An investigation of magneto-acoustic waves in the solar atmosphere

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Contents

1 Abstract

Isothermal and non-isothermal quiet sun atmospheres are modelled and small perturbations are applied. Acoustic wave behaviour is observed and deconstructed both analytically and numerically. Isothermal magnetic regions are then investigated by considering mode conversions.

2 Introduction - Hyperbolic partial differential equations

In this paper, we investigate Partial Differential Equations (PDE) of the form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \tag{2.1}$$

where A, B and C are constants. We describe this PDE as:

(i) hyperbolic if $B^2 - 4AC > 0$

(ii) parabolic if $B^2 - 4AC = 0$

(iii) elliptic if $B^2 - 4AC < 0$

A well-known hyperbolic PDE is the wave equation (2.2):

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{2.2}$$

Whereas a well-known parabolic PDE is the heat equation (2.3):

$$\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = 0 \tag{2.3}$$

In Section 3, we derive a hyperbolic PDE for acoustic gravity waves in the solar atmosphere (from the surface of the sun, through the photosphere, to the chromosphere). These are not the only types of waves found in the solar atmosphere; we also examine magneto-acoustic gravity waves in Section 6. The aim of this paper is to contribute to solving the Solar Coronal heating problem by providing a methodology which accurately models wave speeds.

It is not always possible to find an analytic solution to a problem involving PDEs. The most common numerical techniques used to solve PDEs are finite difference method (FDM), finite element method (FEM), and finite volume method (FVM). We use FDM to evaluate problems numerically in the following sections.

3 Derivation of the acoustic gravity wave in an isothermal atmosphere

We will derive the governing equations for the acoustic gravity wave in the framework of Magnetohydrodynamics (MHD). This requires the equation of continuity (3.1), the equation of motion (3.2), and the adiabatic energy equation (3.3). Additionally we include the magnetic induction equation (3.4) here for completeness as it will be needed for Section 6. Note that the diffusion term has been omitted from (3.4) as we assume infinite electrical conductivity. In the following equations, ρ is the fluid density, $\boldsymbol{v}(x, y, z, t)$ is the fluid velocity, p is the fluid pressure, $\boldsymbol{J}$ is the current density, $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{g}$ is gravity, ν is the coefficient of kinematic viscosity, γ is the ratio of specific heat capacity, and t is time.

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \tag{3.1}$$

$$\rho\frac{d\boldsymbol{v}}{dt} = -\boldsymbol{\nabla} p + \boldsymbol{J} \times \boldsymbol{B} + \rho \boldsymbol{g} + \rho\nu\nabla^2 \boldsymbol{v} \tag{3.2}$$

$$\frac{\partial p}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} p = -\gamma p \boldsymbol{\nabla} \cdot \boldsymbol{v} \tag{3.3}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \tag{3.4}$$

We assume that a *small* perturbation to the equilibrium state is generated and that the equilibrium terms have no time dependence. Thus:

$$\boldsymbol{B} = \boldsymbol{B_0} + \boldsymbol{B_1} \quad \text{where} \quad \boldsymbol{B_1} \ll \boldsymbol{B_0} \tag{3.5}$$
$$\boldsymbol{v} = \boldsymbol{v_0} + \boldsymbol{v_1} \quad \text{where} \quad \boldsymbol{v_1} \ll \boldsymbol{v_0} \tag{3.6}$$
$$\rho = \rho_0 + \rho_1 \quad \text{where} \quad \rho_1 \ll \rho_0 \tag{3.7}$$
$$p = p_0 + p_1 \quad \text{where} \quad p_1 \ll p_0 \tag{3.8}$$

In this section we consider the plasma to be inviscid ($\nu = 0$) and non-magnetic ($\boldsymbol{B} = \boldsymbol{0}$), simulating a quiet sun atmosphere. We assume that gravity is a constant and acts in the negative z-direction, hence:

$$\boldsymbol{g} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} \tag{3.9}$$

We assume that the equilibrium state of the plasma is at rest ($\boldsymbol{v_0} = \boldsymbol{0}$). Additionally, we are only interested in the fluid velocity in the z direction, hence we assume the only non-zero velocity is $v_z = v$. Thus we obtain the following simplified MHD equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho v) = 0 \tag{3.10}$$
$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} + \rho(-g) \tag{3.11}$$
$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial z} = -\gamma p \frac{\partial v}{\partial z} \tag{3.12}$$

We begin by examining the equation of motion (3.11) at equilibrium. We find that:

$$\frac{\partial p_0}{\partial z} = \rho_0(-g) \tag{3.13}$$

Note that sound speed is related to pressure and density via (3.14) [1]:

$$c_s^2 = \frac{\gamma p}{\rho} \tag{3.14}$$

In this section we assume c_s is constant. Using (3.14) to eliminate ρ_0 from equation (3.13) gives:

$$\frac{\partial p_0}{\partial z} = -\frac{\gamma g}{c_s^2} p_0 \tag{3.15}$$

Solving (3.15) yields:

$$p_0 = A \exp\left(-\frac{\gamma g}{c_s^2} z\right) \tag{3.16}$$

Therefore:

$$\rho_0 = \frac{A\gamma}{c_s^2} \exp\left(-\frac{\gamma g}{c_s^2} z\right) \tag{3.17}$$

In Appendix A.1 we linearise (3.10), (3.11) and (3.12) and the results follow:

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \frac{\partial v_1}{\partial z} - v_1 \frac{\partial \rho_0}{\partial z} \tag{3.18}$$
$$\rho_0 \frac{\partial v_1}{\partial t} = -\frac{\partial p_1}{\partial z} + \rho_1(-g) \tag{3.19}$$
$$\frac{\partial p_1}{\partial t} = \rho_0 v_1 g - \gamma p_0 \frac{\partial v_1}{\partial z} \tag{3.20}$$

When the linearised equation of motion (3.19) is differentiated with respect to t we get:

$$\frac{\partial}{\partial t}\left(\rho_0 \frac{\partial v_1}{\partial t}\right) = \frac{\partial}{\partial t}\left(-\frac{\partial p_1}{\partial z}\right) + \frac{\partial}{\partial t}\left(\rho_1(-g)\right) \tag{3.21}$$
$$\rho_0 \frac{\partial^2 v_1}{\partial t^2} = -\frac{\partial}{\partial z}\left(\frac{\partial p_1}{\partial t}\right) - g\frac{\partial \rho_1}{\partial t} \tag{3.22}$$

We use (3.18) to eliminate $\frac{\partial \rho_1}{\partial t}$ from (3.22). This gives:

$$\rho_0 \frac{\partial^2 v_1}{\partial t^2} = -\frac{\partial}{\partial z}\left(\frac{\partial p_1}{\partial t}\right) - g\left(-\rho_0 \frac{\partial v_1}{\partial z} - v_1 \frac{\partial \rho_0}{\partial z}\right) \tag{3.23}$$

$$= -\frac{\partial}{\partial z}\left(\frac{\partial p_1}{\partial t}\right) + g\rho_0 \frac{\partial v_1}{\partial z} + g v_1 \frac{\partial \rho_0}{\partial z} \tag{3.24}$$

We use (3.20) to eliminate $\frac{\partial p_1}{\partial t}$ from (3.24).

$$\rho_0 \frac{\partial^2 v_1}{\partial t^2} = -\frac{\partial}{\partial z}\left(\frac{\partial p_1}{\partial t}\right) + g\rho_0 \frac{\partial v_1}{\partial z} + g v_1 \frac{\partial \rho_0}{\partial z} \tag{3.25}$$

$$= -\frac{\partial}{\partial z}\left(g\rho_0 v_1 - \gamma p_0 \frac{\partial v_1}{\partial z}\right) + g\rho_0 \frac{\partial v_1}{\partial z} + g v_1 \frac{\partial \rho_0}{\partial z} \tag{3.26}$$

$$= -g\frac{\partial}{\partial z}(\rho_0 v_1) + \gamma \frac{\partial}{\partial z}\left(p_0 \frac{\partial v_1}{\partial z}\right) + g\rho_0 \frac{\partial v_1}{\partial z} + g v_1 \frac{\partial \rho_0}{\partial z} \tag{3.27}$$

$$= -g\left(\frac{\partial \rho_0}{\partial z} v_1 + \rho_0 \frac{\partial v_1}{\partial z}\right) + \gamma\left(\frac{\partial p_0}{\partial z}\frac{\partial v_1}{\partial z} + p_0 \frac{\partial^2 v_1}{\partial z^2}\right) + g\rho_0 \frac{\partial v_1}{\partial z} + g v_1 \frac{\partial \rho_0}{\partial z} \tag{3.28}$$

$$= \gamma \frac{\partial p_0}{\partial z}\frac{\partial v_1}{\partial z} + \gamma p_0 \frac{\partial^2 v_1}{\partial z^2} \tag{3.29}$$

Using the sound speed relation (3.14) and the derivative with respect to z of (3.16) gives:

$$\frac{\gamma}{c_s^2} p_0 \frac{\partial^2 v_1}{\partial t^2} = -\frac{\gamma^2 g}{c_s^2} p_0 \frac{\partial v_1}{\partial z} + \gamma p_0 \frac{\partial^2 v_1}{\partial z^2} \tag{3.30}$$

Then dividing through by γp_0 and rearranging yields:

$$\frac{\partial^2 v_1}{\partial z^2} - \frac{\gamma g}{c_s^2}\frac{\partial v_1}{\partial z} - \frac{1}{c_s^2}\frac{\partial^2 v_1}{\partial t^2} = 0 \tag{3.31}$$

Defining the density scale height $H = \frac{c_s^2}{\gamma g}$ completes the derivation:

$$\frac{\partial^2 v_1}{\partial z^2} - \frac{1}{H}\frac{\partial v_1}{\partial z} - \frac{1}{c_s^2}\frac{\partial^2 v_1}{\partial t^2} = 0 \tag{3.32}$$

If we assume that gravity acts in the *positive* z direction instead, we get:

$$\frac{\partial^2 v_1}{\partial z^2} + \frac{1}{H}\frac{\partial v_1}{\partial z} - \frac{1}{c_s^2}\frac{\partial^2 v_1}{\partial t^2} = 0 \tag{3.33}$$

3.1 Semi-Analytical Solution - Laplace transform method

In this section we solve (3.34) semi-analytically following Sutmann et al [2]:

$$\frac{\partial^2 u}{\partial z^2} + \frac{1}{H}\frac{\partial u}{\partial z} - \frac{1}{c_s^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{3.34}$$

Note we have replaced v_1 from (3.33) with u and developed the following initial and boundary conditions from assumptions made during the derivation:

$$u(z,0) = 0 \text{ for } z \neq 0 \tag{3.35}$$

$$\frac{\partial u}{\partial t}(z,0) = 0 \text{ for } z \neq 0 \tag{3.36}$$

$$u(0,t) = u_0 e^{-i\omega t} \tag{3.37}$$

$$\lim_{z\to\infty} u(z,t) = 0 \tag{3.38}$$

Note that (3.34) is a hyperbolic partial differential equation and that boundary condition (3.37) creates the perturbation by driving the system sinusoidally from $z = 0$, the surface of the sun, where ω is the angular frequency of that perturbation. Eventually we will examine the real part of the solution but for now we keep (3.37) as a complex exponential for simplicity.

We cannot solve this problem by a separation of variables so a Laplace transform method is used instead and therefore we denote U to be the Laplace transform of u. Applying Laplace transforms to (3.34) yields:

$$\frac{\partial^2 U}{\partial z^2} + \frac{1}{H}\frac{\partial U}{\partial z} - \frac{1}{c_s^2}\left(s^2 U - s u(z,0) - \frac{\partial u}{\partial t}(z,0)\right) = 0 \tag{3.39}$$

Next we Laplace transform the boundary conditions (3.37) and (3.38). It is not necessary to transfer the initial conditions (3.35) and (3.36). Hence (3.37) and (3.38) become:

$$U(0,s) = \frac{u_0}{s + i\omega} \tag{3.40}$$

$$\lim_{z\to\infty} U(z,s) = 0 \tag{3.41}$$

Applying the initial conditions (3.35) and (3.36) to the transformed equation (3.39) yields:

$$\frac{\partial^2 U}{\partial z^2} + \frac{1}{H}\frac{\partial U}{\partial z} - \frac{s^2}{c_s^2}U = 0 \tag{3.42}$$

Using the method of the characteristic equation to solve (3.42) gives:

$$m^2 + \frac{1}{H}m - \frac{s^2}{c_s^2} = 0 \tag{3.43}$$

Thus the two solutions are:

$$m_{\pm} = -\frac{1}{2H} \pm \frac{1}{c_s}\sqrt{\frac{c_s^2}{4H^2} + s^2} \tag{3.44}$$

Hence we find that:

$$U(z,s) = c_1 \exp\left(-\frac{z}{2H} + \frac{z}{c_s}\sqrt{\frac{c_s^2}{4H^2} + s^2}\right) + c_2 \exp\left(-\frac{z}{2H} - \frac{z}{c_s}\sqrt{\frac{c_s^2}{4H^2} + s^2}\right) \tag{3.45}$$

Note that in Appendix B.1 it is shown that $m_+ > 0$ therefore, in conjunction with (3.41), we find:

$$U(z,s) = c_2 \exp\left(-\frac{z}{2H} - \frac{z}{c_s}\sqrt{\frac{c_s^2}{4H^2} + s^2}\right) \tag{3.46}$$

Using (3.40) we get:

$$U(z,s) = \frac{u_0}{s + i\omega} \exp\left(-\frac{z}{2H} - \frac{z}{c_s}\sqrt{\frac{c_s^2}{4H^2} + s^2}\right) \tag{3.47}$$

Hence:

$$U(z,s) = \frac{u_0}{s + i\omega} \exp\left(-\frac{z}{2H}\right) \exp\left(-\frac{z}{c_s}\sqrt{\frac{c_s^2}{4H^2} + s^2}\right) \tag{3.48}$$

In Appendix B.2 it is derived that if α is constant and if we let J_1 be a Bessel function of the first kind then:

$$\exp\left(-\frac{z}{c_s}\sqrt{\alpha^2 + s^2}\right) = \exp\left(\frac{-sz}{c_s}\right) - \frac{\alpha z}{c_s}\int_{\frac{z}{c_s}}^{\infty} \frac{J_1\left(\alpha\sqrt{t^2 - \left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{t^2 - \left(\frac{z}{c_s}\right)^2}} e^{-st} dt \tag{3.49}$$

Hence, we have:

$$U(z,s) = \frac{u_0}{s + i\omega} \exp\left(-\frac{z}{2H}\right)\left(\exp\left(\frac{-sz}{c_s}\right) - \frac{z}{2H}\int_{\frac{z}{c_s}}^{\infty} \frac{J_1\left(\frac{c_s}{2H}\sqrt{t^2 - \left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{t^2 - \left(\frac{z}{c_s}\right)^2}} e^{-st} dt\right) \tag{3.50}$$

This simplifies to:

$$U(z,s) = \frac{u_0}{s+i\omega}\exp\left(-\frac{z}{2H}\right)\exp\left(\frac{-sz}{c_s}\right) - \frac{u_0}{s+i\omega}\left(\frac{z}{2H}\right)\exp\left(-\frac{z}{2H}\right)\int_{\frac{z}{c_s}}^{\infty}\frac{J_1\left(\frac{c_s}{2H}\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}}e^{-st}dt \tag{3.51}$$

At this stage we focus on the following integral:

$$\int_{\frac{z}{c_s}}^{\infty}\frac{J_1\left(\frac{c_s}{2H}\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}}e^{-st}dt \tag{3.52}$$

This is almost a Laplace transform but the lower limit is $\frac{z}{c_s}$ instead of 0. To remedy this we add the Heaviside Step function $\mathcal{H}\left(t-\frac{z}{c_s}\right)$ (See Appendix B.3 for definition). Thus we can rewrite the integral in (3.52) as:

$$\int_{0}^{\infty}\frac{J_1\left(\frac{c_s}{2H}\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}}\mathcal{H}\left(t-\frac{z}{c_s}\right)e^{-st}dt \tag{3.53}$$

Hence:

$$\begin{aligned} U(z,s) = &\left(\frac{u_0}{s+i\omega}\exp\left(-\frac{z}{2H}\right)\right)\left(\exp\left(\frac{-sz}{c_s}\right)\right) \\ &- \left(\frac{u_0}{s+i\omega}\exp\left(-\frac{z}{2H}\right)\right)\left(\frac{z}{2H}\right)\int_{0}^{\infty}\frac{J_1\left(\frac{c_s}{2H}\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}}\mathcal{H}\left(t-\frac{z}{c_s}\right)e^{-st}dt \end{aligned} \tag{3.54}$$

We now prepare to inverse Laplace transform $U(z,s)$. We observe that:

$$\int_{0}^{\infty}\frac{J_1\left(\frac{c_s}{2H}\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}}\mathcal{H}\left(t-\frac{z}{c_s}\right)e^{-st}dt = \mathcal{L}\left(\frac{J_1\left(\frac{c_s}{2H}\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}}\mathcal{H}\left(t-\frac{z}{c_s}\right)\right) \tag{3.55}$$

Additionally, we let:

$$\mathcal{L}(a_0(t)) = A_0(s) = \frac{u_0}{s+i\omega}\exp\left(-\frac{z}{2H}\right) \tag{3.56}$$

Consequently:

$$a_0(t) = u_0\exp\left(-\frac{z}{2H}\right)e^{-i\omega t} \tag{3.57}$$

Thus:

$$U(z,s) = \mathcal{L}(a_0(t))\exp\left(\frac{-sz}{c_s}\right) - \mathcal{L}(a_0(t))\left(\frac{z}{2H}\right)\mathcal{L}\left(\frac{J_1\left(\frac{c_s}{2H}\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}}\mathcal{H}\left(t-\frac{z}{c_s}\right)\right) \tag{3.58}$$

We use the Second Shifting theorem (See Appendix B.4 for theorem) to find:

$$\mathcal{L}^{-1}\left(\mathcal{L}\left(a_0(t)\right)\exp\left(\frac{-sz}{c_s}\right)\right) = a_0\left(t-\frac{z}{c_s}\right)\mathcal{H}\left(t-\frac{z}{c_s}\right) \tag{3.59}$$

$$= u_0\exp\left(-\frac{z}{2H}\right)\exp\left(-i\omega\left(t-\frac{z}{c_s}\right)\right)\mathcal{H}\left(t-\frac{z}{c_s}\right) \tag{3.60}$$

Thus the first term of (3.54) has been successfully inverse Laplace transformed. To inverse Laplace transform the second term of (3.54) we use the Convolution theorem (See Appendix B.4 for theorem). Hence:

$$\mathcal{L}^{-1}\left(\mathcal{L}(a_0(t))\mathcal{L}\left(\frac{J_1\left(\frac{c_s}{2H}\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}}\mathcal{H}\left(t-\frac{z}{c_s}\right)\right)\right) = \int_0^t a_0(t-\tau)\frac{J_1\left(\frac{c_s}{2H}\sqrt{\tau^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{\tau^2-\left(\frac{z}{c_s}\right)^2}}\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau \tag{3.61}$$

Hence when we inverse Laplace transform (3.58) and then substitute (3.60) and (3.61) into (3.58) we find:

$$u(z,t) = u_0\exp\left(-\frac{z}{2H}\right)\mathcal{H}\left(t-\frac{z}{c_s}\right)\exp\left(-i\omega\left(t-\frac{z}{c_s}\right)\right) - \frac{u_0 z\exp\left(-\frac{z}{2H}\right)e^{-i\omega t}}{2H}\int_0^t \frac{e^{i\omega\tau}J_1\left(\frac{c_s}{2H}\sqrt{\tau^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{\tau^2-\left(\frac{z}{c_s}\right)^2}}\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau \tag{3.62}$$

Consider the integral in (3.62). For integrands involving a dummy function $\phi(z,\tau)$ and the Heaviside Step function $\mathcal{H}\left(\tau-\frac{z}{c_s}\right)$ we find, in Appendix B.5, that:

$$\int_0^t \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau = \int_{\frac{z}{c_s}}^{\infty}\phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau - \int_t^{\infty}\phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau \tag{3.63}$$

This presents three cases:

(i) Assuming that $\frac{z}{c_s} > t$ makes the integrand equal zero.

(ii) Assuming that $\frac{z}{c_s} = t$, gives:

$$\int_0^t \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau = \int_{\frac{z}{c_s}}^{\infty}\phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau - \int_t^{\infty}\phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau = 0 \tag{3.64}$$

(iii) Assuming that $\frac{z}{c_s} < t$ we further simplify to:

$$\int_0^t \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau = \int_{\frac{z}{c_s}}^{\infty}\phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau - \int_t^{\infty}\phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau \tag{3.65}$$

$$= \int_{\frac{z}{c_s}}^{t}\phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau \tag{3.66}$$

Hence for $\frac{z}{c_s} < t$:

$$u(z,t) = u_0\exp\left(-\frac{z}{2H}\right)\exp\left(-i\omega\left(t-\frac{z}{c_s}\right)\right)\mathcal{H}\left(t-\frac{z}{c_s}\right) - \frac{u_0 z\exp\left(-\frac{z}{2H}\right)e^{-i\omega t}}{2H}\int_{\frac{z}{c_s}}^t \frac{e^{i\omega\tau}J_1\left(\frac{c_s}{2H}\sqrt{\tau^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{\tau^2-\left(\frac{z}{c_s}\right)^2}}\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau \tag{3.67}$$

However $\mathcal{H}\left(\tau - \frac{z}{c_s}\right) = 1$ for $\tau \in \left(\frac{z}{c_s}, t\right)$ and therefore:

$$u(z,t) = u_0 \exp\left(-\frac{z}{2H}\right) \exp\left(-i\omega\left(t - \frac{z}{c_s}\right)\right) - \frac{u_0 z \exp\left(-\frac{z}{2H}\right) e^{-i\omega t}}{2H} \int_{\frac{z}{c_s}}^{t} \frac{e^{i\omega\tau} J_1\left(\frac{c_s}{2H}\sqrt{\tau^2 - \left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{\tau^2 - \left(\frac{z}{c_s}\right)^2}} d\tau \tag{3.68}$$

Consider the real part of $u(z,t)$, $\text{Re}(u(z,t))$. Hence:

$$\begin{aligned}\text{Re}(u(z,t)) = u_0 \exp\left(-\frac{z}{2H}\right) \cos\left(\omega\left(t - \frac{z}{c_s}\right)\right) \\ - \left(\frac{u_0 z \exp\left(-\frac{z}{2H}\right)}{2H} \cos(\omega t) \int_{\frac{z}{c_s}}^{t} \cos(\omega\tau) \frac{J_1\left(\frac{c_s}{2H}\sqrt{\tau^2 - \left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{\tau^2 - \left(\frac{z}{c_s}\right)^2}} d\tau \right. \\ \left. + \frac{u_0 z \exp\left(-\frac{z}{2H}\right)}{2H} \sin(\omega t) \int_{\frac{z}{c_s}}^{t} \sin(\omega\tau) \frac{J_1\left(\frac{c_s}{2H}\sqrt{\tau^2 - \left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{\tau^2 - \left(\frac{z}{c_s}\right)^2}} d\tau \right)\end{aligned} \tag{3.69}$$

Thus we must evaluate:

$$\int_{\frac{z}{c_s}}^{t} \cos(\omega\tau) \frac{J_1\left(\frac{c_s}{2H}\sqrt{\tau^2 - \left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{\tau^2 - \left(\frac{z}{c_s}\right)^2}} d\tau \quad \text{and} \quad \int_{\frac{z}{c_s}}^{t} \sin(\omega\tau) \frac{J_1\left(\frac{c_s}{2H}\sqrt{\tau^2 - \left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{\tau^2 - \left(\frac{z}{c_s}\right)^2}} d\tau \tag{3.70}$$

We compute these integrals using Composite Simpson's rule [3].

Definition 3.1 (Composite Simpson's Rule)**.** Composite Simpson's rule states that:

$$\int_a^b f(\tau)\, d\tau \approx \frac{b-a}{3n}\left(f(\tau_0) + 4\sum_{i=1}^{\frac{n}{2}} f(\tau_{2i-1}) + 2\sum_{i=1}^{\frac{n-2}{2}} f(\tau_{2i}) + f(\tau_n)\right) \tag{3.71}$$

where $\tau_i = a + \frac{(b-a)i}{n}$ and n is the number of subintervals.

This allows us to solve (3.70) semi-analytically (See Appendix B.6 for the calculation) in Fortran and we plot $\text{Re}(u(z,t))$ in Section 3.2.

3.2 Semi-analytic Results

Here we show the results for the semi-analytic method from two different perspectives, the left is a 3D image of the entire solution space while a top-down solution with a colour map is shown on the right. We have chosen realistic values for c_s and the other parameters have been chosen for mathematical simplicity. It is important to note that the solar atmosphere is represented by the positive z direction. The plots are coloured to indicate wave speed so here green represents positive wave speed and dark blue represents negative wave speed. Plotting $\text{Re}(u(z,t))$ lets us obtain Figures 3.1, 3.2 and 3.3.

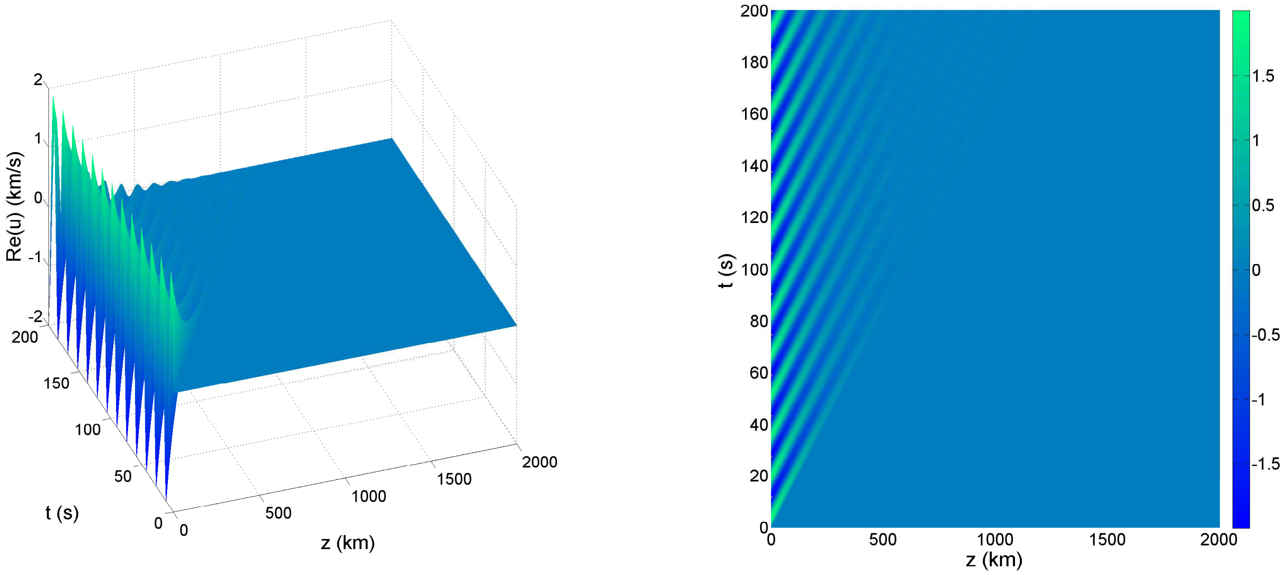


Figure 3.1: $\mathrm{Re}(u(z,t))$ with $c_s = 6.0\ \mathrm{km\,s^{-1}}$, $\omega = 0.4\ \mathrm{s^{-1}}$, $u_0 = 2.0\ \mathrm{km\,s^{-1}}$

Here we can see the system being driven sinusoidally from the surface of the sun. We see the system start at equilibrium and then the wave evolves over time and decays in the stratified medium.

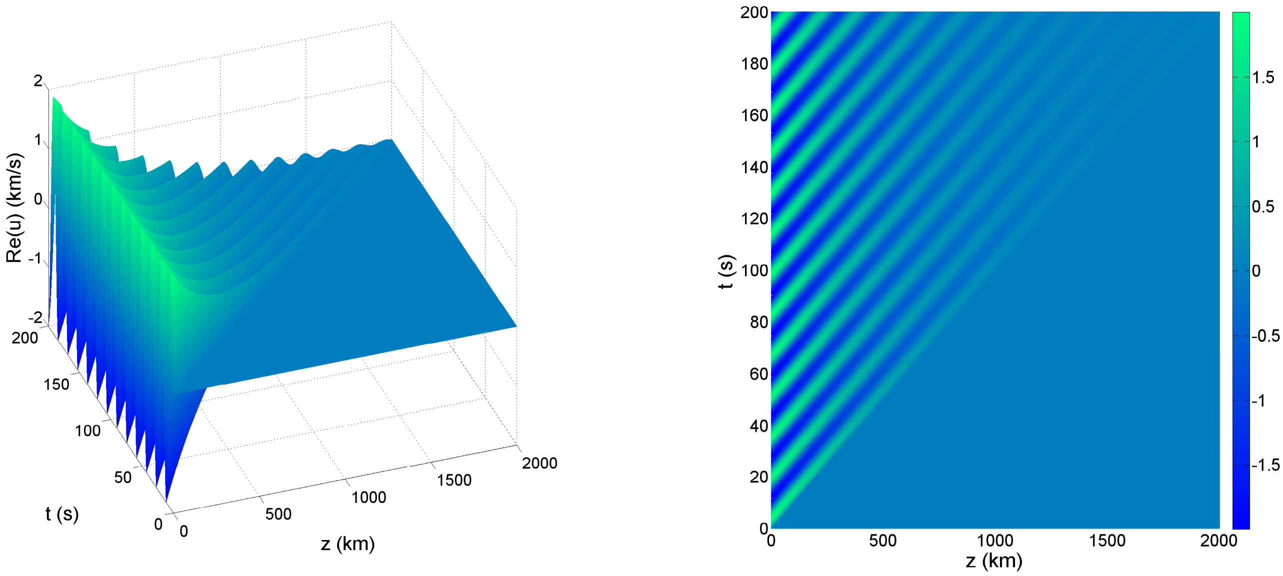


Figure 3.2: $\mathrm{Re}(u(z,t))$ with $c_s = 10.0\ \mathrm{km\,s^{-1}}$, $\omega = 0.4\ \mathrm{s^{-1}}$, $u_0 = 2.0\ \mathrm{km\,s^{-1}}$

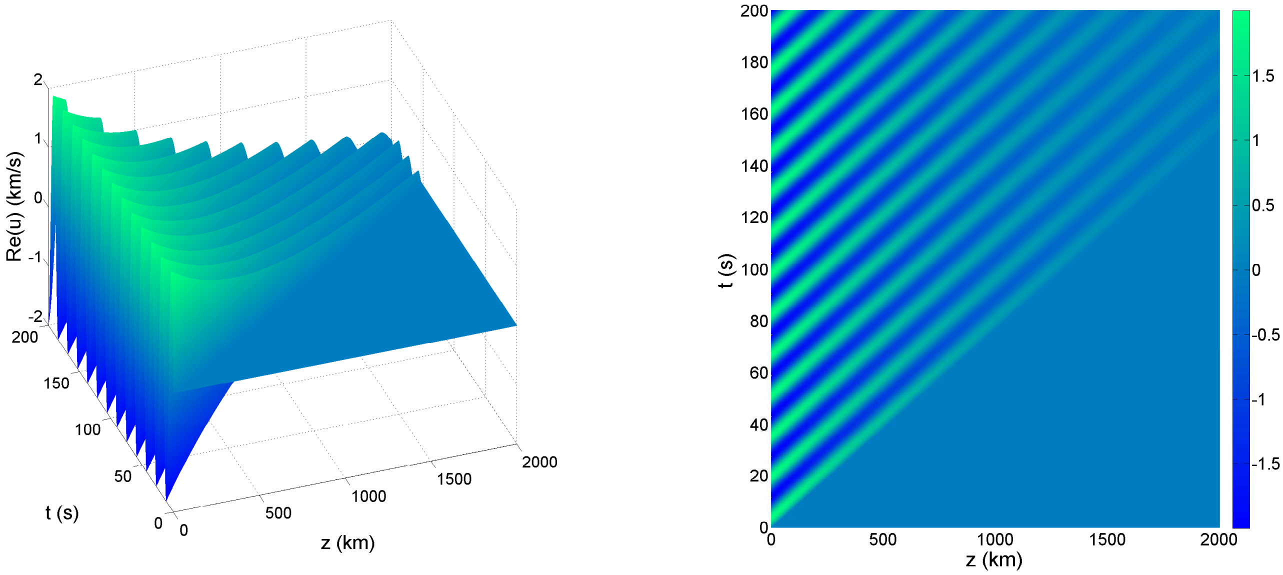


Figure 3.3: Re$(u(z,t))$ with $c_s = 13.0\ \mathrm{km\,s^{-1}}$, $\omega = 0.4\ \mathrm{s^{-1}}$, $u_0 = 2.0\ \mathrm{km\,s^{-1}}$

Figures 3.1, 3.2, and 3.3 show the effect that changing sound speed, c_s, has on the system. We see the wave propagating further with larger amplitudes at greater wave speeds. These results are expected and bolster confidence in the method used.

3.3 Numerical Solution - Finite difference method

In this section we solve (3.34) using numerical techniques. We will use finite difference method (FDM) to transform the problem from a partial differential equation into a system of simultaneous equations. We begin by introducing FDM for first derivatives and then proceed to second derivatives. Here we will state the tools we need to use, but a detailed derivation of the central difference approximation can be found in Appendix C.1.

Definition 3.2 (Central difference approximation)**.** Let h be the step size. Then let $z_{i+1} = z_i + h$ and $z_{i-1} = z_i - h$ then:

$$f'(z_i) \approx \frac{f(z_{i+1}) - f(z_{i-1})}{2h} \tag{3.72}$$

Equation (3.72) is called the *central difference approximation.*

Finite Difference Method enables us to estimate the value of a derivative at a point using surrounding function values.

Note 3.3 (Step size selection)**.** Step size selection can be problematic for more sophisticated functions. If the step size is too small the computation becomes too slow; if the step size is too large then not enough information is gathered.

Definition 3.4 is rewording the conclusion of Definition 3.2 in more appropriate language for solving differential equations numerically.

Definition 3.4 (Central finite difference formula for 1-D first derivative)**.** Let u be a function $u(z)$, $\triangle z$ be the step size and let u_i be $u(z)$ evaluated at $z_i = i\triangle z$. The central second order finite difference for $\frac{du}{dz}$ is:

$$\left.\frac{du}{dz}\right|_i = \frac{u_{i+1} - u_{i-1}}{2\triangle z} \tag{3.73}$$

Definition 3.5 (Finite difference formula for 2-D first derivatives)**.** Let u be a function $u(z,t)$. The central first order finite difference approximations for $\frac{\partial u}{\partial z}$ and $\frac{\partial u}{\partial t}$ are:

$$\left.\frac{\partial u}{\partial z}\right|_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\triangle z} \tag{3.74}$$

and

$$\left.\frac{\partial u}{\partial t}\right|_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2\triangle t} \tag{3.75}$$

Definition 3.6 (Finite difference formula for 2-D second derivatives)**.** Let u be a function $u(z,t)$. The central second order finite difference approximations for $\frac{\partial^2 u}{\partial z^2}$ and $\frac{\partial^2 u}{\partial t^2}$ are:

$$\left.\frac{\partial^2 u}{\partial z^2}\right|_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\triangle z^2} \tag{3.76}$$

and

$$\left.\frac{\partial^2 u}{\partial t^2}\right|_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\triangle t^2} \tag{3.77}$$

The non-dimensionalised form of (3.34) (where $Z = \frac{z}{H}$ and $T = \frac{c_s}{H}t$) is:

$$\frac{\partial^2 u}{\partial Z^2} + \frac{\partial u}{\partial Z} - \frac{\partial^2 u}{\partial T^2} = 0 \tag{3.78}$$

We find the numerical scheme by substituting (3.76), (3.74) and (3.77) into (3.78). This leaves:

$$\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\triangle Z^2}\right) + \left(\frac{u_{i+1,j} - u_{i-1,j}}{2\triangle Z}\right) - \left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\triangle T^2}\right) = 0 \tag{3.79}$$

Expanding, simplifying, and making $u_{i,j+1}$ the subject, gives:

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + \left(\frac{\triangle T^2}{\triangle Z^2}\right)(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \left(\frac{\triangle T^2}{2\triangle Z}\right)(u_{i+1,j} - u_{i-1,j}) \tag{3.80}$$

We make $u_{i,j+1}$ the subject now because we will find that the initial and boundary conditions let us know everything on the right hand side of (3.80). Recall (3.35) to (3.38). Assuming imax steps in the Z direction and jmax steps in the T direction, then the initial and boundary conditions become:

$$u_{i,0} = 0 \text{ for all } i \neq 0 \tag{3.81}$$

$$\left.\frac{\partial u}{\partial T}\right|_{i,0} = 0 \text{ for all } i \neq 0 \tag{3.82}$$

$$u_{0,j} = u_0 \exp\left(-i\left(\frac{\omega H}{c_s}T - \frac{\pi}{2}\right)\right) \tag{3.83}$$

$$u_{\text{imax},j} = 0 \text{ for all } j \tag{3.84}$$

It is important to note that boundary condition (3.84) has been modified so it can be implemented in a finite domain. The Neumann initial condition, (3.82), with $j = 0$ yields:

$$\left.\frac{\partial u}{\partial T}\right|_{i,0} = \frac{u_{i,1} - u_{i,-1}}{2\triangle T} \tag{3.85}$$

Thus from (3.82) we have:

$$\frac{u_{i,j+1} - u_{i,j-1}}{2\triangle T} = 0 \text{ for all } i \neq \text{imax} \tag{3.86}$$

Hence:

$$u_{i,1} = u_{i,-1} \text{ for all } i \neq \text{imax} \tag{3.87}$$

We examine this statement for $i \in \{1, 2 \ldots \text{imax} - 1\}$. Begin by recalling (3.80) with $j = 0$:

$$u_{i,1} = 2u_{i,0} - u_{i,-1} + \left(\frac{\triangle T^2}{\triangle Z^2}\right)(u_{i+1,0} - 2u_{i,0} + u_{i-1,0}) + \left(\frac{\triangle T^2}{2\triangle Z}\right)(u_{i+1,0} - u_{i-1,0}) \tag{3.88}$$

Now let $i = 1$:

$$u_{1,1} = 2u_{1,0} - u_{1,-1} + \left(\frac{\triangle T^2}{\triangle Z^2}\right)(u_{2,0} - 2u_{1,0} + u_{0,0}) + \left(\frac{\triangle T^2}{2\triangle Z}\right)(u_{2,0} - u_{0,0}) \tag{3.89}$$

Substituting (3.87) into this gives:

$$u_{1,1} = 2u_{1,0} - u_{1,1} + \left(\frac{\triangle T^2}{\triangle Z^2}\right)(u_{2,0} - 2u_{1,0} + u_{0,0}) + \left(\frac{\triangle T^2}{2\triangle Z}\right)(u_{2,0} - u_{0,0}) \tag{3.90}$$

Recall that $u_{1,0} = u_{2,0} = 0$ from the Dirichlet initial condition (3.81) and that $u_{0,0} \neq 0$ from the first boundary condition (3.83). Hence:

$$u_{1,1} = \frac{\triangle T^2}{2}\left(\frac{1}{\triangle Z^2} - \frac{1}{2\triangle Z}\right)u_{0,0} \tag{3.91}$$

Next consider $i = 2$:

$$u_{2,1} = 2u_{2,0} - u_{2,-1} + \left(\frac{\triangle T^2}{\triangle Z^2}\right)(u_{3,0} - 2u_{2,0} + u_{1,0}) + \left(\frac{\triangle T^2}{2\triangle Z}\right)(u_{3,0} - u_{1,0}) \tag{3.92}$$

Applying the Dirichlet initial condition (3.81) we find:

$$u_{2,1} = 0 \tag{3.93}$$

This process can be repeated for $i \in \{3, 4 \ldots \text{imax} - 1\}$ and the result is the same. Hence:

$$u_{i,1} = 0 \text{ for all } i \in \{2, 3 \ldots \text{imax} - 1\} \tag{3.94}$$

We now have all the tools we need to iterate (3.80). This is because we know every $u_{i,0}$ and $u_{i,1}$, hence we can use (3.80) to find $u_{i,2}$ and so on. We choose the range of z to be the solar chromosphere and imax and jmax to be as small as possible within the confines of the program. The first results are shown in Figure 3.4.

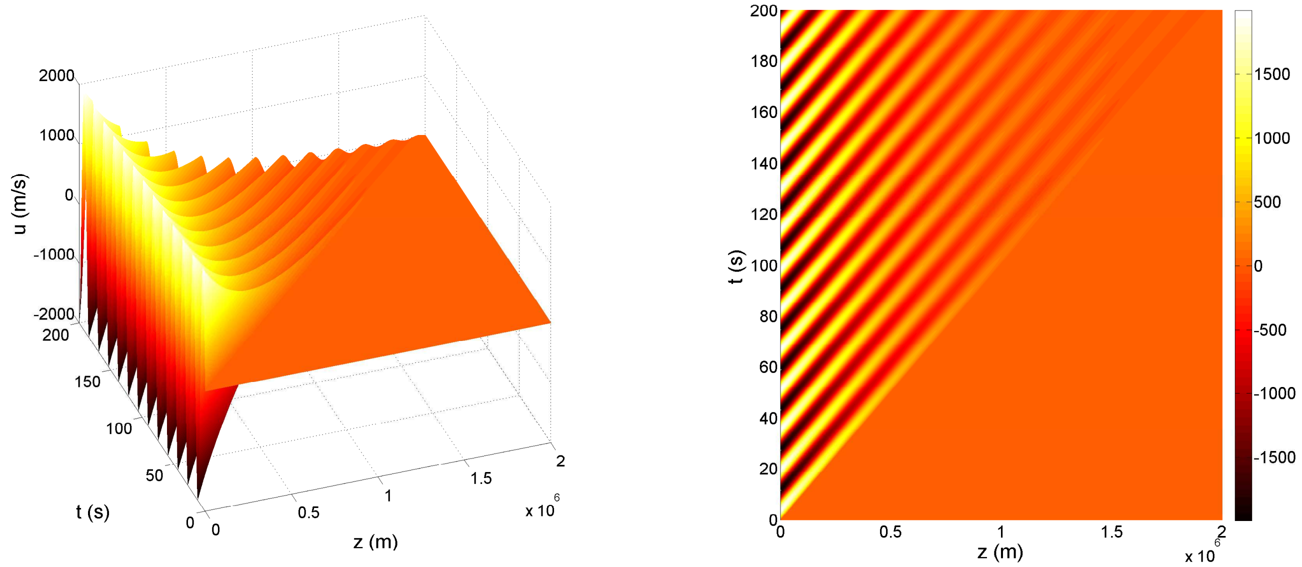


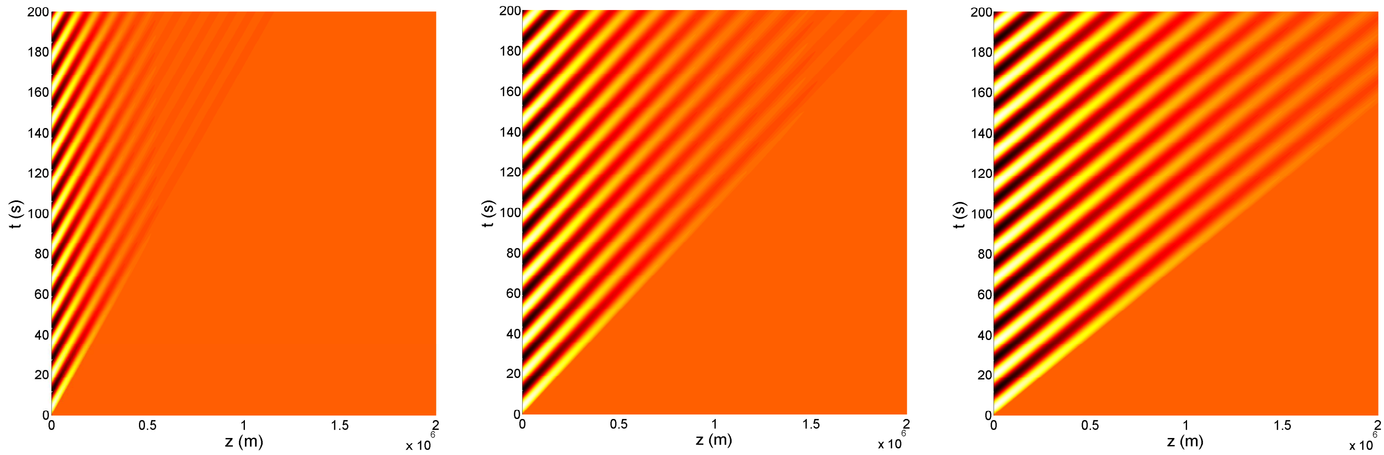
Figure 3.4: Numeric solution with $c_s = 10\,000\ \text{m}\,\text{s}^{-1}$, $\omega = 0.4\ \text{s}^{-1}$ and $u_0 = 2000\ \text{m}\,\text{s}^{-1}$

We see the system starting at equilibrium with the wave generated at $z = 0$, the surface of the sun, propagating upwards through the atmosphere as time passes. We see amplitude fall due to pressure and density decreasing with height and energy dissipating. These results appear reasonable for the model we have examined.

4 Parametric study and a comparison of methods

In this section we examine how changing the parameters c_s and ω affects the system and then compare the semi-analytic and numeric solutions. We begin by varying c_s.

4.1 Varying the wave speed, c_s

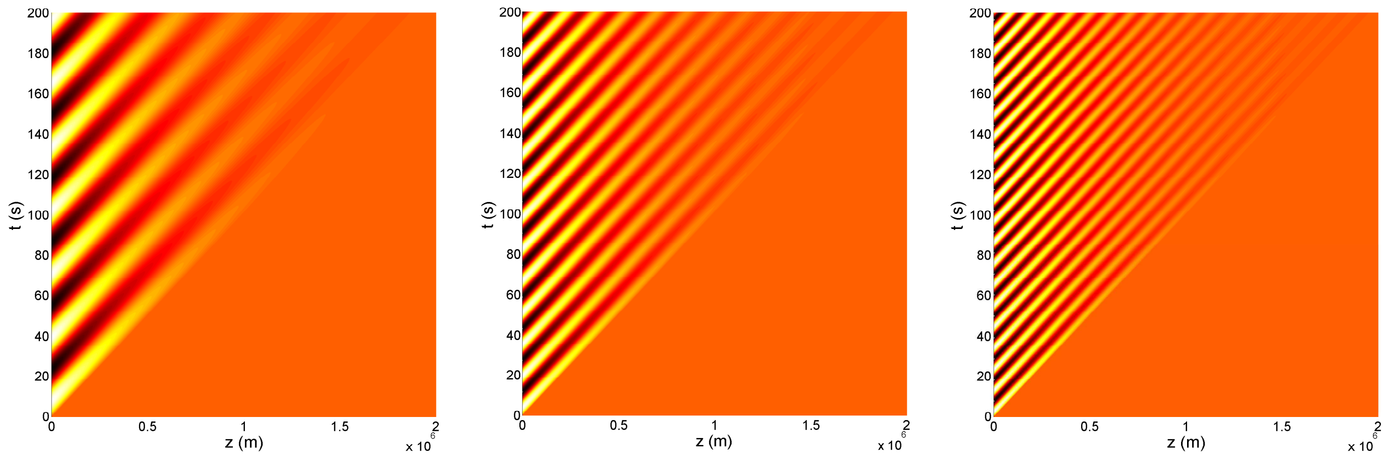


(a) $c_s = 6000\ \mathrm{m\,s^{-1}}$ **(b)** $c_s = 10\,000\ \mathrm{m\,s^{-1}}$ **(c)** $c_s = 13\,000\ \mathrm{m\,s^{-1}}$

Figure 4.1: A collection of numeric solutions with variable c_s where $\omega = 0.4\ \mathrm{s^{-1}}$ and $u_0 = 2000\ \mathrm{m\,s^{-1}}$

We can see that varying c_s affects the angle that the wave makes with the z axis. We expect to see this as increasing the wave speed, c_s, will increase the distance the wave propagates in a set period of time.

4.2 Varying the angular frequency, ω



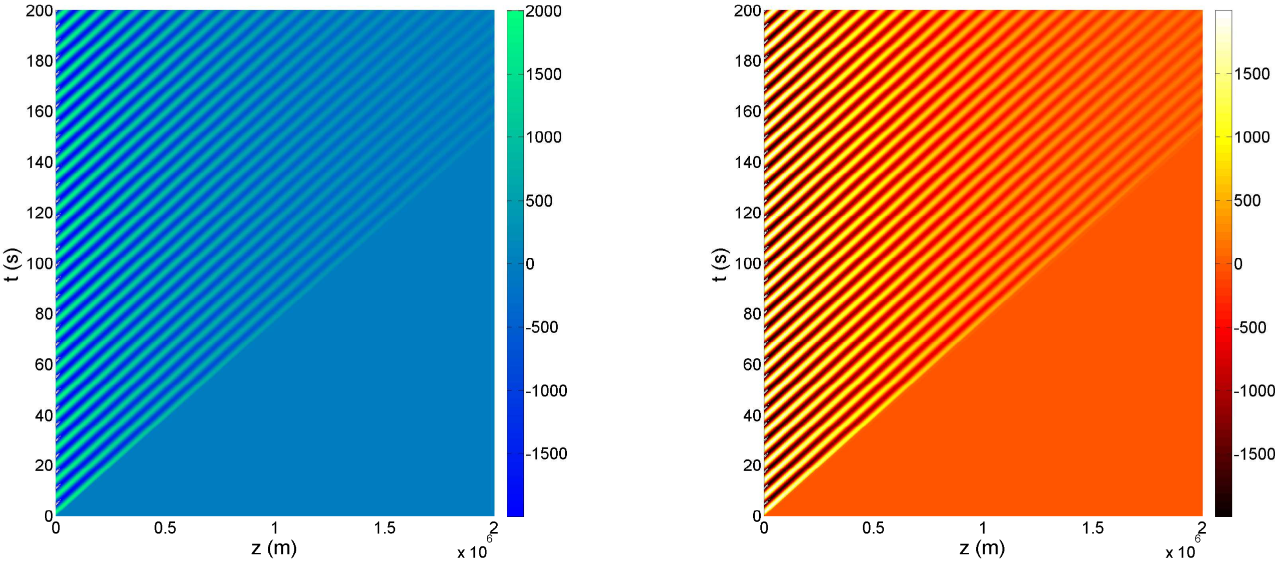
(a) $\omega = 0.2\ \mathrm{s^{-1}}$ **(b)** $\omega = 0.4\ \mathrm{s^{-1}}$ **(c)** $\omega = 0.6\ \mathrm{s^{-1}}$

Figure 4.2: A collection of numeric solutions with variable ω where $c_s = 10\,000\ \mathrm{m\,s^{-1}}$ and $u_0 = 2000\ \mathrm{m\,s^{-1}}$

Here we see ω controlling the rate of input oscillation and therefore the wavelength.

4.3 Comparing semi-analytic and numeric solutions

Here we compare the semi-analytic (Figures 3.1, 3.2, and 3.3) and numeric solutions (Figure 4.1) to ensure that the results are realistic and consistent. It is important to compare these solutions as in the following sections the problem will become more complex due to the addition of a magnetic field.



(a) From the semi-analytic method

(b) From the numeric method

Figure 4.3: Solutions with $c_s = 13\,000\ \mathrm{m\,s^{-1}}$, $\omega = 1.0\ \mathrm{s^{-1}}$ and $u_0 = 2000\ \mathrm{m\,s^{-1}}$

Figure 4.3 demonstrates that the numeric method provides an accurate alternative to the semi-analytic method; this on the condition that values such as imax and jmax are sufficiently large. This conclusion is supported by the solutions when they are subtracted from one another.

5 Modelling a non-isothermal atmosphere

In this section we enhance the model by assuming that c_s in (3.34) varies with z. We use data collected by Harvard Skylab [4] to create a polynomial approximation (using the method of least squares) for $c_s(z)$ to find that, for constants A, B, and C:

$$c_s(z) = Az^2 + Bz + C \tag{5.1}$$

Here:

$$A = 3.179 \times 10^{-9} \tag{5.2}$$

$$B = -3.886 \times 10^{-3} \tag{5.3}$$

$$C = 8.457 \times 10^{3} \tag{5.4}$$

Figure 5.1 shows the polynomial approximation in relation to the raw data.

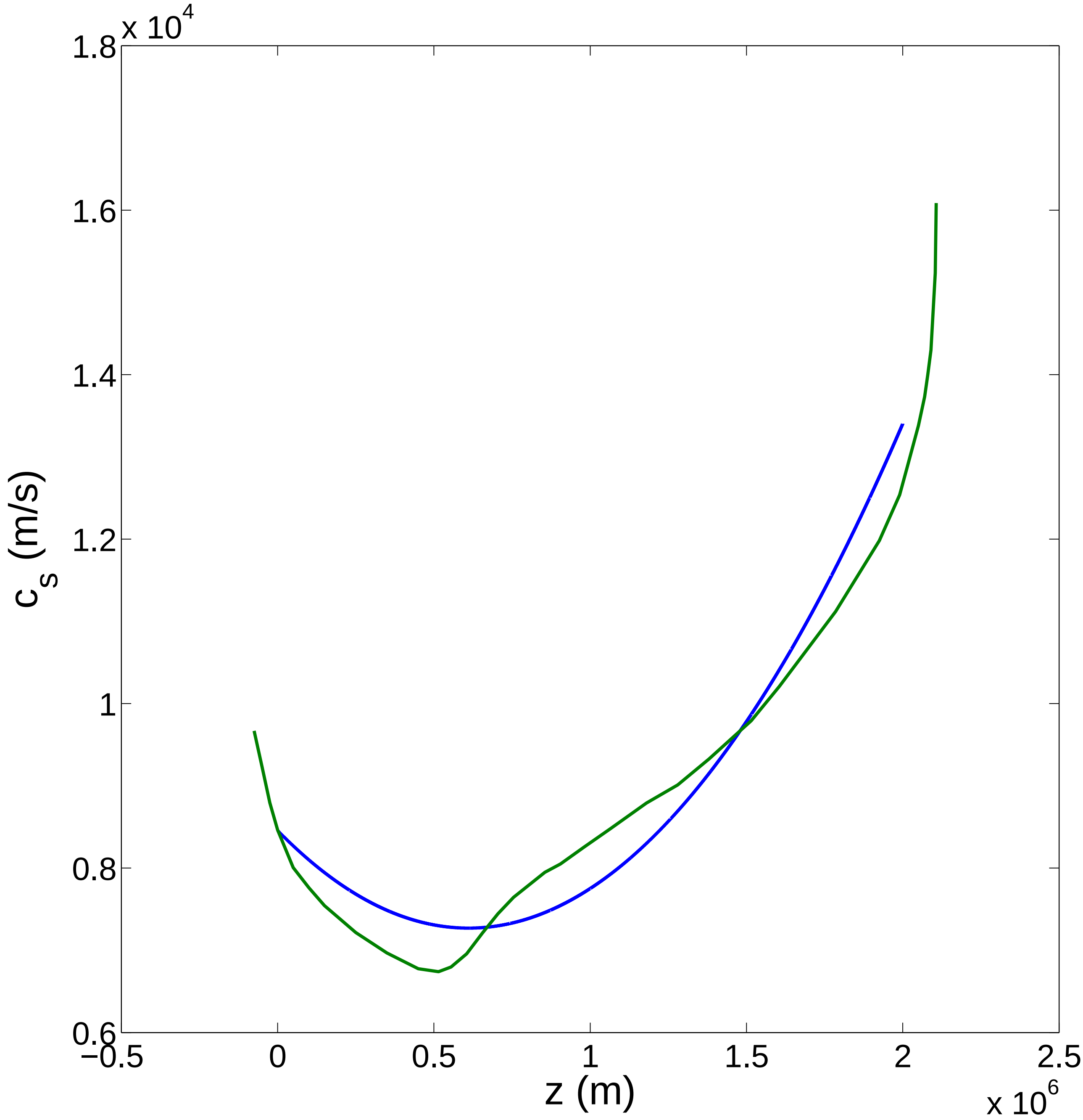


Figure 5.1: The raw sound speed data is shown in green, with a quadratic polynomial of best fit shown in blue

Solving (3.34) numerically, using the polynomial approximation for $c_s(z)$, yields:

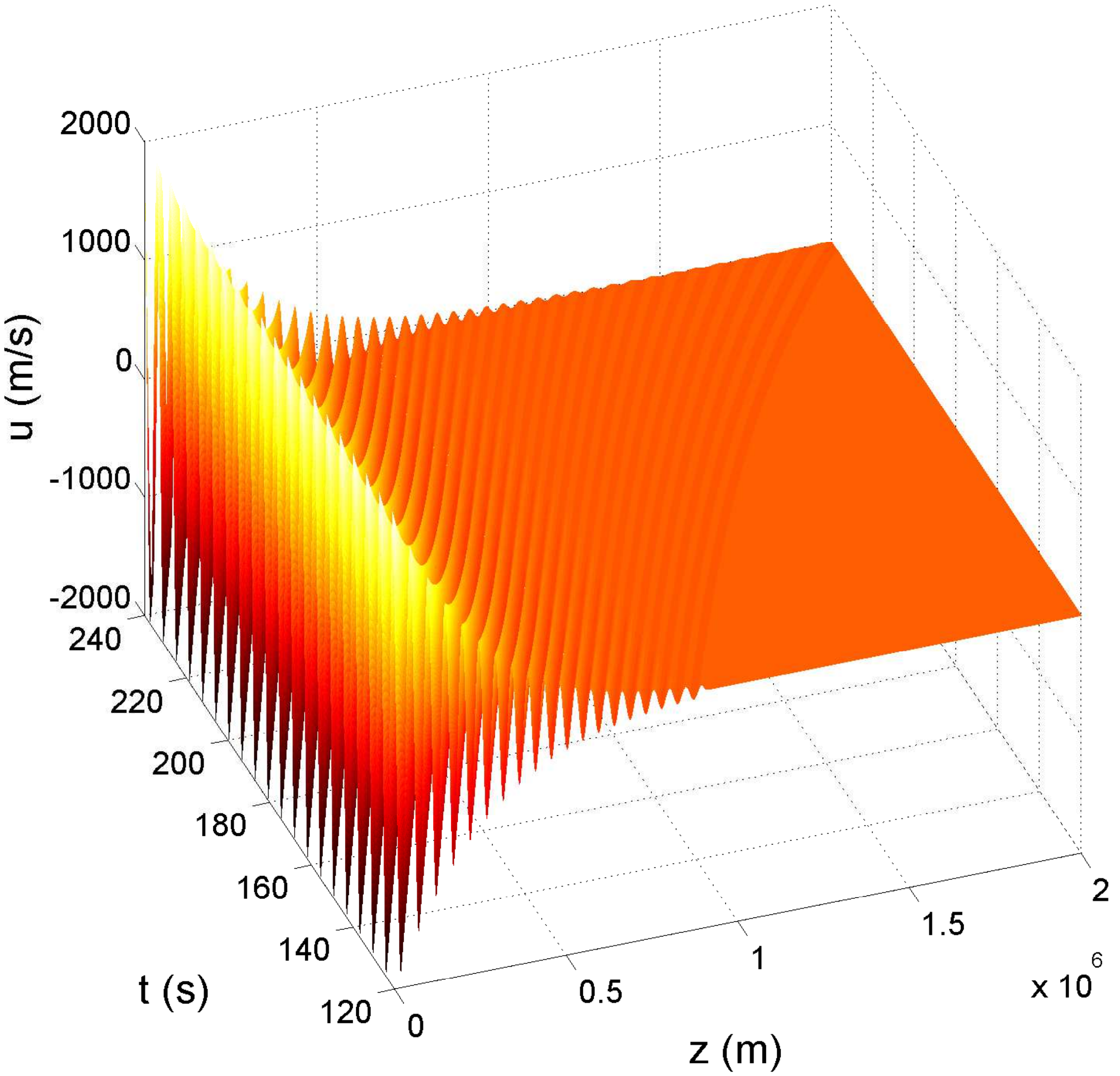
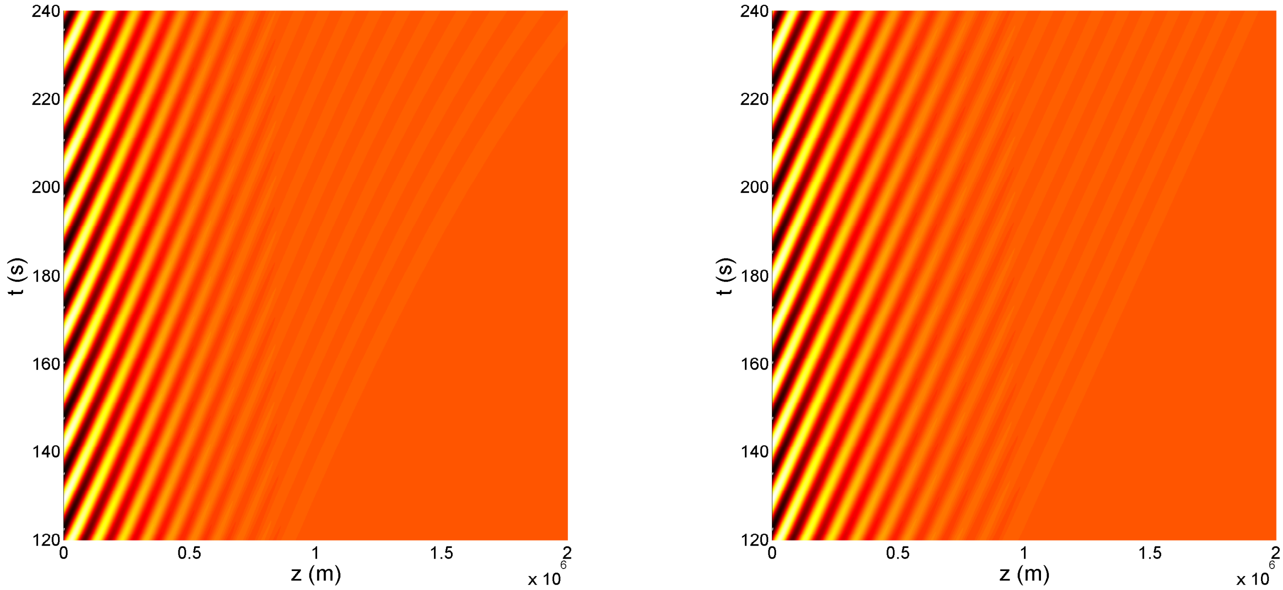


Figure 5.2: Numeric solution u with variable c_s and $u_0 = 2000 \text{ m s}^{-1}$. Here $\omega = 1.0 \text{ s}^{-1}$ has be chosen to highlight the curvature for large values of z

Figure 5.2 shows that as the wave propagates and travels further from the surface of the sun, it disperses. This is due to the sound speed profile and is shown by the curved nature of the plot for large z.



(a) Variable c_s

(b) $c_s = 8000 \text{ m s}^{-1}$

Figure 5.3: Numeric solutions for u with and $u_0 = 2000 \text{ m s}^{-1}$. Here $\omega = 0.5 \text{ s}^{-1}$ has be chosen to highlight the curvature for large values of z

Figure 5.3 highlights the difference that a non-isothermal atmosphere makes. Here the curvature in the variable c_s figure is obvious when compared with the linear nature shown in the constant c_s figure. We conclude that for the z between the surface of the sun, $z = 0$ and $z = 2000$ km, an isothermal model will be sufficient to expect realistic results.

This concludes the investigation into acoustic waves for the quiet sun. In the following chapters we study regions with significant magnetic activity.

6 Derivation of magneto-acoustic gravity waves

In this section we explore the addition of a fixed magnetic field $\boldsymbol{B}$. This is more realistic for modelling the photosphere and chromosphere. We begin by deriving the governing equations. To do this, we require the equation of continuity (3.1), the equation of motion (3.2), the adiabatic energy equation (3.3), and the magnetic induction equation (3.4). We make the same assumptions as made in Section 3 except we consider each velocity component of v to be non-zero and assume that a constant magnetic field acts as follows:

$$\boldsymbol{B_0} = \begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix} \tag{6.1}$$

We get the following set of equations:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \tag{6.2}$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla} p - \boldsymbol{J} \times \boldsymbol{B} - \rho \boldsymbol{g} = 0 \tag{6.3}$$

$$\frac{\partial p}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} p + \gamma p (\boldsymbol{\nabla} \cdot \boldsymbol{v}) = 0 \tag{6.4}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) = 0 \tag{6.5}$$

In Appendix D.3 we linearise these equations. The results are shown below:

$$\frac{\partial \rho_1}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{v_1}) = 0 \tag{6.6}$$

$$\rho_0 \frac{\partial \boldsymbol{v_1}}{\partial t} + \boldsymbol{\nabla} p_0 + \boldsymbol{\nabla} p_1 - \frac{1}{\mu_0}((\boldsymbol{\nabla} \times \boldsymbol{B_1}) \times \boldsymbol{B_0}) - \rho_0 \boldsymbol{g} - \rho_1 \boldsymbol{g} = 0 \tag{6.7}$$

$$\frac{\partial p_1}{\partial t} + \boldsymbol{v_1} \cdot \boldsymbol{\nabla} p_0 + c_s^2 \rho_0 (\boldsymbol{\nabla} \cdot \boldsymbol{v_1}) = 0 \tag{6.8}$$

$$\frac{\partial \boldsymbol{B_1}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v_1} \times \boldsymbol{B_0}) = 0 \tag{6.9}$$

Next we differentiate (6.7) with respect to t:

$$\rho_0 \frac{\partial^2 \boldsymbol{v_1}}{\partial t^2} + \boldsymbol{\nabla} \left(\frac{\partial p_1}{\partial t} \right) - \frac{1}{\mu_0} \left(\left(\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{B_1}}{\partial t} \right) \times \boldsymbol{B_0} \right) - \frac{\partial \rho_1}{\partial t} \boldsymbol{g} = 0 \tag{6.10}$$

Using (6.6) to eliminate $\frac{\partial \rho_1}{\partial t}$, (6.8) to eliminate $\frac{\partial p_1}{\partial t}$, and (6.9) to eliminate $\frac{\partial \boldsymbol{B_1}}{\partial t}$ we find:

$$\rho_0 \frac{\partial^2 \boldsymbol{v_1}}{\partial t^2} - \boldsymbol{\nabla} \Big(\boldsymbol{v_1} \cdot \boldsymbol{\nabla} p_0 + c_s^2 \rho_0 (\boldsymbol{\nabla} \cdot \boldsymbol{v_1}) \Big) - \frac{1}{\mu_0} \left(\Big(\boldsymbol{\nabla} \times \big(\boldsymbol{\nabla} \times (\boldsymbol{v_1} \times \boldsymbol{B_0}) \big) \Big) \times \boldsymbol{B_0} \right) + (\boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{v_1})) \boldsymbol{g} = 0 \tag{6.11}$$

In Appendix D.4 we see that:

$$\left(\Big(\boldsymbol{\nabla} \times \big(\boldsymbol{\nabla} \times (\boldsymbol{v_1} \times \boldsymbol{B_0}) \big) \Big) \times \boldsymbol{B_0} \right) = B_0^2 \begin{pmatrix} \frac{\partial v_x^2}{\partial x^2} + \frac{\partial v_y^2}{\partial x \partial y} + \frac{\partial v_x^2}{\partial z^2} \\ \frac{\partial v_y^2}{\partial y^2} + \frac{\partial v_x^2}{\partial x \partial y} + \frac{\partial v_y^2}{\partial z^2} \\ 0 \end{pmatrix} \tag{6.12}$$

Both the x-direction and the y-direction are unaffected by $\boldsymbol{g}$ and they exhibit similar behaviour, hence we can simplify the problem to a 2D problem by only considering v_x and v_z. This modifies (6.11) to the following:

$$\rho_0 \frac{\partial^2 \boldsymbol{v_1}}{\partial t^2} - \boldsymbol{\nabla} \Big(\boldsymbol{v_1} \cdot \boldsymbol{\nabla} p_0 + c_s^2 \rho_0 (\boldsymbol{\nabla} \cdot \boldsymbol{v_1}) \Big) - \frac{1}{\mu_0} \left(B_0^2 \begin{pmatrix} \frac{\partial v_x^2}{\partial x^2} + \frac{\partial v_x^2}{\partial z^2} \\ 0 \end{pmatrix} \right) + (\boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{v_1})) \boldsymbol{g} = 0 \tag{6.13}$$

Hence we expand and simplify to:

$$\rho_0 \frac{\partial^2 \boldsymbol{v_1}}{\partial t^2} - \boldsymbol{\nabla} \left(v_z \frac{\partial p_0}{\partial z} + c_s^2 \rho_0 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} \right) \right) - \frac{B_0^2}{\mu_0} \begin{pmatrix} \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2} \\ 0 \end{pmatrix} + \left(\rho_0 \frac{\partial v_x}{\partial x} + \frac{\partial}{\partial z} (\rho_0 v_z) \right) \boldsymbol{g} = 0 \tag{6.14}$$

Consider the x component of (6.14):

$$\rho_0 \frac{\partial^2 v_x}{\partial t^2} - \frac{\partial}{\partial x} \left(v_z \frac{\partial p_0}{\partial z} + c_s^2 \rho_0 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} \right) \right) - \frac{B_0^2}{\mu_0} \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2} \right) = 0 \tag{6.15}$$

This simplifies to:

$$\frac{\partial^2 v_x}{\partial t^2} - \left(c_s^2 + \frac{B_0^2}{\mu_0 \rho_0} \right) \frac{\partial^2 v_x}{\partial x^2} - \frac{B_0^2}{\mu_0 \rho_0} \frac{\partial^2 v_x}{\partial z^2} = c_s^2 \frac{\partial^2 v_z}{\partial x \partial z} + \frac{1}{\rho_0} \frac{\partial p_0}{\partial z} \frac{\partial v_z}{\partial x} \tag{6.16}$$

Note that this problem has the same equilibrium conditions as in Section 3 (thus equations (3.13) and (3.15) hold). We define Alfvén speed, v_A, by $v_A^2 = \frac{B_0^2}{\mu_0 \rho_0}$ [5]. Thus (6.16) is transformed into:

$$\left(v_A^2 \frac{\partial^2}{\partial z^2} + \left(c_s^2 + v_A^2 \right) \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) v_x = - \frac{\partial}{\partial x} \left(c_s^2 \frac{\partial}{\partial z} - g \right) v_z \tag{6.17}$$

Next consider the z component of (6.14):

$$\rho_0 \frac{\partial^2 v_z}{\partial t^2} - \frac{\partial}{\partial z} \left(v_z \frac{\partial p_0}{\partial z} + c_s^2 \rho_0 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} \right) \right) - g \left(\rho_0 \frac{\partial v_x}{\partial x} + \frac{\partial}{\partial z} (\rho_0 v_z) \right) = 0 \tag{6.18}$$

After manipulation (See Appendix D.5), this simplifies to:

$$\left(c_s^2\frac{\partial^2}{\partial z^2} - \gamma g\frac{\partial}{\partial z} - \frac{\partial^2}{\partial t^2}\right) v_z = -\frac{\partial}{\partial x}\left(c_s^2\frac{\partial}{\partial z} + (1-\gamma)g\right) v_x \tag{6.19}$$

Hence we have found a coupled system of partial differential equations. We simplify by assuming that both v_x and v_z are periodic in x and t as follows:

$$\boldsymbol{v} = \begin{pmatrix} v_x \\ v_z \end{pmatrix} = \begin{pmatrix} v_x(z)\sin(k_x x)e^{-i\omega t} \\ v_z(z)\cos(k_x x)e^{-i\omega t} \end{pmatrix} \tag{6.20}$$

This assumption simplifies (6.17) and (6.19) to:

$$\left(v_A^2\frac{d^2}{dz^2} - \left(c_s^2 + v_A^2\right)k_x^2 + \omega^2\right) v_x = k_x\left(c_s^2\frac{d}{dz} - g\right) v_z \tag{6.21}$$

$$\left(c_s^2\frac{d^2}{dz^2} - \gamma g\frac{d}{dz} + \omega^2\right) v_z = -k_x c_s^2\left(\frac{d}{dz} + \frac{(1-\gamma)g}{c_s^2}\right) v_x \tag{6.22}$$

We have now simplified to a coupled system of second order differential equations. The goal is to solve this system to find v_x and v_z in terms of z. We first non-dimensionalise (6.21) and (6.22). Recall that $H = \frac{c_s^2}{\gamma g}$, then set:

$$z = Hz_H, \quad k_x = \frac{\lambda}{H}, \quad \omega^2 = \frac{c_s^2}{H^2}\nu^2 \tag{6.23}$$

We also define plasma-β as the ratio of plasma pressure to magnetic pressure:

$$\beta = \frac{p}{p_{mag}} = \frac{2c_s^2}{\gamma v_A^2} = \frac{2c_\rho \mu_0 c_s^2}{\gamma B_0^2}\exp\left(-z_H\right) \tag{6.24}$$

Note that c_ρ is a constant and μ_0 is the permeability of free space. Therefore we see that plasma-β has an exponential relationship with z_H and consequently we transform (6.21) and (6.22) into:

$$\left(\frac{d^2}{dz_H^2} + \frac{\gamma\beta}{2}\left(\nu^2 - \lambda^2\right) - \lambda^2\right) v_x = \frac{\lambda\beta}{2}\left(\gamma\frac{d}{dz_H} - 1\right) v_z \tag{6.25}$$

$$\left(\frac{d^2}{dz_H^2} - \frac{d}{dz_H} + \nu^2\right) v_z = -\lambda\left(\frac{d}{dz_H} + \frac{1-\gamma}{\gamma}\right) v_x \tag{6.26}$$

6.1 Asymptotic solutions for the weak and strong fields

In the subsequent sections we solve the coupled equations (6.21) and (6.22). Recall the definition of plasma-β. We split z into two distinct regions: the weak field ($\beta \ggg 1$) and the strong field ($\beta \lll 1$). There are two modes to consider for each field, the fast mode and the slow mode, both of which appear in magnetic environments. Before we investigate the weak and strong fields, we set $\beta = 1$ at $z = 0$. This makes the region for the weak field $z < 0$ (the solar interior) and the region for the strong field $z > 0$. We assume that $v_z = 0$ for the strong field fast mode and $v_x = 0$ for the strong field slow mode. Additionally, for sections 6.2, 6.3, 6.4, and 6.5 we take:

$$c_s = 8515.9\text{m}\,\text{s}^{-1} \tag{6.27}$$

$$\nu = 0.2 \tag{6.28}$$

$$\lambda = 0.07 \tag{6.29}$$

$$\gamma = \frac{5}{3} \tag{6.30}$$

$$B_0 = 0.1\text{kg}\,\text{s}^{-2}\,\text{A}^{-1} \tag{6.31}$$

$$\mu_0 = 1.2566370 \times 10^{-6}\text{kg}\,\text{m}\,\text{s}^{-2}\,\text{A}^{-2} \tag{6.32}$$

$$g = 277.0\text{m}\,\text{s}^{-2} \tag{6.33}$$

6.2 Weak field fast mode

Assuming that $\beta \ggg 1$, thus $v_A \lll c_s$, we simplify (6.21) to:

$$\left(\omega^2 - k_x^2 c_s^2\right) v_x = k_x \left(c_s^2 \frac{d}{dz} - g\right) v_z \tag{6.34}$$

By rearranging we find that:

$$v_x = \frac{k_x c_s^2}{\left(\omega^2 - k_x^2 c_s^2\right)} \frac{dv_z}{dz} - \frac{g k_x}{\left(\omega^2 - k_x^2 c_s^2\right)} v_z \tag{6.35}$$

Differentiating (6.35) with respect to z yields:

$$\frac{dv_x}{dz} = \frac{k_x c_s^2}{\left(\omega^2 - k_x^2 c_s^2\right)} \frac{d^2 v_z}{dz^2} - \frac{g k_x}{\left(\omega^2 - k_x^2 c_s^2\right)} \frac{dv_z}{dz} \tag{6.36}$$

Substituting (6.35) and (6.36) into (6.22), and then rearranging gives:

$$\left(c_s^2 + \frac{k_x^2 c_s^4}{\left(\omega^2 - k_x^2 c_s^2\right)}\right) \frac{d^2 v_z}{dz^2} - \frac{1}{H}\left(c_s^2 + \frac{k_x^2 c_s^4}{\left(\omega^2 - k_x^2 c_s^2\right)}\right) \frac{dv_z}{dz} + \left(\omega^2 - \frac{(1-\gamma) g^2 k_x^2}{\left(\omega^2 - k_x^2 c_s^2\right)}\right) v_z = 0 \tag{6.37}$$

We non-dimensionalise (6.37) in Appendix D.6. Hence we transform (6.37) into:

$$\frac{d^2 v_z}{dz_H^2} - \frac{dv_z}{dz_H} + \left(\nu^2 - \lambda^2 - \frac{(1-\gamma)\lambda^2}{\gamma^2 \nu^2}\right) v_z = 0 \tag{6.38}$$

This is a linear second order differential equation so we can solve for v_z. Doing so gives us:

$$v_z = c_1 e^{m_+ z_H} + c_2 e^{m_- z_H} \tag{6.39}$$

Here c_1 and c_2 are arbitrary constants. $m_{+,-}$ is represented as follows:

$$m_{+,-} = \frac{1 \pm \sqrt{1 - 4\left(\nu^2 - \lambda^2 - \frac{(1-\gamma)\lambda^2}{\gamma^2 \nu^2}\right)}}{2} \tag{6.40}$$

Using the non-dimensional version of (6.35) allows a solution for v_x to be found:

$$v_x = \frac{\lambda}{\left(\nu^2 - \lambda^2\right)} \left(c_1 \left(m_+ - \frac{1}{\gamma}\right) e^{m_+ z_H} + c_2 \left(m_- - \frac{1}{\gamma}\right) e^{m_- z_H}\right) \tag{6.41}$$

We plot these solutions in Figures 6.1 and 6.2.

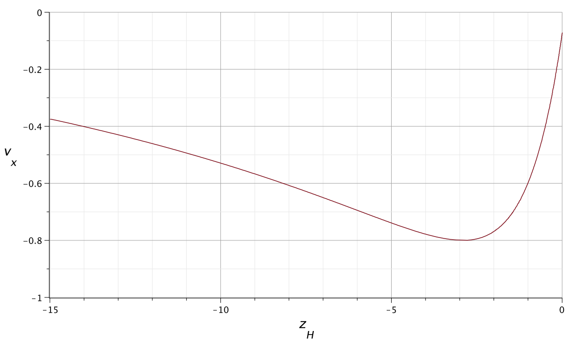


Figure 6.1: v_x weak field fast mode

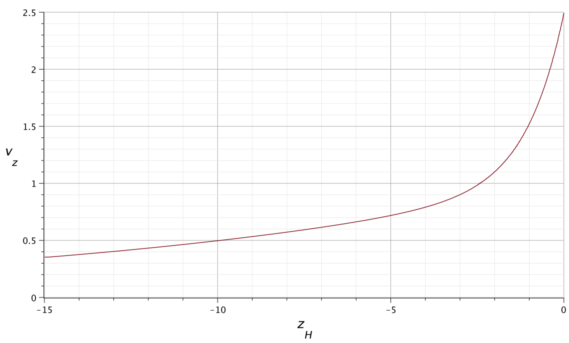


Figure 6.2: v_z weak field fast mode

Here we see v_x decreasing as z_H increases and then at $z_H \approx 3$, v_x grows quickly. This is in contrast to v_z which is increasing throughout.

6.3 Weak field slow mode

The assumption that $\beta \ggg 1$, thus $v_A \lll c_s$, holds here as in Section 6.2, but in this case we cannot cancel the $\frac{d}{dz}$ or $\frac{d^2}{dz^2}$ terms as they could be non-negligible. It is known that the slow mode propagates at approximately Alfvén speed [6], v_A, and we have found that $v_A \to 0$ as $z \to -\infty$. Therefore the wavelength of the slow mode is proportional to the Alfvén speed and so is small when compared with other length scales, therefore terms involving H and g are insignificant. We also omit the $\omega^2 v_z$ term because Alfvén waves (a type of magnetohydrodynamic wave found in solar mediums) are transverse [7]. These assertions simplify (6.21) and (6.22) to:

$$\left(v_A^2 \frac{d^2}{dz^2} - k_x^2 c_s^2 + \omega^2 \right) v_x = k_x c_s^2 \frac{dv_z}{dz} \tag{6.42}$$

$$c_s^2 \frac{d^2 v_z}{dz^2} = -k_x c_s^2 \frac{dv_x}{dz} \tag{6.43}$$

From (6.43), by integrating with respect to z, we calculate:

$$\frac{dv_z}{dz} = -k_x v_x \tag{6.44}$$

Using (6.44) to simplify (6.42) yields:

$$v_A^2 \frac{d^2 v_x}{dz^2} + \omega^2 v_x = 0 \tag{6.45}$$

We non-dimensionalise (6.45) in Appendix D.6. It follows that:

$$\frac{2}{\gamma\beta} \frac{d^2 v_x}{dz_H^2} + \nu^2 v_x = 0 \tag{6.46}$$

The solutions to (6.46) come in the form of Bessel functions. Here we need two types of Bessel function, those of the first type, J_α, and those of the second type, Y_α. Thus:

$$v_x = c_1 J_0\left(\nu\sqrt{2\gamma\beta}\right) + c_2 Y_0\left(\nu\sqrt{2\gamma\beta}\right) \tag{6.47}$$

It is shown in Appendix D.7 that (6.47) is a solution for (6.46). Using the non-dimensional version of (6.44) found below, we find a numerical solution for v_z using Simpson's rule.

$$\frac{dv_z}{dz_H} = -\lambda v_x \tag{6.48}$$

We plot the solutions in Figures 6.3 and 6.4.

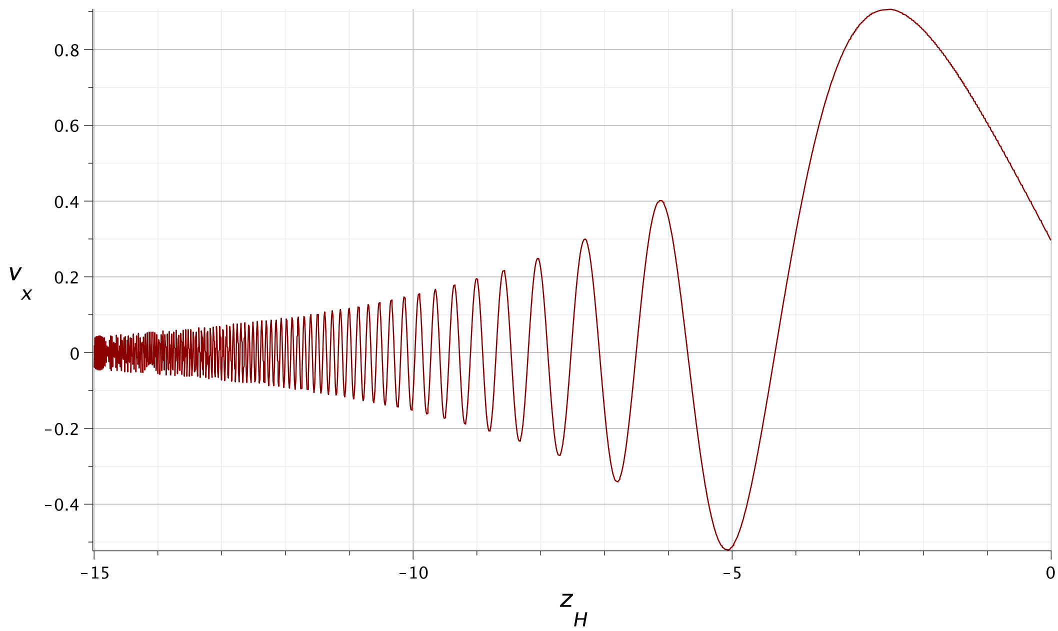


Figure 6.3: v_x weak field slow mode

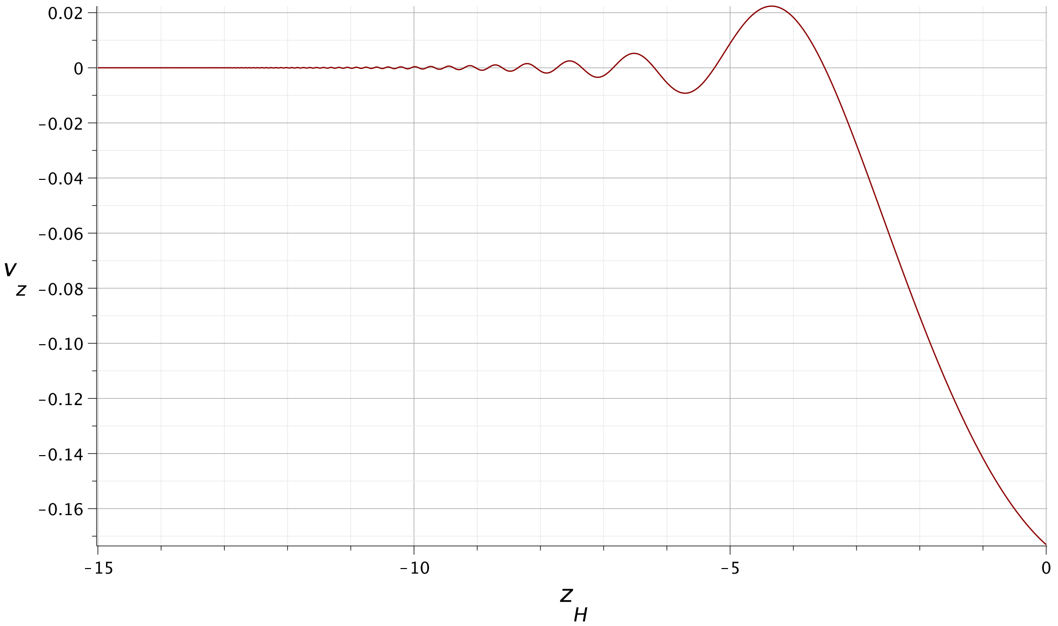


Figure 6.4: v_z weak field slow mode

Here both v_x and v_z exhibit sinusoidal behaviour. We see both oscillating quickly for small z_H and this is in contrast to the high amplitude oscillations as z_H approaches 0.

6.4 Strong field fast mode

For the strong field, the fast mode is primarily driven by magnetic pressure and tension [8], hence $v_x \approx v_A$. Due to Alfvén waves being transverse, we assume that wave propagation in the z direction is negligible, hence $v_z = 0$. Therefore by assuming that $\beta \lll 1$, thus $v_A \ggg c_s$, we find:

$$\frac{d^2 v_x}{dz^2} + \left(\frac{\omega^2}{v_A^2} - k_x^2\right) v_x = 0 \tag{6.49}$$

We non-dimensionalise (6.49) in Appendix D.9. It follows that:

$$\frac{d^2 v_x}{dz_H^2} + \left(\frac{\gamma \nu^2 \beta}{2} - \lambda^2\right) v_x = 0 \tag{6.50}$$

Recall that plasma-β is an exponential function in z_H, hence the solution for v_x involves Bessel functions:

$$v_x = c_1 J_{-2\lambda}\left(\nu\sqrt{2\gamma\beta}\right) + c_2 Y_{-2\lambda}\left(\nu\sqrt{2\gamma\beta}\right) \tag{6.51}$$

We plot the solution in Figure 6.5.

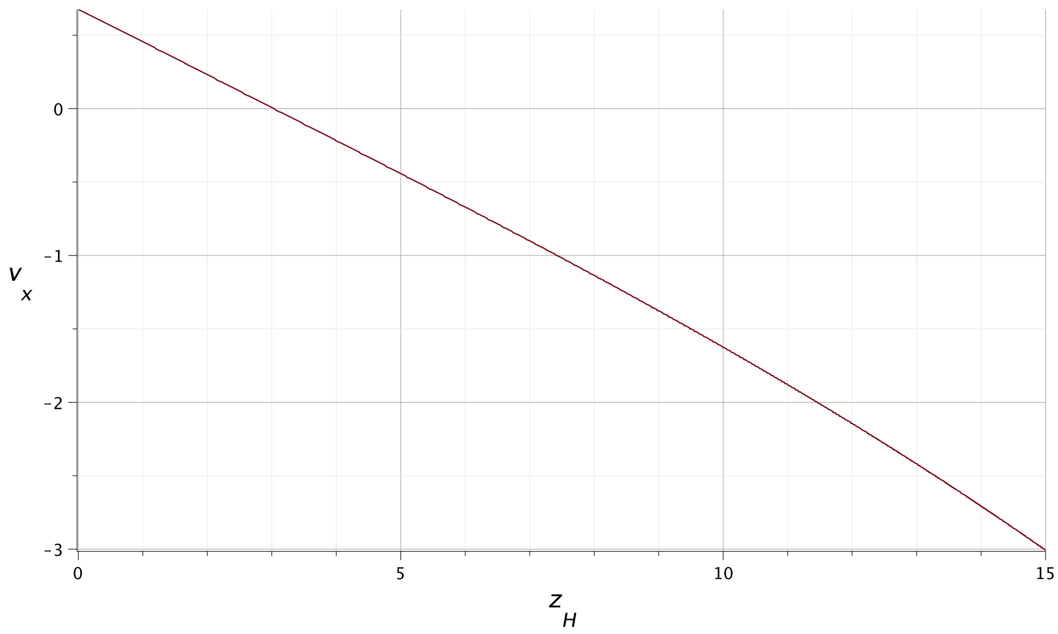


Figure 6.5: v_x strong field fast mode

Here we can see that as z_H increases v_x decreases in a uniform manner. When the same assumptions are made and applied to (6.22) (the other coupled equation), we see that the solution obtained for v_x (See Appendix D.8) is negligible when compared with (6.51), therefore we omit the analysis.

6.5 Strong field slow mode

For the strong field, the slow mode is largely acoustic in nature [8], hence $v_z \approx c_s$. Due to acoustic waves being longitudinal and so travelling along field lines we assume that $v_x = 0$. Therefore by assuming that $\beta \lll 1$, thus $v_A \ggg c_s$, we find:

$$\frac{d^2 v_z}{dz^2} - \frac{1}{H}\frac{dv_z}{dz} + \frac{\omega^2}{c_s^2} v_z = 0 \tag{6.52}$$

We non-dimensionalise (6.52) in Appendix D.9. It follows that:

$$\frac{d^2 v_z}{dz_H^2} - \frac{dv_z}{dz_H} + \nu^2 v_z = 0 \tag{6.53}$$

This is a linear second order differential equation. We solve for v_z to find:

$$v_z = c_1 \exp\left(\frac{1+\sqrt{1-4\nu^2}}{2} z_H\right) + c_2 \exp\left(\frac{1-\sqrt{1-4\nu^2}}{2} z_H\right) \tag{6.54}$$

We plot this solution in Figure 6.6.

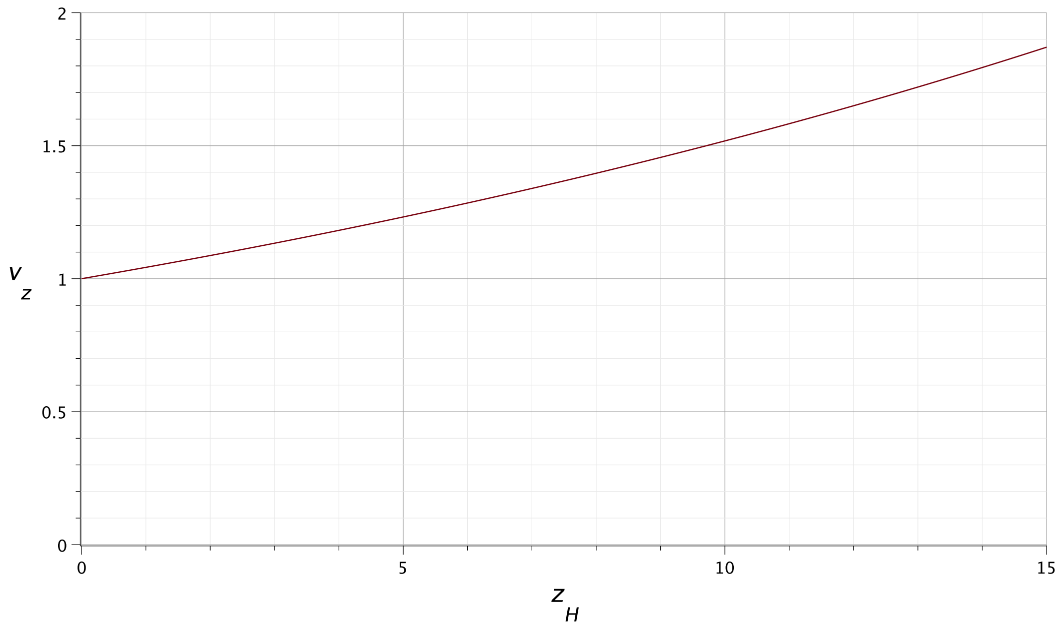


Figure 6.6: v_z strong field slow mode

Similar to Figure 6.5 we see v_z increase as z_H increases. Again, when the same assumptions are applied to (6.21), the solution produced (See Appendix D.8) is negligible when compared with (6.54), therefore we omit the analysis from this section.

6.6 Energy densities and mode conversions

Here we plot $\frac{1}{2}\rho_0 v_x^2$ and $\frac{1}{2}\rho_0 v_z^2$ for both the strong and weak fields, fast and slow modes. This gives us the energy densities for v_x and v_z.

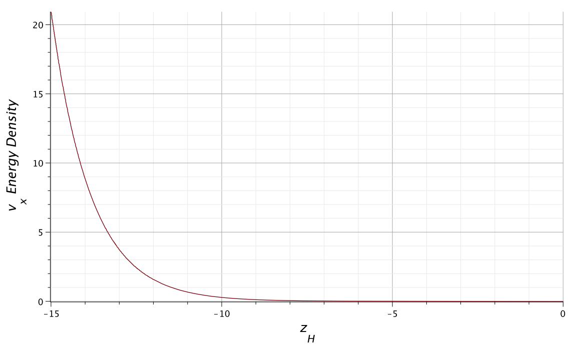


Figure 6.7: $\frac{1}{2}\rho_0 v_x^2$ weak field fast mode

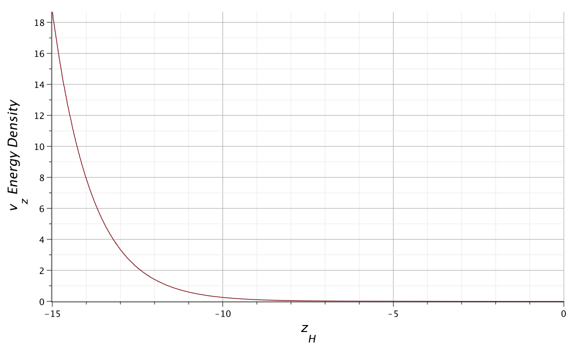


Figure 6.8: $\frac{1}{2}\rho_0 v_z^2$ weak field fast mode

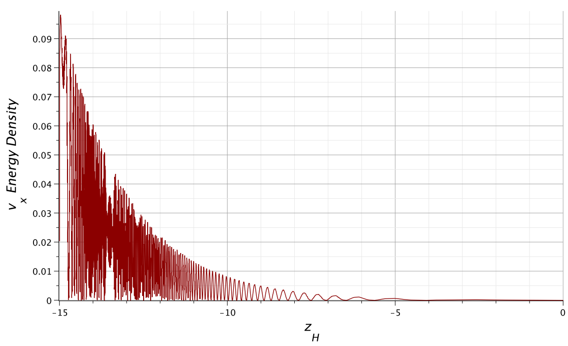


Figure 6.9: $\frac{1}{2}\rho_0 v_x^2$ weak field slow mode

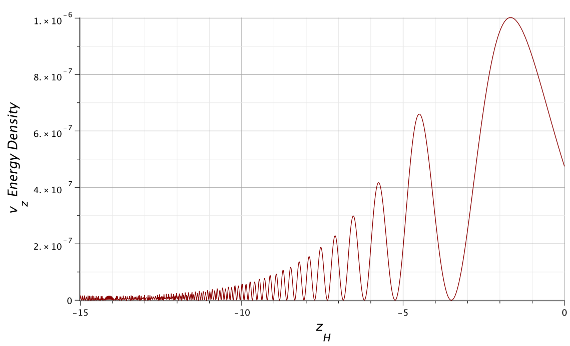


Figure 6.10: $\frac{1}{2}\rho_0 v_z^2$ weak field slow mode

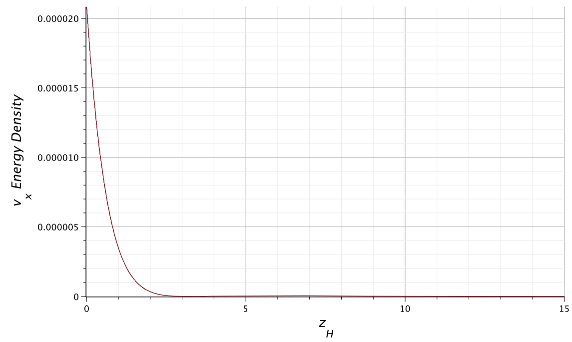


Figure 6.11: $\frac{1}{2}\rho_0 v_x^2$ strong field fast mode

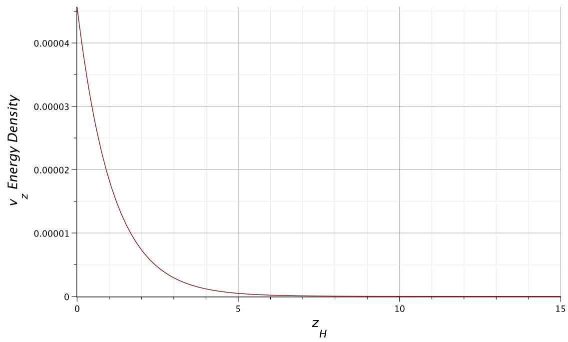


Figure 6.12: $\frac{1}{2}\rho_0 v_z^2$ strong field slow mode

Finally we plot the weak and strong solutions together. We solve for the constants to ensure the solutions meet at $z_H = 0$.

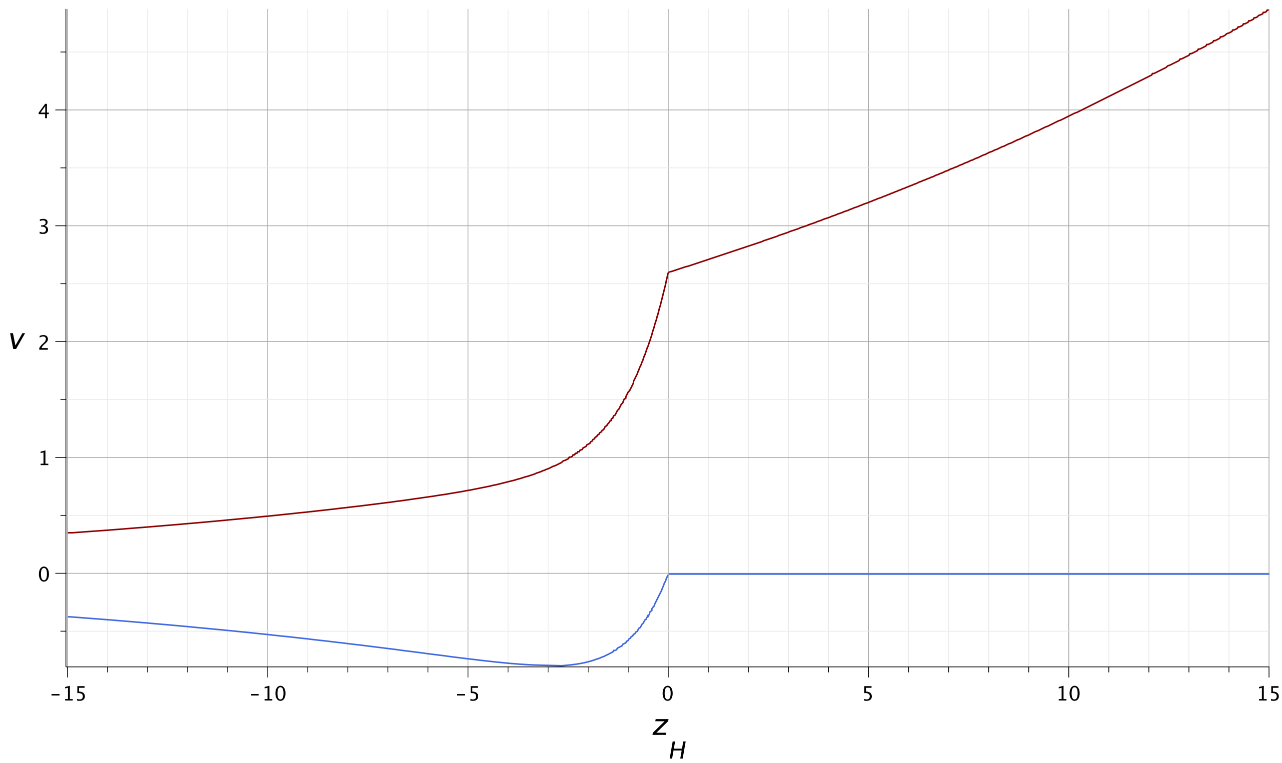


Figure 6.13: v_x (blue) and v_z (red) as they transition from fast to slow

Here the mode conversion, from fast to slow, is most evident when looking at v_z. We also see the v_x weak solution minimalised by the strong solution.

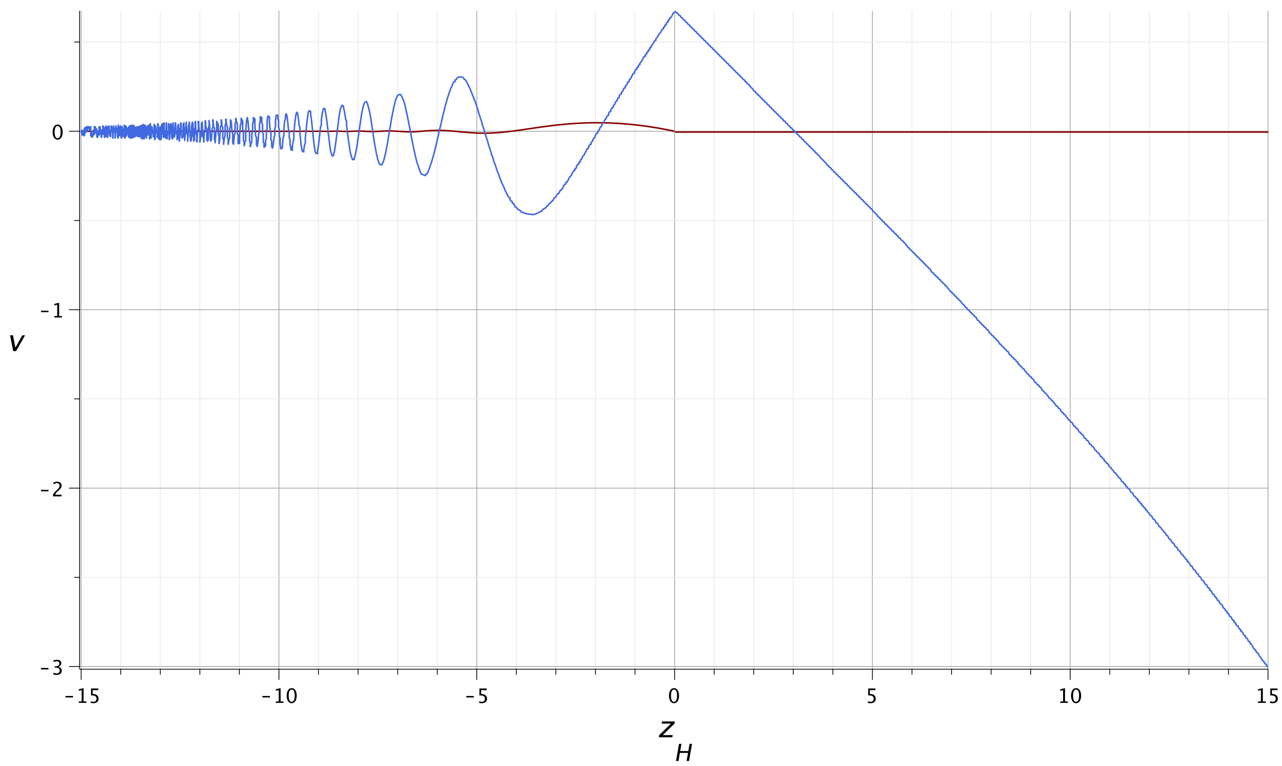


Figure 6.14: v_x (blue) and v_z (red) as they transition from slow to fast

Here we see the mode conversion, from slow to fast, from a sinusoidal oscillation in the solar interior to a linear relationship.

For more about mode conversions please see "Magneto-acoustic-gravity waves on the Sun. I - Exact solution for an oblique magnetic field" by Zhugzhda and Dzhalilov [6]

7 Conclusion

We have successfully modelled both isothermal and non-isothermal quiet sun scenarios. We used analytic and numeric methods to accurately uncover wave propagation behaviours in the photosphere and the chromosphere. We have observed the dampening of acoustic waves in the stratified medium for several different sound speeds. Additionally, we perceived the contrast in the dispersive nature of acoustic waves in isothermal and non-isothermal environments. This enables a much deeper understanding of the quiet sun structure.

We have also investigated magneto-acoustic waves and have attempted to model these waves as they undergo mode conversions. This has yielded interesting results as we have seen wave speeds oscillate as they travel through the stratified magnetic plasma.

Additional research could extend this magnetic model to a non-stratified medium or a non-isothermal scenario which may help further our understanding of magnetohydrodynamic waves in solar plasmas.

A Appendix for derivation of equation (3.34)

A.1 Linearisation of the equation of continuity, motion and energy

Recall that:

$$\frac{\partial p_0}{\partial z} = \rho_0(-g) \tag{A.1}$$

Start by linearising the equation of continuity (3.10):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho v) = 0 \tag{A.2}$$

$$\frac{\partial(\rho_0 + \rho_1)}{\partial t} + \frac{\partial}{\partial z}((\rho_0 + \rho_1)v_1) = 0 \tag{A.3}$$

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial z}(\rho_0 v_1) + \frac{\partial}{\partial z}(\rho_1 v_1) = 0 \tag{A.4}$$

Note that:

(i) ρ_0 has no time dependence so $\frac{\partial \rho_0}{\partial t} = 0$.

(ii) $\frac{\partial}{\partial z}(\rho_1 v_1)$ is small so we ignore it.

Thus:

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \frac{\partial v_1}{\partial z} - v_1 \frac{\partial \rho_0}{\partial z} \tag{A.5}$$

Next we linearise the equation of motion (3.11):

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} + \rho(-g) \tag{A.6}$$

$$(\rho_0 + \rho_1)\frac{\partial v_1}{\partial t} = -\frac{\partial(p_0 + p_1)}{\partial z} + (\rho_0 + \rho_1)(-g) \tag{A.7}$$

$$\rho_0 \frac{\partial v_1}{\partial t} + \rho_1 \frac{\partial v_1}{\partial t} = -\frac{\partial p_0}{\partial z} - \frac{\partial p_1}{\partial z} - \rho_0 g - \rho_1 g \tag{A.8}$$

Using (A.1) this simplifies to:

$$\rho_0 \frac{\partial v_1}{\partial t} + \rho_1 \frac{\partial v_1}{\partial t} = -\frac{\partial p_1}{\partial z} - \rho_1 g \tag{A.9}$$

Note that $\rho_1 \frac{\partial v_1}{\partial t}$ is small therefore we disregard it. Hence:

$$\rho_0 \frac{\partial v_1}{\partial t} = -\frac{\partial p_1}{\partial z} - \rho_1 g \tag{A.10}$$

Finally we linearise the energy equation (3.12):

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial z} = -\gamma p \frac{\partial v}{\partial z} \tag{A.11}$$

$$\frac{\partial(p_0 + p_1)}{\partial t} + v_1 \frac{\partial(p_0 + p_1)}{\partial z} = -\gamma(p_0 + p_1)\frac{\partial v_1}{\partial z} \tag{A.12}$$

$$\frac{\partial p_0}{\partial t} + \frac{\partial p_1}{\partial t} + v_1 \frac{\partial p_0}{\partial z} + v_1 \frac{\partial p_1}{\partial z} = -\gamma p_0 \frac{\partial v_1}{\partial z} - \gamma p_1 \frac{\partial v_1}{\partial z} \tag{A.13}$$

$$\tag{A.14}$$

Note that:

(i) p_0 has no time dependence so $\frac{\partial p_0}{\partial t} = 0$.

(ii) $v_1 \frac{\partial p_1}{\partial t}$ is small so we ignore it.

(iii) $p_1 \frac{\partial v_1}{\partial t}$ is small so we ignore it.

Hence:

$$\frac{\partial p_1}{\partial t} = -v_1 \frac{\partial p_0}{\partial z} - \gamma p_0 \frac{\partial v_1}{\partial z} \tag{A.15}$$

Using (A.1) we get:

$$\frac{\partial p_1}{\partial t} = -v_1(-\rho_0 g) - \gamma p_0 \frac{\partial v_1}{\partial z} \tag{A.16}$$

$$= \rho_0 v_1 g - \gamma p_0 \frac{\partial v_1}{\partial z} \tag{A.17}$$

B Appendix for derivation of analytical solution to equation (3.34)

B.1 Proof that $m_+ > 0$

Assume $m_+ < 0$. Hence:

$$-\frac{1}{2H} + \frac{1}{c_s}\sqrt{\frac{c_s^2}{4H^2} + s^2} < 0 \tag{B.1}$$

$$\frac{1}{c_s}\sqrt{\frac{c_s^2}{4H^2} + s^2} < \frac{1}{2H} \tag{B.2}$$

$$\sqrt{\frac{c_s^2}{4H^2} + s^2} < \frac{c_s}{2H} \tag{B.3}$$

$$\frac{c_s^2}{4H^2} + s^2 < \frac{c_s^2}{4H^2} \tag{B.4}$$

$$s^2 < 0. \tag{B.5}$$

Therefore s is imaginary. This is a contradiction, so the assumption that $m_+ < 0$ is incorrect. So $m_+ \geq 0$. □

B.2 Integral manipulation

We begin by noting that:

$$I = \int_0^\infty \frac{zJ_0(\lambda z)}{\sqrt{z^2 + a^2}} e^{ik\sqrt{z^2+a^2}} dz = \frac{e^{-a\sqrt{\lambda^2 - k^2}}}{\sqrt{\lambda^2 - k^2}} \tag{B.6}$$

for $\lambda^2 > k^2$ and $a > 0$ [9]. We now introduce a substitution. Let:

$$t = \sqrt{z^2 + a^2} \text{ and } s = -ik \tag{B.7}$$

We begin by changing the limits of the integration. When:

$$z = 0 \Rightarrow t = a \tag{B.8}$$

$$z = \infty \Rightarrow t = \infty \tag{B.9}$$

Next we find dz in terms of dt:

$$\frac{dz}{dt} = \frac{t}{\sqrt{t^2 - a^2}} \tag{B.10}$$

$$dz = \frac{t}{\sqrt{t^2 - a^2}} dt \tag{B.11}$$

Lastly we transform the integrand as follows:

$$\frac{zJ_0(\lambda z)}{\sqrt{z^2 + a^2}} e^{ik\sqrt{z^2 - a^2}} = \frac{\sqrt{t^2 - a^2} J_0\left(\lambda\sqrt{t^2 - a^2}\right)}{t} e^{-st} \tag{B.12}$$

Combining the previous results yields:

$$I = \int_a^\infty \frac{\sqrt{t^2 - a^2} J_0\left(\lambda\sqrt{t^2 - a^2}\right)}{t} e^{-st} \frac{t}{\sqrt{t^2 - a^2}} dt \tag{B.13}$$

Hence:

$$I = \int_a^\infty J_0\left(\lambda\sqrt{t^2 - a^2}\right) e^{-st} dt = \frac{e^{-a\sqrt{\lambda^2 + s^2}}}{\sqrt{\lambda^2 + s^2}} \tag{B.14}$$

We now differentiate with respect to a. This requires us to know the following rule about differentiation under an integral sign [10]:

$$\frac{d}{dz}\left(\int_{a(z)}^{b(z)} f(z,t)dt\right) = f(z, b(z))\frac{db}{dz} - f(z, a(z))\frac{da}{dz} + \int_{a(z)}^{b(z)} \frac{df}{dz}(z,t)dt \tag{B.15}$$

Thus when we differentiate (B.14) with respect to a we obtain:

$$\frac{dI}{da} = \frac{d}{da}\int_a^\infty J_0\left(\lambda\sqrt{t^2-a^2}\right)e^{-st}dt \tag{B.16}$$

$$= 0 - J_0(0)e^{-sa} + \int_a^\infty \frac{d}{da}J_0\left(\lambda\sqrt{t^2-a^2}\right)e^{-st}dt \tag{B.17}$$

Using chain rule and the Bessel manipulations $J_0'(z) = -J_1(z)$ and $J_0(0) = 1$ gives:

$$\frac{dI}{da} = -e^{-sa} + \int_a^\infty \frac{\lambda a J_1\left(\lambda\sqrt{t^2-a^2}\right)}{\sqrt{t^2-a^2}}e^{-st}dt = -e^{-a\sqrt{\lambda^2+s^2}} \tag{B.18}$$

Now let $a = \frac{z}{c_s}$, $\lambda = \alpha$. When rearranged it is found that:

$$\exp\left(-\frac{sz}{c_s}\right) - \frac{\alpha z}{c_s}\int_{\frac{z}{c_s}}^\infty \frac{J_1\left(\alpha\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{t^2-\left(\frac{z}{c_s}\right)^2}}e^{-st}dt = \exp\left(-\frac{z}{c_s}\sqrt{\alpha^2+s^2}\right) \tag{B.19}$$

This is the desired substitution.

B.3 Definition of a step function

A step function $\mathcal{H}\left(t-\frac{z}{c_s}\right)$ is defined as follows:

$$\mathcal{H}\left(t-\frac{z}{c_s}\right) = \begin{cases} 0 & \text{if } t \leq \frac{z}{c_s} \\ 1 & \text{if } t > \frac{z}{c_s} \end{cases} \tag{B.20}$$

B.4 Second Shifting theorem and Convolution theorem

Theorem B.1 (Second Shifting theorem)**.** *Let $F(s)$ be the Laplace transform of $f(t)$ and let $\mathcal{H}(t)$ be a step function. Then:*

$$\mathcal{L}\left(f(t-a)\mathcal{H}(t-a)\right) = e^{-as}F(s) \tag{B.21}$$

Hence:

$$\mathcal{L}^{-1}\left(e^{-as}F(s)\right) = f(t-a)\mathcal{H}(t-a) \tag{B.22}$$

Theorem B.2 (Convolution theorem)**.** *Let $F(s)$ be the Laplace transform of $f(t)$ (likewise for $G(s)$ and $g(t)$) and let $\mathcal{H}(t)$ be a step function. Then:*

$$F(s)G(s) = \mathcal{L}\left(\int_0^t f(t-\tau)g(\tau)d\tau\right) \tag{B.23}$$

Hence:

$$\mathcal{L}^{-1}\left(F(s)G(s)\right) = \int_0^t f(t-\tau)g(\tau)d\tau \tag{B.24}$$

B.5 Manipulation of integrals

We begin by splitting an integral into two parts then introduce a dummy function, $\phi(z,\tau)$ and a step function $\mathcal{H}\left(\tau-\frac{z}{c_s}\right)$. We can split the following integral like so:

$$\int_0^\infty \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau = \int_0^t \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau + \int_t^\infty \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau \tag{B.25}$$

Hence:

$$\int_0^t \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau = \int_0^\infty \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau - \int_t^\infty \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau \tag{B.26}$$

Now let's split this $\int_0^\infty$ integral on the right hand side of (B.26):

$$\int_0^\infty \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau = \int_0^{\frac{z}{c_s}} \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau + \int_{\frac{z}{c_s}}^\infty \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau \tag{B.27}$$

Let's substitute (B.27) into (B.26) to get:

$$\int_0^t \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau = \left(\int_0^{\frac{z}{c_s}} \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau + \int_{\frac{z}{c_s}}^\infty \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau\right) - \int_t^\infty \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau \tag{B.28}$$

Consider the $\int_0^{\frac{z}{c_s}}$ integral. We know from Definition B.3 that:

$$\mathcal{H}\left(\tau-\frac{z}{c_s}\right) = 0 \text{ for all } \tau < \frac{z}{c_s} \tag{B.29}$$

Therefore:

$$\int_0^{\frac{z}{c_s}} \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau = 0 \tag{B.30}$$

Hence:

$$\int_0^{\frac{z}{c_s}} \frac{e^{i\omega\tau} J_1\left(\frac{c_s}{2H}\sqrt{\tau^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{\tau^2-\left(\frac{z}{c_s}\right)^2}}\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau = 0 \tag{B.31}$$

This leaves us with:

$$\int_0^t \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau = \int_{\frac{z}{c_s}}^\infty \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau - \int_t^\infty \phi(z,\tau)\mathcal{H}\left(\tau-\frac{z}{c_s}\right)d\tau \tag{B.32}$$

B.6 Using Simpson's rule

Before we begin using Definition 3.1 we change some notation for the sake of clarity. Let:

$$f(\tau,z) = \frac{\cos(\omega\tau)\, J_1\left(\frac{c_s}{2H}\sqrt{\tau^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{\tau^2-\left(\frac{z}{c_s}\right)^2}} \tag{B.33}$$

and let i be an integer then define τ_i as follows:

$$\tau_i = \frac{z}{c_s} - \frac{\left(\frac{z}{c_s}-t\right)i}{n} \tag{B.34}$$

Hence (3.70) becomes:

$$\int_{\frac{z}{c_s}}^t f(\tau,z)\,d\tau \tag{B.35}$$

Using Definition 3.1 equates to finding $f(\tau_i,z)$ for $i=0\ldots n$:

$$f(\tau_i,z) = \frac{\cos(\omega\tau_i)\, J_1\left(\frac{c_s}{2H}\sqrt{\tau_i^2-\left(\frac{z}{c_s}\right)^2}\right)}{\sqrt{\tau_i^2-\left(\frac{z}{c_s}\right)^2}} \tag{B.36}$$

It is a case of computation to calculate $f(\tau_i,z)$ for $i=1\ldots n$. We choose a sufficiently large n by calculating $f(\tau_i,z)$ for many pairs of specific τ_i's and z's and several large n values and calculate the difference. Once this difference becomes arbitrarily small (several decimal places), the largest n value is chosen and used for the simulation.

More care must be taken when $i = 0$. We need to evaluate $f(\tau, z)$ as $\tau \to \frac{z}{c_s}$, or equivalently as $\tau^2 - \left(\frac{z}{c_s}\right)^2 \to 0$. Let:

$$\psi^2 = \tau^2 - \left(\frac{z}{c_s}\right)^2 \tag{B.37}$$

Therefore $\tau^2 - \left(\frac{z}{c_s}\right)^2 \to 0$ is equivalent to $\psi \to 0$. We substitute (B.37) into (B.33) to find:

$$f(\psi, z) = \frac{\cos\left(\omega\sqrt{\psi^2 + \frac{z^2}{c_s^2}}\right) J_1\left(\frac{c_s}{2H}\psi\right)}{\psi} \tag{B.38}$$

We note that when α is fixed and $x \to 0$ [11]:

$$J_\alpha(x) \sim \frac{1}{\Gamma(\alpha+1)}\left(\frac{x}{2}\right)^\alpha \tag{B.39}$$

Therefore when ψ is small we find:

$$f(\psi, z) \sim \frac{\cos\left(\omega\sqrt{\psi^2 + \frac{z^2}{c_s^2}}\right) \frac{c_s\psi}{4H}}{\psi} \tag{B.40}$$

$$= \frac{c_s}{4H}\cos\left(\omega\sqrt{\psi^2 + \frac{z^2}{c_s^2}}\right) \tag{B.41}$$

Hence:

$$\lim_{\tau \to \frac{z}{c_s}} f(\tau, z) = \lim_{\psi \to 0} f(\psi, z) \tag{B.42}$$

$$= \lim_{\psi \to 0}\left(\frac{c_s}{4H}\cos\left(\omega\sqrt{\psi^2 + \frac{z^2}{c_s^2}}\right)\right) \tag{B.43}$$

$$= \frac{c_s}{4H}\cos\left(\frac{\omega z}{c_s}\right) \tag{B.44}$$

We have now successfully calculated all of the necessary components required to use Theorem 3.1.

Lemma B.3 (Derivative of a Bessel function [12])**.** *Let $J_\nu(z)$ be a Bessel function of the first kind or the second kind. Then:*

$$\frac{d}{dz}\left(z^\nu J_\nu(z)\right) = z^\nu J_{\nu-1}(z) \tag{B.45}$$

Lemma B.4 (Derivative of a Bessel function [13])**.** *Let $J_\nu(z)$ be a Bessel function of the first kind or the second kind. Then:*

$$J'_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z) \tag{B.46}$$

Lemma B.5 (Bessel function property [14])**.** *Let $J_\nu(z)$ be a Bessel function of the first kind and ν be a non-negative integer. Then:*

$$J_{-\nu}(z) = (-1)^\nu J_\nu(z) \tag{B.47}$$

C Appendix for numeric methods

C.1 Derivation of Finite Difference Method central difference approximation

Definition C.1 (Forward difference approximation)**.** The Taylor Series expansion for a continuous function $f(z)$ at the point $z = z_0 + h$ is:

$$f(z_0 + h) = f(z_0) + hf'(z_0) + h^2\frac{f''(z_0)}{2!} + \ldots \tag{C.1}$$

Thus, for small h:

$$f'(z_0) \approx \frac{f(z_0+h)-f(z_0)}{h} \tag{C.2}$$

Let $z_{i+1} = z_i + h$. Then generalising gives:

$$f'(z_i) \approx \frac{f(z_{i+1})-f(z_i)}{h} \tag{C.3}$$

Equation (C.3) is called the forward difference approximation.

Definition C.2 (Backwards difference approximation)**.** We use the same machinery as in Definition C.1. Construct the Taylor Series expansion for $f(z)$ at the point $z = z_0 - h$:

$$f(z_0-h) = f(z_0) - hf'(z_0) + h^2\frac{f''(z_0)}{2!} - \dots \tag{C.4}$$

We rearrange to find that:

$$f'(z_0) = \frac{f(z_0)-f(z_0-h)}{h} + h\frac{f''(z_0)}{2!} - \dots \tag{C.5}$$

Thus:

$$f'(z_0) \approx \frac{f(z_0)-f(z_0-h)}{h} \tag{C.6}$$

Let $z_{i-1} = z_i - h$. Then generalising gives:

$$f'(z_i) \approx \frac{f(z_i)-f(z_{i-1})}{h} \tag{C.7}$$

Equation (C.7) is called the backwards difference approximation.

Definition C.3 (Central difference approximation)**.** We use the same machinery as in Definition C.1 and Definition C.2. Construct multiple Taylor Series expansion for $f(z)$ at the points $z = z_0 + h$ and $z = z_0 - h$:

$$f(z_0+h) = f(z_0) + hf'(z_0) + h^2\frac{f''(z_0)}{2!} + h^3\frac{f'''(z_0)}{3!} + \dots \tag{C.8}$$

$$f(z_0-h) = f(z_0) - hf'(z_0) + h^2\frac{f''(z_0)}{2!} - h^3\frac{f'''(z_0)}{3!} + \dots \tag{C.9}$$

We now subtract (C.9) from (C.8) to get:

$$f(z_0+h) - f(z_0-h) = 2hf'(z_0) + 2h^3\frac{f'''(z_0)}{3!} + \dots \tag{C.10}$$

Thus:

$$f'(z_0) = \frac{f(z_0+h)-f(z_0-h)}{2h} - h^2\frac{f'''(z_0)}{3!} - \dots \tag{C.11}$$

So:

$$f'(z_0) \approx \frac{f(z_0+h)-f(z_0-h)}{2h} \tag{C.12}$$

Let $z_{i+1} = z_i + h$ and $z_{i-1} = z_i - h$. Then generalising gives:

$$f'(z_i) \approx \frac{f(z_{i+1})-f(z_{i-1})}{2h} \tag{C.13}$$

Equation (C.13) is called the *central difference approximation.*

D Derivations involving magnetic fields

D.1 Manipulating (6.3) [5]

Begin with:

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla} p - \boldsymbol{J}\times\boldsymbol{B} - \rho\boldsymbol{g} = 0 \tag{D.1}$$

We note that:

$$\boldsymbol{J} = \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B}) \tag{D.2}$$

Substituting this into (D.1) gives:

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla} p - \frac{1}{\mu_0}((\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}) - \rho\boldsymbol{g} = 0 \tag{D.3}$$

D.2 Manipulating (6.4) [5]

Begin with:

$$\frac{\partial p}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} p + \gamma p(\boldsymbol{\nabla} \cdot \boldsymbol{v}) = 0 \tag{D.4}$$

Recall (3.14):

$$c_s^2 = \frac{\gamma p}{\rho} \tag{D.5}$$

Hence:

$$c_s^2 \rho = \gamma p \tag{D.6}$$

Therefore:

$$\gamma p(\boldsymbol{\nabla} \cdot \boldsymbol{v}) = c_s^2 \rho(\boldsymbol{\nabla} \cdot \boldsymbol{v}) \tag{D.7}$$

Hence (D.4) becomes:

$$\frac{\partial p}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} p + c_s^2 \rho(\boldsymbol{\nabla} \cdot \boldsymbol{v}) = 0 \tag{D.8}$$

D.3 Linearising equations (6.2), modified (6.3), modified (6.4) and (6.5)

We linearise the following equations:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \tag{D.9}$$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla} p - \frac{1}{\mu_0}((\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}) - \rho \boldsymbol{g} = 0 \tag{D.10}$$

$$\frac{\partial p}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} p + c_s^2 \rho(\boldsymbol{\nabla} \cdot \boldsymbol{v}) = 0 \tag{D.11}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) = 0 \tag{D.12}$$

Linearising (D.9), (D.11), and (D.12) is straightforward and requires only two facts, that terms such as $\boldsymbol{v_0}$ and $\frac{\partial p_0}{\partial t}$ are equal to zero as the system is begins at rest, and that when two small values are multiplied together, such as $\rho_1 \boldsymbol{v_1}$, the result is too small and is ignored. The linearisation of (D.10) is requires more work and is shown below:

$$\rho_0 \frac{\partial \boldsymbol{v_1}}{\partial t} + \boldsymbol{\nabla} p_0 + \boldsymbol{\nabla} p_1 - \frac{1}{\mu_0}\Big[\big(\boldsymbol{\nabla} \times (\boldsymbol{B_0} + \boldsymbol{B_1})\big) \times (\boldsymbol{B_0} + \boldsymbol{B_1})\Big] - \rho_0 \boldsymbol{g} - \rho_1 \boldsymbol{g} = 0 \tag{D.13}$$

$$\rho_0 \frac{\partial \boldsymbol{v_1}}{\partial t} + \boldsymbol{\nabla} p_0 + \boldsymbol{\nabla} p_1 - \frac{1}{\mu_0}\left[(\boldsymbol{\nabla} \times \boldsymbol{B_0}) \times \boldsymbol{B_0} + (\boldsymbol{\nabla} \times \boldsymbol{B_0}) \times \boldsymbol{B_1} + (\boldsymbol{\nabla} \times \boldsymbol{B_1}) \times \boldsymbol{B_0}\right] - \rho_0 \boldsymbol{g} - \rho_1 \boldsymbol{g} = 0 \tag{D.14}$$

$$\rho_0 \frac{\partial \boldsymbol{v_1}}{\partial t} + \boldsymbol{\nabla} p_0 + \boldsymbol{\nabla} p_1 - \frac{1}{\mu_0}\left[(\boldsymbol{\nabla} \times \boldsymbol{B_1}) \times \boldsymbol{B_0}\right] - \rho_0 \boldsymbol{g} - \rho_1 \boldsymbol{g} = 0 \tag{D.15}$$

This simplifies because $\boldsymbol{\nabla} \times \boldsymbol{B_0} = 0$. Hence the system of linear equations are as follows:

$$\frac{\partial \rho_1}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{v_1}) = 0 \tag{D.16}$$

$$\rho_0 \frac{\partial \boldsymbol{v_1}}{\partial t} + \boldsymbol{\nabla} p_0 + \boldsymbol{\nabla} p_1 - \frac{1}{\mu_0}((\boldsymbol{\nabla} \times \boldsymbol{B_1}) \times \boldsymbol{B_0}) - \rho_0 \boldsymbol{g} - \rho_1 \boldsymbol{g} = 0 \tag{D.17}$$

$$\frac{\partial p_1}{\partial t} + \boldsymbol{v_1} \cdot \boldsymbol{\nabla} p_0 + c_s^2 \rho_0(\boldsymbol{\nabla} \cdot \boldsymbol{v_1}) = 0 \tag{D.18}$$

$$\frac{\partial \boldsymbol{B_1}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v_1} \times \boldsymbol{B_0}) = 0 \tag{D.19}$$

D.4 Simplifying (6.11)

We simplify the following:

$$\Big(\Big(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\boldsymbol{v_1} \times \boldsymbol{B_0}))\Big) \times \boldsymbol{B_0}\Big) \tag{D.20}$$

Begin with:

$$\boldsymbol{v_1} \times \boldsymbol{B_0} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix} \tag{D.21}$$

$$= \begin{pmatrix} v_y B_0 \\ -v_x B_0 \\ 0 \end{pmatrix} \tag{D.22}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{v_1} \times \boldsymbol{B_0}) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} v_y B_0 \\ -v_x B_0 \\ 0 \end{pmatrix} \tag{D.23}$$

$$= \begin{pmatrix} \frac{\partial}{\partial z}(v_x B_0) \\ \frac{\partial}{\partial z}(v_y B_0) \\ -\frac{\partial}{\partial x}(v_x B_0) - \frac{\partial}{\partial y}(v_y B_0) \end{pmatrix} \tag{D.24}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\boldsymbol{v_1} \times \boldsymbol{B_0})) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} \frac{\partial}{\partial z}(v_x B_0) \\ \frac{\partial}{\partial z}(v_y B_0) \\ -\frac{\partial}{\partial x}(v_x B_0) - \frac{\partial}{\partial y}(v_y B_0) \end{pmatrix} \tag{D.25}$$

$$= \begin{pmatrix} \frac{\partial}{\partial y}\left(-\frac{\partial}{\partial x}(v_x B_0) - \frac{\partial}{\partial y}(v_y B_0)\right) - \frac{\partial^2}{\partial^2 z}(v_y B_0) \\ -\frac{\partial}{\partial x}\left(-\frac{\partial}{\partial x}(v_x B_0) - \frac{\partial}{\partial y}(v_y B_0)\right) + \frac{\partial^2}{\partial^2 z}(v_z B_0) \\ \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}(v_y B_0)\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}(v_x B_0)\right) \end{pmatrix} \tag{D.26}$$

$$= \begin{pmatrix} -B_0 \frac{\partial^2 v_x}{\partial x \partial y} - B_0 \frac{\partial^2 v_y}{\partial y^2} - B_0 \frac{\partial^2 v_y}{\partial z^2} \\ B_0 \frac{\partial^2 v_x}{\partial x^2} + B_0 \frac{\partial^2 v_y}{\partial x \partial y} + B_0 \frac{\partial^2 v_x}{\partial z^2} \\ -B_0 \frac{\partial^2 v_x}{\partial y \partial z} + B_0 \frac{\partial^2 v_y}{\partial x \partial z} \end{pmatrix} \tag{D.27}$$

$$(\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\boldsymbol{v_1} \times \boldsymbol{B_0}))) \times \boldsymbol{B_0} = B_0^2 \begin{pmatrix} \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial x \partial y} + \frac{\partial^2 v_x}{\partial z^2} \\ \frac{\partial^2 v_x}{\partial x \partial y} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \\ 0 \end{pmatrix} \tag{D.28}$$

D.5 Simplifying (6.18)

Start with:

$$\rho_0 \frac{\partial^2 v_z}{\partial t^2} - \frac{\partial}{\partial z}\left(v_z \frac{\partial p_0}{\partial z} + c_s^2 \rho_0 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}\right)\right) - g\left(\rho_0 \frac{\partial v_x}{\partial x} + \frac{\partial}{\partial z}(\rho_0 v_z)\right) = 0 \tag{D.29}$$

We expand:

$$\rho_0 \frac{\partial^2 v_z}{\partial t^2} - \frac{\partial}{\partial z}\left(v_z \frac{\partial p_0}{\partial z}\right) - c_s^2 \frac{\partial}{\partial z}\left(\rho_0 \frac{\partial v_x}{\partial x}\right) - c_s^2 \frac{\partial}{\partial z}\left(\rho_0 \frac{\partial v_z}{\partial z}\right) - g\rho_0 \frac{\partial v_x}{\partial x} - g\frac{\partial}{\partial z}(\rho_0 v_z) = 0 \tag{D.30}$$

Next, expand the product rules:

$$\rho_0 \frac{\partial^2 v_z}{\partial t^2} - \left(\frac{\partial v_z}{\partial z}\frac{\partial p_0}{\partial z} + v_z \frac{\partial^2 p_0}{\partial z^2}\right) - c_s^2 \left(\frac{\partial \rho_0}{\partial z}\frac{\partial v_x}{\partial x} + \rho_0 \frac{\partial^2 v_x}{\partial x \partial z}\right)$$
$$- c_s^2 \left(\frac{\partial \rho_0}{\partial z}\frac{\partial v_z}{\partial z} + \rho_0 \frac{\partial^2 v_z}{\partial z^2}\right) - g\rho_0 \frac{\partial v_x}{\partial x} - g\left(\frac{\partial \rho_0}{\partial z} v_z + \rho_0 \frac{\partial v_z}{\partial z}\right) = 0 \tag{D.31}$$

Simplify using (3.13), (3.15) and their derivatives. Then eliminate ρ_0 to get:

$$\frac{\partial^2 v_z}{\partial t^2} + g\frac{\partial v_z}{\partial z} - \frac{\gamma g^2}{c_s^2} v_z + \gamma g \frac{\partial v_x}{\partial x} - c_s^2 \frac{\partial^2 v_x}{\partial x \partial z} + \gamma g \frac{\partial v_z}{\partial z} - c_s^2 \frac{\partial^2 v_z}{\partial z^2} - g\frac{\partial v_x}{\partial x} + \frac{\gamma g^2}{c_s^2} v_z - g\frac{\partial v_z}{\partial z} = 0 \tag{D.32}$$

Cancelling gives:

$$\frac{\partial^2 v_z}{\partial t^2} + \gamma g \frac{\partial v_x}{\partial x} - c_s^2 \frac{\partial^2 v_x}{\partial x \partial z} + \gamma g \frac{\partial v_z}{\partial z} - c_s^2 \frac{\partial^2 v_z}{\partial z^2} - g \frac{\partial v_x}{\partial x} = 0 \tag{D.33}$$

$$\left(\frac{\partial^2}{\partial t^2} + \gamma g \frac{\partial}{\partial z} - c_s^2 \frac{\partial^2}{\partial z^2} \right) v_z = \left(c_s^2 \frac{\partial^2}{\partial x \partial z} - \gamma g \frac{\partial}{\partial x} + g \frac{\partial}{\partial x} \right) v_x \tag{D.34}$$

$$\left(c_s^2 \frac{\partial^2}{\partial z^2} - \gamma g \frac{\partial}{\partial z} - \frac{\partial^2}{\partial t^2} \right) v_z = -\frac{\partial}{\partial x} \left(c_s^2 \frac{\partial}{\partial z} + (1-\gamma) g \right) v_x \tag{D.35}$$

D.6 Non-dimensionalising (6.37) and (6.45)

Here we non-dimensionalise (6.37) using the definitions found in (6.23). Recall that:

$$\left(c_s^2 + \frac{k_x^2 c_s^4}{(\omega^2 - k_x^2 c_s^2)} \right) \frac{d^2 v_z}{dz^2} - \frac{1}{H} \left(c_s^2 + \frac{k_x^2 c_s^4}{(\omega^2 - k_x^2 c_s^2)} \right) \frac{d v_z}{dz} + \left(\omega^2 - \frac{(1-\gamma) g^2 k_x^2}{(\omega^2 - k_x^2 c_s^2)} \right) v_z = 0 \tag{D.36}$$

Therefore:

$$\frac{1}{H^2} \left(c_s^2 + \frac{\frac{\lambda^2}{H^2} c_s^4}{\left(\frac{\nu^2 c_s^2}{H^2} - c_s^2 \frac{\lambda^2}{H^2} \right)} \right) \frac{d^2 v_z}{dz_H^2} - \frac{1}{H^2} \left(c_s^2 + \frac{\frac{\lambda^2}{H^2} c_s^4}{\left(\frac{\nu^2 c_s^2}{H^2} - c_s^2 \frac{\lambda^2}{H^2} \right)} \right) \frac{d v_z}{dz_H} + \left(\frac{\nu^2 c_s^2}{H^2} - \frac{(1-\gamma) \frac{c_s^4}{\gamma^2 H^2} \frac{\lambda^2}{H^2}}{\left(\frac{\nu^2 c_s^2}{H^2} - c_s^2 \frac{\lambda^2}{H^2} \right)} \right) v_z = 0 \tag{D.37}$$

$$\frac{1}{H^2} \left(c_s^2 + \frac{\lambda^2 c_s^2}{(\nu^2 - \lambda^2)} \right) \frac{d^2 v_z}{dz_H^2} - \frac{1}{H^2} \left(c_s^2 + \frac{\lambda^2 c_s^2}{(\nu^2 - \lambda^2)} \right) \frac{d v_z}{dz_H} + \left(\frac{\nu^2 c_s^2}{H^2} - \frac{(1-\gamma) \lambda^2 c_s^2}{\gamma^2 H^2 (\nu^2 - \lambda^2)} \right) v_z = 0 \tag{D.38}$$

$$\frac{c_s^2}{H^2} \left(1 + \frac{\lambda^2}{(\nu^2 - \lambda^2)} \right) \frac{d^2 v_z}{dz_H^2} - \frac{c_s^2}{H^2} \left(1 + \frac{\lambda^2}{(\nu^2 - \lambda^2)} \right) \frac{d v_z}{dz_H} + \frac{c_s^2}{H^2} \left(\nu^2 - \frac{(1-\gamma) \lambda^2}{\gamma^2 (\nu^2 - \lambda^2)} \right) v_z = 0 \tag{D.39}$$

Thus:

$$\frac{d^2 v_z}{dz_H^2} - \frac{d v_z}{dz_H} + \frac{\left(\nu^2 - \frac{(1-\gamma)\lambda^2}{\gamma^2 (\nu^2 - \lambda^2)} \right)}{\left(1 + \frac{\lambda^2}{(\nu^2 - \lambda^2)} \right)} v_z = 0 \tag{D.40}$$

$$\frac{d^2 v_z}{dz_H^2} - \frac{d v_z}{dz_H} + \frac{\left(\frac{\gamma^2 \nu^2 (\nu^2 - \lambda^2)}{\gamma^2 (\nu^2 - \lambda^2)} - \frac{(1-\gamma)\lambda^2}{\gamma^2 (\nu^2 - \lambda^2)} \right)}{\left(\frac{\nu^2 - \lambda^2}{\nu^2 - \lambda^2} + \frac{\lambda^2}{(\nu^2 - \lambda^2)} \right)} v_z = 0 \tag{D.41}$$

$$\frac{d^2 v_z}{dz_H^2} - \frac{d v_z}{dz_H} + \frac{\left(\gamma^2 \nu^2 (\nu^2 - \lambda^2) - (1-\gamma) \lambda^2 \right)}{\gamma^2 \nu^2} v_z = 0 \tag{D.42}$$

Therefore we have:

$$\frac{d^2 v_z}{dz_H^2} - \frac{d v_z}{dz_H} + \left(\nu^2 - \lambda^2 - \frac{(1-\gamma)\lambda^2}{\gamma^2 \nu^2} \right) v_z = 0 \tag{D.43}$$

We non-dimensionalise (6.45) next. Recall that:

$$v_A^2 \frac{d^2 v_x}{dz^2} + \omega^2 v_x = 0 \tag{D.44}$$

We non-dimensionalise here dividing through by c_s^2 and using plasma-β to find that:

$$\frac{2}{\gamma \beta} \frac{d^2 v_x}{dz_H^2} + \nu^2 v_x = 0 \tag{D.45}$$

D.7 Proving that (6.47) is a solution of (6.46)

Begin with the solution:

$$v_x = c_1 J_0\left(\nu\sqrt{2\gamma\beta}\right) + c_2 Y_0\left(\nu\sqrt{2\gamma\beta}\right) \tag{D.46}$$

We rewrite (D.46) as:

$$v_x = c_1 J_0\left(A\exp\left(-\frac{z}{2}\right)\right) + c_2 Y_0\left(A\exp\left(-\frac{z}{2}\right)\right) \tag{D.47}$$

Here:

$$A = 2\nu\sqrt{\frac{c_\rho \mu_0 c_s^2}{B_0^2}} \tag{D.48}$$

Differentiating (D.47) with respect to z requires chain rule and common knowledge of how Bessel functions differentiate. The results follows:

$$\frac{dv_x}{dz} = \frac{A\exp\left(-\frac{z}{2}\right)}{2}\left(c_1 J_1\left(A\exp\left(-\frac{z}{2}\right)\right) + c_2 Y_1\left(A\exp\left(-\frac{z}{2}\right)\right)\right) \tag{D.49}$$

We differentiate again with respect to z. This time we also require product rule. For clarity let:

$$u = \exp\left(-\frac{z}{2}\right) \tag{D.50}$$

This substitution transforms (D.49) into:

$$\frac{dv_x}{dz} = \frac{Au}{2}\left(c_1 J_1(Au) + c_2 Y_1(Au)\right) \tag{D.51}$$

$$= \frac{Ac_1}{2}\left(uJ_1(Au)\right) + \frac{Ac_2}{2}\left(uY_1(Au)\right) \tag{D.52}$$

Differentiating with respect to u and using another Bessel differentiation rule gives:

$$\frac{d}{du}\left(\frac{dv_x}{dz}\right) = \frac{Ac_1}{2}\left(J_1(Au) + u\left(AJ_0(Au) - \frac{1}{u}J_1(Au)\right)\right) + \frac{Ac_2}{2}\left(Y_1(Au) + u\left(AY_0(Au) - \frac{1}{u}Y_1(Au)\right)\right) \tag{D.53}$$

$$= \frac{Ac_1}{2}\left(AuJ_0(Au)\right) + \frac{Ac_2}{2}\left(AuY_0(Au)\right) \tag{D.54}$$

$$= \frac{A^2c_1u}{2}J_0(Au) + \frac{A^2c_2u}{2}Y_0(Au) \tag{D.55}$$

Multiplying by $\frac{du}{dz}$ gives:

$$\frac{d^2v_x}{dz^2} = -\frac{1}{2}\exp\left(-\frac{z}{2}\right)\left[\frac{A^2c_1\exp\left(-\frac{z}{2}\right)}{2}J_0\left(A\exp\left(-\frac{z}{2}\right)\right) + \frac{A^2c_2\exp\left(-\frac{z}{2}\right)}{2}Y_0\left(A\exp\left(-\frac{z}{2}\right)\right)\right] \tag{D.56}$$

$$= -\frac{A^2}{4}\exp(-z)\left[c_1 J_0\left(A\exp\left(-\frac{z}{2}\right)\right) + c_2 Y_0\left(A\exp\left(-\frac{z}{2}\right)\right)\right] \tag{D.57}$$

Multiplying by $\frac{2}{\gamma\beta}$ requires some algebra but yields:

$$\frac{2}{\gamma\beta}\frac{d^2v_x}{dz^2} = -\nu^2\left[c_1 J_0\left(A\exp\left(-\frac{z}{2}\right)\right) + c_2 Y_0\left(A\exp\left(-\frac{z}{2}\right)\right)\right] \tag{D.58}$$

Examining the second term of (6.46) reveals that:

$$\nu^2 v_x = \nu^2\left[c_1 J_0\left(A\exp\left(-\frac{z}{2}\right)\right) + c_2 Y_0\left(A\exp\left(-\frac{z}{2}\right)\right)\right] \tag{D.59}$$

Therefore:

$$\frac{2}{\gamma\beta}\frac{d^2v_x}{dz^2} + \nu^2 v_x = 0 \tag{D.60}$$

Thus (6.47) is a solution of (6.46).

D.8 Solving (6.22) for the strong field fast mode and (6.21) for the strong field slow mode

Begin by considering (6.22) under the assumptions made for the strong field fast mode. These assumptions in conjunction with the non-dimensionalisations given in (6.23) yield:

$$\frac{dv_x}{dz_H} + \frac{1-\gamma}{\gamma} v_x = 0 \tag{D.61}$$

Solving this gives the following solution:

$$v_x = A_1 \exp\left(\frac{\gamma - 1}{\gamma} z_H\right) \tag{D.62}$$

Next consider (6.21) under the assumptions made for the strong field slow mode. Again, these assumptions in conjunction with the non-dimensionalisations given in (6.23) yield:

$$\frac{dv_z}{dz_H} - \frac{1}{\gamma} v_z = 0 \tag{D.63}$$

Solving that gives the following solution:

$$v_z = A_2 \exp\left(\frac{z_H}{\gamma}\right) \tag{D.64}$$

Both of these solutions are positive exponentials so when we calculate the energy densities (by multiplying by ρ_0) we find that they are insignificant in the region we are considering to be the strong field. As such these solutions do not contradict the assumptions made when examining their respective modes. Additionally these solutions are omitted from the analysis due to their triviality, even though they are inconsistent with the primary solutions shown in Sections 6.4 and 6.5.

D.9 Non-dimensionalising (6.49) and (6.52)

Here we non-dimensionalise (6.49) using the definitions found in (6.23). Recall that:

$$\frac{d^2 v_x}{dz^2} + \left(\frac{\omega^2}{v_A^2} - k_x^2\right) v_x = 0 \tag{D.65}$$

Therefore:

$$\frac{1}{H^2}\frac{d^2 v_x}{dz_H^2} + \left(\frac{\frac{\nu^2 c_s^2}{H^2}}{v_A^2} - \frac{\lambda^2}{H^2}\right) v_x = 0 \tag{D.66}$$

$$\frac{d^2 v_x}{dz_H^2} + \left(\frac{\nu^2 c_s^2}{v_A^2} - \lambda^2\right) v_x = 0 \tag{D.67}$$

$$\frac{d^2 v_x}{dz_H^2} + \left(\frac{\gamma \nu^2 \beta}{2} - \lambda^2\right) v_x = 0 \tag{D.68}$$

We non-dimensionalise (6.52) next. Recall that:

$$\frac{d^2 v_z}{dz^2} - \frac{1}{H}\frac{dv_z}{dz} + \frac{\omega^2}{c_s^2} v_z = 0 \tag{D.69}$$

Multiplying through by c_s^2 and using plasma-β yields:

$$\frac{d^2 v_z}{dz_H^2} - \frac{dv_z}{dz_H} + \nu^2 v_z = 0 \tag{D.70}$$

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