ON CERTAIN VARIETIES
OF
METABELIAN GROUPS.

by

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STATEMENT

The results presented in this thesis are my own except where stated otherwise.

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PREFACE

During the time that the work for this thesis was done, I held a Commonwealth Post-graduate Award, which was generously supplemented by the Australian National University.

My thanks are due to Dr. L.G. Kovács, my Supervisor during the past two and a half years. I am most grateful to him for his friendly criticism and encouragement, and his readiness to discuss my problems.

I thank Professor S.J. Tobin, who acted as my Supervisor for six months during Dr. Kovács' absence overseas, and Mrs. F.A. Wickland who typed the stencils for this thesis. Finally, my wife deserves my thanks for her forbearance over the last few months.

CONTENTS

CHAPTER 0

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0.1 Introduction

The problem of classifying varieties of metabelian groups has attracted several authors recently, and partial results have been obtained. For example, Brisley [7] and Weichsel [12] classified all varieties of metabelian p-groups of class at most p-1, and Newman [14] determined all subvarieties of $\mathbf{A}_{p^\alpha}\mathbf{A}_p$. Newman [15] has also classified all metabelian varieties of exponent 4. Getting away from locally nilpotent situations, Cossey [4] classified the varieties of metabelian A-groups, that is, varieties of metabelian groups whose Sylow subgroups are all abelian; in particular this includes all subvarieties of the product varieties $\mathbf{A}_m\mathbf{A}_n$, where m,n are coprime.

The work in the present thesis derives from an attempt to classify the subvarieties of $\mathbf{A}_m\mathbf{A}_n$, without restriction on m,n. The main result is a common extension of the results of Newman and Cossey mentioned above. Call two integers m,n nearly coprime if $p \mid m$ implies $p^2 \nmid n$. We give a complete classification of the subvarieties of $\mathbf{A}_m\mathbf{A}_n$ whenever m,n are nearly coprime; in particular this covers the case $\mathbf{A}_p\mathbf{A}_{pq}$, where p,q are distinct primes. The method can be outlined as follows. A subvariety $\mathbf{V}$ of $\mathbf{A}_m\mathbf{A}_n$ can be written $\mathbf{V} = \mathbf{U}\vee\mathbf{V}_{LN}$ where $\mathbf{U}$ is generated by the non-nilpotent critical groups in $\mathbf{V}$, and where $\underline{\underline{V}}_{LN}$ is generated by the nilpotent critical groups in $\underline{\underline{V}}$. Now $\underline{\underline{V}}_{LN}$ is locally nilpotent, and is covered by Newman's result, so we say no more about it, and concentrate on $\underline{\underline{U}}$. G is a non-nilpotent, metabelian, critical group, its Fitting subgroup F is a Sylow p-subgroup for some prime p, the derived group G' is contained in F, and F is complemented in G by a cycle of order t, say. Let p^{α} be the exponent of G'. Then $G \in \underline{\underline{A}}_m\underline{\underline{A}}_n$ implies $p^{\alpha} | m$, $t | n$, and

$$(*) \qquad \text{var } G = (\text{var } F)\underline{\underline{A}}_t \wedge \underline{\underline{A}}_{p^{\alpha}}\underline{\underline{A}}_n ,$$

at least for m,n nearly coprime. The non-nilpotent critical groups in $\underline{\underline{V}}$ fall into classes determined by the exponents of their derived groups and the orders of their Fitting factor groups; and in a similar manner to (*), each such class generates a variety of the form

$$\underline{\underline{W}}\,\underline{\underline{A}}_t \wedge \underline{\underline{A}}_{p^{\alpha}}\underline{\underline{A}}_n$$

where $\underline{\underline{W}}$ is the p-power exponent variety generated by the Fitting subgroups of the critical groups concerned, and $\underline{\underline{U}}$ is canonically the join of these varieties. This situation is described in Chapter 5.

In proving (*) I have had to introduce varietal concepts which are not concerned with varieties of groups as such. These are the concepts of 'split-group' and 'variety of split-groups'; a split-group is a group with a specified semi-direct decomposition. If G is a non-nilpotent, metabelian critical group as above, then $G' \leq F$ and F splits over G'; thus F may be thought of as a split-group $\underline{F}$, and if, in formula (*), one interprets each side as a statement about split-varieties, it is true without extra conditions on m,n. When m,n are nearly coprime, there is an accidental, very close, relation between the variety generated by F quâ group, and the variety generated by $\underline{F}$ quâ split-group. In the case m,n not nearly coprime, there is no such close relationship in general, and formula (*) is not true as a statement about varieties of groups; even an apparently more restrictive formula fails to hold.

The split-group idea is capable of wider use than this classification problem. In Chapter 4, for example, we prove a finite basis theorem for certain varieties of split-groups, which, by way of application, shows that certain varieties of metabelian groups have a finite basis. Although this is only a special case of D.E. Cohen's finite basis theorem for all metabelian varieties [16], it seems worth doing not only as a demonstration of the strength of the split-group technique, but also for the sake of the additional information obtained about the varieties involved, especially as [16] gives no varietal side results at all. While a complete classification is lacking, even for subvarieties of the product varieties of $\underline{\underline{A}}_p\underline{\underline{A}}_{p^{\alpha}q}$, enough information is obtained to answer several questions concerning the lattice of subvarieties of certain

$\underline{\underline{A}}_m\underline{\underline{A}}_n$, for example, questions of distributivity of the lattice of these varities.

It was pointed out to me by L.G. Kovács, that split-groups of species 2 (that is, groups with a specified decomposition as a semi-direct product of two groups) could be re-interpreted as group pairs, in the terminology of B.I. Plotkin's recent book [19]. In an appendix to that book Plotkin defined varieties of pairs and extended to these some constructions from the theory of varieties of universal algebras. Thus it seems that for split-groups of species 2, some basic definitions and results of a general nature could be obtained by specializing Plotkin's theory. Instead we show that all varieties of split-groups can be interpreted as varieties of universal algebras and so our fundamentals are derived directly from the theory of varieties of universal algebras. However if our results for the case of species 2 are thought of as results in certain varieties of group pairs, they are (so far as we know), the first detailed results on specific varieties of group pairs.

0.2 Notation and terminology.

For results relating to varieties of algebras we refer the reader to B.H. Neumann [21], and for results and notation relating specifically to varieties of groups, to Hanna Neumann [3].

We differ from [3] only in writing $H \leq G$ if H is a subgroup of G. If H is a proper subgroup of the group G, that is $H \neq G$, we write $H < G$. If H is normal in G we write $H \trianglelefteq G$. If G is generated by the subsets $H_1,\ldots,H_r$ then $G = \langle H_1,\ldots,H_r\rangle$.

If G is a group and $x,y \in G$ denote $y^{-1}xy$ by x^y, and the commutator $x^{-1}x^y$ by $[x,y]$. Commutators of higher weight are defined as left-normed: if $x_1,\ldots,x_n \in G$ and $[x_1,\ldots,x_{n-1}]$ has been defined, then

$$[x_1,\ldots,x_n] = [[x_1,\ldots,x_{n-1}],x_n] .$$

Define $[x,0y] = x$, and for $r \geq 0$, $[x,(r+1)y] = [[x,ry],y]$.

If H,K are subgroups of G, then $[H,K]$ is the subgroup generated by the elements $[h,k]$, $h \in H$, $k \in K$. The derived group G' of G is $[G,G]$. A group G is metabelian if $[G',G'] = 1$, where we use 1 to denote the identity of the group as well as the trivial subgroup. The normal closure of H in G is denoted by H^G.

The terms of the lower central series of G are defined inductively by

$$G_{(1)} = G, \quad G_{(c)} = [G_{(c-1)}, G];$$

thus $G_{(2)} = G'$. A group G is nilpotent of class c if $G_{(c+1)} = 1$, $G_{(c)} \neq 1$.

The centralizer of a subgroup H of G is denoted by $C_G(H)$ and the centre of G by $Z(G)$. The Fitting subgroup of a finite group G, the largest normal, nilpotent subgroup of G, is denoted by $F(G)$.

A finite group with a unique minimal normal subgroup is called monolithic, and the unique minimal normal subgroup is called the monolith. The socle of a finite group G is the subgroup generated by all minimal normal subgroups of G, and is denoted by σG.

In late chapters, Chapter 4 in particular, many well-known commutator identities will be used without comment. The ones used are listed here. In any group G the following are identities:

$$\begin{aligned} [x,yz] &= [x,z][x,y][x,y,z], \\ [xy,z] &= [x,z][x,z,y][y,z], \\ [x,y] &= [y,x]^{-1}, \\ [x,y^{-1}] &= [x,y]^{-y^{-1}}. \end{aligned}$$

In a metabelian group G:

$$[x,y,z][y,z,x][z,x,y] = 1;$$

and therefore, if $d \in C_G(G')$, putting $z = d$ we have

$$\begin{aligned}[d,x,y] &= [y,d,x]^{-1} = [[d,y]^{-1},x]^{-1} \\ &= [d,y,x]^{[d,y]} = [d,y,x].\end{aligned}$$

Finally note that we defy convention and write ω for the cardinal of the natural numbers.

CHAPTER 1

VARIETIES OF SPLIT-GROUPS

In this chapter we are concerned with varieties of certain objects called split-groups, which are defined below. A split-group is, suitably interpreted, a universal algebra, and this is pointed out in section 1.2; hence much general theory is applicable to our situation, and it will be called on to eliminate long proofs which would be redundant. However our interest in varieties of split-groups, or split-varieties for short, is the way they can be used to give results about varieties of groups; more insight seems to be gained by developing the theory of split-groups as is done below, then is gained by regarding split-groups and varieties of split-groups as part of a much more general framework. We repeat that our reference for results on varieties of universal algebras is [21].

1.1 Split-groups

(1.1.1) Definition. A split-group of the species n, is an $(n+1)$-tuple $(G, A_1, \ldots, A_n)$ where G is a group, $A_1, \ldots, A_n$ are subgroups generating G such that, if $B_i = \langle A_i, \ldots, A_n \rangle$, $i \in \{1, \ldots, n\}$, then A_i is normal in B_i and is complemented in B_i by B_{i+1}:

$$A_i \trianglelefteq B_i, \quad A_i B_{i+1} = B_i, \quad A_i \cap B_{i+1} = 1 \ .$$

We shall denote the split-group $(G,A_1,\ldots,A_n)$ by $\underline{G}$ when no confusion can arise as to the particular splitting of G involved; also we may write $A_i = A_i(\underline{G})$, $B_i = B_i(\underline{G})$, $i \in \{1,\ldots,n\}$. The group G is called the *carrier* of $\underline{G}$; an *element* of $\underline{G}$ is an element of G.

(1.1.2) *Definition.* A *sub-split-group* of the split-group $(G,A_1,\ldots,A_n)$ is a split-group $(\overline{G},\overline{A}_1,\ldots,\overline{A}_n)$ where $\overline{G}$ is a subgroup of G and where $\overline{A}_i = A_i \cap \overline{G}$, $i \in \{1,\ldots,n\}$. A sub-split-group is *normal* if it is normal as a subgroup.

(1.1.3) *Definition.* A *morphism* μ between two split-groups $(G,A_1,\ldots,A_n)$ and $(\overline{G},\overline{A}_1,\ldots,\overline{A}_n)$ is a group homomorphism $\mu : G \to \overline{G}$ such that $A_i\mu \leq \overline{A}_i$, $i \in \{1,\ldots,n\}$. We write $\mu : \underline{G} \to \underline{\overline{G}}$.

Notice that morphisms are defined only between split-groups of the same species; this dependence on the species will often be left understood, unless it is necessary to clarify the meaning. Note also that, in general, every inner automorphism of G is not a self-morphism of $\underline{G}$.

(1.1.4) *Definition.* A morphism is *epi* or *mono* according as it is onto or one-to-one as a group homomorphism of the carriers.

(1.1.5) Definition. If $\underline{G} = (G, A_1, \dots, A_n)$ is a split-group and $\underline{N}$ is a normal sub-split-group of $\underline{G}$, the quotient split-group $\underline{G}/\underline{N}$ is the split-group

$$\underline{G}/\underline{N} = (G/N,\ A_1N/N, \dots, A_nN/N) .$$

The right-hand side is indeed a split-group: clearly $A_iN/N \trianglelefteq B_iN/N$ and if $a_i \in A_i$, $b_{i+1} \in B_{i+1}$ such that $a_iN = b_{i+1}N$ then $b_{i+1}^{-1}a_i \in N = (N \cap A_1) \dots (N \cap A_n)$ which implies $a_i \in N$ by the uniqueness of the decomposition $g = a_1a_2 \dots a_n$ for any element g of G.

(1.1.6) Lemma. If $\mu : \underline{G} \to \underline{\overline{G}}$ is a morphism between two split-groups then $(\ker \mu, \ker \mu|A_1(\underline{G}), \dots, \ker \mu|A_n(\underline{G}))$ is a normal sub-split-group of $\underline{G}$. (Here $\mu|A_i(\underline{G})$ denotes the restriction of μ to $A_i(\underline{G})$).

Proof. We have only to verify that $\ker \mu$ splits appropriately; indeed if $a_1a_2 \dots a_n \in \ker \mu$ with $a_i \in A_i(\underline{G})$, then $(a_1\mu)(a_2\mu) \dots (a_n\mu) = 1$ so that $a_1\mu = \dots = a_n\mu = 1$, or $a_i \in \ker \mu|A_i(\underline{G})$, $i \in \{1, \dots, n\}$.

(1.1.7) Definition. The cartesian product of a collection of split-groups $\underline{G}_i = (G_i, A_{i1}, \ldots, A_{in})$ of the same species, $i \in I$, is the split-group $\underline{G}$ where $G = \Pi\{G_i : i \in I\}$ and where $A_j(\underline{G}) = \Pi\{A_{ij} : i \in I\}$ is embedded in G in the natural way:

$$A_j(\underline{G}) = \{f \in G \,|\, f(i) \in A_{ij}, i \in I\}.$$

The restricted direct product is defined similarly.

(1.1.8) Definition. A fully-invariant sub-split-group of $\underline{G}$ is one invariant under all self-morphisms of $\underline{G}$.

Note that, as not every inner automorphism of G is a self-morphism of $\underline{G}$, a fully invariant sub-split-group need not be normal. It is easy to see that the intersection of the normal sub-split-groups which contain a given fully invariant sub-split-group is fully invariant (and normal).

(1.1.9) Definition. A generating set $\{a_{ij} \in A_i(\underline{G}) : j \in J_i, 1 \leq i \leq n\}$ of G will be called a generating set of $\underline{G}$. A split-group is finitely generated if it has a finite generating set.

A split-group will be said to have a certain property if its carrier has the property; thus $\underline{G}$ is finite if G is finite. For split-groups of small species, special names will be adopted: a split-group of species 2 is a bigroup, and one of species 3 is a trigroup.

Finally, in this section, we note a few abuses of language that will occur from time to time. The trivial split-group should, of course, be written as $\underline{1} = (1,1,\ldots,1)$, but we will write 1 for it, and also for the trivial sub-split-group of a split-group. A subgroup S of G may be referred to as 'the sub-split-group S' of $\underline{G}$ if it splits appropriately, while a sub-split-group may be referred to as a subgroup if, by doing so, the desired emphasis is conveyed without creating confusion.

1.2 Alternative formulation.

We shall in this section characterize split-groups as certain universal algebras. The operator domain is defined as follows.

(1.2.1) Definition. Ω_n is a commutative semigroup $\{w_0, w_1, \ldots, w_n\}$ of order $n+1$ with multiplication table

$$w_i w_j = w_j \quad \text{for} \quad 0 \leq i \leq j \leq n \; .$$

In the terminology of [6], Ω_n is a commutative band, fully ordered with respect to the relation: $w_i \leq w_j$ if and only if $w_i w_j = w_j$.

(1.2.2) Definition. An Ω_n-group is a triple $(G, \Omega_n, \underline{\underline{e}})$, where G is a group and where the mapping $\underline{\underline{e}} : G \times \Omega_n \to G$ has the properties

$$(xy)w_i\underline{\underline{e}} = (xw_i\underline{\underline{e}})(yw_i\underline{\underline{e}}),$$

$$xw_0\underline{\underline{e}} = x, \qquad xw_n\underline{\underline{e}} = 1,$$

and

$$(xw_i\underline{\underline{e}})w_j\underline{\underline{e}} = x(w_iw_j)\underline{\underline{e}} \; ,$$

for all $x, y \in G$, and $i, j \in \{0, 1, \ldots, n\}$.

Since an Ω_n-group is a universal algebra, the concepts of sub-Ω_n-group, quotient Ω_n-group have standard definitions; we give them here using the well-known correspondence between congruences on groups and normal subgroups.

(1.2.3) <u>Definition.</u> A sub-Ω_n-group of an Ω_n-group $(G,\Omega_n,\underline{\underline{e}})$ is an Ω_n-group $(\overline{G},\Omega_n,\overline{\underline{\underline{e}}})$ where $\overline{G}$ is a subgroup of G and where $\overline{\underline{\underline{e}}} = \underline{\underline{e}}|\overline{G} \times \Omega_n$.

(1.2.4) <u>Definition.</u> If $(G,\Omega_n,\underline{\underline{e}})$ is an Ω_n-group and $(N,\Omega_n,\underline{\underline{e}}')$ is a normal sub-Ω_n-group (that is, a sub-Ω_n-group which is normal quâ subgroup), then the quotient Ω_n-group $(G,\Omega_n,\underline{\underline{e}})/(N,\Omega_n,\underline{\underline{e}}')$ is the Ω_n-group $(G/N,\Omega_n,\underline{\underline{e}}'')$ where $\underline{\underline{e}}'' : G/N \times \Omega_n \rightarrow G/N$ is defined by $xNw_i\underline{\underline{e}}'' = xw_i\underline{\underline{e}}N$.

(1.2.5) <u>Definition.</u> A homomorphism $\mu : (G,\Omega_n,\underline{\underline{e}}) \rightarrow (\overline{G},\Omega_n,\overline{\underline{\underline{e}}})$ between Ω_n-groups is a group homomorphism $\mu : G \rightarrow \overline{G}$ such that for all $x \in G$, $(xw_i\underline{\underline{e}})\mu = (x\mu)w_i\overline{\underline{\underline{e}}}$.

(1.2.6) <u>Definition.</u> The cartesian product of a collection $(G_i,\Omega_n,\underline{\underline{e}}_i)$ $(i \in I)$ of Ω_n-groups is the Ω_n-group $(G,\Omega_n,\underline{\underline{e}})$, where $G = \Pi\{G_i : i \in I\}$ and where $\underline{\underline{e}} : G \times \Omega_n \rightarrow G$ is defined by

$$fw_j\underline{\underline{e}}(i) = f(i)w_j\underline{\underline{e}}_i, \quad f \varepsilon G, \quad i \varepsilon I, \ j \varepsilon \{0,\dots,n\}.$$

(1.2.7) Theorem. There is a functor Φ from the category of all split-groups of species n to the category of all Ω_n-groups, which is one-to-one on both objects and morphisms and which preserves sub-structures, quotient structures and cartesian products.

Proof. Let $(G,A_1,\dots,A_n)$ be a split-group. Define the endomorphisms σ_i of G by

$$(a_1a_2 \dots a_n)\sigma_i = a_{i+1} \dots a_n$$

for all $a_j \varepsilon A_j$, $j \varepsilon \{1,\dots,n\}$, $i \varepsilon \{0,1,\dots,n-1\}$; and define σ_n to be the zero endomorphism of G. We call σ_i the <u>splitting endomorphism</u> of <u>G</u>. Clearly

$$\sigma_i\sigma_j = \sigma_j\sigma_i = \sigma_j, \quad 0 \leq i \leq j \leq n,$$

(1.2.8)

$$\sigma_0 = 1_G, \quad \sigma_n = 0_G.$$

Also $B_{i+1} = G\sigma_i$ and $A_i = \ker\sigma_i \cap B_i$, $i \varepsilon \{0,\dots,n\}$. Conversely, if a group G has endomorphisms σ_i with the properties (1.2.8), then by writing $B_{i+1} = G\sigma_i$, $A_i = \ker\sigma_i \cap B_i$, $i \varepsilon \{0,\dots,n\}$, $(G,A_1,\dots,A_n)$ is a split-group. For, if $x \varepsilon B_{i+1}$ then

$$x = x\sigma_i = ((x\sigma_i)(x\sigma_{i+1})^{-1})(x\sigma_{i+1})$$

and $((x\sigma_i)(x\sigma_{i+1})^{-1})\sigma_{i+1} = (x\sigma_{i+1})(x\sigma_{i+1})^{-1} = 1$, so that $(x\sigma_i)(x\sigma_{i+1})^{-1} \in \ker \sigma_{i+1} \cap B_{i+1} = A_{i+1}$, which shows that $B_{i+1} = A_{i+1}B_{i+2}$. Also $A_{i+1} \trianglelefteq B_{i+1}$, and if $y \in A_{i+1} \cap B_{i+2}$ then there exists $y_1 \in G$ with $y = y_1\sigma_{i+1}$, whence

$$1 = y\sigma_{i+1} = y_1\sigma_{i+1}^2 = y_1\sigma_{i+1} = y$$

and therefore $A_{i+1} \cap B_{i+2} = 1$. This shows that $(G,A_1,\ldots,A_n)$ is a split-group.

If $\underline{G} = (G,A_1,\ldots,A_n)$ is a split-group, define $\underline{G}\Phi = (G,\Omega_n,\underline{\underline{e}})$ where $\underline{\underline{e}} : G \times \Omega_n \to G$ is given by

$$(1.2.9) \qquad xw_i\underline{\underline{e}} = x\sigma_i \ , \quad i \in \{0,\ldots,n\},$$

for all $x \in G$. Conversely, if $(G,\Omega_n,\underline{\underline{e}})$ is an Ω_n-group we use (1.2.9) to define endomorphisms σ_i of G, which may easily be verified to have the properties (1.2.8), and therefore, in this way, $(G,\Omega_n,\underline{\underline{e}})$ defines a unique split-group $(G,\Omega_n,\underline{\underline{e}})\Psi$. Clearly $\Phi\Psi$ is the identity mapping on the class of all split-groups of species n, and $\Psi\Phi$ is the identity mapping in the class of all Ω_n-groups; hence Φ is one-to-one and onto on objects.

If $\mu : \underline{G} \to \underline{\overline{G}}$ is a morphism, then $\mu : \underline{G}\Phi \to \underline{\overline{G}}\Phi$ is a homomorphism; for it is easy to verify that if σ_i, $\overline{\sigma}_i$ are the splitting endomorphisms corresponding to $\underline{G}$, $\underline{\overline{G}}$ respectively, then

$\sigma_i\mu = \mu\overline{\sigma}_i$, $i \in \{0,\dots,n\}$. Hence, from (1.2.9)

$$(xw_i\underline{\underline{e}})\mu = x\sigma_i\mu = x\mu\overline{\sigma}_i = (x\mu)w_i\overline{\underline{\underline{e}}}$$

for all $x \in G$. Conversely every $\mu : \underline{G}\Phi \to \overline{\underline{G}}\Phi$ is a morphism $\mu : \underline{G} \to \overline{\underline{G}}$. If we put $\mu\Phi = \mu$, then clearly Φ is a functor. The rest of the theorem is proved by similar techniques which we omit.

We may use Definition 1.2.2 to appeal to general results: for example the usual homomorphism theorems apply for Ω_n-groups, and therefore, via Theorem 1.2.7, for split-groups also. Because of the application we wish to make, and for convenience in simplifying notation in the calculations of Chapter 4, it is the split-group definition rather than the Ω_n-group definition that we use. In the sequel we shall suppress statements in the Ω_n-group formulation except if the comparison is of interest (for example we are led to different definitions of free objects), or if brevity can be obtained by appeal to more general results.

1.3 Freeness of split-groups.

Let $Y_1,\dots,Y_n$ be free groups of rank $m_1,\dots,m_n$ respectively, on free generators $\{y_{ij} : j \in J_i\}$, $|J_i| = m_i$. We do not suppose that the m_i are finite cardinals. Let $\underline{Q}(m_1,\dots,m_n)$ be the split-group defined as follows: the carrier is to be the free product

$Y_1 * \ldots * Y_n$, and

$$A_i(\underline{Q}(m_1,\ldots,m_n)) = \text{normal closure of } Y_i \text{ in } Y_i * \ldots * Y_n.$$

(1.3.1) Definition. The split-group $\underline{Q}(m_1,\ldots,m_n)$ is the absolutely split-free split-group of rank $(m_1,\ldots,m_n)$ on the split-free generating set $\{y_{ij} : j \in J_i, 1 \leq i \leq n\}$.

The use of the word rank obviously needs justifying and we will cover this in Lemma 1.3.6.

(1.3.2) Theorem. If $\underline{G}$ is a split-group of species n then every set of mappings $\mu_i : \{y_{ij} : j \in J_i\} \to A_i(\underline{G})$ can be extended to a morphism $\mu : \underline{Q}(m_1,\ldots,m_n) \to \underline{G}$.

Proof. Since $Q(m_1,\ldots,m_n)$ is a free group with the y_{ij}'s as a free generating set, certainly a group homomorphism μ, which extends all μ_i, exists; that $A_i(\underline{Q}(m_1,\ldots,m_n))\mu \leq A_i$ follows from the definition of $A_i(\underline{Q}(m_1,\ldots,m_n))$ and the fact that $A_i(\underline{G}) \trianglelefteq \langle A_i(\underline{G}),\ldots,A_n(\underline{G})\rangle$.

As in more general situations, we have the concept of relative freeness, and theorems characterizing it.

(1.3.3) *Definition.* A split-group $\underline{G}$ of species n is *relatively split-free* if it has a generating set $\{a_{ij} : j \in J_i, 1 \leq i \leq n\}$ with $1 \neq a_{ij} \in A_i(\underline{G})$ such that every set of mappings $\mu_i : \{a_{ij} : j \in J_i\} \rightarrow A_i(\underline{G})$ can be extended to a marphism of $\underline{G}$ into $\underline{G}$. Such a generating set is called a *split-free generating set* for $\underline{G}$. If $m_i = |J_i|$, $(m_1, \ldots, m_n)$ is called the *rank* of $\underline{G}$.

Note that in this definition, some of the m_i may be zero; this would occur if $A_i(\underline{G}) = 1$. Invariance of the rank will be proved in Lemma 1.3.6.

(1.3.4) *Theorem.* If $\underline{G}$ is relatively split-free, then $\underline{G}$ has a representation $\underline{Q}/\underline{S}$, where $\underline{Q}$ is absolutely free of the same rank as $\underline{G}$, and $\underline{S}$ is a normal, fully invariant sub-split-group of $\underline{Q}$. Conversely, every such quotient split-group $\underline{Q}/\underline{S}$ is relatively split-free; if the rank of $\underline{Q}$ is $(m_1, \ldots, m_n)$, then that of $\underline{Q}/\underline{S}$ is $(m'_1, \ldots, m'_n)$ where $m'_i = m_i$, unless $A_i(\underline{Q}) \leq S$, in which case $m'_i = 0$.

Proof. Suppose that $\underline{G}$ is relatively split-free on the split-free generating set $\{a_{ij} : j \in J_i, 1 \leq i \leq n\}$. Let $\underline{Q} = \underline{Q}(m_1, \ldots, m_n)$ where $m_i = |J_i|$, $i \in \{1, \ldots, n\}$. Define the epimorphism $\lambda : \underline{Q} \rightarrow \underline{G}$ by

$$y_{ij}\lambda = a_{ij}, \quad j \in J_i, \quad i \in \{1,\ldots,n\},$$

and Theorem 1.3.2. Put $S = \ker\lambda$; then $\underline{S}$ is a normal sub-split-group of $\underline{Q}$ by (1.1.6). To show that $\underline{S}$ is fully-invariant, let α be an arbitrary self-morphism of $\underline{Q}$ and define the mapping $\beta : \{a_{ij} : j \in J_i,\ 1 \leq i \leq n\} \to \underline{G}$ by

$$a_{ij}\beta = (y_{ij}\alpha)\lambda.$$

By definition, β can be extended to a self-morphism of $\underline{G}$. Since the restrictions of $\alpha\lambda$ and $\lambda\beta$ to the set $\{y_{ij} : j \in J_i,\ 1 \leq i \leq n\}$ of generators of $\underline{Q}$ agree, $\alpha\lambda = \lambda\beta$. Hence if $s \in S$, $s\alpha\lambda = s\lambda\beta = 1$, and so $s\alpha \in \ker\lambda = S$.

In order to prove the converse, we need the following lemma, which was proposed to me by L.G. Kovács.

(1.3.5) <u>Lemma</u>. Let $\underline{H}$ be relatively split-free on the generating set $\underset{\sim}{h} = \{h_{ij} : j \in J_i,\ 1 \leq i \leq n\}$. Let $\alpha : \underline{H} \to \underline{K}$ be an epimorphism such that $A_i(\underline{H}) \neq 1$ implies $A_i(\underline{K}) \neq 1$, and such that $\ker\alpha$ is fully invariant. Then $\alpha|\underset{\sim}{h}$ is one-to-one, and $\underline{K}$ is relatively split-free on $\underset{\sim}{h}\alpha$.

Proof. First, $\alpha|\underset{\sim}{h}$ is one-to-one. For, if $h_{ij}\alpha = h_{i\ell}\alpha$, $j \neq \ell$, then $h_{ij} = h_{i\ell}x$, $x \in \ker\alpha$. Define $\eta : \underline{H} \to \underline{H}$ so that $h_{ij}\eta = h_{ij}$, $h_{i\ell}\eta = 1$; then $h_{ij} = h_{ij}\eta = x\eta \in \ker\alpha$ since $\ker\alpha$ is fully invariant. Hence $\{h_{ij} : j \in J_i\} \subseteq \ker\alpha$, or $A_i(\overline{H}) \leq \ker\alpha$ which implies $A_i(\underline{K}) = 1$. It follows that $\alpha|\underset{\sim}{h}$ is one-to-one.

Second, $\underline{H}$ is split-free on $\underset{\sim}{h}\alpha$. For, let $\beta : \underset{\sim}{h}\alpha \to \underline{K}$ be any map such that $h_{ij}\beta \in A_i(\underline{K})$. Define $\eta : \underline{H} \to \underline{H}$ so that $h_{ij}\eta \in h_{ij}\alpha\beta\alpha^{-1}$. Consider the map $\alpha^{-1}\eta\alpha$ from K to the set of non-empty subsets of K. Observe that $1\alpha^{-1}\eta\alpha = (\ker\alpha)\eta\alpha \leq (\ker\alpha)\alpha = 1$; that is $1\alpha^{-1}\eta\alpha = \{1\}$. Also, if $k = k_1^{-1}k_2$ then $k\alpha^{-1} = (k_1^{-1}\alpha^{-1})\cdot(k_2\alpha^{-1})$ in the usual multiplication of subsets of a group, and therefore

$$k\alpha^{-1}\eta\alpha = (k_1^{-1}\alpha^{-1}\eta\alpha)\cdot(k_2\alpha^{-1}\eta\alpha).$$

Thus $\{1\} = (k^{-1}\alpha^{-1}\eta\alpha)\cdot(k\alpha^{-1}\eta\alpha)$ for all $k \in K$ showing that $|k\alpha^{-1}\eta\alpha| = 1$. Hence $\alpha^{-1}\eta\alpha$ is an endomorphism of K, and since it agrees on $\underset{\sim}{h}\alpha$ with β, it is a morphism $\underline{K} \to \underline{K}$.

We return to the proof of (1.3.4). Write $\underline{Q}^*$ for the absolutely split-free split-group of rank $(m_1', \ldots, m_n')$, where $m_i' = m_i$ unless $A_i(\underline{Q}) \leq \underline{S}$, in which case $m_i' = 0$. Then there exists a natural morphism $\gamma : \underline{Q} \to \underline{Q}^*$ such that $\ker\gamma = \langle Y_i : A_i(\underline{Q}) \leq S \rangle^{Q}$.

If $\delta : \underline{Q} \to \underline{Q}/\underline{S}$ is the natural morphism, define $\alpha : \underline{Q}^* \to \underline{Q}/\underline{S}$ by

$$y^*_{ij}\alpha = y_{ij}\delta$$

where y^*_{ij}, y_{ij} are split-free generators of $\underline{Q}^*$ and $\underline{Q}$ respectively. Clearly

$$\gamma\alpha = \delta .$$

Now $\ker\alpha$ is fully invariant in $\underline{Q}^*$, for, if $\xi : \underline{Q}^* \to \underline{Q}^*$ then there exists $\eta : \underline{Q} \to \underline{Q}$ such that $\gamma\xi = \eta\gamma$; and if $q^* \in \ker\alpha$, there exists $q \in \underline{Q}$ with $q\gamma = q^*$. Now $q\gamma\alpha = q^*\alpha = 1 = q\delta$ which means $q \in S$. Therefore

$$q^*\xi\alpha = (q\gamma\xi)\alpha = (q\eta\gamma)\alpha = (q\eta)\delta = 1$$

since $q\eta \in S$. That is, $q^*\xi \in \ker\alpha$, and therefore $\ker\alpha$ is fully invariant. Also $A_i(\underline{Q}^*) \neq 1$ implies $A_i(\underline{Q}/\underline{S}) \neq 1$ and so the conditions of the Lemma 1.3.5 are satisfied, and $\underline{Q}/\underline{S}$ has the asserted properties.

(1.3.6) Lemma. The rank is an invariant of a relatively split-free split-group.

Proof. Let $\underline{G}$ be relatively split-free. If $A_i(\underline{G}) \neq 1$, then $A_i(\underline{G}) \nleq G'$. For, if a_{ij} is an element of a split-free generating set consider the self-morphism $\mu : \underline{G} \to \underline{G}$ such that $a_{ij}\mu = a_{ij}$

with all other split-free generators mapped to 1. Clearly $G' \leq \ker\mu$ and $a_{ij} \notin \ker\mu$. Now G' carries a fully invariant sub-split-group of $\underline{G}$ and the hypotheses of (1.3.5) are satisfied by the natural morphism $\alpha : \underline{G} \to \underline{G}/\underline{G}'$. Hence $\underline{G}/\underline{G}'$ is relatively split-free of the same rank as $\underline{G}$; and since each $A_i(\underline{G}/\underline{G}')$ is a relatively free abelian group, its rank is invariant, and therefore so is that of $\underline{G}$.

To finish off this section we mention that had one treated a split-group as an Ω_n-group as discussed in section 1.2, one would have been led to a smaller class of free split-groups; indeed we can make a distinction between 'free split-group' and 'split-free split-group' as indicated by the following theorem.

(1.3.7) Theorem. Let $(G,\Omega_n,\underline{\underline{e}})$ be a free Ω_n-group in the variety of all Ω_n-groups, say one of rank k. Then $(G,\Omega_n,\underline{\underline{e}})\Phi^{-1}$ is an absolutely split-free split-group of species n, and rank $(k,k,\ldots,k)$.

Proof. Write $\underline{G} = (G,\Omega_n,\underline{\underline{e}})\Phi^{-1}$. Let $\{x_j : j \in J\}$ be a free generating set for $(G,\Omega_n,\underline{\underline{e}})$, $|J| = k$. Put

$$z_{ij} = (x_j w_{i-1}\underline{\underline{e}})(x_j w_i \underline{\underline{e}})^{-1}, \quad i \in \{1,\ldots,n\}, \quad j \in J.$$

Then, for each $j \in J$,

$$x_j = z_{1j} z_{2j} \ldots z_{nj} .$$

It is clear, therefore, that $\{z_{ij} : j \in J_i,\ 1 \leq i \leq n\}$ is a generating set for $\underline{G}$. Let $\underline{H}$ be an arbitrary split-group, $\underline{H}\Phi = (H, \Omega_n, \underline{\underline{e}}^*)$, and $\mu_i : \{z_{ij} : j \in J_i\} \to A_i(\underline{H})$ a set of mappings. Define $\mu : \{x_j : j \in J\} \to \underline{H}\Phi$ by

$$x_j\mu = (z_{1j}\mu_1)(z_{2j}\mu_2)\ \dots\ (z_{nj}\mu_n), \quad j \in J.$$

It follows that μ can be extended to a homomorphism $\mu : \underline{G}\Phi \to \underline{H}\Phi$, and hence, by Theorem 1.2.7, that $\mu : \underline{G} \to \underline{H}$ is a morphism. It is easy to verify that μ does extend the μ_i:

$$\begin{aligned} z_{ij}\mu &= ((x_j w_{i-1}\underline{\underline{e}})(x_j w_i \underline{\underline{e}})^{-1})\mu \\ &= ((x_j\mu)w_{i-1}\underline{\underline{e}}^*)((x_j\mu)w_i\underline{\underline{e}}^*)^{-1} \\ &= (z_{ij}\mu_i)\ \dots (z_{nj}\mu_n).((z_{i+1j}\mu_{i+1})\ \dots\ (z_{nj}\mu_n))^{-1} \\ &= z_{ij}\mu_i\ . \end{aligned}$$

If we choose for $\underline{H}$ the split-free split-group of species n, $\underline{Q}(k,\dots,k)$, define μ as above from $\mu_i : z_{ij} \to y_{ij}$, and $\nu : \underline{H} \to \underline{G}$ by $\nu : y_{ij} \to z_{ij}$ and Theorem 1.3.2, we get that $\mu\nu = 1_{\underline{G}}$ and $\nu\mu = 1_{\underline{H}}$, so $\underline{G} \cong \underline{H} = \underline{Q}(k,\dots,k)$.

1.4 Split-words

(1.4.1) Definition. A split-word is an element of the absolutely split-free split-group $\underline{Q}(\omega,\ldots,\omega)$, the definite species n being understood. We shall often write for a split-word $q \in \underline{Q}(\omega,\ldots,\omega)$,

$$q = q(y_{1i_1},\ldots,y_{1i_r},\ldots,y_{nj_1},\ldots,y_{nj_t}),$$

or, more briefly still,

$$q = q(\underset{\sim}{y}_1,\ldots,\underset{\sim}{y}_n)$$

to indicate the dependence of q on the variables y_{ij}, though all those displayed may not occur explicitly.

(1.4.2) Definition. Two sets S_1, S_2 of split-words of species n are super-equivalent if they have the same fully-invariant closure in the absolutely free split-group $\underline{Q}(\omega,\ldots,\omega)$ of species n.

(1.4.3) Notation. We write $\underline{Q}_n$ for $\underline{Q}(\omega,\ldots,\omega)$, the absolutely free split-group of species n.

We shall need a version of Theorem 33.45 from [3]. To this end, note that the carrier of $\underline{Q}_n$ is a free group of countably infinite rank on the free generating set $\{y_{ij} : j \in J_i, 1 \le i \le n\}$.

Identify this carrier with X_∞ and $\{y_{ij} : j \in J_i, \ 1 \leq i \leq n\}$ with $\{x_k : k = 1,2,\ldots\}$, in the notation of section 3, Chapter 3, of [3]. The deletions δ_k considered there are morphisms of $\underline{Q}_n$, and hence the argument leading to Theorem 33.45 can be transferred to $\underline{Q}_n$. We do not wish to repeat the elaborate formalism which seems necessary to give rigorous meaning to the terms used in Theorem 33.45: intuitively, they may be described as follows. A split-word will be called <u>special</u> if it is equal to a product of powers of commutators $c_1,\ldots,c_k$ whose entries are powers of the free generators y_{ij}, and which have the property that if a power of some y_{ij} occurs as an entry in one c_j, then a power of y_{ij} occurs as an entry in each of $c_1,\ldots,c_k$. Then Theorem 33.45 can be stated in our situation as follows.

(1.4.4) <u>Theorem</u>. Each split-word is super-equivalent to a finite set of special split-words.

(1.4.5) <u>Definition</u>. The <u>split-verbal</u> <u>sub-split-group</u> of a split-group $\underline{G}$ of species n, determined by $S \subseteq \underline{Q}_n$, is the sub-split-group $S(\underline{G})$ whose carrier is the subgroup of G generated by the set

$$\{q\alpha : q \in S, \ \alpha : \underline{Q}_n \to \underline{G}\}.$$

Note that, by definition, this set admits every self-morphisms of $\underline{G}$ hence so does the subgroup of G generated by it. In particular, this subgroup admits the splitting endomorphisms σ_i of $\underline{G}$ and hence carries a sub-split-group: so Definition 1.4.5 is justified. Moreover it follows that every split-verbal sub-split-group is fully invariant. As the carrier of $S(\underline{Q}_n)$ is the least subgroup to contain the images of S under all self-morphisms of $\underline{Q}_n$, the fully invariant closure of S in $\underline{Q}_n$ must contain $S(\underline{Q}_n)$; but as $S(\underline{Q}_n)$ is fully invariant and contains S, it follows that the fully invariant closure of S in $\underline{Q}_n$ is precisely $S(\underline{Q}_n)$.

(1.4.6) Theorem. If $S \subseteq \underline{Q}_n$, then the fully invariant closure of S in $\underline{Q}_n$ is $S(\underline{Q}_n)$.

(1.4.7) Definition. Two sets S_1, S_2 of split-words of the same species n are equivalent if they have the same normalized fully-invariant closure in $\underline{Q}_n$. (It is easily seen that the normal closure quâ subgroup of a sub-split-group is a sub-split-group: if $\underline{U} \leq \underline{G}$, $u \in U$, $g \in G$, then $(u^g)\sigma_i = (u\sigma_i)^{g\sigma_i} \in U^G$).

(1.4.8) Theorem. If S_1, S_2 are super-equivalent, they are equivalent.

(1.4.9) Theorem. Two sets S_1, S_2 of split-words of species n are equivalent if and only if the normalized split-verbal sub-split-groups they determine in every split-group of species n are equal.

Proof. One way around is obvious. For the other, suppose that S_1, S_2 are equivalent, and let $\underline{G}$ be any split-group of species n. We must show that

$$S_1(\underline{G})^{\underline{G}} = S_2(\underline{G})^{\underline{G}}.$$

The following lemma is useful here.

(1.4.10) Lemma. If S is a set of split-words, $\underline{G}$ a split-group and $\underline{N}$ a normal sub-split-group of $\underline{G}$, then $S(\underline{G}/\underline{N}) = S(\underline{G})\underline{N}/\underline{N}$.

Proof. Every morphism $\alpha : \underline{Q}_n \to \underline{G}/\underline{N}$ can be factored through $\underline{G}$ via the natural morphism $\nu : \underline{G} \to \underline{G}/\underline{N}$, say $\alpha = \beta\nu$. Conversely every $\beta : \underline{Q}_n \to \underline{G}$ can be continued to $\alpha : \underline{Q}_n \to \underline{G}/\underline{N}$ by $\alpha = \beta\nu$. Hence $S(\underline{G})\nu = S(\underline{G}/\underline{N})$ which is what we wanted.

The proof of (1.4.9) runs as follows. First note that if $S \subseteq \underline{Q}_n$ and $\alpha : \underline{Q}_n \to \underline{H}$, then $S(\underline{Q}_n)\alpha \leq S(\underline{H})$; hence, with $\underline{H} = \underline{G}/\underline{N}$, we have

$$
\begin{aligned}
S_1(\underline{H}) = 1 &\rightleftarrows S_1(\underline{Q}_n) \leq \cap\{\ker\alpha : \alpha : \underline{Q}_n \to \underline{H}\} \\
&\rightleftarrows S_1(\underline{Q}_n)^{Q_n} \leq \cap\{\ker\alpha : \alpha : \underline{Q}_n \to \underline{H}\} \\
&\rightleftarrows S_2(\underline{Q}_n) \leq \cap\{\ker\alpha : \alpha : \underline{Q}_n \to \underline{H}\} \\
&\rightleftarrows S_2(\underline{H}) = 1 .
\end{aligned}
$$

It follows that $S_1(\underline{G}) \leq S_2(\underline{G})^G$ (putting $\underline{N} = S_2(\underline{G})^G$) and therefore that $S_1(\underline{G})^G \leq S_2(\underline{G})^G$. In a similar way, $S_2(\underline{G})^G \leq S_1(\underline{G})^G$, and this completes the proof.

Theorem 1.4.6 can be stated in a more familiar form for all relatively split-free split-groups as follows.

(1.4.11) Theorem. A sub-split-group of a relatively split-free split-group is fully invariant if and only if it is split-verbal.

Proof. Given a relatively split-free generating set of $\underline{G}$ and an element $h \in \underline{H} \leq \underline{G}$ then there exists a finite subset T of that generating set such that $h \in \langle T \rangle$. There exists a finite subset T' of a free generating set of $\underline{Q}_n$ and a one-to-one map $\mu : T' \to T$ which extends to $\mu^* : \underline{Q}_n \to \underline{G}$.

Now $\langle T' \rangle \mu^* = \langle T \rangle$; hence there exists $q \in \langle T' \rangle$ with $q\mu^* = h$. Given $\alpha : \underline{Q}_n \to \underline{G}$ let $\beta : \underline{G} \to \underline{G}$ be an extension of

$\mu^{-1}\alpha : T \to \underline{G}$. Then as $\mu^*\beta$ and α agree on T', they agree on $\langle T'\rangle$, hence, in particular, $q\alpha = q\mu^*\beta = h\beta \in \underline{H}$ if $\underline{H}$ is fully invariant. This proves that fully invariant sub-split-groups of $\underline{G}$ are split verbal; and the converse is true in any split-group.

(1.4.12) Theorem. There is one-to-one correspondence between the (normalized) fully invariant sub-split-groups of $\underline{Q}_n/S(\underline{Q}_n)^{Q_n}$ and the (normalized) fully invariant sub-split-groups of $\underline{Q}_n$ containing $S(\underline{Q}_n)^{Q_n}$.

Proof. This proof is an easy application of the last theorem.

(1.4.13) Lemma. If $\underline{S}$ is a normal sub-split-group of $\underline{Q}_n$, then $S(\underline{G})$ is normal in $\underline{G}$ for all $\underline{G}$ of species n.

Proof. It is sufficient to show that $(q\alpha)^g \in S(\underline{G})$ whenever $q \in S$, $\alpha : \underline{Q}_n \to \underline{G}$, $g \in \underline{G}$. The proof is similar to that of (1.4.11); there exists $\alpha^* : \underline{Q}_n \to \underline{G}$, $\bar{g} \in \underline{Q}_n$ such that $q\alpha^* = q\alpha$, $\bar{g}\alpha^* = g$, so that

$$(q\alpha)^g = (q\alpha^*)^{\bar{g}\alpha^*} = (q^{\bar{g}})\alpha^* \in S(\underline{G})$$

since $\underline{S} \trianglelefteq \underline{Q}_n$.

Examples of sub-split-groups which are not normal are easy to find, for example each $A_i(\underline{G})$ is split-verbal, but of course not necessarily normal, in $\underline{G}$.

1.5 Split-varieties

(1.5.1) Definition. If $S \subseteq \underline{Q}_n$, the class of all split-groups $\underline{G}$ of species n such that $S(\underline{G}) = 1$ is the variety of split-groups (or, briefly, the split-variety) determined by S.

(1.5.2) Theorem. Equivalent sets of split-words determine the same split-variety.

Proof. If S_1, S_2 are equivalent, then, by Theorem 1.4.9, for any $\underline{G}$, $S_1(\underline{G})^G = 1$ if and only if $S_2(\underline{G})^G = 1$ that is $S_1(\underline{G}) = 1$ if and only if $S_2(G) = 1$.

From this theorem it follows that, in defining split-varieties, we need only consider sets of split-words S which are normal, fully invariant sub-split-groups of $\underline{Q}_n$, since every sub-set of $\underline{Q}_n$ is equivalent, by definition, to its normalized fully invariant closure. The normalized fully invariant closure of S is denoted by $\operatorname{cl} S$.

(1.5.3) Definition. If $\underline{S}$ is a normal, fully invariant sub-split-group of $\underline{Q}_n$, the split-variety determined by S will be denoted by $\underset{\sim}{S}$.

(1.5.4) *Theorem.* The correspondence $\underline{S} \to \underset{\sim}{S}$ between normal, fully invariant sub-split-groups $\underline{S}$ of $\underline{Q}_n$ and the varieties $\underset{\sim}{S}$ of split-groups of species n is one-to-one and reverses inclusions.

Proof. Suppose $\underline{S}_1, \underline{S}_2$ are normal and fully invariant in $\underline{Q}_n$ and $\underset{\sim}{S}_1 \subseteq \underset{\sim}{S}_2$, then by Lemma 1.4.10,

$$\underline{Q}_n/\underline{S}_1 \in \underset{\sim}{S}_1 \subseteq \underset{\sim}{S}_2$$

and so $\underline{S}_2 = \underline{S}_2(\underline{Q}_n) \leq \underline{S}_1$. If follows that if $\underset{\sim}{S}_1 = \underset{\sim}{S}_2$, then $\underline{S}_1 = \underline{S}_2$.

It is clear that a split-variety is closed under the operations of forming sub-split-groups, quotient split-groups and cartesian products of split-groups. The converse of this is also true on account of Theorem 1.2.7, and Birkhoff's corresponding result for varieties of universal algebras. We omit the details of proof.

(1.5.5) *Theorem.* A class of split-groups is closed under the operations of forming sub-split groups, quotient split-groups and cartesian products of split-groups if and only if it is a split-variety.

(1.5.6) *Definition.* A split-word $q \in \underline{Q}_n$ is a *split-law* in $\underline{G}$ if $\{q\}(\underline{G}) = 1$; simply written, $q(\underline{G}) = 1$. If $\underset{\sim}{S}$ is a split-variety determined by the normal, fully invarient sub-split-group $\underline{S}$ of $\underline{Q}_n$ then the elements of $\underline{S}$ are called the *split-laws* of $\underset{\sim}{S}$.

(1.5.7) *Definition*. Given a split-variety $\underset{\sim}{S}$ and a n-triple $\underset{\sim}{m} = (m_1,\ldots,m_n)$ such that $m_i = 0$ if $y_{i1} \in S$, we call $\underline{Q}(\underset{\sim}{m})/S(\underline{Q}(\underset{\sim}{m}))$ the split-free split-group $F_{\underset{\sim}{m}}(\underset{\sim}{S})$ of rank $\underset{\sim}{m}$ of $\underset{\sim}{S}$.

By Theorem 1.3.4 $F_{\underset{\sim}{m}}(\underset{\sim}{S})$ is relatively split-free of rank $\underset{\sim}{m}$ and lies in $\underset{\sim}{S}$. Moreover,

(1.5.8) *Theorem*. Every mapping of a split-free generating set of $F_{\underset{\sim}{m}}(\underset{\sim}{S})$ into a split-group $\underline{G} \in \underset{\sim}{S}$ can be extended to a morphism.

Proof. Let $\underset{\sim}{z} = \{z_{ij} : j \in J_i,\ 1 \le i \le n\}$ be a split-free generating set for $\underline{Q}(\underset{\sim}{m})$; then if $\nu : \underline{Q}(\underset{\sim}{m}) \to F_{\underset{\sim}{m}}(\underset{\sim}{S})$ is the natural morphism, $\{z_{ij}\nu : j \in J_i,\ 1 \le i \le n\} = \underset{\sim}{z}\nu$ is a split-free generating set of $F_{\underset{\sim}{m}}(\underset{\sim}{S})$. Suppose $\beta : \underset{\sim}{z}\nu \to \underline{G} \in \underset{\sim}{S}$ such that $z_{ij}\nu\beta \in A_i(\underline{G})$. Then $\nu\beta : \underset{\sim}{z} \to \underline{G}$ extends to a morphism $\delta : \underline{Q}(\underset{\sim}{m}) \to \underline{G}$. Since $\underline{Q}(\underset{\sim}{m})\delta \le \underline{G} \in \underset{\sim}{S}$ it follows easily that $\ker\delta \ge S(\underline{Q}(\underset{\sim}{m}))$. Hence δ can be factored through ν, say $\delta = \nu\gamma$ and by definition, $(\underset{\sim}{z}\nu)\delta = (\underset{\sim}{z}\nu)\beta$; γ is the extension of β. This completes the proof.

Theorem 1.3.4 yields,

(1.5.9) *Theorem*. Every relatively split-free split-group is split-free in some $\underset{\sim}{S}$.

Finally in this section we note the following results. Imagine $\underline{Q}(m_1,\ldots,m_n)$, where $m_1,\ldots,m_n$ are all countable at most, embedded in $\underline{Q}_n$ in a natural way; then

(1.5.10) Theorem. If S is a fully-invariant sub-split-group of $\underline{Q}_n$ then $\underline{S}(\underset{\sim}{m}) = \underline{S} \cap \underline{Q}(\underset{\sim}{m})$ is fully-invariant in $\underline{Q}(\underset{\sim}{m})$, and

$$\underline{S}(\underset{\sim}{m}) = S(\underline{Q}(\underset{\sim}{m})),$$

and

$$\underline{S}(\underset{\sim}{m})^{Q(\underset{\sim}{m})} = \underline{S}^{Q_n} \cap \underline{Q}(\underset{\sim}{m}).$$

Proof. Clearly $S(\underline{Q}(\underset{\sim}{m})) \leq \underline{Q}(\underset{\sim}{m}) \cap \underline{S} = \underline{S}(\underset{\sim}{m})$. Conversely, if $q \in \underline{Q}(\underset{\sim}{m}) \cap \underline{S}$ and α is a self-morphism of $\underline{Q}_n$ which maps $\underline{Q}(\underset{\sim}{m})$ identically and every thing else to 1, then $q = q\alpha \in S(\underline{Q}(\underset{\sim}{m}))$ which gives us the opposite inequality.

For the second part we have

$$\begin{aligned}\underline{S}(\underset{\sim}{m})^{Q(\underset{\sim}{m})} &= (S \cap \underline{Q}(\underset{\sim}{m}))^{Q(\underset{\sim}{m})} \leq \underline{S}^{Q_n} \cap \underline{Q}(\underset{\sim}{m}) \\ &= S^{Q_n}(\underline{Q}(\underset{\sim}{m})), \text{ by the first part,} \\ &\leq S(\underline{Q}(\underset{\sim}{m}))^{Q(\underset{\sim}{m})} \\ &= \underline{S}(\underset{\sim}{m})^{Q(\underset{\sim}{m})},\end{aligned}$$

again by the first part.

(1.5.11) Definition. The split-variety generated by a set $\{\underline{G}_i : i \in I\}$ of split-groups of the same species n is the smallest split-variety of species n which contains all $\underline{G}_i$; equivalently, the split-variety generated by $\{\underline{G}_i : i \in I\}$ is the class of split-groups satisfying the split-laws which hold in all $\underline{G}_i$. We denote this split-variety by $\mathrm{svar}(\{\underline{G}_i : i \in I\})$.

(1.5.12) Definition. The join of two split-varieties $\underset{\sim}{S}, \underset{\sim}{T}$ of the same species is the split-variety generated by the set $\{\underline{G}_i : \underline{G}_i \in \underset{\sim}{S}$ or $\underline{G}_i \in \underset{\sim}{T}\}$; the intersection of $\underset{\sim}{S}, \underset{\sim}{T}$ is the class intersection of $\underset{\sim}{S}, \underset{\sim}{T}$. We denote join and intersection by $\underset{\sim}{S} \vee \underset{\sim}{T}$ and $\underset{\sim}{S} \wedge \underset{\sim}{T}$ respectively.

(1.5.13) Theorem. The laws of $\underset{\sim}{S} \vee \underset{\sim}{T}$, $\underset{\sim}{S} \wedge \underset{\sim}{T}$ are $\underline{S} \cap \underline{T}$ and ST $\underline{ST}(=\underline{ST})$ respectively.

Proof. The proof follows easily from the definitions and we omit it.

(1.5.14) Theorem. A split-variety $\underset{\sim}{S}$ is generated by its finitely generated split-groups.

Proof. If $\underset{\sim}{T}$ is the sub-split-variety generated by the finitely generated split-groups of $\underset{\sim}{S}$, let q be a split-law of $\underset{\sim}{T}$, and $\underline{H} \in \underset{\sim}{S}$, $\alpha : \underline{Q}_n \to \underline{H}$. As previously, we may suppose that α acts non-trivially on only finitely many free generators y_{ij}, so that $\underline{Q}_n\alpha \leq \underline{H}$ is finitely generated, and therefore $q\alpha = 1$.

1.6 Examples of split-varieties.

Let $\underset{\sim}{S}$ be a split-variety of species n, and for each i consider the variety of groups $\underline{\underline{V}}_i = \mathrm{var}(\{A_i(\underline{G}) : \underline{G} \in S\})$. It is clear then, that $\underline{G} \in \underset{\sim}{S}$ implies $G \in \underline{\underline{V}}_1\underline{\underline{V}}_2 \dots \underline{\underline{V}}_n$. Conversely suppose that $\underline{\underline{V}}_1,\dots,\underline{\underline{V}}_n$ are varieties of groups, and that $\underline{\underline{W}} \subseteq \underline{\underline{V}}_1\underline{\underline{V}}_2 \dots \underline{\underline{V}}_n$. Consider the class $\underline{\underline{W}}\sigma$ obtained in the following way:

$$\underline{\underline{W}}\sigma = \{(G, A_1,\dots,A_n) : G \in \underline{\underline{W}},\ A_i \in \underline{\underline{V}}_i,\ 1 \leq i \leq n\}.$$

Clearly $\underline{\underline{W}}\sigma$ is a split-variety since it is closed under taking sub-split-groups, quotient split-groups, and cartesian products; note that $\underline{\underline{W}}\sigma$ depends on $\underline{\underline{V}}_1,\dots,\underline{\underline{V}}_n$ as well as on $\underline{\underline{W}}$.

(1.6.1) Definition. Denote by $\underline{\underline{V}}_1 \circ \underline{\underline{V}}_2 \circ \dots \circ \underline{\underline{V}}_n$ the split-variety $(\underline{\underline{V}}_1\underline{\underline{V}}_2 \dots \underline{\underline{V}}_n)\sigma$.

(1.6.2) Theorem. To each split-variety of species n there corresponds a unique 'smallest' product variety $\underline{\underline{V}}_1\underline{\underline{V}}_2 \dots \underline{\underline{V}}_n$ such that $\underline{G} \in \underset{\sim}{S}$ implies $G \in \underline{\underline{V}}_1\underline{\underline{V}}_2 \dots \underline{\underline{V}}_n$.

Conversely, there is a meet-homomorphism σ from the lattice of subvarieties of $\underline{\underline{V}}_1\underline{\underline{V}}_2 \ldots \underline{\underline{V}}_n$ to the lattice of sub-split-varieties of $\underline{\underline{V}}_1 \circ \ldots \circ \underline{\underline{V}}_n$. If $\underline{\underline{W}} \subseteq \underline{\underline{V}}_1\underline{\underline{V}}_2 \ldots \underline{\underline{V}}_n$, then the split-laws of $\underline{\underline{W}}\sigma$ are determined by

$$V_i(Y_i),\quad 1 \le i \le n \quad \text{and} \quad W(Q_n)$$

(where $Y_1 * \ldots * Y_n$ is the carrier of $\underline{Q}_n$). The split-free-group of rank $(m_1,\ldots,m_n)$ of $\underline{\underline{V}}_1 \circ \ldots \circ \underline{\underline{V}}_n$ is carried by the iterated verbal wreath product X_1, defined inductively by

$$X_n = F_{m_n}(\underline{\underline{V}}_n),$$

$$X_i = F_{m_i}(\underline{\underline{V}}_i)\mathrm{wr}_{\underline{\underline{V}}_i}\ X_{i+1}, \qquad 1 \le i \le n-1 ,$$

(where, as in (1.5.7), we choose $m_i = 0$ if $y_{i1} \in V_i(Y_i)$).

Proof. To see that σ is a meet-homomorphism, proceed as follows. Let $\underline{\underline{W}}_1, \underline{\underline{W}}_2$ be subvarieties of $\underline{\underline{V}}_1\underline{\underline{V}}_2 \ldots \underline{\underline{V}}_n$; at once we have

$$(\underline{\underline{W}}_1 \wedge \underline{\underline{W}}_2)\sigma \subseteq \underline{\underline{W}}_1\sigma \wedge \underline{\underline{W}}_2\sigma.$$

For the converse, suppose $\underline{G} \in \underline{\underline{W}}_1\sigma \wedge \underline{\underline{W}}_2\sigma$, and then $G \in \underline{\underline{W}}_1 \wedge \underline{\underline{W}}_2$ so that $\underline{G} \in (\underline{\underline{W}}_1 \wedge \underline{\underline{W}}_2)\sigma$.

Now the split-laws of $\underline{\underline{V}}_1 \circ \dots \circ \underline{\underline{V}}_n$ are determined by $V_i(Y_i)$, $i = 1,\dots,n$ since a split-group $\underline{G}$ belongs to $\underline{\underline{V}}_1 \circ \dots \circ \underline{\underline{V}}_n$ if and only if it has these split-laws. The split-free split-group of rank $\underset{\sim}{m}$ in $\underline{\underline{V}}_1 \circ \dots \circ \underline{\underline{V}}_n$ is, by definition, $\underline{Q}(\underset{\sim}{m})/S(\underline{Q}(\underset{\sim}{m}))$ where

$$S = \mathrm{cl}(\{V_1(Y_1),\dots,V_n(Y_n)\}).$$

If $S_i = V_i(Y_i)$, then $S(\underline{Q}(\underset{\sim}{m}))$ is the normal closure in $\underline{Q}(\underset{\sim}{m})$ of all $S_i(\underline{Q}(\underset{\sim}{m}))$. We construct $F_{\underset{\sim}{m}}(\underline{\underline{V}}_1 \circ \dots \circ \underline{\underline{V}}_n)$ by successively factoring out of $\underline{Q}(\underset{\sim}{m})$, the normal closures of the $S_i(\underline{Q}(\underset{\sim}{m}))$. Write $\underline{Q}(\underset{\sim}{m}) = A_1B_2$ in the usual notation; and at the first stage, since

$$A_1 = \Pi^*\{\overline{Y}_1^{\,b} : b \in B_2\}$$

(where $\overline{Y}_1 * \dots * \overline{Y}_n$ is the carrier of $\underline{Q}(\underset{\sim}{m})$), and since $S_1(\underline{Q}(\underset{\sim}{m}))^{Q(\underset{\sim}{m})} = V_1(A_1)$, we get

$$A_1/V_1(A_1) \cong (\Pi^*\{F_{m_1}(\underline{\underline{V}}_1) : b \in B_2\})/V_1(\Pi^*\{F_{m_1}(\underline{\underline{V}}_1) : b \in B_2\})$$

$$= \underline{\underline{V}}_1\Pi\{F_{m_1}(\underline{\underline{V}}_1) : b \in B_2\}$$

(see 18.22, 18.23, 18.31 in [3]). Hence

$$\underline{Q}(\underset{\sim}{m})/S_1(\underline{Q}(\underset{\sim}{m}))^{Q(\underset{\sim}{m})} = F_{m_1}(\underline{\underline{V}}_1)\mathrm{wr}_{\underline{\underline{V}}_1}(\Pi^*\{\overline{Y}_i : 2 \leq i \leq n\}).$$

Using Theorem 1.4.12 and well known properties of verbal wreath products, we arrive by induction, at the assertion of the theorem.

Finally, introduce the free group X_∞ on the free generating set $\{x_j : j = 1,2,\ldots\}$, and the homomorphism $\nu : X_\infty \to Q_n$ defined by

$$x_j\nu = y_{1j}y_{2j} \ldots y_{nj}, \quad j \in \{1,2,\ldots\}.$$

Then it can be proved by standard tricks that $T = W(\underline{Q}_n)$ is the normalized fully invariant closure of $W\nu$ in Q_n. If $\underline{G} \in \underset{\sim}{T} \wedge \underline{\underline{V}}_1 \circ \ldots \circ \underline{\underline{V}}_n$, $w \in W$ and $\beta : X_\infty \to G$, define $\alpha : \underline{Q}_n \to \underline{G}$ so that

$$y_{ij}\alpha = a_{ij}, \quad 1 \leq i \leq n, \quad j \in \{1,2,\ldots\}$$

where $x_j\beta = a_{1j} \ldots a_{nj}$, $a_{ij} \in A_i(\underline{G})$; then $\nu\alpha = \beta$ and

$$w\beta = (w\nu)\alpha = 1$$

since $w\nu \in \underline{T}$ and $\underline{G} \in \underset{\sim}{T}$. We conclude that $G \in \underline{\underline{W}}$ and therefore that $\underline{G} \in \underline{\underline{W}}\sigma$; hence $\underset{\sim}{T} \wedge \underline{\underline{V}}_1 \circ \ldots \circ \underline{\underline{V}}_n \subseteq \underline{\underline{W}}\sigma$. The opposite direction is proved in a similar manner.

Whether or not σ is a join-homomorphism I have been unable to establish. The mapping σ is in general, neither one-to-one nor onto, as the following examples show.

(1.6.3) Example. The mapping σ is in general not onto.

Consider any product variety $\underline{\underline{UV}}$ and the bivariety

$$\underset{\sim}{B} = \{(G,A,B) : G = A \times B,\ A \in \underline{\underline{U}},\ B \in \underline{\underline{V}}\},$$

and let

$$B^* = \{G \in \underline{\underline{UV}} : G = A \times B,\ A \in \underline{\underline{U}},\ B \in \underline{\underline{V}}\}.$$

Now B^* may not be a variety (if it were then clearly $B^*\sigma = \underset{\sim}{B}$), but in any event σ is onto only if $(\text{var } B^*)\sigma = \underset{\sim}{B}$. We construct here an example where this is not the case. As $\text{var } B^* \supseteq \underline{\underline{U}} \vee \underline{\underline{V}}$ it suffices to produce $\underline{\underline{U}}, \underline{\underline{V}}$ such that there exists $\underline{K} \in \underline{\underline{U}} \circ \underline{\underline{V}}$ with $K \in \underline{\underline{U}} \vee \underline{\underline{V}}$ but $K \notin \underset{\sim}{B}$. Put G_1, G_2, G_3 for the following groups: G_1 non-abelian, exponent 3, order 27; G_2 non-abelian, exponent 9, order 27; G_3 cyclic, order 9; and put $\underline{\underline{U}} = \text{var } G_3$, $\underline{\underline{V}} = \text{var } G_1$. Then it is well-known that $G_2 \in \underline{\underline{U}} \vee \underline{\underline{V}}$, $G_2 \notin \underline{\underline{U}}$, $G_2 \notin \underline{\underline{V}}$. As G_2 is a split-extension of G_3 by a cycle of order 3, it carries a bigroup $\underline{G}_2 \in \underline{\underline{U}} \circ \underline{\underline{V}}$ and therefore $\underline{G}_2 \in (\underline{\underline{U}} \vee \underline{\underline{V}})\sigma$. However, $\underline{G}_2 \notin \underset{\sim}{B}$ for G_2 has no proper direct decomposition (since all proper subgroups of G_2 are abelian while G_2 is not).

(1.6.4) <u>Example</u>. The mapping σ is in general not one-to-one.

Put $\underline{\underline{U}} = \underline{\underline{V}} = \underline{\underline{A}}_{2^t}$, $t > 0$, $\underline{\underline{W}} = \underline{\underline{A}}_{2^u}$, $t \leq u \leq 2t$. Then $\underline{\underline{W}}\sigma = \underline{\underline{U}} \times \underline{\underline{V}}$ (see (1.7.1)), for all such u. Other, much less trivial, examples of both situations will occur later, in Chapter 5.

(1.6.5) <u>Remark</u>. In tackling the descending chain condition on subvarieties of the product varieties $\underline{\underline{UV}}$ it would be sufficient to show that

i) $\underline{\underline{U}} \circ \underline{\underline{V}}$ has descending chain condition on sub-split-varieties

and ii) for each $\underline{\underline{W}} \subseteq \underline{\underline{UV}}$, $(\underline{\underline{W}}\sigma)\sigma^{-1}$ has descending chain condition.

It is in situations like this that split-varieties may prove useful.

1.7 <u>Products of split-varieties.</u>

The last section leads us naturally to ask for a product operation on split-varieties similar to that on varieties of groups. Unfortunately it doesn't seem possible to do this inside the variety of all split-groups of the same species. However we can make the following definition, and this suits our purposes later on.

(1.7.1) <u>Definition</u>. If $\underset{\sim}{S}, \underset{\sim}{T}$ are split-varieties of species m, n respectively, then $\underset{\sim}{S} \circ \underset{\sim}{T}$ is the split-variety of species $m+n$:

$$\underset{\sim}{S} \circ \underset{\sim}{T} = \{\underline{G} : A_1(\underline{G}) \ldots A_m(\underline{G}) \in \underset{\sim}{S},\ A_{m+1}(\underline{G}) \ldots A_{m+n}(\underline{G}) \in \underset{\sim}{T}\}.$$

Also define

$$\underset{\sim}{S} \times \underset{\sim}{T} = \{\underline{G} \in \underset{\sim}{S} \circ \underset{\sim}{T} : G = A_1(\underline{G}) \ldots A_m(\underline{G}) \times A_{m+1}(\underline{G}) \ldots A_{m+n}(\underline{G})\}.$$

That $\underset{\sim}{S} \circ \underset{\sim}{T}$ and $\underset{\sim}{S} \times \underset{\sim}{T}$ are split-varieties follows from their closure under taking sub-split-groups, quotients split-groups and cartesian products of split-groups. The split-laws of $\underset{\sim}{S} \circ \underset{\sim}{T}$ are now described. First some terminology.

If m, n are natural numbers, imagine $\underline{Q}_m, \underline{Q}_n$ embedded in $\underline{Q}_{m+n}$ in the natural way: if $Y_1 * \ldots * Y_{m+n}$ is the carrier of $\underline{Q}_{m+n}$ then $\underline{Q}_m$, for example, is the sub-split-group carried by $Y_1 * \ldots * Y_m$. Define a group endomorphism of Q_{m+n}, say τ, by

$$y_{ij}\tau = y_{i+mj}, \quad 1 \leq i \leq n, \ j \in \{1,2,\ldots\}$$

$$y_{ij}\tau = 1, \quad n+1 \leq i, \ j \in \{1,2,\ldots\},$$

where $\{y_{ij} : j = 1,2,\ldots\}$ freely generates Y_i. With this much convention we can now state

(1.7.2) <u>Theorem</u>. The split-laws of $\underset{\sim}{S} \circ \underset{\sim}{T}$, where $\underset{\sim}{S}, \underset{\sim}{T}$ are of species m, n resepctively, are $\mathrm{cl}(S \cup T\tau) = U$, and $U = S(\underline{Q}_{m+n}).T\tau(\underline{Q}_{m+n})^{Q_{m+n}}$.

Proof. If $\underline{G} \in \underset{\sim}{U}$, $q \in S$ and $\alpha : \underline{Q}_m \to A_1(\underline{G}) \ldots A_m(\underline{G})$ then there exists $\beta : \underline{Q}_{m+n} \to \underline{G}$ such that $\beta|\underline{Q}_m = \alpha$; then

$$q\alpha = q\beta = 1$$

whence $A_1(\underline{G}) \dots A_m(\underline{G}) \in \underset{\sim}{S}$. If $r \in T$, and $\gamma : \underline{Q}_n \to B_m(\underline{G})$ then there exists $\delta : \underline{Q}_{m+n} \to \underline{G}$ such that $\tau\delta = \gamma$, and so

$$r\gamma = (r\tau)\delta = 1$$

since $r\tau \in T\tau$. We have shown, therefore, that $\underline{G} \in \underset{\sim}{S} \circ \underset{\sim}{T}$. A similar proofdeals with the opposite direction and thus $\underset{\sim}{U} = \underset{\sim}{S} \circ \underset{\sim}{T}$.

To conclude the proof, note that $\mathrm{cl}(S \cup T\tau) = (\mathrm{cl}S)(\mathrm{cl}T\tau)$. Now observe that it is immaterial whether we regard S as being of species $m+n$, and calculate $S(\underline{Q}_{m+n})$, or of species m, and calculate $S(A_1(\underline{Q}_{m+n}) \dots A_m(\underline{Q}_{m+n}))$ we get the same result in either case. Moreover $S(\underline{Q}_{m+n})$ is normal in $\underline{Q}_{m+n}$: it is certainly normal in $A_1(\underline{Q}_{m+n}) \dots A_m(\underline{Q}_{m+n})$, and if $a_i \in A_i(\underline{Q}_{m+n})$, $m + 1 \leq i \leq m + n$, then a_i induces a self-morphism of $A_1(\underline{Q}_{m+n}) \dots A_m(\underline{Q}_{m+n})$ which is therefore admitted by $S(\underline{Q}_{m+n})$. Also $\mathrm{cl}T\tau = T(\underline{Q}_{m+n})^{\underline{Q}_{m+n}}$. This finishes the proof.

(1.7.3) <u>Theorem</u>. $\underset{\sim}{S}_1 \subseteq \underset{\sim}{S}_2$ if and only if $\underset{\sim}{S}_1 \circ \underset{\sim}{T} \subseteq \underset{\sim}{S}_2 \circ \underset{\sim}{T}$ and $\underset{\sim}{S} \circ \underset{\sim}{T}_1 \subseteq \underset{\sim}{S} \circ \underset{\sim}{T}_2$ if and only if $\underset{\sim}{T}_1 \subseteq \underset{\sim}{T}_2$. The product 'o' is associative.

The proof of this is completely trivial and we omit it. The product we have defined is very similar in its properties, to the product defined for varieties of groups. We note one other result in this direction.

(1.7.4) <u>Theorem</u>. i) $(\underset{\sim}{S}_1 \vee \underset{\sim}{S}_2) \circ \underset{\sim}{T} = \underset{\sim}{S}_1 \circ \underset{\sim}{T} \vee \underset{\sim}{S}_2 \circ \underset{\sim}{T}$;

ii) $(\underset{\sim}{S}_1 \wedge \underset{\sim}{S}_2) \circ \underset{\sim}{T} = \underset{\sim}{S}_1 \circ \underset{\sim}{T} \wedge \underset{\sim}{S}_2 \circ \underset{\sim}{T}$;

iii) $\underset{\sim}{S} \circ (\underset{\sim}{T}_1 \wedge \underset{\sim}{T}_2) = \underset{\sim}{S} \circ \underset{\sim}{T}_1 \wedge \underset{\sim}{S} \circ \underset{\sim}{T}_2$.

By contrast iv) $\underset{\sim}{S} \circ (\underset{\sim}{T}_1 \vee \underset{\sim}{T}_2) \supseteq \underset{\sim}{S} \circ \underset{\sim}{T}_1 \vee \underset{\sim}{S} \circ \underset{\sim}{T}_2$,

and the inclusion may be proper.

Proof. If $\underset{\sim}{T}_o$ is the variety of all split-groups of species n, the first assertion for $\underset{\sim}{T} = \underset{\sim}{T}_o$ is equivalent to

$$(S_1 \cap S_2)(\underline{Q}_{m+n}) = S_1(\underline{Q}_{m+n}) \cap S_2(\underline{Q}_{m+n}).$$

Write $\underline{Q} = \underline{Q}_{m+n}$. Then, as noted in the proof of (1.7.2), $S_i(\underline{Q}) = S_1(A_1(\underline{Q}) \ldots A_m(\underline{Q}))$. We show that $A_1(\underline{Q}) \ldots A_m(\underline{Q})$ is isomorphic quâ split-group, to $\underline{Q}_m$. For, put

$$\overline{Y}_i = \Pi^*\{Y_i^b : b \in B_{m+1}(\underline{Q})\}, \qquad 1 \leq i \leq m,$$

and it is easy to verify, that $A_1(\underline{Q}) \ldots A_m(\underline{Q}) = \overline{Y}_1 * \ldots * \overline{Y}_m$, and that $A_i(\underline{Q})$ is the normal closure of $\overline{Y}_i$ in $\overline{Y}_i, \ldots, \overline{Y}_m$. Hence (i) is true for $\underset{\sim}{T} = \underset{\sim}{T}_o$.

If $\underline{G}$, $\underline{G}_1$, $\underline{G}_2$ are the split-free split-groups of $(\underset{\sim}{S}_1 \vee \underset{\sim}{S}_2) \circ \underset{\sim}{T}_o$, $\underset{\sim}{S}_1 \circ \underset{\sim}{T}_o$, $\underset{\sim}{S}_2 \circ \underset{\sim}{T}_o$ respectively, then $\underline{G}$ can be embedded in $\underline{G}_1 \times \underline{G}_2$ according to the monomorphism $\mu : \underline{G} \to \underline{G}_1 \times \underline{G}_2$ defined by

$$q(S_1(\underline{Q}) \cap S_2(\underline{Q}))\mu = (qS_1(\underline{Q}),\ qS_2(\underline{Q})),\quad q \in \underline{Q}.$$

Now $T\tau(\underline{G}_1 \times \underline{G}_2)^{G_1 \times G_2} \cap \underline{G}\mu = T\tau(\underline{G})^{G}\mu$; for, if $(xS_1(\underline{Q}), yS_2(\underline{Q})) \in T\tau(\underline{G}_1 \times \underline{G}_2)^{G_1 \times G_2} \cap \underline{G}\mu$ then there exists $q \in \underline{Q}$ such that $qS_1 = xS_1$, $qS_2 = yS_2$, and with $q \in T\tau(\underline{Q})^{Q}$. Hence $q(S_1(\underline{Q}) \cap S_2(\underline{Q})) \in T\tau(\underline{G})^{G}$ whence $(xS_1(\underline{Q}), yS_2(\underline{Q})) \in T\tau(\underline{G})^{G}\mu$. It follows that $\underline{G}/T\tau(\underline{G})^{G}$ is embedded monomorphically in $\underline{G}_1/T\tau(\underline{G}_1)^{G_1} \times \underline{G}_2/T\tau(\underline{G}_2^{G_2}$. This shows that

$$(\underset{\sim}{S}_1 \vee \underset{\sim}{S}_2) \circ \underset{\sim}{T} \subseteq \underset{\sim}{S}_1 \circ \underset{\sim}{T} \vee \underset{\sim}{S}_2 \circ \underset{\sim}{T},$$

and as the opposite inclusion is trivial, this completes the proof of (i). The rest are easy: the only non-trivial thing is to show that the inclusion (iv) may be proper. In fact the familiar example which establishes this for products of varieties of groups can be interpreted to settle this (21.25 in [3]):-

Any bigroup in $\underline{\underline{A}} \circ \underline{\underline{A}}_2 \vee \underline{\underline{A}} \circ \underline{\underline{A}}_3$ has the bilaw

(1.7.5) $\quad [y_{11}, y_{21}, y_{22}^2, y_{23}^3]$.

Consider the bigroup $\underline{G} \in \underline{\underline{A}} \circ \underline{\underline{A}}_6$ defined as a 7-cycle $A_1 = \langle a \rangle$ split by its automorphism of order 6, $\langle b \rangle = A_2$ say, with $\underline{G} = (A_1A_2, A_1, A_2)$. Now A_2 is represented fixed point free on A_1, and so

$$[a, b, b^2, b^3] \neq 1,$$

showing that the split-word (1.7.5) is not a split-law of $\underline{G}$. This completes the proof of the theorem.

Note that Definition 1.6.1 is in accord with our definition of produtt, provided that we interpret a variety of groups as a variety of split-groups of species 1.

CHAPTER 2.

MISCELLANEOUS RESULTS

In this brief chapter we record some general results about split-varieties, results related to the lattice of split-varieties, and then introduce the bivarieties with which the remainder of this thesis is principally concerned.

2.1 Lattices of split-varieties.

(2.1.1) Theorem. The split-varieties of the same species n form a modular lattice with respect to (the inclusion order and) the join and intersection defined in (1.5.15). 12

Proof. By virtue of (1.5.4) and (1.5.13) it is sufficient to show that the normal, fully invariant sub-split-groups of $\underline{Q}_n$ form a modular lattice with respect to the inclusion order. This is clear, since if $\underline{S}, \underline{T}$ are normal and fully invariant in $\underline{Q}_n$, $\underline{S} \cap \underline{T}$ and $\underline{ST}$ are also, and therefore the normal, fully invariant sub-split-groups form a sublattice of the modular lattice of the normal subgroups of $\underline{Q}_n$.

Because of this modularity, many results which are essentially lattice-theoretic can be taken over to our situation: all here are quoted without proof. The first is well-known, particularly as a statement about varieties of groups.

(2.1.2) Theorem. If $\underset{\sim}{S}$ is a split-variety which has a finite basis for its split-laws, then every sub-split-variety of $\underset{\sim}{S}$ has a finite basis if and only if every descending chain of sub-split-varieties of $\underset{\sim}{S}$ breaks off.

Of course if there existed an infinite descending chain, $\underline{\underline{V}}_1 \supset \underline{\underline{V}}_2 \supset \dots$ say, of varieties of groups, then we could trivially construct an infinite descending chain of split-varieties of arbitrary species ($\underline{\underline{V}}_1 \circ \underset{\sim}{S} \supset \underline{\underline{V}}_2 \circ \underset{\sim}{S} \supset \dots$ where $\underset{\sim}{S}$ is any split-variety).

The second result noted here I first proved for varieties of groups (see 16.25 in [3]). It is however a much older result about modular lattices, due to Pickert [22].

(2.1.3) Theorem. If $\underset{\sim}{S}, \underset{\sim}{T}$ are split-varieties of the same species, each of which has descending chain condition on sub-split-varieties, then $\underset{\sim}{S} \vee \underset{\sim}{T}$ does also.

By entirely similar methods one also proves

(2.1.4) Theorem. A split-variety $\underset{\sim}{S}$ has descending chain condition on sub-split-varieties if and only if there exists

$\underset{\sim}{S}_o \subseteq \underset{\sim}{S}$ such that $\underset{\sim}{S}_o$ has descending chain condition on sub-split-varieties, and also all descending chains between $\underset{\sim}{S}$ and $\underset{\sim}{S}_o$ break off.

2.2 The bivariety $\underline{\underline{A}} \circ \underline{\underline{A}}$

From now on we will almost exclusively be concerned with varieties of bigroups (bivarieties), mostly, indeed, with subvarieties of $\underline{\underline{A}} \circ \underline{\underline{A}}$. It is convenient to modify our notation to suit this situation. Thus we shall drop double subscripts and write $Y_m{*}Z_n$ for the carrier of the absolutely split-free bigroup of rank (m,n), with split-free generating set $\{y_i : i \in I, |I| = m\}$, $\{z_j : j \in J, |J| = n\}$.

We now restate several results for the case of bivarieties, all of them special cases of Theorem 1.4.4.

(2.2.1) Theorem. If q is a biword, then q is equivalent to a set $U_o \cup V_o \cup S$ of special biwords, where U_o, V_o are contained in Y_ω, Z_ω respectively, and where each element of S is a product of powers of commutators, each of which involves at least one y_i and at least one z_j (and the entries of each are powers of the y_i and z_j). Moreover if $\underline{\underline{V}}_1, \underline{\underline{V}}_2$ are the varieties of groups determined by the laws U_o, V_o respectively, then $\underline{\underline{V}}_1 \circ \underline{\underline{V}}_2$ is the bivariety corresponding to the bivariety determined by q, by Theorem 1.6.2.

(2.2.2) Corollary. Each sub-bivariety of $\underline{\underline{A}} \circ \underline{\underline{V}}$ is determined by the bilaws of $\underline{\underline{A}} \circ \underline{\underline{V}}$ together with a set $\{y_1^m\} \cup \overline{V} \cup S$ of special biwords where $m \geq 0$, $\overline{V} \subseteq Z_\omega - V$, and each element of S is a product of powers of commutators of the type $[y_1,w_1,\dots,w_r]$, with each w_k a commutator whose entries are powers of the z_j but which does not lie in $\mathrm{cl}(V \cup \overline{V})$.

Proof. If $\underset{\sim}{T} \subseteq \underline{\underline{A}} \circ \underline{\underline{V}}$, then by (1.6.2), $\underset{\sim}{T} \subseteq \underline{\underline{A}}_m \circ \underline{\underline{V}}'$, $m \geq 0$, $\underline{\underline{V}}' \subseteq \underline{\underline{V}}$; and if this m is chosen minimal, $\{y_1^m,[y_1,y_2]\}$ is a basis for the laws of all $A_1(\underline{G})$, $\underline{G} \in \underset{\sim}{T}$, as noted in (2.2.1). If $\underline{\underline{V}}'$ is chosen minimal, then write $\overline{V}$ for a set of special biwords which determine V' modulo V.

By (2.2.1) we are left with considering 'genuine' commutator biwords in $\underline{T}$, call one t say. Then t is a product of powers of commutators whose entries are powers of the y_i and z_j. We may assume that each commutator in this product involves only y_1 raised to a power, and no other y_i's, since $[y_1,y_2]$ is a bilaw in $\underline{\underline{A}} \circ \underline{\underline{V}}$. This power of y_1 may be moved to the front of each commutator so that we have t expressed as a product of powers of commutators of the form

$$[y_1^\alpha,w_1,\dots,w_r] = [y_1,w_1,\dots,w_r]^\alpha,$$

as required.

(2.2.3) <u>Corollary</u>. Every sub-bivariety of $\underline{\underline{A}} \circ \underline{\underline{A}}$ is determined by the bilaws of $\underline{\underline{A}} \circ \underline{\underline{A}}$ together with a set $\{y_1^m, z_1^n\} \cup S$ of special bilaws, where $m,n \geq 0$ and where every element $s \in S$ is a product of powers of commutators of the type.

$$[y_1, z_1^{\lambda_1}, \ldots, z_r^{\lambda_r}]$$

where r depends only on s, and $\lambda_1, \ldots, \lambda_r$ are all non-zero, and $\lambda_j < n$ if $n \neq 0$, $j \in \{1, \ldots, r\}$.

Proof. From (2.2.2) we have that every element $s \in S$ can be written as a product of powers of commutators of the type $[y_1, z_{i_1}^{\alpha_1}, \ldots, z_{i_u}^{\alpha_u}]$ where $\alpha_i \neq 0$ and where $\{i_1, \ldots, i_u\} = \{1, \ldots, r\}$. If, for example, $i_1 = i_2$ then since

$$[y_1, z_{i_1}^{\alpha_1+\alpha_2}][y_1, z_{i_1}^{\alpha_1}]^{-1}[y_1, z_{i_2}^{\alpha_2}]^{-1} = [y_1, z_{i_1}^{\alpha_1}, z_{i_2}^{\alpha_2}],$$

we may replace this product by one of the desired type. That the z's can be rearranged into increasing order of their subscripts follows since, modulo the bilaws of $\underline{\underline{A}} \circ \underline{\underline{A}}$, y_1 is in the centralizer of the derived group of a metabelian group.

(2.2.4) <u>Corollary</u>. Every sub-bivariety of $\underline{\underline{A}} \circ \underline{\underline{A}}$ is determined by the bilaws of $\underline{\underline{A}} \circ \underline{\underline{A}}$ together with a set $\{y_1^m, z_1^n\} \cup T$ of special bilaws, where $m,n \geq 0$ and where every element of T is a product of powers of commutators of the type

$$[y_1, \mu_1 z_1^{\varepsilon_1}, \ldots, \mu_r z_r^{\varepsilon_r}]$$

with $\mu_1, \ldots, \mu_r$ natural numbers and $\varepsilon_1, \ldots, \varepsilon_r = \pm 1$; moreover, if $n > 0$ then $\mu_i < n$ and $\varepsilon_i = 1$, $i \in \{1, \ldots, r\}$.

Proof. Use (2.2.3) and the commutator identity

$$[x, y^{\nu}] = \prod_{\mu=1}^{\nu} [x, \mu y]^{\binom{\nu}{\mu}} .$$

Finally, in this chapter, a result of a completely different character. Note that the bivariety $\underline{\underline{A}} \circ \underline{\underline{A}}$ consist of bigroups which are metabelian quâ groups. One of the nice features of such groups, from a varietal standpoint, is that finitely generated ones are residually finite ([8]), and therefore every subvariety of $\underline{\underline{AA}}$ is generated by finite groups. We implicitly adapt this very deep result of Philip Hall to our situation, in the next theorem.

(2.2.5) Theorem. A bigroup $\underline{G}$ is residually finite quâ bigroup if G is residually finite. Consequently every sub-bivariety of $\underline{\underline{A}} \circ \underline{\underline{A}}$ is generated by finite bigroups.

Proof. Let $1 \neq g \in G$. There exists a normal subgroup N of G with $g \notin N$ and $|G:N|$ finite. Write

$$A_1(\underline{G}) \cap N = A_1^*, \quad A_2(\underline{G}) \cap N = A_2^* ,$$

and then $|A_1(\underline{G}) : A_1^*| = |A_1(\underline{G})N:N|$ and $|A_2(\underline{G}):A_2^*| = |A_2(\underline{G})N:N|$ are both finite. Hence

$$|G : A_1^*A_2^*| \leq |A_1(\underline{G}) : A_1^*|.|A_2(\underline{G}) : A_2^*|$$

is finite. Finally put $N^* = (A_1^*A_2^*)^G$, and then $A_1^*A_2^* \leq N^* \leq N$ so that N^* is normal, of finite index and avoids g, and it carries a sub-bigroup of $\underline{G}$, so we are home.

CHAPTER 3.

CRITICAL BIGROUPS IN $\underline{\underline{A}} \circ \underline{\underline{A}}$

In this chapter we define critical split-groups by analogy with critical groups, deduce some elementary facts about them, and then turn our attention to the structure of certain critical bigroups in $\underline{\underline{A}} \circ \underline{\underline{A}}$.

3.1 Critical split-groups

(3.1.1) *Definition*. A finite split-group is *critical* if it is not in the split-variety generated by its proper sub-split-groups and proper quotient split-groups.

Clearly we have

(3.1.2) *Theorem*. If $\underline{G}$ is a split-group and G is a critical group, then $\underline{G}$ is critical.

(3.1.3) *Theorem*. A critical split-group $\underline{G}$ has a unique minimal normal sub-split-group.

Proof. If not, then there exist non-trivial normal sub-split-groups $\underline{N}_1, \underline{N}_2$ of $\underline{G}$ with $\underline{N}_1 \cap \underline{N}_2 = 1$; and then $\underline{G}$ can be embedded in $\underline{G}/\underline{N}_1 \times \underline{G}/\underline{N}_2$ in the usual way.

An example of the situation in Theorem 3.1.2 occurs with $G = S_3$, the symmetric group of permutations on three letters, $A_1(\underline{G})$ the normal 3-cycle and $A_2(\underline{G})$ any 2-cycle. However the converse of (3.1.2) is not true: a critical split-group need not be a critical group. An example of this is the bigroup $\underline{G}$ carried by the wreath product $G = C_p \mathrm{wr}(C_p \times C_p)$ in the natural way: $A_1(\underline{G})$ is the base group of G and $A_2(\underline{G}) \cong C_p \times C_p$.

Clearly a split-group which is monolithic as a group has a unique, minimal normal sub-split-group. In certain cases the converse is true:

(3.1.4) Lemma. If $\underline{G}$ is a bigroup which has a unique minimal normal sub-bigroup, and $A_1(\underline{G})$ is abelian, then G is monolithic.

Proof. Suppose that $1 \neq N$ is a normal subgroup of G. If $N \cap A_1(\underline{G}) > 1$ then we are finished since $N \cap A_1(\underline{G})$ carries a normal sub-bigroup of $\underline{G}$. Hence suppose that $N \cap A_1(\underline{G}) = 1$; then as $A_1(\underline{G}) \lhd G$ we have that $N \leq C_G(A_1(\underline{G}))$ and therefore that $C_G(A_1(\underline{G})) > A_1(\underline{G})$. It follows that $1 < C_G(A_1(\underline{G})) \cap A_2(\underline{G}) \lhd G$. Hence we have a contradiction unless $A_1(\underline{G}) = 1$, in which case the theorem is trivially true.

In the bivariety $\underline{\underline{A}} \circ \underline{\underline{A}}$ the conditions of (3.1.4) are certainly satisfied. In such cases we shall use 'monolithic' for brevity, and denote the monolith of $\underline{G}$ by $\sigma\underline{G}$. Note that the carrier of $\sigma\underline{G}$ is σG.

(3.1.5) Lemma. If a split-variety $\underset{\sim}{S}$ is generated by finite split-groups then it is generated by critical split-groups.

Proof. Let $\underset{\sim}{S}_o$ be the sub-split-variety of $\underset{\sim}{S}$ generated by the critical split-groups in $\underset{\sim}{S}$. If $\underset{\sim}{S}_o \subset \underset{\sim}{S}$, then there exists a finite $\underline{G} \in \underset{\sim}{S} - \underset{\sim}{S}_o$, which we may suppose to have minimal order. Every proper sub-split-group and every proper quotient split-group of $\underline{G}$ then lies in $\underset{\sim}{S}_o$, but $\underline{G}$ does not. This means that $\underline{G}$ is critical. We have thus produced a contradiction and hence $\underset{\sim}{S}_o = \underset{\sim}{S}$.

(3.1.6) Lemma (cf. Theorem 4 in [9]). If $\underline{G}$ is a critical bigroup and $A_1(\underline{G})$ is abelian, then $A_1(\underline{G})$ contains a unique maximal normal subgroup of G.

Proof. If N_1, N_2 are maximal normal subgroups of G in $A_1(\underline{G})$, then N_1A_2, N_2A_2 carry sub-bigroups of $\underline{G}$ (writing $A_i = A_i(\underline{G})$, $i = 1,2$). We shall show that $\underline{G} \in \mathrm{svar}\{N_1,A_2, N_2A_2\}$. Suppose that q is a bilaw in both N_1A_2 and N_2A_2. Since $N_1N_2 = A_1$ and since $A_2(N_1A_2) = A_2$, we may suppose, by virtue of (2.2.2), that q is a product of commutators of the form

$$[y_1, w_1, \ldots, w_t]^{\pm 1}$$

for some words $w_1, \ldots, w_t \in A_2(\underline{Q}_2)$. Let $\alpha : \underline{Q}_2 \to \underline{G}$ be an arbitrary morphism. We write $y_1\alpha = a_1a_2$, $a_1 \in N_1$, $a_2 \in N_2$ (not necessarily uniquely). Define $\alpha_j : \underline{Q}_2 \to N_j A_2$, $j = 1,2$, by

$$y_1\alpha_j = a_j, \quad z_i\alpha_j = z_i\alpha, \quad j = 1,2, \quad i \in \{1,2,\dots\}.$$

Then $[y_1, w_1, \dots, w_t]\alpha = [y_1\alpha, w_1\alpha, \dots, w_t\alpha] = [y_1\alpha_1, w_1\alpha_1, \dots, w_t\alpha_1]$. $[y_1\alpha_2, w_1\alpha_2, \dots, w_t\alpha_2]$. Hence $q\alpha = (q\alpha_1)(q\alpha_2) = 1$, showing that q is a bilaw in $\underline{G}$. This completes the proof.

Finally in this section an analogue of the well-known fact that critical groups which are nilpotent, are p-groups.

(3.1.7) *Theorem*. If $\underline{G}$ is a finite monolithic split-group and G is nilpotent, then for some prime p, G is a p-group.

Proof. If G is nilpotent and finite, its Sylow subgroups are fully invariant, hence carry normal sub-split-groups whose pair-wise intersections are trivial, so $\underline{G}$ cannot be monolithic unless G has only one Sylow subgroup.

Note that 'nilpotent' as used here is a concept related to varieties of groups. As previously, we may give it a split-varietal flavour, if that is thought necessary, by saying that a split-group of species n is nilpotent if it has the split-law.

$$[y_{11}y_{21} \cdots y_{n1}, y_{12}y_{22} \cdots y_{n2}, \dots, y_{1c}y_{2c} \cdots y_{nc}]$$

for some natural number c.

3.2 Non-nilpotent critical bigroups in $\underline{\underline{A}} \circ \underline{\underline{A}}$.

Throughout the remainder of this chapter $\underline{G} = (G,A,B)$ will be a critical, non-nilpotent bigroup contained in $\underline{\underline{A}} \circ \underline{\underline{A}}$; the notation introduced in Theorem 3.2.1 will also be carried through.

(3.2.1) Theorem. If $\underline{G} = (G,A,B) \in \underline{\underline{A}} \circ \underline{\underline{A}}$ is critical and not nilpotent, then

i) A is a p-group, for some prime p, it is self-centralizing in G, and is the derived group $G_{(2)} = G'$ of G.

If $B = H \times K$ where H is the Sylow p-subgroup of B, then

ii) $F = AH$ is the centralizer of the monolith σG of G, and F is the Fitting subgroup of G;

iii) K is a p'-cycle which acts faithfully and irreducibly on σG.

Moreover

iv) Every non-trivial element of K acts fixed point free on A,

and

v) K acts faithfully and irreducibly on A/N

where

vi) $N = A^p[A,H]$ is the unique maximal G-normal subgroup of A.

Proof. Since $\underline{G}$ is critical it has a unique minimal normal sub-bigroup $\sigma\underline{G}$ whose carrier, by Lemma 3.1.4, is the monolith σG of G.

If A were not a p-group, we could write it as a direct product of Sylow subgroups, each of which, being characteristic in A would be normal in G, contradicting the monolithicity of G; hence A is a p-group for some prime p. If A were not self-centralizing, then $A < C_G(A)$ would imply $1 < C_G(A) \cap B \lhd G$, again contradicting the monolithicity of G.

Since B is abelian, $G' \leq A$; and since G is not nilpotent, there exists an integer t such that

$$1 \neq G_{(t)} = G_{(t+1)} = \dots \leq A.$$

By a result of Schenkman [1], G splits over $G_{(t)}$, say

$$G = G_{(t)} . B_0, \quad G_{(t)} \cap B_0 = 1.$$

Therefore $A = G_{(t)} . (A \cap B_0)$; but $A \cap B_0$ is normal in B_0 since A is normal in G, $A \cap B_0$ is normal in A since A is abelian; hence $A \cap B_0$ is normal in G, and so $A \cap B_0 = 1$ because G is monolithic and $A \cap B_0$ avoids $G_{(t)}$. That is,

$$A \leq G_{(t)} \leq G' \leq A,$$

or $G' = A$. This disposes of (i).

We can describe σG more exactly: if F has class c precisely, and if $F_{(c)}$ has exponent p^r, then

$$(3.2.2) \qquad \sigma G = F_{(c)}^{p^{r-1}} = \{z \in Z(F) : z^p = 1\}.$$

For, $1 \neq F_{(c)}^{p^{r-1}}$ is characteristic in F and therefore normal in G, so $\sigma G \leq F_{(c)}^{p^{r-1}}$. If this inclusion were proper then, by Maschke's Theorem, σG would have a non-trivial, K-admissible complement in $F_{(c)}^{p^{r-1}}$ which, being in the centre of F, would be normal in G, a contradiction. A similar argument proves the remainder of (3.2.2).

The same argument can be used to prove that K acts irreducibly on σG. We shall now show not just that K acts faithfully on σG, but that every non-trivial element of K acts fixed point free on A. To this end suppose that there exists $1 \neq k \in K$ and $1 \neq x \in A$ such that

$$x^k = x.$$

If we write

$$\bar{A} = \{a \in A : a^k = a\},$$

then $\bar{A}$ is a non-trivial normal subgroup of G in A and, by a well-known result of representation theory (for example, Lemma, p.455 in [2]), $\bar{A}$ has a B-admissible complement $\bar{\bar{A}}$ in A. But then $\bar{\bar{A}}$ is normal in G

since A is abelian, and therefore $\bar{\bar{A}} = 1$ since G is monolithic; that is $\bar{A} = A$. In this case $\langle k \rangle$ is central in G, contradicting the existence of a monolith in G. It follows that, if $1 \neq k \in K$, then k fixes no non-trivial element of A. Thus F is the centralizer of σG, K acts faithfully (and irreducibly) on σG and so K is cyclic, and F is the Fitting subgroup of G. This completes the proof of (ii), (iii), (iv).

By Lemma 3.1.6 there exists a unique maximal normal subgroup of G contained in A; call it N. Hence $A^p[A,H] \leq N$ since both A^p and $[A,H] = F'$ are proper subgroups of A and both are normal in G. If the inclusion is proper, then $N/A^p[A,H]$ has a non-trivial K-admissible complement $T/A^p[A,H]$ say, in $A/A^p[A,H]$. But then T is normal in G and T is not contained in N, a contradiction to 3.1.6.

To finish the proof of the theorem we have to show that K acts faithfully on A/N, and to do this we use the following lemma which will be useful later on as well.

(3.2.3) Lemma. If $\underline{G} = (G,A,B)$ is as in (3.2.1) and $1 \neq k_0 \in K$, then the mapping $\alpha : A \to A$ defined by

$$a\alpha = [a,k_0]$$

is an automorphism of A which extends to an automorphism of G.

Proof. Define α on the whole of G by

$$(ba)\alpha = b[a,k_0], \quad b \in B, \quad a \in A.$$

This is an endomorphism since

$$((b_1a_1)(b_2a_2))\alpha = (b_1b_2a_1^{b_2}a_2)\alpha = b_1b_2[a_1^{b_2}a_2,k_0]$$

$$= b_1b_2[a_1^{b_2},k_0][a_2,k_0] = b_1b_2[a_1,k_0]^{b_2}[a_2,k_0]$$

$$= b_1[a_1,k_0].b_2[a_2,k_0] = (b_1a_1)\alpha.(b_2a_2)\alpha;$$

and α is an automorphism since G is finite and $b[a,k_0] = 1$ implies $b = 1$ and $[a,k_0] = 1$, which from (iv), gives $a = 1$.

Finally note that if $a \in A$, $k_0 \in K$ and $[a,k_0] \in N$, then, since N is characteristic in G, N admits the inverse of the automorphism α corresponding to k_0 in (3.2.3); that is

$$a = [a,k_0]\alpha^{-1} \in N.$$

Hence K acts faithfully on A/N. The proof of Theorem 3.2.1 is now complete.

The following two lemmas are important in the proof of the crucial Theorem 3.4.4 below.

(3.2.4) Lemma. If $\underline{G} = (G,A,B)$ is as in (3.2.1) with $|K| = r$, and $a_o,\ldots,a_{r-1} \in A$ such that, for all $k \in K$

$$(3.2.5) \qquad \prod_{i=o}^{r-1} [a_i, ik] = 1,$$

then $a_o = \ldots = a_{r-1} = 1$.

Proof. Put $k = 1$ and then $a_o = 1$; we may suppose, therefore, that the product is over the range $1 \leq i \leq r-1$. Let $K = \langle k_o \rangle$. Substitute k_o^j, $1 \leq j \leq r-1$, for k in (3.2.5) in turn, and, using the terminology of (3.2.3) with α_j corresponding to k_o^j, we get

$$\prod_{i=1}^{r-1} a_i \alpha_j^i = 1, \qquad 1 \leq j \leq r-1 .$$

Working in the endomorphism ring of A and utilizing the fact that $\alpha_i\alpha_j = \alpha_j\alpha_i$, $1 \leq i,j \leq r-1$, we deduce that

$$a_t \det(\alpha_j^i) = 1, \qquad 1 \leq t \leq r-1.$$

Now $\det(\alpha_j^i)$ is the van der Monde determinant, and

$$\det(\alpha_j^i) = (\prod_{t=1}^{r-1} \alpha_t).(\prod_{u<v} (\alpha_u - \alpha_v));$$

each α_t is an automorphism of A, and $\det(\alpha_j^i)$ will be an automorphism of A if we can show that for $u < v$, $\alpha_u - \alpha_v$ is an automorphism of A: for $a \in A$,

$$a(\alpha_u-\alpha_v) = (a\alpha_u)(a\alpha_v)^{-1} = [a,k_o^u][a,k_o^v]^{-1}$$

$$= a^{-1}a^{k_o^u}.a^{-k_o^v}a = (a^{-1}a^{k_o^{v-u}})^{-k_o^u} = [a,k_o^{v-u}]^{-k_o^u}$$

and therefore $a(\alpha_u-\alpha_v) = 1$ implies $a = 1$ by (3.2.1) (iv). Hence $a_1 = \dots = a_{r-1} = 1$ as asserted.

(3.2.6) Lemma. Let $\underset{\sim}{G} = (G,A,B)$ be as in (3.2.1) and $|K| = r$. If to each s-tuple $\underset{\sim}{\mu} = (\mu_1,\dots,\mu_s)$, where $0 \leq \mu_i \leq r-1$, $i \in \{1,\dots,s\}$ there is an element $a(\underset{\sim}{\mu})$ of A such that for all $k_1,\dots,k_s \in K$,

$$\prod_{\underset{\sim}{\mu}}[a(\underset{\sim}{\mu}),\mu_1k_1,\dots,\mu_sk_s] = 1,$$

then $a(\underset{\sim}{\mu}) = 1$ for all $\underset{\sim}{\mu}$.

Proof. For each $\nu \in \{0,\dots,r-1\}$ write

$$a_\nu = \prod_{\nu=\mu_s}[a(\underset{\sim}{\mu}),\mu_1k_1,\dots,\mu_{s-1}k_{s-1}];$$

then

$$\prod_{\nu=0}^{r-1}[a_\nu,\nu k_s] = 1$$

for all $k_s \in K$. Hence by (3.2.4), $a_o = \dots = a_{r-1} = 1$. We may now use induction to complete the proof.

3.3 The criticality of G.

We aim to show in this section, that if $\underline{G}$ is as in (3.2.1), then G is a critical group. By Lemma 3.1.4 and (1.2) of [5] it suffices to show that G is not contained in the variety generated by its proper subgroups. To this end we calculate the maximal subgroups of G.

(3.3.1) Lemma. If M is a maximal subgroup of G then either

a) $M = AHK_0$, where K_0 is maximal in K;

b) $M = AH_0K$, where H_0 is maximal in H,

or c) $M \cap F = NH$.

Proof. Suppose that, as in (1.2.8), σ_1 is the retraction of G to B. Then if $M\sigma_1 < B$ we must have $A \leq M$; for, if $A \nleq M$, $AM = G$ and therefore

$$B = G\sigma_1 = (AM)\sigma_1 = M\sigma_1.$$

Hence $M = A(M \cap B)$ and clearly $M \cap B$ must be maximal in B; that is, M has the form (a) or the form (b).

Assume, therefore, that $M\sigma_1 = B$; then $M \cap F = NH$. For, if $N \nleq M$, $G = NM$ and if $a \in A-N$,

$$(3.3.2) \qquad a = xm, \quad x \in N, \quad m \in M$$

and so $x^{-1}a = m \in (A-N) \cap M$. By virtue of (3.2.1)(vi), A is generated quâ B-operator group by any element of $A-N$, and since $M\sigma_1 = B$ and A is abelian,

$$A = \langle m \rangle^B = \langle m \rangle^M \leq M.$$

In other words, $M = G$; hence $N \leq M$. To finish off this case we show that if $a \in A-N$ and $h \in H$, then $ha \notin M$. For, if $1 \neq k \in K$, there exists $a' \in A$ such that $ka' \in M$; and if $ha \in M$,

$$\begin{aligned}[ka',ha] &= [ka',a][ka',h][ka',h,a] \\ &= [k,a][a',h]\end{aligned}$$

belongs to M whence, as $[A,H] \leq N \leq M$, $[k,a]^{-1} = [a,k] \in M$. From (3.2.1)(v), $[a,k] \in (A-N) \cap M$, and an argument similar to that which disposed of (3.3.2) shows that $M = G$. Hence $ha \notin M$. It follows at once that $M\sigma_1 = B$ implies

$$M \cap F = NH,$$

as required in (c).

Note that not all the maximal subgroups of G are sub-bigroups. The ones which are not are those with $M \cap F = NH$ and $ka \in M$, $a \in A-N$, $k \in K$; in these cases, $M = \langle NH, ka \rangle$. A similar argument to the foregoing yields

(3.3.3) Lemma. The maximal sub-bigroups of $\underline{G}$ are precisely FK_0, AH_0K and NHK, where H_0 is maximal in H, K_0 is maximal in K.

We are now ready to prove

(3.3.4) Theorem. If $\underline{G} = (G,A,B) \in \underline{\underline{A}} \circ \underline{\underline{A}}$ is critical and not nilpotent, then G is a critical group.

Proof. Since $\underline{G}$ is critical, there exists a bilaw q of the maximal sub-bigroups of $\underline{G}$ which is not a bilaw in $\underline{G}$ itself. Because of the nature of the maximal sub-bigroups of $\underline{G}$, q must be a genuine commutator biword, and using (2.2.3) we may assume q to take the form

$$q = \prod_{i=1}^{s} [y_1, z_1^{\alpha_{i1}}, \ldots, z_r^{\alpha_{ir}}]^{\varepsilon_i}$$

where $\varepsilon_i = \underline{+}1$, $\alpha_{ij} > 0$, $i \in \{1,\ldots,s\}$, $j \in \{1,\ldots,r\}$. Consider the word

$$w = \prod_{i=1}^{s} [x_1, x_2, x_3^{\alpha_{i1}}, \ldots, x_{r+2}^{\alpha_{ir}}]^{\varepsilon_i} .$$

Then w is a law in every maximal subgroup of G, but not a law in G itself. For, if M is a maximal subgroup of G, then from (3.3.1) it follows that $(M'.M\sigma_1, M', M\sigma_1)$ is a proper sub-bigroup of $\underline{G}$;

and each value of w in M is obtained by choosing arbitrary elements $m_1,\dots,m_{r+2}$ of M and evaluating

$$\prod_{i=1}^{s} [m_1,m_2,m_3^{\alpha_{i1}},\dots,m_{r+2}^{\alpha_{ir}}]^{\varepsilon_i}$$

$$= \prod_{i=1}^{s} [m_1,m_2,(m_3\sigma_1)^{\alpha_{i1}},\dots,(m_{r+2}\sigma_1)^{\alpha_{ir}}]^{\varepsilon_i} ;$$

this is clearly a value of q in a proper sub-bigroup, and is therefore 1. Hence w is a law in M.

On the other hand, since q is not a bilaw in $\underline{G}$, there exist elements $a \in A$, $b_1,\dots,b_r \in B$ such that

$$\prod_{i=1}^{s} [a,b_1^{\alpha_{i1}},\dots,b_r^{\alpha_{ir}}]^{\varepsilon_i} \neq 1.$$

From (3.2.3), if $1 \neq k \in K$, there exists $a' \in A$ with $a = [a',k]$; it follows that $\prod_{i=1}^{s} [a',k,b_1^{\alpha_{i1}},\dots,b_r^{\alpha_{ir}}]^{\varepsilon_i} \neq 1$ and therefore that w is not a law in G. By the remark at the beginning of this section, G is critical. We shall see later that this theorem has a strong converse.

3.4 The bigroup $\underline{F^*}$

In this section we show that, in a sense, the bivariety generated by the critical bigroup $\underline{G}$ is determined by the bivariety generated by a certain sub-bigroup of $\underline{G}$ which turns out to be a little more manageable.

Recall that (3.2.1)(vi) ensures that if $a_o \in A-N$, then A is generated, quâ B-operator group, by a_o. Suppose that one such a_o is chosen and fixed from now on. Write $A_o = \langle a_o \rangle^H$, $F_o = A_oH$ and

$$\underline{F}^* = \underline{F}_o = (F_o, A_o, H).$$

This definition depends on a_o but is unambiguous up to isomorphism, as the following result shows.

(3.4.1) <u>Lemma.</u> If a_o, $a_1 \in A-N$, then the mapping $a_o \to a_1$ can be extended to an isomorphism of the corresponding sub-bigroups $\underline{F}_o$ and $\underline{F}_1$.

Proof. Suppose that $r = r(a_o, h_1, \ldots, h_t) = 1$ is a relation among the generating set $\{a_o\} \cup H$ for F_o. Every relation in H is a relation in both F_o and F_1, so we may assume that r takes the form

$$r = \prod_{i=1}^{t} a_o^{\alpha_i h_i} = 1$$

for some integers α_i. Now there exist $b_1, \ldots, b_u \in B$ such that

$$a_1 = \prod_{i=1}^{u} a_o^{\beta_i b_i}$$

for some integers β_i. Therefore

$$r(a_1,h_1,\dots,h_t) = \prod_{i=1}^{t} \{ \prod_{j=1}^{u} a_o^{\beta_j b_j} \}^{\alpha_i h_i}$$

$$= \prod_{j=1}^{u} \{ \prod_{i=1}^{t} a_o^{\alpha_i h_i} \}^{\beta_j b_j}, \text{ since } A,B \text{ are abelian,}$$

$$= 1.$$

Hence, by von Dyck's Theorem, the mapping $a_o \rightarrow a_1$ and the identity mapping of H extend to a morphism $\underline{F}_o \rightarrow \underline{F}_1$. Similarly, the mapping $a_1 \rightarrow a_o$ and the identity mapping of H extend to a morphism $\underline{F}_1 \rightarrow \underline{F}_o$. Consequently each is an isomorphism.

(3.4.2) <u>Lemma</u>. $\underline{F}$ and $\underline{F}^*$ generate the same bivariety.

The proof of this is similar to that of (3.1.6), and we omit it.

It would be pleasant if it turned out that $\underline{F}^*$ was a critical bigroup. However this is not in general the case. The best that can be said is (3.4.3) below. The trouble comes from the fact that $\underline{F}^*$ need not be monolithic: this topic will be taken up again briefly in Chapter 5.

(3.4.3) <u>Lemma</u>. If $\underline{G}$ is as in (3.2.1), then $\underline{F}^*$ is not in the bivariety generated by its proper sub-bigroups.

This will follow from the next theorem, which is much more important from our point of view in the next two chapters.

(3.4.4) Theorem. Let q be a biword, t a positive integer, and p a prime which does not divide t. There exist biwords $q_1,\dots,q_v$, depending on q,t,p such that if q is a bilaw in a non-nilpotent, critical bigroup $\underline{G} \varepsilon \underline{\underline{A}} \circ \underline{\underline{A}}$ with $|K| = t$, and $\exp\sigma\underline{G} = p$, then $q_1,\dots,q_v$ are bilaws in $\underline{F}^*$. Conversely, if $\underline{G}_1 = (G_1, A_1, H_1 \times K_1) \varepsilon \underline{\underline{A}} \circ \underline{\underline{A}}$ with A_1, H_1 arbitrary and $\exp K_1 | t$, and $q_1,\dots,q_v$ are bilaws in (A_1H_1, A_1, H_1), then q is a bilaw in $\underline{G}_1$.

Proof. If q has one of the forms y_1^m, z_1^n then the theorem is obviously true. Hence, using (2.2.4) we may assume

$$q = \prod_{i=1}^{s} [y_1, \mu_{i1} z_1^{\varepsilon_{i1}}, \dots, \mu_{ir} z_r^{\varepsilon_{ir}}]^{\alpha_i}$$

where μ_{ij} are all natural numbers, and $\varepsilon_{ij} = \pm 1$. Suppose that q is a bilaw of the non-nilpotent critical bigroup $\underline{G}$. Consider the biword

$$q^* = \prod_{i=1}^{s} [y_1, \mu_{i1} z_{1+r}^{\varepsilon_{i1}} z_{1+2r}^{\varepsilon_{i1}}, \dots, \mu_{ir} z_{2r}^{\varepsilon_{ir}} z_{3r}^{\varepsilon_{ir}}]^{\alpha_i} .$$

In this expression for q^* expand each commutator, using repeatedly the identity

$$[x,\mu yz] = \prod_{\lambda=0}^{\mu} \prod_{\nu=\mu-\lambda}^{\mu} [x,\lambda y,\nu z]^{\binom{\mu}{\lambda}\binom{\lambda}{\lambda+\nu-\mu}}$$

modulo the bilaws of $\underline{\underline{A}} \circ \underline{\underline{A}}$. We get a product of powers of commutators each of which has y_1 as first entry and some $z_j^{\pm 1}$, $j \in \{r+1,\ldots,3r\}$ in each other entry. Working modulo the bilaws of $\underline{\underline{A}} \circ \underline{\underline{A}}$ we can collect to the front of each commutator all $z_j^{\pm 1}$ with $j \in \{r+1,\ldots,2r\}$. Hence there exist biwords $q_1^*,\ldots,q_u^*$ such that, modulo $\underline{\underline{A}} \circ \underline{\underline{A}}(\underline{Q}_2)$,

$$q^* = \prod_{i=1}^{u} [q_i^*, \lambda_{i1} z_{1+2r}^{\eta_{i1}}, \ldots, \lambda_{ir} z_{3r}^{\eta_{ir}}]$$

where $q_1^*,\ldots,q_u^*$ are biwords which are products of powers of commutators each of which has as entries, y_1 in the first place, and $z_j^{\pm 1}$, $j \in \{r+1,\ldots,2r\}$ in the other places, and where $\eta_{ij} = \pm 1$, $i \in \{1,\ldots,u\}$, $j \in \{2r+1,\ldots,3r\}$.

Now consider

$$q^{**} = \prod_{i=1}^{u} [q_i^*, \lambda_{i1} z_{1+2r}^{\zeta_{i1}}, \ldots, \lambda_{ir} z_{3r}^{\zeta_{ir}}]$$

where $\zeta_{ij} = \eta_{ij}$ if $\eta_{ij} = 1$, and $\zeta_{ij} = t-1$ if $\eta_{ij} = -1$. Making repeated use of the identity

$$[x,y^N] = \prod_{\mu=1}^{N} [x,\mu y]^{\binom{N}{\mu}}$$

we can write, again modulo $\underline{\underline{A}} \circ \underline{\underline{A}}(\underline{Q}_2)$,

$$q^{**} = \prod_{i=1}^{v} [q_i, \nu_{i1} z_{1+2r}, \ldots, \nu_{ir} z_{3r}] . q'$$

where each q_i is a linear combination of q_j^*'s, where $0 \leq \nu_{ij} \leq t-1$ all i,j, and where q' is a (possibly empty) product of powers of commutators in each of which at least one of $z_{1+2r}, \ldots, z_{3r}$ occurs raised to a power which is a multiple of t.

Now suppose that $\alpha : \underline{Q}_2 \to \underline{F}$ is arbitrary, and for the moment, fixed. With each choice $k_1, \ldots, k_r \in K$ and α, associate a morphism $\beta : \underline{Q}_2 \to \underline{G}$ such that

$$y_i\beta = y_i\alpha, \qquad i \in \{1,2,\ldots\},$$

$$z_{j+r}\beta = z_j\alpha, \quad z_{j+2r}\beta = k_j, \qquad j \in \{1,\ldots,r\}.$$

Then if $\beta^* : \underline{Q}_2 \to \underline{G}$ is such that

$$y_i\beta^* = y_i\alpha, \qquad i \in \{1,2,\ldots\},$$

$$z_j\beta^* = (z_j\alpha).k_j, \qquad j \in \{1,\ldots,r\},$$

we have

$$1 = q\beta^* = q^*\beta = q^{**}\beta = \prod_{i=1}^{v} [q_i\alpha, \nu_{i1}k_1, \ldots, \nu_{ir}k_r],$$

and this for all such β. Hence, by Lemma 3.2.6, $q_i\alpha = 1$, $i \in \{1,\ldots,v\}$, and since α was arbitrary, $q_1, \ldots, q_v$ are bilaws in $\underline{F}$ and so in $\underline{F}^*$.

Conversely suppose that $q_1, \ldots, q_v$ are bilaws in (A_1H_1, A_1, H_1). Then if $\beta^* : \underline{Q}_2 \to \underline{G}_1$ is any morphism we can construct $\alpha : \underline{Q}_2 \to \underline{F}$ and $\beta : \underline{Q}_2 \to \underline{G}_1$ reversing the procedure in the foregoing proof. Then so long as $\exp K_1 | t$ we have $q_1\alpha = \ldots = q_v\alpha = 1$ implies $q\beta^* = 1$ and so q is a bilaw in $\underline{G}_1$.

(3.4.5) Remark. (i) It is clear from the proof of Theorem (3.4.4) that in the case when q is a commutator biword, $q_1, \ldots, q_v$ do not depend at all on p. Also the forward part of the argument works if we assume no more than that K acts fixed point free on A, otherwise the criticallity of $\underline{G}$ is irrelevant.

(ii) The argument above is, of course, essentially a trigroup argument. However it seems easier to treat it as we have done, then to develop the necessary conventions and terminology involved in considering $\underline{G}$ as a trigroup.

Proof of (3.4.3). Since $\underline{G}$ is critical, there is a biword q which is a bilaw in every maximal sub-bigroup of $\underline{G}$, but not in $\underline{G}$ itself. In particular q is a bilaw in the maximal sub-bigroups of the type

$$AH_oK, \qquad NHK, \qquad H_o \text{ maximal in } H.$$

Now in the proof of (3.4.4) the crucial property of $\underline{G}$ was that K acts fixed point free on $A = G'$. It follows therefore, that if $q_1,\dots,q_v$ correspond to q by (3.4.4), then $q_1,\dots,q_v$ are bilaws in all AH_0 and in NH. However $q_1,\dots,q_v$ cannot all be bilaws in AH since q is not a bilaw in $\underline{G}$. It remains to remark that the maximal sub-bigroups of $\underline{F}^*$ are precisely $AH_0 \cap \underline{F}^*$ and $N_0H = NH \cap \underline{F}^*$ by an argument similar to that of (3.3.1), and that they generate the same bivarieties as their counter-parts in AH.

CHAPTER 4

A FINITE BASIS THEOREM

4.0 Introduction

Our aim in this chapter is to prove the following theorems.

(4.0.1) Theorem. If n is a natural number, the bivariety $\underline{\underline{A}} \circ \underline{\underline{A}}_n$ has descending chain condition on sub-bivarieties.

(4.0.2) Theorem. If m is a natural number, the bivariety $\underline{\underline{A}}_m \circ \underline{\underline{A}}$ has descending chain condition on sub-bivarieties.

Then, by virtue of Theorems 2.1.1, 2.1.2, and a relatively simple argument, one has

(4.0.3) Theorem. Every sub-bivariety of $\underline{\underline{A}}_m \circ \underline{\underline{A}} \vee \underline{\underline{A}} \circ \underline{\underline{A}}_n$ has a finite basis for its bilaws.

4.1 $\underline{\underline{A}} \circ \underline{\underline{A}}_n$: reduction to the case $n = p^\nu$.

Suppose that

$$\underset{\sim}{B}_1 \supseteq \underset{\sim}{B}_2 \supseteq \cdots \supseteq \underset{\sim}{B}_i \supseteq \cdots$$

is a descending chain of proper sub-bivarieties of $\underline{\underline{A}} \circ \underline{\underline{A}}_n$. For each $i \in \{1,2,\ldots\}$ write

$$\underset{\sim}{C}_i = \operatorname{svar}\{\underline{G} \in \underset{\sim}{B}_i : \exp A_2(\underline{G}) < n\}$$

and

$$\underset{\sim}{D}_i = \operatorname{svar}\{\underline{G} \in \underset{\sim}{B}_i : \underline{G} \text{ critical}, \exp A_2(\underline{G}) = n\}.$$

Clearly $\underset{\sim}{C}_1 \supseteq \underset{\sim}{C}_2 \supseteq \ldots \supseteq \underset{\sim}{C}_i \supseteq \ldots$ and $\underset{\sim}{D}_1 \supseteq \underset{\sim}{D}_2 \supseteq \ldots \supseteq \underset{\sim}{D}_i \supseteq \ldots$ are descending chains, and

$$(4.1.1) \qquad \underset{\sim}{C}_1 \subseteq v\{\underline{\underline{A}} \circ \underline{\underline{A}}_t : t \neq n, t|n\}.$$

We turn our attention to the chain of the $\underset{\sim}{D}_i$'s.

(4.1.2) Lemma. The chain $\underset{\sim}{D}_1 \supseteq \underset{\sim}{D}_2 \supseteq \ldots \supseteq \underset{\sim}{D}_i \supseteq \ldots$ breaks off if the bivarieties $\underline{\underline{A}} \circ \underline{\underline{A}}_{p^\lambda}$ have descending chain condition on sub-bivarieties, where $p^\lambda || n$.

Proof. With each prime p, each natural number $t|n$, and each $i \in \{1,2,\ldots\}$ associate the bivariety

$$\underset{\sim}{D}_i(p,t) = \operatorname{svar}\{\underline{G} \in \underset{\sim}{B}_i : \underline{G} \text{ critical}, \exp A_2(\underline{G}) = n, \exp\sigma\, \underline{G} = p, |K| = t\}$$

where in the case $\underline{G}$ is critical and nilpotent we interpret $K = 1$. Clearly then

$$\underset{\sim}{D}_1(p,t) \supseteq \underset{\sim}{D}_2(p,t) \supseteq \dots \supseteq \underset{\sim}{D}_i(p,t) \supseteq \dots$$

is a descending chain, and for all $i \in \{1,2,\dots\}$,

$$\underset{\sim}{D}_i = \vee\{\underset{\sim}{D}_i(p,t) : p \text{ prime}, t|n\}.$$

Define

$$\underset{\sim}{D}^*_i(p,t) = \text{svar}\{\underline{F}^* : \underline{G} \in \underset{\sim}{B}_i, \underline{G} \text{ critical}, \exp A_2(\underline{G}) = n,$$
$$\exp\sigma\, \underline{G} = p, |K| = t\}$$

where we interpret $\underline{F}^* = \underline{G}$ in the case $\underline{G}$ critical and nilpotent. Then for each prime p and $t|n$,

$$\underset{\sim}{D}^*_1(p,t) \supseteq \underset{\sim}{D}^*_2(p,t) \supseteq \dots \supseteq \underset{\sim}{D}^*_i(p,t) \supseteq \dots$$

is a descending chain.

Next suppose that the chain $\underset{\sim}{D}^*_1(p,t) \supseteq \underset{\sim}{D}^*_2(p,t) \supseteq \dots$ breaks off; that is, for some natural number ℓ, $\ell \leq i$ implies

$$\underset{\sim}{D}^*_i(p,t) = \underset{\sim}{D}^*_{i+1}(p,t).$$

If q is a bilaw of $\underset{\sim}{D}_{i+1}(p,t)$, let $q_1,\dots,q_v$ be the biwords corresponding to t,p,q according to Theorem 3.4.4. Then $q_1,\dots,q_v$ are bilaws in $\underset{\sim}{D}^*_{i+1}(p,t)$ and therefore in $\underset{\sim}{D}^*_i(p,t)$, whence, using (3.4.2) and the converse part of (3.4.4), q is a bilaw in $\underset{\sim}{D}_i(p,t)$. It follows that for $\ell \leq i$,

$$\underset{\sim}{D}_i(p,t) = \underset{\sim}{D}_{i+1}(p,t)$$

The proof of the lemma is now nearly complete: we need only the following lemma.

(4.1.3) Lemma. If $\{\underline{G}_i : i \in I\}$ is an infinite set of non-isomorphic, non-nilpotent, critical bigroups belonging to $\underline{\underline{A}} \circ \underline{\underline{A}}_n$ such that

i) $A_1(\underline{G}_i) = F(G_i), \quad i \in I,$

ii) $|A_2(\underline{G}_i)| = n > 1, \quad i \in I,$

then $\mathrm{svar}\{\underline{G}_i : i \in I\} = \underline{\underline{A}} \circ \underline{\underline{A}}_n$.

Proof. Under the conditions imposed, each G_i is a critical group, by Theorem 3.3.4. According to Cossey [4, Theorem 4.2.2], G_i is determined uniquely (up to isomorphism) by the invariants $\exp G_i'$, n. Hence, since there are an infinity of non-isomorphic $\underline{G}_i$'s, $\exp G_i'$ is unbounded.

We next employ (3.4.4). Let q be a bilaw in all $\underline{G}_i$. Then we may assume that q is either z_1^N where $n|N$, or a genuine commutator biword. In the first case q is a bilaw of $\underline{\underline{A}} \circ \underline{\underline{A}}_n$, and in the second, note that if $q_1, \ldots, q_v$ correspond to q by (3.4.4), they are independent of p (as noted in (3.4.5)), and $q_1, \ldots, q_v$ are bilaws in every bigroup $(A, A, 1)$. Hence q is a bilaw in every bigroup of $\underline{\underline{A}} \circ \underline{\underline{A}}_n$.

Returning to the proof of (4.1.2), note that if $p \nmid n$, then $\underset{\sim}{D}_i(p,t)$ is trivial unless $m = n$; and by (4.1.3), $\underset{\sim}{D}_i(p,n)$ is non-trivial for only finitely many primes p. Hence there is a finite set Π of primes such that

$$\underset{\sim}{D}_i = \vee\{\underset{\sim}{D}_i(p,t) : p \in \Pi, \ t|n\};$$

and $\underset{\sim}{D}_1 \supseteq \underset{\sim}{D}_2 \supseteq \ldots$ breaks off if and only if all $\underset{\sim}{D}_1(p,t) \supseteq \underset{\sim}{D}_2(p,t) \supseteq \ldots$ break off. This completes the proof.

Now since $\underset{\sim}{B}_i = \underset{\sim}{C}_i \vee \underset{\sim}{D}_i$, the chain $\underset{\sim}{B}_1 \supseteq \underset{\sim}{B}_2 \supseteq \ldots$ breaks off if and only if both the chains $\underset{\sim}{C}_1 \supseteq \underset{\sim}{C}_2 \supseteq \ldots$ and $\underset{\sim}{D}_1 \supseteq \underset{\sim}{D}_2 \supseteq \ldots$ break off. We make the hypothesis

(4.1.4) Inductive Hypothesis. For every natural number $n_o < n$ $\underline{\underline{A}} \circ \underline{\underline{A}}_{n_o}$ has descending chain condition on sub-bivarieties.

Whenever n is not a prime power we have made the inductive step in (4.1.1) and (4.1.2). Since $\underline{\underline{A}} \circ \underline{\underline{A}}_1$ clearly has descending chain condition on sub-bivarieties, it remains to deal with the case when n is a prime power, $n = p^\nu$ say.

4.2 Preliminary lemmas

We change our point of view from now on and consider not descending chains of sub-bivarieties of $\underline{\underline{A}} \circ \underline{\underline{A}}_{p^\nu}$, but ascending chains of normal, fully invariant sub-bigroups of $\underline{F}_{(\omega,\omega)}(\underline{\underline{A}} \circ \underset{p}{\underline{\underline{A}}}_{\nu})$. In fact it suffices to consider the split-free bigroup $\underline{F}_{(1,\omega)}(\underline{\underline{A}} \circ \underset{p}{\underline{\underline{A}}}_{\nu})$:

(4.2.1) Lemma. The lattices of normal, fully invariant sub-bigroups of $\underline{F}_{(\omega,\omega)}(\underline{\underline{A}} \circ \underset{p}{\underline{\underline{A}}}_{\nu})$ and $\underline{F}_{(1,\omega)}(\underline{\underline{A}} \circ \underset{p}{\underline{\underline{A}}}_{\nu})$ are isomorphic.

Proof. We use (1.5.10), imagining $\underline{Q}(1,\omega)$ embedded in $\underline{Q}_2$ in a natural way. Consider the mapping ξ from the lattice of normal, fully invariant sub-bigroups of $\underline{Q}_2$ containing the bilaws $\underline{\underline{A}} \circ \underset{p}{\underline{\underline{A}}}_{\nu}(\underline{Q}_2)$ of $\underline{\underline{A}} \circ \underset{p}{\underline{\underline{A}}}_{\nu}$ to the lattice of normal, fully invariant sub-bigroups of $\underline{Q}(1,\omega)$ containing $\underline{\underline{A}} \circ \underset{p}{\underline{\underline{A}}}_{\nu}(\underline{Q}(1,\omega))$, defined by

$$\underline{S}\xi = \underline{S}(\underline{Q}(1,\omega)).$$

Now ξ is onto by (1.4.11), clearly preserves inclusions, and by (1.5.10) is an intersection-homomorphism; it is easy to see that ξ is then a join-homomorphism if it is one-to-one. If $\underset{\sim}{S}_1, \underset{\sim}{S}_2 \subseteq \underline{\underline{A}} \circ \underset{p}{\underline{\underline{A}}}_{\nu}$ and $\underset{\sim}{S}_1 \neq \underset{\sim}{S}_2$ then there exists $q \in (\underline{Q}(1,\omega) \cap \underline{S}_1) - \underline{S}_2$, say, by virtue of (2.2.3) and so, from (1.5.10), $\underline{Q}(1,\omega) \cap \underline{S}_1 \neq \underline{Q}(1,\omega) \cap \underline{S}_2$ implies $S_1(\underline{Q}(1,\omega)) \neq S_2(\underline{Q}(1,\omega))$. This completes the proof.

(4.2.2) Notation. Write $\underline{W}_{\nu}$ for $\underline{F}_{(1,\omega)}(\underline{\underline{A}} \circ \underset{p}{\underline{\underline{A}}}_{\nu})$, $A = A_1(\underline{W}_{\nu})$, $B = A_2(\underline{W}_{\nu})$. For the split-free generating set of $\underline{W}_{\nu}$ write $\{y_1\} \cup \{z_1, z_2, \ldots, z_i, \ldots\}$: no confusion will result from this.

We will abuse language to the extent of calling elements of $\underline{W}_\nu$ biwords.

From Theorem 1.6.2 we have that

$$W_\nu = C \text{ wr } F_\omega(\underline{\underline{A}}_{p^\nu})$$

where C is an infinite cycle and where A is the base group of W_ν, and $B = F_\omega(\underline{\underline{A}}_{p^\nu})$. Our aim is to prove

(4.2.3) <u>Theorem</u>. All ascending chains of normal, fully invariant sub-bigroups of $\underline{W}_\nu$ break off.

It is worth noting here

(4.2.4) <u>Lemma</u>. Every fully invariant sub-bigroup of $\underline{W}_\nu$ contained in A is normal in $\underline{W}_\nu$.

Proof. This follows since elements of B induce self-morphisms of $\underline{W}_\nu$, and A is abelian.

(4.2.5) <u>Lemma</u>. If $\underline{U}$ is a normal sub-bigroup of $\underline{W}_\nu$, and if for fixed elements $a_1, \ldots, a_m \in A$, and all $b \in B$

$$\prod_{i=1}^{m} [a_i, b^i] \in \underline{U}$$

then for all $b_1,\dots,b_m \in B$, $\prod_{j=u}^{m} [a_j, b_1^j, \dots, b_u^{j-u+1}] \in U$,

$u \in \{1,\dots,m\}$.

Proof. For $u = 1$ the assertion is the hypothesis. Suppose, therefore, that for some $u \in \{1,\dots,m-1\}$ the lemma is true. If $b_1,\dots,b_{u+1} \in B$ are arbitrarily chosen, then

$$\prod_{j=u}^{m} [a_j, b_1^j, \dots, (b_u b_{u+1})^{j-u+1}] \in U.$$

That is,

$$\prod_{j=u}^{m} [a_j, b_1^j, \dots, b_u^{j-u+1}][a_j, b_1^j, \dots, b_{u-1}^{j-u+2}, b_{u+1}^{j-u+1}]$$
$$[a_j, b_1^j, \dots, b_u^{j-u+1}, b_{u+1}^{j-u+1}] \in U$$

and from here, using our inductive hypothesis, we obtain that

$$\prod_{j=u}^{m} [a_j, b_1^j, \dots, b_u^{j-u+1}, b_{u+1}^{j-u+1}] \in U.$$

Since U is normal we have $\prod_{j=u}^{m} [a_j, b_1^j, \dots, b_u^{j-u+1}, b_{u+1}] \in U$ and so

$$\prod_{j=u}^{m} [a_j, b_1^j, \dots, b_u^{j-u+1}, b_{u+1}]^{-1} [a_j, b_1^j, \dots, b_u^{j-u+1}, b_{u+1}^{j-u+1}] \in U .$$

Finally, using the commutator identity $[x,y]^{-1}[x,y^t] = [x,y^{t-1}]^y$ for all integers t, we have

$$\prod_{j=u+1}^{m} [a_j, b_1^j, \dots, b_{u+1}^{j-u}]^{b_{u+1}} \in U,$$

which, since $\underline{U}$ is normal, gives what we want.

This lemma will prove useful in a number of places: first as the initial step of an induction in the proof of Lemma 4.2.10 below, and later in dealing with the structure of certain metabelian varieties.

(4.2.6) <u>Notation</u>. If $\underline{U}$ is normal in $\underline{W}_\nu$ define the sub-bigroups $\underline{U}_i$ of $\underline{W}_\nu$ for $i \in \{0,1,\ldots\}$ by

$$U_i/U = Z_i(W_\nu/U),$$

where $Z_i(W_\nu/U)$ is the i-th term of the upper central series of W_ν/U (see, for example, p.77 in [3]).

Note that if $a \in A$, then $[a,b_1,\ldots,b_r] \in U$ for all $b_1,\ldots,b_r \in B$ if and only if $a \in U_r$.

(4.2.7) <u>Lemma</u>. If to the hypotheses of (4.2.5) we add $m \leq p - 1$, then for $i \in \{1,\ldots,m\}$, $a_i \in U_m$.

Proof. From (4.2.5),

$$[a_m,b_1^m,\ldots,b_{m-1}^2,b_m] \in U$$

for all $b_1,\ldots,b_m \in B$. Since $1,2,\ldots,m$ are all prime to p, we have $a_m \in U_m$.

Assume that it has been proved that $a_{i+1} \varepsilon U_m, \ldots, a_m \varepsilon U_m$ for some $i \geq 1$. Then since $\prod_{j=i}^{m} [a_j, b_1^j, \ldots, b_i^{j-i+1}] \varepsilon U$, we have by commuting with $b_{i+1}, \ldots, b_m$ that $[a_i, b_1^i, \ldots, b_i, b_{i+1}, \ldots, b_m] \varepsilon U$ and hence, as before, $a_i \varepsilon U_m$. This completes the proof.

(4.2.8) Lemma. If $\underline{U}$ is normal in $\underline{W}_\nu$ and if for fixed elements $a_1, \ldots, a_m \varepsilon A$ and all $b \varepsilon B$,

$$\rho = \prod_{i=1}^{m} [a_i, b^i] \varepsilon U$$

then

i) $([a_t, b^p][a_{t+p}, b^{2p}] \ldots) \varepsilon U_{m-1}$, $p \leq t \leq 2p - 1$;

ii) $([a_u, b^u][a_{u+p}, b^{u+p}] \ldots) \varepsilon U_{m+p-2}$, $1 \leq u \leq p - 1$;

iii) $(a_v a_{v+p} \ldots) \varepsilon U_{m+p-2}$, $1 \leq v \leq p - 1$.

In the proof of this lemma we need the following notation and Lemma 4.2.10 below.

(4.2.9) Notation. If $b_1, \ldots, b_m$ are arbitrary elements of B, write

$$c(s,u,v,i) = [a_{s+ip}, b_1^{s+ip}, \ldots, b_{s-up+v}^{up-v+ip+1}, b_{s-(u-1)p+1}^{(u+i-1)p}, \ldots, b_{s-p+1}^{(i+1)p}]$$

where $s \in \{1,\ldots,m\}$, $i \in \{0,\ldots,\ell\}$ where $\ell = [(m-s)/p]$, $v \in \{1,\ldots,p\}$ and where u has the range:

$$u \in \{1,\ldots,s/p\} \quad \text{if} \quad p \mid s;$$

$$u \in \{1,\ldots,[s/p]+1\} \quad \text{if} \quad p \nmid s,$$

with the conventions:

$$s-up+v \le 0 \quad \text{implies} \quad c(s,u,v,i) = [a_{s+ip}, b^{(u+i-1)p}_{s-(u-1)p+1}, \ldots, b^{(i+1)p}_{s-p+1}];$$

$$s-up+v \le s < s-(u-1)p+1 \quad \text{implies}$$

$$c(s,u,v,i) = [a_{s+ip}, b^{s+ip}_{1}, \ldots, b^{up-v+ip+1}_{s-up+v}];$$

$$s < s-up+v \quad \text{implies} \quad c(s,u,v,i) = [a_{s+ip}, b^{s+ip}_{1}, \ldots, b^{ip+1}_{s}] .$$

Also write

$$\rho(s,u,v) = \Pi' c(s,u,v,0) = \prod_{i=0}^{\ell} c(s,u,v,i),$$

and $\Pi' a_s = a_s a_{s+p} \cdots a_{s+\ell p}$.

(4.2.10) Lemma. If ρ is as in (4.2.8) then

$$\rho(s,u,v) \in U_r$$

for all relevant s,u,v, where $r = m - s + u(p-1) - v + 1$.

Proof. From (4.2.5) we have

$$\rho(m,1,p) = [a_m, b_1^m, \ldots, b_{m-1}^2, b_m] \in U;$$

and in this expression we may replace b_i^{m-i+1} by b_i whenever $p \nmid m - i + 1$. Hence

$$\rho(m,u,v) = [a_m, b_1^m, \ldots, b_{m-up+v}^{up-v+1}, b_{m-(u-1)p+1}^{(u-1)p}, \ldots, b_{m-p+1}^p]$$

for all relevant u,v, and therefore

$$\rho(m,u,v) \in U_r$$

where $r = m - (m-up+v+u-1) = u(p-1) - v + 1$. We use this as the start of an induction, the induction being taken over the lexicographically ordered set of triples $(-s,u,-v)$. Suppose, therefore, that for all $(-s,u,-v) < (-t,w,-x+1)$ where $x \in \{2,\ldots,p\}$, the assertion of the lemma is true.

First note that from Lemma 4.2.5 we have

$$\prod_{j=t}^{m} [a_j, b_1^j, \ldots, b_t^{j-t+1}] = \prod_{j=t}^{t+p-1} \rho(j,1,t+p-j)$$
$$\in U.$$

Hence, by the inductive hypothesis we deduce from this that

$$\rho(t,1,p) \in U_{m-t}$$

as required. Second,

$$\rho(t,w,x) = \prod_{i=0}^{\ell} c(t,w,x,i)$$

and

$$c(t,w,x,i) = [a_{t+ip}, b_1^{t+ip}, \ldots, b_{t-wp+x}^{wp-x+ip+1}, b_{t-(w-1)p+1}^{(w+i-1)p}, \ldots, b_{t-p+1}^{(i+1)p}]$$

$$= [a_{t+ip}, b_1^{t+ip}, \ldots, b_{t-wp+x-1}^{wp-x+ip+2}, b_{t-(w-1)p+1}^{(w+i-1)p}, \ldots, b_{t-p+1}^{(i+1)p}, b_{t-wp+x}^{wp-x+ip+1}]$$

$$= [c(t,w,x-1,i), b_{t-wp+x}^{wp-x+1}]$$

$$\times \ [a_{t+ip}, b_1^{t+ip}, \ldots, b_{t-wp+x-1}^{wp-x+ip+2}, b_{t-(w-1)p+1}^{(w+i-1)p}, \ldots, b_{t-p+1}^{(i+1)p}, b_{t-wp+x}^{ip}]^{b_{t-wp+x}^{wp-x+1}}.$$

Therefore

$$\rho(t,w,x) = [\rho(t,w,x-1), b_{t-wp+x}^{wp-x+1}].\rho'(t+p,w+1,x-1)^{b_{t-wp+x}^{wp-x+1}}$$

where $\rho'(t+p,w+1,x-1)$ differs from $\rho(t+p,w+1,x-1)$ only in that the element $b_{(t+p)-p+1}$ occurs as b_{t-wp+x}; in any event

$\rho'(t+p,w+1,x-1)^{b_{t-wp+x}^{wp-x+1}}$ belongs to U_r where

$r = m - (t+p) + (w+1)(p-1) - (x-1) + 1 = m - t + w(p-1) - x + 1$,

by the induction hypothesis. Hence since also $\rho(t,w,x) \in U_r$ by the inductive hypothesis,

$$[\rho(t,w,x-1), b_{t-wp+x}^{wp-x+1}] \in U_r;$$

and the fact that $wp - x + 1$ is prime to p under the assumptions on x, and that b_{t-wp+x} does not occur in $\rho(t,w,x-1)$, means that

$$\rho(t,w,x-1) \in U_{r+1}$$

as required.

Finally, note that for $u \geq 2$,

$$\rho(s,u,p) = \rho(s,u-1,1)$$

and this completes the induction, and the proof of (4.2.10).

Proof of (4.2.8). Put $s = p$, $u = 1$, $v = 1$ in (4.2.10) and we get

$$\prod_{i=0}^{\ell} [a_{(i+1)p}, b_1^{(i+1)p}] \in U_{m-1}.$$

If $p < s \leq 2p-1$, put $u = 2$, $v = 2p-s$ and we get

$$\prod_{i=0}^{\ell} [a_{s+ip}, b_{s-p+1}^{(i+1)p}] \in U_{m-1},$$

and these together are just the assertion (i).

To prove (iii) proceed as follows. Note that for $p \leq j \leq 2p-1$, $[a_j, b_1^j] = c(p,1,1,0)$ if $j = p$ and $[a_j, b_1^j] = c(j-p,1,2p-j+1,1)$ if $p < j$: hence in the following argument, Π' notation can be used. We have

$$\rho = \prod_{i=1}^{p-1} [a_i, b_1^i] . \prod_{j=p}^{2p-1} \{\Pi'[a_j, b_1^j]\}$$

$$= \prod_{i=1}^{p-1} [a_i, b_1^i] . \prod_{j=p}^{2p-1} \{(\Pi'[a_j, b_1^p])^{b_1^{j-p}} . [\Pi' a_j, b_1^{j-p}]\}.$$

(Here $\Pi'[a_j, b_1^p] = [a_j, b_1^p][a_{j+p}, b_1^{2p}]\ldots$ is a harmless abuse of notation). By part (i) we have then

$$\prod_{i=1}^{p-1} [a_i, b_1^i] . \prod_{j=p+1}^{2p-1} [\Pi' a_j, b_1^{j-p}] \in U_{m-1},$$

and therefore

$$\prod_{i=1}^{p-1} [\Pi' a_i, b_1^i] \in U_{m-1}.$$

Then from Lemma 4.2.7,

$$\Pi' a_i \in U_{m+p-2}$$

for all $i \in \{1,\ldots,p-1\}$, and this completes the proof of (iii).

The proof of (ii) uses (i), (iii) and the identity

$$\text{(4.2.11)} \quad \Pi'[a_v, b_1^v] = [\Pi' a_v, b_1^v] . \Pi'[a_{p+v}, b_1^p]^{b_1^v}$$

for $v \in \{1,\ldots,p-1\}$ (where $\Pi'[a_{p+v}, b_1^p] = [a_{p+v}, b_1^p][a_{2p+v}, b_1^{2p}]\ldots$ is again an abuse of notation). The proof of (4.2.8) is now complete.

(4.2.12) *Definition*. An element of W_ν which belongs to the subgroup generated by the set $\{y_1\} \cup \{z_1^p, z_2^p, \ldots, z_i^p, \ldots\}$ will be called a †-*biword*.

(4.2.13) <u>Lemma</u>. If $q \in A$, then there exist †-biwords $q_1,,,,q_d$ and a natural number v such that

$$\mathrm{cl}\{q_1,\dots,q_d\} \geq \mathrm{cl}\{q\} \geq \mathrm{cl}\{[q_i,vB];\ 1 \leq i \leq d\}.$$

Moreover if q is special, so are $q_1,\dots,q_d$. (As usual, $[q_i,vB]$ stands for the subgroup generated by the commutators $[q_i,b_1,\dots,b_v]$, $b_1,\dots,b_v \in B$).

The proof of this lemma depends on the following consideration.

(4.2.14) <u>Lemma</u>. If $q^* \in A$ is a special biword, say involving the variables $y_1,z_1,\dots,z_s$ precisely, then there exist special biwords $q_1^*,\dots,q_r^*$ in which z_s, if it occurs at all, does so raised to a power which is a multiple of p, and $q_1^*,\dots,q_r^*$ involve no variables other than $y_1,z_1,\dots,z_s$; and there exists a natural number v^* such that

$$\mathrm{cl}\{q_1^*,\dots,q_r^*\} \geq \mathrm{cl}\{q^*\} \geq \mathrm{cl}\{[q_i^*,v^*B]:\ 1 \leq i \leq r\}.$$

Proof. We may write

$$q^* = \prod_{i=1}^{t} [y_1,z_1^{\lambda_{i1}},\dots,z_s^{\lambda_{is}}]^{\alpha_i}$$

where $0 < \lambda_{ij} \leq p^{\nu} - 1$ for all i,j. For $j \in \{1,\dots,p^{\nu}-1\}$ define

$$a_j = \begin{cases} \prod\limits_{j=\lambda_{is}} [y_1,z_1^{\lambda_{i1}},\dots,z_{s-1}^{\lambda_{is-1}}]^{\alpha_i}, & \exists i, j = \lambda_{is}, \\ 1, & \text{otherwise.} \end{cases}$$

Then $q^* = \prod_{i=1}^{p^\nu-1} [a_j, z_s^j]$. Since by construction the a_j's do not involve z_s, the hypotheses of Lemma 4.2.8 are satisfied, with $U = \mathrm{cl}\{q^*\}$. Hence

$$\Pi'[a_u, z_s^p] = [a_u, z_s^p][a_{u+p}, z_s^{2p}] \ldots \in U_{p^\nu-2}, u \in \{p, \ldots, 2p-1\},$$

$$\Pi' a_v \in U_{p^\nu+p-3}, \; v \in \{1, \ldots, p-1\}.$$

By virtue of the fact that

$$q^* = \prod_{v=1}^{p} \{[a_v, z_s^v][a_{v+p}, z_s^{v+p}] \ldots\},$$

and (4.2.11), we have

$$q^* \in \mathrm{cl}\{\Pi'[a_u, z_s^p], \Pi' a_v : p \leq u \leq 2p-1, \; 1 \leq v \leq p-1\}.$$

Put $\{q_1^*, \ldots, q_r^*\} = \{\Pi'[a_u, z_s^p \,, \Pi' a_v : p \leq u \leq 2p-1, \; 1 \leq v \leq p-1\}$ and $v^* = p^\nu + p-3$ and we are finished.

Proof of (4.2.13). We can, without loss of generality, assume q to be special. Then apply (4.2.14) to q, say q involves precisely $y_1, z_1, \ldots, z_s$, and obtain $q_1^*, \ldots, q_r^*$ in which z_s occurs either not at all, or to a power which is a multiple of p. Then use (4.2.14) on $q_1^*, \ldots, q_r^*$, first moving z_{s-1} up to the back of each commutator, and making z_{s-1} 'good' according to (4.2.14). Continue this process until we have dealt with $z_s, \ldots, z_1$ in turn, and hence reached a set of

†-biwords $q_1, \dots, q_d$ and a natural number v (the sum of all the relevant v^*'s) which satisfy the assertions of the lemma.

(4.2.15) Lemma. Suppose that $\underline{U}$ is a biverbal sub-bigroup of $\underline{W}_{\mathfrak{v}}$ determined by †-biwords, and suppose $q \in A - U$. Then $q \notin U_r$ for any natural number r.

Proof. We may suppose that the †-biwords determining U are

$$q_i = \prod_{j=1}^{t} [y_1, z_1^{p\lambda_{ij1}}, \dots, z_{s_i}^{p\lambda_{ijs_i}}]^{\alpha_{ij}}, \quad i \in I,$$

where $\lambda_{ijk} > 0$. Clearly it suffices to show that $[q, z_d] \notin U$, where q involves $y_1, z_1, \dots, z_{d-1}$ at most. Suppose to the contrary, that

$$q' = [q, z_d] \in U.$$

Then there are values of the biwords q_i, $v_1, \dots, v_N$ say, such that

(4.2.16) $\quad q' = v_1 v_2 \dots v_N$.

Each v_j is obtained from some q_i by subsituting for y_1 an element of A, and for $z_1, \dots, z_{s_i}$, elements of B. By applying to (4.2.16) the method of Chapter 3, section 3 in [3], we may suppose that each v_j involves z_d. These z_d's entered v_j by substitution in some q_i either for y_1 or for some z_k; in the latter case the relevant z_d's will occur raised to a power which is a multiple of p.

Consider the self-morphism μ of $\underset{\sim}{W}_\nu$ defined by

$$y_1\mu = y_1, \quad z_j\mu = z_j, \quad j \neq d, \quad z_d\mu = z_d^{p^{\nu-1}} .$$

Then, under μ, (4.2.16) becomes

$$(4.2.17) \qquad (q'\mu)(v_{j_1}^{-1}\mu) \ldots (v_{j_w}^{-1}\mu) = 1,$$

where v_{j_k} involves z_d only by virtue of the substitution for y_1 in the relevant q_i. Indeed, since the commutators involved in the expressions for q_i are linear in the first entry, we may suppose, by renaming if necessary, that v_{j_k} is obtained from some q_i by a substitution for y_1 of a power of a single commutator of the form

1

$$[y_1, z_{d_{\cancel{u}}}^{\delta_1}, \ldots, z_{d_u}^{\delta_u}, z_d^{\delta}]$$

where $d_1, \ldots, d_s, d$ are distinct, and where $p \nmid \delta$, and some unspecified substitution for $z_1, \ldots, z_{s_i}$ (though it does not involve z_d). That is, there exist values $v_1', \ldots, v_R'$ of the q_i which do not involve z_d at all, such that

$$(4.2.18) \qquad (q'\mu)[v_1', z_d^{\zeta_1 p^{\nu-1}}] \ldots [v_R', z_d^{\zeta_R p^{\nu-1}}] = 1,$$

with $1 \leq \zeta_1 \leq \ldots \leq \zeta_R \leq p-1$, say.

Lemma 4.2.7, or at any rate the same proof exactly, can now be used to conclude that

$$(4.2.19) \quad [q \prod_{\zeta_i=1} v_i', z_d^{p^{\nu-1}}, \ldots, z_{d+p-2}^{p^{\nu-1}}] = 1.$$

By a result of Baumslag [10] (24.22 in [3]),

$$q \prod_{\zeta_i=1} v_i' = 1 ,$$

and in consequence, $q \varepsilon U$, contrary to hypothesis. Hence $[q,z_d] \notin U$.

4.3 Proof of (4.2.3)

Write Λ_ν for the lattice of normal, fully invariant subbigroups of $\underline{W}_\nu$. We aim to show that, using the lemmas of the previous section and others to be developed here, that the †-biwords provide an embedding of $\Lambda_{\nu-1}$ into Λ_ν in a convenient way.

Suppose, therefore, that $\underline{W}_{\nu-1}$ is free on $\{\overline{y}_1\} \cup \{\overline{z}_1,\overline{z}_2,\ldots\}$, that $\overline{A} = A_1(\underline{W}_{\nu-1})$, $\overline{B} = A_2(\underline{W}_{\nu-1})$ and that the morphism $\xi_\nu : \underline{W}_{\nu-1} \to \underline{W}_\nu$ is defined by

$$\overline{y}_1\xi_\nu = y_1, \quad \overline{z}_j\xi_\nu = z_j^p, \quad j \varepsilon \{1,2,\ldots\}.$$

The morphism ξ_ν induces a mapping $\lambda_\nu : \Lambda_{\nu-1} \to \Lambda_\nu$ in the following natural way: if $\underline{L} \varepsilon \Lambda_{\nu-1}$, that is, if $\underline{L}$ is normal and fully invariant in $\underline{W}_{\nu-1}$, then

(4.3.1) $\underline{L}\lambda_\nu = \mathrm{cl}\{\ell\xi_\nu : \ell \in L\}$.

It is clear at once that λ_ν is a join-homomorphism, but not so clear that it is an intersection-homomorphism. In fact we prove

(4.3.2) Lemma. The mapping $\lambda_\nu : \Lambda_{\nu-1} \to \Lambda_\nu$ is a one-to-one lattice homomorphism.

Proof. First note that λ_ν preserves inclusion. We are left to show that λ_ν is an intersection-homomorphism and that it is one-to-one. To prove the former it suffices to prove that for $\underline{L}_1, \underline{L}_2 \in \Lambda_{\nu-1}$,

(4.3.3) $\underline{L}_1\lambda_\nu \cap \underline{L}_2\lambda_\nu \leq (\underline{L}_1 \cap \underline{L}_2)\lambda_\nu$

since the opposite inclusion is obvious. We need several lemmas to prove what we want.

(4.3.4) Lemma. If $\underline{L} \in \Lambda_{\nu-1}$, then $\underline{L}\lambda_\nu = (\underline{L}\xi_\nu)^{W_\nu}$.

Proof. Let $\alpha : \underline{W}_\nu \to \underline{W}_\nu$, then if $\beta : B \to \overline{B}$ is defined by

$$z_j\beta = \overline{z}_j, \quad j \in \{1,2,\dots\},$$

define $\overline{\alpha} : \underline{W}_{\nu-1} \to \underline{W}_{\nu-1}$ by

$$\overline{y}_1\overline{\alpha} = \overline{y}_1, \quad \overline{z}_j\overline{\alpha} = (z_j\alpha)\beta .$$

Also define $\alpha_1 : \underline{W}_\nu \to \underline{W}_\nu$ by

$$y_1\alpha_1 = y_1\alpha, \quad z_j\alpha_1 = z_j, \quad j \in \{1,2,\ldots\}.$$

Then if $\ell \in \underline{L}$, $(\ell\xi_\nu)\alpha = (\ell\bar{\alpha})\xi_\nu\alpha_1 \in (\underline{L}\xi_\nu)\alpha_1 \leq (\underline{L}\xi_\nu)^{W_\nu}$ as required. This last inclusion is seen from the fact that every normal subgroup of W_ν admits α_1.

(4.3.5) <u>Lemma</u>. If $\underline{L} \in \Lambda_{\nu-1}$, then

$$\underline{L}\lambda_\nu \cap A = (\underline{L} \cap \bar{A})\lambda_\nu, \quad \underline{L}\lambda_\nu \cap B = (\underline{L} \cap \bar{B})\xi_\nu.$$

Proof. For $(\underline{L} \cap \bar{A})\lambda_\nu \leq \underline{L}\lambda_\nu \cap A$ obviously, and if $x \in \underline{L}\lambda_\nu \cap A$ then there exist $\ell_1,\ldots,\ell_t \in L$, $b_1,\ldots,b_t \in B$ such that $x = (\ell_1\xi_\nu)^{b_1} \ldots (\ell_t\xi_\nu)^{b_t}$, whence $1 = x\sigma_1 = (\ell_1\xi_\nu\sigma_1)^{b_1} \ldots (\ell_t\xi_\nu\sigma_1)^{b_t} = (\ell_1\bar{\sigma}_1\xi_\nu)^{b_1} \ldots (\ell_t\bar{\sigma}_t\xi_\nu)^{b_t}$ (where σ_1, $\bar{\sigma}_1$ are the splitting endomorphisms (1.2.8)) and so

$$x = ((\ell_1(\ell_1\bar{\sigma}_1)^{-1})\xi_\nu)^{b_1'} \ldots ((\ell_t(\ell_t\bar{\sigma}_1)^{-1})\xi_\nu)^{b_t'}$$

for some $b_1',\ldots,b_t' \in B$. Hence $x \in (\underline{L} \cap \bar{A})\lambda_\nu$. That $\underline{L}\lambda_\nu \cap B = (\underline{L} \cap \bar{B})\xi_\nu$ is proved similarly.

From this lemma, and from the definition of ξ_ν, we have that

$$(\underline{L}_1\lambda_\nu \cap \underline{L}_2\lambda_\nu) \cap B = (\underline{L}_1\lambda_\nu \cap B) \cap (\underline{L}\lambda_\nu \cap B)$$

$$= (\underline{L}_1 \cap \overline{B})\xi_\nu \cap (\underline{L}_2 \cap \overline{B})\xi_\nu = (\underline{L}_1 \cap \underline{L}_2 \cap \overline{B})\xi_\nu$$

$$= (\underline{L}_1 \cap \underline{L}_2)\lambda_\nu \cap B\,.$$

Hence in order to prove (4.3.3) it suffices to show that $\underline{L}_1\lambda_\nu \cap \underline{L}_2\lambda_\nu \cap A \leq (\underline{L}_1 \cap \underline{L}_2)\lambda_\nu \cap A$, or that

$$(4.3.6) \quad (\underline{L}_1 \cap \overline{A})\lambda_\nu \cap (\underline{L}_2 \cap \overline{A})\lambda_\nu \leq (\underline{L}_1 \cap \underline{L}_2 \cap \overline{A})\lambda_\nu\,.$$

If q belongs to the left hand side of (4.3.6) then, by virtue of (4.2.13) there exist $\dagger$-biwords $q_1,\dots,q_d$, and an integer v such that

$$[q_i, vB] \leq (\underline{L}_1 \cap \overline{A})\lambda_\nu \cap (\underline{L}_2 \cap \overline{A})\lambda_\nu, \quad i \in \{1,\dots,d\}.$$

However $(\underline{L}_1 \cap \overline{A})\lambda_\nu$, $((\underline{L}_2 \cap \overline{A})\lambda_\nu$ are determined by $\dagger$-biwords and therefore Lemma 4.2.15 ensures that for each i, $q_i \in (\underline{L}_1 \cap \overline{A})\lambda_\nu \cap (\underline{L}_2 \cap \overline{A})\lambda_\nu$. The other piece of information from (4.2.13) is that $q \in \mathrm{cl}\{q_1,\dots q_d\}$; hence $(\underline{L}_1 \cap \overline{A})\lambda_\nu \cap (\underline{L}_2 \cap \overline{A})\lambda_\nu$ is determined by $\dagger$-biwords.

In order to finish off the proof of (4.3.3) we need the following lemma. The proof given is due to L.G. Kovács, and replaces my original, much longer, proof.

(4.3.7) Lemma. If $\underline{L} \in \Lambda_{\nu-1}$, $\underline{L} \leq \overline{A}$, and if $q \in \underline{L}\lambda_\nu$ is a $\nvdash$-biword, then $q \in \underline{L}\xi_\nu$.

Proof. By (4.3.4), $q \in (\underline{L}\xi_\nu)^{W_\nu}$ and hence there exist $\ell_i \in L$ and $b_i \in B$ such that

$$q = \prod_{i=1}^{t} (\ell_i \xi_\nu)^{b_i} .$$

Write T for a fixed transversal of B^p in B, with $1 \in T$. Then $b_i = b_i' b_i''$, $b_i' \in B^p$, $b_i'' \in T$ and,

$$q = \prod_{b \in T} \Big(\prod_{b_i''=b} (\ell_i \xi_\nu)^{b_i'} \Big)^b$$

$$= \prod_{b \in T} \Big(\prod_{b_i''=b} (\ell_i^{b_i' \xi_\nu^{-1}}) \xi_\nu \Big)^b$$

$$= \prod_{b \in T} (\ell_b \xi_\nu)^b \quad \text{where } \ell_b \in L .$$

Note that q, ℓ_b all belong to $W_{\nu-1}\xi_\nu \cap A$ and therefore each has its support contained in B^p. However $\mathrm{supp}(\ell_b \xi_\nu)^b$ is contained in $B^p b^{-1}$, and since these cosets are pairwise disjoint,

$$\mathrm{supp}\, q = \bigcup_{b \in T} \mathrm{supp}(\ell_b \xi_\nu)^b \subseteq B^p$$

whence $1 \neq b \in T$ implies $\operatorname{supp} \ell_b \xi_\nu = \emptyset$, or $\ell_b = 1$; thus $q = \ell_1 \xi_\nu \in \underline{L}\xi_\nu$.

To complete the proof of (4.3.3) observe that $(\underline{L}_1 \cap \overline{A})\lambda_\nu \cap (\underline{L}_2 \cap \overline{A})\lambda_\nu$ is determined by $\dagger$-biwords, one of which is $q^\dagger$, say. Since $q^\dagger \in (\underline{L}_1 \cap \overline{A})\lambda_\nu \cap (\underline{L}_2 \cap \overline{A})\lambda_\nu$, Lemma 4.3.7 shows that

$$q^\dagger \in (\underline{L}_1 \cap \overline{A})\xi_\nu \cap (\underline{L}_2 \cap \overline{A})\xi_\nu = (\underline{L}_1 \cap \underline{L}_2 \cap \overline{A})\xi_\nu$$

$$\leq (\underline{L}_1 \cap \underline{L}_2 \cap \overline{A})\lambda_\nu .$$

This completes the proof of (4.3.3).

To finish off the proof of (4.3.2) we need to show that λ_ν is one-to-one. If $\underline{L}_1\lambda_\nu = \underline{L}_2\lambda_\nu$ then $\underline{L}_1\lambda_\nu \cap B = \underline{L}_2\lambda_\nu \cap B$ so that, from (4.3.5) $(\underline{L}_1 \cap \overline{B})\xi_\nu = (\underline{L}_2 \cap \overline{B})\xi_\nu$ whence $\underline{L}_1 \cap \overline{B} = \underline{L}_2 \cap \overline{B}$. Also $\underline{L}_1\lambda_\nu \cap A = \underline{L}_2\lambda_\nu \cap A$ and therefore, by (4.3.5) $(\underline{L}_1 \cap \overline{A})\lambda_\nu = (\underline{L}_2 \cap \overline{A})\lambda_\nu$. Now $(\underline{L}_2 \cap \overline{A})\lambda_\nu$ is determined by $\dagger$-biwords $\ell\xi_\nu$, $\ell \in \underline{L}_2 \cap \overline{A}$, and Lemma 4.3.7 then gives $\ell\xi_\nu \in (\underline{L}_1 \cap \overline{A})\xi_\nu$, or $\ell \in \underline{L}_1 \cap \overline{A}$. That is, $\underline{L}_2 \cap \overline{A} \leq \underline{L}_1 \cap \overline{A}$. In a similar way we prove $\underline{L}_1 \cap \overline{A} \leq \underline{L}_2 \cap \overline{A}$ and therefore $\underline{L}_1 \cap \overline{A} = \underline{L}_2 \cap \overline{A}$, and so $\underline{L}_1 = \underline{L}_2$. This completes the proof of (4.3.2).

We now derive some properties of the embedding λ_ν which are essentially extensions of Lemma 4.2.13, using the Inductive Hypothesis 4.1.4.

(4.3.8) Lemma. To every $\underline{U} \varepsilon \Lambda_\nu$, with $\underline{U} \leq A$ there corresponds a unique $\underline{L} \varepsilon \Lambda_{\nu-1}$, with $\underline{L} \leq \overline{A}$, and an integer $v = v(\underline{U})$ such that

$$[\underline{L}\lambda_\nu, vB] \leq \underline{U} \leq \underline{L}\lambda_\nu.$$

Proof. To each $q \varepsilon U$ associate the $\dagger$-biwords $q_1, \ldots, q_d$ of (4.2.13) and also the integers, v_q say, involved there. If $\underline{S}_q$ is the normalized verbal closure of $\{q_1, \ldots, q_d\}$ then

$$[\underline{S}_q, v_q B] \leq \mathrm{cl}\{q\} \leq \underline{S}_q .$$

As the q_i ~~one~~ are $\dagger$-biwords, there exists $\underline{L}_q \varepsilon \Lambda_{\nu-1}$ with $\underline{L}_q\lambda_\nu = \underline{S}_q$. Write

$$\underline{L} = \Pi\{\underline{L}_q : q \varepsilon \underline{U}\}.$$

Since we have the inductive hypothesis, $\Lambda_{\nu-1}$ has ascending chain condition, and therefore $\underline{L}$ is the join of a finite number of the $\underline{L}_q$'s, say those corresponding to $q^{(1)}, \ldots, q^{(T)} \varepsilon \underline{U}$. Put

$$v = \max\{v_{q^{(i)}} : 1 \leq i \leq T\}.$$

Then $\underline{U} \leq \Pi\{\underline{S}_q : q \varepsilon \underline{U}\} = \Pi\{\underline{L}_q\lambda_\nu : q \varepsilon \underline{U}\} = \Pi\{\underline{L}_{q^{(i)}}\lambda_\nu : 1 \leq i \leq T\}$ $= \underline{L}\lambda_\nu$; and

$$\begin{aligned}
[\underline{L}\lambda_\nu, vB] &= [\Pi\{\underline{L}_{q^{(i)}}\lambda_\nu : 1 \leq i \leq T\}, vB] \\
&= \prod_{i=1}^{T} [\underline{L}_{q^{(i)}}\lambda_\nu, vB] \\
&\leq \prod_{i=1}^{T} [\underline{L}_{q^{(i)}}\lambda_\nu, v_{q^{(i)}}B] \\
&\leq \underline{U},
\end{aligned}$$

which finishes the proof of the theorem except for the uniqueness of $\underline{L}$: if there exists $\underline{L}'$, v' with the asserted properties, then

$$[\underline{L}'\lambda_\nu, v'B] \leq \underline{L}\lambda_\nu \quad \text{and} \quad [\underline{L}\lambda_\nu, vB] \leq \underline{L}'\lambda_\nu,$$

and Lemma 4.2.15 shows that $\underline{L}'\lambda_\nu \leq \underline{L}\lambda_\nu \leq \underline{L}'\lambda_\nu$, or $\underline{L}\lambda_\nu = \underline{L}'\lambda_\nu$ whence $\underline{L} = \underline{L}'$ from (4.3.2).

The last lemma necessary to prove Theorem 4.2.3 is the following.

(4.3.9) Lemma. Let $\underline{L} \in \Lambda_{\nu-1}$, $\underline{L} \leq \overline{A}$, and let v be a natural number. There exists a natural number $s = s(\underline{L}, v)$ such that if $q \in \underline{L}\lambda_\nu$ is special and involves more than s elements of the free generating set $\{z_1, z_2, \ldots\}$ then $q \in [\underline{L}\lambda_\nu, vB]$.

Proof. The proof will be by induction on ν. If $\nu = 1$ then $q \in \underline{L}\lambda_\nu$ can be written

$$q = \prod_{i=1}^{t} [y_1, z_1^{\delta_{i1}}, \ldots, z_u^{\delta_{iu}}]^{\alpha_i}$$

where $1 \leq \delta_{ij} \leq p - 1$ for all i,j, and $(\delta_{i1}, \ldots, \delta_{iu})$ are distinct for distinct i. Employ (4.2.7) u times to deduce that

$$y_1^{\alpha_i} \in (\underline{L}\lambda_\nu)_\delta, \quad i \in \{1, \ldots, t\},$$

where $\delta = \sum_{j=1}^{u} \max_i \delta_{ij}$. Lemma 4.2.15 then yields $y_1^{\alpha_i} \in \underline{L}\lambda_\nu$ whence

$q \varepsilon [\underline{L}\lambda_{\nu}, uB]$. Hence $s = v$ will do, and the proof of the first step is complete.

Assume, therefore, that $v \geq 2$ and that the lemma is proved for $\nu - 1$. Associate with q the special †-biwords $q_1, \dots, q_d$ of (4.2.13). By (4.2.13) and (4.2.15), $q_1, \dots, q_d \varepsilon \underline{L}\lambda_{\nu}$. Suppose that q_i involves s_i variables z_j, $i \varepsilon \{1, \dots, d\}$. Let $\overline{\underline{L}} \varepsilon \Lambda_{\nu-2}$ and $\underline{L} \leq \overline{\underline{L}}\lambda_{\nu-1}$ according to (4.3.8), and define

$$s(\underline{L}, v) = s(\overline{\underline{L}}, v(\underline{L}) + v) + v$$

where $v(\underline{L})$ is defined as in (4.3.8), assuming inductively that s can be defined for $\nu-1$.

Now by (4.2.13) and (4.3.4), q is in the normal closure of $q_1, \dots, q_d$. Hence we may write

$$q = \prod_{j=1}^{t} [q_{i_j}, z_{k_{j1}}^{\alpha_{j1}}, \dots, z_{k_{jr_j}}^{\alpha_{jr_j}}]^{\beta_j}$$

where $1 \leq \alpha_{j\ell} \leq p^{\nu} - 1$, all j, ℓ. We may assume, by using the argument leading to Theorem 33.45 in [3], that if q involves precisely the variables $y_1, z_1, \dots, z_u$ (where $u \geq s(\underline{L}, v)$) then for each j, the set of variables z_w involved in q_{i_j} together with $z_{k_{j1}}, \dots, z_{k_{jr_j}}$ is just $\{z_1, \dots, z_u\}$. (This can also be concluded from a close look at the proof of (4.2.13)). If for some $j \in \{1,\dots,t\}$, $s_{i_j} \leq s(\underline{L},v) - v$ then $|\{k_{j1},\dots,k_{jr_j}\}| \geq v$ and therefore the commutator beginning with q_{i_j} belongs to $[\underline{L}\lambda_\nu, vB]$. If on the other hand $s_{i_j} > s(\underline{L},v) - v$ for some $j \in \{1,\dots,t\}$, then $s_{i_j} > s(\overline{\underline{L}}, v(\underline{L}) + v)$; hence

$$
\begin{aligned}
q_{i_j}\xi_\nu^{-1} &\in [\overline{\underline{L}}\lambda_{\nu-1}, (v(\underline{L}) + v)\overline{B}] \\
&= [[\overline{\underline{L}}\lambda_{\nu-1}, v(\underline{L})\overline{B}], v\overline{B}] \\
&\leq [\underline{L}, v\overline{B}]
\end{aligned}
$$

so that $q_{i_j} \in [\underline{L}, v\overline{B}]\lambda_\nu \leq [\underline{L}\lambda_\nu, vB]$. Clearly, then, the commutator beginning with this q_{i_j} belongs to $[\underline{L}\lambda_\nu, vB]$. Therefore $q \in [\underline{L}\lambda_\nu, vB]$.

Proof of (4.2.3). Suppose that $\underline{U}_1 \leq \underline{U}_2 \leq \dots \leq \underline{U}_i \leq \dots$ is an ascending chain in Λ_ν. Clearly the chain

$$\underline{U}_1 \cap B \leq \underline{U}_2 \cap B \leq \dots \leq \underline{U}_i \cap B \leq \dots$$

terminates in a finite number of steps; hence it suffices to consider the chain of the $\underline{U}_i \cap A$, or, without loss of generality, to assume $\underline{U}_i \leq A$, $i \in \{1,2,\dots\}$. In this case (4.3.8) ensures that there exists to each $i \in \{1,2,\dots\}$ a unique $\underline{L}_i \in \Lambda_{\nu-1}$ and an integer v_i such that

$$[\underline{L}_i\lambda_\nu, v_iB] \leq \underline{U}_i \leq \underline{L}_i\lambda_\nu.$$

Now $i \le j$ implies $\underline{L}_i \le \underline{L}_j$; for

$$[\underline{L}_i\lambda_\nu, v_iB] \le \underline{U}_i \le \underline{U}_j \le \underline{L}_j\lambda_\nu$$

and (4.2.15) and (4.3.7) give $\underline{L}_i \le \underline{L}_j$. Under the inductive hypothesis (4.1.4) it follows that there exists an integer m such that for $m \le i$, $\underline{L}_m = \underline{L}_i$. Hence for $m \le i$

$$[\underline{L}_m\lambda_\nu, v_mB] \le \underline{U}_i \le \underline{L}_m\lambda_\nu \ .$$

By virtue of (4.3.9) there exists an integer $s_0 = s(\underline{L}_m, v_m)$ such that if $q \varepsilon \underline{U}_i$ is special and involves more than s_0 variables z_j, then $q \varepsilon [\underline{L}_m\lambda_\nu, v_mB]$. It follows that $\underline{U}_i$ can be determined, modulo $[\underline{L}_m\lambda_\nu, v_mB]$, by bilaws involving at most $y_1, z_1, \ldots, z_{s_0}$. By the inductive hypothesis (4.1.4), Theorems 2.1.1 and 1.5.4, $\underline{L}_m\lambda_\nu$ is finitely based, and therefore so is $[\underline{L}_m\lambda_\nu, v_mB]$; we may suppose the latter to have a basis involving t_0 variables z_j. Hence $\underline{U}_i$, $m \le i$, is defined by laws involving at most $s_0 + t_0$ variables z_j. It follows that the biverbal sub-bigroup lattice between $[\underline{L}_m\lambda_\nu, v_mB]$ and $\underline{L}_m\lambda_\nu$ is isomorphic to the corresponding one in the free bigroup of rank $(1, s_0 + t_0)$ of $\underline{\underline{A}} \circ {\underline{\underline{A}}_p}_\nu$. This is however, a finitely generated metabelian group, and, by a well-known result of P. Hall [20], has ascending chain condition on normal subgroups. This completes the proof of (4.2.3) and therefore of (4.0.1).

4.4 Descending chain condition for $\underline{\underline{A}}_m \circ \underline{\underline{A}}$

The first lemma proved here is similar to (4.1.3); indeed a similar proof will do. However we give a different one here.

(4.4.1) Lemma. If m,t are coprime, then the bigroup C_m wr C_t generates $\underline{\underline{A}}_m \circ \underline{\underline{A}}_t$. The bigroup C_m wr C generates $\underline{\underline{A}}_m \circ \underline{\underline{A}}$. (Here C_m, C_t are cycles of order m,t, and C is an infinite cycle).

Proof. Let $\underline{G}$ be critical in $\underline{\underline{A}}_m \circ \underline{\underline{A}}_t$; then if either $A_1(\underline{G})$ or $A_2(\underline{G}) = 1$, $\underline{G} \in \text{svar}\{C_m \text{ wr } C_t\}$. If $A_1(\underline{G}), A_2(\underline{G}) \neq 1$ then by (3.2.1), $A_2(\underline{G})$ is cyclic, and $A_1(\underline{G})$ is generated quâ $A_2(\underline{G})$-group by a single element; hence since C_m wr C_t is the split-free bigroup of rank $(1,1)$ in $\underline{\underline{A}}_m \circ \underline{\underline{A}}_t$, $\underline{G}$ is an epimorphic image of C_m wr C_t. That is, $\underline{\underline{A}}_m \circ \underline{\underline{A}}_t$ is generated by C_m wr C_t.

To prove the rest, suppose that $\{t_1,t_2,\dots\}$ is an infinite set of natural numbers all prime to m, with $t_i | t_{i+1}$ for all $i \in \{1,2,\dots\}$. We show that $\underline{\underline{A}}_m \circ \underline{\underline{A}} = \vee\{\underline{\underline{A}}_m \circ \underline{\underline{A}}_{t_i} : i = 1,2,\dots\}$; clearly this implies that C_m wr C generates $\underline{\underline{A}}_m \circ \underline{\underline{A}}$. Consider the descending chain

$$A_2(\underline{W})^{t_1}[A_1(\underline{W}),A_2(\underline{W})^{t_1}] \geq A_2(\underline{W})^{t_2}[A_1(\underline{W}),A_2(\underline{W})^{t_2}] \geq \dots$$

of biverbal sub-bigroups of $\underline{W} = F_{(\omega,\omega)}(\underline{\underline{A}}_m \circ \underline{\underline{A}})$; these biverbal sub-bigroups are those corresponding to the bivarieties $\underline{\underline{A}}_m \circ \underline{\underline{A}}_{t_i}$.

Now the chain

$$A_2(\underline{W})^{t_1} \geq A_2(\underline{W})^{t_2} \geq \ldots$$

has trivial intersection, and if we can show the same for the chain

$$[A_1(\underline{W}), A_2(\underline{W})^{t_1}] \geq [A_1(\underline{W}), A_2(\underline{W})^{t_2}] \geq \ldots$$

then we shall have proved what we want. To this end, let T_i be a fixed set of coset representatives for $A_2(\underline{W})^{t_i}$ in $A_2(\underline{W})$, such that $T_1 \subseteq T_2 \subseteq \ldots$. Now if $a \in [A_1(\underline{W}), A_2(\underline{W})^{t_i}]$ then, by an argument similar to that in (4.3.7), we may write

$$\text{supp } a = \cup\{\Gamma_b^{(i)} : b \in T_i\}$$

where $\Gamma_b^{(i)} \subseteq A_2(\underline{W})^{t_i} b^{-1}$. If $a \in [A_1(\underline{W}), A_2(\underline{W})^{t_i}]$ for all i, then clearly, since supp a is finite, each $\Gamma_b^{(i)} = \emptyset$ and therefore $a = 1$. This completes the proof of (4.4.1).

The next lemma is a trivial adaptation of an unpublished result of L.G. Kovács about varieties of metabelian groups.

(4.4.2) Lemma. If $\underset{\sim}{U}$ is a proper sub-bivariety of $\underline{\underline{A}}_m \circ \underline{\underline{A}}$ then all bigroups in $\underset{\sim}{U}$ satisfy the bilaw

$$[y_1, rz_1^s]^t,$$

for some integers r,s,t with $m \nmid t$.

Proof. Consider the split-free bigroup of rank (ω,ω) in $\underline{\underline{A}}_m \circ \underline{\underline{A}}$, call it $\underline{W}$ say, and the biverbal sub-bigroup $\underline{U}$ of $\underline{W}$. Then

$$(4.4.3) \qquad \prod_{i=0}^{v} y_1^{\alpha_i z_1^i} \in U$$

for some integers $\alpha_0,\ldots,\alpha_v$ with $m \nmid \alpha_0$; for, if there is no such relation holding, then the factor bigroup $\underline{W}/\underline{U}$ has a sub-bigroup $\langle y_1\underline{U}, z_1\underline{U}\rangle$ isomorphic to C_m wr C which generates $\underline{\underline{A}}_m \circ \underline{\underline{A}}$, by (4.4.1).

From (4.4.3) we deduce that

$$\prod_{i=0}^{v} y_1^{\alpha_i z_1^{ij}} \; , \quad j \in \{0,\ldots,v\}$$

are bilaws in $\underline{W}/\underline{U}$ and therefore in $\underset{\sim}{U}$. Working in the endomorphism ring of $A_1(\underline{W}/\underline{U})$ we have

$$\sum_{i=0}^{v} \alpha_i z_1^{ij} = 0 \; , \quad j \in \{0,\ldots,v\}.$$

This implies $\alpha_0 \prod_{j<i} (z_1^i - z_1^j) = 0$ and so $\alpha_0 \prod_{j<i} (z^{i-j} - 1) = 0$.

Hence

$$\alpha_0 \prod_{k=1}^{v} (z_1^k - 1)^{v-k+1} = 0$$

whence

$$\alpha_0 (z_1^{v!} - 1)^{\frac{1}{2}v(v+1)} = 0 \; .$$

Put $r = \frac{1}{2}v(v+1)$, $s = v!$, $t = \alpha_0$ and we have

$$[y_1, rz_1^s]^t \in \underline{U} \; .$$

(4.4.4) Lemma. Every proper sub-bivariety of $\underline{\underline{A}}_p \circ \underline{\underline{A}}$, where p is prime, is contained in some $\underline{\underline{E}} \circ \underline{\underline{A}} \vee \underline{\underline{A}}_p \circ \underline{\underline{A}}_n$.

Proof. If $\underset{\sim}{U} \subset \underline{\underline{A}}_p \circ \underline{\underline{A}}$ then every bigroup in $\underset{\sim}{U}$ has a bilaw $[y_1, rz_1^s]$, since $p \nmid t$. Let ν be a natural number chosen so that $p^\nu \geq r$. Then every bigroup in $\underset{\sim}{U}$ has a bilaw $[y_1, z_1^{sp^\nu}]$, since

$$[y_1, z_1^{sp^\nu}] = \prod_{\mu=1}^{p^\nu} [y_1, z_1^s]^{\binom{p^\nu}{\mu}}$$

modulo the bilaws of $\underline{\underline{A}}_p \circ \underline{\underline{A}}$, and $\mu < r \leq p^\nu$ implies $p \mid \binom{p^\nu}{\mu}$.

In particular the non-abelian critical bigroups $\underline{G}$ of $\underset{\sim}{U}$ satisfy $[y_1, z_1^{sp^\nu}]$. Since $A_1(\underline{G})$ is self-centralizing and not 1 it follows that $z_1^{sp^\nu}$ is a bilaw in $\underline{G}$, and hence $\underline{G} \in \underline{\underline{A}}_p \circ \underline{\underline{A}}_{sp^\nu}$. This concludes the proof.

(4.4.5) Theorem. $\underline{\underline{A}}_{p^\mu} \circ \underline{\underline{A}}$ has descending chain condition on sub-bivarieties.

Proof. The proof is by induction on μ, the previous lemma providing a starting point. We show that all descending chains of bivarieties between $\underline{\underline{A}}_{p^\mu} \circ \underline{\underline{A}}$ and $\underline{\underline{A}}_{p^{\mu-1}} \circ \underline{\underline{A}}$ break off; hence if we assume that $\underline{\underline{A}}_{p^{\mu-1}} \circ \underline{\underline{A}}$ has descending chain condition on sub-bivarieties, Theorem 2.1.3 gives that $\underline{\underline{A}}_{p^\mu} \circ \underline{\underline{A}}$ does also.

Work in the split-free bigroup of rank (ω,ω) in $\underline{\underline{A}}_{p^\mu} \circ \underline{\underline{A}}$, call it $\underline{W}$ say, with $A = A_1(\underline{W})$, $B = A_2(\underline{W})$. The mapping $\alpha : W/A^p \to \langle A^{p^{\mu-1}}, B\rangle$ defined by

$$(baA^p)\alpha = ba^{p^{\mu-1}}$$

is easily checked to be an isomorphism; hence $\langle A^{p^{\mu-1}}, B\rangle$ is isomorphic to $\underline{F}_{(\omega,\omega)}(\underline{\underline{A}}_p \circ \underline{\underline{A}})$. Now if β is a self-morphism of $\langle A^{p^{\mu-1}}, B\rangle$ then $y_i^{p^{\mu-1}}\beta = a_i^{p^{\mu-1}}$, $a_i \in A$. Define $\beta^* : \underline{W} \to \underline{W}$ by

$$y_i\beta^* = a_i, \quad z_j\beta^* = z_j\beta, \quad i,j \in \{1,2,\ldots\}.$$

Clearly $\beta^*|\langle A^{p^{\mu-1}}, B\rangle = \beta$ and therefore a fully invariant sub-bigroup of $\underline{W}$ contained in $\langle A^{p^{\mu-1}}, B\rangle$ is fully invariant in $\langle A^{p^{\mu-1}}, B\rangle$. Therefore all ascending chains of normal, fully invariant sub-bigroups of $\underline{W}$ contained in $\langle A^{p^{\mu-1}}, B\rangle$ break off; in other words, all descending chains of bivarieties between $\underline{\underline{A}}_{p^\mu} \circ \underline{\underline{A}}$ and $\underline{\underline{A}}_{p^{\mu-1}} \circ \underline{\underline{A}}$ break off. This completes the proof of (4.4.5).

It remains to remark that for relatively prime integers u,v:

$$\underline{\underline{A}}_{uv} \circ \underline{\underline{A}} = \underline{\underline{A}}_u \circ \underline{\underline{A}} \vee \underline{\underline{A}}_v \circ \underline{\underline{A}}$$

by (1.7.4), and then (2.1.2) and (4.4.5) give Theorem 4.0.2 .

To prove (4.0.3) the only unproved thing is that $\underline{\underline{A}}_m \circ \underline{\underline{A}} \vee \underline{\underline{A}} \circ \underline{\underline{A}}_n$ is finitely based. This is shown by the following lemma, due to L.G. Kovács.

(4.4.6) Lemma. For natural numbers m,n, $\underline{\underline{A}}_m \circ \underline{\underline{A}} \vee \underline{\underline{A}} \circ \underline{\underline{A}}_n$ has a finite basis for its bilaws.

Proof. In fact $\underline{\underline{A}}_m \circ \underline{\underline{A}} \vee \underline{\underline{A}} \circ \underline{\underline{A}}_n$ is determined by the bilaw $[y_1^m, z_1^n]$, together with the bilaws of $\underline{\underline{A}} \circ \underline{\underline{A}}$. We have to show that if $\underline{W}$ is the split-free bigroup of rank (ω,ω) in $\underline{\underline{A}} \circ \underline{\underline{A}}$, then

$$A_1(\underline{W})^m \cap (A_2(\underline{W})^n)^W = [A_1(\underline{W})^m, A_2(\underline{W})^n].$$

Now if γ is the natural morphism from $\underline{W}$ to the split-free bigroup of rank (ω,ω) of $\underline{\underline{A}} \circ \underline{\underline{A}}_n$, then

$$\ker \gamma \cap A_1(\underline{W}) = [A_1(\underline{W}), A_2(\underline{W})^n].$$

Since $A_1(\underline{W})/[A_1(\underline{W}), A_2(\underline{W})^n]$ is therefore a free abelian group, $[A_1(\underline{W}), A_2(\underline{W})^n]$ is complemented in $A_1(\underline{W})$. Hence

$$A_1(\underline{W})^m \cap (A_2(\underline{W})^n)^W = A_1(\underline{W})^m \cap [A_1(\underline{W}), A_2(\underline{W})^n]$$

$$= [A_1(\underline{W}), A_2(\underline{W})^n]^m = [A_1(\underline{W})^m, A_2(\underline{W})^n].$$

This completes the proof of (4.4.6) and therefore that of (4.0.3).

CHAPTER 5

FURTHER RESULTS AND APPLICATIONS

In this chapter we shall attempt to pin down the structure of the lattice of sub-bivarieties of $\underline{\underline{A}}_m \circ \underline{\underline{A}}_n$ further then we have done already. We shall show that essentially every thing can be described in terms of prime-power exponent sub-bivarieties, and for these we get a complete classification only for the sub-bivarieties of $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p$. Thus when m,n are nearly coprime, a complicated, yet complete description of $\Lambda(\underline{\underline{A}}_m \circ \underline{\underline{A}}_n)$ can be given. In section 5.5 the question of classifying the subvarieties of $\underline{\underline{A}}_m\underline{\underline{A}}_n$ is taken up, and we show how a complete classification can be given in the case m,n nearly coprime, and that this type of classification cannot be extended to general m,n.

A question that has come into vogue recently is that of distributivity of the lattice of varieties of groups. It is known, for example, that the lattice of varieties of A-groups is distributive (Cossey [4]), that the lattice of nilpotent varieties of class at most 3 is distributive (Jónsson [11]), and that certain metabelian varieties form distributive lattices (Brisley [7], Weichsel [12], Newman [14, 15].). On the other hand Higman [23] constructed a non-distributive lattice of varieties of exponent $p(\geq 7)$ and class at most 6. The formulation of some of the results in this chapter is done with the question of distributivity in mind. Among the results proved in this direction is the following: if $\underline{\underline{V}}$ is a variety of metabelian groups of bounded exponent, such that Sylow p-subgroups of groups in $\underline{\underline{V}}$ have class at most $c \leq p$, then $\Lambda(\underline{\underline{V}})$ is distributive provided $\Lambda(\underline{\underline{V}} \wedge \underline{\underline{N}}_c)$ is distributive; and if Sylow p-subgroups of groups in $\underline{\underline{V}}$ have class greater than p, then $\Lambda(\underline{\underline{V}})$ may not be distributive. The results of Brisley, Weichsel and Jónsson mentioned above can then be employed to get positive results about distributivity.

5.1 Further results on critical bigroups in $\underline{\underline{A}} \circ \underline{\underline{A}}$

We saw in Chapter 3 something of the structure of non-nilpotent critical bigroups in $\underline{\underline{A}} \circ \underline{\underline{A}}$; in particular we saw that each such bigroup $\underline{G}$ has a sub-bigroup $\underline{F}^*$, which has p-power exponent and does not belong to the variety of its proper sub-bigroups. Unfortunately $\underline{F}^*$ may be non-monolithic and therefore non-critical: one example of such a situation occurs with $\underline{F}^*$ equal to the central factor group of C_2 wr $(C_4 \times C_2)$. It is easy to prove a general result which implies that if $\underline{F}^*$ is monolithic, then it is critical (cf. (1.2) in [5] of Kovács and Newman):

(5.1.1) Theorem. If $\underline{G} \in \underline{\underline{A}} \circ \underline{\underline{A}}$ is monolithic and not in the bivariety generated by its proper sub-bigroups, then $\underline{G}$ is critical.

Proof. Let $\underline{G} = (G,A,B)$. It follows as in (3.1.6) that there exists a unique maximal normal subgroup of G contained in A, and hence that $N = A^p[A,H]$ where A is a p-group and H is the Sylow p-subgroup of B. Also it is easy to see that the maximal sub-bigroups of $\underline{G}$ are AB_o, NB where B_o is maximal in B.

We show that $\underline{G}/\sigma\underline{G} \varepsilon \text{ svar}\{NB\}$. If q is a bilaw in NB we may assume it to be special, involving $y_1, z_1, \ldots, z_{t-1} (t \geq 1)$. If $\alpha : \underline{Q}_2 \rightarrow \underline{G}$, define $\beta : \underline{Q}_2 \rightarrow NB$ by

$$y_1\beta = (y_1\alpha)^p, \quad z_i\beta = z_i\alpha, \qquad i \varepsilon \{1,2,\ldots\},$$

and $\gamma : \underline{Q}_2 \rightarrow NB$ by

$$y_1\gamma = [y_1, z_t^r], \quad z_i\gamma = z_i\alpha, \qquad i \varepsilon \{1,2,\ldots\}$$

where $r = \exp B/H$. It is easily seen that $(q\alpha)^p = q\beta = 1$, $[q,z_t^r]\alpha = q\gamma = 1$. Thus $q(\underline{G})$ lies in the socle of AH, that is, in $\sigma\underline{G}$. Hence q is a law in $\underline{G}/\sigma\underline{G}$. Since all proper quotient bigroups of $\underline{G}$ are quotient bigroups of $\underline{G}/\sigma\underline{G}$ it follows from the hypotheses that $\underline{G}$ is critical.

(5.1.2) Theorem. If $\underline{P} \varepsilon \underline{\underline{A}} \circ \underline{\underline{A}}$ is nilpotent and critical, $A_1(\underline{P}) \neq 1$, then there exists to each natural number t which is prime to the order of P, a non-nilpotent critical bigroup $\underline{G} \varepsilon \underline{\underline{A}} \circ \underline{\underline{A}}$ with $|K| = t$ and $\underline{F}^* \cong \underline{P}$.

If $\underline{\overline{G}} \in \underline{\underline{A}} \circ \underline{\underline{A}}$, $\overline{G} = (\overline{G},\overline{A},\overline{H} \times \overline{K})$ where $\overline{AH}$ is a p-group, $\overline{A}$ is non-trivial and self-centralizing in $\overline{G}$ and $\overline{K}$ is a p'-cycle which acts fixed point free on $\overline{A}$, then there exist critical bigroups $\underline{G}_1,\dots,\underline{G}_w$ such that each $\underline{F}^*_i$ is critical, each $|K_i| = |\overline{K}|$ and $\mathrm{svar}\{\underline{G}_1,\dots,\underline{G}_w\} = \mathrm{svar}\{\underline{G}\}$.

Proof. From (3.1.6), $A_1(\underline{P})$ is monogenic quâ P operator group; also $\underline{P}$ is monolithic. Choose the natural number s so that $t \mid p^{s\cancel{-1}}-1$ but $t \nmid p^u - 1$ if $u < s$. Let $\underline{P}_1,\dots,\underline{P}_s$ be isomorphic copies of P, say $\lambda_i : \underline{P}_1 \to \underline{P}_i$ is an isomorphism. If $a_i \in A_1(\underline{P}_i)$ is such that

$$\langle a_i\rangle^{A_2(\underline{P}_i)} = A_1(\underline{P}_i)$$

we may suppose $a_i = a_1\lambda_i$, $i \in \{1,\dots,s\}$.

In the direct product $\underline{P}_1 \times \quad \times \underline{P}_s$ write $A = A_1(\underline{P}_1 \times \dots \times \underline{P}_s)$, H for the diagonal of $A_2(\underline{P}_1 \times \dots \times \underline{P}_s)$; that is

$$H = \{f : f(i) = f(1)\lambda_i \in A_2(\underline{P}_i)\},$$

and set $\underline{F} = (AH,A,H)$. We aim to extend $\underline{F}$ by a t-cycle so that the resulting bigroup is critical.

Put $A_o = \langle a_1,\dots,a_s\rangle \leq A$ and let $K = \langle k : k^t = 1\rangle$ be a cycle of order t. According to Cossey (Theorem 4.2.2 in [4]) there exists a unique critical group A_oK; in this group let k induce an automorphism α on A_o. Define the action of α on H to be the identity mapping of H. Then α extends to an automorphism of $\underline{F}$. For, let

$$r = r(a_1,\ldots,a_s,h_1,\ldots,h_u) = 1$$

be a relation among the generating set $\{a_1,\ldots,a_s\} \cup H$ of F. Clearly $r = 1$ is equivalent to a set of relations

$$r_i = r_i(a_i,h_1,\ldots,h_u) = 1, \quad i \in \{1,\ldots,s\}.$$

Because of the way we have constructed F, $r_i = 1$ is a relation in F if and only if $r_i(a_j,h_1,\ldots,h_u) = 1$ is a relation in F, $i,j \in \{1,\ldots,s\}$. If $a_0 = \prod_{i=1}^{s} a_i^{\beta_i}$ is any element of A_0, then

$$r_i(a_0,h_1,\ldots,h_u) = \prod_{j=1}^{s} r_i(a_j^{\beta_j},h_1,\ldots,h_u)$$

$$= \prod_{j=1}^{s} r_i(a_j,h_1,\ldots,h_u)^{\beta_j} = 1.$$

By von Dyck's Theorem, α may be extended to an endomorphism of F. Since $A_0\alpha = A_0$, $F\alpha = F$ and consequently α is an automorphism of $\underline{F}$.

Next we verify that (FK,A,HK) is critical. As a first step we show that K acts fixed point free on A. If $N = A^p[A,H]$, $N_i = A_1(\underline{P}_i)^p[A_1(\underline{P}_i), A_2(\underline{P}_i)]$ then $N_i = N \cap A_1(\underline{P}_i)$ and

$$N = N_1 \times \ldots \times N_s,$$

so that $A/N \cong A_1/N_1 \times \ldots \times A_s/N_s \cong A_0/A_0^p$ where the isomorphisms are K-isomorphisms. Hence K acts faithfully and irreducibly on A/N. Now there exist elements $h_1,\ldots,h_u \varepsilon H$ and an integer $\gamma \geq 0$ such that

$$1 \neq x_i = [a_i,h_1,\ldots,h_u]^{p^\gamma} \varepsilon\ \sigma\underline{P}_i, \quad i \varepsilon \{1,\ldots,s\};$$

and the mapping $a_iN \to x_i$ extends to a K-homomorphism μ of A/N into $\sigma\underline{F}$, the socle of $\underline{F}$. In fact μ is a K-isomorphism since K acts faithfully and irreducibly on A/N and since clearly $\langle x_1,\ldots,x_s\rangle = \sigma F$. It follows that K acts faithfully and irreducibly on σF, and therefore fixed point free on A. Finally a calculation similar to that in the proof of (3.3.1) shows that the maximal sub-bigroups of FK are precisely AH_0K, AHK_0, NHK where H_0, K_0 are maximal in H,K respectively; and, as in the proof of (3.1.6), $\mathrm{svar}(AH_0,A,H_0) = \mathrm{svar}(A_1(\underline{P}_1)H_0,A_1(\underline{P}),H_0)$, and also $\neq \mathrm{svar}(NH,N,H) = \mathrm{svar}(N_1H,N_1,H)$. By hypothesis therefore, there exists a biword q which is a bilaw in AH_0, NH, but not in AH. If q involves the variables $z_1,z_2,\ldots,z_u$ from $\{z_1,z_2,\ldots,\}$ and $t_1,\ldots,t_v$ are the maximal divisors of t, not equal to 1 (if any), consider the biword

$$q'' = [q', z_{u+1}^{t_1},\ldots,z_{n+v}^{t_v}]$$

where q' is obtained from q by replacing z_i, $i \varepsilon \{1,\ldots,u\}$ by z_i^t. Then q'' is a bilaw in all maximal sub-bigroups of FK but not in FK itself. Since FK is monolithic, (5.1.1) concludes the proof of the first part of the theorem.

To prove the second assertion let $\overline{\underline{G}}$ be as stated. Now $\overline{\underline{F}} = (\overline{AH},\overline{A},\overline{H})$ is contained in the bivariety irredundantly generated by some of its critical factors $\underline{F}_1^*,\ldots,\underline{F}_w^*$ say. We may suppose $A_1(\underline{F}_i^*) \neq 1$, $i \in \{1,\ldots,w\}$. For if $A_1(\underline{F}_1^*)$, say, were 1, then $\exp A_2(\underline{F}_1^*) > \exp A_2(\underline{F}_i^*)$, $i \in \{2,\ldots,w\}$ (or else $\underline{F}_1^*$ would be redundant), and then $\underline{F}_1^*,\ldots,\underline{F}_w^*$, and therefore $\overline{\underline{F}}$ would have a bilaw $[y_1,z_1^{p^\beta}]$ where $z_1^{p^\beta}$ is not a bilaw in $\overline{\underline{F}}$. But $A_1(\overline{\underline{F}})$ is non-trivial and self-centralizing in $\underline{F}$ and therefore we would have a contradiction. According to the first part of the theorem, we may construct critical bigroups $\underline{G}_1,\ldots,\underline{G}_w$ from $\underline{F}_1^*,\ldots,\underline{F}_w^*$ respectively, and the same cycle isomorphic to K. Then svar $\overline{\underline{G}}$ = svar$\{\underline{G}_1,\ldots,\underline{G}_w\}$; for if q is a biword, and $q_1,\ldots,q_v$ correspond to q,p,t by Theorem 3.4.4, then, by (3.4.5), q is a bilaw in $\overline{\underline{G}}$ if and only if $q_1,\ldots,q_v$ are bilaws in $\overline{\underline{F}}$, hence if and only if $q_1,\ldots,q_v$ are bilaws in $\underline{F}_1^*,\ldots,\underline{F}_w^*$, and therefore if and only if q is a bilaw in $\underline{G}_1,\ldots,\underline{G}_w$.

We have already seen that non-nilpotent critical bigroups in $\underline{\underline{A}} \circ \underline{\underline{A}}$ are critical quâ groups (3.3.4). The converse, suitably interpreted, is also true.

(5.1.3) Theorem. If G is a non-nilpotent, metabelian, critical group, then G' is complemented in G, say by B, and (G,G',B) is a critical bigroup. Moreover, all such bigroups arising from G are isomorphic.

Proof. Since G is non-nilpotent there exists a natural number u such that $1 \neq G_{(u)} = G_{(u+1)} = \dots$. Since $G_{(u)}$ is abelian, it is complemented in G, and all such complements are conjugate (Shenkman [1]). The same proof as that of (3.2.1) can now be used, together with (3.1.2); the conjugacy of complements ensures that different bigroups (G,G',B) are isomorphic.

5.2 The bivarieties $\underline{\underline{A}}_m \circ \underline{\underline{A}}_n$

We commence with a few remarks of a general character.

(5.2.1) Definition. If $\underset{\sim}{B}$ is a bivariety, define

$$\underset{\sim}{B}\phi = \text{svar}\{\underline{G} \in \underset{\sim}{B} : \underline{G} \text{ critical, } A_1(\underline{G}) \neq 1\},$$

$$\underset{\sim}{B}\psi = \{\underline{G} \in \underset{\sim}{B} : A_1(\underline{G}) = 1\}.$$

Also define

$$\Phi(\underset{\sim}{B}) = \{\underset{\sim}{C}\phi : \underset{\sim}{C} \subseteq \underset{\sim}{B}\},$$

$$\Psi(\underset{\sim}{B}) = \{\underset{\sim}{C}\psi : \underset{\sim}{C} \subseteq \underset{\sim}{B}\}.$$

(5.2.2) <u>Definition</u>. Denote the lattice of sub-split-varieties of a split-variety $\underset{\sim}{S}$ by $\Lambda(\underset{\sim}{S})$.

(5.2.3) <u>Lemma</u>. Each of $\Phi(\underset{\sim}{B})$, $\Psi(\underset{\sim}{B})$ equipped with the inclusion order inherited from $\Lambda(\underset{\sim}{B})$ is a complete lattice. The mappings $\phi : \Lambda(\underset{\sim}{B}) \rightarrow \Phi(\underset{\sim}{B})$, $\psi : \Lambda(\underset{\sim}{B}) \rightarrow \Psi(\underset{\sim}{B})$ are onto lattice-homomorphisms.

Proof. Now $\Psi(B)$ is clearly a sub-lattice of $\underset{\sim}{B}$, in fact equal to $\Lambda(\underset{\sim}{B} \wedge \underline{\underline{E}} \circ \underline{\underline{O}})$ where $\underline{\underline{O}}$ is the variety of all groups). In $\Phi(\underset{\sim}{B})$, the join of any subset is equal to its join in $\Lambda(\underset{\sim}{B})$, and the intersection of any subset is the largest element of $\Phi(\underset{\sim}{B})$ contained in all elements of the subset: indeed if $\underset{\sim}{C}_i \subseteq \underset{\sim}{B}$ $(i \in I)$, then

$$\dot{\wedge}\{\underset{\sim}{C}_i\phi : i \in I\} = (\wedge\{\underset{\sim}{C}_i : i \in I\})\phi.$$

(An instance of $\underset{\sim}{C}_1 \dot{\wedge} \underset{\sim}{C}_2 \neq \underset{\sim}{C}_1 \wedge \underset{\sim}{C}_2$ occurs in the lattice $\Lambda(\underline{\underline{A}}_4 \circ \underline{\underline{A}}_4 \wedge N_3)$ in section 5.4 with $\underset{\sim}{C}_1 = \underline{\underline{A}}_2 \circ \underline{\underline{A}}_4 \wedge \underset{\sim}{N}_3$, $\underset{\sim}{C}_2 = \underset{\sim}{V}_3$).

That ψ is a homomorphism follows since the bilaws defining $\underset{\sim}{C}\psi$ for any $\underset{\sim}{C}$ are precisely $\underline{C} \cap A_2(\underline{Q}_2) = \underline{C}\sigma_1$ (by (1.2.8)), and σ_1 is a lattice homomorphism. To show that ϕ is a homomorphism we need the following lemma.

(5.2.4) <u>Lemma</u>. If $\underline{G}$ is critical with $A_1(\underline{G}) \neq 1$, and if

$$\underline{G} \in \mathrm{svar}\{\underline{G}_j : j \in J\} \vee \underline{\underline{E}} \circ \underline{\underline{Q}}$$

where for each j, $A_1(\underline{G}_j) \neq 1$, then

$$\underline{G} \in \mathrm{svar}\{\underline{G}_j : j \in J\}.$$

Proof. If q is a bilaw in all $\underline{G}_j$ we may assume by virtue of (2.2.1) that either $q \in A_1(\underline{Q}_2)$ or $q \in A_2(\underline{Q}_2)$. Write $q' = q$ in the first case, and $q' = [y_1, q]$ in the second; then q' is a bilaw in all $\underline{G}_j$ and in $\underline{\underline{E}} \circ \underline{\underline{Q}}$, whence in $\underline{G}$. Since $A_1(\underline{G})$ is non-trivial, and the centralizer of $A_1(\underline{G})$ in $A_2(\underline{G})$ is trivial, we deduce that q is a bilaw in $\underline{G}$. This completes the proof.

Returning to the proof of (5.2.3) we note that, if $\underline{G} \in \underset{\sim}{C} \vee \underset{\sim}{D}$ is critical, and $A_1(\underline{G}) \neq 1$, then by (5.2.4), $\underline{G} \in \underset{\sim}{C}\phi \vee \underset{\sim}{D}\phi$, whence

$$(\underset{\sim}{C} \vee \underset{\sim}{D})\phi \subseteq \underset{\sim}{C}\phi \vee \underset{\sim}{D}\phi.$$

As the converse inclusion is obvious this shows that ϕ is a join-homomorphism. By definition, ϕ is an intersection homomorphism, so (5.2.3) is proved.

(5.2.5) <u>Theorem</u>. If $\underset{\sim}{B}$ is a bivariety in which every sub-bivariety is generated by finite bigroups, then $\Lambda(\underset{\sim}{B})$ is a subdirect product of $\Phi(\underset{\sim}{B})$ and $\Psi(\underset{\sim}{B})$.

Proof. In this case, if $\underset{\sim}{C} \subseteq \underset{\sim}{B}$, then

$$\underset{\sim}{C} = \underset{\sim}{C}\phi \vee \underset{\sim}{C}\psi;$$

and therefore $\underset{\sim}{C}\phi = \underset{\sim}{D}\phi$, $\underset{\sim}{C}\psi = \underset{\sim}{D}\psi$ implies $\underset{\sim}{C} = \underset{\sim}{D}$, whence the result.

(5.2.6) *Corollary*. If $\underset{\sim}{B}$ is a bivariety every sub-bivariety of which is generated by finite bigroups, then $\Lambda(\underset{\sim}{B})$ is distributive if and only if $\Phi(\underset{\sim}{B})$, $\Psi(\underset{\sim}{B})$ are distributive.

We start our investigation proper of $\underline{\underline{A}}_m \circ \underline{\underline{A}}_n$ in the special case when $m = p^\alpha$, $n = p^\beta N$ where $p \nmid N$ and p is prime.

(5.2.7) *Theorem*. $\Lambda(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta N})$ can be embedded sub-directly into the lattice.

$$\Lambda(\underline{\underline{E}} \circ \underline{\underline{A}}_N) \times \Lambda(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta}) \times \Phi(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta})^{s-1}$$

where s is the number of divisors $1 = t_1, \ldots, t_s$ of N. Indeed there exist onto lattice-homomorphisms $\lambda_0 : \Lambda(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta N}) \to \Lambda(\underline{\underline{E}} \circ \underline{\underline{A}}_N)$, $\lambda_1 : \Lambda(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta N}) \to \Lambda(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta})$, $\lambda_i : \Lambda(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta N}) \to \Phi(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta})$, $2 \leq i \leq s$, such that if $\underset{\sim}{S} \subseteq \underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta N}$, then

(i) $\qquad t_i | t_j$ implies $\underset{\sim}{S\lambda}_j \subseteq \underset{\sim}{S\lambda}_i$, $1 \leq i, j \leq s$,

(ii) $\qquad \underset{\sim}{S\lambda}_0 = \underline{\underline{E}} \circ \underline{\underline{A}}_{t_i}$ implies $\underset{\sim}{S\lambda}_j = \underline{\underline{E}} \circ \underline{\underline{E}}$, $t_j \nmid t_i$.

Before proving this result we need a lemma similar to (5.2.4), and, if the bigroups involved are thought of as groups, identical with a special case of a result of Kovács and Newman ((1.12) in [5]).

(5.2.8) Lemma. Let $\{\underline{G}_i : i \in I\}$, $\{\underline{H}_j : j \in J\}$ be critical bigroups in $\underline{\underline{A}}_m \circ \underline{\underline{A}}_n$ $(m,n > 0)$, where each $\underline{G}_i$ is non-nilpotent, and each $\underline{H}_j$ is nilpotent. If $\underline{G}$ is critical and not nilpotent and

$$\underline{G} \in \text{svar}\{\underline{G}_i, \underline{H}_j : i \in I,\ j \in J\},$$

then

$$\underline{G} \in \text{svar}\{\underline{G}_i : i \in I,\ |K| \,\Big|\, |K_i|,\ \exp\sigma\underline{G} = \exp\sigma\underline{G}_i\}$$

(in the notation of (3.2.1)).

Proof. Suppose first that q is a bilaw in all $\underline{G}_i$, $\underline{H}_j$ such that $p = \exp\sigma\underline{G}_i = \exp\sigma\underline{H}_j = \exp\sigma\underline{G}$. As usual we may suppose that either $q \in A_1(\underline{Q}_2)$ or $q \in A_2(\underline{Q}_2)$; write q' for q in the first case and for $[y_1,q]$ in the latter. If $m = p^\gamma m'$ where $p \nmid m'$, then $q'^{m'}$ is a bilaw in all $\underline{G}_i$, $\underline{H}_j$ and therefore in $\underline{G}$. Since $p \nmid m'$, q' is a bilaw in $\underline{G}$ and therefore q is a bilaw in $\underline{G}$ since $A_1(G)$ is self-centralizing.

Without loss of generality, then, we may suppose that $\exp\sigma\underline{G}_i = \exp\sigma\underline{H}_j = p$ for all i,j. Then let q be a bilaw in all $\underline{G}_i$ such that $|K|\,\Big|\,|K_i|$: again we may assume $q \in A_1(\underline{Q}_2)$ or $q \in A_2(\underline{Q}_2)$ and define q' as in the last paragraph. If $\{n_1,\dots,n_u\} = \{|K_i| : i \in I, \ |K|\nmid|K_i|\}$ then

$$[q', z_{r+1}^{p^\beta n_1}, \dots, z_{r+u}^{p^\beta n_u}]$$

is a bilaw in all $\underline{G}_i$, $\underline{H}_j$, where $p^\beta||n$ and r is chosen large enough to avoid z's which occur in q. However, since K acts fixed point free on $A_1(\underline{G})$, q' is a bilaw in $\underline{G}$ and, as before, q is a bilaw in $\underline{G}$.

Proof of (5.2.7). Let $\underset{\sim}{S} \subseteq \underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta N}$, and define λ_i as follows:

$$\underset{\sim}{S}\lambda_0 = \underset{\sim}{S} \wedge \underline{\underline{E}} \circ \underline{\underline{A}}_N,$$

$$\underset{\sim}{S}\lambda_i = \text{svar}\{\underline{F}^* : \underline{G} \in \underset{\sim}{S} - \underline{\underline{E}} \circ \underline{\underline{A}}_N \text{ critical}, \ t_i = |K|\},$$

$i \in \{1,\dots,s\}$, where we interpret $\underline{F}^* = \underline{G}$, $K = 1$ in case $\underline{G}$ is a p-group. If $\underline{G} \in \underset{\sim}{S}$ is critical, with $|K| = t_j$ and $t_i|t_j$ write $\overline{\underline{G}}$ for the sub-bigroup $(F\overline{K}, A, H \times \overline{K})$ of $\underline{G}$ where $|\overline{K}| = t_i$. From (5.1.2), there exist critical bigroups $\underline{G}_1,\dots,\underline{G}_w$ with $|K_\ell| = t_i$ such that $\text{svar}\{\underline{G}_1,\dots,\underline{G}_w\} = \text{svar}\{\overline{\underline{G}}\}$, $\text{svar}\{\underline{F}_1^*,\dots,\underline{F}_w^*\} = \text{svar}\{\underline{F}^*\}$. Hence $\underset{\sim}{S}\lambda_j \subseteq \underset{\sim}{S}\lambda_i$. Also if $\underset{\sim}{S}\lambda_0 = \underline{\underline{E}} \circ \underline{\underline{A}}_{t_i}$, then whenever $t_j \nmid t_i$,

$\{\underline{F}^* : \underline{G} \in S - \underline{\underline{E}} \circ \underline{\underline{A}}_N,\ t_j = |K|\}$ is empty, and therefore $\underset{\sim}{S}\lambda_j = \underline{\underline{E}} \circ \underline{\underline{E}}$.

We have to show that the λ_i are homomorphisms. Clearly λ_0 is an intersection-homomorphism; and it is a join-homomorphism since $\underline{H} \in (\underset{\sim}{S} \vee \underset{\sim}{S}') \wedge \underline{\underline{E}} \circ \underline{\underline{A}}_N$ implies $H = A_2(\underline{H}) \in \operatorname{var}\{A_2(\underline{K})^{p^\beta} : \underline{K} \in \underset{\sim}{S}$ or $\underline{K} \in \underset{\sim}{S}'\}$ and therefore $\underline{H} \in (\underset{\sim}{S} \wedge \underline{\underline{E}} \circ \underline{\underline{A}}_N) \vee (\underset{\sim}{S}' \wedge \underline{\underline{E}} \circ \underline{\underline{A}}_N)$. That is $(\underset{\sim}{S} \vee \underset{\sim}{S}')\lambda_0 \subseteq \underset{\sim}{S}\lambda_0 \vee \underset{\sim}{S}'\lambda_0$ and as the opposite inclusion is obvious, we have dealt with λ_0.

Now suppose $\underline{G} \in \underset{\sim}{S} \vee \underset{\sim}{S}'$ is critical and $t_i = |K|$; by (5.2.8) $\underline{G} \in \operatorname{svar}\{\underline{G}_j : \underline{G}_j \in \underset{\sim}{S}$ or $\underline{G}_j \in \underset{\sim}{S}'$, $\underline{G}_j$ critical, $t_i \,\big|\, |K_j|\}$, and so

$$\begin{aligned} \underline{F}^* &\in \operatorname{svar}\{\underline{F}^*_j : \underline{G}_j \in \underset{\sim}{S} \cup \underset{\sim}{S}' \text{ critical},\ t_i \,\big|\, |K_j|\} \\ &= \vee\{\underset{\sim}{S}\lambda_\ell,\ \underset{\sim}{S}'\lambda_\ell : t_i | t_\ell\} \\ &= \underset{\sim}{S}\lambda_i \vee \underset{\sim}{S}'\lambda_i, \end{aligned}$$

whence $(\underset{\sim}{S} \vee \underset{\sim}{S}')\lambda_i \subseteq \underset{\sim}{S}\lambda_i \vee \underset{\sim}{S}'\lambda_i$. The converse inclusion is clear so we have shown that λ_i is a join-homomorphism. To show that λ_i is an intersection-homomorphism, suppose that $\underline{P} \in \underset{\sim}{S}\lambda_i \wedge \underset{\sim}{S}'\lambda_i$ is critical and $A_1(\underline{P}) \neq 1$ (in the case $t_i \neq 1$). By (5.1.2) there exists a critical bigroup $\underline{G}$ with $\underline{F}^* \cong \underline{P}$ and $|K| = t_i$; it follows from (3.4.4) in a routine fashion, that $\underline{G} \in \operatorname{svar}\{\underline{G}_j \in \underset{\sim}{S} : \underline{G}_j$ critical, $t_i = |K_j|\} \wedge \operatorname{svar}\{\underline{G}_j \in \underset{\sim}{S}' : \underline{G}_j$ critical, $t_i = |K_j|\}$, or $\underline{G} \in \underset{\sim}{S} \wedge \underset{\sim}{S}'$. Thus

$\underline{F}^* \in (\underset{\sim}{S} \wedge \underset{\sim}{S}')\lambda_i$, and therefore

$$\underset{\sim}{S}\lambda_i \wedge \underset{\sim}{S}'\lambda_i \subseteq (\underset{\sim}{S} \wedge \underset{\sim}{S}')\lambda_i;$$

as the opposite inclusion is obvious, λ_i is an intersection-homomorphism. Note that the case $t_i = 1$ is easy, since $\underset{\sim}{S}\lambda_1 = \underset{\sim}{S} \wedge \underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta}$.

Finally, note that $\underset{\sim}{S}$ is determined uniquely by the $\underset{\sim}{S}\lambda_i$, $0 \leq i \leq s$; for, if $\underset{\sim}{S}\lambda_i = \underset{\sim}{S}'\lambda_i$ for all i, and if for $i \geq 1$, $\underline{G} \in \underset{\sim}{S}$ is critical with $|K| = t_i$, then $\underline{F}^* \in \underset{\sim}{S}'\lambda_i$, and using (3.4.4) again we deduce $\underline{G} \in \underset{\sim}{S}'$; hence $\underset{\sim}{S} \subseteq \underset{\sim}{S}'$, and, in a similar manner, $\underset{\sim}{S}' \subseteq \underset{\sim}{S}$, or $\underset{\sim}{S} = \underset{\sim}{S}'$. This then shows that the mapping $\underset{\sim}{S} \to (\underset{\sim}{S}\lambda_0, \ldots, \underset{\sim}{S}\lambda_s)$ provides an embedding for $\Lambda(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^{\beta_N}})$ which is clearly sub-direct.

(5.2.9) <u>Theorem</u>. If $m, n > 0$, $m = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ for distinct primes $p_1, \ldots, p_r$, then $\Lambda(\underline{\underline{A}}_m \circ \underline{\underline{A}}_n)$ is a sub-direct product of $\Lambda(\underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n)$, $i \in \{1, \ldots, r\}$ according to homomorphisms

$\mu_i : \Lambda(\underline{\underline{A}}_m \circ \underline{\underline{A}}_n) \to \Lambda(\underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n)$ defined by

$$\underset{\sim}{B}\mu_i = \underset{\sim}{B} \wedge (\underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n),$$

for $\underset{\sim}{B} \subseteq \underline{\underline{A}}_m \circ \underline{\underline{A}}_n$.

Proof. That each μ_i is an intersection homomorphism is obvious. To prove that it is a join-homomorphism we must show that for $\underset{\sim}{B},\underset{\sim}{C} \subseteq \underline{\underline{A}}_m \circ \underline{\underline{A}}_n$,

$$(\underset{\sim}{B} \vee \underset{\sim}{C})\mu_i \subseteq \underset{\sim}{B}\mu_i \vee \underset{\sim}{C}\mu_i,$$

since the converse is clear. If $\underline{G} \in (\underset{\sim}{B} \vee \underset{\sim}{C}) \wedge \underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n$, and $A_1(\underline{G}) \neq 1$ then Lemma 5.2.8 yields $\underline{G} \in (\underset{\sim}{B} \wedge \underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n) \vee (\underset{\sim}{C} \wedge \underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n)$ which is what we want; if $A_1(\underline{G}) = 1$, then

$\underline{G} \in (\underset{\sim}{B} \vee \underset{\sim}{C}) \wedge \underline{\underline{E}} \circ \underline{\underline{A}}_n = (\underset{\sim}{B} \vee \underset{\sim}{C})\psi = \underset{\sim}{B}\psi \vee \underset{\sim}{C}\psi \subseteq \underset{\sim}{B}\mu_i \vee \underset{\sim}{C}\mu_i$,

using (5.2.3). Finally note that for $\underset{\sim}{B} \subseteq \underline{\underline{A}}_m \circ \underline{\underline{A}}_n$,

$$\underset{\sim}{B} = \vee\{\underset{\sim}{B}\mu_i : 1 \leq i \leq r\}$$

and therefore the theorem is proved.

(5.2.10) *Corollary*. If $\underset{\sim}{B} \subseteq \underline{\underline{A}}_m \circ \underline{\underline{A}}_n$, then $\Lambda(\underset{\sim}{B})$ is distributive if and only if for each $p_i | m$, each $\Lambda(\underset{\sim}{B})\mu_i\lambda_j$ is distributive, where λ_j are defined for each i as in (5.2.7).

Proof. Since the homomorphisms μ_i provide a sub-direct decomposition of $\Lambda(\underset{\sim}{B})$ then $\Lambda(\underset{\sim}{B})$ is distributive if and only if each sub-direct factor of it is; that is, if and only if each $\Lambda(\underset{\sim}{B})\mu_i$ is distributive. From (5.2.7), and for the same reason, each $\Lambda(\underset{\sim}{B})\mu_i$ is distributive if and only if $\Lambda(\underset{\sim}{B})\mu_i\lambda_j$ is distributive.

Theorem 5.2.7 can be formulated, a little artificially, but in some respects more naturally, in a different manner using the concept of products of split-varieties introduced in section 1.7. Here we give an informal discussion without proof of how this can be done. Note that if $\underline{G} = (G,A,B) \in \underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^{\beta_N}}$ then B can be written uniquely as $B = H \times K$ with $H \in \underline{\underline{A}}_{p^\beta}$, $K \in \underline{\underline{A}}_N$. The mapping $\chi : \underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^{\beta_N}} \to \underline{\underline{A}}_{p^\alpha} \circ (\underline{\underline{A}}_{p^\beta} \times \underline{\underline{A}}_N)$ defined by

$$\underline{G}\chi = (G,A,H,K)$$

is easily verified to be one-to-one, to take sub-bigroups to sub-trigroups and to take quotient bigroups to quotient trigroups. We can, moreover, easily turn χ into a functor: if $\mu : \underline{G} \to \overline{\underline{G}}$ then clearly μ is a morphism between $\underline{G}\chi$ and $\overline{\underline{G}}\chi$. Define $\mu\chi = \mu$. We can, in these terms, state

(5.2.11) <u>Theorem</u>. To every sub-bivariety $\underset{\sim}{S}$ of $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^{\beta_{\mathbb{N}}}}$ containing $\underline{\underline{E}} \circ \underline{\underline{A}}_{\mathbb{N}}$ there exist unique sub-bivarieties $\underset{\sim}{S}_t$ of $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^{\beta_{\mathbb{N}}}}$ for each $t|\mathbb{N}$, such that

$$\underset{\sim}{S}_\chi = \vee\{\underset{\sim}{S}_t \circ \underline{\underline{A}}_t \wedge \underset{\sim}{T} : t|\mathbb{N}\}$$

where $\underset{\sim}{T} = \underline{\underline{A}}_{p^\alpha} \circ (\underline{\underline{A}}_{p^\beta} \times \underline{\underline{A}}_{\mathbb{N}})$, and such that $\underset{\sim}{S}_1 \subseteq \underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta}$, $\underset{\sim}{S}_t \in \Phi(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta})$, $1 \neq t|\mathbb{N}$, and if $t|t_0|\mathbb{N}$, then $\underset{\sim}{S}_{t_0} \subseteq \underset{\sim}{S}_t$.

The proof is in many respects similar to that of (5.2.7) and we omit it.

Finally in this section, we investigate the nature of join-decompositions of $\underline{\underline{A}}_m \circ \underline{\underline{A}}_n$.

(5.2.12) <u>Theorem</u>. If $m = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ for distinct primes $p_1, \ldots, p_r$ then

$$\underline{\underline{A}}_m \circ \underline{\underline{A}}_n = \vee\{\underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n : 1 \leq i \leq r\}$$

and this is the only way that $\underline{\underline{A}}_m \circ \underline{\underline{A}}_n$ can be written as an irredundant join of join-irreducibles.

Proof. First we show that for prime p, $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta}$ is join irreducible; this is patent for $\beta = 0$. We use induction on β, assuming that $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^{\nu-1}}$ is join irreducible.

Suppose that $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\nu} = \underset{\sim}{B}_1 \vee \underset{\sim}{B}_2$; that is, if $\underline{W}_\nu$ is the free bigroup of rank $(1,\omega)$ in $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\nu}$ (changing the notation of Chapter 4 slightly) then in $\underline{W}_\nu$,

$$\underline{B}_1 \cap \underline{B}_2 = 1 .$$

Clearly we may suppose that $\underline{B}_1, \underline{B}_2$ are contained in $A_1(\underline{W}_\nu)$. Then by (4.3.8) there exist $\underline{L}_1, \underline{L}_2 \in \Lambda(\underline{W}_{\nu-1})$ and integers v_1, v_2 such that

$$[\underline{L}_i\lambda_\nu, v_i\underline{W}_\nu] \leq \underline{B}_i \leq \underline{L}_i\lambda_\nu, \quad i = 1,2.$$

Therefore $[\underline{L}_1\lambda_\nu \cap \underline{L}_2\lambda_\nu, (v_1+v_2)\underline{W}_\nu] \leq \underline{B}_1 \cap \underline{B}_2 = 1$ whence, by (4.2.15), $\underline{L}_1\lambda_\nu \cap \underline{L}_2\lambda_\nu = 1$ which yields $(\underline{L}_1 \cap \underline{L}_2)\lambda_\nu = 1$ or $\underline{L}_1 \cap \underline{L}_2 = 1$ from (4.3.2). By hypothesis, $\underline{L}_1$ say, is trivial and therefore $\underline{B}_1 = 1$, proving what we want.

Next suppose that for $p \nmid N$ (and $\alpha > 0$),

$$\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^{\beta}N} = \underset{\sim}{S} \vee \underset{\sim}{S}' .$$

From (5.2.7) we have that for each $t_i | N$,

$$\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta} = \underset{\sim}{S}\lambda_i \vee \underset{\sim}{S}'\lambda_i \;;$$

in particular, with $t_i = N$, $\underset{\sim}{S}\lambda_N = \underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta}$ say. Hence

$\underset{\sim}{S}\lambda_i = \underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta}$ for all $i = 1,\dots,s$ and therefore $\underset{\sim}{S} = \underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^{\beta_N}}$.

Certainly, then, $\underline{\underline{A}}_m \circ \underline{\underline{A}}_n$ has a decomposition as an irredundant join of join-irreducibles. Suppose that

$$\underline{\underline{A}}_m \circ \underline{\underline{A}}_n = \underset{\sim}{B}_1 \vee \dots \vee \underset{\sim}{B}_t$$

is another such decomposition. Then using (5.2.9) we have for each $i \in \{1,\dots,r\}$

$$\underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n = \underset{\sim}{B}_1\mu_i \vee \dots \vee \underset{\sim}{B}_t\mu_i$$

whence for some $j \in \{1,\dots,t\}$,

$$\underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n = \underset{\sim}{B}_j\mu_i \leq \underset{\sim}{B}_j \;.$$

That is, each $\underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n$ is contained in some $\underset{\sim}{B}_j$; and each $\underset{\sim}{B}_j$ does contain an $\underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n$ since otherwise it is clearly redundant. Also since each $\underset{\sim}{B}_j$ is join irreducible, and $\underset{\sim}{B}_j = \vee\{\underset{\sim}{B}_j\mu_i : 1 \leq i \leq r\}$, $\underset{\sim}{B}_j = \underset{\sim}{B}_j\mu_i$ for some i. That is,

$$\underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n \leq B_j = \underset{\sim}{B}_j\mu_i \leq \underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n$$

whence $\underset{\sim}{B}_j = \underline{\underline{A}}_{p_i^{\alpha_i}} \circ \underline{\underline{A}}_n$, and this completes the proof.

5.3 The bivarieties $\underline{\underline{A}}_m \circ \underline{\underline{A}}_p$.

The problem of determining all sub-bivarieties of $\underline{\underline{A}}_m \circ \underline{\underline{A}}_n$ has been reduced to the case when m,n are powers of the same prime. In general this case seems to be difficult. The results of Chapter 4 show that we can obtain upper and lower bounds for each sub-bivariety of $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta}$ in terms of the sub-bivarieties of $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^{\beta-1}}$, but the fine structure escapes us in general. Only in the case $\beta = 1$ do we get a complete picture. First we prove two lemmas similar to (4.2.7).

(5.3.1) Lemma. If in the notation of (4.2.2), $a_0,\dots,a_{p-1} \in A$ are fixed elements, and if $\underline{U}$ is normal in $\underline{W}_\nu$ such that for all $b \in B$

$$\rho = \prod_{i=0}^{p-1} [a_i, ib] \in \underline{U},$$

then $a_i \in U_i$, $i \in \{0,\dots,p-1\}$.

Proof. Using the identity $[x,y^r] = \prod_{i=1}^{r} [x,iy]^{\binom{r}{i}}$, we may express ρ as

$$\rho = \prod_{i=0}^{p-1} [a_i', b^i] \in \underline{U}$$

where each a'_i is a linear combination of $a_i, \ldots, a_{p-1}$, and $a'_{p-1} = a_{p-1}$. From (4.2.7) we deduce that $a_{p-1} \in U_{p-1}$, whence

$$\prod_{i=0}^{p-2} [a_i, ib] \in \underline{U}.$$

An easy induction is indicated to finish the proof, and we omit the details.

(5.3.2) Lemma. Define $\underset{\sim}{\mu} = (\mu_1, \ldots, \mu_s)$ where $0 \leq \mu_i \leq p-1$ for all i. If $a(\underset{\sim}{\mu})$ are fixed elements of A, and if for all $b_1, \ldots, b_s \in B$

$$\prod_{\underset{\sim}{\mu}} [a(\underset{\sim}{\mu}), \mu_1 b_1, \ldots, \mu_s b_s] \in \underline{U}$$

(where $\underline{U}$ is normal in $\underset{\sim}{W}_\nu$), then $a(\underset{\sim}{\mu}) \in U_\tau$ where $\tau = \mu_1 + \ldots + \mu_s$.

Proof. We proceed by induction on s, the case $s = 1$ being covered by the last lemma. For $i \in \{0, \ldots, p-1\}$ write

$$a_i = \prod_{\mu_s = i} [a(\underset{\sim}{\mu}), \mu_1 b_1, \ldots, \mu_{s-1} b_{s-1}];$$

then $\prod_{i=0}^{p-1} [a_i, ib] \in \underline{U}$ for all $b \in B$. Hence by (5.3.1), $a_i \in U_i$, $i \in \{0, \ldots, p-1\}$. Now $\mu_s = \mu'_s$ implies $(\mu_1, \ldots, \mu_{s-1}) \neq (\mu'_1, \ldots, \mu'_{s-1})$ if $\underset{\sim}{\mu} \neq \underset{\sim}{\mu}'$. We may then, by induction, assume that

$$a(\underset{\sim}{\mu}) \in (\underline{U_i})_j$$

where $j = \mu_1 + \ldots + \mu_{s-1}$. That is, $a(\underset{\sim}{\mu}) \in U_\tau$, $\tau = i+j = \mu_1 + \ldots + \mu_s$, for each $\underset{\sim}{\mu}$ as required.

Before commencing the statement and proof of our main results in this chapter, we introduce the following notation. Write $\underline{X}_\alpha$ for the split-free bigroup of rank $(1,\omega)$ in $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p$ on the split-free generating set $\{y_1\} \cup \{z_1,z_2,\ldots\}$. It is clear from (4.2.1) that the lattice of normal, fully invariant sub-bigroups of $\underline{X}_\alpha$ is dually isomorphic to $\Lambda(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p)$. Write (d,σ) for the fully invariant closure of $[y_1,z_1,\ldots,z_d]^{p^\sigma}$ in $\underline{X}_\alpha$; abusing convention, then

(5.3.3) Notation. For $d \geq 0$, $\sigma \in \{0,1,\ldots,\alpha-1\}$

$$(d,\sigma) = \mathrm{cl}\{[y_1,z_1,\ldots,z_d]^{p^\sigma}\}.$$

(5.3.4) Theorem. Every fully invariant sub-bigroup of $\underline{X}_\alpha$ contained in $A_1(\underline{X}_\alpha)$ can be written as a product of finitely many (d,σ)'s.

Proof. From (2.2.4), every fully invariant $\underline{U}$ contained in $A_1(\underline{X}_\alpha)$ is the closure of special biwords of the type

$$q = \prod_{i=1}^{t} [y_1, \mu_{i1}z_1, \ldots, \mu_{is}z_s]^{\alpha_i}$$

where $1 \le \mu_{ij} \le p-1$, $1 \le \alpha_i \le p^{\alpha}-1$, all i,j, and where $i \neq j$ implies $(\mu_{i1},\ldots,\mu_{is}) \neq (\mu_{j1},\ldots,\mu_{js})$. Lemma 5.3.2 gives that

$$y_i^{\alpha_i} \in (\mathrm{cl}\{q\})_{\tau_i}, \quad \tau_i = \mu_{i1} + \ldots + \mu_{is}.$$

Clearly, then, q is equivalent to a set of (d,σ)'s and therefore so is $\underline{U}$.

With this theorem we can in fact determine all sub-bivarieties of $\underline{\underline{A}}_{p^{\alpha}} \circ \underline{\underline{A}}_{p}$; however we have as yet no way of knowing when two different sets of (d,σ)'s determine different sub-bivarieties. We take up this problem now.

(5.3.5) Theorem. The commutators

$$[y_1,\mu_1 z,\ldots,\mu_r z_r],$$

$r \ge 0$, $0 \le \mu_i \le p-1$ for $i \in \{1,\ldots,r\}$ and $\mu_r > 0$, form a basis for $A_1(\underline{X}_{\alpha})$. If $d > 0$, then a basis for (d,σ) is the set of all $b^{p^{\tau}}$, where b is a basic commutator of weight ≥ 2 and where τ is minimal with respect to $\sigma \le \tau$ and $\mathrm{wt}\, b + (\tau-\sigma)(p-1) \ge d+1$; the set $\{b^{p^{\sigma}} : b \text{ basic}\}$ is a basis for $(0,\sigma)$.

Proof. The set of commutators of the type described certainly generate $A_1(\underline{X}_\alpha)$: the only thing to check is that, using the identity

$$[y_1, pz_1] = \prod_{i=1}^{p-1} [y_1, iz_1]^{-\binom{p}{i}}$$

we can remove p or more repetitions of any variable z_j, replacing the offending commutator by a product of commutators each of which has fewer than p occurences of z_j. That these commutators with few repetitions are basic follows from (5.3.2); for, if

$$\prod_{i=1}^{t} [y_1, \mu_{i1}z_1, \ldots, \mu_{is_i}z_{s_i}]^{\alpha_i} = 1$$

where $(\mu_{i1}, \ldots, \mu_{is_i}) \neq (\mu_{j1}, \ldots, \mu_{js_j})$, $i \neq j$ and $0 \leq \mu_{i\ell} \leq p-1$, $\mu_{is_i} > 0$, $i \in \{1, \ldots, t\}$, $\ell \in \{1, \ldots, s_i\}$, then, if $s = \max\{s_i : 1 \leq i \leq t\}$ we have by defining $\mu_{i\ell} = 0$ for $s_i < \ell \leq s$ where necessary, that

$$\prod_{i=1}^{t} [y_1, \mu_{i1}z_1, \ldots, \mu_{is}z_s]^{\alpha_i} = 1$$

with $(\mu_{i1}, \ldots, \mu_{is}) \neq (\mu_{j1}, \ldots, \mu_{js})$, $i \neq j$. We may therefore apply Lemma 5.3.2 to deduce for each $i \in \{1, \ldots, t\}$, that

$$[y_1, z_1, \ldots, z_\tau]^{\alpha_i} = 1$$

where $\tau = \mu_{i1} + \dots + \mu_{is}$ this would then be a bilaw in $\underline{X}_\alpha$, and therefore $p^\alpha | \alpha_i$. For if not, then $[y_1^{p^{\alpha-1}}, z_1, \dots, z_\tau] = 1$ and therefore $[y_1, z_1, \dots, z_\tau]$ is a bilaw in C_p wr C_p^τ, which is not true (see Liebeck [13]). Hence $p^\alpha | \alpha_i$ for all i, and this shows that the set of commutators $[y_1, \mu_1 z_1, \dots, \mu_r z_r]$ with $r \geq 0$, $0 \leq \mu_i \leq p-1$ and $\mu_r > 0$ is a basis for $A_1(\underline{X}_\alpha)$.

It is quite clear that the set $\{b^{p^\sigma} : b \text{ basic}\}$ is a basis for $(0, \sigma)$, but the remaining assertion of the theorem requires proof. The crucial point is the following result.

(5.3.6) Lemma. $(e, \tau) \leq (d, \sigma)$ if and only if $\sigma \leq \tau$ and $d = 0$ if $e = 0$ and $d \leq e + (\tau-\sigma)(p-1)$ if $e > 0$.

Proof. The first part is easy: if $(e, \tau) \leq (d, \sigma)$ then $[y_1, z_1, \dots, z_e]^{p^\tau}$ can be written as a product of p^σ-th powers, and hence, if $\sigma > \tau$, $[y_1, z_1, \dots, z_e]^{p^{\alpha-1}} = 1$ which, as we have observed, is impossible. Also if $e = 0$ and $d > 0$, then $y_1^{p^\tau}$ can be written as a product of commutators all involving at least one z_j; then by mapping $y_1 \to y_1$ and $z_j \to 1$ for all j we have $y_1^{p^\tau} = 1$ which is a contradiction.

Suppose therefore, that $e > 0$ and $\sigma \leq \tau$. Then

$$(5.3.7) \qquad (e,\tau) \leq (e + (\tau-\sigma)(p-1),\sigma)$$

and

$$(5.3.8) \qquad (e,\tau) \nleq (e + (\tau-\sigma)(p-1) + 1,\sigma).$$

Consider the identity

$$[y_1,z_1,\dots,z_{e+r}]^p = \prod_{i=2}^{p} [y_1,z_1,\dots,z_{e+r-1},iz_{e+r}]^{-\binom{p}{i}};$$

from this one deduces that for $r \leq p-2$

$$(e+r+1,1) \leq (e+p-1,0) \text{ implies } (e+r,1) \leq (e+p-1,0)$$

and therefore, by downward induction on r, $(e,1) \leq (e+p-1,0)$. This then gives by induction on $\tau-\sigma$ ((5.3.7) is trivially true if $\tau = \sigma$),

$$\begin{aligned}(e,\tau) &= (e,1)^{p^{\tau-1}} \leq (e+p-1,0)^{p^{\tau-1}} \\ &= (e+p-1,\tau-1) \leq (e+p-1+(\tau-1-\sigma)(p-1),\sigma) \\ &= (e+(\tau-\sigma)(p-1),\sigma).\end{aligned}$$

This proves (5.3.7). The proof of (5.3.8) is more difficult, and uses the next two lemmas.

(5.3.9) Lemma. If $m > 0$ and

$$[y_1, mz_1] = \prod_{i=1}^{p-1} [y_1, iz_1]^{\delta(m,i)}$$

and if $m = p + (\mu-1)(p-1) + r$, $0 \leq r < p-1$, $0 \leq \mu$, then

i) $\mu = 0$ implies $\delta(m,i) = 1,0$ according as $m = i$ or $m \neq i$;

ii) $r = 0$ implies $p^{\mu} | \delta(m,i)$, $1 \leq i \leq p-1$;

iii) $\mu r \geq 1$ implies $p^{\mu+1} | \delta(m,i)$, $1 \leq i \leq r$

and $p^{\mu} | \delta(m,i)$, $r+1 \leq i \leq p-1$.

Proof. Clearly (i) is a consequence of the uniqueness already proved in (5.3.5). For $\mu = 1$, $r = 0$ (ii) is easily seen to be true. Suppose that the lemma has been proved for some m with $m \geq p$. Then

$$\begin{aligned} [y_1, (m+1)z_1] &= [y_1, z_1, mz_1] \\ &= \prod_{i=1}^{p-1} [y_1, z_1, iz_1]^{\delta(m,i)} \\ &= \prod_{i=1}^{p-2} [y_1, (i+1)z_1]^{\delta(m,i)} \prod_{i=1}^{p-1} [y_1, iz_1]^{-\binom{p}{i}\delta(m,p-1)} \end{aligned}$$

and so, by the uniqueness from (5.3.5),

$$\delta(m+1, i) = \delta(m, i-1) - \binom{p}{i}\delta(m, p-1), \quad 2 \leq i \leq p-1,$$

and $$\delta(m+1, 1) = -p\delta(m, p-1).$$

By assumption $p^{\mu+1}|\delta(m,i)$, $i \le r$ and $p^{\mu}|\delta(m,1)$, $r < i$, whence the proof may be completed.

(5.3.10) <u>Lemma</u>. If $m_1,\ldots,m_d \ge 1$ and

$$[y_1, m_1z_1, \ldots, m_dz_d] = \prod_{\underset{\sim}{i}} [y_1, i_1z_1, \ldots, i_dz_d]^{\beta(\underset{\sim}{i})}$$

where $\underset{\sim}{i} = (i_1,\ldots,i_d)$ with $1 \le i_j \le p-1$, then $m_1 + \ldots + m_d \ge d + \tau(p-1) + 1$ implies $p^{\tau+1}|\beta(1,\ldots,1)$.

Proof. With $d = 1$ we have $d + \tau(p-1) + 1 = p + (\tau-1)(p-1) + 1$ and Lemma 5.3.9 applies. We use this as a starting point for induction on d. Suppose $m_d = \phi(p-1) + \rho \ge 1$, $0 \le \rho < p-1$, $0 \le \phi$. Then

$$m_1 + \ldots + m_{d-1} \ge (d-1) + (\tau-\phi)(p-1) - (\rho-2).$$

Now if $[y_1, m_1z_1, \ldots, m_{d-1}z_{d-1}] = \prod_{\underset{\sim}{i}} [y_1, i_1z_1, \ldots, i_{d-1}z_{d-1}]^{\gamma(\underset{\sim}{i})}$, then we may assume inductively that

$$p^{\tau-\phi+1}|\gamma(1,\ldots,1) \text{ if } \rho \le 1,$$

$$p^{\tau-\phi}|\gamma(1,\ldots,1) \text{ if } 1 < \rho.$$

Also from (5.3.5),

$$\beta(1,\ldots1) = \delta(m_d,1)\gamma(1,\ldots,1);$$

and

$$p^{\phi+1} \mid \delta(m_d,1) \quad \text{if} \quad 1 < \rho ,$$

$$p^{\phi} \mid \delta(m_d,1) \quad \text{if} \quad \rho \leq 1 .$$

In any case, $p^{\tau+1} \mid \beta(1,\ldots,1)$ as required.

Proof of (5.3.8). If $(e,\tau) \leq (e + (\tau-\sigma)(p-1) + 1,\sigma)$, then

$$(*) \qquad [y_1,z_1,\ldots,z_e]^{p^{\tau}} = \prod_{\underset{\sim}{j}} [y_1,j_1z_1,\ldots,j_ez_e]^{p^{\sigma}\beta'(\underset{\sim}{j})}$$

where $j_1 + \ldots + j_e \geq e + (\tau-\sigma)(p-1) + 1$. Now (*) can be re-written by replacing each $[y_1,j_1z_1,\ldots,j_ez_e]$ by a product of powers of basic commutators. Then, using the uniqueness from (5.3.5),

$$[y_1,z_1,\ldots,z_e]^{p^{\tau}} = \prod_{\underset{\sim}{j}} [y_1,z_1,\ldots,z_e]^{p^{\sigma}\beta(\underset{\sim}{j})}$$

where for each $\underset{\sim}{j}$, $p^{\tau-\sigma+1} \mid \beta(\underset{\sim}{j})$ by (5.3.10). Hence

$$p^{\tau} = p^{\sigma}\sum \beta(\underset{\sim}{j}),$$

and since the right-hand side of this equation is divisible by $p^{\tau+1}$ we have a contradiction. This completes the proof of (5.3.8).

Proof of (5.3.5). If $d > 0$ and $b_1,\ldots,b_t$ are distinct basic commutators such that

$$b_1^{\beta_1} \ldots b_t^{\beta_t} \in (d,\sigma),$$

then, from (5.3.2), if b_i has weight $e_i + 1$, and $p^{\tau_i} || \beta_i$,

$$(e_i, \tau_i) \leq (d, \sigma)$$

whence, from the part of (5.3.5) already proved, and (5.3.6),

$$\sigma \leq \tau_i, \quad e_i > 0, \quad d \leq e_i + (\tau_i - \sigma)(p-1).$$

This completes the proof of (5.3.5).

The main result of this section can now be stated. As the proof is of a routine nature using Theorem 5.3.5 we will omit most of the details.

(5.3.11) Theorem. Every normal, fully invariant sub-bigroup $\underline{U} \neq 1$ of $\underline{X}_\alpha$ can be written uniquely as

$$\underline{U} = A_2(\underline{X}_\alpha)^{\varepsilon} . (d_\sigma, \sigma) \ldots (d_{\alpha-1}, \alpha-1)$$

where $\varepsilon = 0,1$ (according as $z_1 \notin \underline{U}$ or $z_1 \in \underline{U}$) and

i) $\varepsilon = 1$ implies $\sigma = 0$, $d_\sigma \leq 1$;

ii) if $\phi \in \{\sigma, \ldots, \alpha-2\}$ then

$$d_{\phi+1} \begin{cases} \leq d_\phi - p + 1, & \text{if } p \leq d_\phi, \\ \leq 1, & \text{if } 1 \leq d_\phi \leq p-1, \\ = 0, & \text{if } 0 = d_\phi. \end{cases}$$

Proof. Theorem 5.3.4 ensures that every $\underline{U} \neq 1$ can be written as a join as indicated; if $z_1 \in U$ then $[y_1,z_1] \in U$ and hence $(1,0) \leq \underline{U}$.

Let σ be the smallest element of $\{0,\ldots,\alpha-1\}$ for which $(d,\sigma) \leq \underline{U}$ for some integer d, and let d_τ be the smallest integer such that $(d_\tau,\tau) \leq \underline{U}$ for $\sigma \leq \tau \leq \alpha-1$. Since by (5.3.6)

$$(d,\tau+1) \leq (d+p-1,\tau)$$

for $d > 0$, we have that $d_\tau \geq p$ implies $d_{\tau+1} \leq d_\tau - p+1$. If $1 \leq d_\tau \leq p-1$ then for all $d > 0$

$$(d,\tau+1) \leq (d+p-1,\tau) \leq (d_\tau,\tau) \leq U,$$

hence $d_{\tau+1} \leq 1$. If $d_\tau = 0$ for some $\tau \in \{\sigma,\ldots,\alpha-2\}$ then clearly $d_{\tau+1} = \ldots = d_{\alpha-1} = 0$. This establishes the existence of such a join decomposition for $\underline{U}$.

The uniqueness is a consequence of the next lemma, whose proof we omit.

(5.3.12) Lemma. If $(d,\tau) \leq (d_\sigma,\sigma) \ldots (d_{\alpha-1},\alpha-1)$ where $d_\sigma,\ldots,d_{\alpha-1}$ satisfy the condition (ii) of (5.3.11), then $\sigma \leq \tau$ and $d_\tau \leq d$.

(5.3.13) Corollary. Let $J = \{0,1,\ldots,i,\ldots\} \cup \{\infty\}$

and $T = \{0,1\}$ have their natural orders, then the lattice

$$T \times J^{\alpha}$$

embeds $\Lambda(\underline{\underline{A}}_{p^{\alpha}} \circ \underline{\underline{A}}_{p})$. $\Lambda(\underline{\underline{A}}_{p^{\alpha}} \circ \underline{\underline{A}}_{p})$ is distributive.

The details of proof are routine and we omit them.

(5.3.14) Corollary. Theorems 5.3.11, 5.2.7 (or 5.2.11) and 5.2.9 afford a complete description of $\Lambda(\underline{\underline{A}}_{m} \circ \underline{\underline{A}}_{n})$ if m,n are nearly coprime. In particular $\Lambda(\underline{\underline{A}}_{m} \circ \underline{\underline{A}}_{n})$ is distributive in such cases.

5.4 The bivarieties $\underline{\underline{A}}_{p^{\alpha}} \circ \underline{\underline{A}}_{p^{\alpha}} \wedge \underset{\sim}{N}_{c}$

In this section we give a classification of another class of bivarieties, and produce an example of a non-distributive bivariety lattice. First note the following:

(5.4.1) Lemma. A bigroup $\underline{G} \varepsilon \underline{\underline{A}} \circ \underline{\underline{A}}$ has the bilaw $[y_1,z_1,\dots,z_d]^m$ if and only if G has the law $[x_1,x_2,\dots,x_{d+1}]^m$.

Proof. Now G has the law $[x_1,x_2,\dots,x_{d+1}]^m$ if and only if $\underline{G}$ has the bilaw $[y_1z_1,\dots,y_{d+1}z_{d+1}]^m$. Modulo the bilaws of $\underline{\underline{A}} \circ \underline{\underline{A}}$

we have

$$[y_1z_1,\ldots,y_{d+1}z_{d+1}] = [y_1,z_2,\ldots,z_{d+1}]^{z_1}[z_1,y_2,z_3,\ldots,z_{d+1}]^{z_2}$$

and therefore $[y_1z_1,\ldots,y_{d+1}z_{d+1}]^m$ is equivalent, modulo the bilaws of $\underline{\underline{A}} \circ \underline{\underline{A}}$, to $[y_1,z_1,\ldots,z_d]^m$.

Note that, in particular, $\underline{G}$ has class c if and only if $\underline{G}$ has the bilaw $[y_1,z_1,\ldots,z_c]$.

(5.4.2) <u>Notation.</u> Denote by $\underset{\sim}{N}_c$ the variety of all bigroups in $\underline{\underline{A}} \circ \underline{\underline{A}}$ of class at most c.

(5.4.3) <u>Notation.</u> Let $\underline{Y}_\alpha$ be the split-free bigroup of rank $(1,\omega)$ in $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\alpha} \wedge \underset{\sim}{N}_p$, and again abuse convention by writing (d,σ) for the normal fully-invariant closure of $[y_1,z_1,\ldots,z_d]^{p^\sigma}$ in $\underline{Y}_\alpha$, $d \in \{0,\ldots,p-1\}$, $\sigma \in \{0,\ldots,\alpha-1\}$.

(5.4.4) <u>Theorem.</u> Every normal, fully invariant sub-bigroup $\underline{U} \neq 1$ of $\underline{Y}_\alpha$ can be written uniquely as

$$\underline{U} = A_2(\underline{Y}_\alpha)^{p^\gamma}(d_\sigma,\sigma)\ \ldots\ (d_{\alpha-1},\alpha-1)$$

where $\gamma \in \{0,\ldots,\alpha\}$, $\sigma \in \{0,\ldots,\alpha-1\}$, $p-1 \geq d_\sigma \geq \ldots \geq d_{\alpha-1} \geq 0$, and if $\gamma < \alpha$ then $\sigma \leq \gamma$ and $d_\gamma \leq 1$. $\Lambda(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\alpha} \wedge \underset{\sim}{N}_p)$ is distributive.

Proof. That every $\underline{U}$ has a decomposition of this form follows from (2.2.4) and (5.3.2): choose σ as the smallest element of $\{0,\dots,\alpha-1\}$ for which there exists $d \in \{0,\dots,p-1\}$ such that $(d,\sigma) \leq \underline{U}$, then choose d_τ as the smallest d for which $(d,\tau) \leq \underline{U}$, $\sigma \leq \tau \leq \alpha-1$. Clearly then $d_\sigma \geq \dots \geq d_{\alpha-1}$. The rest of the proof will follow easily from the next lemma which will also prove useful again in this section.

(5.4.5) Lemma. The split-free bigroup of rank (1,1) in $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\alpha} \wedge \underset{\sim}{N}_{p+1}$ (where $\alpha > 1$) can be presented on the generators $a_0,\dots,a_p$, b subject to the defining relations

$$a_0^{p^\alpha} = \dots = a_{p-1}^{p^\alpha} = a_p^{p^{\alpha-1}} = b^{p^\alpha} = [a_i,a_j] = 1, \quad 0 \leq i,j \leq p,$$

$$a_i^b = a_i a_{i+1}, \quad a_p^b = a_p, \qquad 0 \leq i \leq p-1 .$$

Proof. We omit the details: note that the group presented here is generated by the set $\{a_0,b\}$ and that fairly obviously it is a split-free generating set. The lower exponent on a_p occurs because

$$1 = [a_0, b^{p^\alpha}] = \prod_{i=1}^{p} [a_0, ib]^{\binom{p^\alpha}{i}} = a_p^{\binom{p^\alpha}{p}} .$$

Return to the proof of (5.4.4). If $(d,\tau) \leq \underline{U}$ then $(d,\tau) \leq (d_\sigma,\sigma) \ldots (d_{\alpha-1},\alpha-1)$ and therefore

$$\begin{aligned}(d,\alpha-1) &\leq (d_\sigma,\alpha-\tau-1+\sigma) \ldots (d_\tau,\alpha-1)\\ &\leq (d_\tau,\alpha-\tau-1+\sigma) \ldots (d_\tau,\alpha-1)\\ &= (d_\tau,\alpha-\tau-1+\sigma).\end{aligned}$$

However Lemma 5.4.5 yields, that even in the free bigroup of rank (1,1) in $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\alpha} \wedge \underset{\sim}{N}_p$ (with $\alpha > 1$) this can happen only if

$d \geq d_\tau$, $\alpha-1 \geq \alpha-\tau-1+\sigma$; that is, $d \geq d_\tau$ and $\tau \geq \sigma$, whence $(d,\tau) \leq (d_\tau,\tau)$. Since γ is quite clearly unique, we have shown that this expression for $\underline{U}$ is unique; it only remains to remark, that

$z_1^{p^\gamma} \in \underline{U}$ implies $[y_1,z_1^{p^\gamma}] \in \underline{U}$ and that $[y_1,z_1^{p^\gamma}]$ and $[y_1,z_1]^{p^\gamma}$

are equivalent modulo the bilaws of $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\alpha} \wedge \underset{\sim}{N}_p$, from (5.4.5). As the case $\alpha = 1$ is covered by (5.3.11), this completes the proof of (5.4.4).

(5.4.6) <u>Theorem</u>. $\Lambda(\underline{\underline{A}}_{p^2} \circ \underline{\underline{A}}_{p^2} \wedge \underset{\sim}{N}_{p+1})$ is not distributive.

Proof. We show that in the split-free bigroup of rank (1,1) in $\underline{\underline{A}}_{p^2} \circ \underline{\underline{A}}_{p^2} \wedge \underset{\sim}{N}_{p+1}$, there exist normal, fully invariant sub-bigroups V_1,V_2,V_3 which are pairwise incomparable and whose pariwise joins and intersections are respectively equal. Let V_1, V_2, V_3 be determined by the bilaws

$$[y_1,z_1]^p, \quad [y_1,z_1^p], \quad [y_1,pz_1]$$

respectively, and let V be determined by $[y_1,2z_1]^p$. In the notation of (5.4.5) it is clear that

$$V = \langle a_2^p, \dots, a_{p-1}^p \rangle,$$

$$V_1 = \langle a_1^p, V \rangle, \quad V_3 = \langle a_p, V \rangle.$$

Also since

$$[a_0, b^{kp}] = \prod_{i=1}^{p} [a_0, ib]^{\binom{kp}{i}} = a_1^{kp} a_p^{\binom{kp}{p}}$$

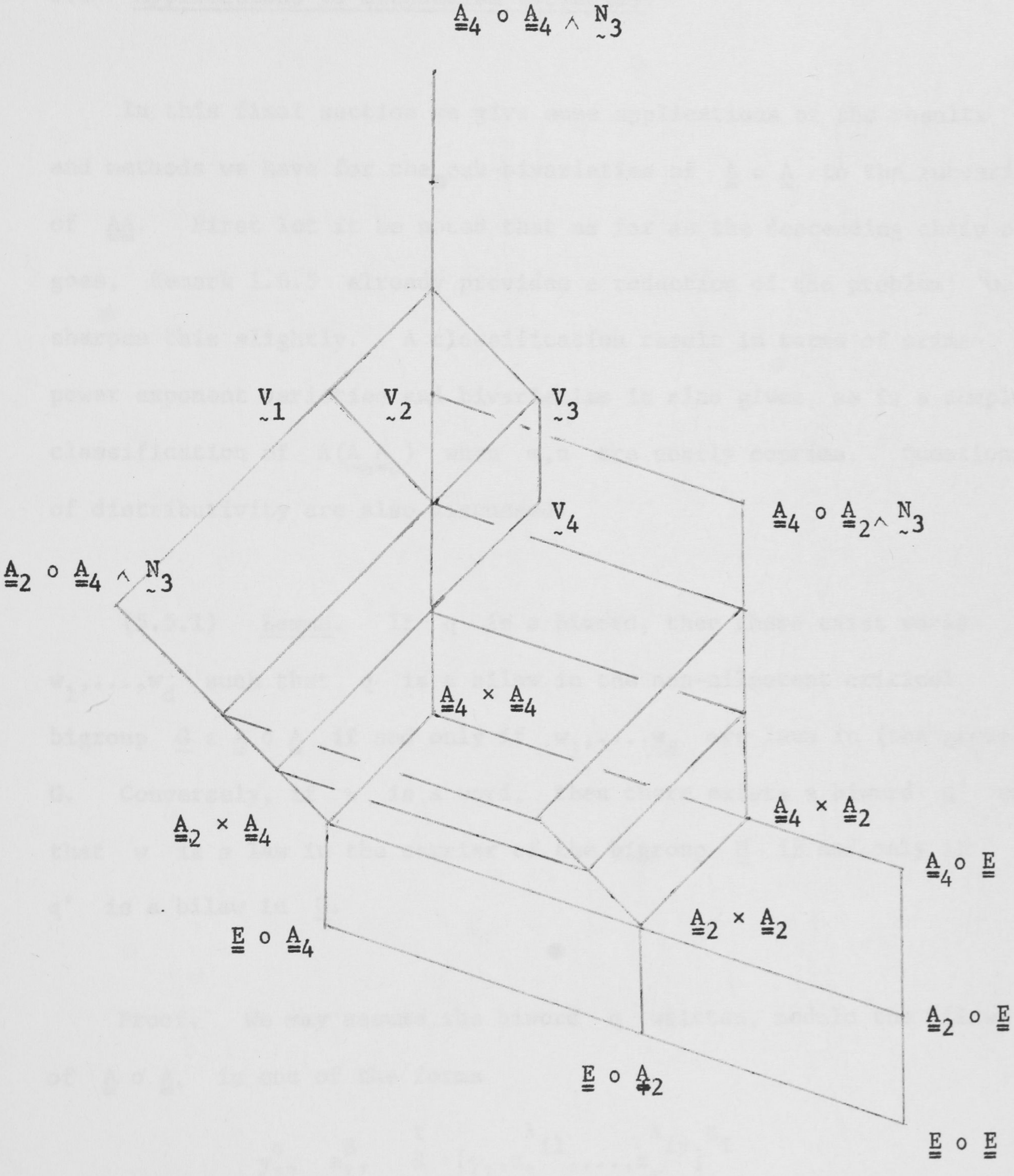
$$= (a_1^p a_p)^k$$

modulo V (using the fact that $\binom{kp}{p} \equiv k(\mathrm{mod}\, p)$), we have that

$$V_2 = \langle a_1^p a_p, V \rangle.$$

Hence (5.4.5) yields that $V_1V_2 = V_2V_3 = V_3V_1 = \langle a_1^p, a_p, V \rangle$, and $V_1 \cap V_2 = V_2 \cap V_3 = V_3 \cap V_1 = V$ and clearly V_1, V_2, V_3, V are all distinct. This completes the proof of (5.4.6); a picture of the lattice $\Lambda(\underline{\underline{A}}_4 \circ \underline{\underline{A}}_4 \wedge \underset{\sim}{N}_3)$ is drawn by way of illustration, but it is not here verified that it has this precise form.

Note that, as we have been working only in the free bigroup of rank (1,1) throughout this section, both results could have been formulated in terms of Engel-type bilaws rather than class bilaws.


$\underline{\underline{A}}_4 \circ \underline{\underline{A}}_4 \wedge \underset{\sim}{N}_3$
$\underset{\sim}{V}_1$
$\underset{\sim}{V}_2$
$\underset{\sim}{V}_3$
$\underset{\sim}{V}_4$
$\underline{\underline{A}}_4 \circ \underline{\underline{A}}_2 \wedge \underset{\sim}{N}_3$
$\underline{\underline{A}}_2 \circ \underline{\underline{A}}_4 \wedge \underset{\sim}{N}_3$
$\underline{\underline{A}}_4 \times \underline{\underline{A}}_4$
$\underline{\underline{A}}_2 \times \underline{\underline{A}}_4$
$\underline{\underline{A}}_4 \times \underline{\underline{A}}_2$
$\underline{\underline{A}}_4 \circ \underline{\underline{E}}$
$\underline{\underline{A}}_2 \times \underline{\underline{A}}_2$
$\underline{\underline{E}} \circ \underline{\underline{A}}_4$
$\underline{\underline{A}}_2 \circ \underline{\underline{E}}$
$\underline{\underline{E}} \circ \underline{\underline{A}}_2$
$\underline{\underline{E}} \circ \underline{\underline{E}}$

5.5 Applications to metabelian varieties.

In this final section we give some applications of the results and methods we have for the sub-bivarieties of $\underline{\underline{A}} \circ \underline{\underline{A}}$ to the subvarieties of $\underline{\underline{AA}}$. First let it be noted that as far as the descending chain condition goes, Remark 1.6.5 already provides a reduction of the problem; we sharpen this slightly. A classification result in terms of prime-power exponent varieties and bivarieties is also given, as is a complete classification of $\Lambda(\underline{\underline{A}}_m \underline{\underline{A}}_n)$ when m, n are nearly coprime. Questions of distributivity are also discussed.

(5.5.1) Lemma. If q is a biword, then there exist words $w_1, \dots, w_d$ such that q is a bilaw in the non-nilpotent critical bigroup $\underline{G} \in \underline{\underline{A}} \circ \underline{\underline{A}}$ if and only if $w_1, \dots, w_d$ are laws in (the group) G. Conversely, if w is a word, then there exists a biword q' such that w is a law in the carrier of the bigroup $\underline{H}$ if and only if q' is a bilaw in $\underline{H}$.

Proof. We may assume the biword q written, modulo the bilaws of $\underline{\underline{A}} \circ \underline{\underline{A}}$, in one of the forms

$$y_1^{\alpha}, \quad z_1^{\beta}, \quad \prod_{i=1}^{t} [y_1, z_1^{\lambda_{i1}}, \dots, z_r^{\lambda_{ir}}]^{\alpha_i}$$

by (2.2.3). The words

$$[x_1,x_2]^{\alpha}, [x_1,x_2,x_3^{\beta}], \quad \prod_{i=1}^{t} [x_1,x_2,x_3^{\lambda_{i1}},\dots,x_{r+2}^{\lambda_{ir}}]^{\alpha_i},$$

respectively, then do what we want. For the converse direction $q' = w(y_1z_1, \ldots, y_sz_s)$ will serve.

(5.5.2) Lemma. There is a one-to-one inclusion preserving correspondence between the set of all subvarieties of $\underline{\underline{AA}}$ generated by non-nilpotent critical groups and the set of all sub-bivarieties of $\underline{\underline{A}} \circ \underline{\underline{A}}$ generated by non-nilpotent critical bigroups; call it θ.

Proof. From (3.3.4) and (5.1.3) there is a one-to-one correspondence between (isomorphism classes of) non-nilpotent critical groups in $\underline{\underline{AA}}$ and (isomorphism classes of) non-nilpotent critical bigroups in $\underline{\underline{A}} \circ \underline{\underline{A}}$ which we may write $G \rightarrow \underline{G}$. If $\underline{\underline{V}} \subseteq \underline{\underline{AA}}$ is generated by non-nilpotent critical groups:

$$\underline{\underline{V}} = \mathrm{var}\{G_i : G_i \text{ non-nilpotent, critical, } i \in I\}$$

define

$$\underset{\sim}{V} = \mathrm{svar}\{\underline{G}_i : i \in I\}.$$

The mapping $\underline{\underline{V}} \rightarrow \underset{\sim}{V}$ may easily be verified to be one-to-one and onto, using (5.5.1).

(5.5.3) Theorem. The variety $\underline{\underline{A}}\,\underline{\underline{A}}_n(\underline{\underline{A}}_m\underline{\underline{A}})$ has descending chain condition on subvarieties if and only if for all $p^\beta | n (p^\alpha | m)$, $\underline{\underline{A}}\,\underline{\underline{A}}_{p^\beta}$ $(\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}})$ does.

Proof. Let $\underline{\underline{V}}_1 \supseteq \underline{\underline{V}}_2 \supseteq \ldots$ be a descending chain in $\Lambda(\underline{\underline{A}}\underline{\underline{A}}_n)$. We may write

$$\underline{\underline{V}}_i = \underline{\underline{V}}'_i \vee \underline{\underline{V}}''_i$$

where $\underline{\underline{V}}'_i$ is generated by the nilpotent critical groups in $\underline{\underline{V}}_i$ and $\underline{\underline{V}}''_i$ by the non-nilpotent critical groups in $\underline{\underline{V}}_i$. Clearly $\underline{\underline{V}}''_1 \supseteq \underline{\underline{V}}''_2 \supseteq \ldots$ is a descending chain, which, by virtue of (5.5.2) and (4.0.1), breaks off. Hence $\underline{\underline{V}}_1 \supseteq \underline{\underline{V}}_2 \supseteq \ldots$ breaks off if and only if $\underline{\underline{V}}'_1 \supseteq \underline{\underline{V}}'_2 \supseteq \ldots$ breaks off. Since

$$\underline{\underline{V}}'_1 \subseteq \vee\{\underline{\underline{A}}\underline{\underline{A}}_{p_i^{\beta_i}} : n = p_1^{\beta_1} \ldots p_t^{\beta_t}, \quad 1 \leq i \leq t\}$$

the theorem follows from (2.1.2).

Now $\underline{\underline{A}}_m\underline{\underline{A}} = \vee\{\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}} : p^\alpha || m\}$ by 21.23 in [3], and therefore (2.1.2) completes the proof.

(5.5.4) Corollary. $\underline{\underline{A}}\underline{\underline{A}}_n$ has descending chain condition on subvarieties if and only if for primes $p|n$ the chains $\underline{\underline{A}}\underline{\underline{A}}_{p^\beta} \supseteq \underline{\underline{V}}_1 \supseteq \underline{\underline{V}}_2 \supseteq \ldots$ with $\underline{\underline{V}}_1\sigma = \underline{\underline{V}}_i\sigma$ for $i \in \{1,2,\ldots\}$ break off. $\underline{\underline{A}}_m\underline{\underline{A}}$ has descending chain condition on subvarieties if for primes $p|n$ the chains $\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}} \supseteq \underline{\underline{V}}_1 \supseteq \underline{\underline{V}}_2 \supseteq \ldots$ with $\underline{\underline{V}}_1\sigma = \underline{\underline{V}}_i\sigma$, $i = 1,2,\ldots$ break off. (The definition of σ is on p.36).

Proof. This follows from (1.6.5). Of course as noted in the Introduction, Cohen [16] has proved descending chain condition for all metabelian varieties, but it is perhaps worth noting that our methods are strong enough to yield such reduction theorems.

We turn our attention now to classification results. Theorem 5.2.7 can be modified in the following way.

(5.5.5) Theorem. $\Lambda(\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_{p^\beta N})$ can be embedded sub-directly into the lattice

$$\Lambda(\underline{\underline{A}}_N) \times \Lambda(\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_{p^\beta}) \times \Phi(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta})^{s-1}$$

where s is the number of divisors $1 = t_1, t_2, \ldots, t_s$ of N. Indeed there exist lattice homomorphisms $\xi_0 : \Lambda(\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_{p^\beta N}) \to \Lambda(\underline{\underline{A}}_N)$,

$\xi_1 : \Lambda(\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_{p^\beta N}) \to \Lambda(\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_{p^\beta})$, $\xi_i : \Lambda(\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_{p^\beta N}) \to \Phi(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_{p^\beta})$, $2 \leq i \leq s$, such that if $\underline{\underline{V}} \subseteq \underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_{p^\beta N}$ then

i) $t_i | t_j$ implies $\underline{\underline{V}}\xi_j \subseteq \underline{\underline{V}}\xi_i$, $2 \leq i,j \leq s$,

ii) $\underline{\underline{V}}\xi_i \subseteq \underline{\underline{V}}\xi_1\sigma$, $2 \leq i \leq s$,

iii) $\underline{\underline{V}}\xi_0 = \underline{\underline{A}}_{t_i}$ implies $\underline{\underline{V}}\xi_j = \underline{\underline{E}} \circ \underline{\underline{E}}$, $t_j \nmid t_i$, $2 \leq j \leq s$.

Proof. We may write

$$\mathbf{V}\xi_0 = \mathbf{V} \wedge \mathbf{A}_N, \quad \mathbf{V}\xi_1 = \mathbf{V} \wedge \mathbf{A}_{p^\alpha}\mathbf{A}_{p^\beta},$$

$$\mathbf{V}\xi = \mathrm{var}\{G \in \mathbf{V} : G \text{ critical, non-nilpotent}\}.$$

The set of subvarieties of $\mathbf{A}_{p^\alpha}\mathbf{A}_{p^{\beta}}{}_{N}$ generated by non-nilpotent critical groups is not a sub-lattice of $\Lambda(\mathbf{A}_{p^\alpha}\mathbf{A}_{p^{\beta}}{}_{N})$, but does form a lattice under the inherited inclusion order. Define $\xi_i = \xi\theta\lambda_i$ (where λ_i is defined in 5.2.7), $2 \leq i \leq s$.

It follows from a result of Kovács and Newman ((1.12) in [5]) and one of Higman (51.1 in [3]), that ξ_0, ξ_1, ξ are lattice homomorphisms; also, in the appropriate sense, θ is a homomorphism (see (5.5.2)). Hence the ξ_i are lattice homomorphisms. Moreover

$$\mathbf{V} = \mathbf{V}\xi_0 \vee \mathbf{V}\xi_1 \vee \mathbf{V}\xi$$

and therefore, using (5.5.2) and (5.2.7) again, $\mathbf{V}$ is determined uniquely by $\{\mathbf{V}\xi_i : i = 0,\ldots,s\}$. That the ξ_i have the properties (i), (ii), (iii) is obvious from their construction and from (5.2.7).

We have the following result similar to (5.2.9), proved by using again (1.12) in [5] and 51.1 in [3].

(5.5.6) Theorem. If $m,n > 0$, $m = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ for distinct primes $p_1,\dots,p_r$, then $\Lambda(\underline{\underline{A}}_m\underline{\underline{A}}_n)$ is a sub-direct product of $\Lambda(\underline{\underline{A}}_{p_i^{\alpha_i}}\underline{\underline{A}}_n)$, $i \in \{1,\dots,r\}$ according to homomorphisms $\eta_i : \Lambda(\underline{\underline{A}}_m\underline{\underline{A}}_n) \to \Lambda(\underline{\underline{A}}_{p_i^{\alpha_i}}\underline{\underline{A}}_n)$ defined by

$$\underline{\underline{V}}\eta_i = \underline{\underline{V}} \wedge \underline{\underline{A}}_{p_i^{\alpha_i}}\underline{\underline{A}}_n ,$$

for $\underline{\underline{V}} \subseteq \underline{\underline{A}}_m\underline{\underline{A}}_n$.

(5.5.7) Corollary. If $\underline{\underline{V}} \subseteq \underline{\underline{A}}_m\underline{\underline{A}}_n$, then $\Lambda(\underline{\underline{V}})$ is distributive if and only if for each $p_i | m$, $\Lambda(\underline{\underline{V}})\eta_i\xi_j$ is distributive, where ξ_j is defined for each i as in (5.5.5)

(5.5.8) Corollary. If m,n are nearly coprime, then $\Lambda(\underline{\underline{A}}_m\underline{\underline{A}}_n)$ is distributive.

(5.5.9) Corollary. Let $\underline{\underline{V}}$ be a variety of metabelian p-groups of bounded exponent in which p-groups have class at most c_p. If $c_p = p$ when $p \leq 3$ and $c_p = p-1$ when $p > 3$, then $\Lambda(\underline{\underline{V}})$ is distributive. On the other hand if $\underline{\underline{W}}$ is the subvariety of $\underline{\underline{A}}_{p^2}\underline{\underline{A}}_{p^2N}$ $(p \nmid N)$ which consists of groups whose Sylow p-subgroups have class at most $p+1$, then $\Lambda(\underline{\underline{W}})$ is not distributive.

Proofs. The proof of (5.5.8) uses (5.5.7), (5.3.13), (5.2.6) and M.F. Newman's unpublished result that $\Lambda(\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_p)$ is distributive. To prove (5.5.9), use (5.5.7), (5.4.4) and Jónsson [11], Weichsel [12] (or Brisley [7]), (5.4.6) and (5.2.6).

Finally we take up the possibility of getting a classification result along the lines of (5.2.11), and prove the following result.

(5.5.10) <u>Theorem</u>. Let $\underline{\underline{V}}$ be a subvariety of $\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_{pN}$. Write $\underline{\underline{V}}_0 = \underline{\underline{V}} \wedge \underline{\underline{A}}_N$, $\underline{\underline{V}}_1 = \underline{\underline{V}} \wedge \underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_p$. There exists a unique subset Δ of the set of divisors of N and to each $\delta \in \Delta$ a subvariety $\underline{\underline{U}}_\delta$ of $\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_p$ with $\underline{\underline{E}} \circ \underline{\underline{E}} \nleq \underline{\underline{U}}_\delta\sigma \in \Phi(\underline{\underline{A}}_{p^\alpha}\underline{\underline{A}}_p)$ and an integer $\alpha(\delta) \leq \min(\alpha, \exp \underline{\underline{U}}_\delta)$ such that

i) $\underline{\underline{V}} = \underline{\underline{V}}_0 \vee \underline{\underline{V}}_1 \vee \{\underline{\underline{U}}_\delta\underline{\underline{A}}_\delta \wedge \underline{\underline{A}}_{p^{\alpha(\delta)}}\underline{\underline{A}}_{pN} : \delta \in \Delta\}$,

ii) $\underline{\underline{U}}_\delta \subseteq \underline{\underline{V}}_1$, $\underline{\underline{A}}_\delta \subseteq \underline{\underline{V}}_0$; and $\delta_0 | \delta$ implies $\underline{\underline{U}}_\delta \subseteq \underline{\underline{U}}_{\delta_0}$ and $\alpha(\delta) \leq \alpha(\delta_0)$,

iii) $\underline{\underline{U}}_\delta\sigma$ and $\alpha(\delta)$ are unique.

Proof. For each $\delta | N$, $\delta > 1$, write

$$\underline{\underline{V}}_\delta = \text{var}\{G \in \underline{\underline{V}} : G \text{ critical, non-nilpotent, } |K| = \delta\};$$

and write Δ for the set of all such δ for which $\underline{\underline{V}}_\delta \neq \underline{\underline{E}}$. Also put for $\delta \varepsilon \Delta$,

$$\underline{\underline{U}}_\delta = \text{var}\{F(G) : G \varepsilon \underline{\underline{V}} \text{ critical, non-nilpotent, } |K| = \delta\},$$

$$\text{and } p^{\alpha(\delta)} = \max\{\exp G' : G \varepsilon \underline{\underline{V}} \text{ critical, non-nilpotent, } |K| = \delta\}.$$

We show that

$$\underline{\underline{V}}_\delta = \underline{\underline{U}}_\delta \underline{\underline{A}}_\delta \wedge \underline{\underline{A}}_{p^{\alpha(\delta)}} \underline{\underline{A}}_{pN} .$$

Now $\underline{\underline{V}}_\delta$ is clearly contained in the right-hand side, and we must show the opposite inclusion. To this end let $\overline{G} \varepsilon \underline{\underline{U}}_\delta \underline{\underline{A}}_\delta \wedge \underline{\underline{A}}_{p^{\alpha(\delta)}} \underline{\underline{A}}_{pN}$ be critical; if $\overline{G} \varepsilon \underline{\underline{A}}_{p^\alpha} \underline{\underline{A}}_p$ or $\overline{G} \varepsilon \underline{\underline{A}}_N$ then clearly $\overline{G} \varepsilon \underline{\underline{V}}_\delta$, and hence we may assume $\overline{G}$ to be non-nilpotent.

Now let $w = w(x_1, \ldots, x_r)$ be a law in $\underline{\underline{V}}_\delta$, that is a law in the generating non-nilpotent critical groups of $\underline{\underline{V}}_\delta$ – call them $\{G_i : i \varepsilon I\}$ say. Now w is a law in a non-nilpotent metabelian critical group G if and only if $q = w(y_1z_1, \ldots, y_rz_r)$ is a bilaw in $\underline{G}$. If $q_1, \ldots, q_d$ correspond to q, p, δ as in Theorem 3.4.4 then $q_1, \ldots q_d$ are bilaws in $\underline{F}^*_i$, $i \varepsilon I$. Now from (5.3.11) each q_i is equivalent, modulo the bilaws of $\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p$, to a set of biwords of the form $y_1^{p^\sigma}, z_1^\varepsilon, [y_1, z_1, \ldots, z_e]^{p^\tau}$ $(0 \leq \sigma, \tau \leq \alpha,\ \varepsilon = 0,1,\ 0 < e)$. We must have $\alpha(\delta) \leq \sigma$, and therefore $y_1^{p^\sigma}$ is a bilaw in $\underline{\overline{F}}$. If z_1 is a bilaw in all $\underline{F}^*_i$ then the F^*_i are abelian and $\underline{\underline{U}}_\delta$ is abelian: hence z_1 is a bilaw in $\underline{\overline{F}}$. In the case of $[y_1,z_1,\ldots,z_e]^{p^\tau}$ we note that, from (5.4.1), it is a bilaw in $\underline{H} \in \underline{\underline{A}} \circ \underline{\underline{A}}$ if and only if $[x_1,x_2,\ldots,x_{e+1}]^{p^\tau}$ is a law in H. Hence since clearly $\overline{F} \in \underline{\underline{U}}_\delta = \mathrm{var}\{F^*_i : i \in I\}$, $q_1,\ldots,q_d$ are all bilaws in $\underline{\overline{F}}$, whence q is a bilaw in $\underline{\overline{G}}$ and thus w is a law in $\overline{G}$. We have proved, therefore, that

$$\underline{\underline{V}}_\delta = \underline{\underline{U}}_\delta\underline{\underline{A}}_\delta \wedge \underline{\underline{A}}_{p^{\alpha(\delta)}}\underline{\underline{A}}_{pN} .$$

Now $\underline{\underline{V}}$ is generated by its critical groups, and therefore,

$$\underline{\underline{V}} = \underline{\underline{V}}_0 \vee \underline{\underline{V}}_1 \vee \vee\{\underline{\underline{V}}_\delta : \delta \in \Delta\}$$

and this disposes of (i). By construction $\alpha(\delta) \leq \min(\alpha, \exp \underline{\underline{U}}_\delta)$ and $\underline{\underline{U}}_\delta\sigma \in \Phi(\underline{\underline{A}}_{p^\alpha} \circ \underline{\underline{A}}_p)$, $\underline{\underline{U}}_\delta \neq \underline{\underline{E}}$. Also if $G \in \underline{\underline{V}}$, G critical, $|K| = \delta$ and $\delta_0|\delta$, consider the subgroup $G_0 = F(G)K_0$, where K_0 is the subgroup of K of order δ_0. From 5.1.2 it follows that there exist critical bigroups $\underline{G}_1,\ldots,\underline{G}_w$ such that $|K_i| = \delta_0$ and $\mathrm{svar}\{\underline{G}_1,\ldots,\underline{G}_w\} = \mathrm{svar}\{\underline{G}_0\}$, and hence that $\mathrm{var}\{G_1,\ldots,G_w\} = \mathrm{var}\{G_0\}$ (from the second part of (5.5.1)). It follows that

$$\mathrm{var}\{F(G_1),\ldots,F(G_w)\} = \mathrm{var}\{F(G)\}$$

and therefore that $\underline{\underline{U}}_\delta \subseteq \underline{\underline{U}}_{\delta_0}$, $\alpha(\delta) \leq \alpha(\delta_0)$. This completes the existence part of the proof.

For the uniqueness, note that if $\underline{\underline{V}}$ has an expression

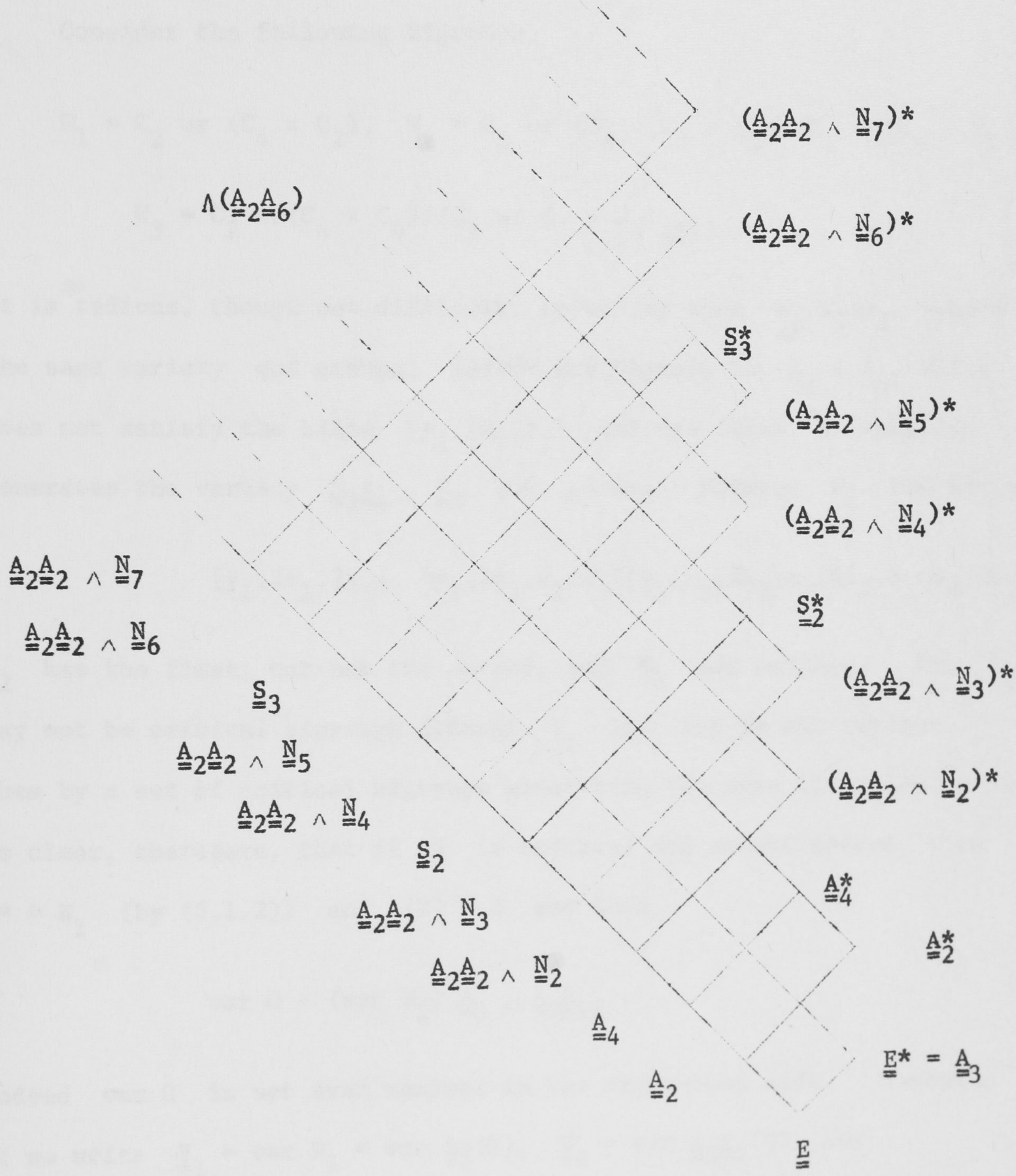
$$\underline{\underline{V}} = \underline{\underline{V}}_0 \vee \underline{\underline{V}}_1 \vee \vee\{\underline{\underline{U}}'_\delta\underline{\underline{A}}_\delta \wedge \underline{\underline{A}}_{p^{\alpha'(\delta)}}\underline{\underline{A}}_{pN} : \delta \in \Delta'\}$$

which satisfies the hypotheses of the theorem, then there exists $1 \neq \underline{P} \in \underline{\underline{U}}'_{\delta}\sigma$, $\underline{P}$ critical. Hence by (5.1.2), (3.3.4) and (1.12) of [5] there exists a critical group G with $|K| = \delta$ and $G \in \underline{\underline{U}}_{\delta}\underline{\underline{A}}_{\delta} \wedge \underline{\underline{A}}_{p^{\alpha(\delta)}}\underline{\underline{A}}_{pN}$. Hence $P \in \underline{\underline{U}}_{\delta}$ and therefore $\underline{P} \in \underline{\underline{U}}_{\delta}\sigma$, or $\underline{\underline{U}}'_{\delta}\sigma \subseteq \underline{\underline{U}}_{\delta}\sigma$. The converse is proved similarly. Finally, since $\alpha'(\delta) \leq \exp \underline{\underline{U}}'_{\delta}$, there exists the critical group $C(p^{\alpha'(\delta)}, \delta)$ of Cossey (Theorem 4.2.2 in [4]) which belongs to $\underline{\underline{V}}$ and therefore to $\underline{\underline{U}}_{\delta}\underline{\underline{A}}_{\delta} \wedge \underline{\underline{A}}_{p^{\alpha(\delta)}}\underline{\underline{A}}_{pN}$; hence $\alpha'(\delta) \leq \alpha(\delta)$. Similarly $\alpha(\delta) \leq \alpha'(\delta)$, and also $\Delta = \Delta'$, and this completes the proof.

The subvarieties of $\underline{\underline{A}}_{p^{\alpha}}\underline{\underline{A}}_{p}$ have been classified by M.F. Newman and thus we have an elaborate, but complete story for $\underline{\underline{A}}_{p^{\alpha}}\underline{\underline{A}}_{pN}$. By way of illustration, the lattice $\Lambda(\underline{\underline{A}}_{2}\underline{\underline{A}}_{6})$ has been drawn. Note that even in this simplest case, the expression (i) in (5.5.10) is not always unique: both the varieties $\underline{\underline{N}}_{2\lambda-1} \wedge \underline{\underline{A}}_{2}\underline{\underline{A}}_{2}$ and $\underline{\underline{S}}_{\lambda}$ give rise to the same non-nilpotent critical groups, that is

$$(\underline{\underline{N}}_{2\lambda-1} \wedge \underline{\underline{A}}_{2}\underline{\underline{A}}_{2})\sigma = \underline{\underline{S}}_{\lambda}\sigma.$$

(5.5.11) Corollary. If m,n are nearly coprime, then (5.5.6) and (5.5.10) give a complete description of $\Lambda(\underline{\underline{A}}_{m}\underline{\underline{A}}_{n})$.



$\underline{\underline{U}}^* = \underline{\underline{U}}\underline{\underline{A}}_3 \wedge \underline{\underline{A}}_2\underline{\underline{A}}_6$

(5.5.12) Example. In general (5.5.10) is not true.

Consider the following bigroups:

$$W_1 = C_2 \text{ wr } (C_4 \times C_2), \quad W_2 = C_2 \text{ wr } (C_4 \times C_2 \times C_2)/(C_2 \text{ wr}(C_4 \times C_2 \times C_2))_{(6)},$$

$$W_3 = C_2 \text{ wr}(C_4 \times C_4)/(C_2 \text{ wr } C_4 \times C_4)_{(6)}:$$

It is tedious, though not difficult, to verify that W_1, W_2, W_3 generate the same variety quâ groups; indeed any bigroup in $\underline{\underline{A}}_2 \circ \underline{\underline{A}}_4$ which does not satisfy the bilaw $[y_1, 3z_1, z_2]$ and has class 5 exactly, generates the variety $\underline{\underline{A}}_2\underline{\underline{A}}_4 \wedge \underline{\underline{N}}_5$ quâ group. However W_1 has the bilaws

$$[y_1, 2z_1, 2z_2], \ [y_1, 2z_1, z_2, z_3][y_1, z_1, 2z_2, z_3][y_1, z_1, z_2, 2z_3],$$

W_2 has the first, but not the second, and W_3 has neither. Now W_2, W_3 may not be critical bigroups (though W_1 is) but we can replace them by a set of critical bigroups generating the same bivariety. It is clear, therefore, that if G is critical and non-nilpotent, with $\underline{F}^* \cong W_1$ (by (5.1.2)) and $|K| = 3$ say then

$$\text{var } G \subset (\text{var } W_1)\, \underline{\underline{A}}_3 \wedge \underline{\underline{A}}_2\underline{\underline{A}}_{12} \, .$$

Indeed var G is not even maximal in the right-hand side. Moreover if we write $\underline{\underline{V}}_1 = \text{var } W_1 = \text{var } \underline{\underline{A}}_3(G)$, $\underline{\underline{V}}_2 = \text{var } \underline{\underline{A}}_3\underline{\underline{A}}_2(G)$ and $\underline{\underline{V}}_3 = \text{var } \underline{\underline{A}}_3\underline{\underline{A}}_2\underline{\underline{A}}_2(G)$ then it is clear on examining the bigroups W_1, W_2, W_3, that var G is at best second maximal in

$$\underline{\underline{V}}_1\underline{\underline{A}}_3 \wedge \underline{\underline{V}}_2\underline{\underline{A}}_6 \wedge \underline{\underline{V}}_3\underline{\underline{A}}_{12} \, .$$

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