SOME PROPERTIES OF WREATH PRODUCTS AND THEIR GENERALISATION

by

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PREFACE

The work for this thesis was done while I held a research scholarship in the Department of Mathematics at the Institute of Advanced Studies of the Australian National University.

I thank my supervisors, Professor B.H. Neumann F.A.A., F.R.S., who suggested my research topic and supervised my work, and Professor Hanna Neumann, who supervised the preparation of this thesis.

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INTRODUCTION

The wreath product of two groups A and B is a well known construction which is particularly useful in proving embedding theorems and which provides a source of counter examples. A precise definition of the standard wreath product will be given in Chapter 1, and a short explanation is sufficient here.

Let B be a permutation group of a set Y and let A be an abstract group. Let H be the product (direct or cartesian) of $|Y|$ isomorphic copies of A . The group B is represented as a group of automorphisms of H and the splitting extension G of H by B is constructed. The group G is called the wreath product of A by B . When the set Y is B , which acts as a right regular representation of itself, then the resulting group is known as the standard wreath product.

Several generalisations of this construction exist. The crown product is a wreath product with central amalgamations. The twisted wreath product [12] is the splitting extension of A^T by B , where T is a right transversal of a subgroup S of B , and where there is a homomorphism α from S into the group of automorphisms of A . Special cases of the twisted wreath product include the standard wreath product of A by B and the splitting extensions of A by B . A further generalisation occurs in a paper by Smel'kin [14] in which the direct or cartesian power of A is replaced by a soluble power.

This thesis concerns the construction of another generalised wreath product in which the direct or cartesian power of A is replaced by a second nilpotent power of A . Certain properties of wreath products and their extension to this generalisation have been considered.

In Chapter 1 a precise definition of a standard wreath product is given together with certain properties which will be needed for the work in this thesis.

Baumslag has given a set of necessary and sufficient conditions for the nilpotency of a wreath product A wr B ; Liebeck has determined the exact nilpotency class when both A and B are abelian. In Chapter 2 upper and lower bounds for the class of A wr B are determined. These bounds depend on the order of B , the class of A and the exponents of the terms of the lower central series of A .

In Chapter 3 the second nilpotent product K of a set $\{A_b ; b \in B\}$ of isomorphic copies of A is constructed. The splitting extension of K by B is formed and the resulting group is the generalised wreath product $A \text{ wr}_2 B$ or $A \text{ Wr}_2 B$, depending on whether K is the restricted or unrestricted product.

In Chapter 4 certain properties of generalised wreath products are considered. It is shown that for some of the results of Chapter 1 there are corresponding results in the case of generalised wreath products. A set of necessary and sufficient conditions for $A \text{ wr}_2 B$ to be nilpotent are determined; in fact $A \text{ wr}_2 B$ is nilpotent if and only if $A \text{ wr } B$ is nilpotent. Finally the results of Chapter 2 are extended to provide bounds for the class of $A \text{ wr}_2 B$, and the exact class is determined when both A and B are cyclic groups of order p .

CHAPTER 1

Introduction.

This chapter contains a brief summary of some known properties of wreath products, which will be of use in the work included in this thesis. The notation used here is the same as that used by the Neumanns in [13]. Most of the results of this chapter are stated without proof, but I have included a proof where I have been unable to find one published in the notation used here.

Notation.

Let $gp\{X ; R\}$ denote the group generated by the set X with the set of defining relations R.

For any group G and $x, y \in G$ let

$$[x, y] = x^{-1}y^{-1}xy ;$$

and if $X, Y \leq G$, then

$$[X, Y] = gp\{[x, y] ; x \in X, y \in Y\} .$$

Left normed commutators of weight n are defined inductively by the rule

$$[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n] ,$$

and if $x_2 = x_3 = \ldots = x_n$, this is written as $[x_1(, x_2)^{n-1}]$.

Definition of a standard wreath product.

Let A and B be abstract groups. The cartesian power A^B of A is the set of functions f from B to A with componentwise multiplication

$$(fg)(b) = f(b)g(b) \quad \text{for all} \quad b \in B .$$

The elements of A^B clearly form a group under this multiplication.

The support $\sigma(f)$ of an element $f \in A^B$ is defined by

$$\sigma(f) = \{ b \in B \; ; \; f(b) \neq 1 \} .$$

The elements f of A^B, which have finite support, form a subgroup $A^{(B)}$ of A^B, and this is the direct power of A .

The group B is represented as a group of automorphisms of A^B by putting

$$f^b(c) = f(cb^{-1}) \quad \text{for all} \quad b, c \in B .$$

The splitting extension of A^B by B is the group of pairs bf , where $b \in B$ and $f \in A^B$, with the following multiplication

$$(bf)(cg) = (bc)(f^c g) \quad \text{for all} \quad b, c \in B \quad \text{and} \quad f, g \in A^B .$$

This group is the unrestricted standard wreath product of A by B , and is denoted by A Wr B .

The direct power $A^{(B)}$ of A admits B as $\sigma(f)$

and $\sigma(f^b)$ have the same cardinal, and thus the splitting extension of $A^{(B)}$ by B is a subgroup of $A \text{ Wr } B$ and is the restricted standard wreath product of A by B, denoted by $A \text{ wr } B$.

For each $b \in B$ there is a component subgroup

$$A_b = \{f \in A^B \, ; \, \sigma(f) \subseteq \{b\}\} \, ;$$

this component subgroup is isomorphic to A.

The set $\{f \in A^B \, ; \, f(b) = f(1) \text{ for each } b \in B\}$ is a subgroup of A^B. It is called the diagonal of A^B and is isomorphic to A. If B is infinite, the diagonal of $A^{(B)}$ is trivial.

<u>Some properties of standard wreath products</u>.

<u>Lemma 1.1</u> (Neumanns' [13], Lemma 3.1) If $A^* \leq A$, then $A^* \text{ wr } B \leq A \text{ wr } B$ and $A^* \text{ Wr } B \leq A \text{ Wr } B$.

<u>Lemma 1.2</u> (Neumanns' [13], Lemma 3.2) Every epimorphism $\varphi : A \longrightarrow A/A^*$ induces epimorphisms

$$\psi : A \text{ Wr } B \longrightarrow (A/A^*)\text{Wr } B \quad \text{and} \quad \psi^{()} : A \text{ wr } B \longrightarrow (A/A^*)\text{wr } B$$

such that $\psi^{()}$ is the restriction of ψ to the restricted wreath product, and such that the restriction of ψ to a component subgroup A_b is an epimorphism $\varphi_b : A_b \longrightarrow A_b/A_b^*$ that corresponds naturally to φ.

In other words ψ is defined by

$$(bf)\psi = bf'$$

where $f' \in (A/A^*)^B$ is given by

$$f'(c) = f(c)\varphi .$$

It is easily shown that the kernel, $\ker\psi$, of ψ is $(A^*)^B$. For let $bf \in \ker\psi$, then

$$1 = (bf)\psi = bf'$$

and hence $b = 1$, and for all $c \in B$

$$f'(c) = f(c)\varphi = 1 ,$$

so that $f(c) \in A^*$ and $f \in (A^*)^B$. Similarly it is shown that $\ker\psi^{()} = (A^*)^{(B)}$, so that

$$(A/A^*) \mathrm{wr}\, B \cong (A \,\mathrm{wr}\, B)/(A^*)^{(B)} ,$$

$$(A/A^*) \mathrm{Wr}\, B \cong (A \,\mathrm{Wr}\, B)/(A^*)^{B} .$$

<u>Lemma 1.3</u> (Neumanns' [13], Lemma 3.4) If $B^* \leq B$, then the standard wreath products $A \,\mathrm{Wr}\, B^*$ and $A \,\mathrm{wr}\, B^*$ are isomorphic to subgroups of the standard wreath products $A \,\mathrm{Wr}\, B$ and $A \,\mathrm{wr}\, B$, respectively.

<u>Lemma 1.4</u> (Gruenberg [5], Lemma 3.2) If A is Abelian and B is an abstract group, then any epimorphism $\varphi : B \longrightarrow B/Z$ may be extended in a natural way to an epimorphism $\psi : A \,\mathrm{wr}\, B \longrightarrow A \,\mathrm{wr}\, B/Z$.

<u>Proof</u>. Extend φ to ψ by setting

$$(bf)\psi = b\varphi f' ,$$

where $f' : B/Z \longrightarrow A$ is defined by

$$f'(b'Z) = \prod_{z\in Z} f(b'z) .$$

Then ψ is a homomorphism, for consider

$$(bfcg)\psi = (bcf^c g)\psi$$
$$= (bc)\varphi f^*$$

where $f^* : B/Z \longrightarrow A$ is defined by

$$f^*(b'Z) = \prod_{z\in Z} (f^c g)(b'z)$$
$$= \prod_{z\in Z} f(b'zc^{-1})g(b'z) .$$

Also

$$(bf)\psi(cg)\psi = b\varphi f' c\varphi g'$$

where $f', g' : B/Z \longrightarrow A$ are defined by

$$f'(b'Z) = \prod_{z\in Z} f(b'z) \quad\text{and}\quad g'(b'Z) = \prod_{z\in Z} g(b'z) ;$$

then

$$(b\varphi f')(c\varphi g') = b\varphi c\varphi (f')^{c\varphi} g'$$
$$= (bc)\varphi g^*$$

where $g^* : B/Z \longrightarrow A$ is defined by

$$g^*(b'Z) = f'(b'Zc^{-1}\varphi)g'(b'Z)$$
$$= f'(b'c^{-1}Z)g'(b'Z)$$
$$= \prod_{z\in Z} f(b'c^{-1}z)g(b'z)$$
$$= f^*(b'Z) .$$

Clearly ψ is an epimorphism, and this completes the proof.

Gruenberg remarks in passing that the kernel of the above epimorphism is easily seen to be the least normal subgroup of $G = A \text{ wr } B$, which contains Z. This will be denoted by $N_G(Z)$. Certainly $\ker\psi$ contains $N_G(Z)$ which is $[A^{(B)}, Z]Z$. Let $bf \in \ker\psi$, then $b \in Z$ and $f\psi = f'$ where

$$f'(b'Z) = \prod_{z \in Z} f(b'z) = 1 ,$$

so that

$$f(b') = \prod_{1 \neq z \in Z} (f(b'z))^{-1} .$$

Let $X = \{b_i \,;\, i \in I \,,\, |I| = |B/Z|\}$ be a transversal of Z in B. For $z \in Z$ define $g_z : X \longrightarrow A$ by

$$g_z(b_i) = f(b_i z) ,$$

and put

$$h = \prod_{1 \neq z \in Z} [g_z, z] .$$

Then for $b_i \in X$,

$$h(b_i) = \prod_{1 \neq z \in Z} (g_z(b_i))^{-1}$$

$$= \prod_{1 \neq z \in Z} (f(b_i z))^{-1}$$

$$= f(b_i) ,$$

and for $1 \neq z \in Z$ and $b_i \in X$

$$h(b_i z) = (g_z)^z(b_i z)$$
$$= g_z(b_i)$$
$$= f(b_i z) .$$

Hence if $f \in \ker \psi$,

$$f = \prod_{1 \neq z \in Z} [g_z, z] \in [A^{(B)}, Z] .$$

Note that if Z is finite then an epimorphism $\varphi : B \longrightarrow B/Z$ can be extended in this way to an epimorphism from $A \text{ Wr } B$ onto $A \text{ Wr } B/Z$, so that

$$A \text{ wr } B/Z \cong (A \text{ wr } B)/[A^{(B)}, Z]Z ,$$

and for a finite subgroup Z

$$A \text{ Wr } B/Z \cong (A \text{ Wr } B)/[A^B, Z]Z .$$

The following theorem was first proved in its present form by Krasner and Kaloujnine [7], but was earlier proved by Frobenius as a theorem in monomial representations. A proof in the notation used here has been published by the Neumanns in [13].

<u>Theorem 1.5</u> Let C be an extension of a group A by a group B . Then C can be embedded in the standard wreath product $A \text{ Wr } B$.

<u>Lemma 1.6</u> The centre of $A \text{ Wr } B$ is the centre of the diagonal subgroup of A^B . If B is infinite,

the centre of A wr B is trivial.

<u>Proof</u>. Let $\overline{G} = A \text{ Wr } B$ and let $\zeta(\overline{G})$ denote the centre of $\overline{G}$. Let $c\,g \in \zeta(\overline{G})$, then for all $b \in B$

$$[cg,b] = 1 = [c,b][c,b,g][g,b] ;$$

therefore $c \in \zeta(B)$ and for all $b' \in B$,

$$g(1) = g(b') .$$

Let $f : B \longrightarrow A$ be such that $\sigma(f) = \{1\}$, then

$$[cg,f] = 1 = g^{-1}f^{-c}gf ,$$

and if $c \neq 1$

$$(g^{-1}f^{-c}gf)(1) = 1 = f(1) .$$

Hence the centre of $\overline{G}$ is the centre of the diagonal subgroup.

If $G = A \text{ wr } B$ then clearly $\zeta(G)$ is the centre of the diagonal subgroup, but if B is infinite, the diagonal subgroup of $A^{(B)}$ is trivial.

CHAPTER 2

Introduction.

The lower central series of a group G is defined as the following series of normal subgroups of G

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots \geq \gamma_i(G) \geq \gamma_{i+1}(G) \geq \ldots ,$$

where, for $i > 1$,

$$\gamma_{i+1}(G) = [\gamma_i(G), G] .$$

For convenience $\gamma_2(G)$ will sometimes be denoted by G' . A group G is said to be nilpotent if its lower central series terminates in the unit subgroup 1 after a finite number of terms, that is if $\gamma_\ell(G) = 1$ for some ~~finite~~ integer ℓ . If $\gamma_{c+1}(G) = 1$ and $\gamma_c(G) \neq 1$, then G is said to be nilpotent of class c .

Baumslag [1] has shown that the restricted wreath product of A by B is nilpotent if and only if A is a nilpotent p-group of finite exponent and B is a finite p-group, for the same prime p .

Liebeck [9] has shown that if A is an abelian group of exponent p^n and B is a direct product of cyclic groups whose orders are $p^{\beta_1}, \ldots, p^{\beta_m}$, with $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_m$, then the class c^* , say, of A wr B

is given by

$$c^* = \sum_{i=1}^{m}(p^{\beta_i}-1) + (n-1)(p-1)p^{\beta_1-1} + 1 .$$

He shows that if B_1 is such that B_1/B'_1 is isomorphic to the group B defined above, and if c_1^* is the class of A wr B_1, then

$$c_1^* \geq \sum_{i=1}^{m}(p^{\beta_i}-1) + (n-1)(p-1)p^{\beta_1-1} + 1 .$$

These results are extended, and upper and lower bounds are determined for the nilpotency class of A wr B when A and B are non-abelian. It is shown that if A has exponent p^n and class c, if $\gamma_\ell(A)$ has exponent p^{n_ℓ} for $1 \leq \ell \leq c$, and B has order p^t, then the class c^*, say, of A wr B has the following bounds

$$c^* \leq cp^{t-1}(np-n+1) \quad \text{when} \quad p > 2 ,$$

$$c^* \leq \max(3n, 4)c2^{t-2} \quad \text{when} \quad p = 2 ;$$

$$c^* \geq \max_{1\leq\ell\leq c} ((\ell t+n_\ell-1)(p-1) + \ell).$$

Thus a lower bound is given by the class when B is elementary abelian; if $p > 2$, the upper bound is attained when B is a cyclic group.

<u>A lower bound for the nilpotency class of A wr B</u>.

The following preliminary lemma is required.

<u>Lemma 2.1</u> Let B be a group of order p^t. There

is a set of generators $b_1, \ldots, b_t$ of B such that, for $1 \leq i \leq t$,

(i) $\text{gp}\{b_1, \ldots, b_i\} = B_i \trianglelefteq B$;

(ii) $b_i B_{i-1} \in \zeta(B/B_{i-1})$, with $B_o = 1$;

(iii) $|B_i| = p^i$.

<u>Proof</u>. This is proved by induction on i . A finite p-group has a non-trivial centre, so b_1 can be chosen to be an element of order p in $\zeta(B)$. Now suppose that the lemma is true for $i < m$. Take $b_m B_{m-1} \in \zeta(B/B_{m-1})$ to be an element of order p, so that $b_m{}^p \in B_{m-1}$. Now, for all $b \in B$

$$b_m{}^{-1} b_m{}^b = [b_m, b] \in B_{m-1} ,$$

and thus $B_m \trianglelefteq B$. Also

$$|B_m| = |B_m/B_{m-1}||B_{m-1}| = p^m .$$

This completes the proof of the lemma.

Using this set of generators, a non-trivial commutator of the required weight will be constructed. First the following lemma and theorem of Liebeck [9] are needed.

Let $B[R]$ be the group ring of B over the integers; let λ_{rs} be the coefficient of b^s in $(1-b)^r$, where b is a fixed element of B .

Lemma 2.2 Let $f \in A^B$ with $\sigma(f) = \{1\}$, and let $g = [f(,b)^r]$. Then

$$\sigma(g) \subseteq \{1, b, \ldots, b^{|b|-1}\} ;$$

$$g(b^i) = f(1)^{-\lambda_{ri}} \text{ if } r \text{ is odd ,}$$

$$g(b^i) = f(1)^{\lambda_{ri}} \text{ if } r \text{ is even .}$$

Theorem 2.3 If $|b| = p^\rho$, then p^{n+1} divides λ_{rs} if $r \geq p^\rho + n(p-1)p^{\rho-1}$; p^{n+1} does not divide λ_{rs} if $r = p^\rho + n(p-1)p^{\rho-1} - 1$.

Lemma 2.4 If B has order p^t and A is a nilpotent p-group of exponent p^n, and if c^* is the class of $A \text{ wr } B$, then

$$c^* \geq (t+n-1)(p-1) + 1 .$$

Proof. Let $b_1, \ldots, b_t$ be the generators of B given by Lemma 2.1, and let $f \in A^B$ with $\sigma(f) = \{1\}$ and $f(1)$ of order p^n. Consider

$$g = [f(,b_2)^{p-1} \ldots (,b_t)^{p-1}] .$$

Now every $b \in B$ can be expressed uniquely in the form

$$b = b_1^{\beta_1} \ldots b_t^{\beta_t} \text{ for } 0 \leq \beta_i \leq p-1 \text{ and } 1 \leq i \leq t ;$$

and it follows from Lemma 2.2 that for all $b \in B$ $\lambda_{(p-1),o} = 1$, so that

$$g(1) = f(1)^{\pm 1} .$$

For all $b \in \sigma(g) = \{b_2^{\beta_2} \ldots b_t^{\beta_t} ; 0 \leq \beta_i \leq p-1, 2 \leq i \leq t\}$

define g_b by

$$g_b(b) = g(b)$$

$$g_b(b^*) = 1 \quad \text{for all } b \neq b^* \in B .$$

Then $\sigma([g_b(,b_1)^r]) = \{b, bb_1, bb_1^2, \ldots, bb_1^{p-1}\}$;

so that, clearly, for $b \neq b^*$ and $b, b^* \in \sigma(g)$,

$$\sigma([g_b(,b_1)^r]) \cap \sigma([g_{b^*}(,b_1)^r]) = \emptyset .$$

Hence $[g(,b_1)^r] \neq 1$ if $[g_1(,b_1)^r] \neq 1$. But $g_1(1) = f(1)^{\pm 1}$ and so has order p^n ; and thus by Theorem 2.3 $[g(,b_1)^r] \neq 1$ if $r = n(p-1)$. This completes the construction of a non-trivial commutator of weight

$$(t+n-1)(p-1) + 1 .$$

<u>Theorem 2.5</u> Let B have order p^t ; let A be a nilpotent p-group of class c and let $\gamma_\ell(A)$ have exponent p^{n_ℓ} for $1 \leq \ell \leq c$. If c^* is the class of A wr B , then

$$c^* \geq \max_{1 \leq \ell \leq c} ((\ell t + n_\ell - 1)(p-1) + \ell) .$$

<u>Proof</u>. For $\ell \in \{1, \ldots, c\}$, let $f_1, \ldots, f_\ell \in A^B$ with $\sigma(f_i) = \{1\}$ for $1 \leq i \leq \ell$, and such that

$$[f_1(1), \ldots, f_\ell(1)] \neq 1 .$$

It can be seen from the proof of Lemma 2.4 that an index $\varepsilon = \pm 1$ can be chosen so that the commutator (of weight

$s = t(\ell-1)(p-1)+\ell$)

$$[f_1{}^{\varepsilon}(,b_1)\overset{p-1}{,\ldots}(,b_t)^{p-1}, f_2{}^{\varepsilon}(,b_1)\overset{p-1}{,\ldots}(,b_t)^{p-1}, f_\ell] = [f_1, f_2, \ldots, f_\ell] ;$$

and hence $\gamma_\ell(A) \leq \gamma_s(A \text{ wr } B)$. Let $g \in A^B$ with $\sigma(g) = \{1\}$ be such that $g(1)$ is an element of order p^{n_ℓ} in $\gamma_\ell(A)$. Then $g \in \gamma_s(A \text{ wr } B)$. By Lemma 2.4

$$[g(,b_2)\overset{p-1}{,\ldots}(,b_t)^{p-1}(,b_1)^r] \neq 1 \quad \text{if} \quad r = n_\ell(p-1) ;$$

and thus

$$c^* \geq (\ell t+n_\ell-1)(p-1) + \ell .$$

But this is true for all $\ell \in \{1,\ldots,c\}$, so that

$$c^* \geq \max_{1\leq\ell\leq c} \{(\ell t+n_\ell-1)(p-1)+\ell\} .$$

An upper bound for the nilpotency class of A wr B .

Before determining an upper bound for the nilpotency class of A wr B some preliminary lemmas are required. Suppose firstly that A is an abelian p-group of exponent p^n .

The following lemma was stated and proved by Liebeck [9] for the case where A and B are abelian.

It is true, and the same proof applies when A is abelian and B is non-abelian.

Lemma 2.6 Let $G = A \text{ wr } B$ with A abelian.

(i) If $f \in A^B$ and $b \in B$, then

$$[b,f] = [f^{-1},b] .$$

(ii) If $f,f' \in A^B$ and $g \in G$, then

$$[f,gf'] = [f,g] \text{ and } [ff',g] = [f,g][f',g] .$$

(iii) If $g_i = b_i f_i$ for $1 \leq i \leq s$ and $b_i \in B$, $f_i, f, f' \in A^B$, then

$$[f,g_1,\ldots,g_s] = [f,b_1,\ldots,b_s]$$

and

$$[ff',g_1,\ldots,g_s] = [f,g_1,\ldots,g_s][f',g_1,\ldots,g_s] .$$

The following lemma, which is stated without proof, follows trivially from Lemma 2.6 and from the fact that A^B is abelian and normal in $A \text{ wr } B$.

Lemma 2.7 If A is an abelian group, any commutator of $A \text{ wr } B$ can be written as a product of commutators of the form

$$[b_{i_1},\ldots,b_{i_r},f,b_{i_{r+1}},\ldots,b_{i_n}] \text{ or } [b_{i_1},\ldots,b_{i_n}] ,$$

where $f \in A^B$ and the b_{i_j} are the generators of B given by Lemma 2.1 .

Note that if B has class k, a commutator of weight less than $k+1$ may lie outside A^B, but $\gamma_{k+1}(A \text{ wr } B) \leq A^B$.

Corollary 2.8 If A is an abelian group of exponent p^n, B is a finite p-group and C is a cyclic group of order p^n, then the class of A wr B is equal to the class of C wr B.

Proof. The class of A wr B is equal to the maximal weight of a non-trivial commutator in A wr B, which is of the form

$$[b_{i_1},\ldots,b_{i_r},f,b_{i_{r+1}},\ldots,b_{i_n}] .$$

For a fixed group B and A abelian this depends only on the exponent of A.

The main result will now be proved, firstly when A is abelian and secondly, by induction on the class of A, when A is a nilpotent p-group of finite exponent.

Lemma 2.9 If C is a cyclic group of order p^n and B is a nilpotent p-group, and $c^*(n,t)$ is the class of C wr B, then

$$c^*(n,t) \leq p^{t-1}(np-n+1) \quad \text{for } p > 2 ,$$

$$c^*(n,t) \leq \max(4,3n)2^{t-2} \quad \text{for } p = 2 .$$

<u>Proof</u>. This is proved firstly by induction on t when $n = 1$, and secondly by induction on n. Suppose first that $n = 1$ and let $B_1 \leq \zeta(B)$ be a subgroup of order p, and let the class of $C \text{ wr } B/B_1$ be r. Then by Lemma 1.4

$$\gamma_{r+1}(C \text{ wr } B/[C^B, B_1]B_1) \cong \gamma_{r+1}(C \text{ wr } B/B_1) = 1,$$

and therefore

$$\gamma_{r+1}(C \text{ wr } B) \leq [C^B, B_1]B_1 .$$

But, by Lemma 2.4, $r \geq (t-1)(p-1) + 1$, which is greater than the class of B, and thus, by Lemma 2.7

$$\gamma_{r+1}(C \text{ wr } B) \leq [C^B, B_1] .$$

Then

$$\begin{aligned}\gamma_{mr+1}(C \text{ wr } B) &\leq [C^B, B_1(, C \text{ wr } B)^{(m-1)r}] \\ &= [C^B, B_1(, B)^{(m-1)r}] .\end{aligned}$$

Now $B_1 \leq \zeta(B)$ and A is abelian and therefore for all $f \in C^B$, $b_1 \in B_1$ and $b \in B$

$$[f, b_1, b] = [f, b, b_1]$$

and hence

$$[C^B, B_1, B] = [C^B, B, B_1] .$$

Thus a simple induction argument on m shows that

$$\gamma_{mr+1}(C \text{ wr } B) \leq [C^B(, B_1)^m] .$$

But by Theorem 2.3

$$[C^B(, B_1)^p] = 1 ,$$

and thus

$$\gamma_{pr+1}(C \text{ wr } B) = 1 .$$

It will now be proved by induction on t that if B is non-cyclic of order p^{t+1}, then

$$c^*(1,t+1) \leq p^{t-1}(2p-1).$$

For $t = 1$ it is known that if D is an elementary abelian group of order p^2, then C wr D has class $2p-1$, and the result follows trivially by induction on t.

Now suppose that B is non-cyclic of order p^t, and that for $n \leq N$

$$c^*(n,t) \leq np^{t-2}(2p-1) .$$

Let C have order p^{N+1} and consider (C/C_1) wr B, where C_1 is the subgroup of C generated by p^Nth powers of elements of C, so that C/C_1 is cyclic of order p^N. By Lemma 1.2

$$(C/C_1) \text{ wr } B \cong C \text{ wr } B/(C_1)^B$$

and so

$$\gamma_{c^*(N,t)+1}((C/C_1)\text{wr } B) = 1 = \gamma_{c^*(N,t)+1}(C \text{ wr} B/C_1{}^B) ;$$

hence

$$\gamma_{c^*(N,t)+1}(C \text{ wr } B) \leq C_1{}^B ,$$

and

$$\begin{aligned}\gamma_{c^*(N,t)+r+1}(C \text{ wr } B) &\leq [C_1{}^B(,C \text{ wr } B)^r] \\ &= [C_1{}^B(,B)^r] \\ &\leq \gamma_{r+1}(C_1 \text{ wr } B) .\end{aligned}$$

But C_1 wr B has class at most $p^{t-2}(2p-1)$ so that

$$c^*(N+1,t) \leq c^*(N,t) + p^{t-2}(2p-1) \leq (N+1)p^{t-2}(2p-1).$$

If C is a cyclic group of order p^n and B is a cyclic group of order p^t, then C wr B has class $p^{t-1}(np-n+1)$. For $p > 2$

$$p^{t-1}(np-n+1) > np^{t-2}(2p-1),$$

but for $p = 2$

$$2^{t-1}(n+1) \geq 3n2^{t-2} \quad \text{if } 2 \geq n,$$

$$2^{t-1}(n+1) \leq 3n2^{t-2} \quad \text{if } 2 \leq n.$$

Therefore

$$c^*(n,t) \leq p^{t-1}(np-n+1) \quad \text{if } p > 2,$$

$$c^*(n,t) \leq 2^{t-2}\max(4,3n) \quad \text{if } p = 2.$$

Theorem 2.10 If A is nilpotent of class c and has exponent p^n and B has order p^t, and if A wr B has class c^*, then

$$c^* \leq cp^{t-1}(np-n+1) \quad \text{if } p > 2,$$

$$c^* \leq \max(4,3n)c2^{t-2} \quad \text{if } p = 2.$$

Proof. This is proved by induction on c. By Corollary 2.8 and Lemma 2.9 it is true when $c = 1$. Suppose that it is true for $c = d$ and let A have class $d + 1$ and $(A/\gamma_d(A))$ wr B have class c_1. Then

$$c_1 \leq dp^{t-1}(np-n+1) \quad \text{if } p > 2,$$

$$c_1 \leq \max(4,3n)d2^{t-2} \quad \text{if } p = 2.$$

Now by Lemma 1.2

$$1 = \gamma_{c_1+1}((A/\gamma_d(A))\text{wr } B) \cong \gamma_{c_1+1}(A \text{ wr } B/(\gamma_d(A))^B) ,$$

and hence

$$\gamma_{c_1+1}(A \text{ wr } B) \leq (\gamma_d(A))^B .$$

Then

$$\gamma_{c_1+r+1}(A \text{ wr } B) \leq [(\gamma_d(A))^B(,A \text{ wr } B)^r] .$$

But

$$(\gamma_d(A))^B \leq \zeta(A^B)$$

and so

$$\gamma_{c_1+r+1}(A \text{ wr } B) \leq [(\gamma_d(A))^B(,B)^r] .$$

By Lemma 1.1

$$\text{sgp}\{(\gamma_d(A))^B, B\} = \gamma_d(A) \text{ wr } B$$

and $\gamma_d(A)$ wr B has class d^*, say, where

$$d^* \leq p^{t-1}(np-n+1) \quad \text{for } p > 2 ,$$

$$d^* \leq \max(4,3n)2^{t-2} \quad \text{for } p = 2 .$$

Hence

$$\gamma_{c_1+d^*+1}(A \text{ wr } B) = 1 ,$$

and so

$$c^* \leq (d+1)p^{t-1}(np-n+1) \quad \text{for } p > 2 ,$$

$$c^* \leq (d+1)\max(4,3n)2^{t-2} \quad \text{for } p = 2 .$$

This completes the proof of the theorem.

CHAPTER 3

Introduction.

Let B be an ordered index set and let $D = \{(b,b') ; b,b' \in B, b > b'\}$. For an arbitrary group A the second nilpotent power of A is defined as a set of pairs of functions ft , where f is a function on B taking values in A and t is a function on D taking values in $A \otimes A$, this set having the appropriate multiplication defined on it. By this means both the restricted, K , and the unrestricted, $\overline{K}$, second nilpotent powers can be defined. This definition differs from those of Golovin and of Moran but it is shown at the end of this chapter that the definitions are equivalent.

The splitting extensions of K and $\overline{K}$ by B are the generalised wreath products $A\ wr_2\ B$ and $A\ Wr_2\ B$ respectively.

Before defining the second nilpotent powers, it is necessary to define the tensor product of groups. Following Wiegold [15] this group is written multiplicatively.

The tensor product of groups.

Let M and N be arbitrary groups. The tensor product $M \otimes N$ of M and N is defined as the group of pairs $m \otimes n$, with $m \in M$ and $n \in N$ subject to the following relations

$$mm' \otimes n = (m \otimes n)(m' \otimes n) ,$$
$$m \otimes nn' = (m \otimes n)(m \otimes n') ,$$

for all $m, m' \in M$ and $n, n' \in N$. The unit element of $M \otimes N$ is $1 \otimes 1$, since

$$(m \otimes n)(1 \otimes 1) = (m \otimes n)(1 \otimes n)(1 \otimes 1) = (m \otimes n)(1 \otimes n)$$
$$= m \otimes n .$$

Also

$$(m \otimes n)(m' \otimes 1) = (mm'^{-1}m' \otimes n)(m' \otimes 1) = (mm'^{-1} \otimes n)(m' \otimes n)$$
$$= m \otimes n ,$$

and hence for all $m \in M$ and $n \in N$,

$$m \otimes 1 = 1 \otimes 1 = 1 \otimes n .$$

The following are well known properties of tensor products.

Lemma 3.1 The tensor product $M \otimes N$ of M and N is abelian.

Proof. Consider $mm' \otimes nn'$;

$$mm' \otimes nn' = (m \otimes nn')(m' \otimes nn')$$
$$= (m \otimes n)(m \otimes n')(m' \otimes n)(m' \otimes n') ;$$
$$mm' \otimes nn' = (mm' \otimes n)(mm' \otimes n')$$
$$= (m \otimes n)(m' \otimes n)(m \otimes n')(m' \otimes n') ,$$

and hence

$$(m \otimes n')(m' \otimes n) = (m' \otimes n)(m \otimes n') .$$

<u>Lemma 3.2</u> For all groups M and N, $M \otimes N \cong M/M' \otimes N/N'$.

<u>Proof</u>. Let $\varphi' : M \longrightarrow M/M'$ and $\varphi^* : N \longrightarrow N/N'$ and define $\varphi : M \otimes N \longrightarrow M/M' \otimes N/N'$ by

$$(m \otimes n)\varphi = m\varphi' \otimes n\varphi^* = mM' \otimes nN' .$$

Then φ is obviously a homomorphism and maps $M \otimes N$ onto $M/M' \otimes N/N'$. Also φ is a monomorphism for suppose that

$$(m \otimes n)\varphi = 1 ,$$

then, since $1 = m \otimes n' = m' \otimes n$ for all $m \in M$, $m' \in M'$, $n \in N$ and $n' \in N'$,

$$1 = M' \otimes N' = mM' \otimes nN'$$

and hence

$$m \otimes n = 1 .$$

<u>Lemma 3.3</u> If $M = \prod_{i \in I} M_i$ and $N = \prod_{j \in J} N_j$, then $M \otimes N = \prod_{i \in I, j \in J} M_i \otimes N_j$, where $\prod_{\lambda \in \Lambda} G_\lambda$ denotes the direct product of the set of groups $\{G_\lambda ; \lambda \in \Lambda\}$.

<u>Proof</u> Clearly $M_i \otimes N_j \cap \prod_{(i,j) \neq (k,\ell)} M_k \otimes N_\ell = 1$, and $\{M_i \otimes N_j ; i \in I \text{ and } j \in J\}$ generates $M \otimes N$. This completes the proof.

To determine the order of an element $m \otimes n$ of $M \otimes N$, it is necessary to consider the order of $M \otimes N$ when M and N are cyclic groups. Suppose that $M = gp\{m ; m^{tr} = 1\}$ and $N = gp\{n ; n^{ts} = 1\}$ and let r and s be co-prime. Then

$$(m \otimes n)^{tr} = m^{tr} \otimes n = 1 ,$$

$$(m \otimes n)^{ts} = m \otimes n^{ts} = 1 .$$

But $M \otimes N$ is obviously a cyclic group generated by $m \otimes n$, and so its order must divide t. It will be shown that the order of $M \otimes N$ is exactly t. A theorem of Curtis and Reiner ([2], page 61) is required and is proved only in the special case needed here. The following two definitions are required.

Let M and N be abelian groups and let R be an abelian group. A balanced map f of the product set $M \times N$ into R assigns to each pair (m,n) of $M \times N$ an element $f(m,n)$ of R, such that

$$f(mm',n) = f(m,n)f(m',n) ,$$

$$f(m,nn') = f(m,n)f(m,n') ,$$

for all $m,m' \in M$ and $n,n' \in N$.

Let $f : M \times N \longrightarrow R$ and $\varphi : M \times N \longrightarrow S$ be balanced maps of $M \times N$ into the abelian groups R and S respectively. Then f is said to be factored through S if there exists a homomorphism $f^* : S \longrightarrow R$ such that

$$f = f^* \varphi .$$

<u>Theorem 3.4</u> Let M and N be abelian groups. Then every balanced map of $M \times N$ into an arbitrary abelian group can be factored through $M \otimes N$.

<u>Proof</u>. Let $F = gp\{(m,n) ; m \in M, n \in N\}$ and let H be the subgroup of F generated by the formal products

$$(mm',n)(m',n)^{-1}(m,n)^{-1} ,$$
$$(m,nn')(m,n')^{-1}(m,n)^{-1} .$$

Then $M \otimes N$ is the factor group F/H. Define a mapping $\sigma : M \times N \longrightarrow M \otimes N$ by

$$\sigma(m,n) = (m,n)H .$$

Let φ be any balanced map of $M \times N$ to R. The mapping φ defines a homomorphism $\varphi' : F \longrightarrow R$ by

$$\varphi' \prod (m_i,n_j)^{r(i,j)} = \prod (\varphi(m_i,n_j))^{r(i,j)} .$$

But φ is a balanced map so that $\varphi'(H) = 1$, and thus φ induces a homomorphism $\varphi^* : F/H \longrightarrow R$ by

$$\varphi^*((m,n)H) = \varphi(m,n) ,$$

or

$$\varphi^* \sigma(m,n) = \varphi(m,n) ,$$

and thus φ has been factored through S.

<u>Corollary 3.5</u> Let $M = gp\{m ; m^{tr} = 1\}$ and let $N = gp\{n ; n^{ts} = 1\}$ with r and s co-prime, then

$$M \otimes N = gp\{m \otimes n \,;\, (m \otimes n)^t = 1\}.$$

<u>Proof</u>. It has been shown above that the order of $M \otimes N$ divides t (p. 27). It is sufficient therefore to find a balanced map from $M \otimes N$ onto a cyclic group of order t.

Let $R = gp\{x \,;\, x^t = 1\}$. Define a map $f : M \times N \rightarrow R$ by

$$f(m^\alpha, n^\beta) = x^{\alpha\beta} \quad \text{for } 0 \le \alpha \le tr \text{ and } 0 \le \beta \le ts.$$

Then

$$f(m^{\alpha+\alpha'}, n^\beta) = x^{(\alpha+\alpha')\beta}$$
$$= x^{\alpha\beta}x^{\alpha'\beta}$$
$$= f(m^\alpha, n^\beta)\, f(m^{\alpha'}, n^\beta),$$

and similarly

$$f(m^\alpha, n^{\beta+\beta'}) = f(m^\alpha, n^\beta)\, f(m^{\alpha'}, n^{\beta'}).$$

Thus f is a balanced map and this completes the proof.

<u>The second nilpotent powers of a group</u>

The second nilpotent power of a group A will now be defined. Let B be an ordered set and let

$D = \{(b,b') \; ; \; b,b' \in B, \; b > b'\}$. For a given group A, let f be a function on B taking values in A and let t be a function on D taking values in $A \otimes A$.
Consider the set $\overline{K}$ of pairs ft with the following multiplication

$$(ft)(gu) = f't'$$

where $f' : B \longrightarrow A$ is given by

$$f'(b) = f(b)g(b) ,$$

and $t' : D \longrightarrow A \otimes A$ is given by

$$t'(b,b') = t(b,b')u(b,b')(f(b) \otimes g(b')) .$$

Note that for all $f : B \longrightarrow A$ and $t : D \longrightarrow A \otimes A$

$$[t, f] = 1 .$$

The multiplication defined above is associative.

<u>Proof</u>. Let $f, g, h : B \longrightarrow A$ and $t, u, v : D \longrightarrow A \otimes A$, then

$$(ft)(gu) = f't'$$

where

$$f'(b) = f(b)g(b) ,$$

$$t'(b,b') = t(b,b')u(b,b')(f(b) \otimes g(b')) ;$$

also

$$(f't')(hv) = f^*t^*$$

where

$$f^*(b) = f'(b)h(b) = f(b)g(b)h(b) ,$$

$$t^*(b,b') = t'(b,b')v(b,b')(f'(b) \otimes h(b'))$$
$$= t(b,b')u(b,b')(f(b) \otimes g(b'))v(b,b')(f(b)g(b) \otimes h(b')).$$

Also

$$(gu)(hv) = g'u'$$

where

$$g'(b) = g(b)h(b) ,$$

$$u'(b,b') = u(b,b')v(b,b')(g(b) \otimes h(b')) ,$$

and

$$(ft)(g'u') = g^*u^*$$

where

$$g^*(b) = f(b)g'(b) = f(b)g(b)h(b) = f^*(b) ,$$

$$u^*(b,b') = t(b,b')u'(b,b')(f(b) \otimes g'(b'))$$
$$= t(b,b')u(b,b')v(b,b')(g(b) \otimes h(b'))(f(b) \otimes g(b')h(b')).$$

It clearly follows from the defining relations of $A \otimes A$ that $t^* = u^*$.

The unit element 1 of $\overline{K}$ is that pair of functions which take the value 1 everywhere.

The inverse of ft is f^*t^* , where

$$f^*(b) = (f(b))^{-1} ,$$

$$t^*(b,b') = (t(b,b'))^{-1}(f(b) \otimes f(b')) .$$

<u>Proof</u>. Consider $gu = (ft)(f^*t^*)$, then

$$g(b) = f(b)f^*(b) = f(b)(f(b))^{-1} = 1 ,$$

$$\begin{aligned} u(b,b') &= t(b,b')t^*(b,b')(f(b) \otimes f^*(b')) \\ &= t(b,b')(t(b,b'))^{-1}(f(b) \otimes f(b'))(f(b) \otimes (f(b'))^{-1}) \\ &= 1 . \end{aligned}$$

Thus the elements of $\overline{K}$ with this multiplication form a group, which is the unrestricted second nilpotent power of A .

The supports $\sigma(f)$ of f and $\sigma(t)$ of t are defined as follows

$$\sigma(f) = \{b \in B ; f(b) \neq 1\} ,$$

$$\sigma(t) = \{d \in D ; t(d) \neq 1\} .$$

Clearly the set of elements of $\overline{K}$ which have finite support forms a subgroup K of $\overline{K}$, and this group is the unrestricted second nilpotent power of A .

The set $\overline{T}$ of functions $t : D \longrightarrow A \otimes A$ is clearly a normal subgroup of $\overline{K}$, and the set T of those $t \in \overline{T}$ with finite support is a normal subgroup of K .

For each $b \in B$ and $(b,b') \in D$ there exist component subgroups

$$A_b = gp\{f : B \longrightarrow A\ ;\ \sigma(f) \subseteq \{b\}\}\ ,$$

$$A_b \otimes A_{b'} = gp\{t : D \longrightarrow A \otimes A\ ;\ \sigma(t) \subseteq \{(b,b')\}\}\ .$$

Clearly A_b is isomorphic to A and $A_b \otimes A_{b'}$ is isomorphic to $A \otimes A$.

The following notation will be used

$$\overline{K} = \prod_{b \in B}^{2} A_b \quad \text{and} \quad K = \prod_{b \in B}^{(2)} A_b\ .$$

Generalised wreath products.

Suppose now that the index set B is a group, which need not be an ordered group. This group B will be represented as an automorphism group of $\overline{K}$.

Define a map $\alpha : A \otimes A \longrightarrow A \otimes A$ by

$(m \otimes n)\alpha = (n \otimes m)^{-1}$ for all $m, n \in A$.

Then $(m \otimes nn')\alpha = (nn' \otimes m)^{-1}$

$= (n \otimes m)^{-1}(n' \otimes m)^{-1}$

$= (m \otimes n)\alpha(m \otimes n')\alpha$

and

$(mm' \otimes n)\alpha = (m \otimes n)\alpha(m' \otimes n)\alpha$.

Thus α can be extended to a homomorphism which is clearly an automorphism of $A \otimes A$.

For $c \in B$ put

$$(ft)^c = f't'\ ,$$

where

$$f'(b) = f(bc^{-1}),$$

$$t'(b,b') = t(bc^{-1}, b'c^{-1}) \qquad \text{if } bc^{-1} > b'c^{-1},$$

$$t'(b,b') = (t(b'c^{-1}, bc^{-1}))\alpha(f(b\, c^{-1}) \otimes f(b'c^{-1}))$$
$$\text{if } bc^{-1} < b'c^{-1}.$$

The elements c of B act as epimorphisms of $\overline{K}$.

Proof. Let $f,g : B \longrightarrow A$ and $t,u : D \longrightarrow A \otimes A$; then

$$(ft)(gu) = hv$$

where

$$h(b) = f(b)g(b),$$
$$v(b,b') = t(b,b')u(b,b')(f(b) \otimes g(b'));$$

and

$$(hv)^c = h'v',$$

where

$$h'(b) = h(bc^{-1}) = f(bc^{-1})g(bc^{-1}),$$

$$v'(b,b') = v(bc^{-1}, b'c^{-1}) \qquad \text{if } bc^{-1} > b'c^{-1},$$

$$v'(b,b') = (v(b'c^{-1}, bc^{-1}))\alpha(h(bc^{-1}) \otimes h(b'c^{-1}))$$
$$\text{if } bc^{-1} < b'c^{-1};$$

$$v'(b,b') = t(bc^{-1}, b'c^{-1})u(bc^{-1}, b'c^{-1})(f(bc^{-1}) \otimes g(b'c^{-1}))$$
$$\text{if } bc^{-1} > b'c^{-1},$$

$$v'(b,b') = (t(b'c^{-1}, bc^{-1}))\alpha(u(b'c^{-1}, bc^{-1}))\alpha(g(bc^{-1}) \otimes f(b'c^{-1}))^{-1}$$
$$(f(bc^{-1})g(bc^{-1}) \otimes f(b'c^{-1})g(b'c^{-1}))$$
$$\text{if } bc^{-1} < b'c^{-1}.$$

Also

$$(ft)^c(gu)^c = (f't')(g'u')$$

where

$$f'(b) = f(bc^{-1}) \;, \quad g'(b) = g(bc^{-1}) \;,$$

$$t'(b,b') = t(bc^{-1},b'c^{-1}) \qquad \text{if} \quad bc^{-1} > b'c^{-1} \;,$$

$$t'(b,b') = (t(b'c^{-1},bc^{-1}))\alpha(f(bc^{-1}) \otimes f(b'c^{-1})) \qquad \text{if} \quad bc^{-1} < b'c^{-1} \;,$$

$$u'(b,b') = u(bc^{-1},b'c^{-1}) \qquad \text{if} \quad bc^{-1} > b'c^{-1} \;,$$

$$u'(b,b') = (u(b'c^{-1},bc^{-1}))\alpha(g(bc^{-1}) \otimes g(b'c^{-1})) \qquad \text{if} \quad bc^{-1} < b'c^{-1} \;;$$

and

$$(f't')(g'u') = h^*v^* \;,$$

where

$$h^*(b) = f'(b)g'(b) = f(bc^{-1})g(bc^{-1}) = h'(b) \;,$$

$$v^*(b,b') = t'(b,b')u'(b,b')(f'(b) \otimes g'(b')) \;;$$

$$v^*(b,b') = t(bc^{-1},b'c^{-1})u(bc^{-1},b'c^{-1})(f(bc^{-1}) \otimes g(b'c^{-1})) \qquad \text{if} \quad bc^{-1} > b'c^{-1} \;,$$

$$= v'(b,b') \qquad \text{if} \quad bc^{-1} > b'c^{-1} \;,$$

$$v^*(b,b') = (t(b'c^{-1},bc^{-1}))\alpha(u(b'c^{-1},bc^{-1}))\alpha(f(bc^{-1}) \otimes f(b'c^{-1}))(g(bc^{-1}) \otimes g(b'c^{-1}))(f(bc^{-1}) \otimes g(b'c^{-1})) \qquad \text{if} \quad bc^{-1} < b'c^{-1} \;,$$

$$= v'(b,b') \qquad \text{if} \quad bc^{-1} < b'c^{-1} \;,$$

and hence

$$((ft)(gu))^c = (ft)^c(gu)^c \;.$$

Clearly c maps $\overline{K}$ onto $\overline{K}$ and thus it acts as an epimorphism of $\overline{K}$.

The elements c of B act as monomorphisms of $\overline{K}$.

Proof. Suppose that $(ft)^c = 1$; then

$$(ft)^c = f't' = 1 ,$$

where

$$f'(b) = f(bc^{-1}) = 1 \quad \text{for all } b \in B ,$$

and hence $f = 1$;

also

$$1 = t'(b,b') = t(bc^{-1},b'c^{-1}) \quad \text{if } bc^{-1} > b'c^{-1} ,$$

$$1 = t'(b,b') = (t(b'c^{-1},bc^{-1}))\alpha \quad \text{if } bc^{-1} < b'c^{-1} .$$

Since α is an automorphism of $A \otimes A$ it follows that $t = 1$, and that c is an automorphism of $\overline{K}$.

For all $c,c' \in B$ and $ft \in \overline{K}$, $((ft)^c)^{c'} = (ft)^{cc'}$.

Proof. Let $ft \in \overline{K}$, and let $(ft)^{cc'} = h'v'$, then

$$(ft)^c = gu$$

where

$$g(b) = f(bc^{-1}) ,$$

$$u(b,b') = t(bc^{-1},b'c^{-1}) \quad \text{if } bc^{-1} > b'c^{-1} ,$$

$$u(b,b') = (t(b'c^{-1},bc^{-1}))\alpha(f(bc^{-1}) \otimes f(b'c^{-1})) \quad \text{if } bc^{-1} < b'c^{-1} ;$$

then

$$(gu)^{c'} = hv$$

where

$$h(b) = g(bc'^{-1}) = f(bc'^{-1}c^{-1}) = h'(b) ,$$

$$v(b,b') = u(bc'^{-1}, b'c'^{-1}) \quad \text{if } bc'^{-1} > b'c'^{-1},$$

$$v(b,b') = (u(b'c'^{-1}, bc'^{-1}))a(g(bc'^{-1}) \otimes g(b'c'^{-1})) \quad \text{if } bc'^{-1} < b'c'^{-1} ;$$

$$v(b,b') = t(bc'^{-1}c^{-1}, b'c'^{-1}c^{-1})$$
$$\text{if } bc'^{-1} > b'c'^{-1} \text{ and } bc'^{-1}c^{-1} > b'c'^{-1}c^{-1} ,$$

$$v(b,b') = (t(b'c'^{-1}c^{-1}, bc'^{-1}c^{-1}))a(f(bc'^{-1}c^{-1}) \otimes f(b'c'^{-1}c^{-1}))$$
$$\text{if } bc'^{-1} > b'c'^{-1} \text{ and } bc'^{-1}c^{-1} < b'c'^{-1}c^{-1} ,$$

$$v(b,b') = (t(b'c'^{-1}c^{-1}, bc'^{-1}c^{-1}))a(f(bc'^{-1}c^{-1}) \otimes f(b'c'^{-1}c^{-1}))$$
$$\text{if } bc'^{-1} < b'c'^{-1} \text{ and } bc'^{-1}c^{-1} < b'c'^{-1}c^{-1} ,$$

$$v(b,b') = t(bc'^{-1}c^{-1}, b'c'^{-1}c^{-1})(f(bc'^{-1}c^{-1}) \otimes f(b'c'^{-1}c^{-1}))^{-1}$$
$$(f(bc'^{-1}c^{-1}) \otimes f(b'c'^{-1}c^{-1}))$$
$$\text{if } bc'^{-1} < b'c'^{-1} \text{ and } bc'^{-1}c^{-1} > b'c'^{-1}c^{-1} ;$$

$$v(b,b') = t(bc'^{-1}c^{-1}, b'c'^{-1}c^{-1}) \quad \text{if } bc'^{-1}c^{-1} > b'c'^{-1}c^{-1} ,$$

$$v(b,b') = (t(b'c'^{-1}c^{-1}, bc'^{-1}c^{-1}))a(f(bc'^{-1}c^{-1}) \otimes f(b'c'^{-1}c^{-1})) \quad \text{if } bc'^{-1}c^{-1} < b'c'^{-1}c^{-1} ;$$

and hence $v = v'$, and

$$((ft)^c)^{c'} = (ft)^{cc'} .$$

The following theorem ([6], Page 88) shows that the set of pairs $\{bk ; b \in B , k \in \overline{K}\}$, with a certain multiplication defined on it, forms a group.

<u>Theorem 3.6</u> If G and H are groups such that for every element $h \in H$ there is an automorphism $g \longleftrightarrow g^h$ of G with $(g^h)^{h'} = g^{hh'}$ for all $h, h' \in H$, then the pairs hg form a group under the product

$$(hg)(h'g') = hh'g^{h'}g' .$$

This group is called the splitting extension of G by H.

Thus the splitting extension of $\overline{K}$ by B is a group $\overline{P}$, the unrestricted generalised wreath product, which is denoted by $A \, Wr_2 \, B$.

If bft, $b'f't' \in \overline{P}$ then

$$(bft)(b'f't') = b^*f^*t^*$$

where

$$b^* = bb' ,$$

$$f^*(c) = f(cb'^{-1})f'(c) ,$$

$$t^*(c,c') = t(cb'^{-1}, c'b'^{-1})t'(c,c')(f(cb'^{-1}) \otimes f'(c')) \quad \text{if } cb'^{-1} > c'b'^{-1} ,$$

$$t^*(c,c') = (t(c'b'^{-1}, cb'^{-1}))at'(c,c')(f(cb'^{-1}) \otimes f'(c')f(c'b'^{-1})) \quad \text{if } cb'^{-1} < c'b'^{-1} .$$

The unit element 1 of $\overline{P}$ is $1_B 1_{\overline{K}}$ where 1_B is the unit element of B and $1_{\overline{K}}$ is the unit element of $\overline{K}$.

The inverse of bft is $b'f't'$, where

$$b' = b^{-1} ,$$

$$f'(c) = (f(cb))^{-1} ,$$

$$t'(c,c') = (t(cb,c'b))^{-1}(f(cb) \otimes f(c'b)) \quad \text{if} \quad cb > c'b ,$$

$$t'(c,c') = ((t(c'b,cb))\alpha)^{-1} \quad \text{if} \quad cb < c'b .$$

<u>Proof</u>. Let

$$(bft)(b'f't') = b^*f^*t^* ,$$

then

$$b^* = bb' = bb^{-1} = 1 ,$$

$$f^*(c) = f(cb)f'(c) = 1 ,$$

$$t^*(c,c') = t(cb,c'b)t'(c,c')(f(cb) \otimes f(c'b))^{-1} \quad \text{if} \quad cb > c'b$$

$$= 1 ,$$

$$t^*(c,c') = (t(c'b,cb))\alpha t'(c,c')(f(cb) \otimes f(c'b)^{-1}f(c'b)) \quad \text{if} \quad cb < c'b$$

$$= 1 .$$

If $\sigma(f)$ and $\sigma(t)$ are finite, then $\sigma(f^b)$ and $\sigma(t^b)$ are finite, for clearly $\sigma(t)$ and $\sigma(t^b)$ have the same cardinal; if $\sigma(f) = \{b_1, \ldots, b_m\}$, then

$$\sigma(f^b) \subseteq \{b_1 b, \ldots, b_m b\} \{(b_i b, b_j b);\ b_i b > b_j b,\ 1 \le i,j \le m\} ,$$

and so $\sigma(f^b)$ has finite order. Thus K admits the automorphisms induced by the elements of B , and the splitting extension P of K by B is a subgroup of $A \text{ Wr}_2 B$. This group is the restricted generalised wreath product and is denoted by $A \text{ wr}_2 B$.

The generalised wreath products $A \, Wr_2 \, B$ and $A \, wr_2 \, B$ are independent of the order on the elements of B.

<u>Proof</u>. Let $<_1$ and $<_2$ be different orders of the elements of B. Let B_1 and B_2 be the sets whose elements are ordered by $<_1$ and $<_2$ respectively. As groups, B_1 and B_2 are identical. Let $\overline{K}_1$ and $\overline{K}_2$ be defined by

$$\overline{K}_1 = \prod^2_{b \in B_1} A_b \quad \text{and} \quad \overline{K}_2 = \prod^2_{b \in B_2} A_b ,$$

and let $\overline{P}_1$ be the splitting extension of $\overline{K}_1$ by B_1 and $\overline{P}_2$ be the splitting extension of $\overline{K}_2$ by B_2. Consider a map $\theta : \overline{P}_1 \longrightarrow \overline{P}_2$ given by

$$(bf_1 t_1)\theta = bf_2 t_2$$

where $f_2 : B_2 \longrightarrow A$ is given by

$$f_2(c) = f_1(c)$$

and $t_2 : \{ (c,c') ; c' <_2 c , c,c' \in B \} \longrightarrow A \otimes A$ is given by

$$t_2(c,c') = t_1(c,c') \qquad \text{if } c' <_1 c ,$$

$$t_2(c,c') = (t_1(c',c))\alpha(f(c) \otimes f(c')) \quad \text{if } c <_1 c' .$$

It is easily proved that θ is an isomorphism and hence the generalised wreath products are independent of the ordering of the elements of B.

<u>The second nilpotent products defined by Golovin and Moran</u>.

The definition and notation for the second nilpotent powers of A, which have been given in this chapter, are not the same as those found in the work of Golovin and Moran.

Let F denote the free product of a set $\{A_\alpha ; \alpha \in N\}$ of groups. Then the free cartesian $[A_\alpha]$ of the A_α is defined to be the subgroup

$$gp\{[a_\alpha, a_\beta]; a_\alpha \in A_\alpha, a_\beta \in A_\beta, \alpha \neq \beta\}.$$

Golovin [3], [4] defines the restricted second nilpotent product K_G of the A_α by

$$K_G = F/[F, [A_\alpha]] .$$

Moran [10] defines the restricted second nilpotent product K_M of the A_α by

$$K_M = F/\gamma_3(F) \cap [A_\alpha]^F ,$$

where $[A_\alpha]^F$ denotes the normal closure of $[A_\alpha]$ in F.

Wiegold [15] remarks that

$$\gamma_3(F) \cap [A_\alpha]^F = [F,[A_\alpha]] ,$$

so that $K_M = K_G$.

Clearly the cartesian of the A_α in K_G is

$$[A_\alpha]^F/[F,[A_\alpha]] = [A_\alpha][F,[A_\alpha]]/[F,[A_\alpha]] ,$$

and this is denoted by $[A_\alpha]_z$. Wiegold [15] proves that

$$[A_\alpha, A_\beta]_Z \cong A_\alpha \otimes A_\beta ,$$

the correspondence

$$[a_\alpha, a_\beta] \longleftrightarrow a_\alpha \otimes a_\beta$$

generates an isomorphism. Golovin [4] proves that

$$[A_\alpha]_Z = \prod_{\alpha \neq \beta} [A_\alpha, A_\beta]_Z .$$

Hence it follows that

$$[A_\alpha]_Z \cong \prod_{\alpha \neq \beta} A_\alpha \otimes A_\beta .$$

Now suppose that K_G is the second nilpotent product of the set $\{A_b ; b \in B\}$ of isomorphic copies of A. Then

$$[A_b]_Z \cong \prod_{b > b'} A_b \otimes A_{b'} \cong T .$$

By Golovin [4], every element $k \in K_G$ has a unique expression of the form

$$k = (a_1)_{b_1} \cdots (a_m)_{b_m} u$$

where $b_1 < b_2 < \cdots < b_m$, $(a_i)_{b_i} \in A_{b_i}$ corresponds to $a_i \in A$, and where $u \in [A_b]_Z$. Consider the following map $\theta : K_G \longrightarrow K$ given by

$$((a_1)_{b_1} \cdots (a_m)_{b_m} u)\theta = ft ,$$

where $f : B \longrightarrow A$ with $\sigma(f) \in \{b_1, \ldots, b_m\}$ and

$$f(b_i) = a_i \quad \text{for } 1 \leq i \leq m ;$$

if the component of u in $[A_b, A_{b'}]$ is $\prod_{i \in I, j \in J} [(a_i)_b, (a_j)_{b'}]$ for some index sets I and J , then

$$t(b,b') = \prod_{i\in I, j\in J} (a_i \otimes a_j) .$$

This map θ is a homomorphism from K_G onto K .

Proof. Consider

$$k = (a_1)_{b_1} \ldots (a_m)_{b_m} u, \; k' = (a_1')_{b_1} \ldots (a_m')_{b_m} u' ,$$

and let $k\theta = ft$, $k'\theta = f't'$ and $ftf't' = gv$.

Then

$$kk' = (a_1 a_1')_{b_1} \ldots (a_m a'_m)_{b_m} uu' \prod_{1\leq j<i\leq m} [(a_i)_{b_i}, (a'_j)_{b_j}] ,$$

and

$$(kk')\theta = g^* v^*$$

where

$$g^*(b_i) = a_i a_i' = f(b_i) f'(b_i) = g(b_i) ,$$

$$\begin{aligned} v^*(b_i, b_j) &= u\theta(b_i, b_j) u'\theta(b_i, b_j)(a_i \otimes a'_j) \\ &= t(b_i, b_j) t'(b_i, b_j)(f(b_i) \otimes f'(b_j)) \\ &= v(b_i, b_j) . \end{aligned}$$

Clearly θ maps K_G onto K .

θ is a monomorphism.

Proof. Let

$$1 = k\theta = ((a_1)_{b_1} \ldots (a_m)_{b_m} u)\theta = ft ,$$

then

$$f(b_i) = a_i = 1 \quad \text{for all } 1 \leq i \leq m ,$$

and

$$u\theta = 1 ,$$

but $[A_b]_Z \cong T$ and θ maps $[A_b]_Z$ onto T so that clearly

$$u = 1 .$$

Hence θ is an isomorphism and $K_G \cong K$.

Golovin [4] shows that when each A_α is such that its factor group by its derived group is a direct product of cyclic groups, then it is possible to construct the second nilpotent product of the A_α and to "know" its structure. This means that the cartesian $[A_\alpha]_Z$ can be determined, not as an abstract group, but as a group generated by commutators. In deriving certain of the properties of $A \text{ wr}_2 B$ in the following chapter, it will be assumed that A/A' is a direct product of cyclic groups.

Before defining the unrestricted second nilpotent product of Moran [11] it is necessary to define the inverse limit of a set of groups. The following definition is from Kurosh ([8], page 227).

Let $\{G_\alpha ;\ \alpha \in N\}$ be a partially ordered set of groups, and let N be directed. Suppose that for all G_α, G_β with $\alpha \leq \beta$ there is a homomorphism $\varphi_{\beta\alpha}$, which maps G_β onto G_α, such that for all $\alpha \leq \beta \leq \gamma$ and all $g_\gamma \in G_\gamma$,

$$g_\gamma \varphi_{\gamma\alpha} = g_\gamma \varphi_{\gamma\beta} \varphi_{\beta\alpha} .$$

A thread is a set of elements $\{g_\alpha\}$ such that for all $\alpha \in N$, $g_\alpha \in G_\alpha$, and for all $\alpha, \beta \in N$ with $\alpha \leq \beta$ the elements g_α and g_β are related by

$$g_\alpha = g_\beta \varphi_{\beta\alpha} .$$

The product of two threads and the inverse of a thread are defined by

$$\{g_\alpha\} \{g'_\alpha\} = \{(gg')_\alpha\} ; \quad \{g_\alpha\}^{-1} = \{g_\alpha{}^{-1}\} .$$

Thus the set of threads becomes a group, which is the inverse limit of the set $\{G_\alpha\}$ with the homomorphisms $\varphi_{\beta\alpha}$.

Moran [11] defines the unrestricted second nilpotent product of groups as an inverse limit. Let B be an index set and let $\mathcal{B}$ be the set of all finite subsets of B. For $\beta \in \mathcal{B}$ let

$$K_\beta = \prod_{b \in \beta}^{(2)} A_b .$$

If $\beta < \beta'$ then there exists a natural homomorphism

$\sigma_{\beta'\beta}$ of $K_{\beta'}$ onto K_β. This homomorphism is simply the projection mapping. Under these homomorphisms the groups K_β form an inverse system whose inverse limit $I\,L_\sigma(K_\beta)$ is the unrestricted second nilpotent product of the set of groups $\{A_b\ ;\ b \in B\}$.

It remains to be shown that this inverse limit is isomorphic to the group $\overline{K}$ defined in this chapter. Since the restricted second nilpotent product is unambiguously defined, let the elements of $I\,L_\sigma(K_\beta)$ be threads $\{f_\gamma t_\gamma\}$, where $f_\gamma : B \longrightarrow A$ and $t_\gamma : D \longrightarrow A \otimes A$. Consider the following map ψ from $I\,L_\sigma(K_\beta)$ to $\overline{K}$ given by

$$\{f_\gamma t_\gamma\}\psi = ft ,$$

where

$$f(b) = f_\gamma(b) \qquad \text{for all } \gamma \text{ with } b \in \gamma ,$$

$$t(b,b') = t_{\gamma'}(b,b') \qquad \text{for all } \gamma' \text{ with } b,b' \in \gamma' .$$

This map ψ is a homomorphism.

<u>Proof</u>. Consider threads $\{f_\gamma t_\gamma\}$ and $\{g_\gamma u_\gamma\}$ and let $\{f_\gamma t_\gamma\}\psi = ft$, $\{g_\gamma u_\gamma\}\psi = gu$. Then

$$\{f_\gamma t_\gamma\}\{g_\gamma u_\gamma\} = \{h_\gamma v_\gamma\} ,$$

where

$$h_\gamma(b) = f_\gamma(b)g_\gamma(b) ,$$

$$v_\gamma(b,b') = t_\gamma(b,b')u_\gamma(b,b')(f_\gamma(b) \otimes g_\gamma(b')) ;$$

and

$$\{h_\gamma v_\gamma\}\Psi = hv ,$$

where

$$\begin{aligned} h(b) &= h_\gamma(b) \qquad \text{for all } \gamma \text{ with } b \in \gamma \\ &= f_\gamma(b) g_\gamma(b) \\ &= f(b)g(b) , \\ v(b,b') &= v_{\gamma'}(b,b') \qquad \text{for all } \gamma' \text{ with } b,b' \in \gamma' \\ &= t_{\gamma'}(b,b')u_{\gamma'}(b,b')(f_{\gamma'}(b) \otimes g_{\gamma'}(b')) \\ &= t(b,b')u(b,b')(f(b) \otimes g(b')) ; \end{aligned}$$

and hence

$$\{f_\gamma t_\gamma\}\Psi\{g_\gamma u_\gamma\}\Psi = (\{f_\gamma t_\gamma\}\{g_\gamma u_\gamma\})\Psi .$$

Clearly Ψ maps $\mathrm{I}\, L_\sigma(K_\beta)$ monomorphically and epimorphically onto $\overline{K}$ so that

$$\mathrm{I}\, L_\sigma(K_\beta) \cong \overline{K} .$$

Note that Moran does not require that the index set B be ordered. When B is non-ordered the elements of the unrestricted second nilpotent product can be uniquely expressed as threads. However in order to obtain a regular representation for an element of $\mathrm{I}\, L_\sigma(K_\beta)$ he requires that the index set B be ordered. In this case every element of the unrestricted second nilpotent product of the $\{G_\alpha ; \alpha \in B\}$ has a unique representation of the form

$$\{g_a\} u ,$$

where $g_a \in G_a$ for all $a \in B$, and u belongs to the unrestricted cartesian subgroup. This unrestricted cartesian is the inverse limit of the restricted cartesians and is proved to be the unrestricted product

$$\prod_{b>b'} [A_b, A_{b'}] .$$

CHAPTER 4

Introduction.

In this chapter certain properties of the generalised wreath products $A \text{ wr}_2 B$ and $A \text{ Wr}_2 B$ are considered and it is shown that some (but not all) of the properties of wreath products stated in Chapter 1 do carry over to generalised wreath products. It is shown first of all that the standard wreath product is a factor group of the generalised wreath product.

Finally conditions for the nilpotency of $A \text{ wr}_2 B$ are determined and the results of Chapter 2 are extended to provide bounds for the nilpotency class of $A \text{ wr}_2 B$. If $A \text{ wr } B$ has class c^* and $A/A' \text{ wr } B$ has class c' and if d is the class of $A \text{ wr}_2 B$, then

$$c^* \leq d \leq c^* + c' .$$

Examples are given of groups A and B which are such that $c^* = d$, and others which are such that $d = c^* + c'$. The exact class of $A \text{ wr}_2 B$ is determined when both A and B are cyclic groups of order p.

Some properties of generalised wreath products.

From the following lemma it is proved that the standard wreath product is a factor group of the generalised wreath product.

<u>Lemma 4.1</u> Let G be a splitting extension of a group X by a group Y and let $\varphi' : X \to X'$ be a homomorphism such that the kernel R of φ' is normal in G. Then φ' can be extended to a homomorphism $\varphi : G \to G'$, where G' is a splitting extension of X' by Y. The kernel of φ is isomorphic to R.

<u>Proof</u>. Every $g \in G$ can be expressed uniquely in the form

$$g = xy \quad \text{for } x \in X \text{ and } y \in Y .$$

Define φ by setting

$$(xy)\varphi = x\varphi' y \quad \text{for all } x \in X , y \in Y .$$

Then φ is a homomorphism, for consider

$$\begin{aligned} ((xy)(x'y'))\varphi &= (xx'^{y^{-1}}yy')\varphi \\ &= (xx'^{y^{-1}})\varphi' yy' \\ &= x\varphi'(x'^{y^{-1}})\varphi' yy' . \end{aligned}$$

Since R is normal in G it follows that

$$(x'^{y^{-1}})\varphi' = (x'\varphi')^{y^{-1}}$$

and so

$$\begin{aligned} ((xy)(x'y'))\varphi &= x\varphi'(x'\varphi)^{y^{-1}}yy' \\ &= x\varphi' yx'\varphi' y' \\ &= (xy)\varphi(x'y')\varphi . \end{aligned}$$

Clearly φ maps G onto G' and its kernel contains R.

Let $x^*y^* \in \ker\varphi$, then

$$(x^*y^*)\varphi = 1 = x^*\varphi' y^*$$

and so

$$x^* \varphi' = 1 \ , \ y^* = 1 \ .$$

Corollary 4.2 For all groups A and B
$A \text{ Wr}_2 B/\overline{T} \cong A \text{ Wr } B \ , \ A \text{ wr}_2 B/T \cong A \text{ wr } B$.

Proof. The proof follows, by direct application of Lemma 4.1, from the facts that

$$A^B \cong \overline{K}/\overline{T} \text{ and } \overline{T} \trianglelefteq A \text{ Wr}_2 B \ ,$$
$$_A(B) \cong K/T \text{ and } T \trianglelefteq A \text{ wr}_2 B \ .$$

The following two lemmas correspond to Lemmas 1.3 and 1.4 of Chapter 1 .

Lemma 4.3 If $B^* \leq B$, then
$A \text{ Wr}_2 B^* \leq A \text{ Wr}_2 B$ and $A \text{ wr}_2 B^* \leq A \text{ wr}_2 B$.

Proof. Let $D^* = \{(c,c') \in D \ ; \ c,c' \in B^*\}$. For $b^* \in B^*$, $f^* : B^* \longrightarrow A$ and $t^* : D^* \longrightarrow A \otimes A$ define μ by setting

$$(b^*f^*t^*)\mu = b^*ft \ ,$$

where $f : B \longrightarrow A$ is given by

$$f(b) = f^*(b) \quad \text{if } b \in B^* \ ,$$
$$f(b) = 1 \quad \text{if } b \notin B^* \ ,$$

and $t : D \longrightarrow A \otimes A$ is given by

$$t(d) = t^*(d) \quad \text{if } d \in D^* \ ,$$
$$t(d) = 1 \quad \text{if } d \notin D^* \ .$$

Then μ is a homomorphism for consider

$$((b^*f^*t^*)(c^*g^*u^*))\mu = (b^*c^*h^*v^*)\mu$$

where

$$h^*(b) = f^*(bc^{*-1})g^*(b) \ ,$$

$$v^*(b,b') = t^*(bc^{*-1},b'c^{*-1})u^*(b,b')(f^*(bc^{*-1}) \otimes g^*(b'))$$
$$\text{if } \ bc^{*-1} > b'c^{*-1} \ ,$$

$$v^*(b,b') = (t^*(b'c^{*-1},bc^{*-1}))\alpha u^*(b,b')(f^*(bc^{*-1}) \otimes g^*(b')f^*(b'c^{*-1}))$$
$$\text{if } \ bc^{*-1} < b'c^{*-1} \ ,$$

and

$$(b^*c^*h^*v^*)\mu = b^*c^*hv$$

where

$$h(b) = f^*(bc^{*-1})g^*(b) \qquad \text{if } \ b \in B^* \ ,$$

$$h(b) = 1 \qquad \text{if } \ b \notin B^* \ ;$$

$$v(b,b') = t^*(bc^{*-1},b'c^{*-1})u^*(b,b')(f^*(bc^{*-1}) \otimes g^*(b'))$$
$$\text{if } \ bc^{*-1} > b'c^{*-1} \text{ and } (b,b') \in D^* \ ,$$

$$v(b,b') = (t^*(b'c^{*-1},bc^{*-1}))\alpha u^*(b,b')(f^*(bc^{*-1}) \otimes g^*(b')f^*(b'c^{*-1}))$$
$$\text{if } \ bc^{*-1} < b'c^{*-1} \text{ and } (b,b') \in D^* \ ,$$

$$v(b,b') = 1 \qquad \text{if } \ (b,b') \notin D^* \ .$$

In other words if $ft = (f^*t^*)\mu$ and $gu = (g^*u^*)\mu$, then

$$h(b) = f(bc^{*-1})g(b) \ ,$$

$$v(b,b') = t(bc^{*-1},b'c^{*-1})u(b,b')(f(bc^{*-1}) \otimes g(b'))$$
$$\text{if } \ bc^{*-1} > b'c^{*-1} \ ,$$

$$v(b,b') = (t(b'c^{*-1},bc^{*-1}))\alpha u(b,b')(f(bc^{*-1}) \otimes g(b')f(b'c^{*-1}))$$
$$\text{if } \ bc^{*-1} < b'c^{*-1} \ ,$$

and hence

$$(b^*ft)(c^*gu) = ((b^*f^*t^*)(c^*g^*u^*))\mu$$
$$= (b^*f^*t^*)\mu(c^*g^*u^*)\mu \ .$$

Clearly μ is a monomorphism. The restriction of μ to the restricted product embeds $A \text{ wr}_2 B^*$ in $A \text{ wr}_2 B$.

Lemma 4.4 Let ψ be a homomorphism from B onto B/Z and let A be an abelian group. Then ψ can be extended to a homomorphism φ from $A \text{ wr}_2 B$ onto $A \text{ wr}_2 B/Z$.

Proof. Let B be ordered in such a way that if $b > b'$ and $bZ \neq b'Z$, then $bz > b'z'$ for all $z, z' \in Z$. Then the order on B can be extended to an order of the elements of B/Z by putting $bZ > b'Z$ if $bZ \neq b'Z$ and $b > b'$.

Extend ψ in the natural way to a homomorphism φ of $A \text{ wr}_2 B$ by putting

$$(cft)\varphi = c\psi f't' ,$$

where $f' : B/Z \longrightarrow A$ is given by

$$f'(bz) = \prod_{z \in Z} f(bz) ,$$

and $t' : Z^* = \{(bZ, b'Z) \ ; \ bZ > b'Z\} \longrightarrow A \otimes A$ is given by

$$t'(bZ, b'Z) = \prod_{z, z' \in Z} t(bz, b'z') .$$

Then φ is a homomorphism of $A \text{ wr}_2 B$ for let $c, c' \in B$, $f, g : B \longrightarrow A$ and $t, u : D \longrightarrow A \otimes A$ and let

$(cft)\varphi = c\psi f't'$, $(c'gu)\varphi = c'\psi g'u'$. Consider

$$((cft)(c'gu))\varphi = (cc'hv)\varphi$$

where

$$h(b) = f(bc'^{-1})g(b) ,$$

$$v(b,b') = t(bc'^{-1}, b'c'^{-1})u(b,b')(f(bc'^{-1}) \otimes g(b')) \quad \text{if } bc'^{-1} > b'c'^{-1} ,$$

$$v(b,b') = (t(b'c'^{-1}, bc'^{-1}))\alpha u(b,b')(f(bc'^{-1}) \otimes g(b')f(b'c'^{-1})) \quad \text{if } bc'^{-1} < b'c'^{-1} ;$$

and

$$(cc'hv)\varphi = (cc')\psi h'v' = c\psi c'\psi h'v'$$

where

$$h'(bZ) = \prod_{z \in Z} h(bz) = \prod_{z \in Z} f(bc'^{-1}z)g(bz) = f'(bc'^{-1}Z)g'(bZ) ,$$

$$v'(bZ, b'Z) = \prod_{z,z' \in Z} v(bz, b'z') ;$$

$$v'(bZ, b'Z) = t'(bc'^{-1}Z, b'c'^{-1}Z)u'(bZ, b'Z)(f'(bc'^{-1}Z) \otimes g'(b'Z)) \quad \text{if } bc'^{-1}Z > b'c'^{-1}Z ,$$

$$v'(bZ,b'Z) = (t'(b'c'^{-1}Z, bc'^{-1}Z))\alpha u'(bZ,b'Z) (f'(bc'^{-1}Z) \otimes g'(b'Z)f'(b'c'^{-1}Z)) \quad \text{if } bc'^{-1}Z < b'c'^{-1}Z .$$

Hence

$$(cft)\varphi\,(c'gu)\varphi = ((cft)(c'gu))\varphi$$

and φ is a homomorphism. Clearly φ maps $A \text{ wr}_2 B$ onto $A \text{ wr}_2 B/Z$.

Note that if Z is finite, this epimorphism $\psi : B \longrightarrow B/Z$ can be extended to an epimorphism $\varphi^* : A\ Wr_2\ B \longrightarrow A\ Wr_2\ B/Z$.

Let $T^* = \{t \in T\ ;\ \sigma(t) \leq \{(b,b') \in D\ ;\ bZ = b'Z\}\}$. Then since A is abelian, $T^*\varphi = 1$, and hence

$$\ker\varphi = T^*Z\{ft\ ;\ \prod_{z\in Z} f(bZ) = 1 = \prod_{z,z'\in Z} t(bz,b'z')\}.$$

As in Lemma 1.4, it follows that if $f \in \ker\varphi$ then $f \in [K,Z]$, and certainly

$$[K,Z]T^*Z \leq \ker\varphi .$$

There are no analogues of Lemmas 1.1 and 1.2 for generalised wreath products. The following theorem of Golovin ([3], Theorem 6.1) proves this, as do a number of counter examples.

<u>Theorem 4.5</u> If $K = \prod^{(2)}_{\alpha\in M} A_\alpha$ and if $A^*_\alpha \leq A_\alpha$ for all $\alpha \in M$, then the subgroup $G^* = gp\{A^*_\alpha\ ;\ \alpha \in M\}$ is isomorphic to a factor group of $\prod^{(2)}_{\alpha\in M} A^*_\alpha$.

Golovin proves, in fact, that if $F^* = \prod^*_{\alpha\in M} A^*_\alpha$,

$T = gp\{[a_\alpha,a_\beta]\ ;\ a_\alpha \in A_\alpha,\ a_\beta \in A_\beta,\ \alpha, \beta \in M, \alpha \neq \beta\}$,

$T^* = gp\{[a^*_\alpha,a^*_\beta]\ ;\ a^*_\alpha \in A^*_\alpha,\ a^*_\beta \in A^*_\beta,\ \alpha, \beta \in M, \alpha \neq \beta\}$

and $V = (F^* \cap [F,T])/[F^*,T^*]$, then

$$G^* = \left(\prod^{(2)}_{\alpha\in M} A_\alpha\right)/V .$$

Suppose for example that A is an arbitrary non-abelian group. Then

$\mathrm{sgp}\{A'_b\ ;\ b \in B\} = \prod_{b \in B} A'_b$ and $\mathrm{sgp}\{A'_b, B\ ;\ b \in B\} \cong A'\ \mathrm{wr}\ B$.

Similarly, let $A = M \times N$ where M is abelian and $N = N' = A/M = A'$. Then

$$\mathrm{sgp}\{(A/M)_b,\ B\ ;\ b \in B\} \cong N\ \mathrm{wr}\ B\ .$$

Let the diagonal of the second nilpotent power $\overline{K}$ of A be

$\mathrm{gp}\{ft\ ;\ f(b) = f(1)$ for each $b \in B,\ t = \prod_{b \in B} u^b$ for some $u \in \overline{T}\}$.

Then the following lemmas prove that the centre of $A\ \mathrm{Wr}_2\ B$ is contained in the diagonal of $\overline{K}$, and if B is infinite then $A\ \mathrm{wr}_2\ B$ has trivial centre.

Lemma 4.6 If A is such that A/A' is a direct product of cyclic groups, then

$\zeta(\overline{K}) = (\zeta(A) \cap A')^B\, \overline{T}$ and $\zeta(K) = (\zeta(A) \cap A')^{(B)}\, T$.

Proof. Clearly $(\zeta(A) \cap A')^B\, \overline{T} \leq \zeta(\overline{K})$. Let $f \in \zeta(\overline{K})$, then for all $g : B \longrightarrow A$

$$f^*t^* = [f,g] = 1\ ,$$

where

$$f^*(b) = [f(b),\ g(b)] = 1$$

and hence $f(b) \in \zeta(A)$. Let $\sigma(g) = \{1\}$ and let $A/A' = \prod_{i \in I} A_i$ where $A_i = \mathrm{gp}\{a_i\ ;\ a_i^{r_i} = 1\}$, and some of the r_i may be zero. Let $f(b) \equiv \prod_{i \in I} a_i^{\alpha_i}$ mod A' and let $g(1) = a_1$, then

$$1 = f(b) \otimes g(1) = (a_1 \otimes a_1)^{\alpha_1}(a_2 \otimes a_1)^{\alpha_2}\dots .$$

By Lemma 3.3 it follows that

$$(a_1 \otimes a_1)^{\alpha_1} = 1 = (a_2 \otimes a_1)^{\alpha_2}\dots ,$$

and hence r_1 divides α_1 . Similarly it is shown that r_i divides α_i for all $i \in I$ and that $f \in (A')^B$. This completes the proof.

<u>Lemma 4.7</u> The centre of $\overline{P} = A\ Wr_2\ B$ is the diagonal subgroup of the centre of $\overline{K}$.

<u>Proof</u>. Clearly the diagonal of $\zeta(\overline{K})$ is central in $\overline{P}$. Let $cft \in \zeta(\overline{P})$ and let $g : B \longrightarrow A$ be such that $\sigma(g) = \{1\}$, then

$$[cft, g] = f^{-1}g^{-c}fg = hv = 1$$

and if $c \neq 1$

$$1 = h(1) = f(1)^{-1}f(1)g(1) ;$$

but $g(1) \neq 1$ and so $\zeta(\overline{P}) \leq \overline{K}$ and hence $\zeta(\overline{P}) \leq \zeta(\overline{K})$.

Also

$$[ft, b'] = f^{-1}f^{b'}t^{-1}t^{b'} = h'v' = 1 \quad \text{for all} \quad b' \in B$$

and

$$1 = h'(b) = f(b)^{-1}f(bb'^{-1})$$

so that

$$f(b) = f(1) \quad \text{for all} \quad b \in B ;$$

also

$$1 = v'(d) = t(d)^{-1}t^{b}(d) \quad \text{for all } d \in D\ ,\ b \in B\ .$$

Hence $\zeta(\overline{P})$ is the diagonal of $\zeta(\overline{K})$ and $\zeta(P)$ is the diagonal of $\zeta(K)$; if B is infinite $\zeta(P) = 1$.

The following result will be needed later and is stated here without proof.

Theorem 4.8 (Golovin [4], Theorem 6.6). Let $K = \prod^{(2)}_{\alpha \in M} A_\alpha$, and let each group A_α be nilpotent of class not greater than ℓ . Then K is nilpotent. If at least one A_α has class exactly ℓ and $\ell > 1$, then K has class ℓ ; if $\ell = 1$, then the class of K is either 1 or 2 .

Nilpotency of $A\ wr_2\ B$.

The following lemma provides a set of necessary and sufficient conditions for $A\ wr_2\ B$ to be a nilpotent group.

Lemma 4.9 $A\ wr_2\ B$ is nilpotent if and only if A is a nilpotent p-group of finite exponent and B is a finite p-group for the same prime p .

Proof. By Corollary 4.2, $(A\ wr_2\ B)/T \cong A\ wr\ B$ and so a necessary condition for the nilpotency of $A\ wr_2\ B$ is that $A\ wr\ B$ be nilpotent, in other words that A be a nilpotent p-group of finite exponent and B be a finite p-group.

Now suppose that these conditions are satisfied. By Lemma 4.8, $K = \prod_{b \in B}^{(2)} A_b$ is nilpotent. By a theorem of Golovin ([4], Theorem 2.2), K has finite exponent. A lemma of Baumslag [1] states that an extension of a nilpotent p-group of finite exponent by a finite p-group is nilpotent. Hence $A \text{ wr}_2 B$ is nilpotent.

An upper bound for the nilpotency class of $A \text{ wr}_2 B$ is given by the following theorem.

Theorem 4.10 Let $A/A' \text{ wr } B$ have class c' and let $A \text{ wr } B$ have class c^*. If d is the class of $P = A \text{ wr}_2 B$, then

$$d \leq c^* + c' .$$

Proof. By Corollary 4.2

$$\gamma_{c^*+1}(A \text{ wr}_2 B/T) \cong \gamma_{c^*+1}(A \text{ wr } B) = 1$$

and hence

$$\gamma_{c^*+1}(A \text{ wr}_2 B) \leq T .$$

But $T \leq \zeta(K)$ so that

$$[T,P] = [T,B] \text{ and } [T(,P)^r] = [T(,B)^r] .$$

By Theorem 1.5

$$gp\{T,B\} \leq T \text{ wr } B$$

and hence

$$[T(,B)^r] \leq \gamma_{r+1}(T \text{ wr } B) .$$

But T is abelian and its exponent is not greater than the exponent of A/A' so that, by Corollary 2.8

$$1 = \gamma_{c'+1}(T \text{ wr } B) \geq \gamma_{c^*+c'+1}(A \text{ wr}_2 B) .$$

Let A have exponent p^n and class c and let B have order p^t, then

$$d \leq (c+1)p^{t-1}(np-n+1) \qquad \text{if } p > 2 ,$$

$$d \leq \max(3n,4)2^{t-2}(c+1) \qquad \text{if } p = 2 .$$

Clearly the class of $A \text{ wr}_2 B$ is not less than the class of $A \text{ wr } B$. There are groups A and B for which the classes d of $A \text{ wr}_2 B$ and c^* of $A \text{ wr } B$ are equal.

Let $B = \text{gp}\{b \, ; \, b^3 = 1\}$ and let

$$A = \text{gp}\{x,y \, ; \, x^3 = y^3 = 1 = [x,y,x] = [x,y,y]\} .$$

Then it can easily be shown that $c^* = d = 6$. However $A \text{ wr } B$ is a proper homomorphic image of $A \text{ wr}_2 B$ as the second nilpotent power of A does not degenerate to the direct power.

There are groups A and B for which

$$d = (c+1)p^{t-1}(np-n+1) .$$

Let $A = \text{gp}\{x,y \, ; \, x^3 = y^3 = 1 = [x,y]\}$ and $B = \text{gp}\{b \, ; \, b^3 = 1\}$. Let $f,g \, ; \, B \longrightarrow A$ be defined by

$$f(1) = x , \quad g(1) = y ,$$

$$f(b^i) = 1 = g(b^i) \quad \text{for } i = 1,2 .$$

Then $[f,b,b,g,b,b] \neq 1$ and so $A \text{ wr}_2 B$ has class exactly 6 .

Finally the exact class of $A \text{ wr}_2 B$ is determined when A and B are cyclic groups of order p .

<u>Lemma 4.11</u> If A and B are cyclic groups of order p , then the class of $A \text{ wr}_2 B$ is $2p-1$.

Proof. Let $A = gp\{a \,;\, a^p = 1\}$ and $B = gp\{b \,;\, b^p = 1\}$ and let $P = A \text{ wr}_2 B$. Let $f,g \,;\, B \longrightarrow A$ be such that $\sigma(f) = \{1\}$, $\sigma(g) = \{b^r\}$ and $f(1) = a = g(b^r)$. Put $x_m = [f(,b)^m , g(,b)^{2p-2-m}]$ for $0 \leq m < 2p-1$ and $x_{2p-1} = [f(,b)^{2p-1}]$. By Theorem 4.10, $\gamma_{2p+1}(P) = 1$ and hence

$$\gamma_{2p}(P) = gp\{x_m \,;\, 0 \leq m \leq 2p-1\} .$$

It is necessary to show that every generator x_m of $\gamma_{2p}(P)$ is trivial.

If $0 \leq m < p-1$, then for some $t \in T$

$$x_m = [t(,b)^{2p-2-m}] \in [T(,B)^p] \leq \gamma_{p+1}(T \text{ wr } B) = 1 .$$

If $m = p-1$, then without loss of generality put $g = f$ and

$$x_{p-1} = [f(,b)^{p-1}, f(,b)^{p-1}]$$
$$= [\prod_{o \leq i < p}[f^{b^i},f](,b)^{p-1}] .$$

By Lemma 2.2

$$x_{p-1} = \prod_{o \leq i < p}[f^{b^{i+k}},f^{b^k}]^{\pm \lambda_{p-1,k}} .$$

Hence $x_{p-1}(b^{i+k},b^k) = (a \otimes a)^{\alpha}$

where

$$\alpha = (-1)^{p-1}\lambda_{p-1,k} - (-1)^{p-1}\lambda_{p-1,i+k}$$

$$= (-1)^{p-1}\left((-1)^k \binom{p-1}{k} - (-1)^{i+k}\binom{p-1}{k+i}\right)$$

$$= (-1)^{p-1+k}(p-1)!\left\{\frac{1}{k!(p-1-k)!} - \frac{(-1)^i}{(i+k)!(p-1-i-k)!}\right\}$$

$$= \frac{(-1)^{p-1+k}(p-1)!}{(i+k)!(p-1-k)!}((k+1)(k+2)\dots(k+i)-(-1)^i(p-(k+1))\dots$$
$$\dots(p-(k+i))),$$

and hence p divides α and $x_{p-1} = 1$.

If $p-1 < m < 2p-1$, then

$$x_m \in [K(,B)^p,K] \leq [T,K] = 1 .$$

Finally when $m = 2p-1$, then

$$x_{2p-1} = [f(,b)^{2p-1}] .$$

Now

$$[f(,b)^{p-1}] \equiv ff^b\dots f^{b^{p-1}} \mod T$$

and hence

$$[f(,b)^{2p-1}] \equiv [ff^b\dots f^{b^{p-1}},b(,b)^{p-1}] \mod [T(,B)^p] ,$$

so that

$$x_{2p-1} = [f^{-b^{p-1}}\dots f^{-b}f^{-1}ff^b f^{b^2}\dots f^{b^{p-1}} f(,b)^{p-1}]$$

$$= [\prod_{1\leq i\leq p-1}[f^{b^i},f](,b)^{p-1}] .$$

But it has already been shown that

$$[\prod_{1\leq i\leq p-1}[f^{b^i},f](,b)^{p-1}] = 1$$

so that

$$x_{2p-1} = 1 .$$

It must now be shown that there is a non-trivial commutator of weight $2p-1$. Suppose that $p > 2$ and consider

$$[f(,b)^{2p-2}] = [\prod_{1\leq i\leq p-1}[f^{b^i},f](,b)^{p-2}] = u ;$$

then

$$u(b^{p-1},1) = (a \otimes a)^{-1} \neq 1$$

and hence

$$u \neq 1 .$$

When $p = 2$ then

$$[f,b,b] = [f^{-b},f] \neq 1$$

and hence for all p, the class of $C_p \; wr_2 \; C_p$ is $2p-1$.

By comparison the class of $C_p \; wr \; C_p$ is p. The order of $C_p \; wr \; C_p$ is p^{p+1}, so that $C_p \; wr \; C_p$ has maximal class, but the order of $C_p \; wr_2 \; C_p$ is $p^{\frac{p^2+p+2}{2}}$, and so the class is not maximal for $p > 2$.

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