad x \in \mathbb{R}. \end{cases} \tag{4.5.7}$$

Noting the terminal value in (4.5.7), we suppose that a solution $\bar{\mathcal{V}}(t,x,y) = \bar{\mathcal{V}}_0(t,x) + \bar{\mathcal{V}}_1(t,y)$ of the auxiliary HJB equation (4.5.7) can be expressed in the form

$$\bar{\mathcal{V}}_0(t,x) = C(t) + A(t)x \text{ and } \bar{\mathcal{V}}_1(t,y) = B(t)y, \tag{4.5.8}$$

where A, B and C are as defined similarly to those in (4.4.13). Then, applying the technique used in part (II) of Section 4.4.1, we see that the functions u_0 required in Theorem 4.4.1 is described by

$$u_0(t,x,y) = -\frac{c_1}{c_3}(A(t) + B(t+\delta)) \tag{4.5.9}$$

which gives an optimal control

$$\bar{u}(s) = -\frac{c_1}{c_3}(A(s) + B(s+\delta)), \quad \forall s \in [t,T],$$

of the control problem (4.2.6) with b, σ, G and g so defined. It is easy to see that $\bar{u}$ is in $\mathcal{U}[t,T] = \mathbb{L}^{22}_{\mathcal{F}_t}([t,T];\mathbb{R})$ since A and B are continuous. To obtain A, B and C, similarly to part (III) of Section 4.4.1, substituting (4.5.8) and u_0 obtained by (4.5.9) into the auxiliary

HJB equation (4.5.7), we see that $\bar{\mathcal{V}}$ defined by (4.5.8) is a solution of (4.5.7) if A and B satisfy the system of ODEs:

$$\begin{cases} \dfrac{dA}{dt}(t) = -\left(A(t) + B(t+\delta)\right) a_1, \quad t \in [0,T], \\ A(T) = a_3, \end{cases} \tag{4.5.10}$$

$$\begin{cases} \dfrac{dB}{dt}(t) = -\left(A(t) + B(t+\delta)\right) b_1, \quad t \in [0,T], \\ B(t) = 0, \quad t \in [T, T+\delta]; \end{cases} \tag{4.5.11}$$

and if

$$C(t) = -\int_t^T \frac{c_1^2}{2c_3} \left(A(s) + B(s+\delta)\right)^2 ds, \quad \forall t \in [0,T].$$

Unlike (4.4.17)-(4.4.18), the above system of ODEs admits a unique solution since the function $F(x,y) = -(x+y)a$ for $a = a_1, b_1$, corresponding to the terms on the right-hand-side of the first equation of (4.5.10) and (4.5.11) respectively, is Lipschitz continuous with respect to (x,y). As for (4.4.17)-(4.4.18), this system of ODEs can be solved numerically by the backward induction algorithm presented in part (IV) of Section 4.4.1.

Figure 4.3 below gives an example of such A, B and C and the corresponding optimal control $\bar{u}$ with b, σ, G and g so defined, where the parameters are the same as those in Example 3.3.3. Note that, since $\frac{\partial^2 \bar{\mathcal{V}}_0}{\partial x^2}(t,x) + \frac{\partial^2 \bar{\mathcal{V}}_1}{\partial y^2}(t+\delta, x) \equiv 0$ under (4.5.8), σ in this case can be any function which satisfies Hypothesis III.

(II) Comparing with the Stochastic Maximum Principle.

For $\bar{\mathcal{V}}_0(t,x) = C(t) + A(t)x$ and $\bar{\mathcal{V}}_1(t,y) = B(t)y$, the conditions required in Theorem 4.5.1 are satisfied. Then, applying Theorem 4.5.1 and setting $t = 0$, we see that the pair $(\bar{P}, H_{\bar{P}})$ defined by

$$\begin{cases} \bar{P}(t) = -\left(A(t) + B(t+\delta)\right), \\ H_{\bar{P}}(t) \equiv 0, \end{cases} \quad t \in [0,T],$$

satisfies the adjoint equation (3.3.20) obtained in Example 3.3.3. Figure 4.4 below gives an example of such a $\bar{P}$ and compares it with the numerical result shown in Figure 3.1. Note that, by increasing the accuracy, Figure 4.4 (b) shows a segment part of this comparsion which indicates that the numerical discrepancy between these two methods for this case can be narrowed down to any fixed $\epsilon > 0$. □

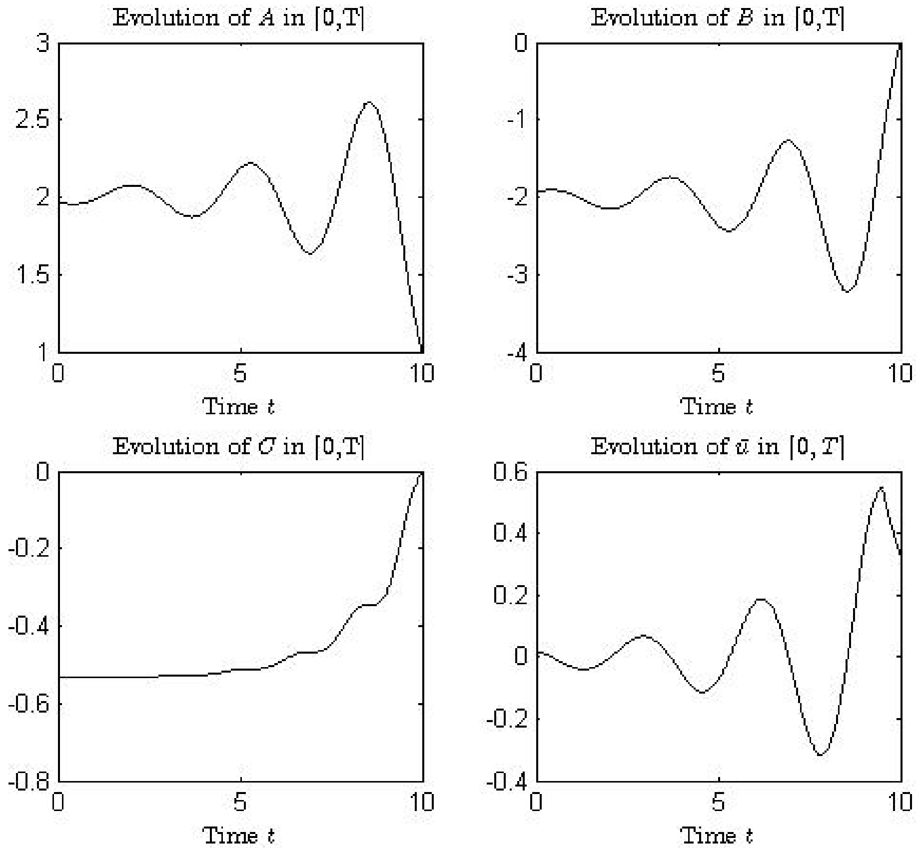

Figure 4.3: Evolution of A, B, C and $\bar{u}$ with parameters $T = 10$, $\delta = 0.5$, $\xi(t) = t + 1$ for $t \in [-\delta, 0]$, $a_1 = a_3 = 1$, $b_1 = -2$, $c_1 = -1$ and $c_3 = 3$.

4.6 Inclusion of the Exponential Moving Average Delay

Similarly to the extension studied in Section 3.4, we can generalize the results obtained in the preceding sections of this chapter to take exponential moving average delays into consideration.

(I) The Control Problem

We continue to work with $(\Omega, \mathcal{F}, \mathbb{P})$, W and $\mathbb{F} = \{\mathcal{F}(s)\}_{s\in[0,T]}$ introduced in Section 4.1.1, and recall the stochastic optimal control problem with both discrete and exponential moving average delays studied in Section 3.4 as follows. For given functions $b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^{n\times m}$, let the $\mathcal{F}(s)$-adapted continuous state process $X : \Omega \times [-\delta, T] \to \mathbb{R}^n$ be described by the controlled SDDE

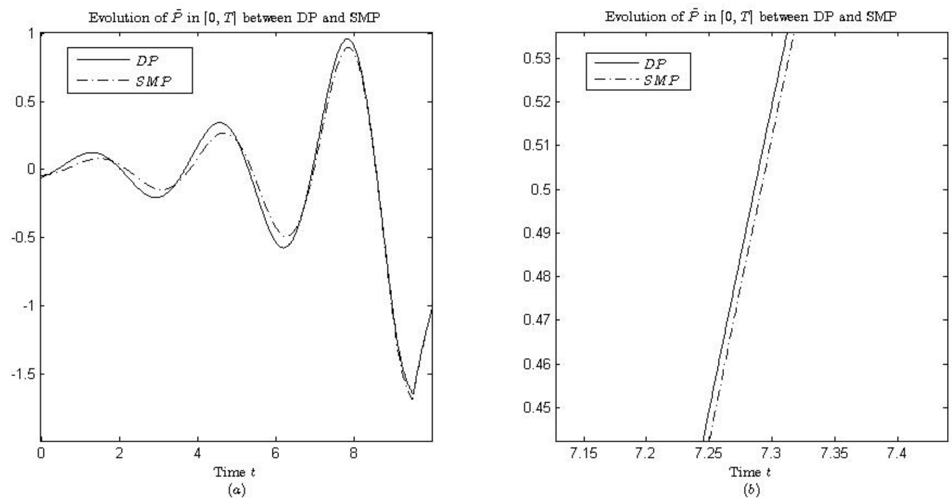

Figure 4.4: Evolution of $\bar{P}$ respectively obtained by the dynamic programming (DP) and the stochastic maximum principle (SMP) with parameters $T = 10$, $\delta = 0.5$, $\xi(t) = t+1$ for $t \in [-\delta, 0]$, $a_1 = a_3 = 1$, $b_1 = -2$, $c_1 = -1$ and $c_3 = 3$.

$$\begin{cases} dX(s) = b(s, X(s), X_a(s), X_\delta(s), u(s))\, ds \\ \qquad\qquad + \sigma(s, X(s), X_a(s), X_\delta(s), u(s))\, dW(s), \quad s \in [0, T], \\ X_a(s) = \int_{-\delta}^{0} e^{\lambda r} X(s+r)\, dr, \quad \lambda > 0, \\ X(s) = \xi_0(s), \quad s \in [-\delta, 0], \end{cases} \tag{4.6.1}$$

where ξ_0, δ and X_δ are as defined before and the control u is selected from the given admissible control set $\mathcal{U}$ as defined in a similar sense to that in (4.2.3). Moreover, for given functions $G : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the cost function J_{ad} is defined by

$$J_{ad}(u) = \mathbb{E}\left[\int_0^T G(s, X(s), X_a(s), X_\delta(s), u(s))\, dt + g(X(T), X_a(T))\right]. \tag{4.6.2}$$

Then, the stochastic optimal control problem with both discrete and exponential moving average delays, associated with the state system (4.6.1) and the cost function (4.6.2), is to find $\bar{u} \in \mathcal{U}$ realizing

$$\inf_{u \in \mathcal{U}} J_{ad}(u). \tag{4.6.3}$$

Similarly to Hypotheses III & IV, we make the following two hypotheses to ensure that, for any $\xi_0 \in \mathbb{C}([-\delta, 0]; \mathbb{R}^n)$ and $u \in \mathcal{U}$, the controlled SDDE (4.6.1) admits a unique strong solution X^u and the cost function J_a is integrable.

Hypothesis* III. The functions b and σ are continuous with respect to $(t,u) \in [0,T] \times \mathbb{U}$ and are Lipschitz continuous with respect to $(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ with the Lipschitz constant independent of $(t,u) \in [0,T] \times \mathbb{U}$.

Hypothesis* IV. The functions G and g are continuous such that, for any $\xi_0 \in \mathbb{C}([-\delta,0];\mathbb{R}^n)$ and $u \in \mathcal{U}$,

$$\mathbb{E}\left[\int_0^T \left|G\left(s, X^u(s), X_a^u(s), X_\delta^u(s), u(s)\right)\right| ds\right] + \mathbb{E}\left[\left|g\left(X^u(T), X_a^u(T)\right)\right|\right] < \infty.$$

To apply dynamic programming, we use a similar argument used for the discrete delay case to associate the control problem (4.6.3) with a family of control problems with different starting times and initial paths. That is, for any $(t,\xi) \in [0,T] \times \mathbb{C}([-\delta,0];\mathbb{R}^n)$, to find $\bar{u} \in \mathcal{U}[t,T]$ realizing

$$\inf_{u \in \mathcal{U}[t,T]} J_{ad}(u;t,\xi), \tag{4.6.4}$$

where the state system is given by

$$\begin{cases} dX^u(s) = b\left(s, X^u(s), X_a^u(s), X_\delta^u(s), u(s)\right) ds \\ \qquad\qquad + \sigma\left(s, X^u(s), X_a^u(s), X_\delta^u(s), u(s)\right) dW(s), \quad s \in [t,T], \\ X^u(s) = \xi(s-t), \quad s \in [t-\delta, t], \end{cases} \tag{4.6.5}$$

and the cost function $J_{ad}(u;t,\xi)$ is given by

$$\begin{aligned} J_{ad}(u;t,\xi) =& \mathbb{E}^{t,\xi}\left[\int_t^T G\left(s, X^u(s), X_a^u(s), X_\delta^u(s), u(s)\right) ds\right] \\ &+ \mathbb{E}^{t,\xi}\left[g\left(X^u(T), X_a^u(T)\right)\right]. \end{aligned} \tag{4.6.6}$$

Note that, following the corresponding argument used for the discrete delay case, X^u is equal to $X^{\tilde{u}}$ in law for $s \in [t,T]$, where $\tilde{u} \in \mathcal{U}$, $u = \tilde{u}|_{[t,T]}$ and $X^{\tilde{u}}$ is the strong solution of the controlled SDDE (4.6.1) with u replaced by $\tilde{u}$. Note also that, for any $\xi \in \mathbb{C}([-\delta,0];\mathbb{R}^n)$, $J_{ad}(u;0,\xi_0) = J_{ad}(u)$ and

$$J_{ad}(u;T,\xi) = g\left(\xi(0), \int_{-\delta}^0 e^{\lambda r}\xi(r)\,dr\right),$$

which is independent to u. Then, similarly to (4.1.14) and (4.2.7), the corresponding value function $V : [0,T] \times \mathbb{C}([-\delta,0];\mathbb{R}^n) \to \mathbb{R}$ is defined by

$$V(t,\xi) = \inf_{u \in \mathcal{U}[t,T]} J_{ad}(u;t,\xi), \quad (t,\xi) \in [0,T) \times \mathbb{C}([-\delta,0];\mathbb{R}^n).$$

(II) The Auxiliary HJB Equation

As mentioned in Section 4.1.1, it is generally difficult to apply the classical Itô formula to the dynamic programming equation

$$V(t,\xi) = \inf_{u\in\mathcal{U}[t,T]} \mathbb{E}^{t,\xi}\left[\int_t^{\hat{t}} G\left(s, X^u(s), X^u_a(s), X^u_\delta(s), u(s)\right) ds + V\left(\hat{t}, X^u_{[\hat{t}-\delta,\hat{t}]}\right)\right], \quad \forall \hat{t}\in[t,T],$$

in a similar way to that for Markovian optimal control problems to get the corresponding HJB equation and also, under the hypothesis (4.1.16), the HJB equation (4.1.17) has a restriction. Thus, similarly to Section 4.3, we investigate a class of the control problems described by (4.6.3), where the value function V is separable in a similar sense to that in Definition 4.3.1.

Hereafter, let $\mathbb{C}^{1,2,1}([0,T]\times\mathbb{R}^{n\times 2})$ be the space of continuous functions $F:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ such that the partial derivatives $\frac{\partial F}{\partial t}$, $\frac{\partial^2 F}{\partial x_i\partial x_j}$ and $\frac{\partial F}{\partial y_i}$, for $i,j\in\{1,2,\ldots,n\}$, exist and are continuous.

Definition 4.6.1. The value function V for the stochastic optimal control problem with both discrete and exponential moving average delays (4.6.3) is called separable if there are two functions $\mathcal{V}_0\in\mathbb{C}^{1,2,1}([0,T]\times\mathbb{R}^{n\times 2})$ and $\mathcal{V}_1\in\mathbb{C}^{1,2}([0,T]\times\mathbb{R}^n)$, with $\mathcal{V}_1(T,\cdot)\equiv 0$, such that, for any $(t,\xi)\in[0,T]\times\mathbb{C}([-\delta,0];\mathbb{R}^n)$,

$$\begin{aligned} V(t,\xi) =& \mathcal{V}_0\left(t,\xi(0),\int_{-\delta}^0 e^{\lambda r}\xi(r)\,dr\right) + \mathcal{V}_1\left(\min\{t+\delta,T\},\xi(0)\right) \\ &- \int_t^{\min\{t+\delta,T\}} \frac{\partial\mathcal{V}_1}{\partial s}\left(s,\xi(s-\delta-t)\right) ds. \end{aligned} \tag{4.6.7}$$

If V is separable, we call

$$\mathcal{V}(t,x,y,z) = \mathcal{V}_0(t,x,y) + \mathcal{V}_1(t,z),$$

an auxiliary function of V.

Similarly to the discrete delay case, we can rewrite (4.6.7) as

$$V(t,\xi) = \mathcal{V}\left(t,\xi(0),\int_{-\delta}^0 e^{\lambda r}\xi(r)\,dr,\xi(-\delta)\right) + \mathcal{C}\left(t,\xi(\cdot - t)\right),$$

where $\mathcal{C}(t,\xi(\cdot-t))$ is defined by (4.3.3) with respect to $\mathcal{V}_1$ above and, for a given separable value function V, the auxiliary function $\mathcal{V}$ is uniquely. Furthermore, having introduced the auxiliary functions, we can re-express the corresponding limit (4.3.4) in terms of the auxiliary function in a similar fashion to (4.3.5). This allows us to obtain the auxiliary HJB equation as follows.

Theorem 4.6.2. *Assume that Hypotheses* III & IV holds. Suppose that the value function V for the stochastic optimal control problem with both discrete and exponential moving average delays* (4.6.3) *is separable associated with an auxiliary function* $\mathcal{V}(t,x,y,z) = \mathcal{V}_0(t,x,y) + \mathcal{V}_1(t,z)$*. Then,* $\mathcal{V}$ *satisfies the second-order PDE:*

$$\begin{cases} \dfrac{\partial \mathcal{V}}{\partial t}(t,x,y,z) + \left\langle \dfrac{\partial \mathcal{V}_0}{\partial y}(t,x,y), \left(x - \lambda y - e^{-\lambda\delta} z\right) \right\rangle \\ \quad + \inf\limits_{u \in \mathbb{U}} \mathcal{G}_{ad}\Big(t,x,y,z,u, \dfrac{\partial \mathcal{V}_0}{\partial x}(t,x,y) + \dfrac{\partial \mathcal{V}_1}{\partial z}(t+\delta,x), \dfrac{\partial^2 \mathcal{V}_0}{\partial x^2}(t,x,y) \\ \qquad\qquad + \dfrac{\partial^2 \mathcal{V}_1}{\partial z^2}(t+\delta,x) \Big) = 0, \quad (t,x,y,z) \in [0,T) \times \mathbb{R}^{n\times 3}, \\ \mathcal{V}(T,x,y,z) = \mathcal{V}_0(T,x,y) = g(x,y), \quad (x,y) \in \mathbb{R}^{n\times 2}, \end{cases} \tag{4.6.8}$$

where $\mathcal{G}_{ad} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}$ *is given by*

$$\begin{aligned} &\mathcal{G}_{ad}(t,x,y,z,u,p,h) \\ =& G(t,x,y,z,u) + \langle b(t,x,y,z,u), p \rangle + \frac{1}{2} \operatorname{tr}\left(\sigma\sigma^\top (t,x,y,z,u)\, h \right). \end{aligned} \tag{4.6.9}$$

The proof of Theorem 4.6.2 uses the essentially same techniques as those in the proofs of both Lemma 4.3.2 and Theorem 4.3.3. The main difference is that, instead of (4.3.7), we apply the Itô formula given in [22, Lemma 5.1] to $\mathcal{V}_0(s, X^u(s), X_a^u(s))$ for $s \in [t, \hat{t}]$ to obtain

$$\begin{aligned} &\mathcal{V}_0\left(\hat{t}, X^u(\hat{t}), X_a^u(\hat{t})\right) - \mathcal{V}_0\left(t, \xi(0), \int_{-\delta}^0 e^{\lambda r} \xi(r)\, dr\right) \\ =& \int_t^{\hat{t}} \Bigg\{ \frac{\partial \mathcal{V}_0}{\partial s}(s, X^u(s), X_a^u(s)) + \left\langle \frac{\partial \mathcal{V}_0}{\partial x}(s, X^u(s), X_a^u(s)), b(s) \right\rangle \\ &\quad + \left\langle \frac{\partial \mathcal{V}_0}{\partial y}(s, X^u(s), X_a^u(s)), \left(X^u(s) - \lambda X_a^u(s) - e^{-\lambda\delta} X_\delta^u(s) \right) \right\rangle \\ &\quad + \frac{1}{2} \operatorname{tr}\left(\sigma\sigma^\top(s) \frac{\partial^2 \mathcal{V}_0}{\partial x^2}(s, X^u(s), X_a^u(s)) \right) \Bigg\} ds \\ &+ \int_t^{\hat{t}} \left\langle \frac{\partial \mathcal{V}_0}{\partial x}(s, X^u(s), X_a^u(s)), \sigma(s)\, dW(s) \right\rangle, \quad \forall \hat{t} \in [t,T], \end{aligned} \tag{4.6.10}$$

where we have used the shorthand notation $b(s)$ and $\sigma(s)$ defined in a similar manner to (4.3.8). Note that the third term on the right-hand-side of (4.6.10) leads to the second term on the right-hand-side of the first equation of (4.6.8). Note also that if $\mathcal{V}$ is independent of z, corresponding to the case where $\mathcal{V}_1$ is a function defined on $[0,T]$ and so $\mathcal{C}\left(t,\xi\left(\cdot-t\right)\right)=0$, the value function V reduces to the one which satisfies (4.1.16). Then, since $\mathcal{V}_1$ only depends on t, the corresponding auxiliary HJB equation (4.6.8) reduces to (4.1.17) obtained in [22, 23].

(III) Stochastic Verification Theorem

Similarly to Theorem 4.4.1, we have the following stochastic verification technique for the control problem (4.6.3).

Theorem 4.6.3. *Assume that Hypotheses* III & IV hold. Suppose that there exist* $\bar{\mathcal{V}}_0 \in \mathbb{C}^{1,2,1}([0,T]\times\mathbb{R}^{n\times 2})$ *and* $\bar{\mathcal{V}}_1 \in \mathbb{C}^{1,2}([0,T]\times\mathbb{R}^{n})$ *with* $\bar{\mathcal{V}}_1(T,\cdot)\equiv 0$ *such that*

$$\bar{\mathcal{V}}\left(t,x,y,z\right)=\bar{\mathcal{V}}_0\left(t,x,y\right)+\bar{\mathcal{V}}_1\left(t,z\right)$$

satisfies the auxiliary HJB equation (4.6.8)*. Then, for any* $(t,\xi)\in[0,T]\times\mathbb{C}([-\delta,0];\mathbb{R}^n)$*,*

$$J_a\left(u;t,\xi\right)\geq\bar{\mathcal{V}}\left(t,\xi\left(0\right),\int_{-\delta}^{0}e^{\lambda r}\xi\left(r\right)dr,\xi\left(-\delta\right)\right)+\bar{\mathcal{C}}\left(t,\xi\left(\cdot-t\right)\right),$$

for all $u\in\mathcal{U}[t,T]$*, where* $\bar{\mathcal{C}}\left(t,\xi\left(\cdot-t\right)\right)$ *is given by* (4.3.3) *with* $\mathcal{V}_1$ *replaced by* $\bar{\mathcal{V}}_1$*. Moreover, if there exists a* $\mathcal{B}([0,T]\times\mathbb{R}^{n\times 3})$*-measurable function* $u_0:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{U}$ *such that the infimum in* (4.6.8) *is attained at* $u_0(t,x,y,z)$ *for all* $(t,x,y,z)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$ *in the sense that*

$$\begin{aligned}&\mathcal{G}_{ad}\bigg(t,x,y,z,u_0\left(t,x,y,z\right),\frac{\partial\bar{\mathcal{V}}_0}{\partial x}\left(t,x,y\right)\\&\qquad+\frac{\partial\bar{\mathcal{V}}_1}{\partial z}\left(t+\delta,x\right),\frac{\partial^2\bar{\mathcal{V}}_0}{\partial x^2}\left(t,x,y\right)+\frac{\partial^2\bar{\mathcal{V}}_1}{\partial z^2}\left(t+\delta,x\right)\bigg)\\&=\inf_{u\in\mathbb{U}}\mathcal{G}_{ad}\bigg(t,x,y,z,u,\frac{\partial\bar{\mathcal{V}}_0}{\partial x}\left(t,x,y\right)\\&\qquad\quad+\frac{\partial\bar{\mathcal{V}}_1}{\partial z}\left(t+\delta,x\right),\frac{\partial^2\bar{\mathcal{V}}_0}{\partial x^2}\left(t,x,y\right)+\frac{\partial^2\bar{\mathcal{V}}_1}{\partial z^2}\left(t+\delta,x\right)\bigg),\end{aligned}\tag{4.6.11}$$

where $\mathcal{G}_{ad}$ *is defined by* (4.6.9)*, then the control* $\bar{u}$*, defined by*

$$\bar{u}\left(s\right)=u_0\left(s,\bar{X}\left(s\right),\bar{X}_a\left(s\right),\bar{X}_\delta\left(s\right)\right),\quad\forall s\in\left[t,T\right],\tag{4.6.12}$$

is an optimal control of (4.6.4)*, where* $\bar{X}$ *denotes the strong solution of the controlled SDDE* (4.6.5) *with* u *in* $b(s,x,y,z,u)$ *and* $\sigma(s,x,y,z,u)$ *replaced by* u_0*.*

Similarly to Corollary 4.4.2, the above theorem also illustrates that if the auxiliary HJB equation (4.6.8) admits a solution $\bar{\mathcal{V}}(t,x,y,z) = \bar{\mathcal{V}}_0(t,x,y) + \bar{\mathcal{V}}_1(t,z)$ and if u_0 attains the infimum in (4.6.8) in the sense of (4.6.11), then $\bar{\mathcal{V}}$ must coincide with the auxiliary function of the value function for the control problem (4.6.3).

The following example demonstrates how to use Theorem 4.6.3 to find an optimal control. Note that this control problem cannot be solved using the HJB equation (4.1.17) obtained in [22, 23]. In fact, as σ below depends on z corresponding to X_δ, the conditions given in [23, Theorem 5.1] cannot be applied to this problem.

Example 4.6.4. As before, we set $m = n = r = 1$. Suppose that $\mathbb{U} = \mathbb{R}$ and $\mathcal{U} = \mathbb{L}^{22}_{\mathcal{F}}$; that b and σ in (4.6.5) are given by

$$\begin{cases} b(s,x,y,z,u) = a_1 x + c_1 u, \\ \sigma(s,x,y,z,u) = \left(a_2 x^2 + b_2 z^2 + p\right)^{\frac{1}{2}}, \end{cases}$$

which are same as those in (4.4.10); that G and g in (4.6.6) are given by

$$\begin{cases} G(s,x,y,z,u) = \dfrac{1}{2}\left(c_3 u^2 + a_3 x^2 + b_3 z^2\right) + f_3 y, \\ g(x,y) = \dfrac{1}{2} a_4 x^2 + a_5 x + f_4 y, \end{cases}$$

where $a_1, a_2, a_3, a_4, b_2, b_3, c_1, c_3, p$ are as given in Section 4.4.1 and a_5, f_3, f_4 are given constants. By the technique used in part (I) of Section 4.4.1, we see that Hypotheses* III & IV are satisfied.

It follows from (4.6.8) that the auxiliary HJB equation for this control problem is given by

$$\begin{cases} \dfrac{\partial \mathcal{V}}{\partial t}(t,x,y,z) + \left(x - \lambda y - e^{-\lambda\delta} z\right) \dfrac{\partial \mathcal{V}_0}{\partial y}(t,x,y) \\ \quad + \inf_{u\in\mathbb{U}} \left\{ \dfrac{1}{2}\left(c_3 u^2 + a_3 x^2 + b_3 z^2\right) + f_3 y + \left(\dfrac{\partial \mathcal{V}_0}{\partial x}(t,x,y) + \dfrac{\partial \mathcal{V}_1}{\partial z}(t+\delta, x) \right)(a_1 x + c_1 u) \right. \\ \qquad \left. + \dfrac{1}{2}\left(\dfrac{\partial^2 \mathcal{V}_0}{\partial x^2}(t,x,y) + \dfrac{\partial^2 \mathcal{V}_1}{\partial z^2}(t+\delta,x) \right)\left(a_2 x^2 + b_2 z^2 + p\right) \right\} = 0, \\ \qquad\qquad\qquad (t,x,y,z) \in [0,T) \times \mathbb{R}^{1\times 3}, \\ \mathcal{V}(T,x,y,z) = \mathcal{V}_0(T,x,y) = \dfrac{1}{2} a_4 x^2 + a_5 x + f_4 y, \quad (x,y) \in \mathbb{R}^{1\times 2}. \end{cases} \tag{4.6.13}$$

Adapting the technique used in part (II) of Section 4.4.1, we suppose that a solution $\bar{\mathcal{V}}(t,x,y,z) = \bar{\mathcal{V}}_0(t,x,y) + \bar{\mathcal{V}}_1(t,z)$ of the above auxiliary HJB equation can be expressed in the form

$$\begin{cases} \bar{\mathcal{V}}_0 (t,x) = \frac{1}{2} A_1 (t) x^2 + A_2 (t) x + B (t) y + D (t), \\ \bar{\mathcal{V}}_1 (t,y) = \frac{1}{2} C_1 (t) z^2 + C_2 (t) z, \end{cases} \tag{4.6.14}$$

where A_1, A_2, B, C_1, C_2 and D are $\mathbb{R}$-valued continuously differentiable functions on $[0,T]$ needing to be determined with $C_i(T) = 0$ for $i = 1,2$. As before, for simplicity of notation, we extend the domain of C_i to $[0, T+\delta]$ by defining $C_i(t) = 0$ for $t > T$. Now, substituting (4.6.14) into the auxiliary HJB equation (4.6.13) and then, following the argument used in part (II) of Section 4.4.1 for deriving (4.4.15), we obtain that an optimal control described by

$$\begin{aligned} \bar{u}(s) = &- \frac{c_1}{c_3} \left(\left(A_1 (s) + C_1 (s+\delta) \right) \bar{X} (s) \right) \\ &- \frac{c_1}{c_3} \left(A_2 (s) + C_2 (s+\delta) \right), \quad \forall s \in [t,T], \end{aligned}$$

where $\bar{X}$ is the strong solution of the controlled SDDE (4.6.5) with respect to $\bar{u}$ as specified in Theorem 4.6.3. Following the argument used at the end of part (II) of Section 4.4.1, $\bar{u} \in \mathcal{U}[t,T] = \mathbb{L}^{22}_{\mathcal{F}_t}([t,T];\mathbb{R})$.

Similarly to Theorem 4.4.3, we obtain that if A_1, A_2, B, C_1 and C_2 are solved by the systems of ODEs

$$\begin{cases} \frac{dA_1}{dt}(t) = \frac{c_1^2}{c_3} \left(A_1 (t) + C_1 (t+\delta) \right)^2 \\ \qquad\qquad - (2a_1 + a_2) \left(A_1 (t) + C_1 (t+\delta) \right) - a_3, \quad t \in [0,T], \\ A_1 (T) = a_4, \end{cases}$$

$$\begin{cases} \frac{dA_2}{dt}(t) = \frac{c_1^2}{c_3} \left(A_1 (t) + C_1 (t+\delta) \right) \left(A_2 (t) + C_2 (t+\delta) \right) \\ \qquad\qquad - a_1 \left(A_2 (t) + C_2 (t+\delta) \right) - B (t), \quad t \in [0,T], \\ A_2 (T) = a_5, \end{cases}$$

$$\begin{cases} \frac{dB}{dt}(t) = \lambda B (t) - f_3, \quad t \in [0,T], \\ B (T) = f_4, \end{cases}$$

$$\begin{cases} \frac{dC_1}{dt}(t) = - \left(A_1 (t) + C_1 (t+\delta) \right) b_2 - b_3, \quad t \in [0,T], \\ C_1 (t) = 0, \quad t \in [T, T+\delta], \end{cases}$$

$$\begin{cases} \dfrac{dC_2}{dt}(t) = e^{-\lambda\delta} B(t), & t \in [0, T], \\ C_2(t) = 0, & t \in [T, T+\delta], \end{cases}$$

and if

$$D(t) = \frac{1}{2} \int_t^T \left\{ (A_1(s) + C_1(s+\delta)) p - \frac{c_1^2}{c_3} (A_2(s) + C_2(s+\delta)) \right\} ds, \quad \forall t \in [0, T],$$

then $\bar{\mathcal{V}}(t, x, y, z)$ described by (4.6.14) is a solution of the auxiliary HJB equation (4.6.13). Figures 4.5 & 4.6 below give an example of such A_1, A_2, B, C_1, C_2 and D and the corresponding pair $(\bar{X}, \bar{u})$ respectively. $\square$

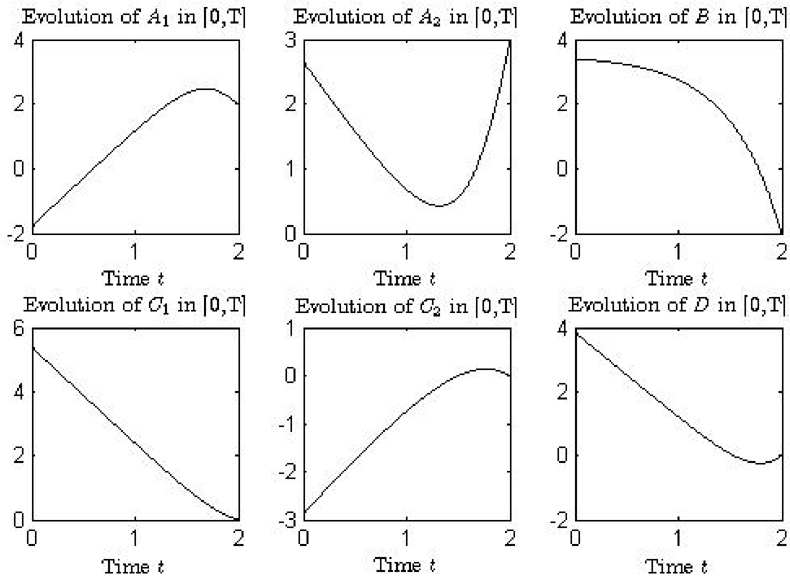

Figure 4.5: Evolution of A_1, A_2, B, C_1, C_2 and D with parameters $T = 2$, $\delta = 0.2$, $\xi(t) = t + 1$ for $t \in [-0.2, 0]$, $\lambda = 2, a_1 = 4, a_2 = 1, a_3 = -3, a_4 = 2, a_5 = 3, b_2 = 2, b_3 = -3, c_1 = -3, c_3 = 3, f_3 = 7, f_4 = -2$ and $p = 2$.

(IV) Connection with the Stochastic Maximum Principle

To explore the connection with the stochastic maximum principle obtained in Section 3.4, similarly to Hypothesis V, we assume the following hypothesis holds in what follows of this section.

Hypothesis* V. The functions b, σ and G are continuously differentiable with respect to $(x, y, z) \in \mathbb{R}^{n \times 3}$; the function g is continuously differentiable with respect to $(x, y) \in \mathbb{R}^{n \times 2}$; and $\mathbb{U} = \mathbb{R}^r$.

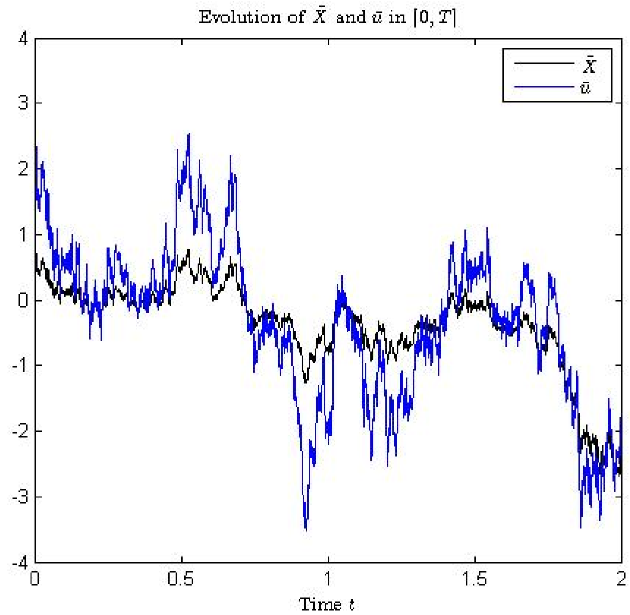

Figure 4.6: Evolution of $\bar{X}$ and $\bar{u}$ with parameters $T = 2$, $\delta = 0.2$, $\xi(t) = t + 1$ for $t \in [-0.2, 0]$, $\lambda = 2$, $a_1 = 4, a_2 = 1, a_3 = -3, a_4 = 2, a_5 = 3, b_2 = 2, b_3 = -3, c_1 = -3, c_3 = 3, f_3 = 7, f_4 = -2$ and $p = 2$.

Let $\mathbb{C}^{1,3,3}([0,T] \times \mathbb{R}^{n\times 2})$ be the space of continuous functions $F : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that the partial derivatives $\frac{\partial^2 \bar{\mathcal{V}}_0}{\partial t \partial x_i}$, $\frac{\partial^2 \bar{\mathcal{V}}_0}{\partial x_i \partial y_j}$, $\frac{\partial^2 \bar{\mathcal{V}}_0}{\partial y_i \partial y_j}$, $\frac{\partial^2 \bar{\mathcal{V}}_0}{\partial t \partial y_i}$, $\frac{\partial^3 \bar{\mathcal{V}}_0}{\partial x_i \partial x_j \partial y_k}$ and $\frac{\partial^3 \bar{\mathcal{V}}_0}{\partial x_i \partial x_j \partial x_k}$, for $i, j, k \in \{1, 2, \ldots, n\}$, exist and are continuous.

Theorem 4.6.5. *Assume that Hypotheses* III, IV & V hold. Suppose that there are $\bar{\mathcal{V}}_0 \in \mathbb{C}^{1,3,3}([0,T] \times \mathbb{R}^{n\times 2})$ and $\bar{\mathcal{V}}_1 \in \mathbb{C}^{1,3}([0,T] \times \mathbb{R}^n)$ with $\bar{\mathcal{V}}_1(T,\cdot) \equiv 0$ such that $\bar{\mathcal{V}}(t,x,y,z) = \bar{\mathcal{V}}_0(t,x,y) + \bar{\mathcal{V}}_1(t,z)$ is a solution of the auxiliary HJB equation* (4.6.8). *Suppose further that the infimum of* (4.6.8) *is at a $\mathcal{B}([0,T] \times \mathbb{R}^{n\times 3})$-measurable function u_0 in the sense of* (4.6.11). *Let $\bar{u}$ be the optimal control defined by* (4.6.12) *and $\bar{X}$ be the strong solution of* (4.6.5) *with u in $b(s,x,y,z,u)$ and $\sigma(s,x,y,z,u)$ replaced by u_0. Then, for any $(t,\xi) \in [0,T] \times \mathbb{C}([-\delta,0];\mathbb{R}^n)$, $(\bar{P}, H_{\bar{P}})$ and $(\bar{P}^a, H_{\bar{P}^a})$ are respectively defined by*

$$\begin{cases} \bar{P}_i(s) = -\left\{ \dfrac{\partial \bar{\mathcal{V}}_0}{\partial x_i}(s, \bar{X}(s), \bar{X}_a(s)) + \dfrac{\partial \bar{\mathcal{V}}_1}{\partial z_i}(s+\delta, \bar{X}(s)) \right\}, \\ H^{ij}_{\bar{P}}(s) = -\displaystyle\sum_{k=1}^{n} \left\{ \dfrac{\partial^2 \bar{\mathcal{V}}_0}{\partial x_i \partial x_k}(s, \bar{X}(s)) + \dfrac{\partial^2 \bar{\mathcal{V}}_1}{\partial z_i \partial z_k}(s+\delta, \bar{X}(s), \bar{X}_a(s)) \right\} \\ \qquad\qquad \times \sigma_{kj}(s, \bar{X}(s), \bar{X}_a(s), \bar{X}_\delta(s), \bar{u}(s)), \end{cases}$$

and

$$\begin{cases} \bar{P^a}_i(s) = -\dfrac{\partial \bar{\mathcal{V}}_0}{\partial y_i}(s, \bar{X}(s), \bar{X}_a(s)), \\ H^{ij}_{\bar{P}^a}(s) = -\displaystyle\sum_{k=1}^{n} \dfrac{\partial^2 \bar{\mathcal{V}}_0}{\partial y_i \partial x_k}(s, \bar{X}(s), \bar{X}_a(s))\, \sigma_{kj}(s, \bar{X}(s), \bar{X}_a(s), \bar{X}_\delta(s), \bar{u}(s)), \end{cases}$$

for $s \in [t, T]$*, where* $i \in \{1, 2, \ldots, n\}$ *and* $j \in \{1, 2, \ldots, m\}$*, satisfy*

$$\begin{cases} d\bar{P}(s) = -\left\{ \dfrac{\partial \bar{\mathcal{H}}_{ad}}{\partial x}(s) + \mathbb{E}\left[\dfrac{\partial \bar{\mathcal{H}}_{ad}}{\partial z}(s+\delta)\, I_{[t,T-\delta]}(s) \,\middle|\, \mathcal{F}(s) \right] \right\} ds \\ \qquad\qquad + H_{\bar{P}}(s)\, dW(s), \quad s \in [t, T], \\ \bar{P}(T) = -\dfrac{\partial g}{\partial x}(\bar{X}(T), \bar{Y}(T)), \end{cases}$$

$$\begin{cases} d\bar{P}^a(s) = -\dfrac{\partial \bar{\mathcal{H}}_{ad}}{\partial y}(s)\, ds + H_{\bar{P}^a}(s)\, dW(s), \quad s \in [t, T], \\ \bar{P}^a(T) = -\dfrac{\partial g}{\partial y}(\bar{X}(T), \bar{Y}(T)). \end{cases}$$

where the Hamiltonian $\mathcal{H}_{ad}$ *is given by* (3.4.9)*; and*

$$\frac{\partial \bar{\mathcal{H}}_{ad}}{\partial x}(s) = \frac{\partial \mathcal{H}_{ad}}{\partial x}(s, \bar{X}(s), \bar{X}_a(s), \bar{X}_\delta(s), \bar{u}(s), \bar{P}(s), \bar{P}^a(s), H_{\bar{P}}(s), H_{\bar{P}^a}(s))$$

and similarly for the partial derivatives $\frac{\partial \bar{\mathcal{H}}_{ad}}{\partial z}(s+\delta)$ *and* $\frac{\partial \bar{\mathcal{H}}_{ad}}{\partial y}(s)$*;*

4.6.1 Discussion

The novelty of Theorem 4.6.3 is that we do not require the solution of the auxiliary HJB equation to be independent of z although $\bar{\mathcal{V}}_0$ does. In particular, this allows us to improve the results in [22, 23, 41].

To show this, we consider the following simple stochastic optimal control problem with both discrete and exponential moving average delays. Note that this control problem usually cannot be solved using the HJB equation (4.1.17) unless the parameters satisfy certain conditions (specified below). For simplicity, we set $m = n = r = 1$ and suppose that $\mathbb{U} = \mathbb{R}$ and $\mathcal{U} = \mathbb{L}^{22}_{\mathcal{F}}$; that b in (4.6.5) is given by

$$b(s, x, y, z, u) = a_1 x + f_1 y + b_1 z + c_1 u;$$

that $\sigma(s, x, y, z, u)$ in (4.6.5) is a function satisfying Hypothesis* III; and that G and g in (4.6.6) are given by

$$G(s, x, y, z, u) = c_3 u^2/2 \ \text{ and } \ g(x, y) = a_3 x + f_3 y,$$

where $a_1, a_3, b_1, c_1, c_3, f_1$ and f_3 are as given in Example 3.4.3. By the argument used in Example 3.4.3, we see that Hypotheses* III & IV are satisfied for this problem.

(I) Using the HJB equation (4.1.17)

Similarly to Example 4.6.4, since σ depends on z and u, the conditions given in [23, Theorem 5.1] cannot be applied here. If $\sigma(s,x,y,z,u) = a_2 x + f_2 y$, where a_2 and f_2 are given constants, then [23, Theorem 5.1] gives that if

$$f_3 e^{-\lambda\delta} = b_1 a_3, \quad f_2 = e^{\lambda\delta} a_1 b_1 \text{ and } e^{-\lambda\delta} f_1 - \lambda b_1 = a_1 b_1 + b_1^2 e^{\lambda\delta}, \tag{4.6.15}$$

the corresponding HJB equation admits a solution $\bar{V}(t,x,y)$. Note that b_1 in (4.6.15) must be nonzero otherwise the model is Markovian. On the other hand, by the technique used in [41, page 27], $\bar{V}(t,x,y)$ has the form

$$\bar{V}(t,x,y) = P(t)\,x + Q(t)\,y + R(t), \tag{4.6.16}$$

where P and Q satisfy the system of ODEs:

$$\begin{cases} \dfrac{dP}{dt}(t) = -a_1 P(t) - Q(t), & t \in [0,T], \\ P(T) = a_3, \end{cases} \tag{4.6.17}$$

$$\begin{cases} \dfrac{dQ}{dt}(t) = -f_1 A(t) + \lambda Q(t), & t \in [0,T], \\ Q(T) = f_3; \end{cases} \tag{4.6.18}$$

and

$$R(t) = -\int_t^T \frac{c_1^2}{2c_3} P^2(s)\,ds, \quad \forall t \in [0,T].$$

In particular, since $\bar{V}(t,x,y)$ described by (4.6.16) is required to be independent of z, P and Q need to satisfy

$$e^{-\lambda\delta} Q(t) = b_1 P(t), \quad \forall t \in [0,T].$$

Consequently, as noted in [41, Theorem 4.2], the parameters in the model satisfy

$$f_3 e^{-\lambda\delta} = b_1 a_3, \quad b_1 \neq 0 \text{ and } \frac{e^{-\lambda\delta} f_1}{b_1} - \lambda = a_1 + b_1 e^{\lambda\delta}, \tag{4.6.19}$$

which are less restrictive than those in (4.6.15), and the corresponding optimal control is expressed by

$$\bar{u}(s) = -\frac{a_3 c_1}{c_3} e^{\left(a_1 + b_1 e^{\lambda\delta}\right)(T-s)}, \quad \forall s \in [t,T].$$

(II) Using the Auxiliary HJB equation (4.6.8)

Adapting the techniques used in part (I) of Example 4.5.2 and Example 4.6.4, we see that $\bar{\mathcal{V}}(t,x,y,z) = \bar{\mathcal{V}}_0(t,x,y) + \bar{\mathcal{V}}_1(t,z)$ with $\bar{\mathcal{V}}_0(t,x,y) = A(t)\,x + B(t)\,y + D(t)$ and $\bar{\mathcal{V}}_1(t,z) = C(t)\,z$ is a solution of the auxiliary HJB equation (4.6.8) with b, σ, G and g so defined, where A, B and C satisfy the system of ODEs:

$$\begin{cases} \dfrac{dA}{dt}(t) = -\left(A(t) + C(t+\delta)\right)a_1 - B(t), \quad t \in [0,T], \\ A(T) = a_3, \end{cases}$$

$$\begin{cases} \dfrac{dC}{dt}(t) = -\left(A(t) + C(t+\delta)\right)b_1 + e^{-\lambda\delta}B(t), \quad t \in [0,T], \\ C(T) = 0, \quad t \in [T, T+\delta], \end{cases}$$

$$\begin{cases} \dfrac{dB}{dt}(t) = -\left(A(t) + C(t+\delta)\right)f_1 + \lambda B(t), \quad t \in [0,T], \\ B(T) = f_3; \end{cases}$$

and where D is obtained by

$$D(t) = -\int_t^T \frac{c_1^2}{2c_3}\left(A(s) + C(s+\delta)\right)^2 ds, \quad \forall t \in [0,T].$$

As for (4.5.10)-(4.5.11), the above system of ODEs always admits a unique solution (A, B, C) and can be solved numerically by the backward induction algorithm described in part (IV) of Section 4.4.1. Then, applying Theorem 4.6.3, the corresponding optimal control $\bar{u}$, specified by (4.6.12), is

$$\bar{u}(s) = -\frac{c_1}{c_3}\left(A(s) + C(s+\delta)\right), \quad \forall s \in [t,T],$$

In particular, if the parameters in the model satisfy (4.6.19), then we see that the above system of ODEs recover (4.6.17)-(4.6.18), so that $A(t) = P(t)$, $B(t) = Q(t)$, $C(t) = 0$ and $D(t) = R(t)$ for $t \in [0,T]$. Therefore, our results improve those in [22, 23, 41] for this problem.

CHAPTER 5

CONCLUSION

This thesis resolves some restrictions in using both the stochastic maximum principle and dynamic programming for stochastic optimal control problems with discrete delay and those with both discrete and exponential moving average delays where we have applied the conjugate duality method for deriving the stochastic maximum principle instead of pure stochastic calculus.

We first study a stochastic convex problem with delay referred to as the primary problem and then obtain the expression for the corresponding dual problem. This generalizes the results obtained in [2, 44] into the stochastic case with delay. Moreover, using the conjugate duality method, we get the conditions for optimality for these problems which, by linking stochastic optimal control problem with delay with a particular type of convex problem, allows us to derive the stochastic maximum principle. In particular, the corresponding adjoint equations and Hamiltonian are derived instead of introduced. Furthermore, if the stochastic optimal control problem involves both the types of delay and is jump-free, the stochastic maximum principle obtained in this thesis improves those obtained in [29, 30]. More importantly, our approach of using the conjugate duality method unifies the Hamiltonian and the associated adjoint equations involved in the stochastic maximum principle for stochastic optimal control problems with either just discrete delay or with both discrete and exponential moving average delays: those for the former are a special case for the latter. The results in this part of the thesis are going to appear in the journal of *Advances in Applied Probability*.

On the other hand, we adapt the technique used in [19, Chapter 3] to the stochastic context which enables us to consider a class of stochastic optimal control problems with delay, where the value functions are separable that they can be expressed in terms of auxiliary functions. This enables us to obtain the auxiliary HJB equation which plays a similar role in the framework of dynamic programming as the classical HJB equation does in the Markovian case. In particular, if both the types of delay are involved, our auxiliary HJB equations generalize the HJB equations obtained in [22, 23] and our verification theorem improves the stochastic verification theorem there. Note that our approach of introducing the auxiliary function not only unifies the auxiliary HJB equations involved in the dynamic programming for stochastic optimal control problems with either just discrete delay or with both discrete and exponential moving average delays, but also has certain connections to our stochastic maximum principles. This work has been submitted to *SIAM Journal on Control and Optimization*.

5.1 Future Research

Although this thesis only considers the stochastic optimal control problems where the model depends on the delayed term of state processes, it is straightforward to generalize our results to the case where the model also depends on the delayed terms of controls. Now, we introduce the following future directions which are much more challenging to investigate.

Restrictions for Using Auxiliary HJB Equations

As discussed in Section 4.6.1, we do require $\mathcal{V}_0$ to be independent of z when we consider the stochastic optimal control problem with both discrete and exponential moving average delays. This still causes certain restrictions for using the corresponding auxiliary HJB equation in applications. Hence, we wonder whether there exists a way to resolve this requirement.

PDEs Versus Anticipated BSDEs

It can be seen from Theorem 4.5.1 that the linear anticipated BSDE with respect to (P, H_P) corresponding to the adjoint equation can be solved by the solution of the PDE corresponding to the auxiliary HJB equation. On the other hand, the general (i.e. nonlinear) classical BSDEs can be solved by the solution of a second-order PDE which is known as the Feynman-Kac formula (see [18, Proposition 4.3]). This plays a crucial role in obtaining numerical solutions of classical BSDEs (see [11]). Hence, we wonder whether there exists a type of PDE which can be connected to general anticipated BSDEs which, more importantly, will motivate us to investigate the corresponding numerical techniques for solving anticipated BSDEs. This has not been studied in the academic literature.

Stochastic Recursive Optimal Controls with Delay

Shi, Xu and Zhang in [42] study the stochastic recursive optimal control problems with both discrete and exponential moving average delays by both the stochastic maximum principle and dynamic programming. In addition to the state process X satisfying the controlled SDDE (4.6.1), the recursive case involves a pair of stochastic processes (Y, Z) which is described by the controlled BSDE

$$\begin{cases} -dY(t) = G(t, X(t), X_a(t), X_\delta(t), Y(t), Z(t), u(t))\,dt \\ \qquad\qquad - Z(t)\,dW(t), \quad t \in [0, T], \\ Y(T) = g(X(T), X_a(T)), \end{cases}$$

where G and g are given functions. Then, the aim of the so-called stochastic recursive optimal control problem with both discrete and exponential moving average delays is to find a $\bar{u}$ minimizing the cost function

$$J(u) = -Y(0) = -\mathbb{E}\left[\int_0^T G(t, X(t), X_a(t), X_\delta(t), Y(t), Z(t), u(t))\,dt \right. \\ \left. + g(X(T), X_a(T))\right].$$

As the adjoint equations and the HJB equation in this paper are generalized from [29] and [23] respectively, they have similar restrictions mentioned before as the classical control problems with delay do. Thus, we wonder whether our approaches are still valid in this context.

Stochastic Differential Games with Delay

We may generalize our approaches to study a stochastic differential game with delay which essentially can be regarded as a control problem with a higher-dimensional control. After that, it might allow us to concern some applications in finance, such as risk minimization problems (see [27]).

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