Continuation Value Methods for Sequential Decisions: A General Theory

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Declaration

Except where otherwise acknowledged, I certify that this thesis is my original work. The thesis is within the 100, 000 word limit set by the Australian National University.

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I dedicate this thesis to my Mother Shuzhen Zhang, Father Shuhua Ma and Sister Qingwen Ma.

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Abstract

After the introductory chapter, this thesis comprises four main chapters before concluding in chapter 6. The thesis undertakes a systematic analysis of the continuation value based method for sequential decision problems originally due to Jovanovic (1982). Although recently this technique is widely employed in a variety of economic applications, its theoretical connections to the traditional value function based method, relative efficiency, and optimality/analytical properties have hitherto received no general investigation. The thesis fills this gap.

On the one hand, the thesis shows that the operator employed by this method (referred to below as the Jovanovic operator) is semiconjugate to the traditional Bellman operator and has essentially equivalent dynamic properties. In particular, under general assumptions, any fixed point of one of the operators is a direct translation of a fixed point of the other. Iterative sequences generated by the operators are also simple translations. After adding topological structure to the generic setting, the thesis shows that the Bellman and Jovanovic operators are both contraction mappings under identical assumptions, and that convergence to the respective fixed points occurs at the same rate.

To ensure sufficient generality for economic applications, the optimality and symmetry analysis has been embedded separately in (a) spaces of potentially unbounded functions endowed with generic weighted supremum norm distances, and (b) spaces of integrable functions with divergence measured by L_p norms. Unbounded rewards are allowed provided that they do not cause continuation values to diverge. Moreover, the theory mentioned above is established for important classes of sequential decision problems, including:

- standard optimal stopping problems (chapter 2),
- repeated optimal stopping problems (chapter 3), and
- dynamic discrete choice problems (chapter 4).

On the other hand, despite these similarities, the thesis shows that there do remain important differences between the continuation value based method and the traditional value function based method in terms of efficiency and analytical convenience.

One of these differences concerns the dimensionality of the effective state spaces associated with the Bellman and Jovanovic operators. First, aside from a class of problems for which the continuation dynamics are trivial, the effective state space of the continuation value function is never larger than that of the value function. Second, for a broad class of sequential problems, the effective state space of the continuation value function is strictly lower dimensional than that of the value function. Another key difference is that continuation value functions are typically smoother than value functions. The relative smoothness comes from taking expectations over stochastic transitions. In each scenario, it is highly advantageous to work with the continuation value method rather than the traditional value function method.

The thesis systematically characterizes these hidden advantages in terms of model primitives and provides a range of important applications (chapters 2 and 5). Moreover, by exploiting these advantages, the thesis develops a general theory for sequential decision problems based around continuation values and obtains a range of new results on optimality, optimal behavior and efficient computation (chapter 5).

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Chapter 1

Introduction

A large variety of decision making problems involve choosing when to act in the face of risk and uncertainty. Examples include deciding if or when to accept a job offer, exit or enter a market, default on a loan, bring a new product to market, exploit some new technology or business opportunity, or exercise a financial or real option (see, e.g., McCall (1970), Jovanovic (1982), Hopenhayn (1992), Dixit and Pindyck (1994), Ericson and Pakes (1995), Peskir and Shiryaev (2006), Arellano (2008), Perla and Tonetti (2014), Fajgelbaum et al. (2017), and Schaal (2017)).

Sequential decision problems regarding optimal timing of decisions can be solved using standard dynamic programming methods based around the Bellman equation. There is, however, an alternative approach—introduced by Jovanovic (1982) in the context of industry dynamics—that focuses on continuation values. The idea involves calculating the continuation value directly, using an operator referred to below as the Jovanovic operator. This technique is now well-known to economists and routinely employed in a variety of economic applications (see, e.g., Gomes et al. (2001), Ljungqvist and Sargent (2008), Lise (2013), Moscarini and Postel-Vinay (2013), Fajgelbaum et al. (2017), and Schaal (2017)).[1]

[1] To our best knowledge, Jovanovic (1982) is the first technically sophisticated economic research that exploits the continuation value structure. Closest early studies that we can track typically focus on exploiting reservation rules (e.g., reservation wage, reservation cost, etc.) rather than continuation values. Their approach covers a narrower range of problems since reservation rules do not exist in many important applications. Moreover, to construct reservation rule structures, extra monotonicity properties are usually required and the models have to be transformed differently in different applications. As to be shown, continuation values avoid these problems while they are closely related to both value functions and reservation rules (if exist). Exploiting the continuation value structure is an important generalization of earlier studies.

Motivation

Despite the existence of these two parallel and commonly used methods, their theoretical connections and relative efficiency have hitherto received no general investigation. One cost of this status quo is that studies using continuation value methods have been compelled to provide their own optimality analysis piecemeal in individual applications (see, e.g., Jovanovic (1982), Moscarini and Postel-Vinay (2013), or Fajgelbaum et al. (2017)), which fosters unnecessary replication and inhabits applied researchers seeking off-the-shelf results. A second cost is that the most effective choice of method vis-a-vis a given application is often unknown ex-ante, and revealed only by experimentation in particular settings.

What is this thesis about?

This thesis undertakes the first systematic analysis of the relationship between these two methods. Within several generic frameworks that cover a broad range of sequential decision problems, we show that the Bellman operator and Jovanovic operator have essentially equivalent dynamic properties in a sense to be made precise. Despite these similarities, we further show that there are important advantages associated with the continuation value based method, both in terms of the dimensionality of effective state spaces associated with the Bellman and Jovanovic operators and in terms of the relative smoothness of their respective fixed points. Finally, we exploit these advantages and develop a general theory for sequential decision problems based around continuation values. A range of new results on optimality, optimal behavior and efficient computation are established.

What is Chapter 2 about?

In chapter 2, we begin the analysis in a generic optimal stopping setting. As a first step, we show that the Bellman operator and the Jovanovic operator are semiconjugate, implying that any fixed point of one of the operators is a direct translation of a fixed point of the other, and that iterative sequences generated by the operators are also simple translations. We then add topological structure to the generic setting and show that, the Bellman and Jovanovic operators are both contraction mappings under identical assumptions, and that convergence to the respective fixed points occurs at the same rate.

To ensure sufficient generality for economic applications, we allow for unbounded reward functions provided that they do not cause continuation values to diverge. To that end, the optimality and symmetry analysis is conducted separately in

- a space of potentially unbounded functions endowed with the weighted supremum norm distance, and
- a space of integrable functions with divergence measured by L_p norm.

Although the results stated above elucidate the natural similarity between the Bellman and Jovanovic operators, there do however remain important differences in terms of efficiency and analytical convenience. One of these differences concerns the dimensionality of the effective state spaces associated with each operator. We show that

(1) aside from a class of problems for which the continuation dynamics are trivial in a sense to be made precise, the effective state space of the continuation value function is never higher dimensional than that of the value function, and

(2) for an important class of problems, referred to below as continuation decomposable problems, the effective state space of the continuation value function is strictly lower dimensional than that of the value function.

Lower dimensionality simplifies both theory and computation. To illustrate, we study the time complexity of iteration with the Jovanovic and Bellman operators and quantify the difference analytically. The efficiency gains of working with the Jovanovic operator are shown to be very large—typically orders of magnitude.

Our theoretical findings are augmented by numerical results. In a typical experiment involving job search, computation time falls from 4.4 days with value function iteration (via Bellman operator) to 24 minutes using continuation value iteration (via Jovanovic operator), in line with the predictions of the time complexity based analysis.

What is Chapter 3 about?

The theory of chapter 2 is developed within a standard optimal stopping framework, where the agent aims to find an optimal stopping time that terminates the sequential decision process permanently. However, in many problems of interest to economists, the choice to stop is only temporary. Typically, agents return to the sequential decision problem with positive probability after termination.

In chapter 3, we extend our theory to address this kind of problem, which we refer to as the repeated optimal stopping problem. As in chapter 2, we show that the Bellman and Jovanovic operators are semiconjugate in a generic setting, so that any fixed point of one of the operators is a direct translation of a fixed point of the other, and that any iterative sequence generated by one of the operators is also a simple translation of that generated by the other.

Topological structure is then added to the generic setting. Similar as in chapter 2, we consider both weighted supremum norm and L_p-norm topologies in order to treat potentially unbounded rewards. Based on the general theory established in the previous step, we show that the Bellman operator and Jovanovic operator are both contraction mappings under identical assumptions, and that convergence to the respective fixed points occurs at the same rate. All these theoretical results are established based on the same assumptions as those of chapter 2.

What is Chapter 4 about?

The theory of chapters 2–3 is developed for sequential decision problems with the key state component (i.e., the state variables that appear in the reward functions) evolving as an exogenous Markov process. Although such frameworks cover a wide range of binary choice sequential problems, there are other cases in which the key state component follows a controlled Markov process (i.e., evolutions of the key states are affected at least partially by some control variables). Such structures are common for sequential decision problems where agents are faced with more than two choices.

In chapter 4, we extend our theory to cover this class of problems, which we refer to as dynamic discrete choice problems. We show that the Bellman and Jovanovic operators are semiconjugate in general, with the same implications as those of chapters 2–3. The optimality and symmetry analysis is then embedded into a space of potentially unbounded functions endowed with a generic weighted supremum norm. Once again, we show that the Bellman and Jovanovic operators are both contraction mappings under identical assumptions, with the same rate of convergence to their respective fixed points. These properties are established by constructing a metric that evaluates the maximum of the weighted supremum norm distances along each dimension of the candidate function space.

What is Chapter 5 about?

Another important advantage associated with the continuation value method not discussed so far is that continuation values are typically smoother than value functions. The relative smoothness comes from taking expectations over stochastic transitions. Like lower dimensionality, increased smoothness helps on both the analytical and the computational side. On the computational side, smoother functions are easier to approximate. On the analytical side, greater smoothness lends itself to sharper results based on derivatives.

In chapter 5, we propose a general theory for sequential decision problems based around continuation values and related Jovanovic operators, heavily exploiting the advantages discussed so far. We obtain:

(1) conditions under which continuation values are: (a) continuous, (b) monotone, and (c) differentiable as functions of the economic environment;

(2) conditions under which parametric continuity holds (often required for proofs of existence of recursive equilibria in many-agent environments);

(3) conditions under which threshold policies are: (a) continuous, (b) monotone, and (c) differentiable.

In the latter case we derive an expression for the derivative of the threshold relative to other aspects of the economic environment and show how it contributes to economic intuition.

The closest counterparts to these results in the existing literature are those concerning individual applications. Our theory generalizes and extends these results in a unified framework. Some results, such as differentiability of threshold policies, are new to the literature to the best of our knowledge.

Chapter 2

Continuation Value Methods for Sequential Decisions: Convergence Properties and Efficiency

2.1 Introduction

In this chapter, we begin a systematic analysis of the relationship between the traditional value function based method and the continuation value based method in a generic optimal stopping setting. In particular, section 2.2 outlines the problem, longer proofs and the characterization of continuation nontriviality are deferred to the appendix, while the rest of the chapter is structurized as follows:

Section 2.3 explores the symmetric theoretical properties of the Bellman and Jovanovic operators in terms of fixed points and convergence. In particular, section 2.3.1 shows that the Bellman operator and the Jovanovic operator are semiconjugate. The implications are mentioned in the previous chapter. In sections 2.3.2–2.3.3, we add topological structure to the generic setting and show that the Bellman operator and Jovanovic operator are both contraction mappings under identical assumptions, and that convergence to the respective fixed points occurs at the same rate.

To ensure sufficient generality for economic applications, we embed our optimality and symmetry analysis separately in (a) a space of potentially unbounded functions endowed with a generic weighted supremum norm distance (section 2.3.2), and (b) a space of integrable functions with divergence measured by L_p norm (section 2.3.3). In particular, unbounded rewards are allowed provided that they do not cause continuation values to diverge. In the first case, we draw on and extend work on dynamic programming with unbounded rewards found in several important studies, including Boyd (1990), Rincón-Zapatero and Rodríguez-Palmero (2003), Martins-da Rocha and Vailakis (2010), Jaśkiewicz and Nowak (2011), Jaśkiewicz et al. (2014), Kamihigashi (2014) and Bäuerle and Jaśkiewicz (2018). The theory we develop in the second case is new to the literature, to the best of our knowledge.

Despite the essentially equivalent dynamic properties between the Bellman and Jovanovic operators established in section 2.3, section 2.4 reveals several important advantages associated with the continuation value based method. One is that, for continuation decomposable problems, the effective state space of the continuation value function is strictly lower dimensional than that of the value function. We characterize this important class of problems in terms of the structure of reward and state transition functions.

Lower dimensionality simplifies both theory and computation. As an illustration, section 2.4 studies the time complexity of iteration with the Jovanovic and Bellman operators and quantifies the difference analytically. These large efficiency gains—typically measured in orders of magnitude—arise because, in the presence of continuation decomposability, continuation value based methods mitigate the curse of dimensionality, one of the primary stumbling blocks for dynamic programming (Rust (1997)).

Section 2.5 provides a range of important applications that are continuation decomposable. In particular, numerical results from these applications are in line with the predictions of the time complexity based analysis of section 2.4.

2.2 Set Up

This section presents a generic optimal stopping problem and the key operators and optimality concepts. As a first step, we introduce some mathematical techniques and notation used in this chapter.

2.2.1 Preliminaries

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. For $a, b \in \mathbb{R}$, let $a \vee b := \max\{a, b\}$. If f and g are functions, then $(f \vee g)(x) := f(x) \vee g(x)$. Given a

Polish space Z and Borel sets $\mathscr{B}$, let $m\mathscr{B}$ be all $\mathscr{B}$-measurable functions from Z to $\mathbb{R}$. Given $\kappa\colon \mathsf{Z} \to (0, \infty)$, the *$\kappa$-weighted supremum norm* of $f\colon \mathsf{Z} \to \mathbb{R}$ is

$$\|f\|_\kappa := \sup_{z \in \mathsf{Z}} \frac{|f(z)|}{\kappa(z)}.$$

If $\|f\|_\kappa < \infty$, we say that f is *κ-bounded*. The symbol $b_\kappa \mathsf{Z}$ denotes all $\mathscr{B}$-measurable functions from Z to $\mathbb{R}$ that are κ-bounded.

Given a probability measure π on $(\mathsf{Z}, \mathscr{B})$ and a constant $p \geq 1$, let

$$\|f\|_p := \left(\int |f|^p \, \mathrm{d}\pi \right)^{1/p}.$$

Let $L_p(\pi)$ be all (equivalence classes of) functions $f \in m\mathscr{B}$ for which $\|f\|_p < \infty$.

Both $(b_\kappa \mathsf{Z}, \|\cdot\|_\kappa)$ and $(L_p(\pi), \|\cdot\|_p)$ form Banach spaces.

A *stochastic kernel* P on Z is a map $P\colon \mathsf{Z} \times \mathscr{B} \to [0, 1]$ such that $z \mapsto P(z, B)$ is $\mathscr{B}$-measurable for each $B \in \mathscr{B}$ and $B \mapsto P(z, B)$ is a probability measure for each $z \in \mathsf{Z}$. For all $t \in \mathbb{N}$, $P^t(z, B) := \int P(z', B) P^{t-1}(z, \mathrm{d}z')$ is the probability of a state transition from z to $B \in \mathscr{B}$ in t steps, where $P^1(z, B) := P(z, B)$. A Z-valued stochastic process $\{Z_t\}$ on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called *P-Markov* if

$$\mathbb{P}\{Z_{t+1} \in B \mid \mathscr{F}_t\} = \mathbb{P}\{Z_{t+1} \in B \mid Z_t\} = P(Z_t, B)$$

$\mathbb{P}$-almost surely for all $t \in \mathbb{N}_0$ and all $B \in \mathscr{B}$. Here $\{\mathscr{F}_t\}$ is the natural filtration induced by $\{Z_t\}$. In what follows, $\mathbb{P}_z$ evaluates probabilities conditional on $Z_0 = z$ and $\mathbb{E}_z$ is the corresponding expectations operator.

2.2.2 Optimal Stopping

Let $(\mathsf{Z}, \mathscr{B})$ be a measurable space. For the purposes of this chapter, an optimal stopping problem is a tuple (β, c, P, r) where

- $\beta \in (0, 1)$ is discount factor,
- $c \in m\mathscr{B}$ is a *flow continuation reward* function,
- P is a stochastic kernel on $(\mathsf{Z}, \mathscr{B})$, and
- $r \in m\mathscr{B}$ is a *terminal reward function*.

The interpretation is as follows: At time t an agent observes Z_t, the current realization of a Z-valued P-Markov process $\{Z_t\}_{t\geq 0}$, and chooses between stopping and continuing. Stopping generates terminal reward $r(Z_t)$ while continuing yields flow continuation reward $c(Z_t)$. If the agent continues, the time $t+1$ state Z_{t+1} is observed and the process repeats. Future rewards are discounted at rate β.

An $\mathbb{N}_0$-valued random variable τ is called a (finite) *stopping time* if $\mathbb{P}\{\tau < \infty\} = 1$ and $\{\tau \leq t\} \in \mathscr{F}_t$ for all $t \geq 0$. Let $\mathscr{M}$ denote all such stopping times. The *value function* v^* for (β, c, P, r) is defined at $z \in \mathsf{Z}$ by

$$v^*(z) := \sup_{\tau \in \mathscr{M}} \mathbb{E}_z \left\{ \sum_{t=0}^{\tau-1} \beta^t c(Z_t) + \beta^\tau r(Z_\tau) \right\}. \tag{2.1}$$

A stopping time $\tau \in \mathscr{M}$ is called *optimal* if it attains the supremum in (2.1). Assume that the value function solves the Bellman equation[1]

$$v^*(z) = \max\left\{ r(z),\, c(z) + \beta \int v^*(z') P(z, dz') \right\}. \tag{2.2}$$

The corresponding Bellman operator is

$$Tv(z) = \max\left\{ r(z), c(z) + \beta \int v(z') P(z, \mathrm{d}z') \right\}.$$

The *continuation value function* associated with this problem is defined at $z \in \mathsf{Z}$ by

$$\psi^*(z) := c(z) + \beta \int v^*(z') P(z, dz'). \tag{2.3}$$

Using (2.2) and (2.3), we observe that ψ^* satisfies

$$\psi^*(z) = c(z) + \beta \int \max\left\{ r(z'), \psi^*(z') \right\} P(z, \mathrm{d}z'). \tag{2.4}$$

Analogous to the Bellman operator, the *continuation value operator* or *Jovanovic operator* Q is constructed such that the continuation value function ψ^* is a fixed point of

$$Q\psi(z) = c(z) + \beta \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z'). \tag{2.5}$$

2.3 Symmetries Between the Operators

In this section we show that Bellman and Jovanovic operators are semiconjugate and discuss the implications. The semiconjugate relationship is most easily

[1]A sufficient condition is that $\mathbb{E}_z \left(\sup_{k \geq 0} \left| \sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k) \right| \right) < \infty$ for all $z \in \mathsf{Z}$, as can be shown by applying theorem 1.11 (claim 1) of Peskir and Shiryaev (2006). Later we provide alternative versions of sufficient conditions that are closer to our primitive set up (see, e.g., appendix 2.E).

shown using operator-theoretic notation. To this end, let $Ph(z) := \int h(z')P(z, \mathrm{d}z')$ for all integrable function $h \in m\mathscr{B}$ and observe that the Bellman operator T can then be expressed as $T = RL$, where

$$R\psi := r \vee \psi \quad \text{and} \quad Lv := c + \beta Pv. \tag{2.6}$$

(For any two operators we write the composition $A \circ B$ more simply as AB.)

2.3.1 General Theory

Let $\mathcal{V}$ be a subset of $m\mathscr{B}$ such that $v^* \in \mathcal{V}$ and $T\mathcal{V} \subset \mathcal{V}$. The set $\mathcal{V}$ is understood as a set of candidate value functions. (Specific classes of functions are considered in the next section.) Let $\mathcal{C}$ be defined by

$$\mathcal{C} := L\mathcal{V} = \{\psi \in m\mathscr{B} \colon \ \psi = c + \beta Pv \ \text{ for some } v \in \mathcal{V}\}. \tag{2.7}$$

By definition, L is a surjective mapping from $\mathcal{V}$ onto $\mathcal{C}$. It is also true that R maps $\mathcal{C}$ into $\mathcal{V}$. Indeed, if $\psi \in \mathcal{C}$, then there exists a $v \in \mathcal{V}$ such that $\psi = Lv$, and $R\psi = RLv = Tv$, which lies in $\mathcal{V}$ by assumption.

Lemma 2.3.1. *On $\mathcal{C}$, the operator Q satisfies $Q = LR$, and $Q\mathcal{C} \subset \mathcal{C}$.*

Proof. The first claim is immediate from the definitions. The second follows from the claims just established (i.e., R maps $\mathcal{C}$ to $\mathcal{V}$ and L maps $\mathcal{V}$ to $\mathcal{C}$). $\square$

The preceding discussion implies that Q and T are *semiconjugate*, in the sense that $LT = QL$ on $\mathcal{V}$ and $TR = RQ$ on $\mathcal{C}$. Indeed, since $T = RL$ and $Q = LR$, we have $LT = LRL = QL$ and $TR = RLR = RQ$ as claimed. This leads to the next result:

Proposition 2.3.1. *The following statements are true:*

(1) If v is a fixed point of T in $\mathcal{V}$, then Lv is a fixed point of Q in $\mathcal{C}$.

(2) If ψ is a fixed point of Q in $\mathcal{C}$, then $R\psi$ is a fixed point of T in $\mathcal{V}$.

Proof. To prove the first claim, fix $v \in \mathcal{V}$. By the definition of $\mathcal{C}$, $Lv \in \mathcal{C}$. Moreover, since $v = Tv$, we have $QLv = LTv = Lv$. Hence, Lv is a fixed point of Q in $\mathcal{C}$. Regarding the second claim, fix $\psi \in \mathcal{C}$. Since R maps $\mathcal{C}$ into $\mathcal{V}$ as shown above, $R\psi \in \mathcal{V}$. Since $\psi = Q\psi$, we have $TR\psi = RQ\psi = R\psi$. Hence, $R\psi$ is a fixed point of T in $\mathcal{V}$. $\square$

The following result says that, at least on a theoretical level, iterating with either T or Q is essentially equivalent.

Proposition 2.3.2. *$T^{t+1} = RQ^tL$ on $\mathcal{V}$ and $Q^{t+1} = LT^tR$ on $\mathcal{C}$ for all $t \in \mathbb{N}_0$.*

Proof. That the claim holds when $t = 0$ has already been established. Now suppose the claim is true for arbitrary t. By the induction hypothesis we have $T^t = RQ^{t-1}L$ and $Q^t = LT^{t-1}R$. Since Q and T are semiconjugate as shown above, we have $T^{t+1} = TT^t = TRQ^{t-1}L = RQQ^{t-1}L = RQ^tL$ and $Q^{t+1} = QQ^t = QLT^{t-1}R = LTT^{t-1}R = LT^tR$. Hence, the claim holds by induction. □

The theory above is based on the primitive assumption of a candidate value function space $\mathcal{V}$ with properties $v^* \in \mathcal{V}$ and $T\mathcal{V} \subset \mathcal{V}$. Similar results can be established if we start with a generic candidate continuation value function space $\mathscr{C}$ that satisfies $\psi^* \in \mathscr{C}$ and $Q\mathscr{C} \subset \mathscr{C}$. Appendix 2.A gives details.

2.3.2 Symmetry under Weighted Supremum Norm

Next we impose a weighted supremum norm on the domain of T and Q in order to compare contractivity, optimality and related properties. The following assumption generalizes the standard weighted supremum norm assumption of Boyd (1990).

Assumption 2.3.1. There exist a $\mathscr{B}$-measurable function $g \colon \mathsf{Z} \to \mathbb{R}_+$ and constants $n \in \mathbb{N}_0$ and $a_1, \cdots, a_4, m, d \in \mathbb{R}_+$ such that $\beta m < 1$, and, for all $z \in \mathsf{Z}$,

$$\int |r(z')| P^n(z, \mathrm{d}z') \leq a_1 g(z) + a_2, \tag{2.8}$$

$$\int |c(z')| P^n(z, \mathrm{d}z') \leq a_3 g(z) + a_4, \tag{2.9}$$

$$\text{and} \int g(z') P(z, \mathrm{d}z') \leq m g(z) + d. \tag{2.10}$$

The interpretation is that both $\mathbb{E}_z |r(Z_n)|$ and $\mathbb{E}_z |c(Z_n)|$ are small relative to some function g such that $\mathbb{E}_z g(Z_t)$ does not grow too fast.[2] Slow growth in $\mathbb{E}_z g(Z_t)$ is imposed by (2.10), which can be understood as a geometric drift condition (see, e.g., Meyn and Tweedie (2009), chapter 15). Note that if both r and c are bounded, then assumption 2.3.1 holds for $n := 0$, $g := \|r\| \vee \|c\|$, $m := 1$ and $d := 0$.

[2]One can show that if assumption 2.3.1 holds for some n, then it must hold for all integer $n' > n$. Hence, to verify assumption 2.3.1, it suffices to find $n_1 \in \mathbb{N}_0$ for which (2.8) holds, $n_2 \in \mathbb{N}_0$ for which (2.9) holds, and that the measurable map g satisfies (2.10).

Assumption 2.3.1 reduces to that of Boyd (1990) if we set $n = 0$. Here we admit consideration of future transitions to enlarge the set of possible weight functions. The value of this generalization is illustrated in section 2.5.

Theorem 2.3.1. *Let assumption 2.3.1 hold. Then there exist positive constants m' and d' such that for $\ell, \kappa \colon \mathsf{Z} \to \mathbb{R}$ defined by*[3]

$$\ell(z) := m' \left(\sum_{t=1}^{n-1} \mathbb{E}_z |r(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z |c(Z_t)| \right) + g(z) + d' \tag{2.11}$$

and $\kappa(z) := \ell(z) + m'|r(z)|$, the following statements hold:

(1) Q is a contraction mapping on $(b_\ell \mathsf{Z}, \|\cdot\|_\ell)$, with unique fixed point $\psi^* \in b_\ell \mathsf{Z}$.

(2) T is a contraction mapping on $(b_\kappa \mathsf{Z}, \|\cdot\|_\kappa)$, with unique fixed point $v^* \in b_\kappa \mathsf{Z}$.

The next result shows that the convergence rates of Q and T are the same. In stating it, L and R are as defined in (2.6), while $\rho \in (0,1)$ is the contraction coefficient of T derived in theorem 2.3.1 (see (2.B.1) in appendix 2.B for details).

Proposition 2.3.3. *If assumption 2.3.1 holds, then*

$$R(b_\ell \mathsf{Z}) \subset b_\kappa \mathsf{Z} \quad \textit{and} \quad L(b_\kappa \mathsf{Z}) \subset b_\ell \mathsf{Z},$$

and for all $t \in \mathbb{N}_0$, the following statements are true:

(1) $\left\|Q^{t+1}\psi - \psi^*\right\|_\ell \leq \rho \left\|T^t R\psi - v^*\right\|_\kappa$ for all $\psi \in b_\ell \mathsf{Z}$.

(2) $\left\|T^{t+1}v - v^*\right\|_\kappa \leq \left\|Q^t L v - \psi^*\right\|_\ell$ for all $v \in b_\kappa \mathsf{Z}$.

Proposition 2.3.3 extends proposition 2.3.2 and lemma 2.A.1 (see appendix 2.A), and their connections can be seen by letting $\mathcal{V} := b_\kappa \mathsf{Z}$ and $\mathscr{C} := b_\ell \mathsf{Z}$. Notably, claim (1) implies that Q converges as fast as T, even when its convergence is weighted by a smaller function (since $\ell \leq \kappa$).

The two operators are also symmetric in terms of continuity of fixed points. The next result illustrates this, when Z is any separable and completely metrizable topological space (e.g., any G_δ subset of $\mathbb{R}^n$) and $\mathscr{B}$ is its Borel sets.

Assumption 2.3.2. (1) The stochastic kernel P is Feller; that is, $z \mapsto \int h(z')P(z, \mathrm{d}z')$ is continuous and bounded on Z whenever h is. (2) c, r, ℓ, $z \mapsto \int |r(z')|P(z, \mathrm{d}z')$, and $z \mapsto \int \ell(z')P(z, \mathrm{d}z')$ are continuous.[4]

[3] If assumption 2.3.1 holds for $n = 0$, then $\ell(z) = g(z) + d'$ and $\kappa(z) = m'|r(z)| + g(z) + d'$. To guarantee that ℓ and κ are real-valued, here and below, we assume that $\mathbb{E}_z|r(Z_t)|$, $\mathbb{E}_z|c(Z_t)| < \infty$ for $t = 1, \cdots, n-1$, which holds trivially in most applications of interest.

[4] A sufficient condition for assumption 2.3.2-(2) is: g and $z \mapsto \mathbb{E}_z g(Z_1)$ are continuous, and $z \mapsto \mathbb{E}_z|r(Z_t)|, \mathbb{E}_z|c(Z_t)|$ are continuous for $t = 0, ..., n$ (with n as defined in assumption 2.3.1).

Proposition 2.3.4. *If assumptions 2.3.1–2.3.2 hold, then ψ^* and v^* are continuous.*

2.3.3 Symmetry in L_p

The results of the preceding section for the most part carry over if we switch the underlying space to L_p. This section provides details.

Assumption 2.3.3. The state process $\{Z_t\}$ admits a stationary distribution π and the reward functions r, c are in $L_q(\pi)$ for some $q \geq 1$.

Theorem 2.3.2. *If assumption 2.3.3 holds, then for all $1 \leq p \leq q$, we have*[5]

(1) Q is a contraction mapping on $(L_p(\pi), \|\cdot\|_p)$ of modulus β, and the unique fixed point of Q in $L_p(\pi)$ is ψ^*.

(2) T is a contraction mapping on $(L_p(\pi), \|\cdot\|_p)$ of modulus β, and the unique fixed point of T in $L_p(\pi)$ is v^*.

The following result implies that Q and T have the same rate of convergence in terms of L_p-norm distance.

Proposition 2.3.5. *If assumption 2.3.3 holds, then for all $1 \leq p \leq q$,*

$$R\left(L_p(\pi)\right) \subset L_p(\pi) \quad \textit{and} \quad L\left(L_p(\pi)\right) \subset L_p(\pi).$$

Moreover, for all $1 \leq p \leq q$ and $t \in \mathbb{N}_0$, the following statements hold:

(1) $\left\|Q^{t+1}\psi - \psi^*\right\|_p \leq \beta \left\|T^t R\psi - v^*\right\|_p$ for all $\psi \in L_p(\pi)$.

(2) $\left\|T^{t+1}v - v^*\right\|_p \leq \left\|Q^t L v - \psi^*\right\|_p$ for all $v \in L_p(\pi)$.

Proposition 2.3.5 is an extension of proposition 2.3.2 and lemma 2.A.1 (see appendix 2.A) in an L_p space, and their connections can be seen by letting $\mathcal{V} = \mathscr{C} := L_p(\pi)$.

2.4 Asymmetries Between the Operators

The preceding results show that T and Q exhibit dynamics that are in many senses symmetric. However, for a large number of economic models, the effective state space for Q is lower dimensional than that of T. This section provides

[5]We typically omit phrases such as "with probability one" or "almost surely" in what follows. Indeed, functional equivalences and uniqueness of fixed points are up to a π-null set.

definitions and analysis, with examples deferred to section 2.5. Throughout, we write

$$\mathsf{Z} = \mathsf{X} \times \mathsf{Y} \quad \text{and} \quad Z_t = (X_t, Y_t)$$

where X is a Borel subset of $\mathbb{R}^k$ and Y is a Borel subset of $\mathbb{R}^n$.

2.4.1 Continuation Decomposability

We call an optimal stopping problem (β, c, P, r) *continuation decomposable* if c and P are such that

(a) (X_{t+1}, Y_{t+1}) and X_t are independent given Y_t and

(b) c is a function of Y_t but not X_t.

Condition (a) implies that $P(z, \mathrm{d}z')$ can be represented by the conditional distribution of (x', y') given y, denoted below by $F_y(x', y')$. On an intuitive level, continuation decomposable problems are those where some state variables matter only for terminal rewards.

The significance of continuation decomposability is that, for such models, the Jovanovic operator can be written as

$$Q\psi(y) = c(y) + \beta \int \max \left\{ r(x', y'), \psi(y') \right\} \mathrm{d}F_y(x', y').$$

Thus, Q acts on functions defined over Y alone. In contrast, assuming that all state variables are non-trivial in the sense that they impact on the value function, T continues to act on functions defined over all of $\mathsf{Z} = \mathsf{X} \times \mathsf{Y}$. The set Y is k dimensions lower than Z.

Remark 2.4.1. Indeed, for most applications of interest (aside from a class of problems for which the continuation dynamics are trivial in a sense to be made precise), the effective state space of the continuation value function is never larger than that of the value function, the fixed point of the Bellman operator. Appendix 2.D characterizes continuation nontriviality in detail and provides a formal proof of the (weakly) lower state dimension of the continuation value function.

2.4.2 Complexity Analysis

One way to compare the efficiency of Q and T is to consider the time complexity of continuation value function iteration (CVI) and value function iteration (VFI). Both finite and infinite space approximations are considered.

Finite Space

Let $\mathsf{X} = \times_{i=1}^{k} \mathsf{X}^i$ and $\mathsf{Y} = \times_{j=1}^{n} \mathsf{Y}^j$, where X^i and Y^j are subsets of $\mathbb{R}$. Each X^i (resp., Y^j) is represented by a grid of K_i (resp., M_j) points. Integration operations in both VFI and CVI are replaced by summations. We use $\hat{P}$ and $\hat{F}$ to denote the transition matrices (i.e., discretized stochastic kernels) for VFI and CVI respectively.[6]

Let $K := \Pi_{i=1}^{k} K_i$ and $M := \Pi_{j=1}^{n} M_j$ with $K = 1$ for $k = 0$. Let $n > 0$. There are KM grid points on $\mathsf{Z} = \mathsf{X} \times \mathsf{Y}$ and M grid points on Y. The matrix $\hat{P}$ is $(KM) \times (KM)$ and $\hat{F}$ is $M \times (KM)$. VFI and CVI are implemented by the operators $\hat{T}$ and $\hat{Q}$ defined respectively by

$$\hat{T}\vec{v} := \vec{r} \vee (\vec{c} + \beta \hat{P} \vec{v}) \quad \text{and} \quad \hat{Q}\vec{\psi}_y := \vec{c}_y + \beta \hat{F}(\vec{r} \vee \vec{\psi}).$$

Here $\vec{q}$ represents a column vector with i-th element equal to $q(x_i, y_i)$, where (x_i, y_i) is the i-th element of the list of grid points on $\mathsf{X} \times \mathsf{Y}$. Let $\vec{q}_y$ denote the column vector with the j-th element equal to $q(y_j)$, where y_j is the j-th element of the list of grid points on Y. The vectors $\vec{v}$, $\vec{r}$, $\vec{c}$ and $\vec{\psi}$ are $(KM) \times 1$, while $\vec{c}_y$ and $\vec{\psi}_y$ are $M \times 1$.

Infinite Space

We use the same number of grid points as before, but now for continuous state function approximation rather than discretization. In particular, we replace the discrete state summation with Monte Carlo integration. Assume that the transition function of the state process follows

$$X_{t+1} = f_1(Y_t, W_{t+1}), \qquad Y_{t+1} = f_2(Y_t, W_{t+1}), \qquad \{W_t\} \overset{\text{IID}}{\sim} \Phi.$$

After drawing $U_1, \cdots, U_N \overset{\text{IID}}{\sim} \Phi$, with N being the MC sample size, CVI and VFI are implemented by

$$\hat{Q}\psi(y) := c(y) + \beta \frac{1}{N} \sum_{i=1}^{N} \max \left\{ r\left(f_1(y, U_i), f_2(y, U_i)\right), h\langle\psi\rangle\left(f_2(y, U_i)\right) \right\}$$

$$\text{and} \quad \hat{T}v(x, y) := \max \left\{ r(x, y),\, c(y) + \beta \frac{1}{N} \sum_{i=1}^{N} g\langle v\rangle\left(f_1(y, U_i), f_2(y, U_i)\right) \right\}.$$

Here $\psi = \{\psi(y)\}$, with y in the set of grid points on Y, and $v = \{v(x, y)\}$, with (x, y) in the set of grid points on $\mathsf{X} \times \mathsf{Y}$. Moreover, $h\langle\cdot\rangle$ and $g\langle\cdot\rangle$ are interpolating functions for CVI and VFI respectively. For example, $h\langle\psi\rangle(z)$ can be understood as interpolating the vector ψ to obtain a function $h\langle\psi\rangle$ and then evaluating at z.

[6] See Tauchen and Hussey (1991) for a general discussion of discretization methods.

Time Complexity

Table 2.1 provides the time complexity of CVI and VFI, estimated by counting the number of floating point operations. Each such operation is assumed to have complexity $\mathcal{O}(1)$.[7] Function evaluations associated with the model primitives are also assumed to be of order $\mathcal{O}(1)$.

Table 2.1: Time complexity: VFI v.s CVI

Cmplx.	VFI: 1-loop	CVI: 1-loop	VFI: n-loop	CVI: n-loop
FS	$\mathcal{O}(K^2M^2)$	$\mathcal{O}(KM^2)$	$\mathcal{O}(nK^2M^2)$	$\mathcal{O}(nKM^2)$
IS	$\mathcal{O}(NKM\log(KM))$	$\mathcal{O}(NM\log(M))$	$\mathcal{O}(nNKM\log(KM))$	$\mathcal{O}(nNM\log(M))$

Note: For IS approximation, binary search is used when we evaluate the interpolating function at a given point. The results hold for linear, quadratic, cubic, and k-nearest neighbors interpolations.

For both finite space (FS) and infinite space (IS) approximations, CVI provides better performance than VFI. For FS, CVI is more efficient than VFI by order $\mathcal{O}(K)$, while for IS, CVI is more efficient than VFI by order $\mathcal{O}\left(K\log(KM)/\log(M)\right)$. For example, if we have 250 grid points in each dimension, then in the FS case, evaluating a given number of loops will take around 250^k times longer via CVI than via VFI, after adjusting for order approximations.

See appendix 2.C for a proof of the results in table 2.1.

2.5 Applications

We consider six applications. For the first two cases, we discuss optimality, continuation decomposability and compare the numerical efficiency of the Bellman and Jovanovic operators. For the remaining cases, we discuss only continuation decomposability. For numerical works we apply infinite space (IS) approximation with $N = 1000$ and use linear interpolation for function approximation. All simulations of this section are processed in a standard Julia environment on a laptop witha a 2.9 GHz Intel Core i7 and 32GB RAM.[8]

[7] Floating point operations are any elementary actions (e.g., $+$, $\times$, $\vee$, $\wedge$) on or assignments with floating point numbers. If f and g are scalar functions on $\mathbb{R}^n$, we write $f(x) = \mathcal{O}(g(x))$ whenever there exist $C, M > 0$ such that $\|x\| \geq M$ implies $|f(x)| \leq C|g(x)|$, where $\|\cdot\|$ is the sup norm.

[8] The Julia code needed to replicate all of the applications discussed in this section, together with alternative versions written in Python, can be found at https://github.com/jstac/continuation_values_public.

2.5.1 Job Search

Consider a worker who receives current wage offer w_t and chooses to either accept and work permanently at that wage, or reject the offer, receive unemployment compensation c_0 and reconsider next period (see, e.g., McCall (1970) or Pissarides (2000)). The wage process $\{w_t\}_{t \geq 0}$ is assumed to be

$$w_t = \eta_t + \theta_t \xi_t, \quad \text{where} \quad \ln \theta_t = \rho \ln \theta_{t-1} + \ln \varepsilon_t \tag{2.12}$$

and $\{\xi_t\}$, $\{\varepsilon_t\}$ and $\{\eta_t\}$ are positive IID innovations that are mutually independent. We interpret θ_t as the persistent component of labor income and allow it to be nonstationary. When η_t is constant it can be interpreted as social security.[9] Viewed as an optimal stopping problem,

- the state is $z = (w, \theta)$, with stochastic kernel P defined by (2.12),
- the terminal reward is $r(w) = u(w)/(1-\beta)$, where u is a utility function,
- and the flow continuation reward c is the constant $u(c_0)$.

The model is continuation decomposable, as can be seen by letting $X_t := w_t$ and $Y_t := \theta_t$. In particular, c does not depend on w_t and (w_{t+1}, θ_{t+1}) is independent of w_t given θ_t. Hence the effective state space for Q is one-dimensional while that of T is two-dimensional. Letting $F_\theta(w', \theta')$ be the distribution of (w_{t+1}, θ_{t+1}) given θ_t, the Bellman operator satisfies

$$Tv(w, \theta) = \max \left\{ \frac{u(w)}{1-\beta}, \; u(c_0) + \beta \int v(w', \theta') \, \mathrm{d}F_\theta(w', \theta') \right\},$$

while the Jovanovic operator is

$$Q\psi(\theta) = u(c_0) + \beta \int \max \left\{ \frac{u(w')}{1-\beta}, \; \psi(\theta') \right\} \mathrm{d}F_\theta(w', \theta').$$

Whether or not assumptions 2.3.1–2.3.3 hold depends on the primitives. Suppose for example that

$$u(w) = \frac{w^{1-\gamma}}{1-\gamma} \quad \text{with} \quad u(w) = \ln w \quad \text{when } \gamma = 1.$$

[9] Similar dynamics appear in many labor market, search-theoretic and real options studies (see e.g., Gomes et al. (2001), Low et al. (2010), Chatterjee and Eyigungor (2012), Bagger et al. (2014), and Kellogg (2014)).

We focus here on the case $\gamma = 1$ and $0 \leq \rho < 1$, although other cases such as $\gamma > 1$ and $-1 < \rho < 0$ can be treated with similar arguments.[10] We take $\varepsilon_t \sim LN(0, \sigma^2)$. Regarding assumptions 2.3.1–2.3.2, we assume that $\{\eta_t\}$, $\{\eta_t^{-1}\}$ and $\{\xi_t\}$ have finite first moments.

The reward function for this dynamic program is unbounded above and below, and the state space is likewise unbounded. Nevertheless, we can establish the key optimality results from section 2.3 as follows. First, choose $n \in \mathbb{N}_0$ such that $\beta \exp(\rho^{2n}\sigma^2) < 1$, and let

$$g(z) = g(w, \theta) = \theta^{\rho^n}.$$

To verify (2.8), we make use of the following technical lemma, which is obtained from the law of motion (2.12), and provides a bound on expected time n wages in terms of initial condition $\theta_0 = \theta$. The proof is in appendix 2.B.

Lemma 2.5.1. *For all $n \in \mathbb{N}_0$, (a) there exist a pair $A_n, B \in \mathbb{R}$ such that $\mathbb{E}_\theta |\ln w_n| \leq A_n \theta^{\rho^n} + B$, and (b) $\theta \mapsto \mathbb{E}_\theta |\ln w_n|$ is continuous.*

Now (2.8) can be established, since, conditioning on $\theta_0 = \theta$,

$$\mathbb{E}_\theta |r(w_n)| = \frac{\mathbb{E}_\theta |\ln w_n|}{1-\beta} \leq \frac{A_n}{1-\beta} \theta^{\rho^n} + \frac{B}{1-\beta} = \frac{A_n}{1-\beta} g(w, \theta) + \frac{B}{1-\beta}.$$

Condition (2.9) is trivial because c is constant. To see that condition (2.10) holds, note that $\rho \in [0, 1)$, so, conditioning on $\theta_0 = \theta$ once more,

$$\mathbb{E}_\theta\, g(w_1, \theta_1) = \mathbb{E}\, (\theta^\rho \varepsilon_1)^{\rho^n} = \theta^{\rho^{n+1}} \exp(\rho^{2n}\sigma^2/2) \leq (\theta^{\rho^n} + 1) \exp(\rho^{2n}\sigma^2).$$

Hence (2.10) holds with $m = d = \exp(\rho^{2n}\sigma^2)$. Assumption 2.3.1 has now been established. By theorem 2.3.1 and proposition 2.3.3, Q and T are contraction mappings with the same rate of convergence. The above analysis also implies that assumption 2.3.2 holds (see footnote 4), so proposition 2.3.4 implies that both v^* and ψ^* are continuous.

We can also embed this problem in $L_p(\pi)$. To verify assumption 2.3.3, we assume that the distributions of $\{\eta_t\}$ and $\{\xi_t\}$ are represented respectively by densities μ and ν, and that $\{\eta_t\}$, $\{\eta_t^{-1}\}$ and $\{\xi_t\}$ have finite q-th moments.

Since $\rho \in [0, 1)$, the state process $\{(w_t, \theta_t)\}$ has stationary density

$$\pi(w, \theta) = f^*(\theta)\, p(w|\theta),$$

[10] One can also treat the nonstationary case $\rho = \pm 1$ under some further parametric assumptions using the weighted supremum norm techniques developed above. Details are provided in chapter 5 of this thesis.

where $f^*(\theta) = LN(0, \sigma^2/(1-\rho^2))$ and $\int_A p(w|\theta)\,\mathrm{d}w = \int_{\{\eta+\theta\xi\in A\}} \mu(\eta)\nu(\xi)\,\mathrm{d}(\eta,\xi)$. Then the next lemma (proved in appendix 2.B) implies that assumption 2.3.3 holds.

Lemma 2.5.2. *The reward functions r and c are in $L_q(\pi)$.*

By theorem 2.3.2 and proposition 2.3.5, Q and T are both contraction mappings with the same rate of convergence (in L_p norm distances, for all $1 \leq p \leq q$).

Finally, we compare the numerical efficiency of the Bellman and Jovanovic operators. Table 2.2 compares the time taken for CVI and VFI under different grid sizes. In tests 1–6 (50 loops), CVI is on average 261 times faster than VFI. Moreover, as we increase the grid size of θ and w, computation time for VFI grows exponentially. Table 2.3 continues the analysis by comparing CVI and VFI under different levels of risk aversion (δ) and income persistency (ρ). Among tests 1–8 (50 loops), CVI is 267 times faster than VFI on average.

Recall from section 2.4.2 that CVI creates an order $\mathcal{O}(K\log(KM)/\log(M))$ speed up over VFI for the MC algorithm. In this model, K and M are respectively the number of grid points for w_t and θ_t. As shown in table 2.2–2.3, CVI is approximately K times faster than VFI in each test, which is broadly in line with the theory.

Table 2.2: Time in seconds under different grid sizes

Time & Size		Test 1	Test 2	Test 3	Test 4	Test 5	Test 6
Grid size (θ, w)		$(200, 200)$	$(200, 400)$	$(300, 200)$	$(300, 400)$	$(400, 200)$	$(400, 400)$
Loop 10	VFI	29.70	62.47	43.78	95.19	59.47	128.73
	CVI	0.325	0.176	0.259	0.268	0.361	0.348
Loop 20	VFI	58.37	125.64	87.44	190.22	118.75	257.32
	CVI	0.493	0.339	0.517	0.529	0.726	0.688
Loop 50	VFI	114.38	314.30	218.62	475.78	297.34	644.57
	CVI	1.014	0.824	1.277	1.289	1.786	1.757

We set $\rho = 0.75$, $\beta = 0.95$, $\tilde{c}_0 = 0.6$, $\delta = 1$, $\gamma_u = 10^{-4}$, $v = LN(0, 10^{-6})$ and $h = LN(0, 5 \times 10^{-4})$. The grid points for (θ, w) lie in $[10^{-4}, 10]^2$.

2.5.2 Search with Learning

Consider a job search problem with learning (see, e.g., McCall (1970), Pries and Rogerson (2005), Nagypál (2007), or Ljungqvist and Sargent (2012)). The setup is

Table 2.3: Time in seconds under different δ and ρ values

Time & Value		Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8
Value (δ,ρ)		$(2,0.8)$	$(2,0.7)$	$(2,0.6)$	$(3,0.8)$	$(3,0.7)$	$(3,0.6)$	$(4,0.8)$	$(4,0.7)$
Loop 10	VFI	69.75	67.86	66.61	69.80	67.68	66.70	69.96	67.84
	CVI	0.205	0.161	0.163	0.375	0.394	0.386	0.372	0.371
Loop 20	VFI	139.23	135.86	133.11	139.17	135.44	132.93	139.44	135.53
	CVI	0.371	0.320	0.320	0.745	0.778	0.765	0.741	0.738
Loop 50	VFI	349.08	339.84	333.24	346.93	339.10	331.86	348.84	338.53
	CVI	0.866	0.834	0.794	1.908	1.894	1.895	1.850	1.837

We set $\beta = 0.95$, $\tilde{c}_0 = 0.6$, $\gamma_u = 10^{-4}$, $v = LN(0, 10^{-6})$ and $h = LN(0, 5 \times 10^{-4})$. The grid points for (θ, w) lie in $[10^{-4}, 10]^2$ with 300 points each.

as in section 2.5.1, except that $\{w_t\}_{t \geq 0}$ follows

$$\ln w_t = \xi + \varepsilon_t, \quad \text{where } \{\varepsilon_t\}_{t\geq 0} \overset{\text{IID}}{\sim} N(0, \delta_\varepsilon).$$

Here ξ is an unobservable mean over which the worker has prior $\xi \sim N(\mu, \delta)$. The worker's current estimate of the next period wage distribution is $f(w'|\mu,\delta) = LN(\mu, \delta + \delta_\varepsilon)$. If the current offer is turned down, the worker updates his belief after observing w'. By Bayes' rule, the posterior satisfies $\xi|w' \sim N(\mu', \delta')$, where $\delta' = v(\delta) := 1/\left(1/\delta + 1/\delta_\varepsilon\right)$ and $\mu' = \phi(\mu, \delta, w') := \delta'\left(\mu/\delta + \ln w'/\delta_\varepsilon\right)$. Viewed as an optimal stopping problem,

- the state is $z = (w, \mu, \delta)$, and for each map h, the stochastic kernel P satisfies

$$\int h(z')P(z, \mathrm{d}z') = \int h\left(w', \phi(\mu,\delta,w'), v(\delta)\right) f(w'|\mu,\delta)\,\mathrm{d}w'$$

- the reward functions are $r(w) = u(w)/(1-\beta)$ and $c \equiv u(c_0)$.

The model is continuation decomposable with $X_t := w_t$ and $Y_t := (\mu_t, \delta_t)$, since r does not depend on (μ_t, δ_t) and the next period state $(w_{t+1}, \mu_{t+1}, \delta_{t+1})$ is independent of w_t once μ_t and δ_t are known. Letting $F_{\mu,\delta}(w', \mu', \delta')$ be the distribution of $(w_{t+1}, \mu_{t+1}, \delta_{t+1})$ given (μ_t, δ_t), the Bellman and Jovanovic operators are, respectively,

$$Tv(w,\mu,\delta) = \max\left\{\frac{u(w)}{1-\beta},\ u(c_0) + \beta\int v(w',\mu',\delta')\,\mathrm{d}F_{\mu,\delta}(w',\mu',\delta')\right\}$$

$$\text{and} \quad Q\psi(\mu,\delta) = u(c_0) + \beta\int \max\left\{\frac{u(w')}{1-\beta},\ \psi(\mu',\delta')\right\}\mathrm{d}F_{\mu,\delta}(w',\mu',\delta').$$

Again, the domain of the candidate function space is one dimension lower for Q than T.

Regarding optimality, suppose, for example, that the CRRA parameter γ is greater than 1. (The case $\gamma = 1$ can be treated along similar lines.) Let $n = 1$ and let

$$g(w, \mu, \delta) = \mathrm{e}^{(1-\gamma)\mu + (1-\gamma)^2 \delta / 2}.$$

Condition (2.8) holds, since, conditioning on $(\mu_0, \delta_0) = (\mu, \delta)$,

$$\mathbb{E}_{\mu,\delta} |r(w_1)| = \frac{\mathbb{E}_{\mu,\delta} w_1^{1-\gamma}}{1-\beta} = \frac{\mathrm{e}^{(1-\gamma)^2 \delta_\varepsilon / 2}}{1-\beta} g(w, \mu, \delta).$$

Condition (2.9) is trivial since c is constant. Condition (2.10) holds, since, conditioning on $(\mu_0, \delta_0) = (\mu, \delta)$, the expressions of μ' and δ' imply that

$$\mathbb{E}_{\mu,\delta}\, g(w_1, \mu_1, \delta_1) = \mathrm{e}^{(1-\gamma)^2 \delta_1 / 2 + (1-\gamma)\delta_1 \mu / \delta}\, \mathbb{E}_{\mu,\delta}\, w_1^{(1-\gamma)\delta_1 / \delta_\varepsilon} = g(w, \mu, \delta).$$

Hence assumption 2.3.1 holds. Theorem 2.3.1 and proposition 2.3.3 imply that Q and T are contraction mappings with the same rate of convergence. The analysis above also implies that assumption 2.3.2 holds (see footnote 4), so v^* and ψ^* are continuous by proposition 2.3.4.

Finally, table 2.4 compares CVI and VFI under different grid sizes. In tests 1–10, CVI is on average 132 times faster than VFI. In test 10, VFI takes more than 4.4 days, while CVI takes 24 minutes. Table 2.5 compares CVI and VFI under different level of risk aversion. CVI is shown to be 98 times faster than VFI on average. Again, these numerical results are close to the prediction of the theory in section 2.4.2.

Table 2.4: Time in seconds under different grid sizes

Time & Size		Test 1	Test 2	Test 3	Test 4	Test 5
Size (w, μ, γ)		$(50, 50, 50)$	$(1, 1, 1) \times 10^2$	$(2, 1, 1) \times 10^2$	$(1, 2, 1) \times 10^2$	$(1, 1, 2) \times 10^2$
Loop 20	VFI	685.2	5242.0	10455.1	10584.9	11443.4
	CVI	14.5	61.3	60.3	134.6	131.7
Loop 50	VFI	1823.2	13020.2	26001.7	26365.2	27149.1
	CVI	35.8	164.6	149.6	342.7	338.3
Time & Size		Test 6	Test 7	Test 8	Test 9	Test 10
Size (w, μ, γ)		$(2, 2, 1) \times 10^2$	$(2, 1, 2) \times 10^2$	$(1, 2, 2) \times 10^2$	$(2, 2, 2) \times 10^2$	$(3, 3, 3) \times 10^2$
Loop 20	VFI	21649.6	22267.9	21042.4	42567.8	152349.0
	CVI	119.3	144.4	246.1	236.7	576.9
Loop 50	VFI	54143.5	55687.4	52578.4	106386.0	380220.0
	CVI	297.0	367.8	679.2	589.9	1430.6

We set $\beta = 0.95$, $\gamma_\varepsilon = 1$, $\tilde{c}_0 = 0.6$ and $\delta = 3$. The grid points for (w, μ, γ) lie in $[10^{-4}, 10] \times [-10, 10] \times [10^{-4}, 10]$.

Table 2.5: Time in seconds under different risk aversion levels

Time & Value		$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$	$\delta = 5$	$\delta = 6$	Mean
Loop 10	VFI	3137.1	2606.9	2615.1	2618.3	2606.5	2625.7	2701.6
	CVI	25.0	25.7	30.4	30.5	30.6	30.5	28.8
Loop 20	VFI	6323.8	5206.8	5220.0	5226.9	5221.5	5242.8	5406.9
	CVI	48.4	48.9	60.8	60.8	61.3	60.9	56.9
Loop 50	VFI	15976.6	13026.4	13066.7	13159.4	13099.6	13135.4	13577.3
	CVI	118.9	118.9	151.9	152.3	152.8	152.6	141.2

We set $\beta = 0.95$, $\gamma_\varepsilon = 1$, and $\tilde{c}_0 = 0.6$. The grid points of (w, μ, γ) lie in $[10^{-4}, 10] \times [-10, 10] \times [10^{-4}, 10]$ with $(100, 100, 100)$ points.

2.5.3 Firm Entry

Consider a condensed version of the firm entry problem in Fajgelbaum et al. (2017). At the start of period t, a firm observes a fixed cost f_t and then decides whether to incur this cost and enter a market, earning stochastic payoff π_t, or wait and reconsider next period. The sequence $\{f_t\}$ is IID, while the current payoff π_t is unknown prior to entry. The firm has prior belief $\phi(\pi; \theta_t)$, where ϕ is a distribution over payoffs that is parameterized by a vector θ_t. If the firm does not enter then θ_t is updated via Bayesian learning. In an optimal stopping format,

- the state is $z = (f, \theta)$, with stochastic kernel P defined by the distribution of $\{f_t\}$ and the Bayesian updating mechanism of $\{\theta_t\}$,
- the terminal reward is the entry payoff $r(f, \theta) = \int \pi \phi(\mathrm{d}\pi; \theta) - f$,
- and the flow continuation reward $c \equiv 0$.

This model is continuation decomposable, as can be seen by letting $X_t := f_t$ and $Y_t := \theta_t$. In particular, since $\{f_t\}$ is IID, (f_{t+1}, θ_{t+1}) is independent of f_t given θ_t. Let $F_\theta(f', \theta')$ be the distribution of (f_{t+1}, θ_{t+1}) given θ_t. The Bellman operator is

$$Tv(f, \theta) = \max \left\{ \int \pi \phi(\mathrm{d}\pi; \theta) - f,\ \beta \int v(f', \theta')\, \mathrm{d}F_\theta(f', \theta') \right\},$$

while the Jovanovic operator is

$$Q\psi(\theta) = \beta \int \max \left\{ \int \pi \phi(\mathrm{d}\pi; \theta') - f',\ \psi(\theta') \right\} \mathrm{d}F_\theta(f', \theta').$$

2.5.4 Research and Development

Firm's R&D decisions are often modeled as a sequential search process for better technologies (see, e.g., Jovanovic and Rob (1989), Bental and Peled (1996), Perla and Tonetti (2014)). Each period, an idea of value s_t is observed, and the firm decides whether to put this idea into productive use, or develop it further by investing in R&D. The former choice yields a payoff $r(s_t, k_t)$, where k_t is the amount of capital input. The latter incurs a fixed cost $c_0 > 0$ (that renders $k_{t+1} = k_t - c_0$) and creates a new technology s_{t+1} next period. Let $\{s_t\} \stackrel{\text{IID}}{\sim} \mu$. Viewed as an optimal stopping problem,

- the state is $z = (s, k)$, and for given map h, the stochastic kernel P satisfies

$$\int h(z')P(z, \mathrm{d}z') = \int h\left(s', k - c_0\right) \mu(\mathrm{d}s'),$$

- the terminal reward is $r(s, k)$ and the flow continuation reward is $c \equiv -c_0$.

This model is also continuation decomposable, as can be seen by letting $X_t := s_t$ and $Y_t := k_t$. Let $F_k(s', k')$ be the distribution of (s_{t+1}, k_{t+1}) given k_t. The Bellman and Jovanovic operators are respectively

$$Tv(s, k) = \max\left\{r(s, k), -c_0 + \beta \int v(s', k') \, \mathrm{d}F_k(s', k')\right\}$$

$$\text{and} \quad Q\psi(s) = -c_0 + \beta \int \max\left\{r(s', k'), \psi(s')\right\} \mathrm{d}F_k(s', k').$$

2.5.5 Real Options

Consider a general financial/real option framework (see, e.g., Dixit and Pindyck (1994), Alvarez and Dixit (2014), and Kellogg (2014)). Let p_t be the current price of a certain financial/real asset and λ_t another state variable. The process $\{\lambda_t\}$ is Φ-Markov and affects $\{p_t\}$ via $p_t = f(\lambda_t, \varepsilon_t)$, where $\{\varepsilon_t\} \stackrel{\text{IID}}{\sim} \mu$ and is independent of $\{\lambda_t\}$. Let K be the strike price of the asset. Each period, the agent decides whether to exercise the option now (i.e., purchase the asset at price K), or wait and reconsider next period. In an optimal stopping format,

- the state is $z = (p, \lambda)$, and for given map h, the stochastic kernel P satisfies

$$\int h(z')P(z, \mathrm{d}z') = \int h\left(f(\lambda', \varepsilon'), \lambda'\right) \mu(\mathrm{d}\varepsilon') \, \Phi(\lambda, \mathrm{d}\lambda'),$$

- the terminal reward (exercise the option now) is $r(p) = (p - K)^+$,
- and the flow continuation reward is $c \equiv 0$.

The model is continuation decomposable, with $X_t := p_t$ and $Y_t := \lambda_t$. Let $F_\lambda(p', \lambda')$ be the distribution of (p_{t+1}, λ_{t+1}) conditional on λ_t. The Bellman and Jovanovic operators are, respectively,

$$Tv(p, \lambda) = \max \left\{ (p - K)^+, \beta \int v(p', \lambda') \, \mathrm{d}F_\lambda(p', \lambda') \right\}$$

$$\text{and} \quad Q\psi(\lambda) = \beta \int \max \left\{ (p' - K)^+, \psi(\lambda') \right\} \mathrm{d}F_\lambda(p', \lambda').$$

2.5.6 Transplants

In health economics, a well-known problem concerns the decision of a surgeon to accept/reject a transplantable organ for the patient (see, e.g., Alagoz et al., 2004). The surgeon aims to maximize the reward of the patient. Each period, she receives an organ offer of quality q_t, where $\{q_t\} \overset{\text{IID}}{\sim} G$. The patient's health h_t evolves according to a H-Markov process if the surgeon rejects the organ. If she accepts this organ for transplant, the operation succeeds with probability $p(q_t, h_t)$, and confers benefit $B(h_t)$ to the patient, while a failed operation results in death. The patient's single period utility when alive is $u(h_t)$. Viewed as an optimal stopping problem,

- the state is $z = (q, h)$, and for a given map f, the stochastic kernel P satisfies
$$\int f(z') P(z, \mathrm{d}z') = \int f(q', h') G(\mathrm{d}q') H(h, \mathrm{d}h'),$$
- the terminal reward (accept the offer) is $r(q, h) = u(h) + p(q, h) B(h)$,
- and the flow continuation reward is $c(h) = u(h)$.

This model is continuation decomposable by letting $X_t := q_t$ and $Y_t := h_t$. Let $F_h(q', h')$ be the distribution of (q_{t+1}, h_{t+1}) given h_t. The Bellman and Jovanovic operators are respectively

$$Tv(q, h) = \max \left\{ u(h) + p(q, h) B(h),\ u(h) + \beta \int v(q', h') \, \mathrm{d}F_h(q', h') \right\}$$

$$\text{and} \quad Q\psi(h) = u(h) + \beta \int \max \left\{ u(h') + p(q', h') B(h'),\ \psi(h') \right\} \mathrm{d}F_h(q', h').$$

Appendix 2.A Some Lemmas

To see the symmetric properties of Q and T from an alternative perspective, we start our analysis with a generic candidate continuation value function space. Let $\mathscr{C}$ be a subset of $m\mathscr{B}$ such that $\psi^* \in \mathscr{C}$ and $Q\mathscr{C} \subset \mathscr{C}$. Let $\mathscr{V}$ be defined by

$$\mathscr{V} := R\mathscr{C} = \{v \in m\mathscr{B} \colon v = r \vee \psi \text{ for some } \psi \in \mathscr{C}\}. \tag{2.A.1}$$

Then R is a surjective map from $\mathscr{C}$ onto $\mathscr{V}$, $Q = LR$ on $\mathscr{C}$ and $T = RL$ on $\mathscr{V}$. The following result parallels the theory of section 2.3.1, and is helpful for deriving important convergence properties once topological structure is added to the generic setting, as to be shown.

Lemma 2.A.1. *The following statements are true:*

(1) *$L\mathscr{V} \subset \mathscr{C}$ and $T\mathscr{V} \subset \mathscr{V}$.*

(2) *If v is a fixed point of T in $\mathscr{V}$, then Lv is a fixed point of Q in $\mathscr{C}$.*

(3) *If ψ is a fixed point of Q in $\mathscr{C}$, then $R\psi$ is a fixed point of T in $\mathscr{V}$.*

(4) *$T^{t+1} = RQ^tL$ on $\mathscr{V}$ and $Q^{t+1} = LT^tR$ on $\mathscr{C}$ for all $t \in \mathbb{N}_0$.*

Proof. The proof is similar to that of propositions 2.3.1–2.3.2 and thus omitted. □

Lemma 2.A.2. *Under assumption 2.3.1, there exist $b_1, b_2 \in \mathbb{R}_+$ such that for all $z \in \mathsf{Z}$,*

(1) $|v^*(z)| \leq \sum_{t=0}^{n-1} \beta^t \mathbb{E}_z[|r(Z_t)| + |c(Z_t)|] + b_1 g(z) + b_2$.

(2) $|\psi^*(z)| \leq \sum_{t=1}^{n-1} \beta^t \mathbb{E}_z|r(Z_t)| + \sum_{t=0}^{n-1} \beta^t \mathbb{E}_z|c(Z_t)| + b_1 g(z) + b_2$.

Proof. Without loss of generality, we assume $m \neq 1$ in assumption 2.3.1. By that assumption, $\mathbb{E}_z|r(Z_n)| \leq a_1 g(z) + a_2$, $\mathbb{E}_z|c(Z_n)| \leq a_3 g(z) + a_4$ and $\mathbb{E}_z g(Z_1) \leq m g(z) + d$ for all $z \in \mathsf{Z}$. For all $t \geq 1$, by the Markov property (see, e.g., Meyn and Tweedie (2009), section 3.4.3),

$$\mathbb{E}_z g(Z_t) = \mathbb{E}_z\left[\mathbb{E}_z\left(g(Z_t)|\mathscr{F}_{t-1}\right)\right] = \mathbb{E}_z\left(\mathbb{E}_{Z_{t-1}} g(Z_1)\right) \leq m\,\mathbb{E}_z g(Z_{t-1}) + d.$$

Induction shows that for all $t \geq 0$,

$$\mathbb{E}_z g(Z_t) \leq m^t g(z) + \frac{1-m^t}{1-m} d. \tag{2.A.2}$$

Moreover, for all $t \geq n$, applying the Markov property again yields

$$\mathbb{E}_z|r(Z_t)| = \mathbb{E}_z\left[\mathbb{E}_z\left(|r(Z_t)||\mathscr{F}_{t-n}\right)\right] = \mathbb{E}_z\left(\mathbb{E}_{Z_{t-n}}|r(Z_n)|\right) \leq a_1\,\mathbb{E}_z g(Z_{t-n}) + a_2.$$

By (2.A.2), for all $t \geq n$, we have

$$\mathbb{E}_z|r(Z_t)| \leq a_1 \left(m^{t-n} g(z) + \frac{1 - m^{t-n}}{1 - m} d \right) + a_2. \tag{2.A.3}$$

Similarly, for all $t \geq n$, we have

$$\mathbb{E}_z|c(Z_t)| \leq a_3 \, \mathbb{E}_z g(Z_{t-n}) + a_4 \leq a_3 \left(m^{t-n} g(z) + \frac{1 - m^{t-n}}{1 - m} d \right) + a_4. \tag{2.A.4}$$

Let $S(z) := \sum_{t \geq 1} \beta^t \mathbb{E}_z \left[|r(Z_t)| + |c(Z_t)| \right]$. Based on (2.A.2)–(2.A.4), we can show that

$$S(z) \leq \sum_{t=1}^{n-1} \beta^t \mathbb{E}_z[|r(Z_t)| + |c(Z_t)|] + \frac{a_1 + a_3}{1 - \beta m} g(z) + \frac{(a_1 + a_3)d + a_2 + a_4}{(1 - \beta m)(1 - \beta)}. \tag{2.A.5}$$

Since $|v^*| \leq |r| + |c| + S$ and $|\psi^*| \leq |c| + S$, the two claims hold by letting $b_1 := \frac{a_1 + a_3}{1 - \beta m}$ and $b_2 := \frac{(a_1 + a_3)d + a_2 + a_4}{(1 - \beta m)(1 - \beta)}$. □

Lemma 2.A.3. *Under assumption 2.3.1, the value function solves the Bellman equation*

$$v^*(z) = \max \left\{ r(z), \, c(z) + \beta \int v^*(z') P(z, dz') \right\} = \max \left\{ r(z), \, \psi^*(z) \right\}.$$

Proof of lemma 2.A.3 (method 1). By theorem 1.11 of Peskir and Shiryaev (2006), it suffices to show that $\mathbb{E}_z \left(\sup_{k \geq 0} \left| \sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k) \right| \right) < \infty$ for all $z \in \mathsf{Z}$. This is true since with probability one we have

$$\sup_{k \geq 0} \left| \sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k) \right| \leq \sum_{t \geq 0} \beta^t [|r(Z_t)| + |c(Z_t)|], \tag{2.A.6}$$

and by the monotone convergence theorem and lemma 2.A.2 (see (2.A.5) in appendix 2.A), the right hand side of (2.A.6) is $\mathbb{P}_z$-integrable for all $z \in \mathsf{Z}$. □

An alternative way of proof of lemma 2.A.3 is provided in appendix 2.E.

Appendix 2.B Main Proofs

Proof of theorem 2.3.1. Let $d_1 := a_1 + a_3$ and $d_2 := a_2 + a_4$. Since $\beta m < 1$ by assumption 2.3.1, we can choose positive constants m' and d' such that

$$m + d_1 m' > 1, \quad \rho := \beta(m + d_1 m') < 1 \quad \text{and} \quad d' \geq (d_2 m' + d)/(m + d_1 m' - 1). \tag{2.B.1}$$

Regarding claim (1), we first show that Q is a contraction mapping on $b_\ell \mathsf{Z}$ with modulus ρ. By the weighted contraction mapping theorem (see, e.g., Boyd (1990), section 3), it suffices to verify: (a) Q is monotone, i.e., $Q\psi \leq Q\phi$ if $\psi, \phi \in b_\ell \mathsf{Z}$ and $\psi \leq \phi$; (b) $Q0 \in b_\ell \mathsf{Z}$ and $Q\psi$ is $\mathscr{B}$-measurable for all $\psi \in b_\ell \mathsf{Z}$; and (c) $Q(\psi + a\ell) \leq Q\psi + a\rho\ell$ for all $a \in \mathbb{R}_+$ and $\psi \in b_\ell \mathsf{Z}$. Obviously, condition (a) holds. By (2.5) and (2.11), we have

$$\frac{|(Q0)(z)|}{\ell(z)} \leq \frac{|c(z)|}{\ell(z)} + \beta \int \frac{|r(z')|}{\ell(z)} P(z, dz') \leq (1 + \beta)/m' < \infty$$

for all $z \in \mathsf{Z}$, so $\|Q0\|_\ell < \infty$. The measurability of $Q\psi$ follows immediately from our primitive assumptions. Hence, condition (b) holds. By the Markov property (see, e.g., Meyn and Tweedie (2009), section 3.4.3), we have

$$\int \mathbb{E}_{z'} |r(Z_t)| P(z, \mathrm{d}z') = \mathbb{E}_z |r(Z_{t+1})| \;\; \text{and} \;\; \int \mathbb{E}_{z'} |c(Z_t)| P(z, \mathrm{d}z') = \mathbb{E}_z |c(Z_{t+1})|.$$

Let $h(z) := \sum_{t=1}^{n-1} \mathbb{E}_z |r(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z |c(Z_t)|$, then we have

$$\int h(z') P(z, \mathrm{d}z') = \sum_{t=2}^{n} \mathbb{E}_z |r(Z_t)| + \sum_{t=1}^{n} \mathbb{E}_z |c(Z_t)|. \tag{2.B.2}$$

By the construction of m' and d', we have $m + d_1 m' > 1$ and $(d_2 m' + d + d')/(m + d_1 m') \leq d'$. Assumption 2.3.1 and (2.B.2) then imply that

$$\begin{aligned}
\int \kappa(z') P(z, dz') &= m' \sum_{t=1}^{n} \mathbb{E}_z [|r(Z_t)| + |c(Z_t)|] + \int g(z') P(z, dz') + d' \\
&\leq m' \sum_{t=1}^{n-1} \mathbb{E}_z [|r(Z_t)| + |c(Z_t)|] + (m + d_1 m') g(z) + d_2 m' + d + d' \\
&\leq (m + d_1 m') \left(\frac{m'}{m + d_1 m'} h(z) + g(z) + d' \right) \\
&\leq (m + d_1 m') \ell(z).
\end{aligned} \tag{2.B.3}$$

Since $\ell \leq \kappa$, this implies that for all $z \in \mathsf{Z}$, we have

$$\int \kappa(z') P(z, \mathrm{d}z') \leq (m + d_1 m') \kappa(z) \quad \text{and} \quad \int \ell(z') P(z, \mathrm{d}z') \leq (m + d_1 m') \ell(z). \tag{2.B.4}$$

Hence, for all $\psi \in b_\ell \mathsf{Z}$, $a \in \mathbb{R}_+$ and $z \in \mathsf{Z}$, we have

$$\begin{aligned} Q(\psi + a\ell)(z) &= c(z) + \beta \int \max\left\{r(z'), \psi(z') + a\ell(z')\right\} P(z, dz') \\ &\leq c(z) + \beta \int \max\left\{r(z'), \psi(z')\right\} P(z, dz') + a\beta \int \ell(z') P(z, dz') \\ &\leq Q\psi(z) + a\beta(m + d_1 m')\ell(z) = Q\psi(z) + a\rho\ell(z). \end{aligned}$$

So condition (c) holds, and $Q : b_\ell \mathsf{Z} \to b_\ell \mathsf{Z}$ is a contraction mapping of modulus ρ.

Moreover, lemma 2.A.3 and the analysis related to (2.4) imply that ψ^* is indeed a fixed point of Q under assumption 2.3.1. Lemma 2.A.2 implies that $\psi^* \in b_\ell \mathsf{Z}$. Hence, ψ^* must coincide with the unique fixed point of Q under $b_\ell \mathsf{Z}$, and claim (1) holds.

The proof of claim (2) is similar. In particular, using (2.B.4) one can show that $T : b_\kappa \mathsf{Z} \to b_\kappa \mathsf{Z}$ is a contraction mapping of the same modulus. We omit the details. □

Proof of proposition 2.3.4. Let $b_\ell c\mathsf{Z}$ be the set of continuous functions in $b_\ell \mathsf{Z}$. Since ℓ is continuous by assumption 2.3.2, $b_\ell c\mathsf{Z}$ is a closed subset of $b_\ell \mathsf{Z}$ (see e.g., Boyd (1990), section 3). To show the continuity of ψ^*, it suffices to verify that $Q(b_\ell c\mathsf{Z}) \subset b_\ell c\mathsf{Z}$ (see, e.g., Stokey et al. (1989), corollary 1 of theorem 3.2). For fixed $\psi \in b_\ell c\mathsf{Z}$, let $h(z) := \max\{r(z), \psi(z)\}$, then there exists $G \in \mathbb{R}_+$ such that $|h(z)| \leq |r(z)| + G\ell(z) =: \tilde{h}(z)$. By assumption 2.3.2, $z \mapsto \tilde{h}(z) \pm h(z)$ are nonnegative and continuous. For all $z \in \mathsf{Z}$ and $\{z_m\} \subset \mathsf{Z}$ with $z_m \to z$, the generalized Fatou's lemma of Feinberg et al. (2014) (theorem 1.1) implies that

$$\int \left(\tilde{h}(z') \pm h(z')\right) P(z, \mathrm{d}z') \leq \liminf_{m \to \infty} \int \left(\tilde{h}(z') \pm h(z')\right) P(z_m, \mathrm{d}z').$$

Since $\lim_{m \to \infty} \int \tilde{h}(z') P(z_m, \mathrm{d}z') = \int \tilde{h}(z') P(z, \mathrm{d}z')$ by assumption 2.3.2, we have

$$\pm \int h(z') P(z, \mathrm{d}z') \leq \liminf_{m \to \infty} \left(\pm \int h(z') P(z_m, \mathrm{d}z') \right),$$

where we have used the fact that for all sequences $\{a_m\}$ and $\{b_m\}$ in $\mathbb{R}$ with $\lim_{m \to \infty} a_m$ exists, we have: $\liminf_{m \to \infty} (a_m + b_m) = \lim_{m \to \infty} a_m + \liminf_{m \to \infty} b_m$. Hence,

$$\limsup_{m \to \infty} \int h(z') P(z_m, \mathrm{d}z') \leq \int h(z') P(z, \mathrm{d}z') \leq \liminf_{m \to \infty} \int h(z') P(z_m, \mathrm{d}z'), \quad (2.B.5)$$

i.e., $z \mapsto \int h(z') P(z, \mathrm{d}z')$ is continuous. Since c is continuous by assumption, $Q\psi \in b_\ell c\mathsf{Z}$. Hence, $Q(b_\ell c\mathsf{Z}) \subset b_\ell c\mathsf{Z}$ and ψ^* is continuous, as was to be shown. The continuity of v^* follows from the continuity of ψ^* and r and the fact that $v^* = r \vee \psi^*$. □

Proof of proposition 2.3.3. Let $\mathscr{C} := b_\ell \mathsf{Z}$ and $\mathcal{V} := b_\kappa \mathsf{Z}$. Let $\mathscr{V} := R\mathscr{C}$ and $\mathcal{C} := L\mathcal{V}$ as defined respectively in (2.A.1) and (2.7). Our first goal is to show that $\mathscr{V} \subset b_\kappa \mathsf{Z}$ and $\mathcal{C} \subset b_\ell \mathsf{Z}$.

For all $v \in \mathscr{V}$, by definition, there exists a $\psi \in \mathscr{C}$ such that $v = R\psi = r \vee \psi$. Since $\mathscr{C} = b_\ell \mathsf{Z}$, we have $|\psi| \leq M\ell$ for some constant $M < \infty$. Without loss of generality, we can let $M > 1/m'$, where m' is defined as in theorem 2.3.1 (see (2.B.1) in the proof of theorem 2.3.1). Hence, $|v| \leq |r| + |\psi| \leq M(m'|r| + \ell) = M\kappa$, i.e., $\|v\|_\kappa < \infty$. Moreover, v is measurable since both r and ψ are. Hence, $v \in b_\kappa \mathsf{Z}$. Since v is arbitrary, we have $\mathscr{V} \subset b_\kappa \mathsf{Z}$.

For all $\psi \in \mathcal{C}$, by definition, there exists $v \in \mathcal{V}$ such that $\psi = Lv = c + \beta P v$. Since $\mathcal{V} = b_\kappa \mathsf{Z}$, we have $|v| \leq M\kappa$ for some constant $M < \infty$. By (2.B.3) in the proof of theorem 2.3.1, $|\psi| \leq |c| + \|v\|_\kappa \ell \leq (1/m' + \|v\|_\kappa)\ell$, i.e., $\|\psi\|_\ell < \infty$. Moreover, ψ is measurable by our primitive assumptions. Hence, $\psi \in b_\ell \mathsf{Z}$. Since ψ is arbitrary, we have $\mathcal{C} \subset b_\ell \mathsf{Z}$.

Regarding claim (1), for all $\psi \in \mathscr{C}$, based on lemma 2.A.1 and theorem 2.3.1, we have

$$\left|Q^{t+1}\psi(z) - \psi^*(z)\right| = \left|LT^t R\psi(z) - Lv^*(z)\right| = \beta \left|P(T^t R\psi)(z) - Pv^*(z)\right|.$$

Since we have shown in the proof of theorem 2.3.1 that $\int \kappa(z')P(z, \mathrm{d}z') \leq (m + m'd_1)\ell(z)$ for all $z \in \mathsf{Z}$ (see equation (2.B.3)), by the definition of operator P, for all $z \in \mathsf{Z}$, we have

$$\begin{aligned}
\left|P(T^t R\psi)(z) - Pv^*(z)\right| &\leq \int \left|(T^t R\psi)(z') - v^*(z')\right| P(z, \mathrm{d}z') \\
&\leq \|T^t R\psi - v^*\|_\kappa \int \kappa(z')P(z, \mathrm{d}z') \\
&\leq (m + m'd_1)\|T^t R\psi - v^*\|_\kappa \, \ell(z).
\end{aligned}$$

Recall $\rho := \beta(m + m'd_1) < 1$ defined in (2.B.1). The above results imply that

$$\|Q^{t+1}\psi - \psi^*\|_\ell \leq \beta(m + m'd_1)\left\|T^t R\psi - v^*\right\|_\kappa = \rho \left\|T^t R\psi - v^*\right\|_\kappa$$

for all $\psi \in \mathscr{C}$. Hence, claim (1) is verified.

Regarding claim (2), for all $v \in \mathcal{V}$, propositions 2.3.1–2.3.2 and theorem 2.3.1 imply that

$$\begin{aligned}
\left|T^{t+1}v(z) - v^*(z)\right| &= \left|(RQ^t L)v(z) - R\psi^*(z)\right| \\
&\leq \left|Q^t Lv(z) - \psi^*(z)\right| \leq \|Q^t Lv - \psi^*\|_\ell \, \ell(z)
\end{aligned}$$

for all $z \in \mathsf{Z}$, where the first inequality is due to the elementary fact that $|a \vee b - c \vee d| \leq |a - c| \vee |b - d|$ for all $a, b, c, d \in \mathbb{R}$. Since $\ell \leq \kappa$ by construction, we have

$$\frac{\left|T^{t+1}v(z) - v^*(z)\right|}{\kappa(z)} \leq \frac{\left|T^{t+1}v(z) - v^*(z)\right|}{\ell(z)} \leq \left\|Q^t Lv - \psi^*\right\|_\ell$$

for all $z \in \mathsf{Z}$. Hence, $\left\|T^{t+1}v - v^*\right\|_\kappa \leq \left\|Q^t Lv - \psi^*\right\|_\ell$ and claim (2) holds. □

Proof of theorem 2.3.2. Since $r, c \in L_q(\pi)$, by the monotonicity of the L_p-norm, we have $r, c \in L_p(\pi)$ for all $1 \leq p \leq q$. Our first goal is to prove claim (1).

Step 1. We show that $Q\psi \in L_p(\pi)$ for all $\psi \in L_p(\pi)$. Notice that for all $z \in \mathsf{Z}$,

$$\begin{aligned}
|Q\psi(z)|^p &\leq 2^p |c(z)|^p + (2\beta)^p \left[\int |r(z')| \vee |\psi(z')| \, P(z, \mathrm{d}z')\right]^p \\
&\leq 2^p |c(z)|^p + (2\beta)^p \int \left[|r(z')| \vee |\psi(z')|\right]^p P(z, \mathrm{d}z') \\
&\leq 2^p |c(z)|^p + (2\beta)^p \left(\int |r(z')|^p \, P(z, \mathrm{d}z') + \int |\psi(z')|^p \, P(z, \mathrm{d}z')\right),
\end{aligned}$$

where for the first and the third inequality we have used the elementary fact that $(a+b)^p \leq 2^p (a \vee b)^p \leq 2^p (a^p + b^p)$ for all $a, b, p \geq 0$, and the second inequality is based on Jensen's inequality. Then we have $\|Q\psi\|_p < \infty$, since the above result implies that

$$\begin{aligned}
\int |Q\psi(z)|^p \, \pi(\mathrm{d}z) &\leq 2^p \int |c(z)|^p \pi(\mathrm{d}z) + (2\beta)^p \int \int |r(z')|^p \, P(z, \mathrm{d}z') \pi(\mathrm{d}z) \\
&\quad + (2\beta)^p \int \int |\psi(z')|^p \, P(z, \mathrm{d}z') \pi(\mathrm{d}z) \\
&= 2^p \int |c(z)|^p \pi(\mathrm{d}z) + (2\beta)^p \int |r(z')|^p \, \pi(\mathrm{d}z') \\
&\quad + (2\beta)^p \int |\psi(z')|^p \, \pi(\mathrm{d}z') \\
&= 2^p \|c\|_p^p + (2\beta)^p \|r\|_p^p + (2\beta)^p \|\psi\|_p^p < \infty,
\end{aligned}$$

where the first equality follows from the Fubini theorem and the fact that π is a stationary distribution. We have thus verified that $Q\psi \in L_p(\pi)$.

Step 2. We show that Q is a contraction mapping on $(L_p(\pi), \|\cdot\|_p)$ of modulus β. For all $\psi, \phi \in L_p(\pi)$, we have

$$\begin{aligned}
|Q\psi(z) - Q\phi(z)|^p &= \beta^p \left|\int \left[r(z') \vee \psi(z') - r(z') \vee \phi(z')\right] P(z, \mathrm{d}z')\right|^p \\
&\leq \beta^p \int |r(z') \vee \psi(z') - r(z') \vee \phi(z')|^p \, P(z, \mathrm{d}z') \\
&\leq \beta^p \int |\psi(z') - \phi(z')|^p \, P(z, \mathrm{d}z'),
\end{aligned}$$

where the first inequality holds by Jensen's inequality, and the second follows from the elementary fact that $|a \vee b - c \vee d| \leq |a - c| \vee |b - d|$ for all $a, b, c, d \in \mathbb{R}$. Hence,

$$\begin{aligned}\int |Q\psi(z) - Q\phi(z)|^p \, \pi(\mathrm{d}z) &\leq \beta^p \int \int |\psi(z') - \phi(z')|^p \, P(z, \mathrm{d}z')\pi(\mathrm{d}z) \\ &= \beta^p \int |\psi(z') - \phi(z')|^p \, \pi(\mathrm{d}z'),\end{aligned}$$

and we have $\|Q\psi - Q\phi\|_p \leq \beta \|\psi - \phi\|_p$. Thus, Q is a contraction on $L_p(\pi)$ of modulus β.

Since $(L_p(\pi), \|\cdot\|_p)$ is a Banach space, based on the contraction mapping theorem, Q admits a unique fixed point in $L_p(\pi)$. In order to prove claim (1), it remains to show that $\psi^* \in L_p(\pi)$ and that ψ^* is a fixed point of Q.

Step 3. We show that $\psi^*, v^* \in L_p(\pi)$. Since $|\psi^*(z)| \vee |v^*(z)| \leq \sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|]$, we have

$$\begin{aligned}&\left[\int |\psi^*(z)|^p \pi(\mathrm{d}z)\right] \vee \left[\int |v^*(z)|^p \pi(\mathrm{d}z)\right] \\ &\leq \int \left(\sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|r(Z_t)| \vee |c(Z_t)|]\right)^p \pi(\mathrm{d}z). \qquad (2.B.6)\end{aligned}$$

Since π is stationary, the Fubini theorem implies that

$$\int \mathbb{E}_z |r(Z_t)|^p \pi(\mathrm{d}z) = \int \int |r(z')|^p P^t(z, \mathrm{d}z') \pi(\mathrm{d}z) = \int |r(z')|^p \pi(\mathrm{d}z') = \|r\|_p^p.$$

Similarly, we have $\int \mathbb{E}_z |c(Z_t)|^p \pi(\mathrm{d}z) = \|c\|_p^p$. Let $\mathbb{E}_{\cdot} f(Z_t)$ denote the function $z \mapsto \mathbb{E}_z f(Z_t)$. By the Minkowski and Jensen inequalities, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}&\left\|\sum_{t=0}^{n} \beta^t \mathbb{E}_{\cdot}[|r(Z_t)| \vee |c(Z_t)|]\right\|_p \leq \sum_{t=0}^{n} \beta^t \left\|\mathbb{E}_{\cdot}[|r(Z_t)| \vee |c(Z_t)|]\right\|_p \\ &\leq \sum_{t=0}^{n} \beta^t \left[\int \mathbb{E}_{\cdot}[|r(Z_t)| \vee |c(Z_t)|]^p \pi(\mathrm{d}z)\right]^{1/p} \\ &\leq \sum_{t=0}^{n} \beta^t \left[\int \left(\mathbb{E}_z |r(Z_t)|^p + \mathbb{E}_z |c(Z_t)|^p\right) \pi(\mathrm{d}z)\right]^{1/p} \\ &= \sum_{t=0}^{n} \beta^t \left(\|r\|_p^p + \|c\|_p^p\right)^{1/p} \leq \frac{\left(\|r\|_p^p + \|c\|_p^p\right)^{1/p}}{1 - \beta} < \infty. \qquad (2.B.7)\end{aligned}$$

Moreover, by the monotone convergence theorem, we have

$$\left\|\sum_{t=0}^{n} \beta^t \mathbb{E}_{\cdot}[|r(Z_t)| \vee |c(Z_t)|]\right\|_p \to \left\|\sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\cdot}[|r(Z_t)| \vee |c(Z_t)|]\right\|_p. \qquad (2.B.8)$$

Together, (2.B.7)–(2.B.8) imply that $\left\|\sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\cdot}[|r(Z_t)| \vee |c(Z_t)|]\right\|_p < \infty$. By (2.B.6), we have $\|\psi^*\|_p \vee \|v^*\|_p < \infty$ and thus $\psi^*, v^* \in L_p(\pi)$.

Step 4. We show that v^* is a fixed point of T and ψ^* is a fixed point of Q, i.e., $\|Tv^* - v^*\|_p = 0$ and $\|Q\psi^* - \psi^*\|_p = 0$. We provide two methods of proof.

- *(Method 1)* It suffices to show that $Tv^* = v^*$ and $Q\psi^* = \psi^*$ π-almost surely. To that end, by theorem 1.11 of Peskir and Shiryaev (2006) and the analysis related to (2.4), it suffices to show that
$$\mathbb{E}_z \left(\sup_{k \geq 0} \left| \sum_{t=0}^{k-1} \beta^t c(Z_t) + \beta^k r(Z_k) \right| \right) < \infty \quad \pi\text{-almost surely.}$$
This obviously holds since the left side is dominated by $\sum_{t=0}^{\infty} \beta^t \mathbb{E}_z [|r(Z_t)| \vee |c(Z_t)|]$, which is finite π-almost surely as step 3 shows that it is an object of $L_p(\pi)$.

- *(Method 2)* Lemma 2.E.1 (see appendix 2.E) shows that $Tv^* = v^*$ π-almost surely, hence $\|Tv^* - v^*\|_p = 0$. Regarding ψ^*, note that
$$\|Q\psi^* - \psi^*\|_p = \|LR\psi^* - Lv^*\|_p = \beta \|P(r \vee \psi^*) - Pv^*\|_p.$$
Since $\psi^* := c + \beta P v^*$ and lemma 2.E.1 implies that $v^* = r \vee (c + \beta P v^*)$ π-almost surely, we have $v^* = r \vee \psi^*$ π-almost surely. Jensen's inequality then implies that
$$\begin{aligned}
&\int |P(r \vee \psi^*)(z) - Pv^*(z)|^p \pi(\mathrm{d}z) \\
&= \int \left| \int [(r \vee \psi^*)(z') - v^*(z')] P(z, \mathrm{d}z') \right|^p \pi(\mathrm{d}z) \\
&\leq \int \int |(r \vee \psi^*)(z') - v^*(z')|^p P(z, \mathrm{d}z') \pi(\mathrm{d}z) \\
&= \int |(r \vee \psi^*)(z') - v^*(z')|^p \pi(\mathrm{d}z') = 0.
\end{aligned}$$
Hence, $\|Q\psi^* - \psi^*\|_p = \beta \|P(r \vee \psi^*) - Pv^*\|_p = 0$. We have thus shown that v^* is a fixed point of T and ψ^* is a fixed point of Q.

Steps 1–4 imply that claim (1) holds. The proof of claim (2) is similar and thus omitted. □

Proof of proposition 2.3.5. Let $\mathscr{C} = \mathcal{V} := L_p(\pi)$, and let $\mathscr{V} := R\mathscr{C}$ and $\mathcal{C} := L\mathcal{V}$ as defined respectively in (2.A.1) and (2.7). Our first goal is to show that $\mathscr{V}, \mathcal{C} \subset L_p(\pi)$.

For all $v \in \mathscr{V}$, there exists a $\psi \in \mathscr{C}$ such that $v = R\psi = r \vee \psi$. Since $\mathscr{C} = L_p(\pi)$ and $r \in L_p(\pi)$ by assumption 2.3.3, we have $v \in L_p(\pi)$. Hence, $\mathscr{V} \subset L_p(\pi)$. For all $\psi \in \mathcal{C}$, there exists $v \in \mathcal{V}$ such that $\psi = Lv = c + \beta P v$. Since $\mathcal{V} = L_p(\pi)$ and π is stationary, Jensen's inequality implies that $Pv \in L_p(\pi)$. Since $c \in L_p(\pi)$, Minkowski's inequality then implies that $\psi \in L_p(\pi)$. Hence, $\mathcal{C} \subset L_p(\pi)$, as claimed.

Regarding claim (1), for all $\psi \in \mathscr{C}$, based on lemma 2.A.1, theorem 2.3.2, Jensen's inequality and Fubini's theorem, we have

$$\begin{aligned}\left\|Q^{t+1}\psi - \psi^*\right\|_p &= \left[\int \left|Q^{t+1}\psi(z) - \psi^*(z)\right|^p \pi(\mathrm{d}z)\right]^{1/p} \\ &= \left[\int \left|LT^t R\psi(z) - Lv^*(z)\right|^p \pi(\mathrm{d}z)\right]^{1/p} \\ &= \beta\left[\int \left|PT^t R\psi(z) - Pv^*(z)\right|^p \pi(\mathrm{d}z)\right]^{1/p} \\ &\leq \beta\left[\int\int \left|T^t R\psi(z') - v^*(z')\right|^p P(z,\mathrm{d}z')\pi(\mathrm{d}z)\right]^{1/p} \\ &= \beta\left[\int \left|T^t R\psi(z') - v^*(z')\right|^p \pi(\mathrm{d}z')\right]^{1/p} = \beta\left\|T^t R\psi - v^*\right\|_p.\end{aligned}$$

Regarding claim (2), for all $v \in \mathcal{V}$, based on propositions 2.3.1–2.3.2 and theorem 2.3.2, we have

$$\begin{aligned}\left\|T^{t+1}v - v^*\right\|_p &= \left[\int \left|T^{t+1}v(z) - v^*(z)\right|^p \pi(\mathrm{d}z)\right]^{1/p} \\ &= \left[\int \left|RQ^t Lv(z) - R\psi^*(z)\right|^p \pi(\mathrm{d}z)\right]^{1/p} \\ &\leq \left[\int \left|Q^t Lv(z) - \psi^*(z)\right|^p \pi(\mathrm{d}z)\right]^{1/p} = \left\|Q^t Lv - \psi^*\right\|_p.\end{aligned}$$

Hence, the second claim holds. This concludes the proof. □

Proof of lemma 2.5.1. Recall that if $X \sim LN(\mu, \sigma^2)$, then $\mathbb{E}\, X^s = \mathrm{e}^{s\mu + s^2\sigma^2/2}$ for all $s \in \mathbb{R}$. By (2.12), the distribution of θ_n given $\theta_0 = \theta$ follows $LN\left(\rho^n \ln\theta,\ \sigma^2 \sum_{i=0}^{n-1} \rho^{2i}\right)$. Hence, $\mathbb{E}_\theta\, \theta_n = \theta^{\rho^n} \exp\left[\frac{\sigma^2(1-\rho^{2n})}{2(1-\rho^2)}\right]$.

Since $w = \eta + \theta\xi$ and $|\ln w| \leq 1/w + w$, we have $|\ln w_n| \leq \eta_n^{-1} + \eta_n + \theta_n\xi_n$. Hence,

$$\mathbb{E}_\theta |\ln w_n| \leq \mathbb{E}_\theta\left(\eta_n^{-1} + \eta_n + \theta_n\xi_n\right) = \mu_1 + \mu_2 + \mu_3\, \mathbb{E}_\theta \theta_n = A_n \theta^{\rho^n} + B, \quad (2.B.9)$$

where μ_1, μ_2 and μ_3 are respectively the mean of η_n^{-1}, η_n and ξ_n, $B := \mu_1 + \mu_2$ and $A_n := \mu_3 \exp\left[\frac{\sigma^2(1-\rho^{2n})}{2(1-\rho^2)}\right]$. Claim (a) is verified.

To verify claim (b), consider a sequence $\theta^{(m)} \to \theta$. By the Fatou's lemma,

$$\mathbb{E}_\theta\left(\eta_n^{-1} + \eta_n + \theta_n\xi_n \pm |\ln w_n|\right) \leq \liminf_m \mathbb{E}_{\theta^{(m)}}\left(\eta_n^{-1} + \eta_n + \theta_n\xi_n \pm |\ln w_n|\right).$$

Since $\theta \mapsto \mathbb{E}_{\theta}\left(\eta_n^{-1} + \eta_n + \theta_n \xi_n\right)$ is continuous by (2.B.9), the above inequality yields

$$\pm\mathbb{E}_{\theta} |\ln w_n| \leq \liminf_{m} \left(\pm\mathbb{E}_{\theta(m)} |\ln w_n|\right),$$

i.e., $\lim_m \mathbb{E}_{\theta(m)} |\ln w_n| = \mathbb{E}_{\theta} |\ln w_n|$. Hence, $\theta \mapsto \mathbb{E}_{\theta} |\ln w_n|$ is continuous, as claimed. $\square$

Proof of lemma 2.5.2. Since $w = \eta + \theta\xi$ and $|\ln w| \leq w + 1/w$, we have $|\ln w|^q \leq 3^q(\eta^{-q} + \eta^q + \theta^q \xi^q)$. By the assumption on $\{\eta_t\}$ and $\{\xi_t\}$, taking expectation (w.r.t π) yields

$$\mathbb{E}\,|\ln w|^q \leq 3^q(\mathbb{E}\,\eta^{-q} + \mathbb{E}\,\eta^q + \mathbb{E}\,\xi^q\,\mathbb{E}\,\theta^q) < \infty.$$

Since $r(w) = \ln w/(1-\beta)$, this inequality implies $r \in L_q(\pi)$. Moreover, $c \in L_q(\pi)$ is trivial since c is constant. $\square$

Appendix 2.C Proof of Time Complexity

To prove the results of table 2.1, we introduce some elementary facts on time complexity:

(a) The multiplication of an $m \times n$ matrix and an $n \times p$ matrix has complexity $\mathcal{O}(mnp)$. See, for example, section 2.5.4 of Skiena (2008).

(b) The binary search algorithm finds the index of an element in a given sorted array of length n in $\mathcal{O}(\log(n))$ time. See, for example, section 4.9 of Skiena (2008).

For finite space (FS) approximation, time complexity of VFI (1-loop) reduces to the complexity of matrix multiplication $\hat{P}\vec{v}$, which is of order $\mathcal{O}(K^2M^2)$ based on the shape of $\hat{P}$ and $\vec{v}$ and fact (a) above. Similarly, time complexity of CVI (1-loop) is determined by $\hat{F}(\vec{r} \vee \vec{\psi})$, which has complexity $\mathcal{O}(KM^2)$. The n-loop complexity is scaled up by $\mathcal{O}(n)$.

For the infinite space (IS) case, let $\mathcal{O}(g)$ and $\mathcal{O}(h)$ denote respectively the complexity (of single point evaluation) of the interpolating functions g and h. Counting the floating point operations associated with all grid points inside the inner loops shows that the one step complexities of VFI and CVI are $\mathcal{O}(NKM)\mathcal{O}(g)$ and $\mathcal{O}(NM)\mathcal{O}(h)$, respectively. Since binary search function evaluation is adopted for the indicated function interpolation mechanisms (see table 2.1 note), and in particular, since evaluating g at a given point uses binary search $k+n$ times, based on fact (b) above, we have

$$\mathcal{O}(g) = \mathcal{O}\left(\sum_{i=1}^{k} \log(K_i) + \sum_{j=1}^{n} \log(M_j)\right) = \mathcal{O}(\log(KM)).$$

Similarly, we can show that $\mathcal{O}(h) = \mathcal{O}(\log(M))$. Combining these results, we see that the claims of the IS case hold, concluding our proof of table 2.1 results.

Appendix 2.D Continuation Nontriviality

In this section, we show that under certain general assumptions, the number of state variables that the continuation value function must track is never larger than the equivalent number for the value function.

For a generic optimal stopping problem, a *policy* is a map $\sigma \colon \mathsf{Z} \to \{0,1\}$, with 0 indicating the decision to continue and 1 indicating the decision to stop. A policy σ is called *optimal* if $\tau^* := \inf\{t \geq 0 \mid \sigma(Z_t) = 1\}$ is an optimal stopping time (recall section 2.2.2). Clearly, σ^* defined as follows is an optimal policy in our set up:

$$\sigma^*(z) := \mathbb{1}\{r(z) \geq \psi^*(z)\} \qquad (z \in \mathsf{Z}).$$

Notably, without further assumptions, we can construct a setting in which the domain of v^* is of strictly lower dimension than the domain of ψ^*, as illustrated by the following example.

Example 2.D.1 (A Heuristic Counterexample)**.** Let (x, y) be the state vector and M be a positive constant. We consider the following optimal stopping problem:

- the stochastic kernel is represented by the probability density $f(x'|x,y)$ with

$$f(x'|x,y) = h(x'|x,y) \text{ if } x \neq x_0 \quad \text{and} \quad f(x'|x,y) = g(x'|x) \text{ if } x = x_0,$$

- the terminal reward satisfies $0 \leq r(x) < M$, and

- the flow continuation reward satisfies

$$c(x) = -\frac{M}{1-\beta} \text{ if } x \neq x_0 \quad \text{and} \quad c(x) = M \text{ if } x = x_0.$$

Under this set up, we have $0 \leq v^*(x) \leq \frac{M}{1-\beta}$. Furthermore,

(1) if $x \neq x_0$, then: $v^*(x) = r(x) > \psi^*(x,y)$ since

$$\psi^*(x,y) = \frac{-M}{1-\beta} + \beta \int v^*(x') h(x'|x,y) \, \mathrm{d}x' \leq \frac{-M}{1-\beta} + \frac{\beta M}{1-\beta} < 0 < r(x);$$

(2) if $x = x_0$, then: ψ^* is a function of x only, and $v^*(x) = \psi^*(x) > r(x)$ since

$$\psi^*(x) = M + \beta \int v^*(x') g(x'|x) \, \mathrm{d}x' \geq M > r(x).$$

Hence, the value function follows

$$\begin{aligned} v^*(x) &= \max\{r(x),\, \psi^*(x,y)\} \\ &= \max\left\{r(x),\, c(x) + \beta \int v^*(x') f(x'|x,y)\, \mathrm{d}x'\right\}, \end{aligned}$$

and the dimensionality of $\mathbf{domain}(\psi^*)$ is strictly higher than that of $\mathbf{domain}(v^*)$. Intuitively, ψ^* is invariant with respect to y at $x = x_0$ (the only point at which ψ^* dominates r), though it is a function of both x and y elsewhere.

Of course, example 2.D.1 above is constructed artificially and has no much economic intuition. However, it clearly indicates the kind of scenarios that might be ignored innocuously when we study an optimal stopping problem. First of all, in example 2.D.1, the state variable y has no effect on the optimal policy, i.e., σ^* is not a function of y. Second, exiting is always the optimal choice expect at the single point $x = x_0$. The following definition rules out these trivial scenarios.

Let E denote the set of state values $z \in \mathsf{Z}$ at which $r > \psi^*$. We call an optimal stopping problem *continuation nontrivial* if[11]

(a) E has non-empty interior, and on E, r is not everywhere constant with respect to each of its arguments.

(b) σ^* is not everywhere constant with respect to each argument of ψ^* that is not an argument of r.

Most applications of interest to economists are continuation nontrivial. If both r and ψ^* are continuous, for example, then E has non-empty interior if and only if there exists a point $z_0 \in \mathsf{Z}$ at which $r > \psi^*$. Intuitively, condition (b) means that any state variable that affects the continuation value but not the terminal reward can also affect the optimal policy. These conditions can be established for all the examples provided in section 2.5.

For continuation nontrivial problems, the effective state space for the continuation value function is never larger than that of the value function:

Proposition 2.D.1. *If an optimal stopping problem is continuation nontrivial, then*

(1) the domain of v^* is of (weakly) higher dimension than the domain of ψ^*, and

(2) v^* is a function of all the state variables.

[11] For a given function $\mathbb{R}^n \ni (x_1, \cdots, x_n) \mapsto f(x_1, \cdots, x_n) \in \mathbb{R}$, we say that f is *not everywhere constant* with respect to x_i if there exist $x_i', x_i'' \in \mathbb{R}$ such that $f(x_1, \cdots, x_{i-1}, x_i', x_{i+1}, \cdots, x_n) \neq f(x_1, \cdots, x_{i-1}, x_i'', x_{i+1}, \cdots, x_n)$ for some $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \mathbb{R}^{n-1}$.

Proof of proposition 2.D.1. Regarding claim (1), since the problem is continuation nontrivial, condition (a) implies that E has non-empty interior and r is not everywhere constant on E with respect to each of its arguments. Since in addition $v^* = r$ on E, any state variable that affects r must also affect v^*.

Let y be an arbitrary state variable that affects ψ^* but not r (if there is no such state variable then the proof is done). Since the problem is continuation nontrivial, condition (b) implies that there exist y_1, y_2 and $\mathbf{z_{-1}}$ such that $\sigma^*(y_1, \cdot) = 1$ and $\sigma^*(y_2, \cdot) = 0$ at $\mathbf{z_{-1}}$, where $\mathbf{z_{-1}}$ denotes a realization of the vector of all state variables excluding y. Hence, $r(\cdot) \geq \psi^*(y_1, \cdot)$ and $r(\cdot) < \psi^*(y_2, \cdot)$ at $\mathbf{z_{-1}}$. Since y has no effect on r, we have $v^*(y_1, \cdot) = r(\cdot) < \psi^*(y_2, \cdot) = v^*(y_2, \cdot)$ at $\mathbf{z_{-1}}$. This implies that y is an argument of v^*. Since y is chosen arbitrarily, we know that any state variable that affects ψ^* must also affect v^*. Hence, claim (1) holds.

Regarding claim (2), since a state variable must affect either r or ψ^* (or both), and we have shown above that any state variable that affects r or ψ^* also affects v^*, we know that v^* must be a function of all the state variables. Claim (2) is verified. $\square$

Appendix 2.E More on Principle of Optimality

In this section, we look at the principle optimality from an alternative perspective. In particular, we show that under assumption 2.3.1, the value function solves the Bellman equation (i.e., (2.2) holds), while under assumption 2.3.3, the value function solves the Bellman equation π-almost surely (i.e., (2.2) holds π-almost surely).

To see the problem from an alternative perspective, suppose that, at date t, an agent is either *active* or *passive*. When active, the agent observes Z_t and chooses whether to continue or exit. Continuation results in a current reward $c(Z_t)$ and the agent remains active at $t+1$. Exit results in a terminal reward $r(Z_t)$ and transition to the passive state. From there the agent has no action available.

We introduce another state process $\{I_t\}_{t\geq 0}$ to indicate the current status of the agent, and a control process $\{J_t\}_{t\geq 0}$ to indicate the agent's action. In particular,

$$I_t = \begin{cases} 0, & \text{if passive} \\ 1, & \text{if active} \end{cases} \quad \text{and} \quad J_t = \begin{cases} 0, & \text{if exit} \\ 1, & \text{if continue} \end{cases}$$

Then the value function of the problem is given by

$$V^*(z_0, i_0) := \sup_{\{J_t\}_{t\geq 0}} \mathbb{E}_{z_0, i_0} \left\{ \sum_{t=0}^{\infty} \beta^t F(Z_t, I_t, J_t) \right\}, \tag{2.E.1}$$

where the reward function F is

$$F(z, i, j) := r(z) \cdot \mathbb{1}(i = 1, j = 0) + c(z) \cdot \mathbb{1}(i = 1, j = 1).$$

To see the connection with the set up of section 2.2.2, notice that

$$V^*(\cdot\,, 0) = 0 \quad \text{and} \quad V^*(\cdot\,, 1) = v^*. \tag{2.E.2}$$

Moreover, let $\tilde{P}$ be the stochastic kernel related to the state process $\{(Z_t, I_t)\}_{t\geq 0}$. The Bellman equation corresponding to the problem stated in (2.E.1) is

$$V(z_0, i_0) = \max_{j_0 \in \{0,1\}} \left\{ F(z_0, i_0, j_0) + \beta \int V(z_1, i_1) \tilde{P}((z_0, i_0), j_0; \mathrm{d}(z_1, i_1)) \right\}. \tag{2.E.3}$$

Equivalently, $V(z_0, 0) = 0$ and[12]

$$V(z_0, 1) = \max \left\{ r(z_0), c(z_0) + \beta \int V(z_1, 1) P(z_0, \mathrm{d}z_1) \right\}. \tag{2.E.4}$$

Note that (2.E.4) is the functional equation corresponding to (2.2). Hence, to show that (2.2) holds (pointwise/π-almost surely), it suffices to show that V^* defined in (2.E.2) solves (2.E.3) (pointwise/π-almost surely).

[12]Note that if $I_t = 0$, then $F(Z_t, I_t, J_t) = 0$. Since the agent is already in the passive state, there is no action available, and $I_{t'} = 0$ with probability one for all $t' \geq t$. Hence, $V(z_0, 0) = 0$.

An alternative proof of lemma 2.A.3. Under assumption 2.3.1, the Bellman equation (2.E.3) is well-defined. To see this, let $\tilde{\psi}$ be the unique fixed point of Q under $b_\ell \mathsf{Z}$ obtained from claim (1) of theorem 2.3.1. Then $\tilde{V}$ defined below solves (2.E.3):

$$\tilde{V}(\cdot, 0) := 0 \quad \text{and} \quad \tilde{V}(\cdot, 1) := r \vee \tilde{\psi} \in b_\kappa \mathsf{Z}.$$

To prove the stated claim, it remains to show that any solution V to the Bellman equation (2.E.3) with $V(\cdot, 1) \in b_\kappa \mathsf{Z}$ satisfies $V = V^*$. Note that for all feasible plan $\{j_t\}_{t \geq 0}$ and $K \in \mathbb{N}$, we have

$$\begin{aligned}
V(z_0, i_0) &\geq F(z_0, i_0, j_0) + \beta \int V(z_1, i_1) \tilde{P}((z_0, i_0), j_0; \mathrm{d}(z_1, i_1)) \\
&\geq F(z_0, i_0, j_0) + \\
&\quad \beta \int \left[F(z_1, i_1, j_1) + \beta \int V(z_2, i_2) \tilde{P}((z_1, i_1), j_1; \mathrm{d}(z_2, i_2)) \right] \tilde{P}((z_0, i_0), j_0; \mathrm{d}(z_1, i_1)) \\
&= F(z_0, i_0, j_0) + \beta \mathbb{E}_{z_0, i_0}^{j_0} F(Z_1, I_1, j_1) + \beta^2 \mathbb{E}_{z_0, i_0}^{j_0, j_1} V(Z_2, I_2) \geq \cdots \\
&\geq \sum_{t=0}^{K} \beta^t \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{t-1}} F(Z_t, I_t, j_t) + \beta^{K+1} \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_K} V(Z_{K+1}, I_{K+1}).
\end{aligned} \tag{2.E.5}$$

Since $V(\cdot, 0) \equiv 0$ and $V(\cdot, 1) \in b_\kappa \mathsf{Z}$, there exists $G \in \mathbb{R}_+$ such that $|V| \leq G\kappa$. The Markov property then implies that

$$\begin{aligned}
&\left| \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{K-1}} V(Z_K, I_K) \right| \leq G \, \mathbb{E}_{z_0} \kappa(Z_K) \\
&= G \left(m' \sum_{t=0}^{n-1} \mathbb{E}_{z_0} [|s(Z_{t+K})| + |c(Z_{t+K})|] + \mathbb{E}_{z_0} g(Z_K) + d' \right).
\end{aligned} \tag{2.E.6}$$

From (2.A.2)–(2.A.4) (the proof of lemma 2.A.2) we know that, for all $z_0 \in \mathsf{Z}$ and $t \in \mathbb{N}_0$,

$$\mathbb{E}_{z_0} g(Z_t) \leq m^t g(z_0) + \frac{1 - m^t}{1 - m} d,$$

and for all $z_0 \in \mathsf{Z}$ and $t \geq n$ (recall that $d_1 := a_1 + a_3$ and $d_2 := a_2 + a_4$),

$$\max\{\mathbb{E}_{z_0} |r(Z_t)|, \mathbb{E}_{z_0} |c(Z_t)|\} \leq d_1 \left(m^{t-n} g(z_0) + \frac{1 - m^{t-n}}{1 - m} d \right) + d_2.$$

Substituting these results into (2.E.6), we can show that, for all $(z_0, i_0) \in \mathsf{Z} \times \{0, 1\}$,

$$\begin{aligned}
\left| \beta^K \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{K-1}} V(Z_K, I_K) \right| &\leq 2 G d_1 m' \sum_{t=0}^{n-1} \left[(\beta m)^K m^{t-n} g(z_0) + \beta^K \frac{1 - m^{K+t-n}}{1 - m} d \right] \\
&\quad + G \left[(\beta m)^K g(z_0) + \beta^K \frac{1 - m^K}{1 - m} d \right] + \beta^K G (d' + 2 n d_2 m').
\end{aligned}$$

Since $\beta m < 1$, this implies that $\lim_{K \to \infty} \beta^K \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{K-1}} V(Z_K, I_K) = 0$ for all $(z_0, i_0) \in \mathsf{Z} \times \{0, 1\}$. Let $K \to \infty$, then (2.E.5) implies that, for all $\{j_t\}_{t \geq 0}$,

$$V(z_0, i_0) \geq \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{t-1}} F(Z_t, I_t, j_t).$$

Hence, $V \geq V^*$. Notice that since (2.E.3) is a binary choice problem, there exists a plan $\{\tilde{j}_t\}_{t\geq 0}$ such that (2.E.5) holds with equality in each step, which implies $V \leq V^*$. Hence, $V = V^*$, as was to be shown. This concludes the proof. □

Lemma 2.E.1. *Under assumption 2.3.3, v^* satisfies (2.2) π-almost surely.*

Proof. Under assumption 2.3.3, the Bellman equation (2.E.3) is well-defined (π-almost surely). To see this, let $\tilde{\psi}$ be the unique fixed point of Q under $(L_q(\pi), \|\cdot\|_q)$ obtained from claim (1) of theorem 2.3.2. Then $\tilde{V}$ defined by

$$\tilde{V}(\cdot, 0) := 0 \quad \text{and} \quad \tilde{V}(\cdot, 1) := r \vee \tilde{\psi} \in L_q(\pi)$$

solves (2.E.3) π-almost surely.

To prove the stated claim, it remains to show that any solution V to the Bellman equation (2.E.3) with $V(\cdot, 1) \in L_q(\pi)$ satisfies $V = V^*$ π-almost surely. For all feasible plan $\{j_t\}_{t\geq 0}$ and $(z_0, i_0) \in \mathsf{Z} \times \{0, 1\}$, since $V(\cdot, 0) \equiv 0$, we have

$$\left| \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{K-1}} V(Z_K, I_K) \right| \leq \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{K-1}} |V(Z_K, I_K)| \leq \mathbb{E}_{z_0} |V(Z_K, 1)|.$$

Hence, for all $(z_0, i_0) \in \mathsf{Z} \times \{0, 1\}$,

$$\sup_{\{j_t\}_{t\geq 0}} \left| \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{K-1}} V(Z_K, I_K) \right| \leq \mathbb{E}_{z_0} |V(Z_K, 1)|. \tag{2.E.7}$$

Since π is stationary, Jensen's inequality yields

$$\begin{aligned}
&\int \left[\mathbb{E}_{z_0} |V(Z_K, 1)|\right]^q \pi(\mathrm{d}z_0) = \int \left[\int |V(z', 1)| P^K(z, \mathrm{d}z') \right]^q \pi(\mathrm{d}z) \\
&\leq \int \int |V(z', 1)|^q P^K(z, \mathrm{d}z') \pi(\mathrm{d}z) = \int |V(z', 1)|^q \pi(\mathrm{d}z') = \|V(\cdot, 1)\|_q^q < \infty.
\end{aligned}$$

Let $\mathbb{E}_{\cdot} f(Z_t)$ denote the function $z \mapsto \mathbb{E}_z f(Z_t)$. The Minkowski inequality then implies that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\left\| \sum_{t=0}^{n} \beta^t \mathbb{E}_{\cdot} |V(Z_t, 1)| \right\|_q &\leq \sum_{t=0}^{n} \beta^t \left\| \mathbb{E}_{\cdot} |V(Z_t, 1)| \right\|_q \\
&= \sum_{t=0}^{n} \beta^t \left[\int \left[\mathbb{E}_z |V(Z_t, 1)|\right]^q \pi(\mathrm{d}z) \right]^{1/q} \\
&\leq \sum_{t=0}^{n} \beta^t \|V(\cdot, 1)\|_q \leq \sum_{t=0}^{\infty} \beta^t \|V(\cdot, 1)\|_q < \infty.
\end{aligned}$$

Moreover, by the monotone convergence theorem,

$$\left\| \sum_{t=0}^{n} \beta^t \mathbb{E}_{\cdot} |V(Z_t, 1)| \right\|_q \to \left\| \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\cdot} |V(Z_t, 1)| \right\|_q.$$

Together, we have $\left\|\sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\cdot}|V(Z_t, 1)|\right\|_q < \infty$, which implies that

$$\lim_{K\to\infty} \beta^K \mathbb{E}_{z_0}|V(Z_K, 1)| = 0 \quad \pi\text{-almost surely.}$$

Then, by (2.E.7), for all $i_0 \in \{0,1\}$,

$$\lim_{K\to\infty} \left[\sup_{\{j_t\}_{t\geq 0}} \left| \beta^{K+1} \mathbb{E}_{z_0,i_0}^{j_0\cdots j_K} V(Z_{K+1}, I_{K+1}) \right| \right] = 0 \quad \pi\text{-almost surely.}$$

Hence,

$$\sup_{\{j_t\}_{t\geq 0}} \left[\lim_{K\to\infty} \left| \beta^{K+1} \mathbb{E}_{z_0,i_0}^{j_0\cdots j_K} V(Z_{K+1}, I_{K+1}) \right| \right] = 0 \quad \pi\text{-almost surely.} \tag{2.E.8}$$

For all feasible plan $\{j_t\}_{t\geq 0}$, $i_0 \in \{0,1\}$ and $K \in \mathbb{N}$, (2.E.5) implies that

$$V(z_0, i_0) \geq \sum_{t=0}^{K} \beta^t \mathbb{E}_{z_0,i_0}^{j_0\cdots j_{t-1}} F(Z_t, I_t, j_t) + \beta^{K+1} \mathbb{E}_{z_0,i_0}^{j_0\cdots j_K} V(Z_{K+1}, I_{K+1}). \tag{2.E.9}$$

Letting $K \to \infty$ and taking supremum with respect to $\{j_t\}_{t\geq 0}$ yield

$$\begin{aligned} V(z_0, i_0) \geq & \sup_{\{j_t\}_{t\geq 0}} \lim_{K\to\infty} \sum_{t=0}^{K} \beta^t \mathbb{E}_{z_0,i_0}^{j_0\cdots j_{t-1}} F(Z_t, I_t, j_t) \\ & + \sup_{\{j_t\}_{t\geq 0}} \lim_{K\to\infty} \beta^{K+1} \mathbb{E}_{z_0,i_0}^{j_0\cdots j_K} V(Z_{K+1}, I_{K+1}). \end{aligned} \tag{2.E.10}$$

Together, (2.E.8) and (2.E.10) imply that $V \geq V^*$ π-almost surely.

Since (2.E.3) is a binary choice problem, there exists a plan $\{\tilde{j}_t\}_{t\geq 0}$ such that (2.E.9) holds with equality for all $K \in \mathbb{N}$. (2.E.8) then implies that $V \leq V^*$ π-almost surely. In summary, we have $V = V^*$ π-almost surely. This concludes the proof.

□

Chapter 3

Extension I: Repeated Optimal Stopping

3.1 Introduction

In the standard optimal stopping framework of chapter 2, the agent aims to find an optimal stopping time that terminates the sequential decision process permanently. However, in many problems of interest, the choice to stop is only temporary. Typically, agents have chances to return to the sequential decision problem once they have terminated.

To see a standard example, in the job search problem, after the agent accepts an offer and gets employed (i.e., terminates the sequential decision process), the resulting job contract could be cancelled later on, in which case the agent is unemployed again and has to search for a new job on the market. See, for example, Rendon (2006), Ljungqvist and Sargent (2008), Poschke (2010), Chatterjee and Rossi-Hansberg (2012), Lise (2013), Moscarini and Postel-Vinay (2013), and Bagger et al. (2014).

Another related example is sovereign default, in which case a country decides whether to default on an international debt (termination) or not (continuation). Default leads to a period of exclusion from international financial markets. The exclusion is not permanent, however. With positive probability, the country exits autarky and begins borrowing from international markets again. See, for example, Choi et al. (2003), Albuquerque and Hopenhayn (2004), Arellano (2008), Alfaro and Kanczuk (2009), Arellano and Ramanarayanan (2012), Bai and Zhang (2012), Chatterjee and Eyigungor (2012), Mendoza and Yue (2012), and Hatchondo et al. (2016).

In this chapter, we extend the theory of chapter 2 to address this kind of problem, referred to below as the repeated optimal stopping problem. In particular, section 3.2 sets up the problem, longer proofs are provided in the appendix, while the rest of the chapter is structurized as follows:

Section 3.3 shows that the Bellman and Jovanovic operators are semiconjugate in a generic setting. The implications are the same as those of chapter 2. Topological structure is then incorporated to the generic setting. Section 3.4 considers a general weighted supremum norm topology, while section 3.5 considers the L_p-norm topology, both of which allow for bounded or unbounded rewards. Based on the general theory of section 3.3, we show that the Bellman operator and Jovanovic operator are both contraction mappings under identical assumptions, and that convergence to the respective fixed points occurs at the same rate.

For weighted supremum norm topology, our goal is achieved by constructing a metric that evaluates the maximum between the weighted supremum norm distance of candidate continuation value functions and the same type of distance for candidate terminal value functions.[1] For the case of L_p-norm topology, the metric is constructed based on the L_p-norm of the maximum between the relative distance of candidate continuation values and the relative distance of candidate terminal values.

Notably, although in the repeated optimal stopping framework the state process is a controlled Markov process (i.e., the evolution of the state process is affected at least partially by a control variable) and complicates the dynamics somewhat, the key state component (i.e., the state variables that appear in the reward functions) evolves as a standard Markov process. We show that, due to this reason, the structures and results of the previous chapter carry over at no additional cost.

3.2 Repeated Optimal Stopping

Unless otherwise specified, we continue to use the notation of chapter 2 throughout this chapter. Recall that for all integrable function $h \in m\mathscr{B}$,

$$Ph(z) := \int h(z')P(z, \mathrm{d}z').$$

[1] In this chapter, *terminal value function* refers to the maximal expected discounted lifetime rewards from choosing termination, while *terminal reward function* refers to the single period reward obtained from terminating. Formal definitions are provided in section 3.2.

To put our problem in a general setting, suppose that, at date t, an agent is either *active* or *passive*. When active, the agent observes Z_t and chooses whether to continue or exit. Continuation results in a *flow continuation reward* $c(Z_t)$ and the agent remains active at $t+1$. Exit results in a *terminal reward* $s(Z_t)$ and transition to the passive state. From there the agent has no action available, but will return to the active state at $t+1$ and all subsequent periods with probability α. The agent aims to maximize the expected discounted lifetime rewards.

Let $\{I_t\}_{t\geq 0}$ and $\{J_t\}_{t\geq 0}$ denote respectively the agent's status and action in each period, where

$$I_t = \begin{cases} 0, & \text{if passive} \\ 1, & \text{if active} \end{cases} \quad \text{and} \quad J_t = \begin{cases} 0, & \text{if exit} \\ 1, & \text{if continue} \end{cases}$$

Then the state process is $\{(Z_t, I_t)\}_{t\geq 0}$ and the state space is $\mathsf{Z} \times \{0,1\}$. The value function of the problem is given by

$$V^*(z_0, i_0) := \sup_{\{J_t\}_{t\geq 0}} \mathbb{E}_{z_0, i_0} \left\{ \sum_{t=0}^{\infty} \beta^t F(Z_t, I_t, J_t) \right\}, \tag{3.2.1}$$

where $V^*(\cdot, 0) =: r^*$ and $V^*(\cdot, 1) =: v^*$ are respectively the maximal expected discounted lifetime rewards in the passive and active states, and F is the reward function defined by

$$F(z, i, j) := s(z) \cdot \mathbb{1}(j = 0) + c(z) \cdot \mathbb{1}(i = 1, j = 1).$$

We call r^* the *terminal value function*. Under certain assumptions, v^* and r^* satisfy[2]

$$v^*(z) = \max \left\{ r^*(z), c(z) + \beta \int v^*(z') P(z, \mathrm{d}z') \right\} \tag{3.2.2}$$

and

$$r^*(z) = s(z) + \alpha\beta \int v^*(z') P(z, \mathrm{d}z') + (1-\alpha)\beta \int r^*(z') P(z, \mathrm{d}z'). \tag{3.2.3}$$

Given v and r, we define

$$\ell(z; v, r) := s(z) + \alpha\beta \int v(z') P(z, \mathrm{d}z') + (1-\alpha)\beta \int r(z') P(z, \mathrm{d}z'). \tag{3.2.4}$$

The corresponding Bellman operator is

$$T \begin{pmatrix} v \\ r \end{pmatrix} (z) = \begin{pmatrix} \max\{\ell(z; v, r),\, c(z) + \beta \int v(z') P(z, \mathrm{d}z')\} \\ \ell(z; v, r) \end{pmatrix}.$$

[2]Later we provide sufficient conditions based on our primitive set up. See, for example, lemmas 3.A.3–3.A.4 in appendix 3.A.

The *continuation value function* associated with this problem is defined at $z \in \mathsf{Z}$ by

$$\psi^*(z) := c(z) + \beta \int v^*(z')P(z, \mathrm{d}z'). \tag{3.2.5}$$

By (3.2.2), (3.2.3) and (3.2.5), ψ^* and r^* satisfy

$$\psi^* = c + \beta P(r^* \vee \psi^*) \quad \text{and} \quad r^* = s + \alpha\beta P(r^* \vee \psi^*) + (1-\alpha)\beta P r^*. \tag{3.2.6}$$

Now we define the *continuation value operator* or *Jovanovic operator* Q by

$$Q \begin{pmatrix} \psi \\ r \end{pmatrix} (z) = \begin{pmatrix} c(z) + \beta \int (r \vee \psi)(z')P(z, \mathrm{d}z') \\ s(z) + \alpha\beta \int (r \vee \psi)(z')P(z, \mathrm{d}z') + (1-\alpha)\beta \int r(z')P(z, \mathrm{d}z') \end{pmatrix}.$$

3.3 General Theory

In this section we show that Bellman and Jovanovic operators are semiconjugate in the repeated optimal stopping framework and discuss the implications. As in chapter 2, we use operator-theoretic notation. Observe that the Bellman operator T can be expressed as $T = RL$, where for each (ψ, r) and (v, r),

$$R \begin{pmatrix} \psi \\ r \end{pmatrix} := \begin{pmatrix} r \vee \psi \\ r \end{pmatrix} \quad \text{and} \quad L \begin{pmatrix} v \\ r \end{pmatrix} := \begin{pmatrix} c + \beta P v \\ s + \alpha\beta P v + (1-\alpha)\beta P r \end{pmatrix}. \tag{3.3.1}$$

(Recall that for any two operators we write the composition $A \circ B$ simply as AB.)

Let $\mathcal{V}$ be a subset of $m\mathscr{B} \times m\mathscr{B}$ such that $(v^*, r^*) \in \mathcal{V}$ and $T\mathcal{V} \subset \mathcal{V}$. The set $\mathcal{V}$ generalizes the candidate value function space of chapter 2 in the sense that it contains an additional candidate terminal value function space. (Specific classes of functions are considered in the next two sections.) Let $\mathcal{C}$ be defined by

$$\mathcal{C} := L\mathcal{V} = \left\{ \begin{pmatrix} \psi \\ r \end{pmatrix} \in m\mathscr{B} \times m\mathscr{B} \colon \begin{pmatrix} \psi \\ r \end{pmatrix} = L \begin{pmatrix} v \\ r \end{pmatrix} \text{ for some } \begin{pmatrix} v \\ r \end{pmatrix} \in \mathcal{V} \right\}.$$

By definition, L is a surjective mapping from $\mathcal{V}$ onto $\mathcal{C}$. It is also true that R maps $\mathcal{C}$ into $\mathcal{V}$. Indeed, if $(\psi, r) \in \mathcal{C}$, then there exists a $(v, r) \in \mathcal{V}$ such that $(\psi, r) = L(v, r)$, and $R(\psi, r) = RL(v, r) = T(v, r)$, which lies in $\mathcal{V}$ by assumption.

Lemma 3.3.1. *On $\mathcal{C}$, the operator Q satisfies $Q = LR$, and $Q\mathcal{C} \subset \mathcal{C}$.*

Proof. The first claim is immediate from the definitions. The second follows from the claims just established (i.e., R maps $\mathcal{C}$ to $\mathcal{V}$ and L maps $\mathcal{V}$ to $\mathcal{C}$). □

The preceding discussion implies that, as in chapter 2, Q and T are *semiconjugate* in the sense that $LT = QL$ on $\mathcal{V}$ and $TR = RQ$ on $\mathcal{C}$. Indeed, since $T = RL$ and $Q = LR$, we have $LT = LRL = QL$ and $TR = RLR = RQ$ as claimed. This leads to the following results, the key results of this section.

Proposition 3.3.1. *The following statements are true:*

(1) If (v, r) is a fixed point of T in $\mathcal{V}$, then $L(v, r)$ is a fixed point of Q in $\mathcal{C}$.

(2) If (ψ, r) is a fixed point of Q in $\mathcal{C}$, then $R(\psi, r)$ is a fixed point of T in $\mathcal{V}$.

Proof. To prove the first claim, fix $(v, r) \in \mathcal{V}$. By the definition of $\mathcal{C}$, $L(v, r) \in \mathcal{C}$. Moreover, since $(v, r) = T(v, r)$, we have $QL(v, r) = LT(v, r) = L(v, r)$. Hence, $L(v, r)$ is a fixed point of Q in $\mathcal{C}$. Regarding the second claim, fix $(\psi, r) \in \mathcal{C}$. Since R maps $\mathcal{C}$ into $\mathcal{V}$ as shown above, $R(\psi, r) \in \mathcal{V}$. Since $(\psi, r) = Q(\psi, r)$, we have $TR(\psi, r) = RQ(\psi, r) = R(\psi, r)$. Hence, $R(\psi, r)$ is a fixed point of T in $\mathcal{V}$. □

The next result, which parallels proposition 2.3.2, implies that iterating with either T or Q is essentially equivalent, at least on a theoretical level.

Proposition 3.3.2. *$T^{t+1} = RQ^t L$ on $\mathcal{V}$ and $Q^{t+1} = LT^t R$ on $\mathcal{C}$ for all $t \in \mathbb{N}_0$.*

Proof. That the claim holds when $t = 0$ has already been established. Now suppose the claim is true for arbitrary t. By the induction hypothesis we have $T^t = RQ^{t-1}L$ and $Q^t = LT^{t-1}R$. Since Q and T are semiconjugate as shown above, we have $T^{t+1} = TT^t = TRQ^{t-1}L = RQQ^{t-1}L = RQ^t L$ and $Q^{t+1} = QQ^t = QLT^{t-1}R = LTT^{t-1}R = LT^t R$. Hence, the claim holds by induction. □

The theory above is based on the primitive assumption of a generic (augmented) candidate value function space $\mathcal{V}$ with properties $(v^*, r^*) \in \mathcal{V}$ and $T\mathcal{V} \subset \mathcal{V}$. Similar results can be established if we start with a generic (augmented) candidate continuation value function space $\mathscr{C}$ that satisfies $(\psi^*, r^*) \in \mathscr{C}$ and $Q\mathscr{C} \subset \mathscr{C}$. Appendix 3.A outlines the main idea.

3.4 Symmetry under Weighted Supremum Norm

In this section, we impose a weighted supremum norm on the domain of T and Q in order to compare contractivity, optimality and related properties. The following assumption generalizes the standard weighted supremum norm assumption of Boyd (1990).[3]

Assumption 3.4.1. There exist a $\mathscr{B}$-measurable function $g : \mathsf{Z} \to \mathbb{R}_+$ and constants $n \in \mathbb{N}_0, a_1, \cdots, a_4, m, d \in \mathbb{R}_+$ such that $\beta m < 1$, and, for all $z \in \mathsf{Z}$,

$$\int |s(z')| P^n(z, \mathrm{d}z') \leq a_1 g(z) + a_2, \tag{3.4.1}$$

$$\int |c(z')| P^n(z, \mathrm{d}z') \leq a_3 g(z) + a_4, \tag{3.4.2}$$

$$\text{and} \quad \int g(z') P(z, \mathrm{d}z') \leq m g(z) + d. \tag{3.4.3}$$

Similar to the previous chapter, we require that the expected rewards $\mathbb{E}_z |s(Z_n)|$ and $\mathbb{E}_z |c(Z_n)|$ are small relative to some function g such that $\mathbb{E}_z g(Z_t)$ does not grow too fast. Moreover, the following statements are true:

- If both s and c are bounded, then assumption 3.4.1 holds for $n := 0$, $g := \|r\| \vee \|c\|$, $m := 1$ and $d := 0$.
- If assumption 3.4.1 holds for some n, then it must hold for all integer $n' > n$.
- To verify assumption 3.4.1, it suffices to find $n_1 \in \mathbb{N}_0$ for which (3.4.1) holds, $n_2 \in \mathbb{N}_0$ for which (3.4.2) holds, and that the measurable map g satisfies (3.4.3).
- Assumption 3.4.1 reduces to that of Boyd (1990) if we set $n = 0$.

Let $d_1 := a_1 + a_3$ and $d_2 := a_2 + a_4$. Choose $m', d' > 0$ such that

$$m + d_1 m' > 1, \quad \rho := \beta(m + d_1 m') < 1 \quad \text{and} \quad d' \geq \frac{d_2 m' + d}{m + d_1 m' - 1}. \tag{3.4.4}$$

Let the weight function $\kappa : \mathsf{Z} \to \mathbb{R}_+$ be defined by

$$\kappa(z) := m' \sum_{t=0}^{n-1} \mathbb{E}_z \left[|s(Z_t)| + |c(Z_t)| \right] + g(z) + d'. \tag{3.4.5}$$

Consider the product space $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$, where ρ_κ is a metric on $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ defined by

$$\rho_\kappa\left((f_1, f_2), (f_1', f_2')\right) = \|f_1 - f_1'\|_\kappa \vee \|f_2 - f_2'\|_\kappa.$$

Lemma 3.A.5 (see appendix 3.A) shows that $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$ is a complete metric space. Recall $\rho \in (0, 1)$ defined in (3.4.4). The following result shows that Q and T are both contraction mappings under identical assumptions.

[3] Indeed, assumption 3.4.1 is equivalent to assumption 2.3.1 of chapter 2, since the terminal reward s of this chapter corresponds to r of the previous chapter. In particular, we admit consideration of future transitions to enlarge the set of possible weight functions. We have seen the value of this generalization in the standard optimal stopping framework in section 2.5.

Theorem 3.4.1. *Under assumption 3.4.1, the following statements hold:*

(1) *Q is a contraction mapping on $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$ with modulus ρ.*

(2) *The unique fixed point of Q in $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ is (ψ^*, r^*).*

(3) *T is a contraction mapping on $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$ with modulus ρ.*

(4) *The unique fixed point of T in $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ is (v^*, r^*).*

The next result shows that the convergence rates of Q and T are the same. In stating it, L and R are as defined in (3.3.1), while again $\rho \in (0, 1)$ is the contraction coefficient defined in (3.4.4).

Proposition 3.4.1. *If assumption 3.4.1 holds, then*

$$R(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}) \subset b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z} \quad \text{and} \quad L(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}) \subset b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z},$$

and for all $t \in \mathbb{N}_0$, the following statements are true:

(1) *$\rho_\kappa(Q^{t+1}(\psi, r), (\psi^*, r^*)) \leq \rho\, \rho_\kappa(T^t R(\psi, r), (v^*, r^*))$ for all $(\psi, r) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$.*

(2) *$\rho_\kappa(T^{t+1}(v, r), (v^*, r^*)) \leq \rho_\kappa(Q^t L(v, r), (\psi^*, r^*))$ for all $(v, r) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$.*

Proposition 3.4.1 extends proposition 3.3.2 and lemma 3.A.1, and their connections can be seen by letting $\mathcal{V} = \mathscr{C} := b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$.

As in chapter 2, the two operators are also symmetric in terms of continuity of fixed points. The next result illustrates this, when Z is any separable and completely metrizable topological space (e.g., any G_δ subset of $\mathbb{R}^n$) and $\mathscr{B}$ is its Borel sets.

Assumption 3.4.2. (1) The stochastic kernel P is Feller; that is, $z \mapsto \int h(z') P(z, \mathrm{d}z')$ is continuous and bounded on Z whenever h is. (2) c, s, κ and $z \mapsto \int \kappa(z') P(z, \mathrm{d}z')$ are continuous.

Proposition 3.4.2. *If assumptions 3.4.1–3.4.2 hold, then ψ^*, r^* and v^* are continuous.*

3.5 Symmetry in L_p

In this section, we show that the results of the preceding section for the most part carry over if we switch the underlying space to L_p.

Consider $(L_p(\pi) \times L_p(\pi), d_p)$, where for all $(f_1, f_2), (f_1', f_2') \in L_p(\pi) \times L_p(\pi)$,

$$d_p\left((f_1, f_2), (f_1', f_2')\right) := \||f_1 - f_1'| \vee |f_2 - f_2'|\|_p.$$

Lemma 3.A.5 (see appendix 3.A) shows that $(L_p(\pi) \times L_p(\pi), d_p)$ is a complete metric space.

Assumption 3.5.1. The state process $\{Z_t\}$ admits a stationary distribution π and the reward functions s, c are in $L_q(\pi)$ for some $q \geq 1$.

The following result shows that Q and T are both contraction mappings on $L_p(\pi) \times L_p(\pi)$ under identical assumptions.[4]

Theorem 3.5.1. *If assumption 3.5.1 holds, then for all $1 \leq p \leq q$, we have*

(1) Q is a contraction mapping on $(L_p(\pi) \times L_p(\pi), d_p)$ of modulus β.

(2) The unique fixed point of Q in $L_p(\pi) \times L_p(\pi)$ is (ψ^*, r^*).

(3) T is a contraction mapping on $(L_p(\pi) \times L_p(\pi), d_p)$ of modulus β.

(4) The unique fixed point of T in $L_p(\pi) \times L_p(\pi)$ is (v^*, r^*).

The next result implies that Q and T have the same rate of convergence in terms of the L_p-norm distance defined above.

Proposition 3.5.1. *If assumption 3.5.1 holds, then for all $1 \leq p \leq q$,*

$$R\left(L_p(\pi) \times L_p(\pi)\right) \subset L_p(\pi) \times L_p(\pi)$$

$$\text{and} \quad L\left(L_p(\pi) \times L_p(\pi)\right) \subset L_p(\pi) \times L_p(\pi).$$

Moreover, for all $1 \leq p \leq q$ and $t \in \mathbb{N}_0$, the following statements are true:

(1) $d_p\left(Q^{t+1}(\psi, r), (\psi^*, r^*)\right) \leq \beta d_p\left(T^t R(\psi, r), (v^*, r^*)\right)$ for all $(\psi, r) \in L_p(\pi) \times L_p(\pi)$.

(2) $d_p\left(T^{t+1}(v, r), (v^*, r^*)\right) \leq d_p\left(Q^t L(v, r), (\psi^*, r^*)\right)$ for all $(v, r) \in L_p(\pi) \times L_p(\pi)$.

Proposition 3.5.1 is an extension of proposition 3.3.2 in an L_p framework, and their connections can be seen by letting $\mathcal{V} = \mathscr{C} := L_p(\pi) \times L_p(\pi)$.

[4] As in section 2.3.3, we omit phrases such as "with probability one" or "almost surely" throughout this section. Indeed, functional equivalences and uniqueness of fixed points are up to a π-null set.

Appendix 3.A Some Lemmas

To see the symmetric properties of Q and T from an alternative perspective, we start our analysis with a generic candidate continuation value function space. Let $\mathscr{C}$ be a subset of $m\mathscr{B} \times m\mathscr{B}$ such that $(\psi^*, r^*) \in \mathscr{C}$ and $Q\mathscr{C} \subset \mathscr{C}$. Let $\mathscr{V}$ be defined by

$$\mathscr{V} := R\mathscr{C} = \left\{ \begin{pmatrix} v \\ r \end{pmatrix} \in m\mathscr{B} \times m\mathscr{B} \colon \begin{pmatrix} v \\ r \end{pmatrix} = R \begin{pmatrix} \psi \\ r \end{pmatrix} \text{ for some } \begin{pmatrix} \psi \\ r \end{pmatrix} \in \mathscr{C} \right\}. \tag{3.A.1}$$

Then R is a surjective map from $\mathscr{C}$ onto $\mathscr{V}$, $Q = LR$ on $\mathscr{C}$ and $T = RL$ on $\mathscr{V}$. The following result parallels the theory of section 3.3, and is helpful for deriving important convergence properties once topological structure is added to the generic setting, as to be shown.

Lemma 3.A.1. *The following statements are true:*

(1) $L\mathscr{V} \subset \mathscr{C}$ and $T\mathscr{V} \subset \mathscr{V}$.

(2) If (v, r) is a fixed point of T in $\mathscr{V}$, then $L(v, r)$ is a fixed point of Q in $\mathscr{C}$.

(3) If (ψ, r) is a fixed point of Q in $\mathscr{C}$, then $R(\psi, r)$ is a fixed point of T in $\mathscr{V}$.

(4) $T^{t+1} = RQ^tL$ on $\mathscr{V}$ and $Q^{t+1} = LT^tR$ on $\mathscr{C}$ for all $t \in \mathbb{N}_0$.

Proof. The proof is similar to that of propositions 3.3.1–3.3.2 and thus omitted. □

Lemma 3.A.2. *Under assumption 3.4.1, there exist $b_1, b_2 \in \mathbb{R}_+$ such that*

$$\max\left\{|v^*(z)|, |r^*(z)|, |\psi^*(z)|\right\} \leq \sum_{t=0}^{n-1} \beta^t \mathbb{E}_z[|s(Z_t)| + |c(Z_t)|] + b_1 g(z) + b_2.$$

Proof of lemma 3.A.2. Without loss of generality, we assume $m \neq 1$ in assumption 3.4.1. By that assumption, $\mathbb{E}_z|s(Z_n)| \leq a_1 g(z) + a_2$, $\mathbb{E}_z|c(Z_n)| \leq a_3 g(z) + a_4$ and $\mathbb{E}_z g(Z_1) \leq m g(z) + d$ for all $z \in \mathsf{Z}$. For all $t \geq 1$, by the Markov property (see, e.g., Meyn and Tweedie (2009), section 3.4.3),

$$\mathbb{E}_z g(Z_t) = \mathbb{E}_z\left[\mathbb{E}_z\left(g(Z_t)|\mathscr{F}_{t-1}\right)\right] = \mathbb{E}_z\left(\mathbb{E}_{Z_{t-1}} g(Z_1)\right) \leq m\,\mathbb{E}_z g(Z_{t-1}) + d.$$

Induction shows that for all $t \geq 0$,

$$\mathbb{E}_z g(Z_t) \leq m^t g(z) + \frac{1 - m^t}{1 - m} d. \tag{3.A.2}$$

Moreover, for all $t \geq n$, applying the Markov property again yields

$$\mathbb{E}_z|s(Z_t)| = \mathbb{E}_z\left[\mathbb{E}_z\left(|s(Z_t)||\mathscr{F}_{t-n}\right)\right] = \mathbb{E}_z\left(\mathbb{E}_{Z_{t-n}}|s(Z_n)|\right) \leq a_1\,\mathbb{E}_z g(Z_{t-n}) + a_2.$$

By (3.A.2), for all $t \geq n$, we have

$$\mathbb{E}_z|s(Z_t)| \leq a_1 \left(m^{t-n} g(z) + \frac{1 - m^{t-n}}{1 - m} d \right) + a_2. \tag{3.A.3}$$

Similarly, for all $t \geq n$, we have

$$\mathbb{E}_z|c(Z_t)| \leq a_3 \, \mathbb{E}_z g(Z_{t-n}) + a_4 \leq a_3 \left(m^{t-n} g(z) + \frac{1 - m^{t-n}}{1 - m} d \right) + a_4. \tag{3.A.4}$$

Let $S(z) := \sum_{t \geq 0} \beta^t \mathbb{E}_z \left[|s(Z_t)| + |c(Z_t)| \right]$. Based on (3.A.2)–(3.A.4), we can show that

$$S(z) \leq \sum_{t=0}^{n-1} \beta^t \mathbb{E}_z[|s(Z_t)| + |c(Z_t)|] + \frac{a_1 + a_3}{1 - \beta m} g(z) + \frac{(a_1 + a_3)d + a_2 + a_4}{(1 - \beta m)(1 - \beta)}. \tag{3.A.5}$$

Since $\max\{|v^*|, |r^*|, |\psi^*|\} \leq S$ by definition, the stated claim holds by letting $b_1 := \frac{a_1 + a_3}{1 - \beta m}$ and $b_2 := \frac{(a_1 + a_3)d + a_2 + a_4}{(1 - \beta m)(1 - \beta)}$. □

Lemma 3.A.3. *Under assumption 3.4.1, v^* and r^* satisfy (3.2.2)–(3.2.3).*

Proof of lemma 3.A.3. We use $\tilde{P}$ to denote the stochastic kernel related to the state process $\{(Z_t, I_t)\}_{t \geq 0}$. The corresponding Bellman equation of the problem stated in (3.2.1) satisfies

$$V(z_0, i_0) = \max_{j_0 \in \{0,1\}} \left\{ F(z_0, i_0, j_0) + \beta \int V(z_1, i_1) \tilde{P}((z_0, i_0), j_0; \mathrm{d}(z_1, i_1)) \right\}. \tag{3.A.6}$$

Note that the Bellman equation defined above is equivalent to the following two equations[5]

$$V(z_0, 1) = \max \left\{ V(z_0, 0), c(z_0) + \beta \int V(z_1, 1) P(z_0, \mathrm{d}z_1) \right\} \tag{3.A.7}$$

and

$$V(z_0, 0) = s(z_0) + \alpha\beta \int V(z_1, 1) P(z_0, \mathrm{d}z_1) + (1 - \alpha)\beta \int V(z_1, 0) P(z_0, \mathrm{d}z_1). \tag{3.A.8}$$

Under assumption 3.4.1, the Bellman equation is well-defined (i.e., a solution exists). To see this, let $(\tilde{\psi}, \tilde{r})$ be the unique fixed point of Q under $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ obtained from claim (1) of theorem 3.4.1. Then $\tilde{V}$ defined by

$$\tilde{V}(\cdot, 0) := \tilde{r} \in b_\kappa \mathsf{Z} \quad \text{and} \quad \tilde{V}(\cdot, 1) := \tilde{r} \vee \tilde{\psi} \in b_\kappa \mathsf{Z} \tag{3.A.9}$$

[5] Note that if $i_t = 0$, then $j_t = 0$ and $F(z_t, 0, j_t) = F(z_t, 0, 0) = s(z_t)$, i.e., the agent has no choice since he/she is already in the passive state. In this case, $i_{t+1} = 1$ with probability α and $i_{t+1} = 0$ with probability $1 - \alpha$. Hence, $V(z_0, 0)$ follows the rule stated in (3.A.8).

solves the equation system (3.A.7)–(3.A.8).

Moreover, since (3.A.7)–(3.A.8) are the functional equations related to (3.2.2)–(3.2.3), and that (3.A.7)–(3.A.8) are equivalent to (3.A.6), to prove the stated claim, it suffices to show that any solution V to (3.A.6) with $(V(\cdot,0), V(\cdot,1)) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ satisfies $V = V^*$. Note that for all feasible plan $\{j_t\}_{t\geq 0}$, we have

$$\begin{aligned}
V(z_0, i_0) &\geq F(z_0, i_0, j_0) + \beta \int V(z_1, i_1) \tilde{P}((z_0, i_0), j_0; \mathrm{d}(z_1, i_1)) \\
&\geq F(z_0, i_0, j_0) + \\
&\quad \beta \int \left[F(z_1, i_1, j_1) + \beta \int V(z_2, i_2) \tilde{P}((z_1, i_1), j_1; \mathrm{d}(z_2, i_2)) \right] \tilde{P}((z_0, i_0), j_0; \mathrm{d}(z_1, i_1)) \\
&= F(z_0, i_0, j_0) + \beta \mathbb{E}_{z_0, i_0}^{j_0} F(Z_1, I_1, j_1) + \beta^2 \mathbb{E}_{z_0, i_0}^{j_0, j_1} V(Z_2, I_2) \geq \cdots \\
&\geq \sum_{t=0}^{K} \beta^t \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{t-1}} F(Z_t, I_t, j_t) + \beta^{K+1} \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_K} V(Z_{K+1}, I_{K+1})
\end{aligned} \tag{3.A.10}$$

for all $K \in \mathbb{N}$. Since $(V(\cdot,0), V(\cdot,1)) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, there exists $G \in \mathbb{R}_+$ such that $|V| \leq G\kappa$. The Markov property then implies that

$$\begin{aligned}
&\left| \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{K-1}} V(Z_K, I_K) \right| \leq G \, \mathbb{E}_{z_0} \kappa(Z_K) \\
&= G \left(m' \sum_{t=0}^{n-1} \mathbb{E}_{z_0} [|s(Z_{t+K})| + |c(Z_{t+K})|] + \mathbb{E}_{z_0} g(Z_K) + d' \right).
\end{aligned} \tag{3.A.11}$$

From (3.A.2)–(3.A.4) (the proof of lemma 3.A.2) we know that, for all $z_0 \in \mathsf{Z}$ and $t \in \mathbb{N}_0$,

$$\mathbb{E}_{z_0} g(Z_t) \leq m^t g(z_0) + \frac{1 - m^t}{1 - m} d,$$

and for all $z_0 \in \mathsf{Z}$ and $t \geq n$,

$$\max\{\mathbb{E}_{z_0} |s(Z_t)|, \mathbb{E}_{z_0} |c(Z_t)|\} \leq d_1 \left(m^{t-n} g(z_0) + \frac{1 - m^{t-n}}{1 - m} d \right) + d_2.$$

Substituting these results into (3.A.11), we can show that, for all (z_0, i_0),

$$\begin{aligned}
\left| \beta^K \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{K-1}} V(Z_K, I_K) \right| &\leq 2 G d_1 m' \sum_{t=0}^{n-1} \left[(\beta m)^K m^{t-n} g(z_0) + \beta^K \frac{1 - m^{K+t-n}}{1 - m} d \right] \\
&\quad + G \left[(\beta m)^K g(z_0) + \beta^K \frac{1 - m^K}{1 - m} d \right] + \beta^K G (d' + 2 n d_2 m').
\end{aligned}$$

Since $\beta m < 1$, this implies that $\lim_{K \to \infty} \beta^K \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{K-1}} V(Z_K, I_K) = 0$ for all $(z_0, i_0) \in \mathsf{Z} \times \{0, 1\}$. Let $K \to \infty$, then (3.A.10) implies that, for all $\{j_t\}_{t \geq 0}$,

$$V(z_0, i_0) \geq \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{t-1}} F(Z_t, I_t, j_t).$$

Hence, $V \geq V^*$. Notice that since (3.A.6) is a binary choice problem, there exists a plan $\{\tilde{j}_t\}_{t \geq 0}$ such that (3.A.10) holds with equality in each step, which implies $V \leq V^*$. Hence, $V = V^*$, as was to be shown. This concludes the proof. □

Lemma 3.A.4. *Under assumption 3.5.1, v^* and r^* satisfy (3.2.2)–(3.2.3) π-almost surely.*

Proof. Recall the two equivalent versions of Bellman equations defined respectively in (3.A.6) and (3.A.7)–(3.A.8) (see the proof of lemma 3.A.3). Under assumption 3.5.1, the Bellman equation is well-defined (π-almost surely). To see this, let $(\tilde{\psi}, \tilde{r})$ be the unique fixed point of Q under $(L_q(\pi) \times L_q(\pi), d_q)$ obtained from claim (1) of theorem 3.5.1. Then $\tilde{V}$ defined by

$$\tilde{V}(\cdot, 0) := \tilde{r} \quad \text{and} \quad \tilde{V}(\cdot, 1) := \tilde{r} \vee \tilde{\psi}$$

solves the equation system (3.A.7)–(3.A.8) π-almost surely.

Since (3.A.7)–(3.A.8) are the functional equations corresponding to (3.2.2)–(3.2.3), and that (3.A.7)–(3.A.8) are equivalent to (3.A.6). It remains to verify that any solution V to the Bellman equation (3.A.6) with $(V(\cdot, 0), V(\cdot, 1)) \in L_q(\pi) \times L_q(\pi)$ satisfies $V = V^*$ π-almost surely. For all feasible plan $\{j_t\}_{t \geq 0}$ and $(z_0, i_0) \in \mathsf{Z} \times \{0, 1\}$, we have

$$\left| \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{K-1}} V(Z_K, I_K) \right| \leq \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{K-1}} |V(Z_K, I_K)| \leq \mathbb{E}_{z_0} \left[|V(Z_K, 0)| \vee |V(Z_K, 1)| \right].$$

Hence, for all $(z_0, i_0) \in \mathsf{Z} \times \{0, 1\}$,

$$\sup_{\{j_t\}_{t \geq 0}} \left| \mathbb{E}_{z_0, i_0}^{j_0 \cdots j_{K-1}} V(Z_K, I_K) \right| \leq \mathbb{E}_{z_0} [|V(Z_K, 0)| \vee |V(Z_K, 1)|]. \tag{3.A.12}$$

Since π is stationary, Jensen's inequality yields

$$\begin{aligned}
&\int \{\mathbb{E}_{z_0} [|V(Z_K, 0)| \vee |V(Z_K, 1)|]\}^q \, \pi(\mathrm{d}z_0) \\
&= \int \left[\int |V(z', 0)| \vee |V(z', 1)| P^K(z, \mathrm{d}z') \right]^q \pi(\mathrm{d}z) \\
&\leq \int \int |V(z', 0)|^q \vee |V(z', 1)|^q P^K(z, \mathrm{d}z') \pi(\mathrm{d}z) \\
&= \int |V(z', 0)|^q \vee |V(z', 1)|^q \pi(\mathrm{d}z') \leq \|V(\cdot, 0)\|_q^q + \|V(\cdot, 1)\|_q^q < \infty.
\end{aligned}$$

Let $\mathbb{E}_{\cdot} f(Z_t)$ denote the function $z \mapsto \mathbb{E}_z f(Z_t)$. The Minkowski inequality then implies that for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\left\| \sum_{t=0}^{n} \beta^t \mathbb{E}_{\cdot} [|V(Z_t, 0)| \vee |V(Z_t, 1)|] \right\|_q \leq \sum_{t=0}^{n} \beta^t \left\| \mathbb{E}_{\cdot} [|V(Z_t, 0)| \vee |V(Z_t, 1)|] \right\|_q \\
&= \sum_{t=0}^{n} \beta^t \left[\int \{\mathbb{E}_z [|V(Z_t, 0)| \vee |V(Z_t, 1)|]\}^q \, \pi(\mathrm{d}z) \right]^{1/q} \\
&\leq \sum_{t=0}^{n} \beta^t \left[\|V(\cdot, 1)\|_q^q + \|V(\cdot, 1)\|_q^q \right]^{1/q} \leq \sum_{t=0}^{\infty} \beta^t \left[\|V(\cdot, 1)\|_q^q + \|V(\cdot, 1)\|_q^q \right]^{1/q} < \infty.
\end{aligned}$$

Moreover, by the monotone convergence theorem,

$$\left\| \sum_{t=0}^{n} \beta^t \mathbb{E}_{\cdot}[|V(Z_t,0)| \vee |V(Z_t,1)|] \right\|_q \to \left\| \sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\cdot}[|V(Z_t,0)| \vee |V(Z_t,1)|] \right\|_q .$$

Together, we have $\left\|\sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\cdot}[|V(Z_t,0)| \vee |V(Z_t,1)|]\right\|_q < \infty$, which implies that

$$\lim_{K\to\infty} \beta^K \mathbb{E}_{z_0}[|V(Z_K,0)| \vee |V(Z_K,1)|] = 0 \quad \pi\text{-almost surely.}$$

Then, by (3.A.12), for all $i_0 \in \{0,1\}$,

$$\lim_{K\to\infty} \left[\sup_{\{j_t\}_{t\geq 0}} \left| \beta^{K+1} \mathbb{E}_{z_0,i_0}^{j_0\cdots j_K} V(Z_{K+1}, I_{K+1}) \right| \right] = 0 \quad \pi\text{-almost surely.}$$

This implies that

$$\sup_{\{j_t\}_{t\geq 0}} \left[\lim_{K\to\infty} \left| \beta^{K+1} \mathbb{E}_{z_0,i_0}^{j_0\cdots j_K} V(Z_{K+1}, I_{K+1}) \right| \right] = 0 \quad \pi\text{-almost surely.} \tag{3.A.13}$$

For all feasible plan $\{j_t\}_{t\geq 0}$, $i_0 \in \{0,1\}$ and $K \in \mathbb{N}$, (3.A.10) implies that

$$V(z_0,i_0) \geq \sum_{t=0}^{K} \beta^t \mathbb{E}_{z_0,i_0}^{j_0\cdots j_{t-1}} F(Z_t, I_t, j_t) + \beta^{K+1} \mathbb{E}_{z_0,i_0}^{j_0\cdots j_K} V(Z_{K+1}, I_{K+1}). \tag{3.A.14}$$

Letting $K \to \infty$ and taking supremum with respect to $\{j_t\}_{t\geq 0}$ yield

$$\begin{aligned} V(z_0,i_0) \geq & \sup_{\{j_t\}_{t\geq 0}} \lim_{K\to\infty} \sum_{t=0}^{K} \beta^t \mathbb{E}_{z_0,i_0}^{j_0\cdots j_{t-1}} F(Z_t, I_t, j_t) \\ & + \sup_{\{j_t\}_{t\geq 0}} \lim_{K\to\infty} \beta^{K+1} \mathbb{E}_{z_0,i_0}^{j_0\cdots j_K} V(Z_{K+1}, I_{K+1}). \end{aligned} \tag{3.A.15}$$

Together, (3.A.13) and (3.A.15) imply that $V \geq V^*$ π-almost surely.

Since (3.A.6) is a binary choice problem, there exists a plan $\{\tilde{j}_t\}_{t\geq 0}$ such that (3.A.14) holds with equality for all $K \in \mathbb{N}$. (3.A.13) then implies that $V \leq V^*$ π-almost surely. In summary, we have $V = V^*$ π-almost surely. This concludes the proof. □

Recall $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$ constructed in section 3.4.

Lemma 3.A.5. *$(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$ is a complete metric space.*

Proof. We first show that ρ_κ is a well-defined metric. We only prove the triangular inequality since the other required properties of a metric hold trivially for ρ_κ. For all $(f_1,f_2), (f_1',f_2'), (f_1'',f_2'') \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, we have

$$\begin{aligned} \rho_\kappa\left((f_1,f_2),(f_1',f_2')\right) &= \|f_1 - f_1'\|_\kappa \vee \|f_2 - f_2'\|_\kappa \\ &\leq \left(\|f_1 - f_1''\|_\kappa + \|f_1'' - f_1'\|_\kappa\right) \vee \left(\|f_2 - f_2''\|_\kappa + \|f_2'' - f_2'\|_\kappa\right) \\ &\leq \|f_1 - f_1''\|_\kappa \vee \|f_2 - f_2''\|_\kappa + \|f_1'' - f_1'\|_\kappa \vee \|f_2'' - f_2'\|_\kappa \\ &= \rho_\kappa\left((f_1,f_2),(f_1'',f_2'')\right) + \rho_\kappa\left((f_1'',f_2''),(f_1',f_2')\right). \end{aligned}$$

Hence, the triangular inequality holds, and ρ_κ is a well-defined metric.

Regarding completeness, let $\{(f_n, h_n)\}$ be a Cauchy sequence of $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$. Then $\{f_n\}$ and $\{h_n\}$ are two Cauchy sequences of the Banach space $(b_\kappa \mathsf{Z}, \|\cdot\|_\kappa)$. Hence, $\|f_n - f\|_\kappa \to 0$ and $\|h_n - h\|_\kappa \to 0$ as $n \to \infty$ for some $f, h \in b_\kappa \mathsf{Z}$. This implies that $(f, h) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ and that $\rho_\kappa\left((f_n, h_n), (f, h)\right) = \|f_n - f\|_\kappa \vee \|h_n - h\|_\kappa \to 0$ as $n \to \infty$. Hence, completeness is established. □

Recall $(L_p(\pi) \times L_p(\pi), d_p)$ constructed in section 3.5.

Lemma 3.A.6. *$(L_p(\pi) \times L_p(\pi), d_p)$ is a complete metric space.*

Proof. We first show that d_p is a well-defined metric. We only verify the triangular inequality since the other required properties of a metric obviously holds for d_p. For all $(f_1, f_2), (f_1', f_2'), (f_1'', f_2'') \in L_p(\pi) \times L_p(\pi)$, we have

$$\begin{aligned} |f_1 - f_1'| \vee |f_2 - f_2'| &\le \left(|f_1 - f_1''| + |f_1'' - f_1'|\right) \vee \left(|f_2 - f_2''| + |f_2'' - f_2'|\right) \\ &\le |f_1 - f_1''| \vee |f_2 - f_2''| + |f_1'' - f_1'| \vee |f_2'' - f_2'|. \end{aligned}$$

The Minkowski inequality then implies that

$$\begin{aligned} d_p\left((f_1, f_2), (f_1', f_2')\right) &= \||f_1 - f_1'| \vee |f_2 - f_2'|\|_p \\ &\le \big\||f_1 - f_1''| \vee |f_2 - f_2''| + |f_1'' - f_1'| \vee |f_2'' - f_2'|\big\|_p \\ &\le \big\||f_1 - f_1''| \vee |f_2 - f_2''|\big\|_p + \big\||f_1'' - f_1'| \vee |f_2'' - f_2'|\big\|_p \\ &= d_p\left((f_1, f_2), (f_1'', f_2'')\right) + d_p\left((f_1'', f_2''), (f_1', f_2')\right). \end{aligned}$$

Hence, the triangular inequality is verified, and d_p is a well-defined metric.

For completeness, let $\{(f_n, h_n)\}$ be a Cauchy sequence of $\left(L_p(\pi) \times L_p(\pi), d_p\right)$. Then $\{f_n\}$ and $\{h_n\}$ are Cauchy sequences of the Banach space $(L_p(\pi), \|\cdot\|_p)$. Hence, $\|f_n - f\|_p \to 0$ and $\|h_n - h\|_p \to 0$ as $n \to \infty$ for some $f, h \in L_p(\pi)$. This implies that $(f, h) \in L_p(\pi) \times L_p(\pi)$ and that

$$\begin{aligned} d_p\left((f_n, h_n), (f, h)\right) = \||f_n - f| \vee |h_n - h|\|_p &\le \||f_n - f| + |h_n - h|\|_p \\ &\le \|f_n - f\|_p + \|h_n - h\|_p \to 0 \end{aligned}$$

as $n \to \infty$. Hence, completeness is established. This concludes the proof. □

Appendix 3.B Main Proofs

Proof of theorem 3.4.1. By the Markov property, we have

$$\int \mathbb{E}_{z'}|s(Z_t)|P(z, \mathrm{d}z') = \mathbb{E}_z|s(Z_{t+1})| \;\text{ and }\; \int \mathbb{E}_{z'}|c(Z_t)|P(z, \mathrm{d}z') = \mathbb{E}_z|c(Z_{t+1})|.$$

Let $h(z) := \sum_{t=0}^{n-1} \mathbb{E}_z[|s(Z_t)| + |c(Z_t)|]$, then we have

$$\int h(z')P(z, \mathrm{d}z') = \sum_{t=1}^{n} \mathbb{E}_z[|s(Z_t)| + |c(Z_t)|]. \tag{3.B.1}$$

By the construction of m' and d' in (3.4.4), we have $m + d_1 m' > 1$ and $(d_2 m' + d + d')/(m + d_1 m') \leq d'$. Assumption 3.4.1 and (3.B.1) then imply that

$$\begin{aligned}\int \kappa(z')P(z, dz') &= m' \sum_{t=1}^{n} \mathbb{E}_z[|s(Z_t)| + |c(Z_t)|] + \int g(z')P(z, dz') + d' \\ &\leq m' \sum_{t=1}^{n-1} \mathbb{E}_z[|r(Z_t)| + |c(Z_t)|] + (m + d_1 m')g(z) + d_2 m' + d + d' \\ &\leq (m + d_1 m') \left(\frac{m'}{m + d_1 m'} h(z) + g(z) + d' \right) \leq (m + d_1 m')\kappa(z).\end{aligned} \tag{3.B.2}$$

In the next, we use this result to prove the stated claims.

Step 1. We prove claim (1). We first show that $Q \colon (b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa) \to (b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$. For all $z \in \mathsf{Z}$ and $(\psi, r) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, we define

$$\jmath(z;\, \psi, r) := c(z) + \beta \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z')$$

and

$$\tilde{\jmath}(z;\, \psi, r) := s(z) + \alpha\beta \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z') + (1 - \alpha)\beta \int r(z')P(z, \mathrm{d}z').$$

Then there exists $G \in \mathbb{R}_+$ such that for all $z \in \mathsf{Z}$,

$$\frac{|\jmath(z;\, \psi, r)|}{\kappa(z)} \leq \frac{|c(z)|}{\kappa(z)} + \frac{\beta G \int \kappa(z')P(z, \mathrm{d}z')}{\kappa(z)} \leq \frac{1}{m'} + \beta(m + d_1 m')G < \infty$$

and

$$\frac{|\tilde{\jmath}(z;\, \psi, r)|}{\kappa(z)} \leq \frac{|s(z)|}{\kappa(z)} + \frac{\beta G \int \kappa(z')P(z, \mathrm{d}z')}{\kappa(z)} \leq \frac{1}{m'} + \beta(m + d_1 m')G < \infty.$$

This implies that $\jmath(\cdot\,;\, \psi, r) \in b_\kappa \mathsf{Z}$ and $\tilde{\jmath}(\cdot\,;\, \psi, r) \in b_\kappa \mathsf{Z}$. Hence, $Q(\psi, r) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, and Q is a self-map on $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$.

Next, we show that Q is a contraction on $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$. For all $(\psi_1, r_1), (\psi_2, r_2) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, we have $\rho_\kappa\left(Q(\psi_1, r_1), Q(\psi_2, r_2)\right) = I \vee J$, where

$$I := \|\beta P(r_1 \vee \psi_1) - \beta P(r_2 \vee \psi_2)\|_\kappa$$

and

$$J := \|\alpha\beta[P(r_1 \vee \psi_1) - P(r_2 \vee \psi_2)] + (1-\alpha)\beta(Pr_1 - Pr_2)\|_\kappa.$$

For all $z \in \mathsf{Z}$, we have

$$\begin{aligned}
|P(r_1 \vee \psi_1)(z) - P(r_2 \vee \psi_2)(z)| &\leq \int |r_1 \vee \psi_1 - r_2 \vee \psi_2|\,(z')P(z, \mathrm{d}z') \\
&\leq \int (|\psi_1 - \psi_2| \vee |r_1 - r_2|)(z')P(z, \mathrm{d}z') \\
&\leq (\|\psi_1 - \psi_2\|_\kappa \vee \|r_1 - r_2\|_\kappa) \int \kappa(z')P(z, \mathrm{d}z') \\
&\leq \rho_\kappa\left((\psi_1, r_1), (\psi_2, r_2)\right)(m + d_1 m')\kappa(z),
\end{aligned}$$

where the second inequality follows from the elementary fact $|a \vee b - a' \vee b'| \leq |a - a'| \vee |b - b'|$. Recall $\rho := \beta(m + d_1 m')$ defined in (3.4.4). We then have

$$I \leq \beta(m + d_1 m')\rho_\kappa\left((\psi_1, r_1), (\psi_2, r_2)\right) = \rho\, \rho_\kappa\left((\psi_1, r_1), (\psi_2, r_2)\right).$$

Similar arguments yield $J \leq \rho\, \rho_\kappa\left((\psi_1, r_1), (\psi_2, r_2)\right)$. In conclusion, we have

$$\rho_\kappa\left(Q(\psi_1, r_1), Q(\psi_2, r_2)\right) = I \vee J \leq \rho\, \rho_\kappa\left((\psi_1, r_1), (\psi_2, r_2)\right).$$

Hence Q is a contraction mapping on $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$ with modulus ρ, and claim (1) is verified.

Step 2. We show that claim (2) holds. Lemma 3.A.3 shows that (3.2.2)–(3.2.3) hold under assumption 3.4.1. Then from (3.2.6) we know that (ψ^*, r^*) is indeed a fixed point of Q. Moreover, lemma 3.A.2 implies that $(\psi^*, r^*) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$. Hence, (ψ^*, r^*) must coincide with the unique fixed point of Q under $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, and claim (2) holds.

Step 3. We prove claim (3). We first show that $T\colon (b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa) \to (b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$. For all $(v, r) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, let $\ell(z;\, v, r)$ be defined as in (3.2.4), and let

$$\tilde{\ell}(z;\, v, r) := \max\left\{\ell(z; v, r),\, c(z) + \beta \int v(z')P(z, \mathrm{d}z')\right\}.$$

Then there exists $G \in \mathbb{R}_+$ such that for all $z \in \mathsf{Z}$,

$$\frac{|\ell(z;\, v, r)|}{\kappa(z)} \leq \frac{|s(z)|}{\kappa(z)} + \frac{\beta G \int \kappa(z')P(z, \mathrm{d}z')}{\kappa(z)} \leq \frac{1}{m'} + \beta(m + d_1 m')G < \infty$$

and

$$\begin{aligned}
\frac{|\tilde{\ell}(z;\, v, r)|}{\kappa(z)} &\leq \max\left\{\frac{|\ell(z;\, v, r)|}{\kappa(z)}, \frac{|c(z)|}{\kappa(z)} + \frac{\beta G \int \kappa(z')P(z, \mathrm{d}z')}{\kappa(z)}\right\} \\
&\leq \frac{1}{m'} + \beta(m + d_1 m')G < \infty.
\end{aligned}$$

This implies that $\tilde{\ell}(\cdot\,;\,v,r) \in b_\kappa \mathsf{Z}$ and $\ell(\cdot\,;\,v,r) \in b_\kappa \mathsf{Z}$. Hence, $T(v,r) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$.

Next, we show that T is a contraction on $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$. For all fixed (v_1, r_1) and (v_2, r_2) in $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, we have $\rho_\kappa \left(T(v_1,r_1), T(v_2,r_2) \right) = I \vee J$, where

$$I := \|\ell(\cdot\,;\,v_1,r_1) \vee (c + \beta P v_1) - \ell(\cdot\,;\,v_2,r_2) \vee (c + \beta P v_2)\|_\kappa$$

and

$$J := \|\ell(\cdot\,;\,v_1,r_1) - \ell(\cdot\,;\,v_2,r_2)\|_\kappa.$$

For all $z \in \mathsf{Z}$, we have

$$\begin{aligned}
&|\ell(z;\,v_1,r_1) - \ell(z;\,v_2,r_2)| \\
&= |\alpha\beta[Pv_1(z) - Pv_2(z)] + (1-\alpha)\beta[Pr_1(z) - Pr_2(z)]| \\
&\leq \alpha\beta \int |v_1(z') - v_2(z')| P(z, \mathrm{d}z') + (1-\alpha)\beta \int |r_1(z') - r_2(z')| P(z, \mathrm{d}z') \\
&\leq \alpha\beta \|v_1 - v_2\|_\kappa \int \kappa(z') P(z, \mathrm{d}z') + (1-\alpha)\beta \|r_1 - r_2\|_\kappa \int \kappa(z') P(z, \mathrm{d}z') \\
&\leq \alpha\beta(m + d_1 m') \|v_1 - v_2\|_\kappa \, \kappa(z) + (1-\alpha)\beta(m + d_1 m') \|r_1 - r_2\|_\kappa \, \kappa(z) \\
&\leq \rho \left(\|v_1 - v_2\|_\kappa \vee \|r_1 - r_2\|_\kappa \right) \kappa(z) = \rho \, \rho_\kappa \left((v_1,r_1), (v_2,r_2) \right) \kappa(z).
\end{aligned}$$

Hence, $J \leq \rho \, \rho_\kappa \left((v_1,r_1), (v_2,r_2) \right)$. Moreover, for all $z \in \mathsf{Z}$, we have

$$\begin{aligned}
&|\ell(z;\,v_1,r_1) \vee (c + \beta P v_1)(z) - \ell(z;\,v_2,r_2) \vee (c + \beta P v_2)(z)| \\
&\leq |\ell(z;\,v_1,r_1) - \ell(z;\,v_2,r_2)| \vee [\beta P v_1(z) - \beta P v_2(z)] \\
&\leq |\ell(z;\,v_1,r_1) - \ell(z;\,v_2,r_2)| \vee \left(\beta \int |v_1(z') - v_2(z')| \, P(z, \mathrm{d}z') \right) \\
&\leq [\rho \, \rho_\kappa \left((v_1,r_1), (v_2,r_2) \right) \kappa(z)] \vee \left(\beta \|v_1 - v_2\|_\kappa \int \kappa(z') P(z, \mathrm{d}z') \right) \\
&\leq [\rho \, \rho_\kappa \left((v_1,r_1), (v_2,r_2) \right) \kappa(z)] \vee [\rho \, \|v_1 - v_2\|_\kappa \kappa(z)] = \rho \, \rho_\kappa \left((v_1,r_1), (v_2,r_2) \right) \kappa(z).
\end{aligned}$$

Hence, $I \leq \rho \rho_\kappa \left((v_1,r_1), (v_2,r_2) \right)$. In conclusion, we have

$$\rho_\kappa \left(T(v_1,r_1), T(v_2,r_2) \right) = I \vee J \leq \rho \, \rho_\kappa \left((v_1,r_1), (v_2,r_2) \right).$$

Hence T is a contraction mapping on $(b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}, \rho_\kappa)$ with modulus ρ, and claim (3) is verified.

Step 4. We show that claim (4) holds. Lemma 3.A.2 and 3.A.3 imply respectively that $(v^*, r^*) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ and that (v^*, r^*) is a fixed point of T. Hence, (v^*, r^*) must coincide with the unique fixed point of T under $b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, and claim (4) holds. This concludes the proof. $\square$

Proof of proposition 3.4.1. Let $\mathcal{V} = \mathscr{C} := b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$. The fact that $R\mathscr{C} \subset b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$ is obvious, and using the Markov property, we can easily show that $L\mathcal{V} \subset b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$.

Regarding claim (1), for all $(\psi, r) \in \mathscr{C}$, we have

$$\rho_\kappa \left[Q \begin{pmatrix} \psi \\ r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right] = \rho_\kappa \left[LR \begin{pmatrix} \psi \\ r \end{pmatrix}, L \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right] = \rho_\kappa \left[L \begin{pmatrix} r \vee \psi \\ r \end{pmatrix}, L \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right] = I \vee J,$$

where

$$I := \beta \| P(r \vee \psi) - Pv^* \|_\kappa$$

and

$$J := \beta \| \alpha [P(r \vee \psi) - Pv^*] + (1-\alpha)(Pr - Pr^*) \|_\kappa.$$

Notice that for all $z \in \mathsf{Z}$, (3.B.2) implies that

$$\begin{aligned} |P(r \vee \psi)(z) - Pv^*(z)| &\leq \int |(r \vee \psi)(z') - v^*(z')| P(z, \mathrm{d}z') \\ &\leq \| r \vee \psi - v^* \|_\kappa \int \kappa(z') P(z, \mathrm{d}z') \\ &\leq \| r \vee \psi - v^* \|_\kappa (m + d_1 m') \kappa(z). \end{aligned}$$

Hence, $\beta \| P(r \vee \psi) - Pv^* \|_\kappa \leq \beta(m + d_1 m') \| r \vee \psi - v^* \|_\kappa = \rho \, \| r \vee \psi - v^* \|_\kappa$, and

$$I \leq \rho \, \| r \vee \psi - v^* \|_\kappa \leq \rho \, \left(\| r \vee \psi - v^* \|_\kappa \vee \| r - r^* \|_\kappa \right) = \rho \, \rho_\kappa \left(R(\psi, r), (v^*, r^*) \right)$$

Similarly, for all $z \in \mathsf{Z}$, we have

$$|Pr(z) - Pr^*(z)| \leq \| r - r^* \|_\kappa (m + d_1 m') \kappa(z).$$

Hence, $\beta \| Pr - Pr^* \|_\kappa \leq \beta(m + d_1 m') \| r - r^* \|_\kappa = \rho \, \| r - r^* \|_\kappa$, and

$$\begin{aligned} J &\leq \alpha \beta \| P(r \vee \psi) - Pv^* \|_\kappa + (1 - \alpha) \beta \| Pr - Pr^* \|_\kappa \\ &\leq \alpha \rho \, \| r \vee \psi - v^* \|_\kappa + (1 - \alpha) \rho \, \| r - r^* \|_\kappa \\ &\leq \rho \, \left(\| r \vee \psi - v^* \|_\kappa \vee \| r - r^* \|_\kappa \right) = \rho \, \rho_\kappa \left(R(\psi, r), (v^*, r^*) \right). \end{aligned}$$

In summary, we have

$$\rho_\kappa \left(Q(\psi, r), (\psi^*, r^*) \right) = I \vee J \leq \rho \, \rho_\kappa \left(R(\psi, r), (v^*, r^*) \right)$$

and claim (1) holds for $t = 0$. Now suppose claim (1) is true for arbitrary t. By the induction hypothesis we have

$$\rho_\kappa (Q^t(\psi, r), (\psi^*, r^*)) \leq \rho \, \rho_\kappa (T^{t-1} R(\psi, r), (v^*, r^*))$$

for all $(\psi, r) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$. Since Q and T are semiconjugate as shown in section 3.3, we have

$$\rho_k \left[Q^{t+1} \begin{pmatrix} \psi \\ r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right] = \rho_k \left[Q^t Q \begin{pmatrix} \psi \\ r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right] \leq \rho\, \rho_k \left[T^{t-1} R Q \begin{pmatrix} \psi \\ r \end{pmatrix}, \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right]$$
$$= \rho\, \rho_k \left[T^{t-1} T R \begin{pmatrix} \psi \\ r \end{pmatrix}, \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right] = \rho\, \rho_k \left[T^t R \begin{pmatrix} \psi \\ r \end{pmatrix}, \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right]$$

for all $(\psi, r) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$. Hence, claim (1) holds by induction.

Regarding claim (2), for all $(v, r) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$, we have

$$\rho_\kappa \left[T \begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right] = \rho_\kappa \left[R L \begin{pmatrix} v \\ r \end{pmatrix}, R \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right]$$
$$= \rho_\kappa \left[R \begin{pmatrix} c + \beta P v \\ s + \alpha \beta P v + (1 - \alpha) \beta P r \end{pmatrix}, R \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right] = \tilde{I} \vee \tilde{J},$$

where

$$\tilde{I} := \| s + \alpha \beta P v + (1 - \alpha) \beta P r - r^* \|_\kappa$$

and

$$\tilde{J} := \| [s + \alpha \beta P v + (1 - \alpha) \beta P r] \vee (c + \beta P v) - r^* \vee \psi^* \|_\kappa.$$

Since

$$|[s + \alpha \beta P v + (1 - \alpha) \beta P r](z) \vee (c + \beta P v)(z) - r^*(z) \vee \psi^*(z)|$$
$$\leq |[s + \alpha \beta P v + (1 - \alpha) \beta P r](z) - r^*(z)| \vee |(c + \beta P v)(z) - \psi^*(z)|$$

for all $z \in \mathsf{Z}$, we have

$$\tilde{J} \leq \| s + \alpha \beta P v + (1 - \alpha) \beta P r - r^* \|_\kappa \vee \| c + \beta P v - \psi^* \|_\kappa,$$

and thus

$$\rho_\kappa \left[T \begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right] = \tilde{I} \vee \tilde{J} \leq \| s + \alpha \beta P v + (1 - \alpha) \beta P r - r^* \|_\kappa \vee \| c + \beta P v - \psi^* \|_\kappa$$
$$= \rho_\kappa \left[\begin{pmatrix} c + \beta P v \\ s + \alpha \beta P v + (1 - \alpha) \beta P r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right] = \rho_\kappa \left[L \begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right].$$

Hence, claim (2) holds for $t = 0$. Now suppose that the claim holds for arbitrary t. By the induction hypothesis we have

$$\rho_\kappa \left(T^t (v, r), (v^*, r^*) \right) \leq \rho_\kappa \left(Q^{t-1} L (v, r), (\psi^*, r^*) \right)$$

for all $(v, r) \in b_\kappa \mathsf{Z} \times b_\kappa \mathsf{Z}$. Since Q and T are semiconjugate as shown in section 3.3, we have

$$\rho_\kappa \left[T^{t+1} \begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right] = \rho_\kappa \left[T^t T \begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right] = \rho_\kappa \left[Q^{t-1} L T \begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right]$$
$$= \rho_\kappa \left[Q^{t-1} Q L \begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right] = \rho_\kappa \left[Q^t L \begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right].$$

Hence, claim (2) holds by induction. This concludes the proof. □

Proof of proposition 3.4.2. Let $b_\kappa c\mathsf{Z}$ be the set of continuous functions in $b_\kappa \mathsf{Z}$. Since κ is continuous by assumption 3.4.2, $b_\kappa c\mathsf{Z}$ is a closed subset of $b_\kappa \mathsf{Z}$ (see e.g., Boyd (1990), section 3). To show the continuity of ψ^* and r^*, it suffices to verify that $Q(b_\kappa c\mathsf{Z} \times b_\kappa c\mathsf{Z}) \subset b_\kappa c\mathsf{Z} \times b_\kappa c\mathsf{Z}$ (see, e.g., Stokey et al. (1989), corollary 1 of theorem 3.2). For all $(\psi, r) \in b_\kappa c\mathsf{Z} \times b_\kappa c\mathsf{Z}$, there exists $G \in \mathbb{R}_+$ such that $|r(z) \vee \psi(z)| \leq G\kappa(z)$. By assumption 3.4.2, $z \mapsto G\kappa(z) \pm r(z) \vee \psi(z)$ are nonnegative and continuous. For all $z \in \mathsf{Z}$ and $\{z_m\} \subset \mathsf{Z}$ with $z_m \to z$, the generalized Fatou's lemma of Feinberg et al. (2014) (theorem 1.1) implies that (note that $P(z_m, \cdot) \stackrel{w}{\to} P(z, \cdot)$ since P is Feller)

$$\int [G\kappa(z') \pm r(z') \vee \psi(z')] \, P(z, \mathrm{d}z') \leq \liminf_{m \to \infty} \int [G\kappa(z') \pm r(z') \vee \psi(z')] \, P(z_m, \mathrm{d}z').$$

Since $\lim_{m \to \infty} \int \kappa(z') P(z_m, \mathrm{d}z') = \int \kappa(z') P(z, \mathrm{d}z')$ by assumption 3.4.2, we have

$$\pm \int r(z') \vee \psi(z') P(z, \mathrm{d}z') \leq \liminf_{m \to \infty} \left[\pm \int r(z') \vee \psi(z') P(z_m, \mathrm{d}z') \right],$$

where we have used the fact that for all sequences $\{a_m\}, \{b_m\} \subset \mathbb{R}$ with $\lim_{m \to \infty} a_m$ exists, we have: $\liminf_{m \to \infty} (a_m + b_m) = \lim_{m \to \infty} a_m + \liminf_{m \to \infty} b_m$. Hence,

$$\limsup_{m \to \infty} \int r(z') \vee \psi(z') P(z_m, \mathrm{d}z') \leq \int r(z') \vee \psi(z') P(z, \mathrm{d}z')$$
$$\leq \liminf_{m \to \infty} \int r(z') \vee \psi(z') P(z_m, \mathrm{d}z'),$$

i.e., $z \mapsto \int r(z') \vee \psi(z') P(z, \mathrm{d}z')$ is continuous. Similarly, one can show that $z \mapsto \int r(z') P(z, \mathrm{d}z')$ is continuous. Since s and c are continuous by assumption 3.4.2, we have $Q(\psi, r) \in b_\kappa c\mathsf{Z} \times b_\kappa c\mathsf{Z}$. Hence, $Q(b_\kappa c\mathsf{Z} \times b_\kappa c\mathsf{Z}) \subset b_\kappa c\mathsf{Z} \times b_\kappa c\mathsf{Z}$ and (ψ^*, r^*) is continuous, as was to be shown. The continuity of v^* follows from the continuity of ψ^* and r^* and the fact that $v^* = r^* \vee \psi^*$. □

Proof of theorem 3.5.1. For all $z \in \mathsf{Z}$ and $(\psi, r) \in L_p(\pi) \times L_p(\pi)$, define

$$\jmath(z; \psi, r) := c(z) + \beta \int (r \vee \psi)(z') P(z, \mathrm{d}z') \tag{3.B.3}$$

and

$$\tilde{\jmath}(z;\,\psi,r) := s(z) + \alpha\beta\int (r\vee\psi)(z')P(z,\mathrm{d}z') + (1-\alpha)\beta\int r(z')P(z,\mathrm{d}z'). \quad (3.B.4)$$

Then, by definition, the Jovanovic operator Q satisfies

$$Q\begin{pmatrix}\psi\\ r\end{pmatrix}(z) = \begin{pmatrix}\jmath(z;\,\psi,r)\\ \tilde{\jmath}(z;\,\psi,r)\end{pmatrix}.$$

For all $z\in\mathsf{Z}$ and $(v,r)\in L_p(\pi)\times L_p(\pi)$, recall $\ell(z;\,v,r)$ defined by (3.2.4), and let

$$\tilde{\ell}(z;\,v,r) := \max\left\{\ell(z;\,v,r),\, c(z)+\beta\int v(z')P(z,\mathrm{d}z')\right\}. \quad (3.B.5)$$

Then, by definition, the Bellman operator T satisfies

$$T\begin{pmatrix}v\\ r\end{pmatrix}(z) = \begin{pmatrix}\tilde{\ell}(z;\,v,r)\\ \ell(z;\,v,r)\end{pmatrix}.$$

Since $s,c\in L_q(\pi)$, by the monotonicity of the L_p-norm, we have $s,c\in L_p(\pi)$ for all $1\le p\le q$.

Proof of claim (1). **Step 1.** We show that $Q(\psi,r)\in L_p(\pi)\times L_p(\pi)$ for all $(\psi,r)\in L_p(\pi)\times L_p(\pi)$. For all $z\in\mathsf{Z}$ and $(\psi,r)\in L_p(\pi)\times L_p(\pi)$, we have

$$\begin{aligned}
|\tilde{\jmath}(z;\,\psi,r)|^p &\le 3^p|s(z)|^p + (3\alpha\beta)^p\left[\int |r(z')|\vee|\psi(z')|P(z,\mathrm{d}z')\right]^p\\
&\quad + [3(1-\alpha)\beta]^p\left[\int |r(z')|P(z,\mathrm{d}z')\right]^p\\
&\le 3^p|s(z)|^p + (3\alpha\beta)^p\int\left[|r(z')|\vee|\psi(z')|\right]^p P(z,\mathrm{d}z')\\
&\quad + [3(1-\alpha)\beta]^p\int |r(z')|^pP(z,\mathrm{d}z')\\
&\le 3^p|s(z)|^p + (3\alpha\beta)^p\int |\psi(z')|^pP(z,\mathrm{d}z')\\
&\quad + (3\beta)^p[\alpha^p+(1-\alpha)^p]\int |r(z')|^pP(z,\mathrm{d}z'),
\end{aligned}$$

where for the first and third inequality we have used the elementary fact that $\left(\sum_{i=1}^n a_i\right)^p \le n^p\left(\vee_{i=1}^n a_i\right)^p \le n^p\sum_{i=1}^n a_i^p$ for all positive $\{a_i\}_{i=1}^n$ and p, and the second inequality is due to Jensen's inequality. Since $s,\psi,r\in L_p(\pi)$ and π is the stationary distribution of P, Fubini theorem then implies that

$$\int\left[\int |\psi(z')|^pP(z,\mathrm{d}z')\right]\pi(\mathrm{d}z) = \int |\psi(z')|^p\pi(\mathrm{d}z') = \|\psi\|_p^p < \infty$$

and

$$\int\left[\int |r(z')|^pP(z,\mathrm{d}z')\right]\pi(\mathrm{d}z) = \int |r(z')|^p\pi(\mathrm{d}z') = \|r\|_p^p < \infty.$$

The above inequalities then imply that $\int |\tilde{\jmath}(z;\, \psi, r)|^p \pi(\mathrm{d}z) < \infty$, i.e, $\|\tilde{\jmath}(\cdot\,;\, \psi, r)\|_p < \infty$. Similarly, we can show that $\int |\jmath(z;\, \psi, r)|^p \pi(\mathrm{d}z) < \infty$, i.e, $\|\jmath(\cdot\,;\, \psi, r)\|_p < \infty$. Hence, $Q(\psi, r) \in L_p(\pi) \times L_p(\pi)$, as was to be shown.

Step 2. We show that Q is a contraction mapping on $(L_p(\pi) \times L_p(\pi), d_p)$ of modulus β. For all $z \in \mathsf{Z}$ and $(\psi_1, r_1), (\psi_2, r_2) \in L_p(\pi) \times L_p(\pi)$, we have

$$
\begin{aligned}
& |\tilde{\jmath}(z;\, \psi_1, r_1) - \tilde{\jmath}(z;\, \psi_2, r_2)|^p \\
& = \left| \alpha\beta \int (r_1 \vee \psi_1 - r_2 \vee \psi_2)(z') P(z, \mathrm{d}z') + (1-\alpha)\beta \int (r_1 - r_2)(z') P(z, \mathrm{d}z') \right|^p \\
& \leq \left[\alpha\beta \int |r_1 \vee \psi_1 - r_2 \vee \psi_2|(z') P(z, \mathrm{d}z') + (1-\alpha)\beta \int |r_1 - r_2|(z') P(z, \mathrm{d}z') \right]^p \\
& \leq \left[\beta \int |r_1(z') - r_2(z')| \vee |\psi_1(z') - \psi_2(z')| P(z, \mathrm{d}z') \right]^p \\
& \leq \beta^p \int \left[|r_1(z') - r_2(z')| \vee |\psi_1(z') - \psi_2(z')| \right]^p P(z, \mathrm{d}z').
\end{aligned}
$$

Similarly, for all $z \in \mathsf{Z}$ and $(\psi_1, r_1), (\psi_2, r_2) \in L_p(\pi) \times L_p(\pi)$, we have

$$
\begin{aligned}
|\jmath(z;\, \psi_1, r_1) - \jmath(z;\, \psi_2, r_2)|^p & = \left| \beta \int (r_1 \vee \psi_1 - r_2 \vee \psi_2)(z') P(z, \mathrm{d}z') \right|^p \\
& \leq \left[\beta \int |r_1 \vee \psi_1 - r_2 \vee \psi_2|(z') P(z, \mathrm{d}z') \right]^p \\
& \leq \left[\beta \int |r_1(z') - r_2(z')| \vee |\psi_1(z') - \psi_2(z')| P(z, \mathrm{d}z') \right]^p \\
& \leq \beta^p \int \left[|r_1(z') - r_2(z')| \vee |\psi_1(z') - \psi_2(z')| \right]^p P(z, \mathrm{d}z').
\end{aligned}
$$

By the definition of d_P and Fubini theorem, we have

$$
\begin{aligned}
& d_p \left(Q(\psi_1, r_1), Q(\psi_2, r_2) \right) \\
& = \left[\int |\jmath(z;\, \psi_1, r_1) - \jmath(z;\, \psi_2, r_2)|^p \vee |\tilde{\jmath}(z;\, \psi_1, r_1) - \tilde{\jmath}(z;\, \psi_2, r_2)|^p \pi(\mathrm{d}z) \right]^{1/p} \\
& \leq \left[\int \beta^p \int [|r_1(z') - r_2(z')| \vee |\psi_1(z') - \psi_2(z')|]^p P(z, \mathrm{d}z') \pi(\mathrm{d}z) \right]^{1/p} \\
& = \beta \left[\int [|r_1(z') - r_2(z')| \vee |\psi_1(z') - \psi_2(z')|]^p \pi(\mathrm{d}z') \right]^{1/p} = \beta d_p \left((\psi_1, r_1), (\psi_2, r_2) \right).
\end{aligned}
$$

Hence, we have shown that Q is a contraction on $L_p(\pi) \times L_p(\pi)$ of modulus β. Claim (1) is now established.

Proof of claim (2). Since $\left(L_p(\pi) \times L_p(\pi), d_p\right)$ is a complete metric space, based on the contraction mapping theorem, Q admits a unique fixed point in $L_p(\pi) \times L_p(\pi)$. In order to prove claim (2), it suffices to show that $(\psi^*, r^*) \in L_p(\pi) \times L_p(\pi)$ and that (ψ^*, r^*) is a fixed point of Q.

Step 1. We show that $v^*, r^*, \psi^* \in L_p(\pi)$. Notice that

$$|v^*(z)| \vee |r^*(z)| \vee |\psi^*(z)| \leq \sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|s(Z_t)| \vee |c(Z_t)|].$$

Hence, we have

$$\begin{aligned}
&\left[\int |v^*(z)|^p \pi(\mathrm{d}z)\right] \vee \left[\int |r^*(z)|^p \pi(\mathrm{d}z)\right] \vee \left[\int |\psi^*(z)|^p \pi(\mathrm{d}z)\right] \\
&\leq \int \left(\sum_{t=0}^{\infty} \beta^t \mathbb{E}_z[|s(Z_t)| \vee |c(Z_t)|]\right)^p \pi(\mathrm{d}z). \qquad (3.B.6)
\end{aligned}$$

By Jensen's inequality, we have

$$\begin{aligned}
\|\mathbb{E}_z[|s(Z_t)| \vee |c(Z_t)|]\|_p &= \left[\int \left(\int |s(z')| \vee |c(z')| P^t(z, \mathrm{d}z')\right)^p \pi(\mathrm{d}z)\right]^{1/p} \\
&\leq \left[\int \int [|s(z')| \vee |c(z')|]^p P^t(z, \mathrm{d}z') \pi(\mathrm{d}z)\right]^{1/p} \\
&= \left[\int \mathbb{E}_z[|s(Z_t)| \vee |c(Z_t)|]^p \pi(\mathrm{d}z)\right]^{1/p}.
\end{aligned}$$

Since π is stationary, the Fubini theorem implies that

$$\int \mathbb{E}_z |s(Z_t)|^p \pi(\mathrm{d}z) = \int \int |s(z')|^p P^t(z, \mathrm{d}z') \pi(\mathrm{d}z) = \int |s(z')|^p \pi(\mathrm{d}z') = \|s\|_p^p.$$

Similarly, we have $\int \mathbb{E}_z |c(Z_t)|^p \pi(\mathrm{d}z) = \|c\|_p^p$. Let $\mathbb{E}_{\cdot} f(Z_t)$ denote the function $z \mapsto \mathbb{E}_z f(Z_t)$. The Minkowski inequality and the results established above then imply that for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\left\|\sum_{t=0}^{n} \beta^t \mathbb{E}_{\cdot}[|s(Z_t)| \vee |c(Z_t)|]\right\|_p \leq \sum_{t=0}^{n} \beta^t \left\|\mathbb{E}_{\cdot}[|s(Z_t)| \vee |c(Z_t)|]\right\|_p \\
&\leq \sum_{t=0}^{n} \beta^t \left[\int \mathbb{E}_z[|s(Z_t)| \vee |c(Z_t)|]^p \pi(\mathrm{d}z)\right]^{1/p} \\
&\leq \sum_{t=0}^{n} \beta^t \left[\int \left(\mathbb{E}_z |s(Z_t)|^p + \mathbb{E}_z |c(Z_t)|^p\right) \pi(\mathrm{d}z)\right]^{1/p} \\
&= \sum_{t=0}^{n} \beta^t \left(\|s\|_p^p + \|c\|_p^p\right)^{1/p} \leq \frac{\left(\|s\|_p^p + \|c\|_p^p\right)^{1/p}}{1-\beta} < \infty. \qquad (3.B.7)
\end{aligned}$$

Moreover, by the monotone convergence theorem, we have

$$\left\|\sum_{t=0}^{n} \beta^t \mathbb{E}_{\cdot}[|s(Z_t)| \vee |c(Z_t)|]\right\|_p \to \left\|\sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\cdot}[|s(Z_t)| \vee |c(Z_t)|]\right\|_p. \qquad (3.B.8)$$

Together, (3.B.7)–(3.B.8) imply that $\left\|\sum_{t=0}^{\infty} \beta^t \mathbb{E}_{\cdot}[|s(Z_t)| \vee |c(Z_t)|]\right\|_p < \infty$. By (3.B.6), we have $\|v^*\|_p \vee \|r^*\|_p \vee \|\psi^*\|_p < \infty$ and thus $v^*, r^*, \psi^* \in L_p(\pi)$.

Step 2. We show that (v^*, r^*) is a fixed point of T and (ψ^*, r^*) is a fixed point of Q, i.e., $d_p\left(T(v^*, r^*), (v^*, r^*)\right) = 0$ and $d_p\left(Q(\psi^*, r^*), (\psi^*, r^*)\right) = 0$. The former obviously holds since lemma 3.A.4 shows that $T(v^*, r^*) = (v^*, r^*)$ π-almost surely. Regarding the latter, note that

$$d_p\left(Q(\psi^*, r^*), (\psi^*, r^*)\right) = d_p\left(LR(\psi^*, r^*), L(v^*, r^*)\right) = d_p\left(L(r^* \vee \psi^*, r^*), L(v^*, r^*)\right)$$
$$= \| |\beta P(r^* \vee \psi^*) - \beta P v^*| \vee |\alpha\beta P(r^* \vee \psi^*) - \alpha\beta P v^*| \|_p = \beta \| P(r^* \vee \psi^*) - P v^* \|_p.$$

Since $\psi^* := c + \beta P v^*$ and lemma 3.A.4 implies that $v^* = r^* \vee (c + \beta P v^*)$ π-almost surely, we have $v^* = r^* \vee \psi^*$ π-almost surely. By Jensen's inequality,

$$\int |P(r^* \vee \psi^*)(z) - Pv^*(z)|^p \pi(\mathrm{d}z) \leq \int \int |(r^* \vee \psi^*)(z') - v^*(z')|^p P(z, \mathrm{d}z') \pi(\mathrm{d}z)$$
$$= \int |(r^* \vee \psi^*)(z') - v^*(z')|^p \pi(\mathrm{d}z') = 0.$$

Hence, $d_p\left(Q(\psi^*, r^*), (\psi^*, r^*)\right) = \beta \| P(r^* \vee \psi^*) - Pv^* \|_p = 0$. The second claim is verified.

Proof of claim (3). **Step 1.** We show that $T(v, r) \in L_p(\pi) \times L_p(\pi)$ for all $(v, r) \in L_p(\pi) \times L_p(\pi)$. For all $z \in \mathsf{Z}$ and $(v, r) \in L_p(\pi) \times L_p(\pi)$, we have

$$\begin{aligned} |\ell(z; v, r)|^p &\leq 3^p |s(z)|^p + (3\alpha\beta)^p \left[\int |v(z')| P(z, \mathrm{d}z') \right]^p \\ &\quad + [3(1-\alpha)\beta]^p \left[\int |r(z')| P(z, \mathrm{d}z') \right]^p \\ &\leq 3^p |s(z)|^p + (3\alpha\beta)^p \int |v(z')|^p P(z, \mathrm{d}z') \\ &\quad + [3(1-\alpha)\beta]^p \int |r(z')|^p P(z, \mathrm{d}z'), \end{aligned}$$

where again the first inequality is based on the elementary fact that $(\sum_{i=1}^n a_i)^p \leq n^p \left(\vee_{i=1}^n a_i\right)^p \leq n^p \sum_{i=1}^n a_i^p$ for all positive $\{a_i\}_{i=1}^n$ and p, and the second inequality is due to Jensen's inequality. Since $s, v, r \in L_p(\pi)$ and π is the stationary distribution of P, Fubini theorem implies that

$$\int \left[\int |v(z')|^p P(z, \mathrm{d}z') \right] \pi(\mathrm{d}z) = \int |v(z')|^p \pi(\mathrm{d}z') = \|v\|_p^p < \infty$$

and

$$\int \left[\int |r(z')|^p P(z, \mathrm{d}z') \right] \pi(\mathrm{d}z) = \int |r(z')|^p \pi(\mathrm{d}z') = \|r\|_p^p < \infty.$$

Based on the above results, we have $\int |\ell(z; v, r)|^p \pi(\mathrm{d}z) < \infty$, i.e., $\|\ell(\cdot; v, r)\|_p < \infty$. Similarly, since $c, v \in L_p(\pi)$, one can show that $c + \beta P v \in L_p(\pi)$. Then

$$\int |\tilde{\ell}(z; v, r)|^p \pi(\mathrm{d}z) \leq \int |\ell(z; v, r)|^p \pi(\mathrm{d}z) + \int |c(z) + \beta P v(z)|^p \pi(\mathrm{d}z) < \infty,$$

i.e., $\|\tilde{\ell}(\cdot\,; v, r)\|_p < \infty$. Hence, $T(v,r) \in L_p(\pi) \times L_p(\pi)$, as was to be shown.

Step 2. We show that T is a contraction mapping on $(L_p(\pi) \times L_p(\pi), d_p)$ of modulus β. For all $z \in \mathsf{Z}$ and $(v_1, r_1), (v_2, r_2) \in L_p(\pi) \times L_p(\pi)$, we have

$$\begin{aligned}
&|\ell(z; v_1, r_1) - \ell(z; v_2, r_2)|^p \\
&= \left| \alpha\beta \int (v_1 - v_2)(z') P(z, \mathrm{d}z') + (1-\alpha)\beta \int (r_1 - r_2)(z') P(z, \mathrm{d}z') \right|^p \\
&\leq \left[\alpha\beta \int |v_1 - v_2|(z') P(z, \mathrm{d}z') + (1-\alpha)\beta \int |r_1 - r_2|(z') P(z, \mathrm{d}z') \right]^p \\
&\leq \left[\beta \int |v_1(z') - v_2(z')| \vee |r_1(z') - r_2(z')| P(z, \mathrm{d}z') \right]^p \\
&\leq \beta^p \int \left[|v_1(z') - v_2(z')| \vee |r_1(z') - r_2(z')| \right]^p P(z, \mathrm{d}z').
\end{aligned}$$

Similarly, for all $z \in \mathsf{Z}$ and $(v_1, r_1), (v_2, r_2) \in L_p(\pi) \times L_p(\pi)$, we have

$$\begin{aligned}
&|\tilde{\ell}(z; v_1, r_1) - \tilde{\ell}(z; v_2, r_2)|^p \\
&\leq \left[|\ell(z; v_1, r_1) - \ell(z; v_2, r_2)| \vee \left(\beta \int |v_1 - v_2|(z') P(z, \mathrm{d}z') \right) \right]^p \\
&\leq |\ell(z; v_1, r_1) - \ell(z; v_2, r_2)|^p \vee \left[\beta^p \int |v_1(z') - v_2(z')|^p P(z, \mathrm{d}z') \right] \\
&\leq \beta^p \int \left[|v_1(z') - v_2(z')| \vee |r_1(z') - r_2(z')| \right]^p P(z, \mathrm{d}z').
\end{aligned}$$

The definition of d_p and Fubini theorem then imply that

$$\begin{aligned}
&d_p \left(T(v_1, r_1), T(v_2, r_2) \right) \\
&= \left[\int |\ell(z; v_1, r_1) - \ell(z; v_2, r_2)|^p \vee |\tilde{\ell}(z; v_1, r_1) - \tilde{\ell}(z; v_2, r_2)|^p \pi(\mathrm{d}z) \right]^{1/p} \\
&\leq \left[\int \beta^p \int [|v_1(z') - v_2(z')| \vee |r_1(z') - r_2(z')|]^p P(z, \mathrm{d}z') \pi(\mathrm{d}z) \right]^{1/p} \\
&= \beta \left[\int [|v_1(z') - v_2(z')| \vee |r_1(z') - r_2(z')|]^p \pi(\mathrm{d}z') \right]^{1/p} = \beta d_p \left((v_1, r_1), (v_2, r_2) \right).
\end{aligned}$$

Hence, T is a contraction on $L_p(\pi) \times L_p(\pi)$ of modulus β. Claim (3) is verified.

Proof of claim (4). We only need to verify that $(v^*, r^*) \in L_p(\pi) \times L_p(\pi)$ and that (v^*, r^*) is a fixed point of T. These results have been established in the proof of claim (2). The proof is now complete. □

Proof of proposition 3.5.1. We first prove claim (1). For all $(\psi, r) \in L_p(\pi) \times L_p(\pi)$,

$$d_p \left[Q \begin{pmatrix} \psi \\ r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right] = d_p \left[Q \begin{pmatrix} \psi \\ r \end{pmatrix}, L \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right] = \|I \vee J\|_p,$$

where

$$I := \beta|P(r \vee \psi) - Pv^*| \quad \text{and} \quad J := \beta|\alpha[P(r \vee \psi) - Pv^*] + (1-\alpha)(Pr - Pr^*)|.$$

Notice that

$$\begin{aligned} I \vee J &\leq \beta \left\{ (P|r \vee \psi - v^*|) \vee [\alpha P|r \vee \psi - v^*| + (1-\alpha)P|r - r^*|] \right\} \\ &\leq \beta P \left(|r \vee \psi - v^*| \vee |r - r^*| \right). \end{aligned}$$

Hence, by Jensen's inequality and Fubini theorem, we have

$$\begin{aligned} \|I \vee J\|_p &\leq \beta \left\{ \int \left[\int |r(z') \vee \psi(z') - v^*(z')| \vee |r(z') - r^*(z')| P(z, \mathrm{d}z') \right]^p \pi(\mathrm{d}z) \right\}^{1/p} \\ &\leq \beta \left\{ \int \int [|r(z') \vee \psi(z') - v^*(z')| \vee |r(z') - r^*(z')|]^p P(z, \mathrm{d}z') \pi(\mathrm{d}z) \right\}^{1/p} \\ &= \beta \left\{ \int [|r(z') \vee \psi(z') - v^*(z')| \vee |r(z') - r^*(z')|]^p \pi(\mathrm{d}z') \right\}^{1/p} \\ &= \beta \, d_p \left(R(\psi, r), (v^*, r^*) \right). \end{aligned}$$

Hence, $d_p\left(Q(\psi,r),(\psi^*,r^*)\right) \leq \beta\, d_p\left(R(\psi,r),(v^*,r^*)\right)$, and claim (1) holds for $t = 0$. Now suppose claim (1) holds for arbitrary t. By the induction hypothesis we have $d_p\left(Q^t(\psi,r),(\psi^*,r^*)\right) \leq \beta\, d_p\left(T^{t-1}R(\psi,r),(v^*,r^*)\right)$ for all $(\psi,r) \in L_p(\pi) \times L_p(\pi)$. Since Q and T are semiconjugate as shown in section 3.3, we have

$$\begin{aligned} d_p \left[Q^{t+1} \begin{pmatrix} \psi \\ r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right] &= d_p \left[Q^t Q \begin{pmatrix} \psi \\ r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right] \leq \beta d_p \left[T^{t-1} R Q \begin{pmatrix} \psi \\ r \end{pmatrix}, \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right] \\ &= \beta d_p \left[T^{t-1} T R \begin{pmatrix} \psi \\ r \end{pmatrix}, \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right] = \beta d_p \left[T^t R \begin{pmatrix} \psi \\ r \end{pmatrix}, \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right]. \end{aligned}$$

Hence, claim (1) holds by induction.

Regarding claim (2), for all $(v,r) \in L_p(\pi) \times L_p(\pi)$, we have

$$d_p \left[T \begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} v^* \\ r^* \end{pmatrix} \right] = d_p \left[T \begin{pmatrix} v \\ r \end{pmatrix}, R \begin{pmatrix} \psi^* \\ r^* \end{pmatrix} \right] = \|\tilde{I} \vee \tilde{J}\|_p,$$

where (recall ℓ and $\tilde{\ell}$ defined respectively in (3.2.4) and (3.B.5))

$$\tilde{I} := |\ell(\cdot\,; v, r) - r^*| \quad \text{and} \quad \tilde{J} := |\tilde{\ell}(\cdot\,; v, r) - r^* \vee \psi^*|.$$

Hence,

$$\begin{aligned} \tilde{I} \vee \tilde{J} &= |\ell(\cdot\,; v, r) - r^*| \vee |\tilde{\ell}(\cdot\,; v, r) - r^* \vee \psi^*| \\ &\leq |\ell(\cdot\,; v, r) - r^*| \vee [|\ell(\cdot\,; v, r) - r^*| \vee |c + \beta P v - \psi^*|] \\ &= |\ell(\cdot\,; v, r) - r^*| \vee |c + \beta P v - \psi^*| \end{aligned}$$

and we have

$$d_p\left(T(v,r),(v^*,r^*)\right) \leq \| |\ell(\cdot\,;v,r) - r^*| \vee |c + \beta P v - \psi^*| \|_p = d_p\left(L(v,r),(\psi^*,r^*)\right).$$

Thus, claim (2) holds for $t = 0$. Suppose claim (2) holds for arbitrary t. Then the induction hypothesis implies that $d_p\left(T^t(v,r),(v^*,r^*)\right) \leq d_p\left(Q^{t-1}L(v,r),(\psi^*,r^*)\right)$ for all $(v,r) \in L_p(\pi) \times L_p(\pi)$. Since Q and T are semiconjugate as shown in section 3.3, we have

$$\begin{aligned} d_p\left[T^{t+1}\begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} v^* \\ r^* \end{pmatrix}\right] &= d_p\left[T^t T\begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} v^* \\ r^* \end{pmatrix}\right] \leq d_p\left[Q^{t-1}LT\begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix}\right] \\ &= d_p\left[Q^{t-1}QL\begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix}\right] = d_p\left[Q^t L\begin{pmatrix} v \\ r \end{pmatrix}, \begin{pmatrix} \psi^* \\ r^* \end{pmatrix}\right]. \end{aligned}$$

Hence, claim (2) holds by induction. This concludes the proof. □

Chapter 4

Extension II: Dynamic Discrete Choices

4.1 Introduction

In chapters 2–3, the key state component (i.e., the state variables that appear in the reward functions) of the sequential decision problems evolves as an exogenous Markov process. Although such frameworks cover a wide range of binary choice sequential problems, there are other cases in which evolution of the key states follows a controlled Markov process (i.e., the evolution is affected at least partially by some control variables). Such settings are common for sequential decision problems where agents have more than two choices.

A standard example in economics is on-the-job search, where an employed worker can choose from quitting the job market and taking the unemployment compensation, staying in the current job for a stochastic wage return, or searching for a new job. In general, the worker's productivity (key state component) evolves according to different transition laws depending on whether the worker stays in the current job or search for a new one. See, for example, Jovanovic (1987), Bull and Jovanovic (1988), and Gomes et al. (2001). Other examples in economics can be found at Crawford and Shum (2005), Cooper et al. (2007), Vereshchagina and Hopenhayn (2009), Low et al. (2010), and Moscarini and Postel-Vinay (2013).

In econometrics, such problems are generally called dynamic discrete choice models, and the framework is widely employed in both theoretical and empirical studies. See, for example, Rust (1988), Eckstein and Wolpin (1989), Rust (1994), Aguirregabiria and Mira (2002), Bajari et al. (2007), Norets (2009, 2010), Aguirregabiria and Mira (2010), and Su and Judd (2012).

In this chapter, we extend our theory to cover this class of problems, which we refer to as dynamic discrete choice problems in line with the econometric literature. Section 4.3 shows that the Bellman and Jovanovic operators are semiconjugate in general, with the same implications as those of chapters 2–3. In section 4.4, we add a generic weighted supremum norm topology and show that the Bellman and Jovanovic operators are both contraction mappings under identical assumptions, and that convergence to the respective fixed points occurs at the same rate. These properties are established by constructing a metric that evaluates the maximum of the weighted supremum norm distances along each dimension of the candidate function space.

On the other hand, the dynamics of the current setting are more complicated to some degree. Since the key state component is in general a controlled Markov process with its evolution governed by an N-choice control variable, it is hard to exploit the ergodicity structures of the state process. As a result, establishing the symmetric properties under the L_p-norm topology is more challenging. For that reason, the L_p framework is left for future research.

4.2 Dynamic Discrete Choices

Unless otherwise specified, the notation of chapter 2 will continue to be used throughout this chapter.

To treat this type of problem generally, suppose that in period t, the agent observes Z_t and makes choices among N alternatives. Let $\{I_t\}_{t\geq 0}$ be the sequence of control variables with $I_t = i$ if the agent chooses alternative $i \in \{1, \cdots, N\}$. A selection of I_t results in a current reward $F(Z_t, I_t)$, or $r_{I_t}(Z_t)$ for simplicity. The state process $\{Z_t\}_{t\geq 0}$ evolves according to a controlled Markov process with stochastic kernel $P(Z_t, I_t; \mathrm{d}Z_{t+1})$, or more simply $P_{I_t}(Z_t, \mathrm{d}Z_{t+1})$. In particular, $P_i(z, \mathrm{d}z')$ can be interpreted as the transition probability of $\{Z_t\}$ if alternative i is selected in the current period. The agent aims to find an optimal policy $\{I_t^*\}_{t\geq 0}$ that maximizes the expected discounted lifetime rewards.

The value function of the problem is defined by

$$v^*(z) := \sup_{\{I_t\}_{t\geq 0}} \mathbb{E}_z \left\{ \sum_{t=0}^{\infty} \beta^t F(Z_t, I_t) \right\}. \tag{4.2.1}$$

Under certain assumptions, v^* satisfies[1]

$$v^*(z) = \max\{\psi_1^*(z), \cdots, \psi_N^*(z)\}, \tag{4.2.2}$$

where for $i = 1, \cdots, N$,

$$\psi_i^*(z) = r_i(z) + \beta \int v^*(z') P_i(z, \mathrm{d}z'). \tag{4.2.3}$$

We call ψ_i^* the *continuation value associated with alternative* i, which can be interpreted as the maximal expected discounted lifetime reward from choosing i. The Bellman operator corresponding to this problem is

$$Tv(z) = \max_{i \in \{1, \cdots, N\}} \left\{ r_i(z) + \beta \int v(z') P_i(z, \mathrm{d}z') \right\}.$$

Define the *continuation value function* as $\psi^* := (\psi_1^*, \cdots, \psi_N^*)$. Then (4.2.2)–(4.2.3) imply that ψ_i^* can be written as

$$\psi_i^*(z) = r_i(z) + \beta \int \max\{\psi_1^*(z'), \cdots, \psi_N^*(z')\} P_i(z, \mathrm{d}z') \tag{4.2.4}$$

for $i = 1, \cdots, N$. For each $\psi := (\psi_1, \cdots, \psi_N)$ and $z \in \mathsf{Z}$, the *continuation value operator* or *Jovanovic operator* is defined as

$$Q\psi(z) = Q \begin{pmatrix} \psi_1 \\ \cdots \\ \psi_N \end{pmatrix} (z) = \begin{pmatrix} r_1(z) + \beta \int (\psi_1 \vee \cdots \vee \psi_N)(z') P_1(z, \mathrm{d}z') \\ \cdots \\ r_N(z) + \beta \int (\psi_1 \vee \cdots \vee \psi_N)(z') P_N(z, \mathrm{d}z') \end{pmatrix}. \tag{4.2.5}$$

4.3 General Theory

In this section, we show that Bellman and Jovanovic operators are semiconjugate in a generic framework and discuss the implications. As in chapters 2–3, the semiconjugate relationship is shown using operator-theoretic notation. To this end, for all integrable function $h \in m\mathscr{B}$ and $i \in \{1, \cdots, N\}$, let

$$P_i h(z) := \int h(z') P_i(z, \mathrm{d}z').$$

Observe that the Bellman operator T can then be expressed as $T = RL$, where for each $\psi = (\psi_1, \cdots, \psi_N)$ and v,

$$R\psi := \vee_{i=1}^N \psi_i \quad \text{and} \quad Lv := (r_1 + \beta P_1 v, \cdots, r_N + \beta P_N v). \tag{4.3.1}$$

(Recall that for any two operators we write the composition $A \circ B$ simply as AB.)

[1]Later we provide sufficient conditions based on our primitive set up. See, for example, lemma 4.A.2 in appendix 4.A.

Let $\mathcal{V}$ be a subset of $m\mathscr{B}$ such that $v^* \in \mathcal{V}$ and $T\mathcal{V} \subset \mathcal{V}$. The set $\mathcal{V}$ is understood as the set of candidate value functions. (Specific classes of functions are considered in the next section.) Let $\mathcal{C}$ be defined by[2]

$$\mathcal{C} := L\mathcal{V} = \left\{ \psi = (\psi_1, \cdots, \psi_N) \in \times_{i=1}^N m\mathscr{B} \colon \psi = Lv \text{ for some } v \in \mathcal{V} \right\}.$$

By definition, L is a surjective mapping from $\mathcal{V}$ onto $\mathcal{C}$. It is also true that R maps $\mathcal{C}$ into $\mathcal{V}$, as in chapters 2–3. Indeed, if $\psi \in \mathcal{C}$, then there exists a $v \in \mathcal{V}$ such that $\psi = Lv$, and $R\psi = RLv = Tv$, which lies in $\mathcal{V}$ by assumption.

Lemma 4.3.1. *On $\mathcal{C}$, the operator Q satisfies $Q = LR$, and $Q\mathcal{C} \subset \mathcal{C}$.*

Proof. The first claim is immediate from the definitions. The second follows from the claims just established (i.e., R maps $\mathcal{C}$ to $\mathcal{V}$ and L maps $\mathcal{V}$ to $\mathcal{C}$). □

The preceding discussion implies that Q and T are *semiconjugate*, in the sense that $LT = QL$ on $\mathcal{V}$ and $TR = RQ$ on $\mathcal{C}$. Indeed, since $T = RL$ and $Q = LR$, we have $LT = LRL = QL$ and $TR = RLR = RQ$ as claimed. This leads to the key results of this section listed and proved in the following:

Proposition 4.3.1. *The following statements are true:*

(1) If v is a fixed point of T in $\mathcal{V}$, then Lv is a fixed point of Q in $\mathcal{C}$.

(2) If ψ is a fixed point of Q in $\mathcal{C}$, then $R\psi$ is a fixed point of T in $\mathcal{V}$.

Proof. To prove the first claim, fix $v \in \mathcal{V}$. By the definition of $\mathcal{C}$, $Lv \in \mathcal{C}$. Moreover, since $v = Tv$, we have $QLv = LTv = Lv$. Hence, Lv is a fixed point of Q in $\mathcal{C}$. Regarding the second claim, fix $\psi \in \mathcal{C}$. Since R maps $\mathcal{C}$ into $\mathcal{V}$ as shown above, $R\psi \in \mathcal{V}$. Since $\psi = Q\psi$, we have $TR\psi = RQ\psi = R\psi$. Hence, $R\psi$ is a fixed point of T in $\mathcal{V}$. □

The next result, which parallels propositions 2.3.2 and 3.3.2, implies that, at least on a theoretical level, iterating with either T or Q is essentially equivalent.

Proposition 4.3.2. *$T^{t+1} = RQ^tL$ on $\mathcal{V}$ and $Q^{t+1} = LT^tR$ on $\mathcal{C}$ for all $t \in \mathbb{N}_0$.*

Proof. That the claim holds when $t = 0$ has already been established. Now suppose the claim is true for arbitrary t. By the induction hypothesis we have

[2] We define $\times_{i=1}^N m\mathscr{B} := m\mathscr{B} \times \cdots \times m\mathscr{B}$. Moreover, the definition of $\mathcal{C}$ implies that for all $\psi = (\psi_1, \cdots, \psi_N) \in \mathcal{C}$, there exists a fixed element $v \in \mathcal{V}$ such that $\psi_i = r_i + \beta P_i v$ for all $i = 1, \cdots, N$.

$T^t = RQ^{t-1}L$ and $Q^t = LT^{t-1}R$. Since Q and T are semiconjugate as shown above, we have $T^{t+1} = TT^t = TRQ^{t-1}L = RQQ^{t-1}L = RQ^tL$ and $Q^{t+1} = QQ^t = QLT^{t-1}R = LTT^{t-1}R = LT^tR$. Hence, the claim holds by induction. $\square$

The theory above is based on the primitive assumption of a candidate value function space $\mathcal{V}$ with properties $v^* \in \mathcal{V}$ and $T\mathcal{V} \subset \mathcal{V}$. Similar results can be established if we start with a generic candidate continuation value function space $\mathscr{C}$ that satisfies $\psi^* \in \mathscr{C}$ and $Q\mathscr{C} \subset \mathscr{C}$. Appendix 4.A outlines the main idea.

4.4 Symmetry under Weighted Supremum Norm

As in chapters 2–3, we next impose a weighted supremum norm on the domain of T and Q in order to compare contractivity, optimality and related properties. The following assumption is a generalization of the standard weighted supremum norm assumption of Boyd (1990).

Assumption 4.4.1. There exist a $\mathscr{B}$-measurable function $g : \mathsf{Z} \to \mathbb{R}_+$ and constants $a, b, m, d \in \mathbb{R}_+$ such that $\beta m < 1$, and, for all $z \in \mathsf{Z}$ and $i, j = 1, \cdots, N$,

$$\int |r_i(z')| P_j(z, \mathrm{d}z') \leq a\, g(z) + b \tag{4.4.1}$$

$$\text{and} \quad \int g(z') P_i(z, \mathrm{d}z') \leq m g(z) + d. \tag{4.4.2}$$

The interpretation is that $\mathbb{E}_z |r_i(Z_1)|$ is small relative to some function g such that $\mathbb{E}_z g(Z_t)$ does not grow too fast. Slow growth of $\mathbb{E}_z g(Z_t)$ is imposed by the geometric drift condition (4.4.2) (see, e.g., Meyn and Tweedie (2009), chapter 15). Note that the following statements hold:

(a) If both r and c are bounded, then assumption 4.4.1 holds for $g := \vee_{i=1}^N \|r_i\|$, $m := 1$ and $d := 0$.

(b) Assumption 4.4.1 reduces to the standard weighted supremum norm assumption of Boyd (1990) if instead of imposing condition (4.4.1), we assume that there exist $a, b \in \mathbb{R}_+$ such that $\vee_{i=1}^N |r_i(z)| \leq a g(z) + b$.

Regarding claim (b), notice that the latter condition implies the former since condition (4.4.2) holds. Here we admit consideration of one-step future transition to enlarge the set of possible weight functions.

Choose $m', d' \in \mathbb{R}_{++}$ such that

$$Nm'a + m > 1, \quad \rho := \beta(Nm'a + m) < 1 \quad \text{and} \quad d' \geq \frac{Nm'b + d}{Nm'a + m - 1}. \tag{4.4.3}$$

Let the weight function $\kappa\colon \mathsf{Z} \to \mathbb{R}_+$ be defined by

$$\kappa(z) := m' \sum_{i=1}^{N} |r_i(z)| + g(z) + d'. \tag{4.4.4}$$

As mentioned in chapter 2, $(b_\kappa \mathsf{Z}, d_\kappa)$ is a complete metric space, where

$$d_\kappa(v, \tilde{v}) := \|v - \tilde{v}\|_\kappa \quad \text{for all} \quad v, \tilde{v} \in b_\kappa \mathsf{Z}.$$

Consider the product space $\left(\times_{i=1}^N b_\kappa \mathsf{Z},\, \rho_\kappa\right)$, where for all $\psi = (\psi_1, \cdots, \psi_N)$ and $\tilde{\psi} = (\tilde{\psi}_1, \cdots, \tilde{\psi}_N)$ in $\times_{i=1}^N b_\kappa \mathsf{Z}$, the distance ρ_κ is defined by

$$\rho_\kappa(\psi, \tilde{\psi}) := \vee_{i=1}^N \|\psi_i - \tilde{\psi}_i\|_\kappa.$$

Lemma 4.A.3 (see appendix 4.A) shows that $\left(\times_{i=1}^N b_\kappa \mathsf{Z},\, \rho_\kappa\right)$ is a complete metric space.

Recall $\rho \in (0, 1)$ defined in (4.4.3). The following result shows that Q and T are both contraction mappings under identical assumptions.

Theorem 4.4.1. *Under assumption 4.4.1, the following statements hold:*

(1) Q is a contraction mapping on $\left(\times_{i=1}^N b_\kappa \mathsf{Z},\, \rho_\kappa\right)$ of modulus ρ.

(2) The unique fixed point of Q in $\times_{i=1}^N b_\kappa \mathsf{Z}$ is $\psi^* = (\psi_1^*, \cdots, \psi_N^*)$.

(3) T is a contraction mapping on $(b_\kappa \mathsf{Z}, d_\kappa)$ of modulus ρ.

(4) The unique fixed point of T in $b_\kappa \mathsf{Z}$ is v^*.

The next result shows that the rate of convergence of Q and T to their respective fixed points is the same. In stating it, R and L are as defined in (4.3.1), and $\rho \in (0, 1)$ is the contraction coefficient defined in (4.4.3).

Proposition 4.4.1. *If assumption 4.4.1 holds, then*

$$R\left(\times_{i=1}^N b_\kappa \mathsf{Z}\right) \subset b_\kappa \mathsf{Z} \quad \textit{and} \quad L\left(b_\kappa \mathsf{Z}\right) \subset \times_{i=1}^N b_\kappa \mathsf{Z},$$

and for all $t \in \mathbb{N}_0$, the following statements are true:

(1) $\rho_\kappa\left(Q^{t+1}\psi,\, \psi^*\right) \leq \rho\, d_\kappa\left(T^t R \psi,\, v^*\right)$ for all $\psi \in \times_{i=1}^N b_\kappa \mathsf{Z}$.

(2) $d_\kappa\left(T^{t+1} v,\, v^*\right) \leq \rho_\kappa\left(Q^t L v,\, \psi^*\right)$ for all $v \in b_\kappa \mathsf{Z}$.

Proposition 4.4.1 extends proposition 4.3.2 and lemma 4.A.1, and their connections can be seen by letting $\mathcal{V} := b_\kappa \mathsf{Z}$ and $\mathscr{C} := \times_{i=1}^N b_\kappa \mathsf{Z}$.

Similar to chapters 2–3, the Bellman and Jovanovic operators are also symmetric in terms of continuity of fixed points, as illustrated by the following result.

Assumption 4.4.2. The following conditions hold for all $i \in \{1, \cdots, N\}$:

(1) The stochastic kernel P_i is Feller; that is, $z \mapsto \int h(z') P_i(z, \mathrm{d}z')$ is continuous and bounded on Z whenever h is.

(2) r_i, κ and $z \mapsto \int \kappa(z') P_i(z, \mathrm{d}z')$ are continuous.

Proposition 4.4.2. *If assumptions 4.4.1–4.4.2 hold, then ψ^* and v^* are continuous.*

4.5 Application: On-the-Job Search

Consider a standard on-the-job search framework (see, e.g., Bull and Jovanovic (1988) and Gomes et al. (2001)). Each period, an employee has three choices: quit the job market, stay in the current job, or search for a new job. Let c_0 be the value of leisure and θ be the worker's productivity at a given firm, with $(\theta_t)_{t \geq 0} \overset{\text{IID}}{\sim} G(\theta)$. Let p be the current price. The price sequence $(p_t)_{t \geq 0}$ is Markov with transition probability $F(p'|p)$ and stationary distribution $F^*(p)$. It is assumed that there is no aggregate shock so that F^* is the distribution of prices over firms. The current wage of the worker is $p\theta$. The value function satisfies $v^* = \psi_1^* \vee \psi_2^* \vee \psi_3^*$, where

$$\psi_1^*(p, \theta) := c_0 + \beta \int v^*(p', \theta') \, \mathrm{d}F^*(p') \, \mathrm{d}G(\theta')$$

denotes the expected value of quitting the job, while

$$\psi_2^*(p, \theta) := p\theta + \beta \int v^*(p', \theta) \, \mathrm{d}F(p'|p)$$

is the expected value of staying in the current firm, and

$$\psi_3^*(p, \theta) := p\theta + \beta \int v^*(p', \theta') \, \mathrm{d}F^*(p') \, \mathrm{d}G(\theta')$$

represents the expected value of searching for a new job. Bull and Jovanovic (1988) assumes that there are compact supports $[\underline{\theta}, \bar{\theta}]$ and $[\underline{p}, \bar{p}]$ for the state processes $(\theta_t)_{t \geq 0}$ and $(p_t)_{t \geq 0}$, where $0 < \underline{\theta} < \bar{\theta} < \infty$ and $0 < \underline{p} < \bar{p} < \infty$. This assumption can be relaxed based on our theory. Let the state space be $\mathsf{Z} := \mathbb{R}_+^2$. Let $\mu_p := \int p \, \mathrm{d}F^*(p)$ and $\mu_\theta := \int \theta \, \mathrm{d}G(\theta)$.

Assumption 4.5.1. There exist a Borel measurable map $\tilde{g} : \mathbb{R}_+ \to \mathbb{R}_+$, and constants $\tilde{m}, \tilde{d} \in \mathbb{R}_+$ such that $\beta \tilde{m} < 1$, and, for all $p \in \mathbb{R}_+$,

(1) $\int p' \, \mathrm{d}F(p'|p) \leq \tilde{g}(p)$,

(2) $\int \tilde{g}(p') \, \mathrm{d}F(p'|p) \leq \tilde{m}\tilde{g}(p) + \tilde{d}$,

(3) $\mu_p, \mu_\theta < \infty$ and $\mu_{\tilde{g}} := \int \tilde{g}(p) \, \mathrm{d}F^*(p) < \infty$.

Let $\tilde{m} > 1$ and $\tilde{m}' \geq d/(\tilde{m} - 1)$, then assumption 4.4.1 holds by letting $g(p, \theta) := \theta(\tilde{g}(p) + \tilde{m}')$, $m := \tilde{m}$ and $d := \mu_\theta(\mu_{\tilde{g}} + m')$. By theorem 4.4.1, Q is a contraction mapping on $\left(\times_{i=1}^{3} b_\ell Z, \rho_\ell\right)$. Obviously, assumption 4.5.1 is weaker than the assumption of compact supports.

Appendix 4.A Some Lemmas

To see the symmetric properties of Q and T from an alternative perspective, we start our analysis with a generic candidate continuation value function space. Let $\mathscr{C}$ be a subset of $\times_{i=1}^{N} m\mathscr{B}$ such that $\psi^* = (\psi_1^*, \cdots, \psi_N^*) \in \mathscr{C}$ and $Q\mathscr{C} \subset \mathscr{C}$. Let $\mathscr{V}$ be defined by

$$\mathscr{V} := R\mathscr{C} = \left\{ v \in m\mathscr{B} \colon v = \vee_{i=1}^{N} \psi_i \text{ for some } \psi = (\psi_1, \cdots, \psi_N) \in \mathscr{C} \right\}. \tag{4.A.1}$$

Then R is a surjective map from $\mathscr{C}$ onto $\mathscr{V}$, $Q = LR$ on $\mathscr{C}$ and $T = RL$ on $\mathscr{V}$. The following result parallels the theory of section 4.3, and is helpful for deriving important convergence properties once topological structure is added to the generic setting, as to be shown.

Lemma 4.A.1. *The following statements are true:*

(1) $L\mathscr{V} \subset \mathscr{C}$ *and* $T\mathscr{V} \subset \mathscr{V}$.

(2) If v *is a fixed point of* T *in* $\mathscr{V}$*, then* Lv *is a fixed point of* Q *in* $\mathscr{C}$.

(3) If ψ *is a fixed point of* Q *in* $\mathscr{C}$*, then* $R\psi$ *is a fixed point of* T *in* $\mathscr{V}$.

(4) $T^{t+1} = RQ^tL$ *on* $\mathscr{V}$ *and* $Q^{t+1} = LT^tR$ *on* $\mathscr{C}$ *for all* $t \in \mathbb{N}_0$.

Proof. The proof is similar to that of propositions 4.3.1–4.3.2 and thus omitted. □

Lemma 4.A.2. *Under assumption 4.4.1,* v^* *and* ψ^* *satisfy* (4.2.2)–(4.2.3).

Proof of lemma 4.A.2. The Bellman equation corresponding to the problem stated in (4.2.1) is

$$v(z_0) = \max_{i_0 \in \{1, \cdots, N\}} \left\{ r_{i_0}(z_0) + \beta \int v(z_1) P_{i_0}(z_0, \mathrm{d}z_1) \right\}. \tag{4.A.2}$$

Under assumption 4.4.1, this Bellman equation is well-defined. In particular, if $\tilde{\psi} := (\tilde{\psi}_1, \cdots, \tilde{\psi}_N)$ is the unique fixed point of Q under $\times_{i=1}^{N} b_\kappa \mathsf{Z}$ obtained from theorem 4.4.1, then $\tilde{v} := \vee_{i=1}^{N} \tilde{\psi}_i \in b_\kappa \mathsf{Z}$ solves the Bellman equation (4.A.2). It remains to verify that any solution $v \in b_\kappa \mathsf{Z}$ to the Bellman equation defined by (4.A.2) satisfies $v = v^*$. Note that for all feasible plan $\{j_t\}_{t \geq 0}$, we have

$$\begin{aligned}
v(z_0) &\geq r_{j_0}(z_0) + \beta \int v(z_1) P_{j_0}(z_0, i_0) \\
&\geq r_{j_0}(z_0) + \beta \int \left[r_{j_1}(z_1) + \beta \int v(z_2) P_{j_1}(z_1, \mathrm{d}z_2) \right] P_{j_0}(z_0, \mathrm{d}z_1) \\
&= r_{j_0}(z_0) + \beta \mathbb{E}_{z_0}^{j_0} r_{j_1}(Z_1) + \beta^2 \mathbb{E}_{z_0}^{j_0 j_1} v(Z_2) \geq \cdots \\
&\geq \sum_{t=0}^{K} \beta^t \mathbb{E}_{z_0}^{j_0 \cdots j_{t-1}} r_{j_t}(Z_t) + \beta^{K+1} \mathbb{E}_{z_0}^{j_0 \cdots j_K} v(Z_{K+1})
\end{aligned} \tag{4.A.3}$$

for all $K \in \mathbb{N}$. Since $v \in b_\kappa \mathsf{Z}$, there exists $G \in \mathbb{R}_+$ such that $|v| \leq G\kappa$. Thus,

$$\begin{aligned}\left|\mathbb{E}_{z_0}^{j_0\cdots j_{K-1}} v(Z_K)\right| &\leq G\,\mathbb{E}_{z_0}^{j_0\cdots j_{K-1}} \kappa(Z_K)\\ &= G\cdot\left(m'\sum_{i=1}^{N}\mathbb{E}_{z_0}^{j_0\cdots j_{K-1}}|r_i(Z_K)| + \mathbb{E}_{z_0}^{j_0\cdots j_{K-1}} g(Z_K) + d'\right). \end{aligned} \tag{4.A.4}$$

Based on the Markov property, for all $z_0 \in \mathsf{Z}$ and $t \in \mathbb{N}_0$,

$$\begin{aligned}\mathbb{E}_{z_0}^{j_0\cdots j_{t-1}} g(Z_t) &= \mathbb{E}_{z_0}^{j_0\cdots j_{t-2}}\mathbb{E}_{Z_{t-1}}^{j_{t-1}} g(Z_t) \leq m\mathbb{E}_{z_0}^{j_0\cdots j_{t-2}} g(Z_{t-1}) + d\\ &\leq \cdots \leq m^t g(z_0) + \frac{1-m^t}{1-m}d,\end{aligned}$$

and for all $t \geq 1$ and $i = 1, \cdots, N$,

$$\begin{aligned}\mathbb{E}_{z_0}^{j_0\cdots j_{t-1}}|r_i(Z_t)| &= \mathbb{E}_{z_0}^{j_0\cdots j_{t-2}}\mathbb{E}_{Z_{t-1}}^{j_{t-1}}|r_i(Z_t)| \leq a\,\mathbb{E}_{z_0}^{j_0\cdots j_{t-2}} g(Z_{t-1}) + b\\ &\leq \cdots \leq a\left(m^{t-1}g(z_0) + \frac{1-m^{t-1}}{1-m}d\right) + b.\end{aligned} \tag{4.A.5}$$

Substituting these results into (4.A.4), we can show that $\lim_{K\to\infty}\beta^K\mathbb{E}_{z_0}^{j_0\cdots j_{K-1}} v(Z_K) = 0$ for all $z_0 \in \mathsf{Z}$. Let $K \to \infty$, from (4.A.3) we know that, for all $\{j_t\}_{t\geq 0}$,

$$v(z_0) \geq \sum_{t=0}^{\infty}\beta^t\mathbb{E}_{z_0}^{j_0\cdots j_{t-1}}\, r_{j_t}(Z_t).$$

Hence, $v \geq v^*$. Notice that since (4.A.2) is a discrete choice problem, there exists a plan $\{i_t\}_{t\geq 0}$ such that (4.A.3) holds with equality in each step, which implies $v \leq v^*$. Hence, $v = v^*$, as was to be shown. □

Recall $\left(\times_{i=1}^N b_\kappa \mathsf{Z},\, \rho_\kappa\right)$ constructed in section 4.4.

Lemma 4.A.3. $\left(\times_{i=1}^N b_\kappa \mathsf{Z},\, \rho_\kappa\right)$ *is a complete metric space.*

Proof. We first show that ρ_κ is a well-defined metric. We only prove the triangular inequality since the other required properties of a metric hold trivially for ρ_κ. For all $f = (f_1, \cdots, f_N), f' = (f_1', \cdots, f_N'), f'' = (f_1'', \cdots, f_N'') \in \times_{i=1}^N b_\kappa \mathsf{Z}$, we have

$$\begin{aligned}\rho_\kappa\left(f, f'\right) &= \vee_{i=1}^N\|f_i - f_i'\|_\kappa \leq \vee_{i=1}^N\left(\|f_i - f_i''\|_\kappa + \|f_i'' - f_i'\|_\kappa\right)\\ &\leq \vee_{i=1}^N\|f_i - f_i''\|_\kappa + \vee_{i=1}^N\|f_i'' - f_i'\|_\kappa = \rho_\kappa\left(f, f''\right) + \rho_\kappa\left(f'', f'\right).\end{aligned}$$

Hence, the triangular inequality holds and ρ_κ is a well-defined metric.

To show that the space is complete, let $\{f_n\} = \{(f_n^1, \cdots, f_n^N)\}$ be a Cauchy sequence of $(\times_{i=1}^N b_\kappa \mathsf{Z}, \rho_\kappa)$. Then for all $i \in \{1, \cdots, N\}$, $\{f_n^i\}$ is a Cauchy sequence of the Banach space $(b_\kappa \mathsf{Z}, \|\cdot\|_\kappa)$, and thus $\|f_n^i - f^i\|_\kappa \to 0$ as $n \to \infty$ for some $f^i \in b_\kappa \mathsf{Z}$. This implies that $f := (f^1, \cdots, f^N) \in \times_{i=1}^N b_\kappa \mathsf{Z}$ and that $\rho_\kappa(f_n, f) = \vee_{i=1}^N\|f_n^i - f^i\|_\kappa \to 0$ as $n \to \infty$. Hence, completeness is established. □

Appendix 4.B Main Proofs

Proof of theorem 4.4.1. **Step 1.** We prove claim (1). By assumption 4.4.1 and the construction of m' and d' in (4.4.3), for all $z \in \mathsf{Z}$ and $i = 1, \cdots, N$, we have

$$\int \kappa(z')P_i(z, \mathrm{d}z') = m' \sum_{j=1}^{N} \int |r_j(z')| P_i(z, \mathrm{d}z') + \int g(z')P_i(z, \mathrm{d}z') + d'$$
$$\leq (Nm'a + m)\left[g(z) + \frac{Nm'b + d + d'}{Nm'a + m}\right] \leq (Nm'a + m)\kappa(z). \tag{4.B.1}$$

Next, we show that Q maps $\left(\times_{i=1}^N b_\kappa \mathsf{Z},\, \rho_\kappa\right)$ into itself. For all $\psi = (\psi_1, \cdots, \psi_N) \in \times_{i=1}^N b_\kappa \mathsf{Z}$ and $i \in \{1, \cdots, N\}$, we define

$$h_i(z) := r_i(z) + \beta \int \max\{\psi_1(z'), ..., \psi_N(z')\} P_i(z, \mathrm{d}z'). \tag{4.B.2}$$

Then there exists $G \in \mathbb{R}_+$ such that for all $z \in \mathsf{Z}$ and $i \in \{1, \cdots, N\}$,

$$\left|\frac{h_i(z)}{\kappa(z)}\right| \leq \left|\frac{r_i(z)}{\kappa(z)}\right| + \frac{\beta G \int \kappa(z') P_i(z, \mathrm{d}z')}{\kappa(z)} \leq \frac{1}{m'} + \beta G(Nm'a + m) < \infty.$$

Hence, $Q\psi \in \times_{i=1}^N b_\kappa \mathsf{Z}$ and Q is a self-map on $\times_{i=1}^N b_\kappa \mathsf{Z}$, as was to be shown.

We then show that Q is a contraction on $\left(\times_{i=1}^N b_\kappa \mathsf{Z}, \rho_\kappa\right)$. For all $\psi, \tilde{\psi} \in \times_{i=1}^N b_\kappa \mathsf{Z}$, we have $\rho_\kappa(Q\psi, Q\tilde{\psi}) = \vee_{j=1}^N J_j$, where

$$J_j = \left\| \beta P_j \left(\vee_{i=1}^N \psi_i\right) - \beta P_j \left(\vee_{i=1}^N \tilde{\psi}_i\right) \right\|_\kappa \qquad (j = 1, \cdots, N)$$

For all $z \in \mathsf{Z}$ and $j \in \{1, \cdots, N\}$, we have

$$\begin{aligned}
&\left| P_j \left(\vee_{i=1}^N \psi_i\right)(z) - P_j\left(\vee_{i=1}^N \tilde{\psi}_i\right)(z) \right| \\
&\leq \int \left| \left(\vee_{i=1}^N \psi_i\right)(z') - \left(\vee_{i=1}^N \tilde{\psi}_i\right)(z') \right| P_j(z, \mathrm{d}z') \\
&\leq \int \left(\vee_{i=1}^N |\psi_i - \tilde{\psi}_i|\right)(z') P_j(z, \mathrm{d}z') \\
&\leq \left(\vee_{i=1}^N \|\psi_i - \tilde{\psi}_i\|_\kappa\right) \int \kappa(z') P_j(z, \mathrm{d}z') \\
&\leq (Nm'a + m)\left(\vee_{i=1}^N \|\psi_i - \tilde{\psi}_i\|_\kappa\right) \kappa(z).
\end{aligned}$$

Hence, $J_j \leq \beta(Nm'a + m)\left(\vee_{i=1}^N \|\psi_i - \tilde{\psi}_i\|_\kappa\right) = \rho\, \rho_\kappa(\psi, \tilde{\psi})$ for all $j \in \{1, \cdots, N\}$, and we have

$$\rho_\kappa(Q\psi, Q\tilde{\psi}) = J_1 \vee \cdots \vee J_N \leq \rho\, \rho_\kappa(\psi, \tilde{\psi}).$$

Thus, Q is a contraction mapping on $\left(\times_{i=1}^N b_\kappa \mathsf{Z},\, \rho_\kappa\right)$ of modulus ρ.

Step 2. The proof of claim (2). Lemma 4.A.2 shows that under assumption 4.4.1, (4.2.2)–(4.2.3) hold, which implies that v^* is a fixed point of T. From (4.2.4) we know that ψ^* is a fixed point of Q. To prove claim (2), it remains to show that $\psi^* \in \times_{i=1}^N b_\kappa \mathsf{Z}$. Notice that for all $z \in \mathsf{Z}$ and $i \in \{1, \cdots, N\}$, there exists a feasible plan $\{j_t\}_{t \geq 1}$ such that

$$|\psi_i^*(z)| = \left| r_i(z) + \sum_{t=1}^{\infty} \beta^t \mathbb{E}_z^{i j_1 \cdots j_{t-1}} r_{j_t}(Z_t) \right| \leq |r_i(z)| + \sum_{t=1}^{\infty} \beta^t \mathbb{E}_z^{i j_1 \cdots j_{t-1}} |r_{j_t}(Z_t)|.$$

Notice that (4.A.5) (see the proof of lemma 4.A.2) implies that for all $t \geq 1$ and $\{j_t\}_{t \geq 1}$, we have

$$\mathbb{E}_z^{i j_1 \cdots j_{t-1}} |r_{j_t}(Z_t)| \leq a \left[m^{t-1} g(z) + \frac{1 - m^{t-1}}{1 - m} d \right] + b.$$

Since $\beta m < 1$, there exist constants $a_1, a_2 \geq 0$ such that, for all $z \in \mathsf{Z}$,

$$|\psi_i^*(z)| \leq |r_i(z)| + a_1 g(z) + a_2.$$

This implies that $\psi_i^* \in b_\kappa \mathsf{Z}$ for all $i \in \{1, \cdots, N\}$ and $\psi^* \in \times_{i=1}^N b_\kappa \mathsf{Z}$. Hence, the unique fixed point of Q in $\times_{i=1}^N b_\kappa \mathsf{Z}$ must be ψ^*. Claim (2) is verified.

Step 3. We prove claim (3). We first show that T maps $(b_\kappa \mathsf{Z}, d_\kappa)$ into itself. For all $v \in b_\kappa \mathsf{Z}$ and $z \in \mathsf{Z}$, assumption 4.4.1 and (4.B.1) imply that

$$\begin{aligned} \frac{|Tv(z)|}{\kappa(z)} &\leq \max_{i \in \{1, \cdots, N\}} \left\{ \frac{|r_i(z)|}{\kappa(z)} + \frac{\beta \int |v(z')| P_i(z, \mathrm{d}z')}{\kappa(z)} \right\} \\ &\leq \max_{i \in \{1, \cdots, N\}} \left\{ \frac{1}{m'} + \frac{\beta \|v\|_\kappa \int \kappa(z') P_i(z, \mathrm{d}z')}{\kappa(z)} \right\} \leq \frac{1}{m'} + \beta(Nm'a + m) \|v\|_\kappa. \end{aligned}$$

Hence, $\|Tv\|_\kappa < \infty$ and $Tv \in b_\kappa \mathsf{Z}$. We have shown that T is a self-map on $(b_\kappa \mathsf{Z}, d_\kappa)$. Next, we show that T is a contraction mapping on $(b_\kappa \mathsf{Z}, d_\kappa)$ of modulus ρ. For all $v, \tilde{v} \in b_\kappa \mathsf{Z}$, we have

$$\begin{aligned} d_\kappa(Tv, T\tilde{v}) &= \left\| \vee_{i=1}^N (r_i + \beta P_i v) - \vee_{i=1}^N (r_i + \beta P_i \tilde{v}) \right\|_\kappa \\ &\leq \vee_{i=1}^N \|\beta P_i v - \beta P_i \tilde{v}\|_\kappa = \beta \vee_{i=1}^N \|P_i v - P_i \tilde{v}\|_\kappa, \end{aligned} \tag{4.B.3}$$

where the inequality is due to the fact that for all $z \in \mathsf{Z}$,

$$\left| \vee_{i=1}^N (r_i + \beta P_i v)(z) - \vee_{i=1}^N (r_i + \beta P_i \tilde{v})(z) \right| \leq \vee_{i=1}^N |\beta P_i v(z) - \beta P_i \tilde{v}(z)|.$$

Note that for all $z \in \mathsf{Z}$ and $i \in \{1, \cdots, N\}$, (4.B.1) implies that

$$\begin{aligned} |P_i v(z) - P_i \tilde{v}(z)| &\leq \int |v(z') - \tilde{v}(z')| P_i(z, \mathrm{d}z') \\ &\leq \|v - \tilde{v}\|_\kappa \int \kappa(z') P_i(z, \mathrm{d}z') \leq (Nm'a + m) \|v - \tilde{v}\|_\kappa \, \kappa(z). \end{aligned}$$

Hence, $\beta \|P_i v - P_i \tilde{v}\|_\kappa \leq \rho \|v - \tilde{v}\|_\kappa = \rho \, d_\kappa(v, \tilde{v})$ for all $i \in \{1, \cdots, N\}$. Then (4.B.3) implies that $d_\kappa(Tv, T\tilde{v}) \leq \rho d_\kappa(v, \tilde{v})$ for all $v, \tilde{v} \in b_\kappa \mathsf{Z}$, i.e., T is a contraction mapping on $(b_\kappa \mathsf{Z}, d_\kappa)$ of modulus ρ. Claim (3) is verified.

Step 4. Regarding claim (4), it suffices to show that v^* is a fixed point of T and that $v^* \in b_\kappa \mathsf{Z}$. The former has been established in step 2. Regarding the latter, since in step 2 we have shown that $\psi_i^* \in b_\kappa \mathsf{Z}$ for all $i \in \{1, \cdots, N\}$, we then have $v^* = \vee_{i=1}^N \psi_i^* \in b_\kappa \mathsf{Z}$. This concludes the proof. $\square$

Proof of proposition 4.4.1. Let $\mathcal{V} := b_\kappa \mathsf{Z}$ and $\mathscr{C} := \times_{i=1}^N b_\kappa \mathsf{Z}$. The fact that $R\mathscr{C} \subset \mathcal{V}$ is obvious, and we can easily verify by applying the Markov property that $L\mathcal{V} \subset \mathscr{C}$.

Regarding claim (1), for all $\psi \in \mathscr{C}$, based on lemma 4.A.1 and theorem 4.4.1, we have

$$\rho_\kappa(Q^{t+1}\psi, \psi^*) = \rho_\kappa(LT^t R\psi, Lv^*) = \beta \vee_{i=1}^N \|P_i(T^t R\psi) - P_i v^*\|_\kappa.$$

Since we have shown in the proof of theorem 4.4.1 that $\int \kappa(z')P_i(z, \mathrm{d}z') \leq (Nm'a + m)\kappa(z)$ for all $z \in \mathsf{Z}$ and $i \in \{1, \cdots, N\}$ (see equation (4.B.1)), by the definition of operator P, for all $z \in \mathsf{Z}$ and $i \in \{1, \cdots, N\}$, we have

$$\begin{aligned} \left|P_i(T^t R\psi)(z) - P_i v^*(z)\right| &\leq \int \left|(T^t R\psi)(z') - v^*(z')\right| P_i(z, \mathrm{d}z') \\ &\leq \|T^t R\psi - v^*\|_\kappa \int \kappa(z')P_i(z, \mathrm{d}z') \\ &\leq (Nm'a + m)\|T^t R\psi - v^*\|_\kappa \, \kappa(z). \end{aligned}$$

Hence, $\|P_i(T^t R\psi) - P_i v^*\|_\kappa \leq (Nm'a + m)\|T^t R\psi - v^*\|_\kappa$ for all $i \in \{1, \cdots, N\}$. Recall $\rho := \beta(Nm'a + m) < 1$ defined in (4.4.3). We then have

$$\rho_\kappa(Q^{t+1}\psi, \psi^*) \leq \beta(Nm'a + m) \left\|T^t R\psi - v^*\right\|_\kappa = \rho \, d_\kappa(T^t R\psi, v^*)$$

for all $\psi \in \mathscr{C}$. Hence, claim (1) is verified.

Regarding claim (2), for all $v \in \mathcal{V}$, propositions 4.3.1–4.3.2 and theorem 4.4.1 imply that

$$\begin{aligned} \left|T^{t+1}v(z) - v^*(z)\right| = \left|(RQ^t L)v(z) - R\psi^*(z)\right| &\leq \vee_{i=1}^N \left|(Q^t Lv)_i(z) - \psi_i^*(z)\right| \\ &\leq \vee_{i=1}^N \|(Q^t Lv)_i - \psi_i^*\|_\kappa \, \kappa(z) = \rho_\kappa(Q^t Lv, \psi^*) \, \kappa(z) \end{aligned}$$

for all $z \in \mathsf{Z}$, where the first inequality is due to the elementary fact that $| \vee_{i=1}^N a_i - \vee_{i=1}^N b_i| \leq \vee_{i=1}^N |a_i - b_i|$. Hence, $d_\kappa(T^{t+1}v, v^*) = \left\|T^{t+1}v - v^*\right\|_\kappa \leq \rho_\kappa(Q^t Lv, \psi^*)$ for all $v \in \mathcal{V}$ and claim (2) holds. $\square$

Proof of proposition 4.4.2. Let $b_\kappa c\mathsf{Z}$ be the set of continuous functions in $b_\kappa \mathsf{Z}$. Since κ is continuous by assumption 4.4.2, $b_\kappa c\mathsf{Z}$ is a closed subset of $b_\kappa \mathsf{Z}$ (see e.g., Boyd (1990), section 3). To show the continuity of ψ^*, it suffices to verify: $Q(\times_{i=1}^N b_\kappa c\mathsf{Z}) \subset$

$\times_{i=1}^N b_\kappa c\mathsf{Z}$ (see, e.g., Stokey et al. (1989), corollary 1 of theorem 3.2). For each fixed $\psi = (\psi_1, \cdots, \psi_N) \in \times_{i=1}^N b_\kappa c\mathsf{Z}$, there exists $G \in \mathbb{R}_+$ such that $\left|\vee_{j=1}^N \psi_j(z)\right| \leq G\kappa(z)$ for all $z \in \mathsf{Z}$. By assumption 4.4.2, $z \mapsto G\kappa(z) \pm \vee_{j=1}^N \psi_j(z)$ are nonnegative and continuous. For all $z \in \mathsf{Z}$ and $\{z_m\} \subset \mathsf{Z}$ with $z_m \to z$, since P is Feller, we have $P_i(z_m, \cdot) \xrightarrow{w} P_i(z, \cdot)$ for all $i \in \{1, \cdots, N\}$. The generalized Fatou's lemma of Feinberg et al. (2014) (theorem 1.1) then implies that, for all $i \in \{1, \cdots, N\}$,

$$\int \left[G\kappa(z') \pm \vee_{j=1}^N \psi_j(z')\right] P_i(z, \mathrm{d}z') \leq \liminf_{m\to\infty} \int \left[G\kappa(z') \pm \vee_{j=1}^N \psi_j(z')\right] P_i(z_m, \mathrm{d}z').$$

Since $\lim_{m\to\infty} \int \kappa(z')P_i(z_m, \mathrm{d}z') = \int \kappa(z')P_i(z, \mathrm{d}z')$ by assumption 4.4.2, we have

$$\pm \int \vee_{j=1}^N \psi_j(z')P_i(z, \mathrm{d}z') \leq \liminf_{m\to\infty} \left[\pm \int \vee_{j=1}^N \psi_j(z')P_i(z_m, \mathrm{d}z')\right],$$

where we have used the fact that for all sequences $\{a_m\}, \{b_m\} \subset \mathbb{R}$ with $\lim_{m\to\infty} a_m$ exists, we have: $\liminf_{m\to\infty}(a_m + b_m) = \lim_{m\to\infty} a_m + \liminf_{m\to\infty} b_m$. Hence,

$$\begin{aligned}\limsup_{m\to\infty} \int \vee_{j=1}^N \psi_j(z')P_i(z_m, \mathrm{d}z') &\leq \int \vee_{j=1}^N \psi_j(z')P_i(z, \mathrm{d}z') \\ &\leq \liminf_{m\to\infty} \int \vee_{j=1}^N \psi_j(z')P_i(z_m, \mathrm{d}z'),\end{aligned}$$

i.e., $z \mapsto \int \vee_{j=1}^N \psi_j(z')P_i(z, \mathrm{d}z')$ is continuous for all $i \in \{1, \cdots, N\}$. Since r_i is continuous by assumption, $Q\psi \in \times_{i=1}^N b_\kappa c\mathsf{Z}$. Hence, $Q\left(\times_{i=1}^N b_\kappa c\mathsf{Z}\right) \subset \times_{i=1}^N b_\kappa c\mathsf{Z}$ and ψ^* is continuous, as was to be shown. The continuity of v^* follows from the continuity of $\psi^* = (\psi_1^*, \cdots, \psi_N^*)$ and the fact that $v^* = \vee_{j=1}^N \psi_j^*$. $\square$

Chapter 5

Optimal Timing of Decisions: A General Theory Based on Continuation Values

5.1 Introduction

In the previous few chapters, we have shown that the Bellman operator and Jovanovic operator have essentially equivalent dynamic properties, while for most problems of interest to economists, the dimensionality of the effective state space associated with the former is at least as large as that related to the latter, and often strictly so (recall section 2.4 and appendix 2.D of chapter 2).

Another important asymmetry between the value function based method and the continuation value based method not discussed so far is that continuation values are typically smoother than value functions. For example, in job search problems, the value function is usually kinked at the reservation wage, while the continuation value function is smooth. In this and other settings, the relative smoothness comes from taking expectations over stochastic transitions, since integration is a smoothing operation.

Like lower dimensionality, increased smoothness help on both the analytical and the computational side. On the computational side, smoother functions are easier to approximate. On the analytical side, greater smoothness lends itself to sharper results based on derivatives, as elaborated on below.

In this chapter, we provide a general theory for sequential decision problems based around continuation value functions and the Jovanovic operator, heavily exploiting the advantages discussed so far. The theory is established for the optimal stopping framework. Extensions can be made to accommodate other classes of sequential decision problems. Moreover, our analysis is conducted in the weighted supremum norm topological framework constructed in chapter 2. Analysis in the L_p-norm topological framework is left for future research.

Section 5.2 reviews the general optimality theory and provides a variety of examples. Section 5.3 discusses the properties of continuation values. In particular, we obtain conditions under which continuation values are continuous (section 5.3.1), monotone (section 5.3.2), and differentiable (section 5.3.3) as functions of economic environment, and conditions under which parametric continuity holds (section 5.3.4). The latter is often required for proofs of existence of recursive equilibria in many-agent environments.

Section 5.4 then discusses the properties of optimal policies. After formulating the threshold state problems in section 5.4.1, section 5.4.2 provides conditions under which threshold policies are (a) continuous, (b) monotone, and (c) differentiable. In the latter case we derive an expression for the derivative of the threshold relative to other aspects of the economic environment and show how it contributes to economic intuition.

In terms of connections to the existing literature, these results are related to the theoretical results of Norets (2010), which provides sufficient conditions for continuity and differentiability of the expected value functions in a dynamic discrete choice framework.[1] Closest counterparts to the theory of this chapter in the existing literature are those concerning individual applications.

For example, Jovanovic (1982) shows that the continuation value function associated with an incumbent firm's exit decision is monotone and continuous (theorem 1). Chatterjee and Rossi-Hansberg (2012) shows that the continuation value

[1]The difference between this framework and the optimal stopping framework is discussed in chapter 4. To compare with the theory of this chapter, aside from the different sequential decision frameworks, Norets provides sufficient conditions for differentiability with respect to the model parameters, an important topic not covered by this chapter. On the other hand, the results of this chapter are sharper. While Norets treats unbounded rewards via the standard weighted supremum norm method of Boyd (1990), our theory extends that by exploiting extra future transition structures (recall assumptions 2.3.1, 3.4.1 and 4.4.1). Moreover, while Norets (2010) assumes that the stochastic kernel related to the state process admits a density representation, our theory of optimality and continuity works for more general state transitions (recall theorem 2.3.1 and proposition 2.3.4). Finally, this chapter explores a range of properties of the threshold policies, a topic not treated in Norets (2010).

function (the value of not using a project) is continuous and increasing in the average revenue (lemma 1, section 2.1). In a model of oil drilling investment, Kellogg (2014) provides sufficient conditions for the existence of a threshold policy (a reservation productivity at which the firm is indifferent between drilling a well or not), and conditions under which this policy is decreasing in the average oil price and increasing in the dayrate (conditions (i)–(v), section B).

The theory of this chapter generalizes and extends these results in a unified framework. Some results, such as differentiability of threshold policies, are new to the literature to the best of our knowledge.

Section 5.5 provides a list of important applications that illustrate the advantages of working with the continuation value based method over the traditional value function based method. Finally, longer proofs are deferred to the appendix.

5.2 Optimality Results Revisit

This section reviews the key optimality result derived in chapter 2 and provides a range of examples.

5.2.1 Preliminaries

Unless otherwise specified, we continue to use the notation of chapter 2 throughout this chapter. Moreover, the following definitions are required for our analysis.

Let $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_{++} := (0, \infty)$. For a stochastic kernel P on $(\mathsf{Z}, \mathscr{B})$ and a $\mathscr{B}$-measurable function $h : \mathsf{Z} \to \mathbb{R}$, let

$$(P^n h)(z) :=: \mathbb{E}_z h(Z_n) := \int h(z') P^n(z, \mathrm{d}z') \quad \text{for all } n \in \mathbb{N}_0,$$

with $P^0 h := h$ and $Ph := P^1 h$. We say that P is *stochastically increasing* if Ph is increasing for all increasing function $h \in b\mathsf{Z}$. When Z is a Borel subset of $\mathbb{R}^m$, a *density kernel* on Z is a measurable map $f : \mathsf{Z} \times \mathsf{Z} \to \mathbb{R}_+$ such that $\int_{\mathsf{Z}} f(z'|z) dz' = 1$ for all $z \in \mathsf{Z}$. We say that P *has a density representation* if there exists a density kernel f such that $P(z, B) = \int_B f(z'|z)\, \mathrm{d}z'$ for all $z \in \mathsf{Z}$ and $B \in \mathscr{Z}$.

5.2.2 Optimality Results: Review and Examples

Recall the benchmark set up and the theoretical results established in section 2.2.2, section 2.3.2 and appendix 2.D of chapter 2. In particular, for a generic optimal stopping problem, a *policy* is a map $\sigma \colon \mathsf{Z} \to \{0, 1\}$, with 0 indicating the decision to continue and 1 indicating the decision to stop. A policy σ is called *optimal* if $\tau^* := \inf\{t \geq 0 \mid \sigma(Z_t) = 1\}$ is an optimal stopping time.

Theorem 5.2.1. *Let assumption 2.3.1 hold. Then there exist positive constants m' and d' such that for $\ell \colon \mathsf{Z} \to \mathbb{R}$ defined by*

$$\ell(z) := m' \left(\sum_{t=1}^{n-1} \mathbb{E}_z |r(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z |c(Z_t)| \right) + g(z) + d', \tag{5.2.1}$$

the following statements hold:

(1) Q is a contraction mapping on $(b_\ell \mathsf{Z}, \|\cdot\|_\ell)$.

(2) The unique fixed point of Q in $b_\ell \mathsf{Z}$ is ψ^*.

(3) The policy defined by $\sigma^*(z) = \mathbb{1}\{r(z) \geq \psi^*(z)\}$ is an optimal policy.

Proof. The first two claims are established in theorem 2.3.1. Moreover, a simple extension of theorem 1.11 of Peskir and Shiryaev (2006) shows that

$$\tau^* := \inf\{t \geq 0 \colon v^*(Z_t) = r(Z_t)\}$$

is an optimal stopping time. Claim (3) then follows from the definition of the optimal policy and the fact that $v^* = r \vee \psi^*$. $\square$

In the following, we provide some typical examples.

Example 5.2.1. Consider a job search problem where a worker aims to maximize expected lifetime rewards (see, e.g., Jovanovic (1987), Cooper et al. (2007), Ljungqvist and Sargent (2008), Robin (2011), Moscarini and Postel-Vinay (2013), Bagger et al. (2014)). She can accept current wage offer w_t and work permanently at that wage, or reject the offer, receive unemployment compensation $\tilde{c}_0$ and reconsider next period. Let $w_t = w(Z_t)$ for some idiosyncratic or aggregate state process $\{Z_t\}_{t \geq 0}$. The terminal reward is $r(z) = u(w(z))/(1-\beta)$, the lifetime reward associated with stopping at state z. Here u is a utility function and β is the discount factor. The flow continuation reward is the constant $c_0 := u(\tilde{c}_0)$.

A common specification for the state process $\{Z_t\}_{t \geq 0} \subset \mathsf{Z} := \mathbb{R}$ is

$$Z_{t+1} = \rho Z_t + b + \varepsilon_{t+1}, \quad \{\varepsilon_t\}_{t \geq 1} \overset{\text{IID}}{\sim} N(0, \sigma^2), \quad \rho \in [-1, 1]. \tag{5.2.2}$$

The Jovanovic operator for this problem is

$$Q\psi(z) = c_0 + \beta \int \max\left\{ \frac{u(w(z'))}{1-\beta}, \psi(z') \right\} f(z'|z) \, \mathrm{d}z'. \tag{5.2.3}$$

where $f(z'|z) = N(\rho z + b, \sigma^2)$. Let $w(z) := \mathrm{e}^z$ and let utility have the CRRA form

$$u(w) = \frac{w^{1-\delta}}{1-\delta} \quad (\text{if } \delta \geq 0 \text{ and } \delta \neq 1) \quad \text{and} \quad u(w) = \ln w \quad (\text{if } \delta = 1). \tag{5.2.4}$$

Since the terminal reward is unbounded, traditional solution method based on the Bellman operator and sup norm contractions are not generally valid. Moreover, since $\{\varepsilon_t\}$ has unbounded support, the local contraction method fails.[2] But theorem 5.2.1 can be applied. Consider, for example, $\delta \geq 0$, $\delta \neq 1$ and $\rho \in [0,1)$. Choose $n \in \mathbb{N}_0$ such that $\beta\,\mathrm{e}^{\rho^n \xi} < 1$, where $\xi := \xi_1 + \xi_2$ with $\xi_1 := (1-\delta)b$ and $\xi_2 := (1-\delta)^2\sigma^2/2$. Observe that[3]

$$\int \mathrm{e}^{(1-\delta)z'} P^n(z, \mathrm{d}z') = b_n\, \mathrm{e}^{\rho^n(1-\delta)z}, \text{ where } b_n := \mathrm{e}^{\xi_1 \sum_{i=0}^{t-1} \rho^i + \xi_2 \sum_{i=0}^{t-1} \rho^{2i}}. \tag{5.2.5}$$

It follows that assumption 2.3.1 holds when $g(z) = \mathrm{e}^{\rho^n(1-\delta)z}$,

$$m = d = \mathrm{e}^{\rho^n \xi}, \quad a_1 = \frac{b_n}{(1-\beta)(1-\delta)}, \quad a_2 = a_3 = 0, \;\text{ and }\; a_4 = c_0.$$

Specifically, since $r(z) = \mathrm{e}^{(1-\delta)z}/((1-\beta)(1-\delta))$, an application of (5.2.5) gives

$$\int |r(z')| P^n(z, \mathrm{d}z') = b_n\, \mathrm{e}^{\rho^n(1-\delta)z} \frac{1}{(1-\beta)(1-\delta)} = a_1 g(z) + a_2,$$

which is (2.8). Condition (2.9) is trivial. Condition (2.10) holds because

$$\int g(z') P(z, \mathrm{d}z') = \mathrm{e}^{\rho^{n+1}(1-\delta)z} \mathrm{e}^{\rho^n \xi_1 + \rho^{2n} \xi_2} \leq (g(z) + 1)\, \mathrm{e}^{\rho^n \xi} = m g(z) + d. \tag{5.2.6}$$

Hence, theorem 5.2.1 applies. The cases $\rho = 1$, $\delta = 1$ and $\rho \in [-1, 0]$ can be treated using similar methods. Appendix 5.B provides details.

Remark 5.2.1. The use of n-step transitions in assumption 2.3.1 has certain benefits. For example, if $\{Z_t\}_{t \geq 0}$ is mean-reverting, as time iterates forward, the initial effect tends to die out, making the conditional expectations $\mathbb{E}_z |r(Z_n)|$ and $\mathbb{E}_z |c(Z_n)|$ flatter than the original rewards. As a result, finding an appropriate g-function with geometric drift property becomes easier. Typically, if $\rho \in (-1, 1)$ in example 5.2.1, without using future transitions (i.e., $n = 0$ is imposed),[4] one need a further assumption $\beta \mathrm{e}^{|\xi_1| + \xi_2} < 1$ (see appendix 5.B), which in some circumstances puts nontrivial restrictions on the key parameters β and δ. Using n-step transitions, however, such restrictions are completely removed.

[2] The method requires an increasing sequence of compact sets $\{K_j\}$ such that $\mathsf{Z} = \cup_{j=1}^{\infty} K_j$ and $\Gamma(K_j) \subset K_{j+1}$ with probability one, where $\Gamma : \mathsf{Z} \mapsto 2^{\mathsf{Z}}$ is the feasibility correspondence of the state process $\{Z_t\}$ (see, e.g., Rincón-Zapatero and Rodríguez-Palmero (2003), theorems 3–4). This fails in the current case, since Γ corresponds to (5.2.2) and shocks have unbounded support.

[3] Recall that for $X \sim N(\mu, \sigma^2)$, we have $\mathbb{E}\,\mathrm{e}^{sX} = \mathrm{e}^{s\mu + s^2\sigma^2/2}$ for any $s \in \mathbb{R}$. Based on (5.2.2), the distribution of Z_t given $Z_0 = z$ follows $N\left(b \sum_{i=0}^{t-1} \rho^i, \sigma^2 \sum_{i=0}^{t-1} \rho^{2i}\right)$.

[4] Indeed, in this case, our assumption reduces to the standard weighted supremum norm assumption. See, e.g., section 4 of Boyd (1990), or assumptions 1-4 of Durán (2003).

Example 5.2.2. Recall the job search problem with learning discussed in section 2.5.2 of chapter 2.[5] Basically, the set up is as in example 5.2.1, except that $\{w_t\}_{t\geq 0}$ follows

$$\ln w_t = \xi + \varepsilon_t, \quad \{\varepsilon_t\}_{t\geq 0} \overset{\text{IID}}{\sim} N(0, \gamma_\varepsilon),$$

where ξ is the unobservable mean of the wage process, over which the worker has prior $\xi \sim N(\mu, \gamma)$. The worker's current estimate of the next period wage distribution is $f(w'|\mu,\gamma) = LN(\mu, \gamma + \gamma_\varepsilon)$. If the current offer is turned down, the worker updates his belief after observing w'. By the Bayes' rule, the posterior satisfies $\xi | w' \sim N(\mu', \gamma')$, where

$$\gamma' = 1/\left(1/\gamma + 1/\gamma_\varepsilon\right) \quad \text{and} \quad \mu' = \gamma'\left(\mu/\gamma + \ln w'/\gamma_\varepsilon\right). \tag{5.2.7}$$

Recall that the utility of the worker follows (5.2.4). For any integrable function h, the stochastic kernel satisfies

$$\int h(z')P(z, \mathrm{d}z') = \int h(w', \mu', \gamma') f(w'|\mu, \gamma)\,\mathrm{d}w', \tag{5.2.8}$$

where μ' and γ' are defined by (5.2.7). Moreover, the Jovanovic operator follows

$$Q\psi(\mu,\gamma) = c_0 + \beta \int \max\left\{\frac{u(w')}{1-\beta}, \psi(\mu', \gamma')\right\} f(w'|\mu,\gamma)\,\mathrm{d}w'. \tag{5.2.9}$$

We have shown that if $\delta \geq 0$ and $\delta \neq 1$, then assumption 2.3.1 holds with $n := 1$, $g(\mu, \gamma) := \mathrm{e}^{(1-\delta)\mu + (1-\delta)^2\gamma/2}$, $m := 1$ and $d := 0$. In particular, we have

$$\int w'^{1-\delta} f(w'|\mu,\gamma)\,\mathrm{d}w' = \mathrm{e}^{(1-\delta)^2\gamma_\varepsilon/2} \cdot \mathrm{e}^{(1-\delta)\mu + (1-\delta)^2\gamma/2} \tag{5.2.10}$$

and

$$\mathbb{E}_{\mu,\gamma} g(\mu', \gamma') := \int g(\mu', \gamma') f(w'|\mu,\gamma)\,\mathrm{d}w' = g(\mu,\gamma) = mg(\mu,\gamma) + d. \tag{5.2.11}$$

Indeed, the case $\delta = 1$ can also be treated. Appendix 5.B gives details.

Remark 5.2.2. From (5.2.8) we know that the conditional expectation of the reward functions in example 5.2.2 is defined on a space of lower dimension than the state space. Although there are 3 states, $\mathbb{E}_z|r(Z_1)|$ is a function of only 2 arguments: μ and γ. Hence, using future transitions in assumption 2.3.1 makes it easier to find an appropriate g-function. Indeed, if the standard weighted supnorm method were applied, one need to find a $\tilde{g}(w, \mu, \gamma)$ with geometric drift property that dominates $|r|$ (see, e.g., section 4 of Boyd (1990), or, assumptions 1–4 of Durán (2003)), which is not as obvious due to the higher state dimension.

[5] The notation will be changed a little bit in order to maintain coherence of the current chapter.

Remark 5.2.3. From (5.2.9), we see that the continuation value is a function of (μ, γ). However, since current rewards depend on w, the value function depends on (w, μ, γ). Thus, the former is lower dimensional than the latter.

With unbounded rewards and shocks, solution methods based on the Bellman operator with respect to the supremum norm or local contractions fail in the next few examples. However, theorem 5.2.1 can be applied. (See appendix 5.B for proofs.)

Example 5.2.3. Consider an infinite-horizon American call option (see, e.g., Peskir and Shiryaev (2006) or Duffie (2010)) with state process be as in (5.2.2) and state space $\mathsf{Z} := \mathbb{R}$. Let $p_t = p(Z_t) = \mathrm{e}^{Z_t}$ be the current price of the underlying asset, and $\gamma > 0$ be the riskless rate of return (i.e., $\beta = \mathrm{e}^{-\gamma}$). With strike price K, the terminal reward is $r(z) = (p(z) - K)^+$, the reward of exercising the option, while the flow continuation reward is $c \equiv 0$. The Jovanovic operator for the option satisfies

$$Q\psi(z) = \mathrm{e}^{-\gamma} \int \max\{(p(z') - K)^+, \psi(z')\} f(z'|z) \, \mathrm{d}z'.$$

If $\rho \in (-1, 1)$, we can let $\xi := |b| + \sigma^2/2$ and fix $n \in \mathbb{N}_0$ such that $\mathrm{e}^{-\gamma + |\rho^n|\xi} < 1$, so assumption 2.3.1 holds with $g(z) := \mathrm{e}^{\rho^n z} + \mathrm{e}^{-\rho^n z}$ and $m = d := \mathrm{e}^{|\rho^n|\xi}$. (If $\mathrm{e}^{-\gamma+\xi} < 1$, then assumption 2.3.1 holds with $n = 0$ for all $\rho \in [-1, 1]$.)

Example 5.2.4. Suppose that, in each period, a firm observes an idea with value $Z_t \in \mathsf{Z} := \mathbb{R}_+$ and decides whether to put this idea into productive use or develop it further, by investing in R&D (see, e.g., Jovanovic and Rob (1989), Bental and Peled (1996), Perla and Tonetti (2014)). The first choice gives reward $r(Z_t) = Z_t$. The latter incurs fixed cost $c_0 > 0$. Let the R&D process be governed by the exponential law (with rate parameter $\theta > 0$)

$$F(z'|z) := \mathbb{P}(Z_{t+1} \leq z' | Z_t = z) = 1 - \mathrm{e}^{-\theta(z'-z)} \quad (z' \geq z). \tag{5.2.12}$$

The Jovanovic operator satisfies

$$Q\psi(z) = -c_0 + \beta \int \max\{z', \ \psi(z')\} \, \mathrm{d}F(z'|z).$$

In this case, assumption 2.3.1 is satisfied with $n := 0$, $g(z) := z$, $m := 1$ and $d := 1/\theta$.

Example 5.2.5. Consider a firm exit problem (see, e.g., Hopenhayn (1992), Ericson and Pakes (1995), Asplund and Nocke (2006), Dinlersoz and Yorukoglu (2012), Coşar et al. (2016)). In each period, an incumbent firm observes a productivity shock a_t, where $a_t = a(Z_t) = \mathrm{e}^{Z_t}$ and $Z_t \in \mathsf{Z} := \mathbb{R}$ obeys (5.2.2), and decides whether or not to exit the market next period. A fixed cost $c_f > 0$ is paid each period and the firm's output is $q(a, l) = al^{\alpha}$, where $\alpha \in (0, 1)$ and l is labor demand. Given output and input prices p and w, the reward functions are $r(z) = c(z) = Ga(z)^{\frac{1}{1-\alpha}} - c_f$, where $G = (\alpha p/w)^{\frac{1}{1-\alpha}}(1-\alpha)w/\alpha$. The Jovanovic operator is

$$Q\psi(z) = \left(Ga(z)^{\frac{1}{1-\alpha}} - c_f\right) + \beta \int \max\left\{Ga(z')^{\frac{1}{1-\alpha}} - c_f, \psi(z')\right\} f(z'|z)\,\mathrm{d}z'.$$

For $\rho \in [0, 1)$, choose $n \in \mathbb{N}_0$ such that $\beta \mathrm{e}^{\rho^n \xi} < 1$, where $\xi := \frac{b}{1-\alpha} + \frac{\sigma^2}{2(1-\alpha)^2}$. Then assumption 2.3.1 holds with $g(z) := \mathrm{e}^{\rho^n z/(1-\alpha)}$ and $m := d := \mathrm{e}^{\rho^n \xi}$. Other parameterizations (such as the unit root case $\rho = 1$) can also be handled.

5.3 Properties of Continuation Values

This section studies some further properties of the continuation value function.

5.3.1 Continuity

We begin by stating conditions under which the continuation value function is continuous. Recall that we have established a general result on continuity in section 2.3.2 of chapter 2—proposition 2.3.4. The next result treats the special case when P admits a density representation. Note that continuity of r is not required.

Corollary 5.3.1. *If assumption 2.3.1 holds, P admits a density representation $f(z'|z)$ that is continuous in z, and that c, ℓ and $z \mapsto \int |r(z')| f(z'|z)\,\mathrm{d}z'$, $\int \ell(z') f(z'|z)\,\mathrm{d}z'$ are continuous, then ψ^* is continuous.*

Remark 5.3.1. If r and c are bounded, then assumption 2.3.2-(1) and the continuity of r and c are sufficient for the continuity of ψ^* (by proposition 2.3.4). If in addition P has a density representation f, then the continuity of c and $z \mapsto f(z'|z)$ is sufficient for ψ^* to be continuous by corollary 5.3.1.

Example 5.3.1. In the job search model of example 5.2.1, ψ^* is continuous. Assumption 2.3.1 holds, as shown. P has a density representation $f(z'|z) = N(\rho z + b, \sigma^2)$ that is continuous in z. Moreover, $c \equiv c_0$, g and $z \mapsto \mathbb{E}_z g(Z_1)$ are continuous, and $z \mapsto \mathbb{E}_z |r(Z_t)|$ is continuous for all $t \in \mathbb{N}$ (see (5.2.5)–(5.2.6)). Hence, ℓ and $z \mapsto \mathbb{E}_z \ell(Z_1)$ are continuous, and the conditions of corollary 5.3.1 are satisfied.

Example 5.3.2. In the adaptive search model of example 5.2.2, assumption 2.3.1 holds for $n = 1$, as shown. By (5.2.8) and lemma 5.A.1, P is Feller. Moreover, $c \equiv c_0$, and r, g and $(\mu, \gamma) \mapsto \mathbb{E}_{\mu,\gamma}|r(w')|, \mathbb{E}_{\mu,\gamma} g(\mu', \gamma')$ are continuous (see (5.2.10)–(5.2.11)), where we define $\mathbb{E}_{\mu,\gamma}|r(w')| := \int |r(w')| f(w'|\mu,\gamma) \, \mathrm{d}w'$. Hence, assumption 2.3.2 holds (recall footnote 4 of chapter 2). By proposition 2.3.4, ψ^* is continuous.

Example 5.3.3. Recall the option pricing model of example 5.2.3. By corollary 5.3.1, we can show that ψ^* is continuous. The proof is similar to example 5.3.1, except that we use $|r(z)| \leq \mathrm{e}^z + K$, the continuity of $z \mapsto \int (\mathrm{e}^{z'} + K) f(z'|z) \, \mathrm{d}z'$, and lemma 5.A.1 to show that $z \mapsto \mathbb{E}_z |r(Z_1)|$ is continuous. Appendix 5.B provides details.

Example 5.3.4. Recall the R&D decision problem of example 5.2.4. Assumption 2.3.1 holds for $n = 0$. For all bounded continuous function $h : \mathsf{Z} \to \mathbb{R}$, lemma 5.A.1 shows that $z \mapsto \int h(z') \, \mathrm{d}F(z'|z)$ is continuous, so P is Feller. Moreover, r, c and g are continuous, and $z \mapsto \mathbb{E}_z g(Z_1)$ is continuous since

$$\int |z'| P(z, \mathrm{d}z') = \int_{[z,\infty)} z' \theta \mathrm{e}^{-\theta(z'-z)} \, \mathrm{d}z' = z + 1/\theta.$$

Hence, assumption 2.3.2 holds. By proposition 2.3.4, ψ^* is continuous.

Example 5.3.5. Recall the firm exit model of example 5.2.5. Through similar analysis to examples 5.3.1 and 5.3.3, we can show that ψ^* is continuous. Appendix 5.B gives details.

5.3.2 Monotonicity

We now study monotonicity under the following assumption.[6] In the assumption, the latter statement holds whenever r is increasing and P is stochastically increasing (recall section 5.2.1).

Assumption 5.3.1. The function c is increasing, as is

$$z \mapsto \int \max\{r(z'), \psi(z')\} P(z, \mathrm{d}z')$$

for all increasing $\psi \in b_\ell \mathsf{Z}$.

Proposition 5.3.1. *If assumptions 2.3.1 and 5.3.1 hold, then ψ^* is increasing.*

[6] We focus on the monotone increasing case. The monotone decreasing case is similar.

Example 5.3.6. In example 5.2.1, assumption 2.3.1 holds. If $\rho \geq 0$, the density kernel $f(z'|z) = N(\rho z + b, \sigma^2)$ is stochastically increasing in z. Since r and c are increasing, assumption 5.3.1 holds. By proposition 5.3.1, ψ^* is increasing.

Similarly, for the option pricing model of example 5.2.3 and the firm exit model of example 5.2.5, if $\rho \geq 0$, then ψ^* is increasing. Moreover, ψ^* is increasing in example 5.2.4. The details are omitted.

Example 5.3.7. For the adaptive search model of example 5.2.2, $r(w)$ is increasing, μ' is increasing in μ, and $f(w'|\mu,\gamma) = N(\mu, \gamma + \gamma_\varepsilon)$ is stochastically increasing in μ, so $\mathbb{E}_{\mu,\gamma}(r(w') \vee \psi(\mu',\gamma'))$ is increasing in μ for all candidate ψ that is increasing in μ. Since $c \equiv c_0$, by proposition 5.3.1, ψ^* is increasing in μ.

5.3.3 Differentiability

Suppose $\mathsf{Z} = \mathsf{Z}^1 \times \cdots \times \mathsf{Z}^m \subset \mathbb{R}^m$, with typical element $z = (z^1, \cdots, z^m)$. Given $h : \mathsf{Z} \to \mathbb{R}$, let $D_i^j h(z)$ be the j-th partial derivative of h with respect to z^i. For a density kernel f, let $D_i^j f(z'|z) := \partial^j f(z'|z)/\partial(z^i)^j$.

Assumption 5.3.2. $D_i c(z)$ exists for all $z \in \text{int}(\mathsf{Z})$ and $i = 1, \cdots, m$.

Let $z^{-i} := (z^1, \cdots, z^{i-1}, z^{i+1}, \cdots, z^m)$. Given $z_0 \in \mathsf{Z}$ and $\delta > 0$, let $B_\delta(z_0^i) := \{z^i \in \mathsf{Z}^i : |z^i - z_0^i| < \delta\}$ and let $\bar{B}_\delta(z_0^i)$ be its closure.

Assumption 5.3.3. P has a density representation f, and, for $i = 1, \cdots, m$,

(1) $D_i^2 f(z'|z)$ exits for all $(z, z') \in \text{int}(\mathsf{Z}) \times \mathsf{Z}$;

(2) $(z, z') \mapsto D_i f(z'|z)$ is continuous;

(3) There are finite solutions of z^i to $D_i^2 f(z'|z) = 0$ (denoted by $z_i^*(z', z^{-i})$), and, for all $z_0 \in \text{int}(\mathsf{Z})$, there exist $\delta > 0$ and a compact subset $A \subset \mathsf{Z}$ such that $z' \notin A$ implies $z_i^*(z', z_0^{-i}) \notin B_\delta(z_0^i)$.

Remark 5.3.2. When the state space is unbounded above and below, a sufficient condition for assumption 5.3.3-(3) is: there are finite solutions of z^i to $D_i^2 f(z'|z) = 0$, and, for all $z_0 \in \text{int}(\mathsf{Z})$, $\|z'\| \to \infty$ implies $|z_i^*(z', z_0^{-i})| \to \infty$.

Assumption 5.3.4. k is continuous and $\int |k(z') D_i f(z'|z)| \, \mathrm{d}z' < \infty$ for all $z \in \text{int}(\mathsf{Z})$, $k \in \{r, \ell\}$ and $i = 1, \cdots, m$.

The following provides a general result for the differentiability of ψ^*.

Proposition 5.3.2. *If assumptions 2.3.1 and 5.3.2–5.3.4 hold, then ψ^* is differentiable at interior points, and, for all $z \in \mathrm{int}(\mathsf{Z})$ and $i = 1, \cdots, m$,*

$$D_i \psi^*(z) = D_i c(z) + \int \max\{r(z'), \psi^*(z')\} D_i f(z'|z) \, \mathrm{d}z'.$$

To obtain continuous differentiability we add the following:

Assumption 5.3.5. For $i = 1, \cdots, m$, the following conditions hold:

(1) $z \mapsto D_i c(z)$ is continuous on $\mathrm{int}(\mathsf{Z})$;

(2) k and $z \mapsto \int |k(z') D_i f(z'|z)| \, \mathrm{d}z'$ are continuous on $\mathrm{int}(\mathsf{Z})$ for $k \in \{r, \ell\}$.

Proposition 5.3.3. *If assumptions 2.3.1, 5.3.3 and 5.3.5 hold, then $z \mapsto D_i \psi^*(z)$ is continuous on $\mathrm{int}(\mathsf{Z})$ for $i = 1, \cdots, m$.*

Example 5.3.8. Recall the job search model of example 5.2.1. It can be shown that, with $h(z, a) := \mathrm{e}^{a(\rho z + b) + a^2 \sigma^2/2} / \sqrt{2\pi\sigma^2}$,

(a) the two solutions of $\frac{\partial^2 f(z'|z)}{\partial z^2} = 0$ are $z^*(z') := \frac{z' - b \pm \sigma}{\rho}$;

(b) $\int \left| \frac{\partial f(z'|z)}{\partial z} \right| \mathrm{d}z' = \frac{|\rho|}{\sigma} \sqrt{\frac{2}{\pi}}$;

(c) $\mathrm{e}^{az'} \left| \frac{\partial f(z'|z)}{\partial z} \right| \leq h(z, a) \exp\left\{ -\frac{[z' - (\rho z + b + a\sigma^2)]^2}{2\sigma^2} \right\} \frac{|\rho z'| + |\rho(\rho z + b)|}{\sigma^2}$;

(d) the two terms on both sides of (c) are continuous in z;

(e) the integration (w.r.t. z') of the right side of (c) is continuous in z.

By remark 5.3.2 and (a), assumption 5.3.3-(3) holds. Based on (5.2.5), conditions (b)–(e), and lemma 5.A.1, one can show that assumption 5.3.5-(2) holds. The other conditions of proposition 5.3.3 are straightforward. Hence, ψ^* is continuously differentiable.

Example 5.3.9. Recall the option pricing problem of example 5.2.3. Through similiar analysis, one can show that ψ^* is continuously differentiable (see appendix 5.B for proofs). Figure 5.1 illustrates. While ψ^* is smooth, v^* is kinked at around $z = 3$ in both cases.[7]

Example 5.3.10. Recall the firm exit model of example 5.2.5. Through similar analysis to examples 5.3.8–5.3.9, one can show that ψ^* is continuously differentiable (see appendix 5.B for proofs). Figure 5.2 illustrates. While ψ^* is smooth, v^* is kinked at around $z = 1.5$ when $\rho = 0.7$, and has two kinks when $\rho = -0.7$.[8]

[7] We set $\gamma = 0.04$, $K = 20$, $b = -0.2$, $\sigma = 1$, and consider $\rho = \pm 0.65$.

[8] We set $\beta = 0.95$, $\sigma = 1$, $b = 0$, $c_f = 5$, $\alpha = 0.5$, $p = 0.15$, $w = 0.15$, and consider respectively $\rho = 0.7$ and $\rho = -0.7$.

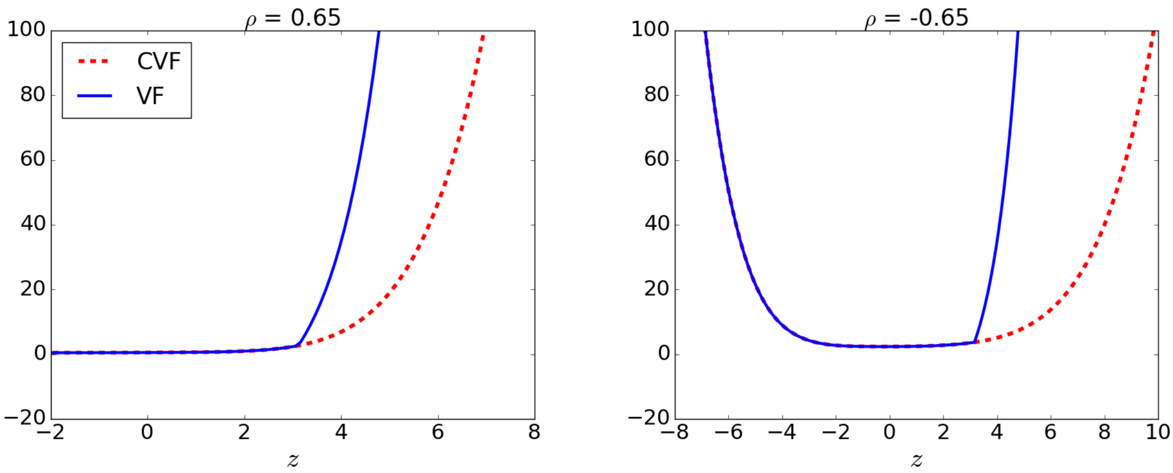

Figure 5.1: Comparison of ψ^* and v^* (Option Pricing)

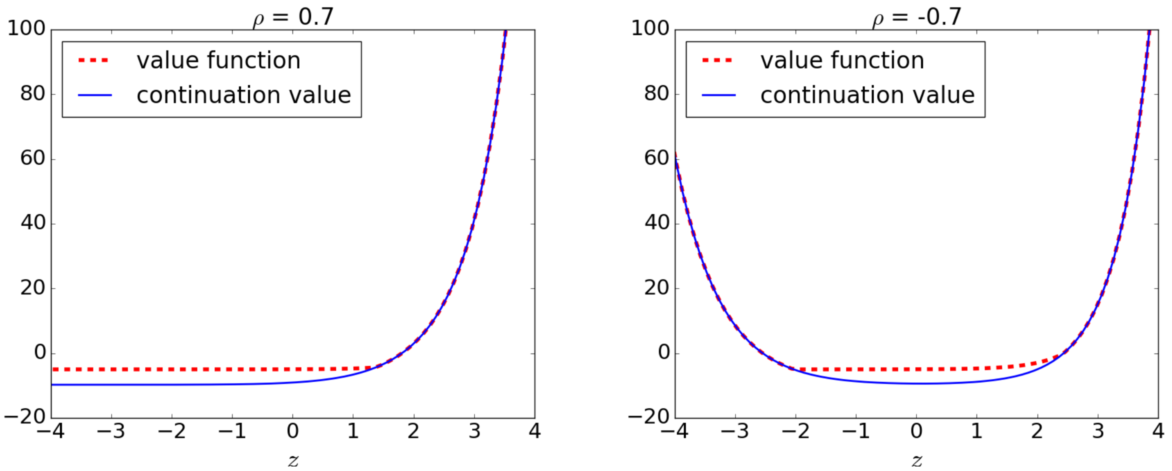

Figure 5.2: Comparison of ψ^* and v^* (Firm Exit)

5.3.4 Parametric Continuity

Consider the parameter space $\Theta \subset \mathbb{R}^k$. Let $P_\theta, r_\theta, c_\theta, v^*_\theta$ and ψ^*_θ denote the stochastic kernel, exit and flow continuation rewards, value and continuation value functions with respect to the parameter $\theta \in \Theta$, respectively. Similarly, let n_θ, $a_{i\theta}$, m_θ, d_θ and g_θ denote the key elements of assumption 2.3.1 with respect to θ. Define $n := \sup_{\theta \in \Theta} n_\theta$, $m := \sup_{\theta \in \Theta} m_\theta$, $d := \sup_{\theta \in \Theta} d_\theta$ and $\bar{a} := \sum_{i=1}^4 \sup_{\theta \in \Theta} a_{i\theta}$.

Assumption 5.3.6. Assumption 2.3.1 holds at all $\theta \in \Theta$, with $\beta m < 1$ and $n, d, \bar{a} < \infty$.

Similar to theorem 5.2.1, one can show that if assumption 5.3.6 holds, then there exist positive constants m' and d' such that for $\ell : \mathsf{Z} \times \Theta \to \mathbb{R}$ defined by

$$\ell(z, \theta) := m' \left(\sum_{t=1}^{n-1} \mathbb{E}_z^\theta |r_\theta(Z_t)| + \sum_{t=0}^{n-1} \mathbb{E}_z^\theta |c_\theta(Z_t)| \right) + g_\theta(z) + d',$$

Q is a contraction mapping on $b_\ell(\mathsf{Z} \times \Theta)$ with unique fixed point $(z, \theta) \mapsto \psi_\theta^*(z)$, where $\mathbb{E}_z^\theta$ denotes the conditional expectation with respect to $P_\theta(z, \cdot)$.

Assumption 5.3.7. The following conditions hold:

(1) $P_\theta(z, \cdot)$ is Feller; that is, $(z, \theta) \mapsto \int h(z') P_\theta(z, \mathrm{d}z')$ is continuous for all continuous bounded function $h : \mathsf{Z} \to \mathbb{R}$.

(2) $(z, \theta) \mapsto c_\theta(z), r_\theta(z), \ell(z, \theta), \int |r_\theta(z')| P_\theta(z, \mathrm{d}z'), \int \ell(z', \theta) P_\theta(z, \mathrm{d}z')$ are continuous.

The following result is an extension of proposition 2.3.4.

Proposition 5.3.4. *If assumptions 5.3.6–5.3.7 hold, then $(z, \theta) \mapsto \psi_\theta^*(z)$ is continuous.*

Example 5.3.11. Recall the job search problem of example 5.2.1. Let $\Theta := [-1, 1] \times A \times B \times C$, where A, B are bounded subsets of $\mathbb{R}_{++}, \mathbb{R}$, respectively, and $C \subset \mathbb{R}$. A typical element of Θ is $\theta = (\rho, \sigma, b, c_0)$. Proposition 5.3.4 implies that $(\theta, z) \mapsto \psi_\theta^*(z)$ is continuous. The proof is similar to example 5.3.1.

5.4 Optimal Policies

This section provides a systematic study of optimal timing of decisions when there are threshold states, and explores the key properties of the optimal policies. We begin in the next section by imposing assumptions under which the optimal policy follows a reservation rule.

5.4.1 Set Up

Let Z be a subset of $\mathbb{R}^m$ with $\mathsf{Z} = \mathsf{X} \times \mathsf{Y}$, where X is a convex subset of $\mathbb{R}$, Y is a convex subset of $\mathbb{R}^{m-1}$, and the state process $\{Z_t\}_{t \geq 0}$ takes the form $\{(X_t, Y_t)\}_{t \geq 0}$. Here $\{X_t\}_{t \geq 0}$ and $\{Y_t\}_{t \geq 0}$ are stochastic processes on X and Y respectively. We assume throughtout section 5.4 that the optimal stopping problem is continuation decomposable (recall section 2.4.1 of chapter 2). The stochastic kernel $P(z, \mathrm{d}z')$ can then be represented by the conditional distribution of (x', y') on y, denoted as $F_y(x', y')$, i.e., $P(z, \mathrm{d}z') = P((x, y), \mathrm{d}(x', y')) = \mathrm{d}F_y(x', y')$.

Assumption 5.4.1. r is strictly monotone on X. Moreover, for all $y \in \mathsf{Y}$, there exists $x \in \mathsf{X}$ such that $r(x, y) = c(y) + \beta \int v^*(x', y') \, \mathrm{d}F_y(x', y')$.

With assumption 5.4.1 in force, we call X_t the *threshold state* and Y_t the *environment*. We call X the *threshold state space* and Y the *environment space*. Under assumption 5.4.1, the reservation rule property holds: there is a *decision threshold* $\bar{x} : \mathsf{Y} \to \mathsf{X}$ such that when x attains $\bar{x}$, the agent is indifferent between stopping and continuing, i.e., $r(\bar{x}(y), y) = \psi^*(y)$ for all $y \in \mathsf{Y}$. The optimal policy then follows

$$\sigma^*(x,y) = \begin{cases} \mathbb{1}\{x \geq \bar{x}(y)\}, & \text{if } r \text{ is strictly increasing in } x \\ \mathbb{1}\{x \leq \bar{x}(y)\}, & \text{if } r \text{ is strictly decreasing in } x \end{cases} \tag{5.4.1}$$

5.4.2 Results

The next few results mainly rely on the implicit function theorem.

Proposition 5.4.1. *Suppose assumption 5.4.1 holds, and that either the assumptions of proposition 2.3.4 or of corollary 5.3.1 (plus the continuity of r) hold. Then $\bar{x}$ is continuous.*

Regarding parametric continuity, let $\bar{x}_\theta$ be the decision threshold w.r.t. $\theta \in \Theta$.

Proposition 5.4.2. *If assumptions 5.3.6–5.3.7 and 5.4.1 hold, then $(y, \theta) \mapsto \bar{x}_\theta(y)$ is continuous.*

A typical element of Y is $y = (y^1, \cdots, y^{m-1})$. Given $h : \mathsf{Y} \to \mathbb{R}$ and $l : \mathsf{X} \times \mathsf{Y} \to \mathbb{R}$, we define $D_i h(y) := \partial h(y)/\partial y^i$, $D_i l(x,y) := \partial l(x,y)/\partial y^i$ and $D_x l(x,y) := \partial l(x,y)/\partial x$. The next result shows that the decision threshold is continuously differentiable with respect to the environment under certain assumptions, and provides an expression for the derivative.

Proposition 5.4.3. *Let assumptions 2.3.1, 5.3.3, 5.3.5 and 5.4.1 hold. If r is continuously differentiable on* $\operatorname{int}(\mathsf{Z})$, *then $\bar{x}$ is continuously differentiable on* $\operatorname{int}(\mathsf{Y})$, *with*

$$D_i \bar{x}(y) = -\frac{D_i r(\bar{x}(y), y) - D_i \psi^*(y)}{D_x r(\bar{x}(y), y)} \quad \textit{for all } y \in \operatorname{int}(\mathsf{Y}) \textit{ and } i = 1, \cdots, m.$$

The intuition behind this expression is as follows: Since $(x, y) \mapsto r(x, y) - \psi^*(y)$ is the premium of terminating the sequential decision process, which is null at the decision threshold, the change in this premium in response to instantaneous changes of x and y cancel out. Hence, the rate of change of $\bar{x}(y)$ with respect to y^i is equivalent to minus the ratio of the marginal premiums of y^i and x. See (5.5.3) for an application in the context of job search.

The next result considers monotonicity and applications are given below.

Proposition 5.4.4. *Let assumptions 2.3.1, 5.3.1 and 5.4.1 hold. If r is defined on X and is strictly increasing, then $\bar{x}$ is increasing.*

5.5 Applications

Let us now look at applications in some more detail, including how the preceding results can be applied and what their implications are. In contrast with chapter 2, all simulations of this section are processed in a standard Python environment on a laptop with a 2.9 GHz Intel Core i7 and 32GB RAM, unless otherwise specified. Moreover, integration is computed via Monte Carlo with 1000 draws, and function approximation is via linear interpolation. For unbounded problems, we set ψ^* outside the grid range to its value at the boundary.[9]

5.5.1 Search with Learning

Recall the adaptive search model of example 5.2.2 (see also examples 5.3.2 and 5.3.7). The value function satisfies

$$v^*(w, \mu, \gamma) = \max\left\{\frac{u(w)}{1-\beta}, c_0 + \beta \int v^*(w', \mu', \gamma') f(w'|\mu, \gamma)\, \mathrm{d}w'\right\},$$

while the Jovanovic operator is given by (5.2.9). This is a threshold state sequential decision problem, with threshold state $x := w \in \mathbb{R}_{++} =: \mathsf{X}$ and environment $y := (\mu, \gamma) \in \mathbb{R} \times \mathbb{R}_{++} =: \mathsf{Y}$. By the intermediate value theorem, assumption 5.4.1 holds. Hence, the optimal policy is represented by a reservation wage $\bar{w} : \mathsf{Y} \to \mathbb{R}$ at which the worker is indifferent between accepting and rejecting the offer. By examples 5.3.2 and 5.3.7 and propositions 5.4.1 and 5.4.4, $\bar{w}$ is increasing in μ and continuous. The intuition behind this monotonicity is that more optimistic agent (higher μ) expects higher offers to be realized.

By theorem 5.2.1, we can compute the reservation wage by iterating with Q. In doing so we set $\beta = 0.95$, $\gamma_\varepsilon = 1.0$, $\tilde{c}_0 = 0.6$, and consider different levels of risk aversion: $\delta = 3, 4, 5, 6$. The grid points of (μ, γ) lie in $[-10, 10] \times [10^{-4}, 10]$ with 200 points for the μ grid and 100 points for the γ grid.

In Figure 5.3, the reservation wage is increasing in μ, which agrees with our analysis (see example 5.3.7). The reservation wage is increasing in γ when μ is small and decreasing in γ when μ is large. Intuitively, although a pessimistic worker (low μ) expects to obtain low offers on average, the downside risks are mitigated because compensation is obtained when the offer is turned down. A higher level

[9] Changing the number of Monte Carlo samples, the grid range and grid density produces similar results in our simulations. See, for example, Fukushima and Waki (2013) and Pál and Stachurski (2013), for systematic analysis of related numerical algorithms.

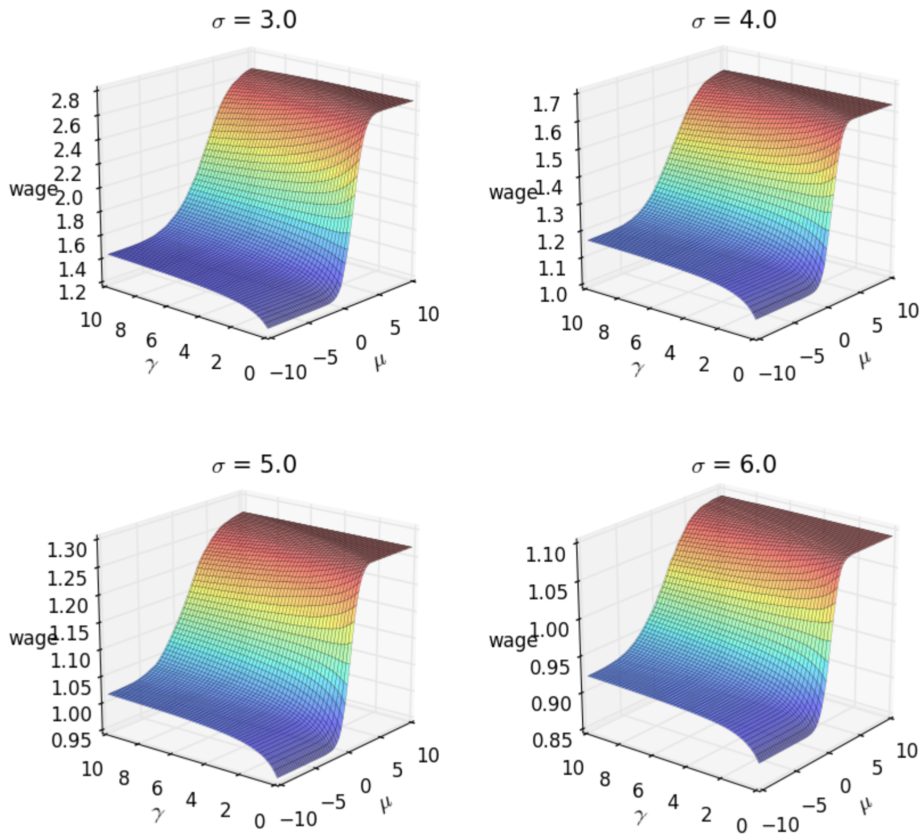

Figure 5.3: The reservation wage

of uncertainty (higher γ) provides a better chance of high offers. For an optimistic (high μ) worker, however, the insurance of compensation has less impact. With higher level uncertainty, the risk-averse worker has incentive to enter the labor market at an earlier stage so as to avoid downside risks.

In computation, value function iteration (VFI) takes more than one week, while continuation value iteration (CVI), being only 2-dimensional, takes 178 seconds.[10]

5.5.2 Firm Entry

Consider a firm entry problem in the style of Fajgelbaum et al. (2017). Each period, an investment cost $f_t \in \mathbb{R}$ is observed, where $\{f_t\} \overset{\text{IID}}{\sim} h$ with $\int |f| h(f) \, \mathrm{d}f <$

[10]We terminate the iteration at precision 10^{-4}. The time of CVI is calculated as the average of the four cases ($\sigma = 3, 4, 5, 6$). Moreover, to implement VFI, we set the grid points of w in $[10^{-4}, 10]$ with 50 points, and combine them with the grid points for μ and γ to run the simulation.

∞.[11] The firm then decides whether to incur this cost and enter the market to earn a stochastic dividend x_t or wait and reconsider. Let

$$x_t = \xi_t + \varepsilon_t^x, \quad \{\varepsilon_t^x\} \overset{\text{IID}}{\sim} N(0, \gamma_x),$$

where ξ_t and ε_t^x are respectively a persistent and a transient component, with

$$\xi_t = \rho \xi_{t-1} + \varepsilon_t^\xi, \quad \{\varepsilon_t^\xi\} \overset{\text{IID}}{\sim} N(0, \gamma_\xi).$$

A public signal y_{t+1} is released at the end of each period t, where

$$y_t = \xi_t + \varepsilon_t^y, \quad \{\varepsilon_t^y\} \overset{\text{IID}}{\sim} N(0, \gamma_y).$$

The firm has prior $\xi \sim N(\mu, \gamma)$ that is updated after observing y' if the firm chooses to wait. The posterior satisfies $\xi | y' \sim N(\mu', \gamma')$, with

$$\gamma' = 1/\left(1/\gamma + \rho^2/(\gamma_\xi + \gamma_y)\right) \quad \text{and} \quad \mu' = \gamma'\left(\mu/\gamma + \rho y'/(\gamma_\xi + \gamma_y)\right). \tag{5.5.1}$$

The firm has utility $u(x) = \left(1 - \mathrm{e}^{-ax}\right)/a$, where $a > 0$ is the absolute risk aversion coefficient. The terminal and flow continuation rewards are respectively

$$r(f, \mu, \gamma) := \mathbb{E}_{\mu,\gamma}[u(x)] - f = \left(1 - \mathrm{e}^{-a\mu + a^2(\gamma+\gamma_x)/2}\right)/a - f$$

and $c \equiv 0$.[12] This is a threshold state problem, with threshold state $x := f \in \mathbb{R} =: \mathsf{X}$ and environment $y := (\mu, \gamma) \in \mathbb{R} \times \mathbb{R}_{++} =: \mathsf{Y}$. The Jovanovic operator is

$$Q\psi(\mu,\gamma) = \beta \int \max\left\{\mathbb{E}_{\mu',\gamma'}[u(x')] - f', \psi(\mu', \gamma')\right\} p(f', y' | \mu, \gamma)\, \mathrm{d}(f', y'),$$

where

$$p(f', y' | \mu, \gamma) = h(f') l(y' | \mu, \gamma) \quad \text{with} \quad l(y' | \mu, \gamma) = N(\rho\mu, \rho^2\gamma + \gamma_\xi + \gamma_y).$$

As our primitive set up, we let

$$n := 1, \quad g(\mu, \gamma) := \mathrm{e}^{-\mu + a^2\gamma/2}, \quad m := 1 \quad \text{and} \quad d := 0,$$

and define ℓ according to (5.2.1). Moreover, let $\bar{f} : \mathsf{Y} \to \mathbb{R}$ be the reservation cost.

Proposition 5.5.1. *The following statements are true:*

(1) Q is a contraction mapping on $(b_\ell \mathsf{Y}, \|\cdot\|_\ell)$ with unique fixed point ψ^*.

[11] Generally, we allow for $f_t < 0$, which can be interpreted as investment compensation.

[12] Since in general $\{f_t\}$ can be supported on $\mathbb{R}$ and the terminal reward is unbounded, solution methods based on the Bellman operator with respect to the supremum norm or local contractions fail on the theoretical side. However, the theory we develop can be applied.

(2) *The reservation cost is* $\bar{f}(\mu,\gamma) = \mathbb{E}_{\mu,\gamma}[u(x)] - \psi^*(\mu,\gamma)$.

(3) ψ^* *and* $\bar{f}$ *are continuous functions.*

(4) *If* $\rho \geq 0$, *then* ψ^* *is increasing in* μ.

Remark 5.5.1. Note that the first three claims of proposition 5.5.1 place no restriction on the range of ρ values, the correlation coefficient of $\{\xi_t\}$.

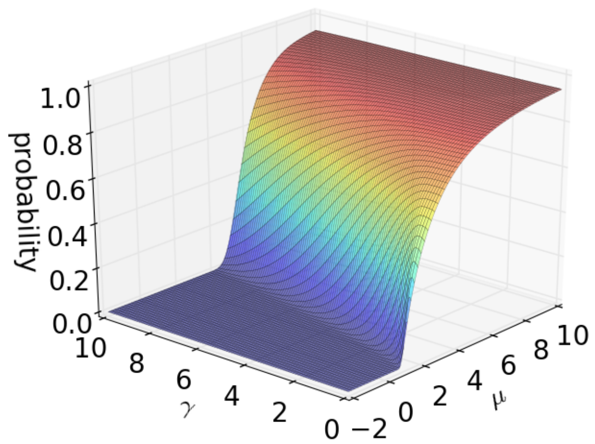

Figure 5.4: The perceived probability of investment

In simulation, we set $\beta = 0.95$, $a = 0.2$, $\gamma_x = 0.1$, $\gamma_y = 0.05$, and $h = LN(0, 0.01)$. Let $\rho = 1$, $\gamma_\xi = 0$, and let the grid points of (μ, γ) lie in $[-2, 10] \times [10^{-4}, 10]$ with 100 points for each variable.

Figure 5.4 plots the perceived probability of investment $\mathbb{P}\left\{f \leq \bar{f}(\mu,\gamma)\right\}$. As expected, this probability is increasing in μ and decreasing in γ, since a more optimistic firm (higher μ) is more likely to invest, and with higher level uncertainty (higher γ) the risk averse firm prefers to delay investment so as to avoid downside risks.[13] In terms of computation time, VFI takes more than one week, while CVI takes 921 seconds.[14]

[13]This result parallels propositions 1–2 of Fajgelbaum et al. (2017).

[14]We terminate the iteration at precision 10^{-4}. To implement VFI, we set the grid points of f in $[10^{-4}, 10]$ with 50 points, and combine them with the grid points for μ and γ to run the simulation.

5.5.3 Search with Permanent and Transitory Components

Recall the job search problem introduced in section 2.5.1 of chapter 2.[15] In particular, the state process follows

$$w_t = \eta_t + \theta_t \xi_t, \quad \ln \theta_t = \rho \ln \theta_{t-1} + \ln u_t, \quad \rho \in [-1, 1]. \tag{5.5.2}$$

Here $\{\xi_t\} \stackrel{\text{IID}}{\sim} h$ and $\{\eta_t\} \stackrel{\text{IID}}{\sim} v$ are positive processes with finite first moments, $\int \eta^{-1} v(\eta) \, \mathrm{d}\eta < \infty$, $\{u_t\} \stackrel{\text{IID}}{\sim} LN(0, \gamma_u)$, and $\{\xi_t\}$, $\{\eta_t\}$ and $\{\theta_t\}$ are independent. We interpret θ_t and ξ_t respectively as the persistent and transitory components of income, and η_t as social security. The Jovanovic operator is

$$Q\psi(\theta) = c_0 + \beta \int \max\left\{\frac{u(w')}{1-\beta}, \psi(\theta')\right\} f(\theta'|\theta) h(\xi') v(\eta') \, \mathrm{d}(\theta', \xi', \eta'),$$

where

$$w' = \eta' + \theta' \xi' \quad \text{and} \quad f(\theta'|\theta) = LN(\rho \ln \theta, \gamma_u).$$

This is a threshold state problem, with threshold state $w \in \mathbb{R}_{++} =: \mathsf{X}$ and environment $\theta \in \mathbb{R}_{++} =: \mathsf{Y}$. Let $\bar{w}$ be the reservation wage. Recall the risk aversion coefficient δ in (5.2.4).

Proposition 5.5.2. *Suppose that $\delta = 1$ and $\rho \in (-1, 1)$.*

- *Choose $n \in \mathbb{N}_0$ such that $\beta e^{\rho^{2n} \gamma_u} < 1$.*
- *Let $g(\theta) := \theta^{\rho^n} + \theta^{-\rho^n}$ and $m = d := e^{\rho^{2n} \gamma_u}$.*
- *Let ℓ be defined as in (5.2.1).*

Then the following statements hold:

(1) Q is a contraction mapping on $(b_\ell \mathsf{Y}, \|\cdot\|_\ell)$ with unique fixed point ψ^*.

(2) The reservation wage is $\bar{w}(\theta) = e^{(1-\beta)\psi^*(\theta)}$.

(3) ψ^* and $\bar{w}$ are continuously differentiable, and

$$\bar{w}'(\theta) = (1-\beta)\bar{w}(\theta)\psi^{*\prime}(\theta). \tag{5.5.3}$$

(4) If $\rho \geq 0$, then ψ^* and $\bar{w}$ are increasing in θ.

Remark 5.5.2. If $\beta e^{\gamma_u/2} < 1$, claims (1)–(3) of proposition 5.5.2 remain true for $|\rho| = 1$, and claim (4) is true for $\rho = 1$.

[15] The notation will be changed a little bit in order to maintain coherence of the current chapter.

The intuition behind the expression (5.5.3) for the derivative of $\bar{w}$ is as follows: Since the terminating premium is zero at the reservation wage, the overall effect of changes in w and θ cancel out. Hence, the rate of change of $\bar{w}$ with respect to θ equals the minus ratio of the marginal premiums of θ and w at the decision threshold, denoted respectively by $-\psi^{*\prime}(\theta)$ and $[(1-\beta)\bar{w}(\theta)]^{-1}$.

Proposition 5.5.3. *Suppose that $\delta \geq 0$, $\delta \neq 1$ and $\rho \in (-1, 1)$.*

- *Choose $n \in \mathbb{N}_0$ such that $\beta \, \mathrm{e}^{\rho^{2n}(1-\delta)^2 \gamma_u} < 1$.*
- *Let $g(\theta) := \theta^{(1-\delta)\rho^n} + \theta^{-(1-\delta)\rho^n}$ and $m = d := \mathrm{e}^{\rho^{2n}(1-\delta)^2 \gamma_u}$.*
- *Let ℓ be defined as in (5.2.1).*

Then the following statements hold:

(1) Q is a contraction mapping on $(b_\ell \mathsf{Y}, \| \cdot \|_\ell)$ with unique fixed point ψ^*.

(2) The reservation wage $\bar{w}(\theta) = [(1-\beta)(1-\delta)\psi^*(\theta)]^{\frac{1}{1-\delta}}$.

(3) ψ^* and $\bar{w}$ are continuously differentiable, and

$$\bar{w}'(\theta) = (1-\beta)[\bar{w}(\theta)]^{\delta} \, \psi^{*\prime}(\theta).$$

(4) If $\rho \geq 0$, then ψ^* and $\bar{w}$ are increasing in θ.

Remark 5.5.3. If $\beta e^{(1-\delta)^2 \gamma_u / 2} < 1$, then claims (1)–(3) of proposition 5.5.3 remain true for $|\rho| = 1$, and claim (4) remains true for $\rho = 1$.

Remark 5.5.4. Since the terminating premium is 0 at the reservation wage, the overall effect of changes in w and θ cancel out. Hence, the rate of change of $\bar{w}$ w.r.t. θ equals the ratio of the marginal premiums of θ and w at the decision threshold, denoted respectively by $\psi^{*\prime}(\theta)$ and $\bar{w}(\theta)^{-\delta}/(1-\beta)$, as documented by claim (3).

Group–1 Experiments

In simulation, we set $\beta = 0.95$, $\delta = 1$, $\tilde{c}_0 = 0.6$, $\gamma_u = 10^{-4}$, $v = LN(0, 10^{-6})$, $h = LN(0, 5 \times 10^{-4})$, and consider the parametric class problem of $\rho \in [0, 1]$, with 100 grid points. Grid points of θ lie in $[10^{-4}, 10]$ with 200 points, and are scaled to be more dense when θ is smaller.

When $\rho = 0$, $\{\theta_t\} \stackrel{\text{IID}}{\sim} LN(0, \gamma_u)$, in which case each realized θ will be forgotten at future stages. As a result, the continuation value is independent of θ, yielding

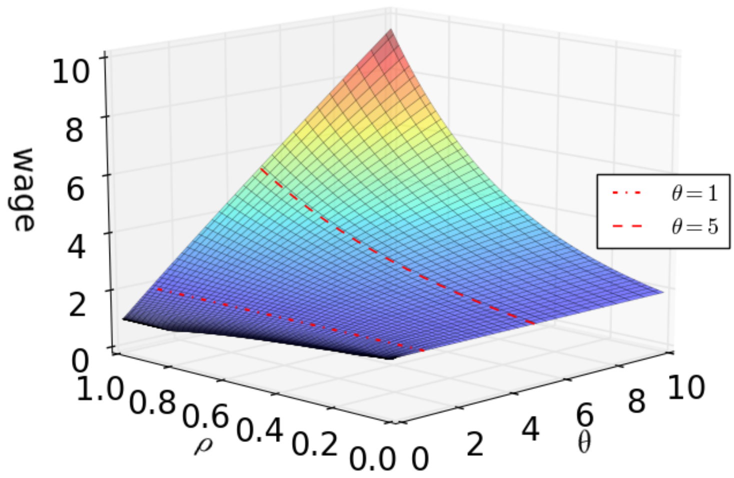

Figure 5.5: The reservation wage

a reservation wage parallel to the θ-axis, as shown in figure 5.5. When $\rho > 0$, the reservation wage is increasing in θ, which is intuitive because higher θ implies a better current situation. Since a higher degree of income persistence (higher ρ) prolongs the mean-reverting process, the reservation wage tends to decrease in ρ in bad states ($\theta < 1$) and increase in ρ in good states ($\theta > 1$).

Table 5.1: Time in seconds under different grid sizes (group–1 expr.)

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9
CVI	0.141	0.125	0.125	0.813	0.812	0.812	1.062	1.062	1.063
VFI	171.63	284.01	440.75	277.19	1078.27	1488.78	1075.22	1622.09	2696.09

We set $\rho = 0.75$, $\beta = 0.95$, $\tilde{c}_0 = 0.6$, $\delta = 1$, $\gamma_u = 10^{-4}$, $v = LN(0, 10^{-6})$ and $h = LN(0, 5 \times 10^{-4})$. Grid points for (θ, w) lie in $[10^{-4}, 10]^2$, and the grid sizes for (θ, w) in each test are: **Test 1:** $(200, 200)$, **Test 2:** $(200, 300)$, **Test 3:** $(200, 400)$, **Test 4:** $(300, 200)$, **Test 5:** $(300, 300)$, **Test 6:** $(300, 400)$, **Test 7:** $(400, 200)$, **Test 8:** $(400, 300)$, and **Test 9:** $(400, 400)$. For both CVI and VFI, we terminate the iteration at a precision level 10^{-4}. We run the simulation 10 times for CVI, 5 times for VFI, and calculate the average time (in seconds).

Table 5.1 provides a numerical comparison between CVI and VFI under different grid sizes. In tests 1–9, CVI is 1733 times faster than VFI on average, and outperforms VFI more strongly as we increase the grid size. For example, as we increase the grid size of w and π, there is a slight decrease in the speed of CVI, while the speed of VFI drops exponentially (see, e.g., tests 1, 5 and 9).

Group–2 Experiments

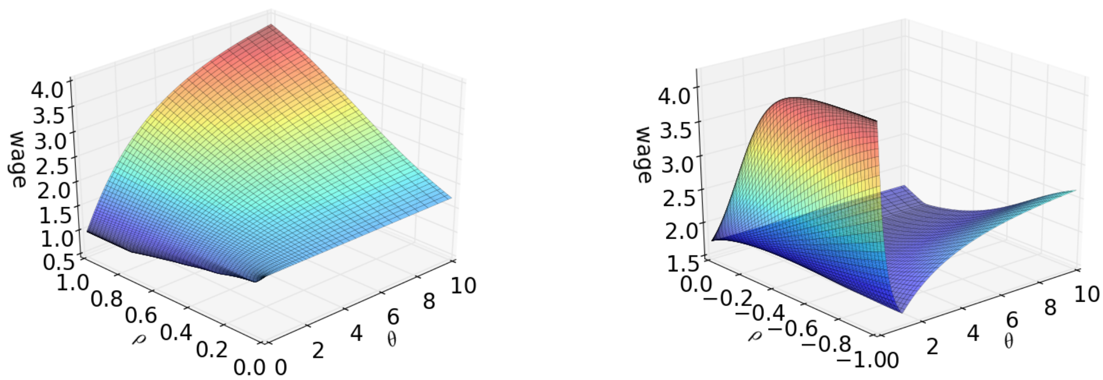

Figure 5.6: The reservation wage

The benchmark set up is the same as in group–1 experiment of section 5.5.3, except that we set $\delta = 2.5$. We consider parametric class problems of ρ. $\rho \in [0, 1]$ and $\rho \in [-1, 0]$ is treated separately, with 100 grid points in each case. Grid points of θ lie in $[10^{-4}, 10]$ with 200 points, and are scaled to be more dense when θ is smaller.

When $\rho = 0$, $\{\theta_t\} \overset{\text{IID}}{\sim} LN(0, \gamma_u)$, in which case each realized θ will be forgotten in future stages. As a result, the continuation value is independent of θ, yielding a reservation wage parallel to the θ-axis, as shown in figure 5.6.

When $\rho > 0$, the reservation wage is increasing in θ, which is natural since a higher θ implies a better current situation. Further, since a higher degree of income persistence (higher ρ) prolongs the mean-reverting process, the reservation wage tends to decrease in ρ in bad states ($\theta < 1$) and increase in ρ in good states ($\theta > 1$).

When $\rho < 0$, the reservation wage decreases in θ initially and then starts to increase in θ afterwards. Intuitively, a low or a high θ is more favorable than a medium level θ since it allows the agent to take advantage of the countercyclical patterns.

The relative numerical efficiency of VFI and CVI is shown in table 5.2. The interpretation is similar to group–1 experiment of section 5.5.3.

Table 5.2: Time in seconds under different grid sizes (group–2 expr.)

Time	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9
CVI	0.300	0.295	0.294	0.453	0.450	0.448	0.620	0.618	0.622
VFI	277.33	364.68	527.83	355.92	558.06	870.41	451.06	795.90	1191.43

We set $\rho = 0.75$, $\beta = 0.95$, $\tilde{c}_0 = 0.6$, $\delta = 2.5$, $\mu_\eta = 0$, $\gamma_\eta = 10^{-6}$, $\gamma_\xi = 5 \times 10^{-4}$, and $\gamma_u = 10^{-4}$. The grid points for (θ, w) lie in $[10^{-4}, 10]^2$, and the grid sizes for (θ, w) in each test are **Test 1:** $(200, 200)$, **Test 2:** $(200, 300)$, **Test 3:** $(200, 400)$, **Test 4:** $(300, 200)$, **Test 5:** $(300, 300)$, **Test 6:** $(300, 400)$, **Test 7:** $(400, 200)$, **Test 8:** $(400, 300)$, and **Test 9:** $(400, 400)$. For both CVI and VFI, we terminate the iteration at a precision level 10^{-4}. We run the simulation 10 times for CVI, 5 times for VFI, and calculate the average time (in seconds).

5.5.4 Firm Exit with Learning

Consider agent's learning in a firm exit framework (see, e.g., Jovanovic (1982), Pakes and Ericson (1998), Mitchell (2000), Timoshenko (2015)). Let q be firm's output, $C(q)$ be a cost function, and $C(q)x$ be the total cost, where the state process $\{x_t\}_{t\geq 0}$ satisfies

$$\ln x_t = \xi + \varepsilon_t, \quad \{\varepsilon_t\}_{t\geq 0} \overset{\text{IID}}{\sim} N(0, \gamma_\varepsilon),$$

where ξ denotes the firm's type. At the beginning of each period, the firm observes x and decides whether to exit the industry or not. The prior belief is $\xi \sim N(\mu, \gamma)$, so the posterior after observing x' is $\xi | x' \sim N(\mu', \gamma')$, where

$$\gamma' = (1/\gamma + 1/\gamma_\varepsilon)^{-1} \quad \text{and} \quad \mu' = \gamma' \left(\mu/\gamma + (\ln x')/\gamma_\varepsilon\right).$$

Let $\pi(p, x) = \max_q [pq - C(q)x]$ be the maximal profit (the flow continuation reward), and $r(p, x)$ be the profit from other industries (the terminal reward, or the opportunity cost of staying in the current industry), where p is price. Consider, for example, $C(q) := q^2$, and the price sequence $\{p_t\}_{t\geq 0}$ satisfies

$$\ln p_{t+1} = \rho \ln p_t + b + \varepsilon^p_{t+1}, \quad \{\varepsilon^p_t\}_{t\geq 0} \overset{\text{IID}}{\sim} N(0, \gamma_p).$$

Let $z := (p, x, \mu, \gamma) \in \mathbb{R}^2_+ \times \mathbb{R} \times \mathbb{R}_+ =: \mathsf{Z}$. Then the Jovanovic operator satisfies

$$Q\psi(z) = \pi(p, x) + \beta \int \max\{r(p', x'), \psi(z')\} l(p', x' | p, \mu, \gamma) \, \mathrm{d}(p', x'),$$

where

$$l(p', x' | p, \mu, \gamma) := h(p'|p) f(x'|\mu, \gamma), \quad \text{with}$$

$$h(p'|p) := LN(\rho \ln p + b, \gamma_p) \quad \text{and} \quad f(x'|\mu, \gamma) := LN(\mu, \gamma + \gamma_\varepsilon).$$

Unbounded rewards can be treated via our methodology. Notice that

$$\pi(p, x) = p \cdot q^*(p, x) - x \cdot C\left[q^*(p, x)\right],$$

where $q^*(p,x) := (C')^{-1}(p/x)$. Since $C(q) := q^2$, we have $q^*(p,x) = p/2x$ and $\pi(p,x) = p^2/4x$. The following results can be established via our methodology.

Proposition 5.5.4. *Suppose $\rho \in (-1,1)$ and that for some $h_1, h_2 \in \mathbb{R}_+$,*

$$|r(p,x)| \leq h_1 p^2/x + h_2.$$

We set up the model primitives as follows:

- *Define $\xi := 2(|b| + \gamma_p)$.*
- *Choose $n \in \mathbb{N}_0$ such that $\beta e^{|\rho^n|\xi} < 1$.*
- *Choose $\delta > 0$ such that $\delta \geq e^{|\rho^n|\xi} / \left(e^{|\rho^n|\xi} - 1\right)$.*
- *Let $g(p,\mu,\gamma) := \left(p^{2\rho^n} + p^{-2\rho^n} + \delta\right) e^{-\mu+\gamma/2}$.*
- *Let $m := e^{|\rho^n|\xi}$ and $d := 0$.*
- *Let ℓ be defined as in (5.2.1).*

Then the following statements hold:

(1) Q is a contraction mapping on $(b_\ell \mathsf{Z}, \|\cdot\|_\ell)$ with unique fixed point ψ^*.

(2) If in addition r is continuous, then ψ^* is continuous.

The next result shows that the assumptions on r can be further relaxed.

Proposition 5.5.5. *Suppose $\rho \in (-1,1)$ and for some $h_1, \cdots, h_5 \in \mathbb{R}_+$,*

$$|r(p,x)| \leq h_1 p^2/x + h_2 p^2 + h_3 x^{-1} + h_4 x + h_5.$$

We set up the model primitives as follows:

- *Define $\xi := 2(|b| + \gamma_p)$.*
- *Choose $n \in \mathbb{N}_0$ such that $\beta e^{|\rho^n|\xi} < 1$.*
- *Choose $\delta > 0$ such that $\delta \geq e^{|\rho^n|\xi} / \left(e^{|\rho^n|\xi} - 1\right)$.*
- *Let $g(p,\mu,\gamma) := \left(p^{2\rho^n} + p^{-2\rho^n} + \delta\right) (e^{-\mu+\gamma/2} + 1) + e^{-\mu+\gamma/2} + e^{\mu+\gamma/2}$.*
- *Let $m := e^{|\rho^n|\xi}$ and $d := 0$.*
- *Let ℓ be defined as in (5.2.1).*

Then the following statements are true:

(1) Q is a contraction mapping on $(b_\ell \mathsf{Z}, \|\cdot\|_\ell)$ with unique fixed point ψ^*.

(2) If in addition r is continuous, then ψ^* is continuous.

5.5.5 Search with Learning II

Consider the adaptive search model of Ljungqvist and Sargent (2012) (section 6.6). The model is as example 5.2.1, apart from the fact that the distribution of the wage process h is unknown. The worker knows that there are two possible densities f and g, and puts prior probability π_t on f being chosen. If the current offer w_t is rejected, a new offer w_{t+1} is observed at the beginning of next period, and, by the Bayes' rule, π_t updates via

$$\pi_{t+1} = \pi_t f(w_{t+1}) / [\pi_t f(w_{t+1}) + (1 - \pi_t) g(w_{t+1})] =: q(w_{t+1}, \pi_t). \tag{5.5.4}$$

The state space is $\mathsf{Z} := \mathsf{X} \times [0, 1]$, where X is a compact interval of $\mathbb{R}_+$. Let $u(w) := w$. The value function of the unemployed worker satisfies

$$v^*(w, \pi) = \max \left\{ \frac{w}{1 - \beta}, c_0 + \beta \int v^*(w', q(w', \pi)) h_\pi(w') \, \mathrm{d}w' \right\},$$

where

$$h_\pi(w') := \pi f(w') + (1 - \pi) g(w').$$

This is a typical threshold state problem, with threshold state $x := w \in \mathsf{X}$ and environment $y := \pi \in [0, 1] =: \mathsf{Y}$. As to be shown, the optimal policy is determined by a reservation wage $\bar{w} : [0, 1] \to \mathbb{R}$ such that when $w = \bar{w}(\pi)$, the worker is indifferent between accepting and rejecting the offer. Consider the candidate space $(b[0, 1], \| \cdot \|)$. The Jovanovic operator is

$$Q\psi(\pi) = c_0 + \beta \int \max \left\{ \frac{w'}{1 - \beta}, \psi \circ q(w', \pi) \right\} h_\pi(w') \, \mathrm{d}w'. \tag{5.5.5}$$

Proposition 5.5.6. *Let $c_0 \in \mathsf{X}$. The following statements are true:*

(1) Q is a contraction on $(b[0, 1], \| \cdot \|)$ of modulus β, with unique fixed point ψ^*.

(2) The reservation wage $\bar{w}(\pi) = (1 - \beta)\psi^*(\pi)$.

(3) ψ^* and $\bar{w}$ are continuous.

Since the computation is 2-dimensional via value function iteration (VFI), and is only 1-dimensional via continuation value function iteration (CVI), we expect the computation via CVI to be much faster. We run several groups of tests and compare the time taken by the two methods. All tests are processed in a standard Python environment on a laptop with a 2.5 GHz Intel Core i5 and 8GB RAM.

Group-1 Experiments

This group documents the time taken to compute the fixed point across different parameter values and at different precision levels. Table 5.3 provides the list of experiments performed and table 5.4 shows the result.

Table 5.3: Group-1 experiments

Parameter	Test 1	Test 2	Test 3	Test 4	Test 5
β	0.9	0.95	0.98	0.95	0.95
c_0	0.6	0.6	0.6	0.001	1

Note: Different parameter values in each experiment.

Table 5.4: Time in seconds under different parameter values (group–1 expr.)

Test/Method/Precision		10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
Test 1	VFI	114.17	140.94	174.91	201.77	228.59	255.67
	CVI	0.67	0.92	1.16	1.43	1.71	1.94
Test 2	VFI	181.78	234.58	271.89	323.22	339.87	341.55
	CVI	0.95	1.49	1.80	2.27	2.69	3.11
Test 3	VFI	335.78	335.87	335.28	335.91	338.70	334.21
	CVI	1.77	2.68	3.08	3.03	3.03	3.06
Test 4	VFI	154.18	201.05	247.72	294.90	335.32	335.00
	CVI	0.79	1.22	1.65	2.06	2.50	2.91
Test 5	VFI	275.41	336.02	326.33	327.41	327.11	327.71
	CVI	1.33	2.12	2.79	2.99	2.97	2.97

Note: We set $\mathsf{X} = [0, 2]$, $f = \text{Beta}(1, 1)$ and $g = \text{Beta}(3, 1.2)$. The grid points of (w, π) lie in $[0, 2] \times [10^{-4}, 1 - 10^{-4}]$ with 100 points for w and 50 for π. For each given test and level of precision, we run the simulation 50 times for CVI, 20 times for VFI, and calculate the average time (in seconds).

As shown in table 5.4, CVI performs much better than VFI. On average, CVI is 141 times faster than VFI. In the best case, CVI is 207 times faster (in test 5, VFI takes 275.41 seconds to achieve a level of accuracy 10^{-3}, while CVI takes only 1.33 seconds). In the worst case, CVI is 109 times faster (in test 5, CVI takes 2.99 seconds as opposed to 327.41 seconds by VFI to attain a precision level 10^{-6}).

Group-2 Experiments

In applications, increasing the number of grid points provides more accurate numerical approximations. This group of tests compares how the two approaches perform under different grid sizes. The setup and result are summarized in table 5.5 and table 5.6, respectively.

Table 5.5: Group-2 experiments

Variable	Test 2	Test 6	Test 7	Test 8	Test 9	Test 10
π	50	50	50	100	100	100
w	100	150	200	100	150	200

Note: Different grid sizes of the state variables in each experiment.

Table 5.6: Time in seconds under different grid sizes (group–2 expr.)

Test/Precision/Method		10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
Test 2	VFI	181.78	234.58	271.89	323.22	339.87	341.55
	CVI	0.95	1.49	1.80	2.27	2.69	3.11
Test 6	VFI	264.34	336.20	407.52	476.01	508.05	509.05
	CVI	0.96	1.39	1.82	2.30	2.73	3.14
Test 7	VFI	355.40	449.55	545.51	641.05	679.93	678.28
	CVI	0.92	1.37	1.79	2.22	2.84	3.07
Test 8	VFI	352.76	447.36	541.75	639.73	678.91	677.52
	CVI	1.94	2.74	3.58	4.42	5.30	6.14
Test 9	VFI	526.72	670.19	812.66	951.78	1017.29	1015.15
	CVI	1.81	2.68	3.68	4.33	5.23	6.08
Test 10	VFI	706.34	897.07	1086.15	1278.27	1354.37	1360.07
	CVI	1.83	2.72	3.51	4.40	5.21	6.10

Note: We set $\mathsf{X} = [0, 2]$, $\beta = 0.95$, $c_0 = 0.6$, $f = \text{Beta}(1, 1)$ and $g = \text{Beta}(3, 1.2)$. The grid points of (w, π) lie in $[0, 2] \times [10^{-4}, 1 - 10^{-4}]$. For each given test and precision level, we run the simulation 50 times for CVI, 20 times for VFI, and calculate the average time (in seconds).

CVI outperforms VFI more obviously as the grid size increases. In table 5.6 we see that as we increase the number of grid points for w, the speed of CVI is not affected. However, the speed of VFI drops significantly. Amongst tests 2, 6 and 7, CVI is 219 times faster than VFI on average. In the best case, CVI is 386 times faster (while it takes VFI 355.40 seconds to achieve a precision level 10^{-3} in test

7, CVI takes only 0.92 second). As we increase the grid size of w from 100 to 200, CVI is not affected, but the time taken for VFI almost doubles.

As we increase the grid size of both w and π, there is a slight decrease in the speed of CVI. Nevertheless, the decrease in the speed of VFI is exponential. Among tests 2 and 8–10, CVI is 223.41 times as fast as VFI on average. In test 10, VFI takes 706.34 seconds to obtain a level of precision 10^{-3}, instead, CVI takes only 1.83 seconds, which is 386 times faster.

Group-3 Experiments

Since the total number of grid points increases exponentially with the number of states, the speed of computation will drop dramatically with an additional state. To illustrate, consider a parametric class problem with respect to c_0. We set $\mathsf{X} = [0,2]$, $\beta = 0.95$, $f = \text{Beta}(1,1)$ and $g = \text{Beta}(3,1.2)$. Let (w, π, c_0) lie in $[0,2] \times [10^{-4}, 1 - 10^{-4}] \times [0,1.5]$ with 100 grid points for each. In this case, VFI is 3-dimensional and suffers the "curse of dimensionality": the computation takes more than 7 days. However, CVI is only 2-dimensional and the computation finishes within 171 seconds (with precision 10^{-6}).

In figure 5.7, we see that the reservation wage is increasing in c_0 and decreasing in π. Intuitively, a higher level of compensation hinders the agent's incentive of entering into the labor market. Moreover, since f is a less attractive distribution than g and larger π means more weight on f and less on g, a larger π depresses the worker's assessment of future prospects, and relatively low current offers become more attractive.

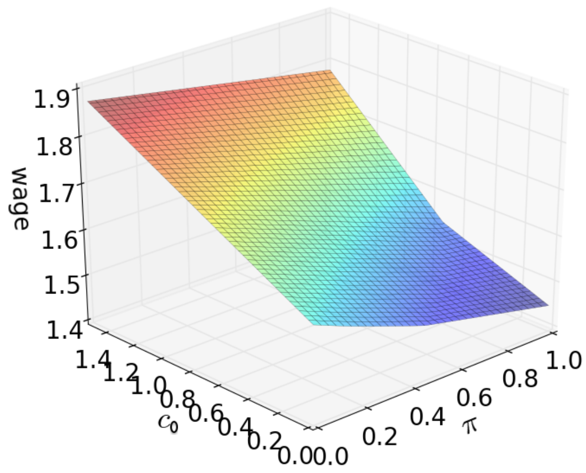

Figure 5.7: The reservation wage

Appendix 5.A A Continuity Lemma

Let $(X, \mathcal{X}, \nu)$ and $(Y, \mathcal{Y}, u)$ two measure spaces. Lemma 5.A.1 below can be shown by the Fatou's lemma. The idea of proof is similar to proposition 2.3.4 below.

Lemma 5.A.1. *Let $p : Y \times X \to \mathbb{R}$ be a measurable map that is continuous in x. If there exists a measurable map $q : Y \times X \to \mathbb{R}$ that is continuous in x with $q \geq |p|$ on $Y \times X$, and that $x \mapsto \int q(y,x)u(\mathrm{d}y)$ is continuous, then $x \mapsto \int p(y,x)u(\mathrm{d}y)$ is continuous.*

Proof. Since $q(y,x) \geq |p(y,x)|$ for all $(y,x) \in Y \times X$, we know that $(y,x) \mapsto q(y,x) \pm p(y,x)$ are nonnegative measurable functions. Let (x_n) be a sequence of X with $x_n \to x$. By Fatou's lemma, we have

$$\int \liminf_{n\to\infty} [q(y,x_n) \pm p(y,x_n)] u(\mathrm{d}y) \leq \liminf_{n\to\infty} \int [q(y,x_n) \pm p(y,x_n)] u(\mathrm{d}y).$$

From the given assumptions we know that $\lim_{n\to\infty} \int q(y,x_n)u(\mathrm{d}y) = q(y,x)$. Combine this result with the above inequality, we have

$$\pm \int p(y,x)u(\mathrm{d}y) \leq \liminf_{n\to\infty} \left(\pm \int p(y,x_n)u(\mathrm{d}y) \right),$$

where we have used the fact that for any two given sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ of $\mathbb{R}$ with $\lim_{n\to\infty} a_n$ exists, we have: $\liminf_{n\to\infty}(a_n + b_n) = \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$. So

$$\limsup_{n\to\infty} \int p(y,x_n)u(\mathrm{d}y) \leq \int p(y,x)u(\mathrm{d}y) \leq \liminf_{n\to\infty} \int p(y,x_n)u(\mathrm{d}y).$$

Therefore, the mapping $x \mapsto \int p(y,x)u(\mathrm{d}y)$ is continuous. $\square$

Appendix 5.B Main Proofs

5.B.1 Proof of Section 5.2 Results.

In this section, we prove examples 5.2.1–5.2.5. Note that in examples 5.2.1, 5.2.3 and 5.2.5, the stochastic kernel P has a density representation $f(z'|z) = N(\rho z + b, \sigma^2)$.

Proof of example 5.2.1. ***Case I:*** $\delta \geq 0$ and $\delta \neq 1$.

In this case, the terminal reward is $r(z) := \mathrm{e}^{(1-\delta)z}/((1-\beta)(1-\delta))$. Since

$$\int \mathrm{e}^{(1-\delta)z'} f(z'|z)\,\mathrm{d}z' = a_1 \mathrm{e}^{\rho(1-\delta)z}$$

for some constant $a_1 > 0$, induction shows that for all $t \in \mathbb{N}$,

$$\int \mathrm{e}^{(1-\delta)z'} P^t(z, \mathrm{d}z') = a_t\, \mathrm{e}^{\rho^t(1-\delta)z} \leq a_t \left(\mathrm{e}^{\rho^t(1-\delta)z} + \mathrm{e}^{\rho^t(\delta-1)z} \right) \tag{5.B.1}$$

for some constant $a_t > 0$. Recall ξ_1 and ξ_2 defined in example 5.2.1, and let $\zeta := |\xi_1| + \xi_2$.

- If $\rho \in (-1,1)$, then we can choose $n \in \mathbb{N}_0$ such that $\beta \mathrm{e}^{|\rho^n|\zeta} < 1$. Let $g(z) := \mathrm{e}^{\rho^n(1-\delta)z} + \mathrm{e}^{\rho^n(\delta-1)z}$ and $m = d := \mathrm{e}^{|\rho^n|\zeta}$. Then condition (2.8) holds. Condition (2.9) holds trivially since c is constant. It remains to verify condition (2.10). Since $\xi_1 + \xi_2 \leq \zeta$, we have[16]

$$\begin{aligned}
\int g(z') f(z'|z)\,\mathrm{d}z' &= \mathrm{e}^{\rho^{n+1}(1-\delta)z}\, \mathrm{e}^{\rho^n \xi_1 + \rho^{2n}\xi_2} + \mathrm{e}^{\rho^{n+1}(\delta-1)z}\, \mathrm{e}^{-\rho^n \xi_1 + \rho^{2n}\xi_2} \\
&\leq \left(\mathrm{e}^{\rho^{n+1}(1-\delta)z} + \mathrm{e}^{\rho^{n+1}(\delta-1)z} \right) \mathrm{e}^{|\rho^n|\zeta} \\
&\leq \left(\mathrm{e}^{\rho^{n}(1-\delta)z} + \mathrm{e}^{\rho^{n}(\delta-1)z} + 1 \right) \mathrm{e}^{|\rho^n|\zeta} = mg(z) + d.
\end{aligned} \tag{5.B.2}$$

  Since $\beta m = \beta \mathrm{e}^{|\rho^n|\zeta} < 1$, g satisfies the geometric drift condition (2.10), and assumption 2.3.1 is verified.

- If $\rho \in [-1,1]$ and $\beta \mathrm{e}^{\zeta} < 1$, by (5.B.1)–(5.B.2) one can show that assumption 2.3.1 holds with $n := 0$, $g(z) := \mathrm{e}^{(1-\delta)z} + \mathrm{e}^{(\delta-1)z}$ and $m := d := \mathrm{e}^{\zeta}$.

[16] To obtain the second inequality of (5.B.2), note that either $\rho^{n+1}(1-\delta)z \leq 0$ or $\rho^{n+1}(\delta-1)z \leq 0$. Assume without loss of generality that the former holds, then $\mathrm{e}^{\rho^{n+1}(1-\delta)z} \leq 1$ and $0 \leq \rho^{n+1}(\delta - 1)z \leq [\rho^n(\delta-1)z] \vee [\rho^n(1-\delta)z]$. The latter implies that $\mathrm{e}^{\rho^{n+1}(\delta-1)z} \leq \mathrm{e}^{\rho^n(1-\delta)z} + \mathrm{e}^{\rho^n(\delta-1)z}$. Combine this with $\mathrm{e}^{\rho^{n+1}(1-\delta)z} \leq 1$ yields the second inequality of (5.B.2).

- If $\rho \in [0,1]$ and $\beta e^{\xi} < 1$, by (5.2.5)–(5.2.6) one can show that assumption 2.3.1 holds with $n := 0$, $g(z) := e^{(1-\delta)z}$ and $m = d := e^{\xi}$, where $\xi := \xi_1 + \xi_2 \leq \zeta$.

The latter two scenarios show that we can treat nonstationary state process at the cost of some additional restrictions on parameter values.

Case II: $\delta = 1$. In this case, we assume further that $\beta|\rho| < 1$.

The terminal reward is $r(z) = z/(1-\beta)$. Let $n := 0$, $g(z) := |z|$, $m := |\rho|$ and $d := \sigma + |b|$. Since $\{\varepsilon_t\}_{t\geq 0} \overset{\text{IID}}{\sim} N(0, \sigma^2)$, by Jensen's inequality,

$$\int g(z')f(z'|z)\,\mathrm{d}z' = \mathbb{E}_z|Z_1| \leq |\rho||z| + |b| + \mathbb{E}\,|\varepsilon_1|$$
$$\leq |\rho||z| + |b| + \sqrt{\mathbb{E}\,(\varepsilon_1^2)} = |\rho||z| + |b| + \sigma = mg(z) + d.$$

Since $\beta m = \beta|\rho| < 1$, assumption 2.3.1 holds. This concludes the proof. (Notably, since $|\rho| \geq 1$ is not excluded, wages can be nonstationary provided that they do not grow too fast.) $\square$

Proof of example 5.2.2. It remains to prove that assumption 2.3.1 holds for $\delta = 1$. Let $n := 1$, $g(\mu, \gamma) := e^{-\mu+\gamma/2} + e^{\mu+\gamma/2}$, $m := 1$ and $d := 0$.

Since $|\ln a| \leq a + 1/a$ for all $a > 0$, we have: $|u(w')| \leq w' + w'^{-1}$, and then

$$\int |u(w')|f(w'|\mu,\gamma)\,\mathrm{d}w' \leq e^{\mu+(\gamma+\gamma_\varepsilon)/2} + e^{-\mu+(\gamma+\gamma_\varepsilon)/2} = e^{\gamma_\varepsilon} g(\mu,\gamma). \tag{5.B.3}$$

Similarly as in the case of $\delta \geq 0$ and $\delta \neq 1$, one can show that

$$\mathbb{E}_{\mu,\gamma} g(\mu', \gamma') = \int g(\mu',\gamma')f(w'|\mu,\gamma)\,\mathrm{d}w' = g(\mu,\gamma).$$

Hence, assumption 2.3.1 holds in both cases. This concludes the proof. $\square$

Proof of example 5.2.3. The terminal reward is $r(z) = (e^z - K)^+$. Notice that

$$\int e^{z'} f(z'|z)\,\mathrm{d}z' = e^{\rho z}\, e^{b+\sigma^2/2}.$$

Induction shows that for all $t \in \mathbb{N}$,

$$\int e^{z'} P^t(z, \mathrm{d}z') = a_t\, e^{\rho^t z} \leq a_t \left(e^{\rho^t z} + e^{-\rho^t z}\right)$$

for some constant $a_t > 0$.

- If $\rho \in (-1,1)$, we can let $\xi := |b| + \sigma^2/2$ and fix $n \in \mathbb{N}_0$ such that $\mathrm{e}^{-\gamma+|\rho^n|\xi} < 1$. Let $g(z) := \mathrm{e}^{\rho^n z} + \mathrm{e}^{-\rho^n z}$ and $m = d := \mathrm{e}^{|\rho^n|\xi}$. The above inequality implies that condition (2.8) holds. Condition (2.9) holds trivially. Moreover,
$$\begin{aligned}\int g(z')f(z'|z)\,\mathrm{d}z' &= \int \mathrm{e}^{\rho^n z'} f(z'|z)\,\mathrm{d}z' + \int \mathrm{e}^{-\rho^n z'} f(z'|z)\,\mathrm{d}z' \\ &= \mathrm{e}^{\rho^{n+1}z}\,\mathrm{e}^{\rho^n b+\rho^{2n}\sigma^2/2} + \mathrm{e}^{-\rho^{n+1}z}\,\mathrm{e}^{-\rho^n b+\rho^{2n}\sigma^2/2} \\ &\leq \left(\mathrm{e}^{\rho^{n+1}z} + \mathrm{e}^{-\rho^{n+1}z}\right)\mathrm{e}^{|\rho^n b|+\rho^{2n}\sigma^2/2} \\ &\leq \left(\mathrm{e}^{\rho^n z} + \mathrm{e}^{-\rho^n z} + 1\right)\mathrm{e}^{|\rho^n b|+\rho^{2n}\sigma^2/2} \\ &\leq \left(\mathrm{e}^{\rho^n z} + \mathrm{e}^{-\rho^n z} + 1\right)\mathrm{e}^{|\rho^n|\xi} = mg(z) + d.\end{aligned} \tag{5.B.4}$$
Hence, condition (2.10) holds. Assumption 2.3.1 is true for $\rho \in (-1,1)$.

- If $\rho \in [-1,1]$ and $\mathrm{e}^{-\gamma+\xi} < 1$, then similar analysis shows that assumption 2.3.1 holds with $n = 0$, $g(z) := \mathrm{e}^{z} + \mathrm{e}^{-z}$ and $m = d := \mathrm{e}^{\xi}$.

Hence, assumption 2.3.1 holds in example 5.2.3. □

Proof of example 5.2.4. Notice that the density kernel corresponding to the transition probability F is $f(z'|z) = \theta\,\mathrm{e}^{-\theta(z'-z)}$ for $z' \geq z$.

Let $n := 0$, $g(z) := |z|$, $m := 1$ and $d := 1/\theta$. Obviously, condition (2.8) holds since $|r(z)| \vee |c(z)| \leq |z| + c_0$ for all $z \in \mathsf{Z}$. Condition (2.9) holds trivially. Moreover, by the elementary properties of the exponential distribution,
$$\int g(z')P(z,\mathrm{d}z') = \int |z'|f(z'|z)\,\mathrm{d}z' = \int_{[z,\infty)} z'\theta\,\mathrm{e}^{-\theta(z'-z)}\,\mathrm{d}z' = z + \frac{1}{\theta}.$$
Hence, condition (2.10) holds, and assumption 2.3.1 is verified. □

Proof of example 5.2.5. The reward functions are $r(z) = c(z) = G\mathrm{e}^{\frac{1}{1-\alpha}z} - c_f$. Note that
$$\int \mathrm{e}^{\frac{1}{1-\alpha}z'} f(z'|z)\,\mathrm{d}z' = \mathrm{e}^{\frac{\rho}{1-\alpha}z}\,\mathrm{e}^{\frac{b}{1-\alpha}+\frac{\sigma^2}{2(1-\alpha)^2}}.$$
Induction shows that for all $t \in \mathbb{N}$,
$$\int \mathrm{e}^{\frac{1}{1-\alpha}z'} P^t(z,\mathrm{d}z') = a_t\,\mathrm{e}^{\frac{\rho^t}{1-\alpha}z} \tag{5.B.5}$$
for some constant $a_t > 0$. Let $\xi := \frac{b}{1-\alpha} + \frac{\sigma^2}{2(1-\alpha)^2}$ and $\zeta := \frac{|b|}{1-\alpha} + \frac{\sigma^2}{2(1-\alpha)^2}$.

- If $\rho \in [0,1)$, then we can choose $n \in \mathbb{N}_0$ such that $\beta\mathrm{e}^{\rho^n\xi} < 1$. Let $g(z) := \mathrm{e}^{\frac{\rho^n}{1-\alpha}z}$ and $m = d := \mathrm{e}^{\rho^n\xi}$. Then conditions (2.8)–(2.9) hold. Moreover, condition (2.10) and thus assumption 2.3.1 hold since
$$\begin{aligned}\int g(z')f(z'|z)\,\mathrm{d}z' &= \mathrm{e}^{\frac{\rho^{n+1}}{1-\alpha}z}\,\mathrm{e}^{\frac{\rho^n b}{1-\alpha}+\frac{\rho^{2n}\sigma^2}{2(1-\alpha)^2}} \leq \left(\mathrm{e}^{\frac{\rho^n}{1-\alpha}z} + 1\right)\mathrm{e}^{\frac{\rho^n b}{1-\alpha}+\frac{\rho^{2n}\sigma^2}{2(1-\alpha)^2}} \\ &\leq [g(z)+1]\,\mathrm{e}^{\rho^n\xi} = mg(z) + d.\end{aligned} \tag{5.B.6}$$

- Indeed, if $\rho \in [0,1]$ and $\beta e^{\xi} < 1$, then similar analysis shows that assumption 2.3.1 holds for $n = 0$, $g(z) := e^{\frac{1}{1-\alpha}z}$ and $m = d := \beta e^{\xi}$.

- If $\rho \in (-1,1)$, then we can choose $n \in \mathbb{N}_0$ such that $\beta e^{|\rho^n|\zeta} < 1$. Let $g(z) := e^{\frac{\rho^n}{1-\alpha}z} + e^{\frac{\rho^n}{\alpha-1}z}$ and $m = d := e^{|\rho^n|\zeta}$. Then conditions (2.8)–(2.9) hold. Moreover, condition (2.10) and thus assumption 2.3.1 hold since

$$\begin{aligned}
\int g(z')f(z'|z)\,\mathrm{d}z' &\leq e^{\frac{\rho^{n+1}}{1-\alpha}z}\, e^{\frac{\rho^n b}{1-\alpha}+\frac{\rho^{2n}\sigma^2}{2(1-\alpha)^2}} + e^{\frac{\rho^{n+1}}{\alpha-1}z}\, e^{\frac{\rho^n b}{\alpha-1}+\frac{\rho^{2n}\sigma^2}{2(1-\alpha)^2}} \\
&\leq \left(e^{\frac{\rho^{n+1}}{1-\alpha}z} + e^{\frac{\rho^{n+1}}{\alpha-1}z}\right) e^{\frac{|\rho^n b|}{1-\alpha}+\frac{\rho^{2n}\sigma^2}{2(1-\alpha)^2}} \\
&\leq \left(e^{\frac{\rho^n}{1-\alpha}z} + e^{\frac{\rho^n}{\alpha-1}z} + 1\right) e^{|\rho^n|\zeta} = mg(z) + d.
\end{aligned}$$

- Indeed, if $\rho \in [-1,1]$ and $\beta e^{\zeta} < 1$, then similar analysis shows that assumption 2.3.1 holds for $n = 0$, $g(z) := e^{\frac{1}{1-\alpha}z} + e^{\frac{1}{\alpha-1}z}$ and $m = d := \beta e^{\zeta}$.

Hence, assumption 2.3.1 holds in example 5.2.5. □

5.B.2 Proof of Section 5.3 Results.

Proof of corollary 5.3.1. Let $b_\ell c\mathsf{Z}$ be the set of continuous functions in $b_\ell\mathsf{Z}$. Since ℓ is continuous by assumption, it is easy to show that $b_\ell c\mathsf{Z}$ is a closed subset of $b_\ell\mathsf{Z}$. To verify the continuity of ψ^*, it suffices to show that $Q(b_\ell c\mathsf{Z}) \subset b_\ell c\mathsf{Z}$ (see, e.g., Stokey et al. (1989), corollary 1 of theorem 3.2).

For all $\psi \in b_\ell c\mathsf{Z}$, there exists $G \in \mathbb{R}_+$ such that $|\psi| \leq G\ell$, so we have:

$$|\max\{r(z'), \psi(z')\} f(z'|z)| \leq [|r(z')| + G\ell(z')] f(z'|z).$$

Based on the given assumptions, $z \mapsto [|r(z')| + G\ell(z')]f(z'|z)$ is nonnegative and continuous for all $z' \in \mathsf{Z}$, and $z \mapsto \int [|r(z')| + G\ell(z')]f(z'|z)dz'$ is continuous. By lemma 5.A.1, $z \mapsto \int \max\{r(z'), \psi(z')\} f(z'|z)dz'$ is continuous. Combined with the fact that c is continuous, we know that $Q\psi \in b_\ell c\mathsf{Z}$. So ψ^* is continuous. □

Proof of example 5.3.3. For primitive set ups, recall option pricing model given by example 5.2.3.

- P admits a density representation $f(z'|z) = N(\rho z + b, \sigma^2)$ such that $z \mapsto f(z'|z)$ is continuous in z for all $z' \in \mathsf{Z}$.

- Since $|r(z)| \leq e^z + K$ and $z \mapsto \int e^{z'} f(z'|z) \, dz'$ is continuous, lemma 5.A.1 shows that $z \mapsto \mathbb{E}_z|r(Z_1)|$ is continuous. Also, $\mathbb{E}_z|r(Z_1)| \leq a_1 e^{\rho z} + K$ for some constant $a_1 > 0$. To apply induction, suppose that $z \mapsto \mathbb{E}_z|r(Z_{t-1})|$ is continuous and that

$$\mathbb{E}_z|r(Z_{t-1})| \leq a_{t-1} e^{\rho^{t-1} z} + K$$

for some constant $a_{t-1} > 0$. Since $z \mapsto \int a_{t-1} e^{\rho^{t-1} z} f(z'|z) \, dz' = a_t e^{\rho^t z}$ for some constant $a_t > 0$, which is continuous, applying lemma 5.A.1 again yields: $z \mapsto \mathbb{E}_z|r(Z_t)| = \mathbb{E}_z\left(\mathbb{E}_{Z_1}|r(Z_{t-1})|\right)$ is continuous and $\mathbb{E}_z|r(Z_t)| \leq a_t e^{\rho^t z} + K$. Hence, $z \mapsto \mathbb{E}_z|r(Z_t)|$ is continuous for all $t \in \mathbb{N}$.

- In addition, g is continuous, and (5.B.4) implies that $z \mapsto \mathbb{E}_z g(Z_1)$ is continuous. Hence, ℓ and $z \mapsto \int \ell(z') f(z'|z) \, dz'$ are continuous.

- The reward function c is continuous.

Based on corollary 5.3.1, ψ^* is continuous. This concludes the proof. □

Proof of example 5.3.5. For primitive set ups, recall the firm exit model of example 5.2.5.

- P admits a density representation $f(z'|z) = N(\rho z + b, \sigma^2)$ such that $z \mapsto f(z'|z)$ is continuous in z for all $z' \in \mathsf{Z}$.

- From equation (5.B.5) we know that $z \mapsto \mathbb{E}_z|r(Z_t)|, \mathbb{E}_z|c(Z_t)|$ are continuous for all $t \in \mathbb{N}$.

- Moreover, g is continuous, and (5.B.6) implies that $z \mapsto \mathbb{E}_z g(Z_1)$ is continuous. Thus, ℓ and $z \mapsto \int \ell(z') f(z'|z) \, dz'$ are continuous.

- The reward function c is continuous.

Based on corollary 5.3.1, ψ^* is continuous, as was to be shown. □

Proof of proposition 5.3.1. Standard argument shows that $b_\ell i\mathsf{Z}$, the set of increasing functions in $b_\ell \mathsf{Z}$, is a closed subset. To show that ψ^* is increasing, it suffices to verify that $Q(b_\ell i\mathsf{Z}) \subset b_\ell i\mathsf{Z}$ (see, e.g., Stokey et al. (1989), corollary 1 of theorem 3.2). The assumptions of the proposition guarantee that this is the case. □

In the next, we prove differentiability and smoothness. We define

$$\mu(z) := \int \max\{r(z'), \psi^*(z')\} f(z'|z)\, \mathrm{d}z'$$

$$\text{and} \quad \mu_i(z) := \int \max\{r(z'), \psi^*(z')\} D_i f(z'|z)\, \mathrm{d}z'.$$

Lemma 5.B.1. *Suppose assumption 2.3.1 holds, and, for $i = 1, \cdots, m$*

(1) P has a density representation f such that $D_i f(z'|z)$ exists, $\forall (z, z') \in \mathrm{int}(\mathsf{Z}) \times \mathsf{Z}$.

(2) For all $z_0 \in \mathrm{int}(\mathsf{Z})$, there exists $\delta > 0$, such that for $k \in \{r, \ell\}$,

$$\int |k(z')| \sup_{z^i \in \bar{B}_\delta(z_0^i)} \left| D_i f(z'|z) \right| \mathrm{d}z' < \infty \qquad (z^{-i} = z_0^{-i}).$$

Then: $D_i \mu(z) = \mu_i(z)$ for all $z \in \mathrm{int}(\mathsf{Z})$ and $i = 1, \cdots, m$.

Proof of lemma 5.B.1. For all $z_0 \in \mathrm{int}(\mathsf{Z})$, let $\{z_n\}$ be an arbitrary sequence of $\mathrm{int}(\mathsf{Z})$ such that $z_n^i \to z_0^i$, $z_n^i \neq z_0^i$ and $z_n^{-i} = z_0^{-i}$ for all $n \in \mathbb{N}$. For the $\delta > 0$ given by (2), there exists $N \in \mathbb{N}$ such that $z_n^i \in \bar{B}_\delta(z_0^i)$ for all $n \geq N$. Holding $z^{-i} = z_0^{-i}$, by the mean value theorem, there exists $\xi^i(z', z_n, z_0) \in \bar{B}_\delta(z_0^i)$ such that

$$\begin{aligned} \left| \triangle^i(z', z_n, z_0) \right| &:= \left| \frac{f(z'|z_n) - f(z'|z_0)}{z_n^i - z_0^i} \right| \\ &= \left| D_i f(z'|z)_{z^i = \xi^i(z', z_n, z_0)} \right| \leq \sup_{z^i \in \bar{B}_\delta(z_0^i)} \left| D_i f(z'|z) \right|. \end{aligned}$$

Since in addition $|\psi^*| \leq G\ell$ for some $G \in \mathbb{R}_+$, we have: for all $n \geq N$,

(a) $\left|\max\{r(z'), \psi^*(z')\} \triangle^i(z', z_n, z_0)\right| \leq (|r(z')| + G\ell(z')) \sup_{z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)|$,

(b) $\int (|r(z')| + G\ell(z')) \sup_{z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)|\, dz' < \infty$, and

(c) $\max\{r(z'), \psi^*(z')\} \triangle^i(z', z_n, z_0) \to \max\{r(z'), \psi^*(z')\} D_i f(z'|z_0)$ as $n \to \infty$,

where (b) follows from condition (2). By the dominated convergence theorem,

$$\begin{aligned} \frac{\mu(z_n) - \mu(z_0)}{z_n^i - z_0^i} &= \int \max\{r(z'), \psi^*(z')\} \triangle^i(z', z_n, z_0)\, \mathrm{d}z' \\ &\to \int \max\{r(z'), \psi^*(z')\} D_i f(z'|z_0)\, \mathrm{d}z' = \mu_i(z_0). \end{aligned}$$

Hence, $D_i \mu(z_0) = \mu_i(z_0)$, and the claim of the lemma is verified. $\square$

Proof of proposition 5.3.2. Fix $z_0 \in \text{int}(\mathsf{Z})$. By assumption 5.3.3 (2)–(3), there exist $\delta > 0$ and a compact subset $A \subset \mathsf{Z}$ such that $z' \notin A$ implies $z_i^*(z', z_0^{-i}) \notin B_\delta(z_0^i)$, hence, for $z^{-i} = z_0^{-i}$,

$$\sup_{z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)| = |D_i f(z'|z)|_{z^i = z_0^i + \delta} \vee |D_i f(z'|z)|_{z^i = z_0^i - \delta} =: h^\delta(z', z_0).$$

By assumption 5.3.3-(2), there exists $G \in \mathbb{R}_+$, such that for $z^{-i} = z_0^{-i}$,

$$\begin{aligned}\sup_{z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)| &\leq \sup_{z' \in A, z^i \in \bar{B}_\delta(z_0^i)} |D_i f(z'|z)| \cdot \mathbb{1}(z' \in A) + h^\delta(z', z_0) \cdot \mathbb{1}(z' \in A^c) \\ &\leq G \cdot \mathbb{1}(z' \in A) + \left(|D_i f(z'|z)|_{z^i = z_0^i + \delta} + |D_i f(z'|z)|_{z^i = z_0^i - \delta} \right) \cdot \mathbb{1}(z' \in A^c).\end{aligned}$$

Assumption 5.3.4 then shows that condition (2) of lemma 5.B.1 holds. By assumption 5.3.2 and lemma 5.B.1, $D_i \psi^*(z) = D_i c(z) + \mu_i(z)$ for all $z \in \text{int}(\mathsf{Z})$, as was to be shown. This concludes the proof. □

Proof of proposition 5.3.3. Since assumption 5.3.5 implies assumptions 5.3.2 and 5.3.4, by proposition 5.3.2, $D_i \psi^*(z) = D_i c(z) + \mu_i(z)$ on $\text{int}(\mathsf{Z})$. Since $D_i c(z)$ is continuous by assumption 5.3.5-(1), to show that ψ^* is continuously differentiable, it remains to verify: $z \mapsto \mu_i(z)$ is continuous on $\text{int}(\mathsf{Z})$. Since $|\psi^*| \leq G\ell$ for some $G \in \mathbb{R}_+$,

$$\left|\max\{r(z'), \psi^*(z')\} D_i f(z'|z)\right| \leq (|r(z')| + G\ell(z')) |D_i f(z'|z)|, \ \forall z', z \in \mathsf{Z}. \quad (5.B.7)$$

By assumptions 5.3.3-(2) and 5.3.5-(2), both sides of (5.B.7) are continuous in z, and $z \mapsto \int [|r(z')| + G\ell(z')] |D_i f(z'|z)| \, \mathrm{d}z'$ is continuous. Then $z \mapsto \mu_i(z)$ is continuous by lemma 5.A.1. This concludes the proof. □

Proof of example 5.3.9. For primitive set ups, recall the option pricing problem given by example 5.2.3. For all $a \in \mathbb{R}$, let

$$h(z, a) := \mathrm{e}^{a(\rho z + b) + a^2 \sigma^2 / 2} / \sqrt{2\pi\sigma^2}. \quad (5.B.8)$$

- Assumption 2.3.1 holds, as shown in example 5.2.3.
- The solutions to $\frac{\partial^2 f(z'|z)}{\partial z^2} = 0$ are $z^*(z') = \frac{z' - b \pm \sigma}{\rho}$. This implies that assumption 5.3.3 holds.
- r is continuous, and we have shown in the proof of example 5.3.3 that ℓ is continuous. Note that

$$\mathrm{e}^{az'} \left| \frac{\partial f(z'|z)}{\partial z} \right| \leq h(z)\, \mathrm{e}^{-[z' - (\rho z + b + a\sigma^2)]^2 / (2\sigma^2)} \frac{|\rho z'| + |\rho(\rho z + b)|}{\sigma^2}. \quad (5.B.9)$$

Both sides of (5.B.9) are continuous in z and the integration of the right side term (w.r.t. z') is continuous. Lemma 5.A.1 implies that $z \mapsto \int e^{az'} \left|\frac{\partial f(z'|z)}{\partial z}\right| dz'$ is continuous. Hence, $z \mapsto \int g(z') \left|\frac{\partial f(z'|z)}{\partial z}\right| dz'$ is continuous (by the definition of g). Moreover, the proof of example 5.3.3 shows that

$$\mathbb{E}_{z'}|r(Z_t)| \left|\frac{\partial f(z'|z)}{\partial z}\right| \leq \left(a_t e^{\rho^t z'} + K\right) \left|\frac{\partial f(z'|z)}{\partial z}\right|.$$

Lemma 5.A.1 shows that $z \mapsto \int \mathbb{E}_{z'}|r(Z_t)| \left|\frac{\partial f(z'|z)}{\partial z}\right| dz'$ is continuous for all $t \in \mathbb{N}$. Hence, assumption 5.3.4 and condition (2) of assumption 5.3.5 hold.

- Since $c \equiv 0$ is obviously continuously differentiable, condition (1) of assumption 5.3.5 holds.

Based on proposition 5.3.3, ψ^* is continuously differentiable. □

Proof of example 5.3.10. For primitive set ups, recall the firm exit model of example 5.2.5. For all $(z, a) \in Z \times \mathbb{R}$, let $h(z, a)$ be defined as in (5.B.8).

- Assumption 2.3.1 holds, as was shown in example 5.2.5.
- The solutions to $\frac{\partial^2 f(z'|z)}{\partial z^2} = 0$ are $z^*(z') = \frac{z' - b \pm \sigma}{\rho}$. This implies that assumption 5.3.3 holds.
- r is continuous, and we have shown in the proof of example 5.3.5 that ℓ is continuous. Recall (5.B.9) in the proof of example 5.3.9. Both sides of (5.B.9) are continuous in z and the integration of the right side term (w.r.t. z') is continuous. Lemma 5.A.1 implies that $z \mapsto \int e^{az'} \left|\frac{\partial f(z'|z)}{\partial z}\right| dz'$ is continuous. Hence, $z \mapsto \int g(z') \left|\frac{\partial f(z'|z)}{\partial z}\right| dz'$ is continuous (by the definition of g). Moreover, (5.B.5) implies that

$$\mathbb{E}_z|r(Z_t)| = \mathbb{E}_z|c(Z_t)| = G a_t e^{\rho^t z/(1-\alpha)} - c_f.$$

Hence, $z \mapsto \int \mathbb{E}_{z'}|r(Z_t)| \left|\frac{\partial f(z'|z)}{\partial z}\right| dz'$ and $z \mapsto \int \mathbb{E}_{z'}|c(Z_t)| \left|\frac{\partial f(z'|z)}{\partial z}\right| dz'$ are continuous for all $t \in \mathbb{N}$. Hence, assumption 5.3.4 and condition (2) of assumption 5.3.5 hold.

- Since c is continuously differentiable, condition (1) of assumption 5.3.5 holds.

Based on proposition 5.3.3, ψ^* is continuously differentiable. □

Proof of proposition 5.3.4. Consider the Banach space $(b_\ell(\mathsf{Z}\times\Theta), \|\cdot\|_\ell)$ and the continuation value operator $Q : b_\ell(\mathsf{Z}\times\Theta) \to b_\ell(\mathsf{Z}\times\Theta)$ defined by

$$Q\psi_\theta(z) = c_\theta(z) + \beta \int \max\{r_\theta(z'), \psi_\theta(z')\} P_\theta(z, \mathrm{d}z').$$

Based on theorem 5.2.1, $(z,\theta) \mapsto \psi^*_\theta(z)$ is the unique fixed point of Q in $b_\ell(\mathsf{Z}\times\Theta)$. Let $b_\ell c(\mathsf{Z}\times\Theta)$ be the set of continuous functions in $b_\ell(\mathsf{Z}\times\Theta)$. Since ℓ is continuous, $b_\ell c(\mathsf{Z}\times\Theta)$ is a closed subset. To show the continuity of $(z,\theta) \mapsto \psi^*_\theta(z)$, it suffices to show that $Q : b_\ell c(\mathsf{Z}\times\Theta) \to b_\ell c(\mathsf{Z}\times\Theta)$.

For all candidate $(z,\theta) \mapsto \psi_\theta(z)$ in $b_\ell c(\mathsf{Z}\times\Theta)$, there exists $G \in \mathbb{R}_+$ such that $|\psi_\theta(z)| \leq G\ell(z,\theta)$ for all $(z,\theta) \in \mathsf{Z}\times\Theta$, so $|r_\theta(z)| + G\ell(\theta,z) \pm \max\{r_\theta(z), \psi_\theta(z)\} \geq 0$. Based on assumptions 5.3.7 and Feinberg et al. (2014) (see theorem 1.1), for all $(z_m, \theta_m) \to (z,\theta)$, we have

$$\begin{aligned}
&\int \left[|r_\theta(z')| + G\ell(\theta,z') \pm \max\{r_\theta(z'), \psi_\theta(z')\}\right] P_\theta(z, \mathrm{d}z') \\
&\leq \liminf_{m\to\infty} \int \left[|r_{\theta_m}(z')| + G\ell(\theta_m,z') \pm \max\{r_{\theta_m}(z'), \psi_{\theta_m}(z')\}\right] P_{\theta_m}(z_m, \mathrm{d}z').
\end{aligned}$$

Since assumptions 5.3.7 implies that

$$\lim_{m\to\infty} \int \left[|r_{\theta_m}(z')| + G\ell(\theta_m,z')\right] P_{\theta_m}(z_m, \mathrm{d}z') = \int \left[|r_\theta(z')| + G\ell(\theta,z')\right] P_\theta(z, \mathrm{d}z'),$$

we have

$$\begin{aligned}
&\limsup_{m\to\infty} \int \max\{r_{\theta_m}(z'), \psi_{\theta_m}(z')\} P_{\theta_m}(z_m, \mathrm{d}z') \\
&\leq \int \max\{r_\theta(z'), \psi_\theta(z')\} P_\theta(z, \mathrm{d}z') \leq \liminf_{m\to\infty} \int \max\{r_{\theta_m}(z'), \psi_{\theta_m}(z')\} P_{\theta_m}(z_m, \mathrm{d}z'),
\end{aligned}$$

where we have used the elementary fact that for given sequences $(a_m)_{m\geq 0}$ and $(b_m)_{m\geq 0}$ of $\mathbb{R}$ with $\lim_{m\to\infty} a_m$ exists, we have: $\liminf_{m\to\infty}(a_m + b_m) = \lim_{m\to\infty} a_m + \liminf_{m\to\infty} b_m$. So we have

$$\lim_{m\to\infty} \int \max\{r_{\theta_m}(z'), \psi_{\theta_m}(z')\} P_{\theta_m}(z_m, \mathrm{d}z') = \int \max\{r_\theta(z'), \psi_\theta(z')\} P_\theta(z, \mathrm{d}z').$$

Since $(z,\theta) \mapsto c_\theta(z)$ is continuous by assumption, $(z,\theta) \mapsto Q\psi_\theta(z)$ is continuous. This implies that Q is a self-map on $b_\ell c(\mathsf{Z}\times\Theta)$. Hence, $(z,\theta) \mapsto \psi^*_\theta(z)$ is continuous. This concludes the proof. □

5.B.3 Proof of Section 5.4 Results

Proof of proposition 5.4.2. Define $F : \mathsf{X}\times\mathsf{Y}\times\Theta \to \mathbb{R}$ by $F(x,y,\theta) := r_\theta(x,y) - \psi^*_\theta(y)$. Without loss of generality, assume that $(x,y,\theta) \mapsto r_\theta(x,y)$ is strictly increasing in x, then F is strictly increasing in x and continuous. For all fixed

$(y_0, \theta_0) \in \mathsf{Y} \times \Theta$ and $\varepsilon > 0$, since F is strictly increasing in x and $F(\bar{x}_{\theta_0}(y_0), y_0, \theta_0) = 0$, we have

$$F(\bar{x}_{\theta_0}(y_0) + \varepsilon, y_0, \theta_0) > 0 \quad \text{and} \quad F(\bar{x}_{\theta_0}(y_0) - \varepsilon, y_0, \theta_0) < 0.$$

Since F is continuous with respect to (y, θ), there exists $\delta > 0$ such that for all $(y, \theta) \in B_\delta((y_0, \theta_0)) := \{(y, \theta) \in \mathsf{Y} \times \Theta : \|(y, \theta) - (y_0, \theta_0)\| < \delta\}$, we have

$$F(\bar{x}_{\theta_0}(y_0) + \varepsilon, y, \theta) > 0 \quad \text{and} \quad F(\bar{x}_{\theta_0}(y_0) - \varepsilon, y, \theta) < 0.$$

Since $F(\bar{x}_\theta(y), y, \theta) = 0$ and F is strictly increasing in x, we have

$$\bar{x}_\theta(y) \in \left(\bar{x}_{\theta_0}(y_0) - \varepsilon, \bar{x}_{\theta_0}(y_0) + \varepsilon\right), \text{ i.e., } |\bar{x}_\theta(y) - \bar{x}_{\theta_0}(y_0)| < \varepsilon.$$

Hence, $(y, \theta) \mapsto \bar{x}_\theta(y)$ is continuous, as was to be shown. □

5.B.4 Proof of Section 5.5 Results

Proof of proposition 5.5.1. Regarding claims (1)–(2), the terminal reward satisfies

$$\left|r(f', \mu', \gamma')\right| \leq 1/a + \left(\mathrm{e}^{a^2\gamma_x/2}/a\right) \cdot \mathrm{e}^{-a\mu' + a^2\gamma'/2} + |f'|. \tag{5.B.10}$$

Using (5.5.1), we can show that (recall the first claim of footnote 3)

$$\int \mathrm{e}^{-a\mu' + a^2\gamma'/2} P(z, \mathrm{d}z') = \int \mathrm{e}^{-a\mu' + a^2\gamma'/2} l(y'|\mu, \gamma)\, \mathrm{d}y' = \mathrm{e}^{-a\mu + a^2\gamma/2}. \tag{5.B.11}$$

Let $\mu_f := \int |f| h(f)\, \mathrm{d}f$. Combining (5.B.10)–(5.B.11) yields

$$\int \left|r(f', \mu', \gamma')\right| P(z, \mathrm{d}z') \leq (1/a + \mu_f) + \left(\mathrm{e}^{a^2\gamma_x/2}/a\right) \cdot \mathrm{e}^{-a\mu + a^2\gamma/2}. \tag{5.B.12}$$

By (5.B.11)–(5.B.12), assumption 2.3.1 holds for $n := 1$, $g(\mu, \gamma) := \mathrm{e}^{-a\mu + a^2\gamma/2}$, $m := 1$ and $d := 0$. Moreover, the intermediate value theorem shows that assumption 5.4.1 holds. By theorem 5.2.1 and (5.4.1) (section 5.4.1), claims (1)–(2) hold.

Regarding claim (3), P is Feller by (5.5.1) and lemma 5.A.1. By (5.5.1), both sides of (5.B.10) are continuous in (μ, γ). By (5.B.11), the conditional expectation of the right side of (5.B.10) is continuous in (μ, γ). Lemma 5.A.1 implies that $(\mu, \gamma) \mapsto \mathbb{E}_{\mu,\gamma}|r(Z_1)|$ is continuous. Since in addition g is continuous and $g(\mu, \gamma) = \mathbb{E}_{\mu,\gamma} g(\mu', \gamma')$ by (5.B.11), assumption 2.3.2 holds. Claim (3) then follows from propositions 2.3.4 and 5.4.1.

Since l is stochastically increasing in μ for $\rho \geq 0$, claim (4) holds by proposition 5.3.1. This concludes the proof. □

Proof of proposition 5.5.2. For claims (1)–(2), since $w = \eta + \theta\xi$ and $|\ln w| \leq 1/w + w$, we have

$$\int |\ln w'| P(z, \mathrm{d}z') \leq \int (1/\eta' + \eta') v(\eta') \, \mathrm{d}\eta' + \int \xi' h(\xi') \, \mathrm{d}\xi' \cdot \int \theta' f(\theta'|\theta) \, \mathrm{d}\theta'$$
$$= \mu_\eta^- + \mu_\eta^+ + \mu_\xi \cdot e^{\gamma_u/2}\theta^\rho, \tag{5.B.13}$$

where $\mu_\eta^+ := \int \eta v(\eta) \, \mathrm{d}\eta$, $\mu_\eta^- := \int \eta^{-1} v(\eta) \, \mathrm{d}\eta$ and $\mu_\xi := \int \xi h(\xi) \, \mathrm{d}\xi$. Hence,[17]

$$\int |\ln w'| P^t(z, \mathrm{d}z') \leq a_1^{(t)} \, \theta^{\rho^t} + a_2^{(t)} \leq a_1^{(t)} \left(\theta^{\rho^t} + \theta^{-\rho^t} \right) + a_2^{(t)} \tag{5.B.14}$$

for some $a_1^{(t)}, a_2^{(t)} > 0$ (do not depend on θ) and all $t \in \mathbb{N}$. Let n, g, m and d be defined as in section 5.5.3. Then since $\theta^{\rho^{n+1}} + \theta^{-\rho^{n+1}} \leq \theta^{\rho^n} + \theta^{-\rho^n} + 1$ for $\theta > 0$ and $\rho \in [-1, 1]$, we have

$$\int g(\theta') f(\theta'|\theta) \, \mathrm{d}\theta' = \left(\theta^{\rho^{n+1}} + \theta^{-\rho^{n+1}} \right) e^{\rho^{2n}\gamma_u/2} \leq m g(\theta) + d. \tag{5.B.15}$$

Hence, assumption 2.3.1 holds. Assumption 5.4.1 holds by the intermediate value theorem. Claims (1)–(2) then follow from theorem 5.2.1 and (5.4.1) (section 5.4.1).

Regarding claim (3), it is straightforward to show that $\theta \mapsto f(\theta'|\theta)$ is twice differentiable for all θ', that $(\theta, \theta') \mapsto \partial f(\theta'|\theta)/\partial\theta$ is continuous, and that

$$\partial^2 f(\theta'|\theta)/\partial\theta^2 = 0 \;\text{ has two solutions: }\; \theta = \theta^*(\theta') = \tilde{a}_i \mathrm{e}^{\ln\theta'/\rho}, \; i = 1, 2$$

where $\tilde{a}_1, \tilde{a}_2 > 0$ are constants. If $\rho > 0 \, (< 0)$, then $\theta^*(\theta') \to \infty \, (0)$ as $\theta' \to \infty$, and $\theta^*(\theta') \to 0 \, (\infty)$ as $\theta' \to 0$. Hence, assumption 5.3.3 holds. Based on (5.B.13)–(5.B.15) and lemma 5.A.1, assumption 5.3.5 holds. Claim (3) then holds by propositions 5.3.3 and 5.4.3.

As $f(\theta'|\theta)$ is stochastically increasing ($\rho > 0$), claim (4) holds by propositions 5.3.1 and 5.4.4. This concludes the proof. □

Proof of proposition 5.5.3. Proof of claim (1). Since

$$w'^{1-\delta} = (\eta' + \theta'\xi')^{1-\delta} \leq 2 \left(\eta'^{1-\delta} + \theta'^{1-\delta}\xi'^{1-\delta} \right), \tag{5.B.16}$$

we have

$$\int w'^{1-\delta} P(z, \mathrm{d}z') \leq 2 \int \eta'^{1-\delta} v(\eta') \, \mathrm{d}\eta' + 2 \int \xi'^{1-\delta} h(\xi') \, \mathrm{d}\xi' \int \theta'^{1-\delta} f(\theta'|\theta) \, \mathrm{d}\theta'$$
$$= 2\mathrm{e}^{(1-\delta)\mu_\eta + (1-\delta)^2\gamma_\eta/2} + 2\mathrm{e}^{(1-\delta)^2(\gamma_\xi + \gamma_u)/2} \, \theta^{(1-\delta)\rho}. \tag{5.B.17}$$

[17] Recall that if $X \sim LN(\mu, \sigma^2)$, then $\mathbb{E}\, X^s = \mathrm{e}^{s\mu + s^2\sigma^2/2}$ for all $s \in \mathbb{R}$.

Induction shows that

$$\int w'^{1-\delta} P^t(z, \mathrm{d}z') \leq a_1^{(t)} + a_2^{(t)} \theta^{(1-\delta)\rho^t} \leq a_1^{(t)} + a_2^{(t)} \left(\theta^{(1-\delta)\rho^t} + \theta^{-(1-\delta)\rho^t} \right) \tag{5.B.18}$$

for some $a_1^{(t)}, a_2^{(t)} > 0$ and all $t \in \mathbb{N}$. Define g as in the assumption, then

$$\begin{aligned} \int g(\theta') f(\theta'|\theta)\,\mathrm{d}\theta' &= \left(\mathrm{e}^{(1-\delta)\rho^{n+1} \ln \theta} + \mathrm{e}^{-(1-\delta)\rho^{n+1} \ln \theta} \right) \mathrm{e}^{(1-\delta)^2 \rho^{2n} \gamma_u /2} \\ &\leq \left(\mathrm{e}^{(1-\delta)\rho^{n} \ln \theta} + \mathrm{e}^{-(1-\delta)\rho^{n} \ln \theta} + 1 \right) \mathrm{e}^{(1-\delta)^2 \rho^{2n} \gamma_u /2} \\ &= [g(\theta) + 1]\, \mathrm{e}^{(1-\delta)^2 \rho^{2n} \gamma_u /2} \leq m g(\theta) + d. \end{aligned} \tag{5.B.19}$$

Hence, assumption 2.3.1 holds. Claim (1) then follows from theorem 5.2.1.

Proof of claim (2). Assumption 5.4.1 holds by the intermediate value theorem. Claim (2) then follows from theorem 5.2.1, assumption 5.4.1 and (5.4.1).

Proof of claim (3). Note that the stochastic kernel P has a density representation in the sense that for all $z \in \mathsf{Z}$ and $B \in \mathscr{Z}$,

$$P(z, B) = \int \mathbb{1} \left\{ (\eta' + \xi'\theta', \theta') \in B \right\} v(\eta') h(\xi') f(\theta'|\theta)\, \mathrm{d}(\eta', \xi', \theta').$$

Moreover, it is straightforward (though tedious) to show that $\theta \mapsto f(\theta'|\theta)$ is twice differentiable for all θ', that $(\theta, \theta') \mapsto \partial f(\theta'|\theta)/\partial\theta$ is continuous, and that

$$\partial^2 f(\theta'|\theta)/\partial\theta^2 = 0 \quad \text{if and only if} \quad \theta = \theta^*(\theta') = \tilde{a}_i\, \mathrm{e}^{\ln \theta'/\rho}, i = 1, 2 \tag{5.B.20}$$

where $\tilde{a}_1, \tilde{a}_2 = \mathrm{e}^{\frac{\gamma_u}{\rho}\left(-\frac{1}{2\rho} \pm \sqrt{\frac{1}{4\rho^2} + \frac{1}{\gamma_u}} \right)}$. If $\rho > 0$, then $\theta^*(\theta') \to \infty$ as $\theta' \to \infty$ and $\theta^*(\theta') \to 0$ as $\theta' \to 0$. If $\rho < 0$, then $\theta^*(\theta') \to 0$ as $\theta' \to \infty$ and $\theta^*(\theta') \to \infty$ as $\theta' \to 0$. Hence, assumption 5.3.3 holds. Based on (5.B.16)–(5.B.19) and lemma 5.A.1, we can show that assumption 5.3.5 holds. By proposition 5.3.3, ψ^* is continuously differentiable. Since assumption 5.4.1 holds and r is continuously differentiable, by proposition 5.4.3, $\bar{w}$ is continuously differentiable.

Proof of claim (4). Note that $c \equiv c_0$ is a constant, and that

$$r(w) = r(\eta + \xi\theta) = (\eta + \xi\theta)^{1-\delta} / [(1-\beta)(1-\delta)]$$

is increasing in θ, and, when $\rho > 0$, $f(\theta'|\theta)$ is stochastically increasing in θ. Hence, assumption 5.3.1 holds. By propositions 5.3.1 and 5.4.4, ψ^* and $\bar{w}$ are increasing in θ. □

In the next, we are going to proof the results of section 5.5.4. Based on our theory, it suffices to verify that there exist a measurable map $g(p, \mu, \gamma)$ and $n \in \mathbb{N}_0$, $a_1, \cdots, a_4, m, d \in \mathbb{R}_+$ such that $\beta m < 1$ and

- $\int |\pi(p',x')|P^n(z,\mathrm{d}z') \le a_1 g(p,\mu,\gamma) + a_2$,
- $\int |r(p',x')|P^n(z,\mathrm{d}z') \le a_3 g(p,\mu,\gamma) + a_4$, and
- $\int g(p',\mu',\gamma') l(p',x'|p,\mu,\gamma)\,\mathrm{d}(p',x') \le mg(p,\mu,\gamma) + d$.

We first provide a useful lemma.

Lemma 5.B.2. *For all $a \in \mathbb{R}$, we have*

$$\int \mathrm{e}^{a\mu' + a^2\gamma'/2} f(x'|\mu,\gamma)\,\mathrm{d}x' = \mathrm{e}^{a\mu + a^2\gamma/2}.$$

Proof. Notice that $\mu' = b_1\mu + b_2 \ln x'$, where $b_1 = \frac{\gamma_\varepsilon}{\gamma+\gamma_\varepsilon}$ and $b_2 = \frac{\gamma}{\gamma+\gamma_\varepsilon}$.

$$\begin{aligned}
&\int \mathrm{e}^{a\mu' + a^2\gamma'/2} f(x'|\mu,\gamma)\,\mathrm{d}x' \\
&= \mathrm{e}^{a^2\gamma'/2} \int \mathrm{e}^{ab_1\mu + ab_2 \ln x'} f(x'|\mu,\gamma)\,\mathrm{d}x' \le \mathrm{e}^{a^2\gamma'/2}\mathrm{e}^{ab_1\mu} \int x'^{ab_2} f(x'|\mu,\gamma)\,\mathrm{d}x' \\
&= \mathrm{e}^{a^2\gamma'/2}\,\mathrm{e}^{ab_1\mu}\,\mathrm{e}^{ab_2\mu + a^2b_2^2(\gamma+\gamma_\varepsilon)/2} = \mathrm{e}^{a\mu}\,\mathrm{e}^{a^2\gamma\gamma_\varepsilon/[2(\gamma+\gamma_\varepsilon)]}\,\mathrm{e}^{a^2\gamma^2/[2(\gamma+\gamma_\varepsilon)]} = \mathrm{e}^{a\mu + a^2\gamma/2}.
\end{aligned}$$

Hence, the claim holds, completing our proof. □

Proof of proposition 5.5.4. Regarding claim (1), note that we have

$$\begin{aligned}
&\int |\pi(p',x')| l(p',x'|p,\mu,\gamma)\,\mathrm{d}(p',x') = \frac{1}{4}\int p'^2 h(p'|p)\,\mathrm{d}p' \int x'^{-1} f(x'|\mu,\gamma)\,\mathrm{d}x' \\
&= \frac{1}{4}\,\mathrm{e}^{2(\rho\ln p + b) + 2\gamma_p}\,\mathrm{e}^{-\mu + (\gamma+\gamma_\varepsilon)/2} = p^{2\rho}\,\mathrm{e}^{-\mu+\gamma/2} \cdot \frac{1}{4}\,\mathrm{e}^{2b + 2\gamma_p + \gamma_\varepsilon/2}.
\end{aligned}$$

Induction then shows that for all $t \in \mathbb{N}$,

$$\int |\pi(p',x')| P^t(z,\mathrm{d}z') = a_t\, p^{2\rho^t} \mathrm{e}^{-\mu+\gamma/2} \le a_t \left(p^{2\rho^t} + p^{-2\rho^t} + \delta\right) \mathrm{e}^{-\mu+\gamma/2} \tag{5.B.21}$$

for some constant $a_t > 0$. Similar properties can be obtained for r. So g can chosen as in the set up and conditions (2.8)–(2.9) of assumption 2.3.1 hold. Moreover,

$$\begin{aligned}
&\int g(z') l(p',x'|p,\mu,\gamma)\,\mathrm{d}(p',x') \\
&= \int \left(p'^{2\rho^n} + p'^{-2\rho^n} + \delta\right) h(p'|p)\,\mathrm{d}p' \cdot \int \mathrm{e}^{-\mu' + \gamma'/2} f(x'|\mu,\gamma)\,\mathrm{d}x' \\
&= \left(p^{2\rho^{n+1}}\,\mathrm{e}^{2\rho^n b + 2\rho^{2n}\gamma_p} + p^{-2\rho^{n+1}}\,\mathrm{e}^{-2\rho^n b + 2\rho^{2n}\gamma_p} + \delta\right) \mathrm{e}^{-\mu+\gamma/2} \\
&\le \left[\mathrm{e}^{|\rho^n|\xi}\left(p^{2\rho^{n+1}} + p^{-2\rho^{n+1}}\right) + \delta\right] \mathrm{e}^{-\mu+\gamma/2} \\
&\le \left[\mathrm{e}^{|\rho^n|\xi}\left(p^{2\rho^n} + p^{-2\rho^n} + 1\right) + \delta\right] \mathrm{e}^{-\mu+\gamma/2} \\
&\le \mathrm{e}^{|\rho^n|\xi}\left(p^{2\rho^n} + p^{-2\rho^n} + \delta\right) \mathrm{e}^{-\mu+\gamma/2} = mg(p,\mu,\gamma) + d,
\end{aligned} \tag{5.B.22}$$

where to get the last inequality we have used the definition of δ. Hence, assumption 2.3.1 holds. Claim (1) then follows from theorem 5.2.1.

Regarding claim (2), by the dominated convergence theorem, we can show that P has the Feller property. The rewards π and r are continuous by assumption. Based on equation (5.B.21), $(p,\mu,\gamma) \mapsto \int |\pi(p',x')|P^t(z,\mathrm{d}z')$ is continuous for all $t \in \mathbb{N}$. Since $|r(p,x)| \leq h_1 p^2/x + h_2$ and $(p,\mu,\gamma) \mapsto \int (p'^2/x')P(z,\mathrm{d}z')$ is continuous, from lemma 5.A.1 we know that $(p,\mu,\gamma) \mapsto \int |r(p',x')|P(z,\mathrm{d}z')$ is continuous. Also,

$$\int |r(p',x')|P(z,\mathrm{d}z') \leq h_1^{(1)} p^{2\rho}\,\mathrm{e}^{-\mu+\gamma/2} + h_2^{(1)}$$

for some constants $h_1^{(1)}, h_2^{(1)} \geq 0$. To apply induction, suppose that

$$z \mapsto \int |r(p',x')|P^{t-1}(z,\mathrm{d}z') \leq h_1^{(t-1)} p^{2\rho^{t-1}}\mathrm{e}^{-\mu+\gamma/2} + h_2^{(t-1)}$$

for some constants $h_1^{(t-1)}, h_2^{(t-1)} \geq 0$, and that it is continuous in (p,μ,γ). Since for some constant $a_t > 0$, we have

$$\int p'^{2\rho^{t-1}}\mathrm{e}^{-\mu'+\gamma'/2}P(z,\mathrm{d}z') = a_t p^{2\rho^t}\mathrm{e}^{-\mu+\gamma/2},$$

which is continuous, applying lemma 5.A.1 again yields: $z \mapsto \int |r(p',x')|P^t(z,\mathrm{d}z')$ is continuous and $\int |r(p',x')|P^t(z,\mathrm{d}z') \leq h_1^{(t)} p^{2\rho^t}\,\mathrm{e}^{-\mu+\gamma/2} + h_2^{(t)}$ for some constants $h_1^{(t)}, h_2^{(t)} \geq 0$. Moreover, g is continuous and from equation (5.B.22) we know that $(p,\mu,\gamma) \mapsto \int g(p',\mu',\gamma')P(z,\mathrm{d}z')$ is continuous. Hence, assumption 2.3.2 holds. Based on proposition 2.3.4, ψ^* is continuous. □

Proof of proposition 5.5.5. Regarding claim (1), note that for some $a_1, \cdots, a_6 \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\int |r(p',x')| l(p',x'|p,\mu,\gamma)\,\mathrm{d}(p',x') \\
&\leq a_1 p^{2\rho}\,\mathrm{e}^{-\mu+\gamma/2} + a_2 p^{2\rho} + a_3\,\mathrm{e}^{-\mu+\gamma/2} + a_4\,\mathrm{e}^{\mu+\gamma/2} + a_5.
\end{aligned}$$

Induction then shows that for all $t \in \mathbb{N}$,

$$\begin{aligned}
&\int |r(p',x')|P^t(z,\mathrm{d}z') \\
&\leq a_1^{(t)} p^{2\rho^t}\mathrm{e}^{-\mu+\gamma/2} + a_2^{(t)} p^{2\rho^t} + a_3^{(t)}\mathrm{e}^{-\mu+\gamma/2} + a_4^{(t)}\mathrm{e}^{\mu+\gamma/2} + a_5^{(t)} \\
&\leq a_6^{(t)}\left[\left(p^{2\rho^t} + p^{-2\rho^t} + \delta\right)\left(\mathrm{e}^{-\mu+\gamma/2} + 1\right) + \mathrm{e}^{-\mu+\gamma/2} + \mathrm{e}^{\mu+\gamma/2} + 1\right]
\end{aligned}$$

for some constants $a_1^{(t)}, \cdots, a_5^{(t)} > 0$ and $a_6^{(t)} = a_1^{(t)} \vee \cdots \vee a_5^{(t)}$. Hence, g can be chosen as in the set up, and conditions (2.8)–(2.9) of assumption 2.3.1 hold.

Moreover,

$$
\begin{aligned}
&\int g(p,\mu,\gamma) l(p',x'|p,\mu,\gamma)\,\mathrm{d}(p',x') \\
&= \left(p^{2\rho^{n+1}} \mathrm{e}^{2\rho^n b + 2\rho^{2n}\gamma_p} + p^{-2\rho^{n+1}} \mathrm{e}^{-2\rho^n b + 2\rho^{2n}\gamma_p} + \delta\right)\left(\mathrm{e}^{-\mu+\gamma/2}+1\right) + \mathrm{e}^{-\mu+\gamma/2} + \mathrm{e}^{\mu+\gamma/2} \\
&\leq \left[\mathrm{e}^{|\rho^n|\xi}\left(p^{2\rho^{n+1}} + p^{-2\rho^{n+1}}\right) + \delta\right]\left(\mathrm{e}^{-\mu+\gamma/2}+1\right) + \mathrm{e}^{-\mu+\gamma/2} + \mathrm{e}^{\mu+\gamma/2} \\
&\leq \left[\mathrm{e}^{|\rho^n|\xi}\left(p^{2\rho^{n}} + p^{-2\rho^{n}} + 1\right) + \delta\right]\left(\mathrm{e}^{-\mu+\gamma/2}+1\right) + \mathrm{e}^{-\mu+\gamma/2} + \mathrm{e}^{\mu+\gamma/2} \\
&\leq \mathrm{e}^{|\rho^n|\xi}\left(p^{2\rho^{n}} + p^{-2\rho^{n}} + \delta\right)\left(\mathrm{e}^{-\mu+\gamma/2}+1\right) + \mathrm{e}^{-\mu+\gamma/2} + \mathrm{e}^{\mu+\gamma/2} \leq m g(p,\mu,\gamma) + d,
\end{aligned}
$$

where we have used the definition of δ in the third inequality. Hence, assumption 2.3.1 holds. Claim (1) then follows from theorem 5.2.1.

The proof of claim (2) is similar to that of proposition 5.5.4 and thus omitted. □

Proof of proposition 5.5.6. Regarding claim (1), notice that assumption 2.3.1 holds trivially due to bounded rewards. Then claim (1) holds by theorem 5.2.1.

Regarding claim (2), we let $\mathsf{X} = [w_l, w_h] \subset \mathbb{R}_+$. Since $c_0 \in \mathsf{X}$, we have

$$v^*(w,\pi) \in [w_l/(1-\beta), w_h/(1-\beta)],$$

and then

$$c_0 + \beta \int v^*(w',\pi') h_\pi(w')\,\mathrm{d}w' \in [w_l/(1-\beta), w_h/(1-\beta)].$$

By the intermediate value theorem, assumption 5.4.1 holds. By theorem 5.2.1 and (5.4.1), claim (2) holds.

Regarding claim (3), P satisfies the Feller property by lemma 5.A.1. Since the reward functions are continuous, the continuity of ψ^* follows from proposition 2.3.4 (or remark 5.3.1). Then the continuity of $\bar{w}$ follows from proposition 5.4.1. Hence, claim (3) is verified. This concludes the proof. □

Chapter 6

Conclusions

This thesis undertakes the first systematic analysis of the continuation value based method for sequential decision problems.

On the one hand, we show that the Jovanovic and Bellman operators are semiconjugate, implying that any fixed point of one of the operators is a direct translation of a fixed point of the other. Iterative sequences generated by the operators are also simple translations. We then add topological structure to the generic setting, and embed our optimality and symmetry analysis respectively in (a) spaces of potentially unbounded functions endowed with generic weighted supremum norm distances, and (b) spaces of integrable functions with divergence measured by L_p norms. In each setting, we show that the Bellman and Jovanovic operators are both contraction mappings under identical assumptions, and that convergence to the respective fixed points occurs at the same rate. The theory is established for important classes of sequential decision problems, including:

(1) standard optimal stopping problems (chapter 2),

(2) repeated optimal stopping problems (chapter 3), and

(3) dynamic discrete choice problems (chapter 4).

On the other hand, the thesis reveals several important differences between the continuation value based method and the traditional value function based method in terms of efficiency and analytical convenience. The former is shown to be highly advantageous over the latter in a wide range of problems.

In chapter 2, we show that, for most problems of interest to economists, the dimensionality of the effective state space associated with the Jovanovic operator is no larger than that related to the Bellman operator (see appendix 2.D), while for continuation decomposable problems, the dimensionality of the effective state space associated with the Jovanovic operator is strictly smaller than that related to the Bellman operator (see section 2.4). In the latter case, continuation value iteration obtains an $\mathcal{O}(K)$ speed up over value function iteration for finite space approximation, and an $\mathcal{O}(K \log(KM) / \log(M))$ speed up for infinite space approximation (see table 2.1).

In chapter 5, we show that continuation value functions are typically smoother than value functions, yielding sharper analytical properties related to derivatives. We propose a general theory for sequential decision problems based around continuation value functions and the Jovanovic operator, heavily exploiting the advantages discussed above. We obtain:

(1) conditions under which continuation values are: (a) continuous, (b) monotone, and (c) differentiable as functions of the economic environment;

(2) conditions under which parametric continuity holds (often required for proofs of existence of recursive equilibria in many-agent environments);

(3) conditions under which threshold policies are: (a) continuous, (b) monotone, and (c) differentiable.

In the latter case we derive an expression for the derivative of the threshold relative to other aspects of the economic environment and show how it contributes to economic intuition.

The closest counterparts to these results in the existing literature are those concerning individual applications. Our theory generalizes and extends these results in a unified framework. Some results, such as differentiability of threshold policies, are new to the literature to our best knowledge.

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