ON SOME ASPECTS

OF

FINITE SOLUBLE GROUPS

by

A.R. MAKAN

A thesis presented for the degree of Doctor of Philosophy in the Australian National University

Canberra, April, 1971.

STATEMENT

The results presented in this thesis are my own except where stated otherwise.

Amirali Makan

ACKNOWLEDGMENTS

The work for the present thesis was done while I held a post-graduate research scholarship in the Department of Mathematics in the Institute of Advanced Studies of the Australian National University.

The major part of the work was supervised by Dr H. Lausch and after his departure my supervisor was Dr L.G. Kovács. I am deeply indebted to them for the great interest they have shown in my work throughout and also for their patience and constant encouragement. They were always accessible and their suggestions and friendly criticisms have always been invaluable. Dr L.G. Kovács helped considerably with the final stages of this thesis. On this occasion, his readiness to discuss problems and read the drafts at very short notice, was invaluable.

Dr M.F. Newman allowed me access to his unpublished results used in this thesis. I am indebted to Dr M.F. Newman for this, and to him and Dr R.M. Bryant for helpful discussions related to the work of the thesis.

I am also indebted to Professor B.H. Neumann and Professor Hanna Neumann for their readiness to discuss my problems and for their very helpful advice.

My thanks go to the members of the department for their friendliness, and to Mrs Barbara Geary for the excellent typing of this thesis and for her helpfulness which made the production of the thesis easier than it would have been otherwise.

My final, but most, thanks are due to my wife Sultan for her patience and encouragement over the past three years.

CONTENTS

CHAPTER 1

1.1 Introduction

Various characteristic conjugacy classes of subgroups having covering/avoidance properties with respect to chief factors have recently played a major role in the study of finite soluble groups. Apart from the subgroups which are now called Hall subgroups, P. Hall [18] also considered the system normalizers of finite soluble groups and showed that these form a characteristic conjugacy class, cover the central chief factors and avoid the rest. The system normalizers were later shown by Carter and Hawkes [4] to be the simplest example of a wealth of characteristic conjugacy classes of subgroups of finite soluble groups which arise naturally as a consequence of the theory of formations. They show that a finite soluble group has, corresponding to each saturated formation $\underline{\underline{X}}$ containing the class $\underline{\underline{N}}$ of all finite nilpotent groups, a characteristic conjugacy class of subgroups called the $\underline{\underline{X}}$-normalizers which have properties closely analogous to the system normalizers of P. Hall and coincide with the latter when $\underline{\underline{X}} = \underline{\underline{N}}$. This part of the theory of formations has been extended by Wright [38] for the case when a saturated formation does not necessarily contain $\underline{\underline{N}}$.

Dual to the concept of formations is the concept of Fitting classes introduced by Fischer [7]. As Fischer, Gaschütz and Hartley [9] have shown a finite soluble group has, corresponding to each Fitting class $\underline{\underline{Y}}$, a characteristic conjugacy class of subgroups called the $\underline{\underline{Y}}$-injectors.

On the other hand, Gaschütz [10] also considered what he called the Prefrattini subgroups of finite soluble groups. In particular these cover the Frattini chief factors and avoid the complemented ones. Working in this direction, Hawkes [21], in turn, obtained further characteristic conjugacy classes, one class in each group corresponding to each saturated formation, of subgroups of finite soluble groups which have properties closely analogous to the Prefrattini subgroups and which coincide with the latter in the case when the saturated formation under consideration is the trivial one. Hawkes' subgroups corresponding to a saturated formation $\underline{\underline{X}}$ are called the $\underline{\underline{X}}$-Prefrattini subgroups. Besides possessing an interesting covering/avoidance property, an $\underline{\underline{X}}$-Prefrattini subgroup of a finite soluble group can be expressed as a product of a Prefrattini subgroup and an $\underline{\underline{X}}$-normalizer of the group.

This latter fact suggests the study in a finite soluble group of the lattice $\underline{\underline{L}}$ of subgroups generated by the Prefrattini subgroups of the group, the $\underline{\underline{F}}$-normalizers of the group corresponding to a saturated formation $\underline{\underline{F}}$, and the $\underline{\underline{H}}$-injectors of the group corresponding to a Fitting class $\underline{\underline{H}}$. The idea is to find within $\underline{\underline{L}}$ further characteristic conjugacy classes of subgroups with covering/avoidance properties.

The work in the Chapters 3 and 4 of this thesis derives from our attempt to study $\underline{\underline{L}}$. We had to restrict ourselves to $\underline{\underline{H}}$ being a Fischer class since in this case more is known about the behaviour of the Sylow subgroups of the $\underline{\underline{H}}$-injectors, and this information is vital in our investigation. The results of our investigation may be summarized as follows (see Theorem 4.0.1).

With a Sylow system in a finite soluble group, one can naturally associate a Prefrattini subgroup, an $\underline{\underline{F}}$-normalizer and an $\underline{\underline{H}}$-injector, of the group and the sublattice $\underline{\underline{L}}^*$ generated by these three subgroups (in the full subgroup lattice of the group) is distributive, the conjugacy classes of its elements are characteristic, the elements of $\underline{\underline{L}}^*$ are pairwise permutable, they all have interesting covering/avoidance properties and the given Sylow system reduces into each of them. As an example at the end of Chapter 4 shows, $\underline{\underline{L}}^*$ can be as large as the preceding statement allows, namely it can have eighteen distinct elements (in which case $\underline{\underline{L}}^*$ is a free distributive lattice of rank 3). Fourteen elements of $\underline{\underline{L}}^*$ then belong to distinct characteristic conjugacy classes which have not been known to exist before.

The latter part of this thesis, namely, Chapters 5 and 6, deals with a particular instance of the general problem of obtaining information about the global structure of a group from the information about the local structure. In particular, we consider the conjugacy classes of $\underline{\underline{N}}^k$-maximal subgroups of a finite soluble group for $k \geq 1$, where $\underline{\underline{N}}^k$ is the class of all finite groups of nilpotent length at most k, and investigate the restriction imposed by their number on a particular invariant of the group, namely its Fitting length.

The case $k = 1$ is investigated in Chapter 5 and a logarithmic upper bound on the Fitting length of a finite soluble group in terms of the number of conjugacy classes of maximal nilpotent subgroups of the group is obtained (see Theorem 5.2.8). In certain special cases the bound obtained is shown to be the best possible.

The case $k \geq 2$ is investigated in Chapter 6, the last chapter of this thesis, in a slightly more general set up. The bound on the Fitting length of a finite soluble group in terms of the number of conjugacy classes of $\underline{\underline{N}}^k$-maximal subgroups of the group which we obtain there is a linear one.

We conclude here with a remark that the Fitting length of a finite soluble group, on the other hand, imposes no restriction on the number of conjugacy classes of maximal nilpotent subgroups of the group, as has been shown by Rose [33] using wreath product constructions.

1.2 Notation and terminology

Throughout this thesis, the word "group" means "finite soluble group", except when stated otherwise.

The letters p, q and r always denote primes.

Given a set π of primes, π' denotes the complement of π in the set of all primes, and an integer n is called a π-number if each of its prime divisors is an element of π.

We denote the trivial subgroup of a group by $\{1\}$ and the identity of a group by 1. If G is a group and H a subgroup of G, we write $H \leq G$ or $G \geq H$. If H is a proper subgroup of G, that is, if $H \neq G$, then we write $H < G$. Similarly, we write $H \trianglelefteq G$ if H is a normal subgroup of G and we write $H \triangleleft G$ if, moreover, $H < G$. If H is a subnormal subgroup of G, we write $H \triangleleft\triangleleft G$.

The order of a group G is denoted by $|G|$ and, if $H \leq G$, $|G : H|$ denotes the index of H in G. For any subsets $X_1, X_2, \ldots, X_n$ of G, $\langle X_1, X_2, \ldots, X_n \rangle$ denotes the subgroup of G generated by these n subsets. The minimal number of generators of G is denoted by $d(G)$.

If K and H are any two subsets of G, then $K \backslash H$ denotes the set of elements of K not contained in H, $H \cap K$ is the intersection of H with K and HK is the product set $\{hk \mid h \in H, k \in K\}$.

If $g, h \in G$, then the conjugate $h^{-1}gh$ of g by h is denoted by g^h and $g^{-1}g^h$ is denoted by $[g, h]$. Similarly, if $H \leq G$ and $g \in G$, H^g denotes the conjugate $g^{-1}Hg$ of H in G. For any subgroups K

and H of G, $[K, H] = \langle [k, h] \mid k \in K, h \in H \rangle$.

The normal closure in G of a subgroup H of G is denoted by H^G and its core in G by $\mathrm{Core}_G(H)$. $C_G(H)$ denotes the centralizer of H in G and $N_G(H)$ denotes the normalizer of H in G. If $H = G$, then $C_G(H) = Z(G)$, the centre of G.

If G has a unique minimal normal subgroup, then G is called a monolithic group and its unique minimal normal subgroup is called its monolith.

A subgroup H of G is said to be pronormal in G if any two conjugates H^x, H^y of H in G are conjugate by an element of $\langle H^x, H^y \rangle$, and it is said to be abnormal in G, if $g \in \langle H, H^g \rangle$ for each $g \in G$.

The Frattini subgroup of G, which is the intersection of all maximal subgroups of G, is denoted by $\Phi(G)$.

If $K, H \leq G$ and $K \trianglelefteq H$, then H/K is called a factor of G. If, moreover, H and K are both normal subgroups of G, $K < H$ and H/K is a minimal normal subgroup of G/K, then H/K is called a chief factor of G. Every chief factor of G is of order a power of some prime. We will call a chief factor of G a p-chief factor, if its order is a power of p. A series of normal subgroups $\{1\} = K_0 < K_1 < \ldots < K_n = G$ is called a chief series of G if K_i/K_{i-1} is a chief factor of G for each $i = 1, 2, \ldots, n$.

A subgroup L of G is said to cover a factor H/K of G if $(H \cap L)K = H$ and it is said to avoid H/K if $H \cap L \leq K$. If each chief factor of G is either covered or avoided by a given subgroup of G, then the subgroup is said to have a covering/avoidance property. In general, a subgroup of G need not cover or avoid a chief factor of G. It is easy to see that L covers (avoids) a p-chief factor of G if and only if a Sylow p-subgroup of L covers (avoids) it.

A chief factor H/K of G is said to be Frattini in G if $H/K \leq \Phi(G/K)$ and complemented otherwise, since then some maximal subgroup of G/H complements K/H in G/H. K, H/K, K

If A is a group of automorphisms of G and H a subgroup of G which is mapped into itself by every element of A, then H is said to be A-invariant in G. If A is the full automorphism group of G, then an A-invariant subgroup of G is called a characteristic subgroup of G. On the other hand, a family of subgroups of G is said to be characteristic if the subgroups in the family are permuted by the automorphisms of G.

A representation of G of dimension n over a field K is a (group) homomorphism from G into the general linear group $GL(n, K)$ of dimension n over K, that is, the group of all ~~endo~~morphisms of an n-dimensional vector space over K. auto

If $G_1, G_2, \ldots, G_n$ are groups, $\prod_{i=1}^{n} G_i$ denotes their direct product. The standard wreath product of a group H by a group K is denoted by $H \text{ wr } K$. For basic definition and properites of wreath products our main reference is Schenkman [34]. s, ti

Throughout the thesis, the statement "$\underline{\underline{X}}$ is a class of groups" implies that $\underline{\underline{X}}$ consists of finite soluble groups and contains all isomorphic copies of its members.

Groups in a class $\underline{\underline{X}}$ are called $\underline{\underline{X}}$-groups and those factors of a group which are members of $\underline{\underline{X}}$ are called $\underline{\underline{X}}$-factors. An $\underline{\underline{X}}$-subgroup of a group which is not a proper subgroup of any $\underline{\underline{X}}$-subgroup of the group is said to be $\underline{\underline{X}}$-maximal in the group.

A class $\underline{\underline{X}}$ of groups is said to be

(i) s-closed if subgroups of $\underline{\underline{X}}$-groups are $\underline{\underline{X}}$-groups;

(ii) Q-closed if homomorphic images of $\underline{\underline{X}}$-groups are $\underline{\underline{X}}$-groups;

(iii) R_0-closed if subdirect products of $\underline{\underline{X}}$-groups are $\underline{\underline{X}}$-groups;

(iv) E_{φ}-closed if $G/\Phi(G) \in \underline{\underline{X}}$ implies $G \in \underline{\underline{X}}$;

(v) E_p-closed if an extension of a p-group by an $\underline{\underline{X}}$-group is an $\underline{\underline{X}}$-group;

(vi) S_N-closed if normal subgroups of $\underline{\underline{X}}$-groups are $\underline{\underline{X}}$-groups; and

(vii) N_0-closed if the product of any two normal $\underline{\underline{X}}$-subgroups of a group is an $\underline{\underline{X}}$-group.

Classes of groups and various other families and sets are denoted by capital Roman letters with a double underline. Occassionally, we also use capital Greek letters to denote sets. The empty class of a set is, however, an exception and is denoted by φ .

The rest of the notation and terminology required in the course of this thesis will be introduced as and when required.

CHAPTER 2

PRELIMINARIES

2.1 Sylow systems and their reducibility

In this section, we introduce P. Hall's concept of Sylow systems of a finite soluble group (see [17], for example) and discuss their properties which will be required in the course of this thesis. In order to explain the term "Sylow system" we need to make the following definition.

(2.1.1) Definition. Let G be a group and π a set of primes.

(i) G is called a *π-group* if $|G|$ is a π-number.

(ii) A subgroup H of G is called a *Hall π-subgroup* of G if H is a π-subgroup of G and $|G : H|$ is a π'-number.

We will denote a Hall π-subgroup of a group G by G_π and, if at any time more than one Hall π-subgroup of G is considered, they will be distinguished by superscripts such as $*$.

A non-soluble group may not necessarily possess Hall π-subgroups corresponding to a set π of primes. But

(2.1.2) Theorem (P. Hall [15]). *A soluble group G has Hall π-subgroups corresponding to every set π of primes, and any two Hall π-subgroups of G are conjugate in G. Moreover, every π-subgroup of G is contained in some Hall π-subgroup of G.* //

It is an immediate consequence of Theorem 2.1.2 that a soluble group possesses a Hall p'-subgroup, that is, a Sylow p-complement, corresponding to each p. In [16], P. Hall has shown that this property, in fact, characterizes soluble groups.

(2.1.3) Definition. Let G be a group and $\{G_{p'}\}$ a set consisting of a Hall p'-subgroup of G, one for each p, that is, a complete set of Sylow complements of G. Then the set consisting of all the possible intersections of subgroups in the set $\{G_{p'}\}$, together with G, is called *the Sylow system of* G *generated by* $\{G_{p'}\}$.

Clearly, a Sylow system Σ of a group G contains a unique Hall π-subgroup of G corresponding to each set π of primes. We will denote the Hall π-subgroup of G in Σ by Σ_π.

Any two Sylow systems of a group are conjugate in the group in the following sense.

(2.1.4) Theorem (P. Hall [17]). *Let* $\{G_{p'}\}$ *and* $\{G^*_{p'}\}$ *be any two complete sets of Sylow complements of a group* G. *Then there is an element* g *of* G *such that* $G^g_{p'} = G^*_{p'}$ *for each* p. *In particular, if* Σ *and* Σ^* *are the Sylow systems of* G *generated by* $\{G_{p'}\}$ *and* $\{G^*_{p'}\}$, *respectively, then, for every set* π *of primes,* $\Sigma^g_\pi = \Sigma^*_\pi$. //

It is clear from Theorem 2.1.4 that the Sylow systems of a group are transitively permuted by the inner automorphisms of the group.

Extensions and Reductions of Sylow systems of a group were first considered systematically by Carter in his paper [2].

(2.1.5) Definition. Let Σ be a Sylow system of a group G and $H \leq G$. Then Σ is said to *reduce* into H if the intersections of H with the subgroups in Σ form a Sylow system, denoted by $H \cap \Sigma$, of H. On the other hand, a Sylow system Σ^* of H is said to *extend* to Σ if $\Sigma \cap H = \Sigma^*$.

By a result of P. Hall [17], every Sylow system of a subgroup of a group can be extended, though not necessarily uniquely, to a Sylow system of the group; consequently, by Theorem 2.1.4,

(2.1.6) Lemma. *If Σ is a Sylow system of a group G and $H \leq G$, then Σ reduces into at least one conjugate of H.* //

Occassionally, we will require the following result of Shamash [36] on the reducibility of Sylow systems.

(2.1.7) Lemma (Shamash [36]). *Let H and K be any two subgroups of a group G and Σ a Sylow system of G. If Σ reduces into both H and K, then Σ reduces into $H \cap K$.* //

A Sylow system of a group may reduce into more than one conjugate of a subgroup of the group. But for pronormal subgroups we have

(2.1.8) Lemma (Mann [30]). *A Sylow system of a group reduces into* *precisely one conjugate of a pronormal subgroup of the group.* //

2.2 Formation theory and the Prefrattini subgroups

We begin this section by surveying briefly the theory of formations which was orginated by Gaschütz [11].

(2.2.1) Definition. A class of groups is called a *formation* if it is both Q-closed and R_0-closed.

A non-empty formation is called a *saturated formation* if it is, in addition, E_φ-closed.

The empty class $\emptyset$ is clearly a formation. If $\underline{\underline{X}}$ is a formation, we will denote by $G^{\underline{\underline{X}}}$ the intersection of all normal subgroups N of a group G such that $G/N \in \underline{\underline{X}}$. If no such N exists, that is, if $\underline{\underline{X}} = \emptyset$, the empty class, we will put $G^{\emptyset} = G$. Clearly $G/G^{\underline{\underline{X}}} \in \underline{\underline{X}}$ unless $\underline{\underline{X}} = \emptyset$, the empty class, and $G^{\underline{\underline{X}}}$ is always characteristic in G. For the particular case $\underline{\underline{X}} = \underline{\underline{S}}_\pi$, the class of all finite soluble π-groups, which can be easily shown to be a saturated formation for any set π of primes, we write $O^\pi(G)$ to denote $G^{\underline{\underline{S}}_\pi}$.

(2.2.2) Lemma (Barnes, Kegel [1]). *Let $\underline{\underline{X}}$ be a saturated formation of characteristic π. Then every $\underline{\underline{X}}$-group is a π-group and all nilpotent π-groups are $\underline{\underline{X}}$-groups.*

Apart from the class $\underline{\underline{S}}_\pi$ that was mentioned above, other familiar examples of saturated formations include the class $\underline{\underline{S}}$ of all finite soluble groups (see Theorem 2.4.1 (i) - (iii) in [14]) and the class $\underline{\underline{N}}$ of all finite nilpotent groups (see Satz III. 2.5 (a) and (c), and Satz III. 3.7 in [25]).

If $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are saturated formations, then, by Proposition 7.16 [13], $\underline{\underline{XY}}$ is a saturated formation. Thus, in particular, $\underline{\underline{N}}^n$ is a saturated formation for every non-negative integer n .

In [11], Gaschütz describes a method of constructing a wealth of saturated formations:

Assign to each p a formation $\underline{\underline{X}}(p)$, which may possibly be empty, and let $\underline{\underline{X}}$ be the class of all groups G with the property:

If $p \,\Big|\, |G|$ and H/K is a p-chief factor of G , then

$G/C_G(H/K) \in \underline{\underline{X}}(p)$.

It is shown in [11] that the class $\underline{\underline{X}}$ so defined is a saturated formation. It is called *the saturated formation defined locally by the family* $\{\underline{\underline{X}}(p)\}$ *of formations.*

In fact, every saturated formation can be defined locally by some suitable family of formations. This has been shown by Lubeseder [28] (see Satz VI. 7.25 in [25], for example). In particular, it is easy to check that, if $\underline{\underline{X}}(p) = \{1\}$ for each p , then $\underline{\underline{X}}$ is precisely $\underline{\underline{N}}$.

If $\underline{\underline{X}}$ is a saturated formation defined locally by the family $\{\underline{\underline{X}}(p)\}$ of formations, then it can be easily verified that the characteristic of $\underline{\underline{X}}$ is precisely the set of those p for which $\underline{\underline{X}}(p) \neq \emptyset$.

(2.2.3) Definition. Let $\underline{\underline{X}}$ be a saturated formation defined locally by a family $\{\underline{\underline{X}}(p)\}$ of formations. Then

(i) the family $\{\underline{\underline{X}}(p)\}$ is said to be *integrated* if $\underline{\underline{X}}(p) \leq \underline{\underline{X}}$ for each p , and it is said to be *full* if $\underline{\underline{X}}(p)$ is ε_p-closed for each p .

Henceforth, assume that the family $\{\underline{\underline{X}}(p)\}$ is integrated and full.

(ii) A chief factor H/K of a group G is called $\underline{\underline{X}}$-*central* if $G/C_G(H/K) \in \underline{\underline{X}}(p)$, where p is the prime which divides $|H/K|$, and $\underline{\underline{X}}$-*eccentric* otherwise;

(iii) a maximal subgroup M of G is called $\underline{\underline{X}}$-*normal* if $M/\mathrm{Core}_G(M) \in \underline{\underline{X}}(p)$, where p is the prime which divides $|G : M|$, and $\underline{\underline{X}}$-*abnormal* otherwise;

(iv) a maximal subgroup M of G is called $\underline{\underline{X}}$-*critical* if it is $\underline{\underline{X}}$-abnormal and supplements in G the Fitting subgroup of G .

By a result of Carter, Fischer and Hawkes [5], every saturated formation can be defined locally by a unique full, integrated family of formations.

Now, corresponding to each saturated formation $\underline{\underline{X}}$, every group has a characteristic conjugacy class of subgroups called the $\underline{\underline{X}}$-normalizers of the group, which are defined as follows (see [38]).

(2.2.4) Definition. Let G be a group and Σ a Sylow system of G , and let $\underline{\underline{X}}$ be a saturated formation which is defined locally by the full, integrated family $\{\underline{\underline{X}}(p)\}$ of formations. If π is the characteristic of $\underline{\underline{X}}$, then *the* $\underline{\underline{X}}$-*normalizer of* G *corresponding to*

Σ is defined to be the subgroup

$$\Sigma_\pi \cap \bigcap_{p \in \pi} N_G\left(\Sigma_{p'} \cap G^{\underline{\underline{X}}(p)}\right)$$

of G .

Since $G^{\underline{\underline{X}}(p)}$ is a characteristic subgroup of G for each p and since, by Theorem 2.1.4, the Sylow systems of G are transitively permuted by the inner automorphisms of G , the $\underline{\underline{X}}$-normalizers of G indeed constitute a characteristic conjugacy class of subgroups of G .

(2.2.5) Lemma (Wright [38]). *A Sylow system of a group reduces into the corresponding* $\underline{\underline{X}}$*-normalizer of the group.* //

The following theorem describes the properties of the $\underline{\underline{X}}$-abnormal maximal subgroups and the $\underline{\underline{X}}$-normalizers of a group, which we will need in the latter chapters. Though the theorem has been proved by Carter and Hawkes [4] for the special case when $\underline{\underline{N}}$ is a subclass of $\underline{\underline{X}}$, their proof holds in the general case.

(2.2.6) Theorem (Carter and Hawkes [4]). *Let* G *be a group and* Σ *a Sylow system of* G .

(i) A maximal subgroup of G *is* $\underline{\underline{X}}$*-abnormal in* G *if and only if it complements an* $\underline{\underline{X}}$*-eccentric chief factor of* G .

(ii) A maximal subgroup of G *contains an* $\underline{\underline{X}}$*-normalizer of* G *if and only if it is* $\underline{\underline{X}}$*-abnormal in* G .

(iii) Let M *be an* $\underline{\underline{X}}$*-abnormal maximal subgroup of* G *into which* Σ *reduces and let* D *and* D^* *be the* $\underline{\underline{X}}$*-normalizers of* G *and*

M *corresponding to* Σ *and* $\Sigma \cap M$ *, respectively. Then* $D \leq D^*$.

(iv) An $\underline{\underline{X}}$*-normalizer of an* $\underline{\underline{X}}$*-critical maximal subgroup of* G *is an* $\underline{\underline{X}}$*-normalizer of* G .

(v) The $\underline{\underline{X}}$*-normalizers of* G *are invariant under homomorphisms of* G .

(vi) An $\underline{\underline{X}}$*-normalizer of* G *covers the* $\underline{\underline{X}}$*-central chief factors of* G *and avoids the rest.* //

Besides the $\underline{\underline{X}}$-normalizers. a group has, corresponding to any saturated formation $\underline{\underline{X}}$, a unique conjugacy class of subgroups called the $\underline{\underline{X}}$-projectors of the group. These were first defined by Gaschütz [11] as follows.

(2.2.7) Definition. Let $\underline{\underline{Y}}$ be a class of groups. A subgroup E of a group G is called a $\underline{\underline{Y}}$*-projector* of G if

(i) $E \in \underline{\underline{Y}}$;

(ii) $H = KE$ whenever $E \leq H \leq G$ and $K \trianglelefteq H$ such that $H/K \in \underline{\underline{Y}}$.

Apart from the saturated formations a group has $\underline{\underline{Y}}$-projectors corresponding to each Schunck class $\underline{\underline{Y}}$ (see [35], for example).

A saturated formation is necessarily a Schunck class (see [13]) though the converse is not, to our knowledge, known to hold.

If $\underline{\underline{X}}$ is a saturated formation which contains the class $\underline{\underline{N}}$, then it has been shown by Gaschütz [11] that the $\underline{\underline{X}}$-projectors of a group G are abnormal subgroups of G . Consequently, by Lemma 2.1.8, a Sylow system of G reduces into precisely one conjugate of an $\underline{\underline{X}}$-projector of G .

The following two lemmas describe some of the properties of the $\underline{\underline{X}}$-projectors of a group corresponding to a Q-closed class $\underline{\underline{X}}$.

(2.2.8) <u>Lemma</u> (Gaschütz [11], Schunck [35]). *Let* G *be a group and* E *an* $\underline{\underline{X}}$*-projector of* G .

(i) If $E \leq H \leq G$ *, then* E *is an* $\underline{\underline{X}}$*-projector of* H .

(ii) If $N \trianglelefteq G$ *, then* NE/N *is an* $\underline{\underline{X}}$*-projector of* G/N . //

(2.2.9) <u>Lemma</u> (Gaschütz [11], Schunck [35]). *Let* G *be a group and* $N \trianglelefteq G$. *If* E^*/N *is an* $\underline{\underline{X}}$*-projector of* G/N *and* E *is an* $\underline{\underline{X}}$*-projector of* E^* *, then* E *is an* $\underline{\underline{X}}$*-projector of* G . //

The following theorem describes a relation between the $\underline{\underline{X}}$-projectors and the $\underline{\underline{X}}$-normalizers of a group corresponding to a saturated formation $\underline{\underline{X}}$ which contains the class $\underline{\underline{N}}$.

(2.2.10) <u>Theorem</u> (Carter and Hawkes [4], Hawkes [22]). *Let* G *be a group and* Σ *a Sylow system of* G . *Let* E *be the* $\underline{\underline{X}}$*-projector of* G *into which* Σ *reduces and* D *the* $\underline{\underline{X}}$*-normalizer of* G *corresponding to* Σ . *Then*

(i) $D \leq E$. *Consequently, every* $\underline{\underline{X}}$*-normalizer of* G *is contained in some* $\underline{\underline{X}}$*-projector of* G *and every* $\underline{\underline{X}}$*-porjector of* G *contains an* $\underline{\underline{X}}$*-normalizer of* G .

(ii) If $G \in \underline{\underline{NX}}$, $D = E$; *that is to say, the* $\underline{\underline{X}}$*-normalizers and the* $\underline{\underline{X}}$*-projectors of* G *coincide.*

Next, we define the Prefrattini subgroups of a group and their analogues. The Prefrattini subgroups of a group were first constructed by Gaschütz [10] who showed that these cover the Frattini chief factors of the group and avoid the complemented ones. In [21], Hawkes described a method of constructing a wealth of characteristic conjugacy classes of subgroups of a group, one class corresponding to each saturated formation, which have properties analogous to the Prefrattini subgroups of the group.

In the special case when the saturated formation under consideration is the trivial formation $\{1\}$, the corresponding subgroups obtained by Hawkes are precisely the Prefrattini subgroups of the group.

(2.2.11) Definition. Let $\underline{\underline{X}}$ be a saturated formation, not necessarily containing the class $\underline{\underline{N}}$, let G be a group and let Σ be a Sylow system of G . Then, the $\underline{\underline{X}}$-Prefrattini subgroup of G corresponding to Σ is defined as the intersection of all those $\underline{\underline{X}}$-abnormal maximal subgroups of G into which Σ reduces.

Since, by Theorem 2.1.4, the Sylow systems of G are transitively permuted by the inner automorphisms of the group, its $\underline{\underline{X}}$-Prefrattini subgroups clearly constitute a characteristic conjugacy class of subgroups. Also, it is immediate from the definition and a repeated application of Lemma 2.1.7 that

(2.2.12) Theorem. *A Sylow system of a group reduces into the corresponding* $\underline{\underline{X}}$*-Prefrattini subgroup of the group.* //

The following theorems describe some of the properties of the $\underline{\underline{X}}$-Prefrattini subgroups of a group.

(2.2.13) Theorem (Hawkes [21]). *Let* G *be a group,* Σ *a Sylow system of* G *and* W *the* $\underline{\underline{X}}$*-Prefrattini subgroup of* G *corresponding to* Σ . *Then*

(i) W *avoids the* $\underline{\underline{X}}$*-eccentric, complemented chief factors of* G *and covers the rest ;*

(ii) the $\underline{\underline{X}}$*-Prefrattini subgroups of* G *are invariant under the homomorphisms of* G . //

(2.2.14) Theorem (Hawkes [21]). *Let* G *be a group,* Σ *a Sylow system of* G *and* D, W* *and* W *the* $\underline{\underline{X}}$*-normalizer of* G *, the Prefrattini subgroup of* G *and the* $\underline{\underline{X}}$*-Prefrattini subgroup of* G *, respectively, each corresponding to* Σ . *Then*

(i) W = DW* *;*

(ii) $D \cap W^*$ *covers the* $\underline{\underline{X}}$*-central, Frattini chief factors of* G *and avoids the rest. Moreover, as* Σ *runs through the Sylow systems of* G *,* $(D \cap W^*)$*'s constitute a characteristic conjugacy class of subgroups of* G . //

2.3 Fitting classes and the Injectors

The concept of Fitting classes was introduced by Fischer in [7] and is dual to that of formations.

(2.3.1) Definition. A class $\underline{\underline{X}}$ of groups is called a *Fitting* *class* if it is both s_N-closed and N_0-closed.

A Fitting class $\underline{\underline{X}}$ is a *Fischer class* if it has the following additional property:

If $G \in \underline{\underline{X}}$, $N \trianglelefteq G$ and $N \leq H \leq G$ such that $H/N \in \underline{\underline{S}}_p$ for some p, then $H \in \underline{\underline{X}}$.

By Lemma 6.1.1 and Theorem 2.3.3 (i) [14], the class $\underline{\underline{N}}$ is an s-closed Fitting class. Consequently, by a repeated application of Theorem 8.2 [13], $\underline{\underline{N}}^n$ is an s-closed Fitting class for every non-negative integer n. Also by the remarks (iii) and (v) following Definition IV. 1.b in [34], the class $\underline{\underline{S}}_\pi$ is an s-closed Fitting class.

Clearly, corresponding to any N_0-closed class $\underline{\underline{X}}$ of groups, a group G has a unique largest normal $\underline{\underline{X}}$-subgroup which we will denote by $G_{\underline{\underline{X}}}$. $G_{\underline{\underline{X}}}$ is certainly a characteristic subgroup of G and if $\underline{\underline{X}} \geq \underline{\underline{N}}$, then by Theorem 6.1.3 [14], $C_G(G_{\underline{\underline{X}}}) \leq G_{\underline{\underline{X}}}$.

For $\underline{\underline{X}} = \underline{\underline{S}}_\pi, \underline{\underline{S}}_{\pi'}, \underline{\underline{S}}_{\pi'}\underline{\underline{S}}_\pi$, where π is any set of primes, we will denote $G_{\underline{\underline{X}}}$ by $O_\pi(G)$, $O_{\pi'}(G)$ and $O_{\pi'\pi}(G)$, respectively. Observe that $\underline{\underline{S}}_{\pi'}\underline{\underline{S}}_\pi$ is, by Theorem 8.2 [13], a Fitting class, and so $O_{\pi'\pi}(G)$ is well-defined.

It is clear from the definition of a Fischer class that every s-closed Fitting class is a Fischer class, though the converse is not true as was pointed by Fischer in [7]. Thus, in particular, the class $\underline{\underline{S}}_\pi$ for any set π of primes and the class $\underline{\underline{N}}^n$ for every non-negative integer n are both Fischer classes.

A recent work of Hawkes [23] shows that a meta-nilpotent Fitting formation, that is, a formation which is also a Fitting class and is contained in $\underline{\underline{N}}^2$, is always s-closed. We show here that

(2.3.2) Theorem. *Every meta-nilpotent Fischer class is* s-*closed.*

Proof. Assume the result is false and let $\underline{\underline{X}}$ be a meta-nilpotent Fischer class which is not s-closed. Let G be a group of minimal order among those $\underline{\underline{X}}$-groups which have subgroups not belonging to $\underline{\underline{X}}$. Choose among such subgroups of G one, say H , of maximal order. Then it is clear from our choice of G and H , that H is a maximal subgroup of G . Consequently, $H \cap G_{\underline{\underline{N}}} \triangleleft G$ since $H \cap G_{\underline{\underline{N}}} \triangleleft H$ and $H \cap G_{\underline{\underline{N}}} \triangleleft\triangleleft G_{\underline{\underline{N}}}$. Also, since $G \in \underline{\underline{N}}^2$ and $H/H \cap G_{\underline{\underline{N}}} \cong HG_{\underline{\underline{N}}}/G_{\underline{\underline{N}}}$, $H/H \cap G_{\underline{\underline{N}}} \in \underline{\underline{N}}$, so that $H/H \cap G_{\underline{\underline{N}}}$ is a direct product of its Sylow subgroups. In particular, $H = H_1H_2 \ldots H_n$, where, for each $i = 1, 2, \ldots, n$, $H_i \trianglelefteq H$, $H_i \geq H \cap G_{\underline{\underline{N}}}$ and $H_i/H \cap G_{\underline{\underline{N}}} \in \underline{\underline{S}}_p$ for some p . But now, since $H \cap G_{\underline{\underline{N}}} \triangleleft G \in \underline{\underline{X}}$ and $\underline{\underline{X}}$ is a Fischer class, it follows that $H_i \in \underline{\underline{X}}$ for each $i = 1, 2, \ldots, n$. Hence, finally, since $\underline{\underline{X}}$ is N_0-closed and $H_i \trianglelefteq H$ for each $i = 1, 2, \ldots, n$, $H \in \underline{\underline{X}}$ and we have a contradiction. With this contradiction the proof is complete. //

Now, it has been shown by Fischer, Gaschütz and Hartley [9] that corresponding to each Fitting class $\underline{\underline{X}}$, each group has a unique conjugacy class of subgroups called $\underline{\underline{X}}$-injectors. These are defined as follows:

(2.3.3) Definition. A subgroup V of a group G is called an $\underline{\underline{X}}$-*injector* of G if

(i) $V \in \underline{\underline{X}}$;

(ii) $V \cap N$ is $\underline{\underline{X}}$-maximal in N whenever $N \lhd\lhd G$.

The following theorem describes some of the properties of $\underline{\underline{X}}$-injectors of a group G .

(2.3.4) Theorem (Fischer, Gaschütz, Hartley [9], Hartley [20]).

(i) All conjugates of an $\underline{\underline{X}}$-*injector of* G *are* $\underline{\underline{X}}$-*injectors of* G *, and any two* $\underline{\underline{X}}$-*injectors of* G *are conjugate in* G .

(ii) $\underline{\underline{X}}$-*injectors of* G *are pronormal subgroups of* G .

(iii) An $\underline{\underline{X}}$-*injector of* G *either covers or avoids a chief factor of* G .

(iv) An $\underline{\underline{X}}$-*injector of* G *is an* $\underline{\underline{X}}$-*injector of every subgroup of* G *which contains it.*

In view of Theorem 2.3.4 *(i) - (ii)* and Lemma 2.1.8, a Sylow system of a group reduces into precisely one $\underline{\underline{X}}$-injector of the group. Consequently, we have

(2.3.5) Lemma. *Let* G *be a group,* Σ *a Sylow system of* G *and* V *the* $\underline{\underline{X}}$-*injector of* G *into which* Σ *reduces. Let* M *be a subgroup of* G *which contains an* $\underline{\underline{X}}$-*injector of* G *. If* Σ *reduces into* M *, then* $V \leq M$.

Proof. By hypothesis, $\Sigma \cap M$ is a Sylow system of M. Let W be an $\underline{\underline{X}}$-injector of G which is contained in M. By Lemma 2.1.6, there is a conjugate U of W in M into which $\Sigma \cap M$, and hence Σ, reduces. Clearly, U is, by Theorem 2.3.4 *(i)*, an $\underline{\underline{X}}$-injector of G, and hence, by the same result, a conjugate of V. Thus, it follows from the remark preceding this lemma that $U = V$, and the lemma is proved. //

(2.3.6) Definition. A factor of a group G is called $\underline{\underline{X}}$-*covered* if it is covered by an $\underline{\underline{X}}$-injector of G and $\underline{\underline{X}}$-*avoided* if it is avoided by an $\underline{\underline{X}}$-injector of G.

By Theorem 2.3.4 *(iii)*, a chief factor of a group G is either $\underline{\underline{X}}$-covered or $\underline{\underline{X}}$-avoided.

We conclude this section by mentioning a result which will be used repeatedly in the course of this thesis.

(2.3.7) Theorem (Fischer [7]). *Let* $\underline{\underline{X}}$ *be a Fischer class and* V *an* $\underline{\underline{X}}$-*injector of a group* G. *Then, for each* p *dividing* $|V|$, V_p *is a Sylow* p-*subgroup of* V_p^G.

It is an immediate consequence of this theorem that

(2.3.8) Corollary. *A* p-*chief factor of* G *is* $\underline{\underline{X}}$-*covered if and only if it is covered by* V_p^G. *Consequently, the number of* $\underline{\underline{X}}$-*covered* p-*chief factors of any given* G-*isomorphism type in a chief series of* G *is independent of the series.*

2.4 Generalized nilpotent length

In this section, we discuss briefly a generalization of the concepts of the nilpotent length and the p-length of a group. Our general reference for this section is [8].

Let $\underline{\underline{X}}$ be a saturated Fitting formation, that is, a saturated formation which is also a Fitting class, and let π be the characteristic of $\underline{\underline{X}}$.

(2.4.1) Definition. An ascending series $1 = G_0 < G_1 < \ldots < G_m = G$ of normal subgroups of a group G is called an *$\underline{\underline{X}}$-series* of G if for each $i = 1, 2, \ldots, m$, either $G_i/G_{i-1} \in \underline{\underline{X}}$ or $G_i/G_{i-1} \in \underline{\underline{S}}_{\pi'}$.

The upper $\underline{\underline{X}}$-series of G is the series of normal subgroups of G defined inductively as follows: $R_0 = \{1\}$ and, for $i \geq 1$, $T_i/R_{i-1} = O_{\pi'}(G/R_{i-1})$ and $R_i/T_i = (G/T_i)_{\underline{\underline{X}}}$.

Similarly, *the lower $\underline{\underline{X}}$-series* of G is the series of normal subgroups of G defined inductively as follows: $R_0^* = G$ and, for $i \geq 1$, $T_i^* = O^{\pi'}(R_{i-1}^*)$ and $R_i^* = (T_i^*)^{\underline{\underline{X}}}$.

The $\underline{\underline{X}}$-length of G is defined to be the smallest number of $\underline{\underline{X}}$-factors in any $\underline{\underline{X}}$-series of G and is written as $h_{\underline{\underline{X}}}(G)$.

It is clear, in view of Lemma 2.2.2, that the factors R_i/T_i and T_i^*/R_i^* are always non-trivial for $i \geq 1$ unless $T_i = G$ or $T_i^* = 1$, respectively.

We now state without proof the following elementary fact about the $\underline{\underline{X}}$-length of G .

(2.4.2) Theorem. *The invariant* $h_{\underline{\underline{X}}}(G)$ *of a group* G *is the number of* $\underline{\underline{X}}$*-factors in both the upper and the lower* $\underline{\underline{X}}$*-series of* G . //

We end this section with a remark that, if $\underline{\underline{X}} = \underline{\underline{N}}$, then π is the whole prime set and the invariant $h_{\underline{\underline{N}}}(G)$ of a group G is the familiar nilpotent length $h(G)$ of G . On the other hand, if $\underline{\underline{X}} = \underline{\underline{S}}_p$, then $\pi = \{p\}$ and $h_{\underline{\underline{X}}}(G)$ is the p-length $\ell_p(G)$ of G .

2.5 Miscellaneous Results

First of all, we show

(2.5.1) Lemma. *Let* $G = \prod_{i=1}^{n} H_i$ *be the direct product of the groups* $H_1, H_2, \ldots, H_n$ *and let, for* $i = 1, 2, \ldots, n$, $\pi_i : G \twoheadrightarrow H_i$ *be the projection map of* G *onto* H_i *and* $\mu_i : H_i \rightarrowtail G$ *the injection map of* H_i *into* G . *Let* $K \leq G$. *Then*

(i) $C_G(K) = \prod_{i=1}^{n} \left(C_{H_i}(K\pi_i)\right)\mu_i$;

(ii) $N_G(K) \leq N_G\left(\prod_{i=1}^{n} K\pi_i\mu_i\right)$;

(iii) K *is a Sylow* p*-subgroup or a maximal nilpotent subgroup*

of G *if and only if* $K = \prod_{i=1}^{n} K\pi_i\mu_i$ *and, for each* $i = 1, 2, \ldots, n$, $K\pi_i$ *is a Sylow* p*-subgroup of* H_i *in the first case and a maximal nilpotent subgroup of* H_i *in the second case.*

Proof. *(i)* An element g of G centralizes K if and only if $[g\pi_i, K\pi_i] = \{1\}$ for all $i = 1, 2, \ldots, n$, that is, if and only if $g\pi_i \in C_{H_i}(K\pi_i)$ for all $i = 1, 2, \ldots, n$. Hence *(i)* holds.

(ii) Let $g \in N_G(K)$. Since $g = \prod_{i=1}^{n} g\pi_i\mu_i$ and, for all $j \neq i$, $g\pi_j\mu_j$ centralizes $K\pi_i\mu_i$, we have that

$$(K\pi_i\mu_i)^g = (K\pi_i\mu_i)^{g\pi_i\mu_i} = (K^g)\pi_i\mu_i = K\pi_i\mu_i .$$

Thus g normalizes each $K\pi_i\mu_i$ and so it normalizes their product.

(iii) Suppose first that K is a maximal nilpotent subgroup of G . Then clearly $K\pi_i$ is a nilpotent subgroup of H_i for each $i = 1, 2, \ldots, n$. Let, for each $i = 1, 2, \ldots, n$, K_i^* be a maximal nilpotent subgroup of H_i which contains $K\pi_i$. Then

$$K \leq \prod_{i=1}^{n} K\pi_i\mu_i \leq \prod_{i=1}^{n} K_i^*\mu_i = L .$$

Hence, since L is a nilpotent subgroup of G , it follows by the maximality of K that $K = L$, and, therefore, also $K\pi_i = K_i^*$ for each $i = 1, 2, \ldots, n$, as required.

Suppose next that $K\pi_i$ is a maximal nilpotent subgroup of H_i for each $i = 1, 2, \ldots, n$, and that $K = \prod_{i=1}^{n} K\pi_i\mu_i$. Let K^* be a maximal nilpotent subgroup of G which contains K . Then from what has been just shown above, together with our assumptions,

$$K^* = \prod_{i=1}^{n} K^*\pi_i\mu_i = \prod_{i=1}^{n} K\pi_i\mu_i = K ,$$

as required.

The other half of *(iii)* can be proved similarly. //

Next we show

(2.5.2) <u>Lemma</u>. *Let* H *and* K *be any two groups, let* $G = H \text{ wr } K$ *and let* B *be the base group of* G . *Let, for each* $k \in K$, $\pi_k : B \twoheadrightarrow H$ *be the projection map of* B *onto* H *and* $\mu_k : H \rightarrowtail B$ *the the injection map of* H *into* B . *If* $\{1\} \neq L \leq B$ *and* $L = \prod_{k \in K} L\pi_k\mu_k$, *then* $C_G(L) = C_B(L)$. *Also, if* S *is a Sylow subgroup of* L , *then* $S = \prod_{k \in K} S\pi_k\mu_k$.

Proof. By hypothesis, $L\pi_k\mu_k \neq \{1\}$ for some $k \in K$. Let $g = k'b$, $k' \in K$, $b \in B$, be an element of $C_G(L)$. We show that $k' = 1$. Clearly, $(L\pi_k\mu_k)^{k'b} = (L\pi_k\mu_{kk'})^b \leq (H\mu_{kk'})^b = H\mu_{kk'}$. Thus $(L\pi_k\mu_k)^g \leq H\mu_{kk'}$. But, also $(L\pi_k\mu_k)^g = L\pi_k\mu_k \leq H\mu_k$. Hence

$(L\pi_k\mu_k)^g \le H\mu_k \cap H\mu_{kk'}$, and so, since $L\pi_k\mu_k \ne \{1\}$, it follows that $k = kk'$, whence $k' = 1$ and $g \in B$, as required. The final statement of the lemma now follows from 2.5.1 *(iii)*. //

The following result of Huppert [26], which we quote here without proof, will prove very useful in establishing the main result of the first part of this thesis.

(2.5.3) Lemma (Huppert [26]). *Let* A, B *and* C *be any three subgroups of a group* G . *Then* $A \cap BC = (A \cap B)(A \cap C)$ *if and only if* $AB \cap AC = A(B \cap C)$. //

Finally, we have

(2.5.4) Lemma. *Let* A *and* B *be any two permutable subgroups of a group* G *such that each of* A *and* B *either covers or avoids each chief factor of* G . *Then* $A \cap B$ *covers all those chief factors of* G *which are covered simultaneously by* A *and* B *if and only if* AB *avoids all those chief factors of* G *which are avoided simultaneously by* A *and* B .

Proof. Let η be a chief series of G , α the product of the orders of those chief factors of G in η which are covered by both A and B , β the product of orders of those chief factors in η which are covered by A but avoided by B and γ the product of the orders of those chief factors in η which are avoided by A but covered by B . Then it is easy to check that $|A| = \alpha\beta$ and $|B| = \alpha\gamma$. Also $|A \cap B| \le \alpha$ since every chief factor of G which is not avoided by $A \cap B$ is covered by both A and B .

Suppose first that $A \cap B$ covers all those chief factors of G which are covered by both A and B. Then from what has been just said $|A \cap B| = \alpha$ and so $|AB| = |A||B|/|A \cap B| = \alpha\beta\gamma$. Consequently AB avoids all those chief factors of G in η which are avoided by both A and B.

On the other hand, if AB avoids all those chief factors of G which are avoided simultaneously by A and B, then, since AB covers all those chief factors of G which are covered either by A or B, $\alpha\beta\gamma = |AB|$, whence $|A \cap B| = \alpha$, that is, $A \cap B$ covers all those chief factors of G in η which are covered simultaneously by A and B.

Since η was an arbitrary chief series of G, the lemma is finally proved. //

CHAPTER 3

INJECTORS AND ANALOGUES OF THE PREFRATTINI SUBGROUPS

Throughout this chapter and the next, $\underline{\underline{F}}$ will denote a saturated formation, $\underline{\underline{H}}$ a Fischer class, G a group, Σ a Sylow system of G and V the $\underline{\underline{H}}$-injector of G into which Σ reduces. Moreover, D, W and W^* will denote the $\underline{\underline{F}}$-normalizer of G, the $\underline{\underline{F}}$-Prefrattini subgroup of G and the Prefrattini subgroup of G, respectively, all three corresponding to the same Sylow system Σ.

Our main aim in this chapter is to show that W and V, and hence, W^* and V, are permutable subgroups of G, and that WV, W^*V, $W \cap V$ and $W^* \cap V$ each have a covering/avoidance property with respect to the chief factors of G. This occupies the whole of Section 3.3.

The first two sections of this chapter are preliminaries to Section 3.3. In Section 3.1, we consider those $\underline{\underline{H}}$-avoided, complemented chief factors of G at least one of whose complements in G contains an $\underline{\underline{H}}$-injector of G. In Section 3.2, we establish a relation between the $\underline{\underline{F}}$-eccentric, $\underline{\underline{H}}$-avoided, complemented chief factors of G which are covered by WV and the $\underline{\underline{F}}$-eccentric, $\underline{\underline{H}}$-covered, Frattini chief factors of G which are avoided by $W \cap V$.

The Corollaries 3.3.3 and 3.3.7 have been published in my paper [29], where my approach in proving these results was different from the present one. The alternative approach taken here relies on the main result of Section 3.2, namely Theorem 3.2.2.

3.1 Partially $\underline{\underline{H}}$-complemented chief factors

In [20], B. Hartley calls a complemented factor H/K of G an $\underline{\underline{H}}$-complemented factor if every complement of H/K in G contains some $\underline{\underline{H}}$-injector of G. $\underline{\underline{H}}$-complemented chief factors of G are among those chief factors of G which are of particular interest to us in the last section. In fact we will need to consider a slightly more general situation which necessitates the following definition.

(3.1.1) Definition. A complemented factor H/K of G is said to be *partially* $\underline{\underline{H}}$*-complemented* in G if at least one of its complements in G contains an $\underline{\underline{H}}$-injector of G.

Clearly, an $\underline{\underline{H}}$-complemented factor of G is partially $\underline{\underline{H}}$-complemented in G, though the converse is not true as the following example shows.

(3.1.2) Example. Let

$$G = \langle x, y, z \mid x^2 = y^5 = z^2 = 1, z^{-1}yz = y^{-1}, xy = yx, xz = zx \rangle ,$$

a dihedral group of order 20, and let $\underline{\underline{H}} = \underline{\underline{N}}$, the class of all finite nilpotent groups. In G, $\langle x, y \rangle$ is the $\underline{\underline{H}}$-injector of G and it complements the chief factor $\langle y, z \rangle / \langle y \rangle$ of G. But $\langle y, xz \rangle$ is also a complement for the latter in G, and clearly $\langle y, xz \rangle \neq \langle x, y \rangle$. Thus, $\langle y, z \rangle / \langle y \rangle$ is a partially $\underline{\underline{H}}$-complemented factor of G which is not $\underline{\underline{H}}$-complemented in G.

It is also not true that an $\underline{\underline{H}}$-avoided, complemented factor of G is necessarily partially $\underline{\underline{H}}$-complemented in G as the following example shows, though a partially $\underline{\underline{H}}$-complemented factor of G is obviously $\underline{\underline{H}}$-avoided and complemented.

(3.1.3) Example. Let G be the semidirect product of a cyclic group $\langle z\rangle$ of order 5 by the direct product H of a cyclic group $\langle x\rangle$ of order 2 and a cyclic group $\langle y\rangle$ of order 4, with the action of H on $\langle z\rangle$ given by: $x^{-1}zx = z^{-1}$ and $y^{-1}zy = z^2$. Let $\underline{\underline{H}} = \underline{\underline{N}}$, the class of all finite nilpotent groups. In G, xy^2 acts trivially on z, and so $\langle z\rangle \times \langle xy^2\rangle$ is a nilpotent subgroup of G. In fact, it is the $\underline{\underline{H}}$-injector of G. Consider the chief factor $\langle z, x\rangle/\langle z\rangle$ of G. It is certainly avoided by $\langle z\rangle \times \langle xy^2\rangle$ and complemented in G by $\langle z, xy\rangle$ and $\langle z, y\rangle$, which are all its complements in G. But neither $\langle z, xy\rangle$ nor $\langle z, y\rangle$ contains $\langle xy^2\rangle$. Thus, $\langle z, x\rangle/\langle z\rangle$ is an $\underline{\underline{H}}$-avoided, complemented chief factor of G which is not partially $\underline{\underline{H}}$-complemented in G.

The following theorem gives a necessary and sufficient condition for a complement of an $\underline{\underline{H}}$-avoided, complemented chief factor of G to contain an $\underline{\underline{H}}$-injector of G.

(3.1.4) Theorem. *Let* H/K *be a complemented* p*-chief factor of* G *and* M *a complement for* H/K *in* G. *Then* M *contains an* $\underline{\underline{H}}$*-injector of* G *if and only if it contains* V_p^G.

Proof. Let $B = \mathrm{Core}_G(M)$ and let A/B be the unique minimal normal subgroup of G/B. As is well known (see for instance Theorem 3.1 of [10]), $A = C_G(H/K)$.

Assume first that M contains an $\underline{\underline{H}}$-injector of G. Then clearly both H/K and A/B are $\underline{\underline{H}}$-avoided in G, and, therefore, by Corollary 2.3.8, they are avoided by V_p^G. Thus

$[H, V_p^G] \leq H \cap V_p^G = K \cap V_p^G \leq K$, and so $V_p^G \leq C_G(H/K) = A$. But then

$V_p^G = V_p^G \cap A = V_p^G \cap B \leq B \leq M$, as required.

Conversely, if $M \geq V_p^G$, then clearly A/B is avoided by V_p^G, and, therefore, is $\underline{\underline{H}}$-avoided in G. Since it is also self-centralizing in G/B, it follows, by Lemma 4 of Hartley [20], that M contains an $\underline{\underline{H}}$-injector of G, as required. //

As immediate consequences of Theorem 3.1.4, we have:

(3.1.5) <u>Corollary</u>. *A complemented* p-*chief factor of* G *is partially* $\underline{\underline{H}}$-*complemented in* G *if and only if at least one of its complements in* G *contains* V_p^G. //

(3.1.6) <u>Corollary</u>. *If* H/K *is a complemented* p-*chief factor of* G *such that* $K \geq V_p^G$, *then* H/K *is* $\underline{\underline{H}}$-*complemented in* G. //

Another simple consequence of Theorem 3.1.4 is the following corollary.

(3.1.7) <u>Corollary</u>. *A complemented* p-*chief factor* H/K *of* G *is partially* $\underline{\underline{H}}$-*complemented in* G *if and only if* HV_p^G/KV_p^G *is non-trivial and complemented in* G.

Proof. Assume first that H/K is partially $\underline{\underline{H}}$-complemented in G and let M be a complement for H/K in G which contains an $\underline{\underline{H}}$-injector of G. Then, by Theorem 3.1.4, $M \geq V_p^G$, and so $M \geq KV_p^G$. But $M \ngeq HV_p^G$ since $M \ngeq H$, and hence M complements HV_p^G/KV_p^G.

Conversely, if HV_p^G/KV_p^G is non-trivial and complemented in G, then, by Corollary 3.1.6, it is $\underline{\underline{H}}$-complemented in G. Thus, since a complement for HV_p^G/KV_p^G is also a complement for H/K in G, H/K is partially $\underline{\underline{H}}$-complemented in G, as required. //

The next result gives a necessary and sufficient condition for G to have a partially $\underline{\underline{H}}$-complemented chief factor.

(3.1.8) Theorem. G *has a partially* $\underline{\underline{H}}$-*complemented* p-*chief factor if and only if some* p-*chief factor of* G *is* $\underline{\underline{H}}$-*avoided in* G.

Proof. Assume first that G has an $\underline{\underline{H}}$-avoided p-chief factor and consider $G_{p'}V_p^G$. If $G_{p'}V_p^G = G$, then G/V_p^G is a p'-group, and so V_p^G contains a Sylow p-subgroup of G. But then, by Theorem 2.3.7, V_p is a Sylow p-subgroup of G, whence every p-chief factor of G is $\underline{\underline{H}}$-covered in G, contrary to our assumption. Thus, $G_{p'}V_p^G < G$. Now, if M is a maximal subgroup of G which contains $V_p^G G_{p'}$, then, by Theorem 3.1.4, the unique minimal normal subgroup of $G/\mathrm{Core}_G(M)$ is partially $\underline{\underline{H}}$-complemented in G.

The converse of this theorem is, on the other hand, obvious, since a partially $\underline{\underline{H}}$-complemented chief factor of G is necessarily $\underline{\underline{H}}$-avoided. //

We end this section with a result which describes a relation between the partially $\underline{\underline{H}}$-complemented chief factors of G in any two chief series of G .

(3.1.9) Theorem. *Given any two chief series of* G *, there is a one-one correspondence between partially* $\underline{\underline{H}}$*-complemented chief factors of* G *in one and those in the other, corresponding chief factors being* G*-isomorphic.*

Proof. Let η_1 and η_2 be any two chief series of G . Consider the chief series μ_1 and μ_2 of G which are obtained by multiplying each member of η_1 and η_2 , respectively, by V_p^G and refining V_p^G . By Corollary 3.1.7, there is one-one correspondence between partially $\underline{\underline{H}}$-complemented p-chief factors of G in η_i and the complemented p-chief factors of G in μ_i above V_p^G , for each $i = 1, 2$, corresponding chief factors being G-isomorphic. But, by Lemma 2.6 of Carter, Fischer and Hawkes [5], the complemented p-chief factors of G in μ_1 and μ_2 above V_p^G are in one-one correspondence, corresponding chief factors being G-isomorphic. Hence, the result clearly holds for partially $\underline{\underline{H}}$-complemented p-chief factors of G in η_1 and η_2 . Since p was an arbitrary prime, the result holds for all partially $\underline{\underline{H}}$-complemented chief factors. //

3.2 Relation between $\hat{\underline{\underline{H}}}$-chief factors and $\Phi\underline{\underline{H}}$-chief factors

In order to formulate the main result of this section we require the following definition.

(3.2.1) Definition. (i) A p-chief factor H/K of G is said to be $\underline{\underline{H}}$-*Frattini* in G if it is $\underline{\underline{H}}$-covered and $H \cap V_p^G / K \cap V_p^G$ is Frattini in G.

(ii) A chief factor of G is called an $\hat{\underline{\underline{H}}}$-*chief factor* if it is $\underline{\underline{H}}$-avoided, complemented but not partially $\underline{\underline{H}}$-complemented in G, and a $\Phi\underline{\underline{H}}$-*chief factor* if it is $\underline{\underline{H}}$-covered, Frattini but not $\underline{\underline{H}}$-Frattini in G.

Our main result of this section is:

(3.2.2) Theorem. *Given any chief series of* G, *there is a one-one correspondence between the* $\hat{\underline{\underline{H}}}$-*chief factors and the* $\Phi\underline{\underline{H}}$-*chief factors in the series, corresponding chief factors being* G-*isomorphic*.

Proof. Let

$$\eta : 1 = G_0 < G_1 < \ldots < G_m = G$$

be a chief series of G and let Λ be the set of all chief factors of G in η which are G-isomorphic to G_i/G_{i-1} for some i, $1 \leq i \leq m$. Let α be the number of $\underline{\underline{H}}$-covered, complemented chief factors of G in Λ, β the number of $\Phi\underline{\underline{H}}$-chief factors in Λ, γ the number of $\hat{\underline{\underline{H}}}$-chief factors in Λ, and δ the number of partially

$\underline{\underline{H}}$-complemented chief factors of G in Λ. Clearly, the number of complemented chief factors of G in η which are G-isomorphic to G_i/G_{i-1} is $\alpha + \gamma + \delta$ in view of Theorem 2.3.4 *(iii)* and Corollary 2.3.8.

Consider the chief series

$$\eta' : 1 = G_0 \cap V_p^G \leq G_1 \cap V_p^G \leq \ldots \leq G_m \cap V_p^G = V_p^G =$$

$$V_p^G G_0 \leq V_p^G G_1 \leq \ldots \leq V_p^G G_m = G$$

of G through V_p^G, where p is the prime dividing $|G_i/G_{i-1}|$, and let $H/K \in \Lambda$. If H/K is either an $\underline{\underline{H}}$-covered, complemented chief factor of G or a $\Phi\underline{\underline{H}}$-chief factor, then $H \cap V_p^G/K \cap V_p^G$ is non-trivial and complemented in G since in the first case a complement of H/K is also a complement of $H \cap V_p^G/K \cap V_p^G$ and in the second case $H \cap V_p^G/K \cap V_p^G$ is complemented in G, by definition. On the other hand, if H/K is partially $\underline{\underline{H}}$-complemented in G, then HV_p^G/KV_p^G is, by Corollary 3.1.7, non-trivial and complemented in G.

Thus, since H/K is G-isomorphic to $H \cap V_p^G/K \cap V_p^G$ if V_p^G covers H/K and H/K is G-isomorphic to HV_p^G/KV_p^G otherwise, it follows, in view of Theorem 2.3.4 *(iii)* and Corollary 2.3.8, that the number of complemented chief factors of G in η' which are G-isomorphic to G_i/G_{i-1} is $\alpha + \beta + \delta$. Consequently, by Satz 4.1 of Gaschütz [10], $\alpha + \beta + \delta = \alpha + \gamma + \delta$, whence $\beta = \gamma$.

Since G_i/G_{i-1} was an arbitrary chief factor in η , the theorem is finally proved. //

(3.2.3) Corollary. *Given any chief series of* G *, there is a one-one correspondence between the* $\underline{\underline{F}}$*-eccentric* $\hat{\underline{\underline{H}}}$*-chief factors and the* $\underline{\underline{F}}$*-eccentric* $\Phi\underline{\underline{H}}$*-chief factors in the series, corresponding chief factors being* G*-isomorphic.* //

3.3 $\underline{\underline{FH}}\Phi$-subgroups

In this section, we establish the permutability in G of V and W , and hence that of V and W* , and describe the covering/avoidance properties of VW , VW* , $V \cap W$ and $V \cap W^*$, respectively.

First of all, we show:

(3.3.1) Theorem. $W \cap V$ *covers the* $\underline{\underline{H}}$*-Frattini chief factors and the* $\underline{\underline{H}}$*-covered,* $\underline{\underline{F}}$*-central chief factors of* G *, and avoids the rest.*

In order to prove Theorem 3.3.1, we need the following lemma.

(3.3.2) Lemma. $W \cap V \cap \Sigma_p$ *is a Sylow* p*-subgroup of both* $W \cap V$ *and* $V_p^G \cap W$.

Proof. Since Σ reduces into W according to Theorem 2.2.12 and into V according to our assumption, it follows, by Lemma 2.1.7, that Σ reduces into $V \cap W$. Consequently, $W \cap V \cap \Sigma_p$ is a Sylow p-subgroup of $V \cap W$, as required. Moreover, since Σ also reduces into V_p^G, we have, by the same lemma, that Σ reduces into $W \cap V_p^G$, whence $W \cap V_p^G \cap \Sigma_p$ is a Sylow p-subgroup of $W \cap V_p^G$. But, by our assumption and by Theorem 2.3.7, $V_p^G \cap \Sigma_p = V \cap \Sigma_p$. Thus, finally, $W \cap V \cap \Sigma_p = W \cap V_p^G \cap \Sigma_p$ is a Sylow p-subgroup of $V_p^G \cap W$, and the lemma is proved. //

We can now proceed with the proof of Theorem 3.3.1.

Proof of Theorem 3.3.1. Let H/K be a p-chief factor of G for some p. Suppose first that H/K is either $\underline{\underline{H}}$-covered, $\underline{\underline{F}}$-central or $\underline{\underline{H}}$-Frattini in G. Then, since H/K is $\underline{\underline{H}}$-covered in either case, $H \cap V_p^G/K \cap V_p^G$ is, by Corollary 2.3.8, non-trivial. Thus $H \cap V_p^G/K \cap V_p^G$ is covered by W since, in the first case, it is $\underline{\underline{F}}$-central in G, being G-isomorphic to H/K, and, in the second case, it is Frattini in G, by definition. Consequently,
$H = K\left(H \cap V_p^G\right) = K\left(K \cap V_p^G\right)\left(H \cap V_p^G \cap W\right) = K\left(H \cap V_p^G \cap W\right)$, and so H/K is covered by $V_p^G \cap W$. By Lemma 3.3.2, it is then covered by $W \cap V$, as required.

Suppose next that H/K is an $\underline{\underline{F}}$-eccentric $\Phi\underline{\underline{H}}$-chief factor of G. Then, since H/K is $\underline{\underline{H}}$-covered in particular, once again $H \cap V_p^G/K \cap V_p^G$ is, by Corollary 2.3.8, non-trivial. But this time $H \cap V_p^G/K \cap V_p^G$ is avoided by W since, in the first place, it is, by definition, complemented in G and, moreover, it is $\underline{\underline{F}}$-eccentric in G, being G-isomorphic to H/K. Thus $W \cap V_p^G$ avoids H/K and hence, by Lemma 3.3.2, $W \cap V$ avoids H/K, as required.

Since p was an arbitrary prime dividing $|G|$, and since $W \cap V$ avoids the $\underline{\underline{H}}$-avoided chief factors of G and, in view of Theorem 2.2.13 *(i)*, also the $\underline{\underline{F}}$-eccentric, complemented chief factors of G, the theorem is finally proved. //

In particular, when $\underline{\underline{F}}$ is the trivial saturated formation $\{1\}$, $W = W^*$ by Theorem 2.2.14 *(i)*, and all chief factors of G are $\underline{\underline{F}}$-eccentric. Thus, putting $\underline{\underline{F}} = \{1\}$ in Theorem 3.3.1, we have

(3.3.3) Corollary. $W^* \cap V$ *covers the* $\underline{\underline{H}}$*-Frattini chief factors of* G *and avoids the rest.*

With the help of Theorems 3.2.2 and 3.3.1 we can now calculate $|VW|$ as follows:

Let α be the product of the orders of the $\underline{\underline{F}}$-eccentric, Frattini chief factors of G in a chief series η of G, and β the product of the orders of the $\underline{\underline{F}}$-central chief factors of G in η. Then, by Theorem 2.2.13 *(i)* and the order argument, $|W| = \alpha\beta$. Similarly, if γ is the product of the orders of the $\underline{\underline{F}}$-eccentric, $\underline{\underline{H}}$-Frattini chief factors of G in η and δ the product of the orders of the $\underline{\underline{H}}$-covered, $\underline{\underline{F}}$-central chief factors of G in η, then, by Theorem 3.3.1 and the order argument, $|V \cap W| = \delta\gamma$. Furthermore, if λ is the product of the orders of the $\underline{\underline{F}}$-eccentric, $\underline{\underline{H}}$-avoided, Frattini chief factors of G in η, and ψ the product of the orders of the $\underline{\underline{F}}$-eccentric, $\Phi\underline{\underline{H}}$-chief factors of G in η, then $\alpha/\gamma = \lambda\psi$. Thus, $|VW| = |V|.|W|/|V \cap W| = |V|.\beta\lambda\psi/\delta$. But β/δ is the product of the orders of the $\underline{\underline{F}}$-central, $\underline{\underline{H}}$-avoided chief factors of G in η, and, by Corollary 3.2.3, ψ is that of the $\underline{\underline{F}}$-eccentric, $\hat{\underline{\underline{H}}}$-chief factors of G. Hence, clearly, we have shown

(3.3.4) Lemma. $|VW|$ *is the product of the orders of those chief factors in a chief series of* G *which are not* $\underline{\underline{F}}$-*eccentric, partially* $\underline{\underline{H}}$-*complemented in* G. //

(3.3.5) Definition. The $\underline{\underline{FH}}\Phi$-subgroup ($\underline{\underline{H}}\Phi$-subgroup) of G corresponding to Σ is defined to be the intersection of all those $\underline{\underline{F}}$-abnormal maximal (maximal) subgroups of G each of which contains an $\underline{\underline{H}}$-injector of G and into each of which Σ reduces.

We now show

(3.3.6) Theorem. *If* Z *is the* $\underline{\underline{FH}}\Phi$-*subgroup of* G *corresponding to* Σ, *then* $Z = VW$. *In particular,* V *and* W *permute in* G *and* VW *avoids the* $\underline{\underline{F}}$-*eccentric, partially* $\underline{\underline{H}}$-*complemented chief factors of* G *and covers the rest.*

Proof. From the definitions of Z and W, we have that $W \leq Z$. Also, from the definition of Z and Lemma 2.3.5 we have that $V \leq Z$. Thus, it will be sufficient to show that Z avoids the $\underline{\underline{F}}$-eccentric, partially $\underline{\underline{H}}$-complemented chief factors of G since then, in view of Lemma 3.3.4, $|Z| \leq |VW|$, and, therefore, $Z = VW$.

Let H/K be an $\underline{\underline{F}}$-eccentric, partially $\underline{\underline{H}}$-complemented p-chief factor of G for some p. By Corollary 3.1.7, HV_p^G/KV_p^G is non-trivial and complemented in G. Let M be a complement of

HV_p^G/KV_p^G in G into which Σ reduces. By Theorem 2.2.6 *(i)*, M is $\underline{\underline{F}}$-abnormal in G since HV_p^G/KV_p^G is $\underline{\underline{F}}$-eccentric in G, being G-isomorphic to H/K. Also, by Corollary 3.1.5 and Lemma 2.3.5, $M \geq V$. Thus, by the definition of Z, $Z \leq M$. But then, since M complements H/K also, Z avoids H/K, as required. From the arbitrariness of p and H/K, it follows thus that Z avoids the $\underline{\underline{F}}$-eccentric, partially $\underline{\underline{H}}$-complemented chief factors of G and hence $Z = VW$. The rest of the ~~Lemma~~ theorem now follows, by Lemma 3.3.4. //

In the special case when $\underline{\underline{F}} = \{1\}$, the trivial saturated formation, the $\underline{\underline{FH}}\Phi$-subgroups of G coincide with the $\underline{\underline{H}}\Phi$-subgroups of G and $W^* = W$, by Theorem 2.2.14 *(i)*. Moreover, all chief factors of G are $\underline{\underline{F}}$-eccentric. Thus, putting $\underline{\underline{F}} = \{1\}$ in Theorem 3.3.6, we have

(3.3.7) Corollary. *W^*V is the $\underline{\underline{H}}\Phi$-subgroup of G corresponding to Σ. In particular, W^* and V permute in G and VW^* avoids the partially $\underline{\underline{H}}$-complemented chief factors of G and covers the rest.* //

CHAPTER 4

SOME PROPERTIES OF THE LATTICE $\underline{\underline{L}}(D, W^*, V)$

4.0 Introduction

In this chapter, we prove our main result of the first part of our thesis. This can be stated as follows. Let $\underline{\underline{L}}(D, W^*, V)$ be the lattice of subgroups of G generated by D, W^* and V.

(4.0.1) Theorem. *(i) The lattice* $\underline{\underline{L}}(D, W^*, V)$ *is distributive.*

(ii) Any two subgroups of G *in* $\underline{\underline{L}}(D, W^*, V)$ *are permutable in* G.

(iii) Each subgroup of G *in* $\underline{\underline{L}}(D, W^*, V)$ *has a covering/avoidance property with respect to the chief factors of* G.

(iv) Σ *reduces into each subgroup of* G *in* $\underline{\underline{L}}(D, W^*, V)$.

(v) If A *is a subgroup of* G *in* $\underline{\underline{L}}(D, W^*, V)$, *then the family* $\{A^\alpha \mid \alpha$ *is an automorphism of* $G\}$ *of subgroups constitutes a characteristic conjugacy class of subgroups of* G.

At the end of Section 4.2 we give an example of G in which $\underline{\underline{L}}(D, W^*, V)$ is a free distributive lattice on the three generators.

Throughout this chapter, $V_p = \Sigma_p \cap V$, $D_p = \Sigma_p \cap D$ and $W^*_p = \Sigma_p \cap W^*$ for each p. Since Σ reduces into V, V_p is a Sylow p-subgroup of V. Also, by Lemma 2.2.5 and by Theorem 2.2.12

with the trivial saturated formation $\{1\}$ in the role of $\underline{\underline{X}}$ in the theorem, D_p and W^*_p are Sylow p-subgroups of D and W^*, respectively, for each p.

4.1 The sublattice generated by D and V

In this section, we establish Theorem 4.0.1 for the sublattice of $\underline{\underline{L}}(D, W^*, V)$ generated by D and V, except for part *(v)* which will be dealt with in a single general step in the next section. Note that every 2-generator lattice is distributive, so we need not do anything about *(i)* at this stage.

The first four lemmas concern the subgroup $\langle V_p, D_p\rangle$ of G for each p.

(4.1.1) Lemma. $\langle V_p, D_p\rangle = V_pD_p$.

Proof. By Theorem 2.3.7, V_p is a Sylow p-subgroup of a normal subgroup of G. Thus, by 1.1 of Rose [32], V_p is pronormal in G, and hence, by 1.2 of Rose [32], also in Σ_p. Since V_p is, moreover, subnormal in Σ_p, it follows by 1.5 of Rose [32], that $V_p \trianglelefteq \Sigma_p$. Hence, finally, $V_pD_p = D_pV_p = \langle V_p, D_p\rangle$, as required. //

(4.1.2) Lemma. *Let* V_p *be a Sylow* p-*subgroup of a normal subgroup* N *of* G. *Then* V_pD_p *is a Sylow* p-*subgroup of* ND.

Proof. Since $V_pD_p \leq \Sigma_p$, V_pD_p is a p-subgroup of ND. Let

P be a Sylow p-subgroup of ND which contains V_pD_p. Clearly $ND_p/N = NP/N$. Thus, by the modular law, $P = D_p(N \cap P)$. But $N \cap P \geq V_p$ and V_p is a Sylow p-subgroup of N. Hence $N \cap P = V_p$ and so $V_pD_p = P$, as required. //

(4.1.3) Lemma. *V_pD_p covers all those p-chief factors of G which are not simultaneously $\underline{\underline{F}}$-eccentric and $\underline{\underline{H}}$-avoided and avoids the rest. Consequently, $V_p \cap D_p$ covers the $\underline{\underline{F}}$-central, $\underline{\underline{H}}$-covered p-chief factors of G and avoids the rest.*

Proof. Clearly V_pD_p covers the $\underline{\underline{H}}$-covered p-chief factors of G since, by Definition 2.3.6, V_p does so. Also, by Theorem 2.2.6 *(vi)*, D_p, and hence V_pD_p covers the $\underline{\underline{F}}$-central p-chief factors of G. Let H/K be an $\underline{\underline{H}}$-avoided, $\underline{\underline{F}}$-eccentric p-chief factor of G. By Corollary 2.3.8, V_p^G avoids H/K. Thus HV_p^G/KV_p^G is a non-trivial p-chief factor of G, and hence $\underline{\underline{F}}$-eccentric in G, being G-isomorphic to H/K which is $\underline{\underline{F}}$-eccentric in G. By Theorem 2.2.6 *(v)* and *(vi)*, it follows now that $V_p^GD_p \cap V_p^GH = V_p^GD_p \cap V_p^GK$. Equivalently,
$V_p^G\left(H \cap V_p^GD_p\right) = V_p^G\left(K \cap V_p^GD_p\right)$, using the modular law. Thus, since, moreover,

$$V_p^G \cap \left(H \cap V_p^GD_p\right) = V_p^G \cap H = V_p^G \cap K = V_p^G \cap \left(K \cap V_p^GD_p\right)$$

and

$$H \cap V_p^GD_p \geq K \cap V_p^GD_p ,$$

we have that $H \cap V_p^G D_p = K \cap V_p^G D_p$. Therefore, $V_p^G D_p$, and hence $V_p D_p$, avoids H/K . Since H/K was an arbitrary $\underline{\underline{H}}$-avoided, $\underline{\underline{F}}$-eccentric p-chief factor of G , we have shown that $V_p D_p$ has the required covering/avoidance property.

A simple order argument now shows that $|V_p \cap D_p|$ is the product of the orders of the $\underline{\underline{F}}$-central, $\underline{\underline{H}}$-covered p-chief factors of G in any chief series of G . Since, in view of Theorems 2.2.6 *(vi)* and 2.3.4 *(iii)* and Definition 2.3.6, $V_p \cap D_p$ avoids all those chief factors of G which are not simultaneously $\underline{\underline{F}}$-central and $\underline{\underline{H}}$-covered, it follows finally that $V_p \cap D_p$, too, has the required covering/avoidance property and we are done. //

(4.1.4) Lemma. *$V_p D_p$ is a Sylow p-subgroup of* $\langle V, D \rangle = A$.

Proof. We proceed by induction on $|G|$. Let $J/V_p^G = O_{p'}\left(G/V_p^G\right)$ and $I/V_p^G = O_{p'p}\left(G/V_p^G\right)$. By Theorem 2.3.7, V_p is a Sylow p-subgroup of V_p^G and hence that of J too. Assume first of all that every p-chief factor of G/V_p^G is $\underline{\underline{F}}$-central in G . Then clearly $G/I \in \underline{\underline{F}}(p) \le \underline{\underline{F}}$, where $\{\underline{\underline{F}}(p)\}$ is the integrated and full family of formations which defines $\underline{\underline{F}}$ locally. In particular, $G/J \in \underline{\underline{F}}$, so that $G = JD$ by Theorem 2.2.6 *(vi)*. But now it follows, by Lemma 4.1.2, that $V_p D_p$ is a Sylow p-subgroup of G and hence certainly that of A , as required.

Thus, assume next that not every p-chief factor of G/V_p^G is $\underline{\underline{F}}$-central in G/V_p^G and let T/V_p^G be the intersection of all maximal subgroups of G/V_p^G of index a power of p. Clearly, T/V_p^G is a characteristic subgroup of G/V_p^G, and, by Theorem 2.5 of Carter and Hawkes [4], some minimal normal p-subgroup S/T of G/T is an $\underline{\underline{F}}$-eccentric chief factor of G. Since T/J is the Frattini subgroup of G/J (*cf.* last sentence on page 179 in Carter and Hawkes [4]), S/T is also complemented in G. Let M be a complement of S/T in G into which Σ reduces and let D^* be the $\underline{\underline{F}}$-normalizer of M corresponding to $\Sigma \cap M$. By Theorem 2.2.6 *(i)*, M is an $\underline{\underline{F}}$-abnormal maximal subgroup of G, and so, by Theorem 2.2.6 *(iii)*, $D \leq D^*$. Moreover, since M/J supplements the Fitting subgroup I/J of G/J, M/J is, by definition, an $\underline{\underline{F}}$-critical maximal subgroup of G/J. Hence, by Theorems 2.2.6 *(iv)* and *(v)*, $DJ = D^*J$.

Now, let $D_p^* = \Sigma_p \cap D^* = (\Sigma_p \cap M) \cap D^*$. D_p^* is, by Lemma 2.2.5, a Sylow p-subgroup of D^*. Also, $D_p = \Sigma_p \cap D \leq \Sigma_p \cap D^* = D_p^*$. On the other hand, by Corollary 3.1.6 and Lemma 2.3.5, $V \leq M$, so that V is, by Theorem 2.3.4 *(iv)*, an $\underline{\underline{H}}$-injector of M. Hence, by the induction hypothesis, $D_p^*V_p$ is a Sylow p-subgroup of $\langle D^*, V\rangle$. But $D_p^*V_p = D_pV_p$ since, by Lemma 4.1.2, D_pV_p is a Sylow p-subgroup of $DJ = D^*J$, and, moreover, $D_p^*V_p \geq D_pV_p$. Thus, finally, D_pV_p is a Sylow p-subgroup of $\langle D^*, V\rangle$, and hence certainly that of A, as required. //

Since p was an arbitrary prime dividing $|G|$ in the preceding four lemmas, it follows from Lemma 4.1.4 that

$$|\langle V, D\rangle| = \prod_{p \mid |G|} |V_p D_p| = |V|.|D| / \prod_{p \mid |G|} |V_p \cap D_p| .$$

But $|V_p \cap D_p| = |(V \cap D)_p|$ since, by Lemma 4.1.3, $|V_p \cap D_p|$ is the product α of the orders of the $\underline{\underline{F}}$-central, $\underline{\underline{H}}$-covered p-chief factors in any given chief series of G, while on account of the avoidance properties of V and D, $|(V \cap D)_p| \leq \alpha$. Hence,

$$|\langle V, D\rangle| \leq |V|.|D| / \prod_{p \mid |G|} |(V \cap D)_p| = |VD| ,$$

and we have shown that

(4.1.5) <u>Theorem</u>. $DV = VD$.

It follows now from Lemmas 4.1.3 and 4.1.4 that

(4.1.6) <u>Theorem</u>. DV *avoids the* $\underline{\underline{F}}$*-eccentric,* $\underline{\underline{H}}$*-avoided chief factors of* G *and covers the rest, while* $V \cap D$ *covers the* $\underline{\underline{F}}$*-central,* $\underline{\underline{H}}$*-covered chief factors of* G *and avoids the rest.* //

Finally, we show

(4.1.7) <u>Theorem</u>. Σ *reduces into* DV *and into* $D \cap V$.

Proof. By Lemma 4.1.4, $\Sigma_p \cap DV$ is a Sylow p-subgroup of DV for each p. Let π be any set of primes. Clearly $\langle \Sigma_p \cap DV \mid p \in \pi\rangle \leq \Sigma_\pi \cap DV$, and so $\Sigma_\pi \cap DV$ is a Hall π-subgroup of DV. It remains to note that Σ reduces into $D \cap V$ because of Lemma 2.1.7.

4.2 Distributivity of $\underline{\underline{L}}(D, W^*, V)$

We begin this section by describing the covering/avoidance property of $V \cap W^* \cap D$.

(4.2.1) Theorem. *$V \cap W^* \cap D$ covers the $\underline{\underline{F}}$-central, $\underline{\underline{H}}$-Frattini chief factors of G and avoids the rest.*

We need the following lemma in order to prove this result.

(4.2.2) Lemma. *Σ reduces into both $V \cap W^* \cap D$ and $V_p^G \cap W^* \cap D$. Moreover, $\Sigma_p \cap (V \cap W^* \cap D)$ is a Sylow p-subgroup of $V_p^G \cap W^* \cap D$ as well as that of $V \cap W^* \cap D$, for each p .*

Proof. By Theorem 2.2.12 with $\{1\}$, the trivial saturated formation, in the role of $\underline{\underline{X}}$ in the theorem, Σ reduces into W^* , and, by Lemma 2.2.5, Σ reduces into D . Also, by our assumption, Σ reduces into V , and, since $V_p^G \trianglelefteq G$, Σ reduces into V_p^G . Thus, it follows, by a repeated application of Lemma 2.1.7, that Σ reduces into both $V \cap W^* \cap D$ and $V_p^G \cap S^* \cap D$. Consequently, $V_p \cap D_p \cap W_p^*$ is a Sylow p-subgroup of $V \cap W^* \cap D$ and $\left(\Sigma_p \cap V_p^G\right) \cap W_p^* \cap D_p$ is a Sylow p-subgroup of $V_p^G \cap D \cap W^*$, for each p . But, by Theorem 2.3.7, $\Sigma_p \cap V_p^G = V_p$; hence the result. //

We can now prove Theorem 4.2.1 as follows.

Proof of Theorem 4.2.1. In view of Theorems 2.2.6 *(vi)*, 2.2.13 *(i)* and 2.3.4 *(iii)*, $V \cap W^* \cap D$ clearly avoids the complemented, the $\underline{\underline{H}}$-avoided and the $\underline{\underline{F}}$-eccentric chief factors of G. Let H/K be a Frattini, $\underline{\underline{H}}$-covered, $\underline{\underline{F}}$-central p-chief factor of G for some p dividing $|G|$ and assume first that H/K is $\underline{\underline{H}}$-Frattini in G. Then, by our assumption and Definition 3.2.1 (i), $H \cap V_p^G/K \cap V_p^G$ is a non-trivial Frattini p-chief factor of G. Moreover, $H \cap V_p^G/K \cap V_p^G$ is $\underline{\underline{F}}$-central in G, being G-isomorphic to H/K, which is $\underline{\underline{F}}$-central in G. Thus, by Theorem 2.2.14 *(ii)*, $W^* \cap D$ covers $H \cap V_p^G/K \cap V_p^G$. But now

$$\left(H \cap V_p^G \cap W^* \cap D\right)K = \left(H \cap V_p^G \cap W^* \cap D\right)\left(K \cap V_p^G\right)K = \left(H \cap V_p^G\right)K = H$$

since H/K is $\underline{\underline{H}}$-covered in G and hence, by Corollary 2.3.8, is covered by V_p^G. Hence $V_p^G \cap W^* \cap D$ covers H/K. By Lemma 4.2.2, it follows then that $V \cap W^* \cap D$, too, covers H/K.

Assume next that H/K is a $\Phi\underline{\underline{H}}$-chief factor of G. Then, by Definition 3.2.1 (ii), $H \cap V_p^G/K \cap V_p^G$ is complemented in G. Thus, by Theorem 2.2.14 *(ii)*, $H \cap V_p^g/K \cap V_p^G$ is avoided by $W^* \cap D$. In particular, $V_p^G \cap W^* \cap D$ avoids H/K. Hence, once again by Lemma 4.2.2, H/K is avoided by $V \cap W^* \cap D$. The proof is complete. //

Having proved Theorem 4.2.1, we proceed to show

(4.2.3) Lemma. *$V \cap W^*D = (V \cap W^*)(V \cap D)$ and $VD \cap VW^* = V(D \cap W^*)$.*

Proof. By Theorem 2.2.14 *(i)*, $W^*D = W$. Thus, by Theorems 3.3.1, 4.1.6 and 4.2.1, and Corollary 3.3.3,

$$|V \cap W^*D| = |V \cap W^*||V \cap D|/|V \cap W^* \cap D| = |(V \cap W^*)(V \cap D)| .$$

Moreover, $\langle V \cap W^*, V \cap D\rangle \leq V \cap W^*D$. Thus, the first of the two distributive equalities of the theorem clearly holds. The second distributive equality of the theorem is now a consequence of Lemma 2.5.3. //

The following theorem describes the covering/avoidance property of $VD \cap VW^*$.

(4.2.4) Theorem. *$VD \cap VW^*$ covers (i) the $\underline{\underline{H}}$-covered, (ii) the $\underline{\underline{F}}$-central, Frattini and (iii) the $\underline{\underline{F}}$-central $\hat{\underline{\underline{H}}}$-chief factors of G , and avoids the rest.*

Proof. By Theorems 4.1.5 and 2.2.14 *(i)*, we have

$$(VD)(VW^*) = V(DW^*) = VW .$$

Thus, by Theorem 3.3.6, $(VD)(VW^*)$ avoids the $\underline{\underline{F}}$-eccentric, partially $\underline{\underline{H}}$-complemented chief factors of G . In particular, in view of Theorem 4.1.6 and Corollary 3.3.7, it avoids all chief factors of G which are avoided simultaneously by VD and VW^* . Hence, by Lemma 2.5.4, $VD \cap VW^*$ covers all those chief factors of G which are covered simultaneously by VD and VW^* , and avoids the rest. The former are, according to Theorem 4.1.6 and Corollary 3.3.7, precisely the chief factors *(i)*, *(ii)* and *(iii)* mentioned in the theorem. The proof is complete. //

The next theorem describes the covering/avoidance property of $D \cap W*V$.

(4.2.5) <u>Theorem</u>. $D \cap W*V$ *covers (i) the* $\underline{\underline{F}}$*-central,* $\underline{\underline{H}}$*-covered, (ii) the* $\underline{\underline{F}}$*-central, Frattini and (iii) the* $\underline{\underline{F}}$*-central,* $\hat{\underline{\underline{H}}}$*-chief factors of* G *, and avoids the rest.*

Proof. By Theorem 2.2.14 *(i)*, $D(W*V) = WV$. Thus, by Theorem 3.3.6, $D(W*V)$ avoids the $\underline{\underline{F}}$-eccentric, partially $\underline{\underline{H}}$-complemented chief factors of G . These are, in view of Theorem 2.2.6 *(vi)* and Corollary 3.3.7, precisely those chief factors of G which are avoided simultaneously by D and $W*V$. Hence, by Lemma 2.5.4, it follows now that $D \cap W*V$ covers all those chief factors of G which are covered simultaneously by D and $W*V$, and avoids the rest. It is easy to see that the former are, according to Theorem 2.2.6 *(vi)* and Corollary 3.3.7, precisely the chief factors *(i)*, *(ii)* and *(iii)* mentioned in the theorem. This remark completes the proof. //

As an easy consequence, we have

(4.2.6) <u>Lemma</u>. $D \cap VW* = (D \cap V)(D \cap W*)$ *and* $DV \cap DW* = D(V \cap W*)$.

Proof. Let α be the product of the orders of the $\underline{\underline{F}}$-central $\hat{\underline{\underline{H}}}$-chief factors of G in a chief series η of G and let β be the product of the orders of the $\underline{\underline{F}}$-central, $\underline{\underline{H}}$-covered, Frattini chief factors of G in η . Then, by Theorems 2.2.14 *(ii)*, 4.1.6 and 4.2.5,

$$|D \cap VW^*| = (\alpha/\beta).|D \cap V|.|D \cap W^*| .$$

But, by Theorem 3.2.2, α is the product of the orders of the $\underline{\underline{F}}$-central $\Phi\underline{\underline{H}}$-chief factors of G in η . Thus, by Theorem 4.2.1, $\alpha/\beta = 1/|V \cap W^* \cap D|$, and so, $|D \cap VW^*| = |(D \cap V)(D \cap W^*)|$. Since, moreover, $\langle D \cap V, D \cap W^*\rangle \leq D \cap VW^*$, the first distributive equality of the theorem is proved. The second distributive equality of the theorem is now a consequence of Lemma 2.5.3. //

The following theorem describes the covering/avoidance property of $D(V \cap W^*)$.

(4.2.7) Theorem. $D(V \cap W^*)$ *covers (i) the* $\underline{\underline{F}}$*-central and (ii) the* $\underline{\underline{H}}$*-Frattini chief factors of* G *and avoids the rest.*

Proof. In view of Theorems 2.2.6 *(vi)* and 4.2.1, and Corollary 3.3.3, $D \cap (V \cap W^*)$ covers all those chief factors of G which are covered simultaneously by D and $V \cap W^*$. Thus, by Lemma 2.5.4, and Theorems 2.2.6 *(vi)* and 4.2.1, $D(V \cap W^*)$ has the required covering/avoidance property. //

We now show

(4.2.8) Lemma. $W^* \cap DV = (W^* \cap D)(W^* \cap V)$ *and* $W^*D \cap W^*V = W^*(D \cap V)$.

Proof. Let α be the product of the orders of the $\underline{\underline{F}}$-central, $\underline{\underline{H}}$-avoided, Frattini chief factors of G in a chief series η of G , β the product of the orders of the $\underline{\underline{F}}$-eccentric $\underline{\underline{H}}$-chief factors of G in η , γ the product of the orders of the $\underline{\underline{H}}$-covered, Frattini chief factors of G in η, δ the product of the orders of the $\underline{\underline{F}}$-central, $\underline{\underline{H}}$-covered, Frattini chief factors of G in η and ψ the product of the orders of the $\underline{\underline{F}}$-central $\Phi\underline{\underline{H}}$-chief factors of G in η. By Theorem 2.2.14 *(i)*, $W^*D = W$. Thus, by Theorem 3.3.6,

$$|W^* \cap DV| = \alpha\gamma/\beta .$$

But, by Corollary 3.2.3, β is the product of the orders of the $\underline{\underline{F}}$-eccentric, $\Phi\underline{\underline{H}}$-chief factors of G in η, so that, by Corollary 3.3.3,

$$\gamma/\beta = \psi.|W^* \cap V| .$$

Also, by Theorem 2.2.14 *(ii)*, $\alpha = |W^* \cap D|/\delta$ and, by Theorem 4.2.1, $\psi/\delta = 1/|W^* \cap D \cap V|$. Hence, finally,

$$|W^* \cap DV| = |(W^* \cap D)(W^* \cap V)| ,$$

and so, the first distributive equality of the theorem holds. The second distributive equality of the theorem is now a consequence of Lemma 2.5.3. //

The following two theorems describe the covering/avoidance properties of $W^* \cap DV$ and $W^*D \cap W^*V$, respectively.

(4.2.9) Theorem. $W^* \cap DV$ *covers (i) the* $\underline{\underline{F}}$*-central, Frattini and (ii) the* $\underline{\underline{H}}$*-Frattini chief factors of* G*, and avoids the rest.*

Proof. By Theorem 2.2.14 *(ii)*, $W^* \cap D$ covers the $\underline{\underline{F}}$-central, Frattini chief factors of G and avoids the rest, while, by Corollary 3.3.3, $W^* \cap V$ covers the $\underline{\underline{H}}$-Frattini chief factors of G and avoids the rest. Thus, by Theorem 4.2.1, $W^* \cap D \cap V$ covers all those chief factors of G which are covered simultaneously by $W^* \cap D$ and $W^* \cap V$. It follows now, by Lemmas 2.5.4 and 4.2.8, that $(W^* \cap D)(W^* \cap V) = W^* \cap DV$ has the required covering/avoidance property and the proof is complete. //

(4.2.10) Theorem. $W^*D \cap W^*V$ *covers (i) the Frattini, (ii) the* $\underline{\underline{F}}$*-central,* $\underline{\underline{H}}$*-covered and (iii) the* $\underline{\underline{F}}$*-central* $\hat{\underline{\underline{H}}}$*-chief factors of* G , *and avoids the rest.*

Proof. By Corollary 3.3.7, W^*V avoids the partially $\underline{\underline{H}}$-complemented chief factors of G and covers the rest. On the other hand, by Theorem 2.2.13 *(i)*, W^*D avoids the $\underline{\underline{F}}$-eccentric, complemented chief factors of G and covers the rest. But, by Theorem 2.2.14 *(i)*, $(W^*D)(W^*V) = W^*DV = WV$, so that, by Theorem 3.3.6, $(W^*D)(W^*V)$ avoids the $\underline{\underline{F}}$-eccentric, partially $\underline{\underline{H}}$-complemented chief factors of G which are precisely those chief factors of G which are avoided simultaneously by W^*D and W^*V . Thus, by Lemma 2.5.4, it follows now that $W^*D \cap W^*V$ covers all those chief factors of G which are covered simultaneously by W^*D and W^*V , and avoids the rest. Since the former are precisely the chief factors *(i)*, *(ii)* and *(iii)* of the lemma, the theorem is clearly proved. //

Next we show

(4.2.11) Theorem. $DV \cap DW^* \cap VW^* = (D \cap V)(D \cap W^*)(V \cap W^*)$ *covers (i) the* $\underline{\underline{F}}$*-central,* $\underline{\underline{H}}$*-covered, (ii) the* $\underline{\underline{F}}$*-central, Frattini, (iii) the* $\underline{\underline{F}}$*-central* $\hat{\underline{\underline{H}}}$*-chief factors and (iv) the* $\underline{\underline{H}}$*-Frattini chief factors of* G *and avoids the rest.*

Proof. By Lemma 4.2.6 and the modular law,

$$DV \cap DW^* \cap VW^* = D(V \cap W^*) \cap VW^* = (D \cap VW^*)(V \cap W^*)$$
$$= (D \cap V)(D \cap W^*)(V \cap W^*) .$$

Now, in view of Corollary 3.3.3 and Theorems 4.2.5 and 4.2.1, $(V \cap W^*) \cap (D \cap VW^*) = V \cap W^* \cap D$ covers all those chief factors of G which are covered simultaneously by $V \cap W^*$ and $D \cap VW^*$. Thus, by Lemma 2.5.4, $DV \cap DW^* \cap VW^*$ has the required covering/avoidance property. The proof is complete. //

In order to complete the proof of Theorem 4.0.1 we need to refer to the following elementary and probably well-known result which was brought to ~~our~~ attention by L.G. Kovács and which ~~we~~ quote here without proof. my, I

(4.2.12) Theorem (L.G. Kovács). *If in a lattice* $\underline{L}$ *generated by three elements* x, y *and* z *all the seven relations*

$$(x \cup y) \cap z = (x \cap z) \cup (y \cap z)$$
$$(y \cup z) \cap x = (y \cap x) \cup (z \cap x)$$
$$(x \cup z) \cap y = (x \cap y) \cup (z \cap y)$$
$$(x \cap y) \cup z = (x \cup z) \cap (y \cup z)$$
$$(y \cap z) \cup x = (y \cup x) \cap (z \cup x)$$
$$(x \cap z) \cup y = (x \cup y) \cap (z \cup y)$$
$$(x \cup y) \cap (y \cup z) \cap (x \cup z) = (x \cap y)(y \cap z)(x \cap z)$$ ∪, ∪

hold, then $\underline{L}$ *is distributive.*

It is an easy consequence of Theorem 4.2.12 that

(4.2.13) <u>Corollary</u>. *(i) Every element of* $\underline{\underline{L}}$ *can be expressed as the meet of suitable elements of the join-semilattice generated by* x, y *and* z .

(ii) Every element of $\underline{\underline{L}}$ *can be expressed as the join of suitable elements of the meet-semilattice generated by* x, y *and* z .

(iii) $\underline{\underline{L}}$ *has at most eighteen elements.*

In view of Lemmas 4.2.3, 4.2.6 and 4.2.8 and Theorem 4.2.11, Theorem 4.0.1 *(i)* now follows from Theorem 4.2.12. On the other hand, since, by Theorems 2.2.14 *(i)* and 4.1.5, Lemmas 4.2.3, 4.2.6 and 4.2.8 and Corollary 3.3.7, the elements of the meet-semilattice generated by D, W* and V are pairwise permutable, Theorem 4.0.1 *(ii)* follows from Corollary 4.2.13 *(ii)*.

Similarly, part *(iv)* of Theorem 4.0.1 follows from Corollary 4.2.13 *(i)* and Lemma 2.1.7, since, by Lemma 2.2.5, Theorems 2.2.12 and 4.1.7 and our assumption concerning Σ and V , Σ reduces into D, W*, DW*, DV and V , respectively, and also, by Definition 3.3.5, Theorem 3.3.6, Corollary 3.3.7 and Lemma 2.1.7, Σ reduces into VW = VDW* and into VW* .

Next, we deduce part *(iii)* of Theorem 4.0.1 from the preceding information. In view of Corollary 4.2.13, the following list of the subgroups in $\underline{\underline{L}}$(D, W*, V) is complete (although some subgroups may occur repeatedly). We set against each subgroup in $\underline{\underline{L}}$(D, W*, V) the reference number of the result which establishes (and specifies) its covering/avoidance property.

D	Theorem 2.2.6 *(vi)*.
W*	Theorem 2.2.13 *(i)* with $\underline{\underline{X}} = \{1\}$, the trivial saturated formation in the theorem.
V	Theorem 2.3.4 *(iii)*.
D ∩ W*	Theorem 2.2.14 *(ii)*.
D ∩ V	Theorem 4.1.6.
W* ∩ V	Corollary 3.3.3.
D ∩ W* ∩ V	Theorem 4.2.1.
DW*	Theorem 2.2.13 *(i)*.
DV	Theorem 4.1.6.
VW*	Corollary 3.3.7.
DW*V	Theorem 3.3.6.
D ∩ W*V	Theorem 4.2.5.
W* ∩ DV	Theorem 4.2.9.
V ∩ DW*	Theorem 3.3.1.
DW* ∩ DV	Theorem 4.2.7.
VW* ∩ DW*	Theorem 4.2.10.
DV ∩ VW*	Theorem 4.2.4.
DW* ∩ DV ∩ VW*	Theorem 4.2.11.

Finally, since the stabilizer B of Σ in the group A of automorphisms of G also stabilizes each of D, W* and V , B stabilizes every element in the lattice $\underline{\underline{L}}$(D, W*, V) . Moreover, in view of Theorem 2.1.4, B supplements in A the group of inner automorphisms of G . Hence the statement of part *(v)* of Theorem 4.0.1 is proved. //

We conclude this section with an example G in which $\underline{\underline{L}}$(D, W*, V) has eighteen distinct elements and includes neither {1} nor G .

Let H be the semidirect product of a cyclic group $\langle a \rangle$ of order 25 by a cyclic group $\langle b \rangle$ of order 4, with the action of $\langle b \rangle$ on $\langle a \rangle$ given by $a^b = a^7$.

Now, let $K = H \text{ wr } \langle c \rangle$, where $\langle c \rangle$ is a cyclic group of order 5 and let $G = \langle d \rangle \times K$, the direct product of K and a cyclic group $\langle d \rangle$ of order 4.

It is easy to verify that for $\underline{\underline{F}} = \underline{\underline{N}} = \underline{\underline{H}}$, the lattice $\underline{\underline{L}}(D, W^*, V)$ corresponding to G has eighteen distinct elements and so is a free distributive lattice on the three generators.

CHAPTER 5

THE FITTING LENGTH OF A GROUP AND THE NUMBER OF CONJUGACY CLASSES OF ITS MAXIMAL NILPOTENT SUBGROUPS

In this chapter we continue the investigation begun in [27] of a relation between the Fitting length of a group and the number of conjugacy classes of its maximal nilpotent subgroups. The main result of [27], which is due to H. Lausch, shows that the Fitting length $h(G)$ of a group G of odd order is bounded above in terms of the number $\nu(G)$ of conjugacy classes of its maximal nilpotent subgroups. In Section 5.2 we establish this result without the restriction on the group order and with a much better bound than that obtained in [27]. The precise form of the bound we obtain (though not its order of magnitude) relies on an unpublished result of M.F. Newman (see Theorem 5.2.6). Another unpublished result of his (see Theorem 5.3.13) is essential for our purposes in Section 5.3, where we obtain the best possible bounds on the Fitting length of a group G with $\nu(G)$ small.

In Section 5.1 we prove two preparatory results the first one of which provides a basis for induction argument throughout the rest of the thesis, and in the last section we obtain a lower estimate for a general upper bound.

Throughout this Chapter and the next, for a group G and a saturated formation $\underline{\underline{F}}$, $\nu_{\underline{\underline{F}}}(G)$ denotes the number of conjugacy classes of $\underline{\underline{F}}$-maximal subgroups of G, and $\nu(G) = \nu_{\underline{\underline{N}}}(G)$.

5.1 Two preliminary results

The following lemma is a straightforward generalization of Lemma 1 of [27] due to H. Lausch.

(5.1.1) Lemma. *Let* $\underline{\underline{F}}$ *be a saturated formation,* G *any group and* $N \trianglelefteq G$. *Then every* $\underline{\underline{F}}$*-maximal subgroup of* G/N *is the image in* G/N *of a suitable* $\underline{\underline{F}}$*-maximal subgroup of* G . *In particular,* $\nu_{\underline{\underline{F}}}(G/N) \leq \nu_{\underline{\underline{F}}}(G)$. *Moreover, if* $\nu_{\underline{\underline{F}}}(G/N) = \nu_{\underline{\underline{F}}}(G)$ *then the image in* G/N *of every* $\underline{\underline{F}}$*-maximal subgroup of* G *is an* $\underline{\underline{F}}$*-maximal subgroup of* G/N .

Proof. Let W/N be an $\underline{\underline{F}}$-maximal subgroup of G/N . Since $\underline{\underline{F}}$ is a saturated formation, W has an $\underline{\underline{F}}$-projector V , say (see Section 2.2). Also, since $W/N \in \underline{\underline{F}}$, W = VN . Let V* be an $\underline{\underline{F}}$-maximal subgroup of G which contains V . Clearly $W/N = NV/N \leq NV^*/N \in \underline{\underline{F}}$. Thus, since W/N is $\underline{\underline{F}}$-maximal in G/N , it follows that NV*/N = W/N . In particular, since V is $\underline{\underline{F}}$-maximal in W , V* = V . The rest of the lemma now follows easily. //

Next, we give a slight extension of Lemma 6 of [27] which was due to H. Lausch.

(5.1.2) Lemma. *Let* G *be a monolithic group with its Fitting subgroup as its monolith. If* h(G) > 1 , *there is a normal subgroup* S *of* G *such that* h(G/S) = h(G) - 1 *and the Fitting subgroup* R/S *of* G/S *is the monolith of* G/S . *Moreover, if* V *is a maximal nilpotent subgroup of* G *such that* $VF \geq G_{\underline{\underline{N}}^2}$, *then* $VS/S \geq R/S$.

Proof. Let $\underline{\underline{F}}$ be the class of all groups of Fitting length at most $h(G) - 2$, $H = G^{\underline{\underline{F}}}$ and H/K a chief factor of G. Since $\underline{\underline{F}}$ is a saturated formation, H/K is clearly complemented in G. Let M be a complement of H/K in G, $R = C_G(H/K)$ and $S = R \cap M$. By a well known result (see for example Satz 3.1 of Gaschütz [10]), $R = C_G(R/S)$. Thus, clearly R/S is the monolith of G/S. By Lemma 2.2 of Carter, Fischer and Hawkes [5], it follows now that R/S is also the Fitting subgroup of G/S. Also $h(G/S) < h(G) - 1$ would mean that $h(G/S) \leq h(G) - 2$, and so $h(G/H \cap S) \leq h(G) - 2$. Since $H \cap S = K$, it would then follow that $h(G/K) \leq h(G) - 2$, contrary to $H = G^{\underline{\underline{F}}}$. Hence $h(G/S) = h(G) - 1$.

Next, let V be a maximal nilpotent subgroup of G such that $VF \geq G_{\underline{\underline{N}}^2}$. Since F is the monolith and $H \neq F$, we have that $F \leq K < H \leq G_{\underline{\underline{N}}^2}$, and so V clearly covers H/K. Thus, as $K \leq S$, $VS = (VK)S \geq HS = R$, as required. //

5.2 The Main Result

In this section we will prove (as Theorem 5.2.8) the main result of this chapter. We begin with the following elementary lemma.

(5.2.1) Lemma. *Let* G *be a group whose Fitting subgroup* F *is a* p-*group and let* Q *be a* q-*subgroup of* G, $q \neq p$. *Then there is a maximal nilpotent subgroup* W *of* G *such that* $C_F(Q) = W \cap F$. *Moreover, if* $Q \neq \{1\}$, *then* $W \cap F < F$.

Proof. Let W be a maximal nilpotent subgroup of G which contains $C_F(Q) \times Q$. Clearly $W \cap F = C_F(Q)$. Also, since $C_G(F) \leq F$, $F \nleq W$ unless $Q = \{1\}$. Thus $W \cap F < F$ if $Q \neq \{1\}$, and so we are done. //

Next we prove a lemma.

(5.2.2) Lemma. *Let* G *be a group whose Fitting subgroup* F *is an elementary abelian* p*-group, let* Q *be a non-trivial* q*-subgroup of* G *,* $q \neq p$ *, and let* $\overline{Q}$ *be a maximal element in the set*

$$\underline{\underline{T}} = \{Q^* \triangleleft Q \mid C_F(Q) < C_F(Q^*)\}$$

which is ordered by inclusion. Then $C_Q\{C_F(\overline{Q})\} = \overline{Q}$ *and, moreover,* $Z(Q/\overline{Q})$ *is cyclic.*

Proof. We regard F as a vector space over $GF(p)$, the field with p elements. Then, it follows from Theorem 2.6.1 and Lemma 2.6.2 in Gorenstein [14], that $C_F(\overline{Q})$ is Q-invariant, whence $H = C_Q\{C_F(\overline{Q})\} \triangleleft Q$. However, $H \geq \overline{Q}$ and also $C_F(H) \geq C_F(\overline{Q}) > C_F(Q)$. Thus $\overline{Q} = H$ since $\overline{Q}$ is a maximal element in $\underline{\underline{T}}$, and so $C_Q\{C_F(\overline{Q})\} = \overline{Q}$, as required.

In order to show that $Z(Q/\overline{Q})$ is cyclic, we proceed as follows. Since $C_F(\overline{Q})$ is Q-invariant, we observe first that, by Theorem 3.3.2 in Gorenstein [14], $C_F(\overline{Q}) = C_F(Q) \times L$, where L is Q-invariant. Clearly $L \neq \{1\}$, since $C_F(\overline{Q}) > C_F(Q)$. Next, let L^* be a non-trivial Q-invariant subgroup of L of minimal order and let $K = C_Q(L^*)$. Since L^* is Q-invariant, $K \triangleleft Q$. Moreover, $K \leq C_Q\{L^* C_F(Q)\}$. In particular, $C_F(K) > C_F(Q)$. However, $K \geq \overline{Q}$.

Hence, since $\overline{Q}$ is a maximal element of $\underline{\underline{T}}$, $K = \overline{Q}$, and so $Q/\overline{Q}$ is represented faithfully and irreducibly on L^*. By Theorem 3.2.2 in Gorenstein [14], it follows then that $Z(Q/\overline{Q})$ is cyclic, and the proof is complete. //

Part *(i)* of our next result occured as Lemma 3 in [27] and was due to H. Lausch.

(5.2.3) Lemma. *Let* G, F *and* Q *be as in Lemma* 5.2.2 *and let* ℓ *be the largest integer for which there exists a chain of subgroups*

$$(5.2.4) \qquad C_F(Q) = V_\ell \cap F < V_{\ell-1} \cap F < \dots < V_1 \cap F < F ,$$

where V_i *is a maximal nilpotent subgroup of* G *for* $i = 1, 2, \dots, \ell$ *and* $V_\ell \geq C_F(Q) \times Q$.

(i) If Q *is abelian,* $d(Q) \leq \ell$.

(ii) If $Q/Z(Q)$ *is elementary abelian,* $d(Q) \leq 2\ell$.

Proof. We proceed by induction on $|Q|$. Let R be a maximal element in the set $\underline{\underline{T}}$ of Lemma 5.2.2. Then, by the same lemma, $Z(Q/R) = \overline{Z}/R$ is cyclic. Also, by hypothesis, a chain of subgroups of the type (5.2.4) which joins $C_F(R)$ to F has length at most $\ell - 1$ since $C_F(R) > C_F(Q)$ and, by Lemma 5.2.1, there is a maximal nilpotent subgroup W of G such that $W \cap F = C_F(R)$.

Suppose first of all that Q is abelian. Then clearly $\overline{Z} = Q$, and so Q/R is cyclic. Moreover, by the induction hypothesis, $d(R) \leq \ell - 1$. Thus $d(Q) \leq \ell$ and hence *(i)* is proved.

Suppose next that $Q/Z(Q)$ is elementary abelian. If $Q = \overline{Z}$, then once again Q/R is cyclic, and, moreover, by induction, $d(R) \le 2(\ell-1)$, whence $d(Q) \le 2\ell - 1 \le 2\ell$ and we are done. Hence assume $\overline{Z} \ne Q$. In particular, Q/R is non-abelian. Let A/R be a maximal abelian normal subgroup of Q/R. Since $Q/Z(Q)$, and therefore $Q/\overline{Z}$, is elementary abelian, it follows from Satz III. 13.7 in Huppert [25] that there is a maximal abelian normal subgroup B/R of Q/R such that $AB = Q$, $A \cap B = \overline{Z}$ and $D(A/\overline{Z}) = d(B/\overline{Z})$; consequently, $d(Q/R) \le 2d(A/R)$. It remains now to show that $d(A/R) \le \ell$ and that a chain of subgroups of the type (5.2.4) which joins $C_F(R)$ to F has length at most $\ell - d(A/R)$, for, then, by the inductive hypothesis, $d(R) \le 2\ell - 2d(A/R) \le 2\ell - d(Q/R)$ and hence $d(Q) \le d(R) + d(Q/R) \le 2\ell$. We show this as follows.

Let $A = A_0$ and, for $i = 1, 2, \dots$, define A_i to be a maximal element in the set

$$\{Q^* \le A_{i-1} \mid Q^* \ge R \text{ and } C_F(A_{i-1}) < C_F(Q^*)\} .$$

For some integer $n \ge 1$, $A_n = R$. Let $\overline{G}$ be the semidirect product of $X = C_F(R)$ by $Y = A/R$. Since, by Lemma 5.2.2, $C_Y(X) = \{1\}$, X is clearly the Fitting subgroup of $\overline{G}$. Thus, by the same lemma and the fact that A/R is abelian, it follows now that A_{i-1}/A_i is cyclic for $i = 1, \dots, n$. In particular, $d(A/R) \le n$. On the other hand, by Lemma 5.2.1, there exist maximal nilpotent subgroups $W_0, W_1, \dots, W_n$ of G such that

$$C_F(A_0) = W_0 \cap F < C_F(A_1) = W_1 \cap F < \dots < C_F(A_n) = C_F(R) = W_n \cap F .$$

Hence, by hypothesis, $d(A/R) \leq n \leq \ell$. Also, the chain of subgroups of the type (5.2.4) joining $C_F(R)$ to F has length certainly at most $\ell - n \leq \ell - d(A/R)$. This completes the proof. //

(5.2.5) Remark. In lemma 5.2.3 we have have $\ell \leq \nu(G) - 1$, since each member of at least one conjugacy class of maximal nilpotent subgroups of G contains F , and, trivially, if V and W are conjugate maximal nilpotent subgroups of G , neither $V \cap F < W \cap F$ nor $W \cap F < V \cap F$.

It has been well known for some time that the Fitting length of a soluble linear group is bounded in terms of its degree. The best possible bound has been obtained in recent unpublished work of M.F. Newman:

(5.2.6) Theorem (M.F. Newman). *Let* G *be a soluble linear group of degree* $n \geq 1$. *Then*

$$h(G) \leq \begin{cases} 1 & , \textit{if } n = 1 ; \\ 3 & , \textit{if } n = 2 ; \\ 2s+4 & , \textit{if } 2.3^s < n \leq 4.3^s ; \\ 2s+5 & , \textit{if } 4.3^s < n \leq 2.3^{s+1} . \end{cases}$$

In particular, $h(G) \leq 2 \log_3 \left(\frac{9n-1}{2}\right)$. //

We will deduce the main result of this section from the following lemma.

(5.2.7) Lemma. *Let* G *be a group whose Fitting subgroup* F *is an elementary abelian* p-*group. If* H/K *is a* q-*chief factor of* G , *where* $q \neq p$, *then*

$$h\left(G/C_G(H/K)\right) \leq 2 \log_3 \left(\frac{18\nu(G)-19}{2}\right) .$$

Proof. Let Q be a Sylow q-subgroup of H and $N = N_G(Q)$. Then, by the Frattini argument, $G = NH$, and hence $G/C_G(H/K) \cong N/C_N(H/K) = N/C_N(Q/Q \cap K)$. Thus, clearly it will be sufficient to show that $h\left(N/C_N(Q/Q \cap K)\right) \leq 2 \log_3 \left(\frac{18\nu(G)-19}{2}\right)$.

Let C^* be a characteristic subgroup of Q given by Lemma 8.2 of Feit and Thompson [6]. Then, by the same lemma, C^* has, among others, the following two properties of interest to us:

(i) $C^*/Z(C^*)$ is elementary abelian;

(ii) every non-trivial q'-automorphism of Q induces a non-trivial automorphism of C^* .

In particular, by (ii), $C_N(C^*)/C_N(Q)$ is a q-group. Also, since C^* is characteristic subgroup of $Q \trianglelefteq N$, we have C^* , and hence $C_N(C^*)$, is a normal subgroup of N . Now, by Theorem 3.1.3 in Gorenstein [14], $G/C_G(H/K)$, and hence $N/C_N(Q/Q \cap K)$, has no non-trivial normal q-subgroups. Thus, since the normal subgroup $C_N(C^*)C_N(Q/Q \cap K)/C_N(Q/Q \cap K)$ of $N/C_N(Q/Q \cap K)$ is isomorphic to $C_N(C^*)/C_N(C^*) \cap C_N(Q/Q \cap K)$ which is a factor group of the q-group $C_N(C^*)/C_N(Q)$, it must be trivial, and so $C_N(C^*) \leq C_N(Q/Q \cap K)$. For similar reasons, $C_N\left(C^*/\Phi(C^*)\right) \leq C_N(Q/Q \cap K)$ since, by a result of

Burnside (see Theorem 5.1.4 in Gorenstein [14], for example), $C_N\{C^*/\Phi(C^*)\}/C_N(C^*)$ is a q-group, and moreover, $C_N\{C^*/\Phi(C^*)\} \trianglelefteq N$ as $\Phi(C^*) \triangleleft N$, being a characteristic subgroup of $C^* \triangleleft N$.

Thus, it suffices to show that $h\left[N/C_N\{C^*/\Phi(C^*)\}\right]$ is below the upper bound claimed. However, $N/C_N\{C^*/\Phi(C^*)\}$ is a soluble linear group of degree at most the dimension of $C^*/\Phi(C^*)$, regarded as a vector space over $GF(q)$, and the latter is, in view of *(i)*, Lemma 5.2.3 and Remark 5.2.5, at most $2\{\nu(G)-1\}$. Thus, by the preceding result of M.F. Newman, namely Theorem 5.2.6,
$h\left[N/C_N\{C^*/\Phi(C^*)\}\right] \le 2 \log_3 \left\{\frac{18\nu(G)-19}{2}\right\}$, and so the proof is complete. //

We now prove the main result of this chapter.

(5.2.8) <u>Theorem</u>. *For any group* G *,*

$$h(G) \le \begin{cases} 1 & , \text{ if } \nu(G) = 1 , \\ 2\left\{1 + \log_3 \left[\frac{18\nu(G)-19}{2}\right]\right\} & , \text{ if } \nu(G) > 1 . \end{cases}$$

Proof. Since each Sylow subgroup of G is contained in some maximal nilpotent subgroup of G, it is immediate that $h(G) \le 1$ when $\nu(G) = 1$. For a proof by contradiction let G be a counterexample of minimal order. Then $\nu(G) > 1$, and in view of Lemma 5.1.1 and the fact that $\underline{\underline{N}}^k$, where $k = 2\left\{1 + \log_3 \left[\frac{18\nu(G)-19}{2}\right]\right\}$, is a saturated formation, G is a monolithic group with its Fitting subgroup F as it monolith. In particular, F is an elementary abelian p-group for some p. Since $O_{p'p}(G)$ is the intersection of the centralizers of the q-chief factors of G , Lemma 5.2.7 gives that

$$h\left(G/ \bigcap_{q\neq p} O_{q'q}(G)\right) \leq 2 \log_3 \left(\frac{18\nu(G)-19}{2}\right) = k - 2 .$$

As $\bigcap_{q\neq p} O_{q'q}(G)$ is q-nilpotent for every q other than p , it is an extension of a p-group by a nilpotent group. Thus $h(G) \leq k$ and G is not a counterexample after all. This contradiction completes the proof. //

We conclude this section with the remark that the bound of Theorem 5.2.8 is, at least for certain values of $\nu(G)$, not the best possible as will be clear from the following section. However, it is certainly an improvement of the bound obtained in [27] for groups of odd order.

5.3 <u>Some special cases</u>

Here, in this section, we will obtain sharp bounds on the Fitting length of a group G for $\nu(G) = 2, 3$ respectively. The bound in the case when $\nu(G) = 2$ (see the following proposition) is due to H. Lausch.

(5.3.1) <u>Proposition</u>. *For* $\nu(G) = 2$, $h(G) \leq 3$.

Proof. We proceed by induction on $|G|$. Thus, in view of Lemma 5.1.1 with $\underline{\underline{F}} = \underline{\underline{N}}$ in the lemma, Theorem 5.2.8 and the fact that $\underline{\underline{N}}^3$ is a saturated formation, we can assume that G is monolithic with its Fitting subgroup F as its monolith and $\nu(G/F) = 2$.

Let $|F| = p^{\alpha}$, $\alpha > 0$. Then, since $C_G(F) = F$, a Sylow p-subgroup P of G is clearly a maximal nilpotent subgroup of G. Moreover, since $\nu(G/F) = \nu(G)$, it follows, by Lemma 5.1.1, that the conjugates of P/F are maximal nilpotent subgroups of G/F. On the other hand, it is clear that $G_{\underline{\underline{N}}^2}/F$ is a p'-group.

Now, in view of Lemma 5.2.3 *(i)*, Remark 5.2.5 and our hypothesis, every abelian p'-subgroup of G is cyclic. Therefore, if $2 \,\Big|\, |G_{\underline{\underline{N}}^2}/F|$, then $p \neq 2$, $G_{\underline{\underline{N}}^2}/F$ has a unique element of order 2, and this element must be central in G/F. This contradicts the fact that P/F is a maximal nilpotent subgroup of G/F. Thus, $G_{\underline{\underline{N}}^2}/F$ is a nilpotent group of odd order with all its abelian subgroups cyclic. Hence, it follows that all Sylow subgroups of $G_{\underline{\underline{N}}^2}/F$ are cyclic (see for instance Theorem 5.4.10 (ii) of Gorenstein [14]), whence $G_{\underline{\underline{N}}^2}/F$ itself is cyclic. Finally, the facts that $G_{\underline{\underline{N}}^2} \geq C_G(G_{\underline{\underline{N}}^2}/F)$ and that the automorphism groups of cyclic groups are abelian (see for instance Theorem 1.3.10 (i) of Gorenstein [14]) imply that $G/G_{\underline{\underline{N}}^2}$ is abelian, and so $h(G) \leq 3$, as required. //

The bound obtained in Proposition 5.3.1 is certainly the best possible. The symmetric group S_4 on four letters provides an example of groups in which the number of conjugacy classes of maximal nilpotent subgroups is two and whose Fitting length is three.

Next, we consider the case when $\nu(G) = 3$ and show that

(5.3.2) <u>Proposition</u>. *For* $\nu(G) = 3$, $h(G) \leq 4$.

Proof. Suppose the result is false and let G be a minimal counter-example. Then, in view of Lemma 5.1.1 with $\underline{\underline{F}} = \underline{\underline{N}}$ in the lemma, Proposition 5.3.1 and the fact that $\underline{\underline{N}}^4$ is a saturated formation, it follows that

(5.3.3) G is monolithic with its Fitting subgroup F, say, as its monolith, $\nu(G/F) = 3 = \nu(G)$ and $h(G) = 5$.

Let $|F| = p^\alpha$, $\alpha > 0$. Then, as in the proof of Proposition 5.3.1, a Sylow p-subgroup P of G is a maximal nilpotent subgroup of G. Let V and W be representatives of the remaining two conjugacy classes of maximal nilpotent subgroups of G, respectively, and assume without loss of generality that $VF \geq G_{\underline{\underline{N}}^2}$. Then, since $G_{\underline{\underline{N}}^2}/F$ is a p'-group and $C_G(G_{\underline{\underline{N}}^2}/F) \leq G_{\underline{\underline{N}}^2}$, it follows that

(5.3.4) $$VF/F \text{ is a } p'\text{-group.}$$

Consequently,

(5.3.5) $$V \cap F = \{1\}.$$

For, assume to the contrary that $V \cap F > \{1\}$. Since F is an abelian p-group (see (5.3.3)) and $V/V \cap F$ is a p'-group (see (5.3.4)), $V \cap F \leq Z(G_{\underline{\underline{N}}^2})$, so that by our assumption, $Z(G_{\underline{\underline{N}}^2}) > \{1\}$. But $Z(G_{\underline{\underline{N}}^2}) \trianglelefteq G$, being a characteristic subgroup of a normal subgroup, namely $G_{\underline{\underline{N}}^2}$, of G. Therefore, since F is the monolith of G (see (5.3.3)), $Z(G_{\underline{\underline{N}}^2}) \geq F$. However, since $C_G(F) = F$, this can only happen if $F = G$. Thus, since $h(G) = 5$ (see (5.3.3)), we must have $V \cap F = \{1\}$.

Now, since G is monolithic with its Fitting subgroup as its monolith and $h(G) > 1$ {see (5.3.3)}, G has, by Lemma 5.1.2, a normal subgroup S such that $h(G/S) = h(G) - 1$ and the Fitting subgroup R/S of G/S is the monolith of G/S. Clearly $h(G/S) > 3$, since otherwise $h(G) \leq h(G/S) + 1 \leq 4$, contrary to G being a minimal counter-example. Thus, it follows from (5.3.3) that

(5.3.6) $$h(G/R) = 3 \text{ and } h(G/S) = 4 .$$

In particular,

(5.3.7) $$\nu(G/S) = \nu(G) = 3 .$$

For, if $\nu(G/S) \neq \nu(G)$, then, by Lemma 5.1.1, $\nu(G/S) \leq 2$ and hence, by Proposition 5.3.1, $h(G/S) \leq 3$, contrary to (5.3.6).

Let $|R/S| = q^{\beta}$, $\beta > 0$. Since, by Lemma 5.1.2, $VS/S \geq R/S$, it follows from (5.3.4) that

(5.3.8) $$q \neq p .$$

Also,

(5.3.9) $$q \neq 2 .$$

For, suppose to the contrary that $q = 2$. Then, the proof of Lemma 5.2.7 shows that $G/C_G(R/S) = G/R$ is a factor of $GL(4, GF(2))$. Since $|GL(4, GF(2))| = 2^6 . 3^2 . 5 . 7$, it follows, therefore, that G/R is a soluble group of order dividing $2^6 . 3^2 . 5 . 7$. Thus, in view of Theorem 1.3.10 (ii) of Gorenstein [14], the group of automorphisms induced by G on a 5-chief factor or a 7-chief factor of G/R is cyclic, and clearly that induced on a 3-chief factor of G/R is a 2-group. In particular, the group of automorphisms induced by G on each odd-ordered chief factor of G/R is nilpotent. But then, since $\underline{\underline{N}}$ is a formation and, by Theorem 3.1.3 of Gorenstein [14], G/R has no non-trivial normal 2-subgroups so that $(G/R)_{\underline{\underline{N}}}$ is the intersection of the centralizers of the odd-ordered chief factors of G/R (see proof of Theorem 5.2.8), we have $G/R \in \underline{\underline{N}}^2$. However, this is impossible because of (5.3.6), and so we conclude that q cannot be 2.

But, in view of (5.3.3), Lemma 5.2.3 *(i)* and Remark 5.2.5,

(5.3.10) G has no elementary abelian r-subgroups of order r^3, for each $r \neq p$,

so that, by a result of Thompson, namely Lemma 5.24 in [37], every odd-ordered r-chief factor of G, for each $r \neq p$, is of rank at most 2. Thus, since G/R is represented faithfully and irreducibly on R/S and $h(G/R) = 3$ $\{$see (5.3.6)$\}$, we have, by Theorem 3.2.5 of Gorenstein [14], that

(5.3.11) $|R/S| = q^2$; consequently, G/R is isomorphic to a subgroup of $GL\{2, GF(q)\}$.

It follows then that

(5.3.12) $$q \neq 3 .$$

For, otherwise, $G/R \cong GL\{2, GF(3)\}$ since every proper subgroup of $GL\{2, GF(3)\}$ has Fitting length at most, while $h(G/R) = 3$ $\{$see (5.3.6)$\}$. But then, contrary to (5.3.7), $\nu(G/S) \geq 4$, the Sylow 2-subgroups, the Sylow 3-subgroups and the two distinct conjugacy classes of 6-cycles of G/S constituting four distinct conjugacy classes of maximal nilpotent subgroups of G/S. Hence, q cannot be 3 .

In what follows, we need the following consequence, which we state without proof, of a deep result (unpublished) of M.F. Newman on the structure of soluble subgroups of $GL(2, K)$, where K is an arbitrary field.

(5.3.13) Theorem (M.F. Newman). *Let* K *be an arbitrary field. If* H *is a soluble subgroup of* $GL(2, K)$ *of Fitting length greater than* 2 *, then* $H/Z(H) \cong S_4$ *and* 2 *divides* $|Z(H)|$.

Now, let Z/R be the centre of G/R . Since G/R is isomorphic to a subgroup of $GL\left(2, GF(q)\right)$ of Fitting length three (see (5.3.6) and (5.3.11)), it follows from Theorem 5.3.13 that

(5.3.14) $G/Z \cong S_4$ and 2 divides $|Z/R|$. Moreover, if $K \triangleleft G$ such that $K \leq R$ and G/K is isomorphic to a subgroup of $GL\left(2, GF(q)\right)$, then Z/K is the centre of G/K .

Consequently, in view of (5.3.9), (5.3.12) and the fact that G/R has no non-trivial normal q-subgroups (see for instance Theorem 3.1.3 of Gorenstein [14]), we have

(5.3.15) $q \nmid |G/R|$, and so R/S is a maximal nilpotent subgroup of G/S .

But,

(5.3.16) V contains a Sylow q-subgroup of G .

For, otherwise, W contains one, since a Sylow q-subgroup of G is contained in some maximal nilpotent subgroup of G . Since, by Lemma

5.1.2, $VS/S \geq R/S$, it follows then that $R/S \leq WS/S \cap VS/S$. Thus, since $C_{G/S}(R/S) = R/S$, both WS/S and VS/S are q-groups, and so $\nu(G/S) \leq 2$, contrary to (5.3.7).

Next, let $L = O_{q'}(G)$, N/L a minimal normal q-subgroup of G/L and $C = C_G(N/L)$. Since $C_{G/S}(R/S) = R/S$ and L avoids R/S , the latter because R/S is a q-group, we have $L \leq S$. On the other hand, since V contains a Sylow q-subgroup of G {see (5.3.16)}, and since the Fitting subgroup of G/L is a q-group and contains its own centralizer in G/L , it is clear that

(5.3.17) VL/L is a Sylow q-subgroup of G/L .

Thus, as G/Z is a q'-group {see (5.3.9), (5.3.12) and (5.3.14)}, it follows that

(5.3.18) $$VL \leq Z .$$

Suppose first that $WL \geq N$. Then clearly $(WL/L)_{q'} \leq C/L$, and so WC/C is a q-group. Hence, since, in view of (5.3.17), VC/C is a Sylow q-subgroup of G/C , $WC/C \leq V^gC/C$ for some g in G . But now, since $V \leq Z$ {see (5.3.18)}, we may conclude that $W \leq V^gC \leq ZC$, whence, by Lemma 5.1.1, G/ZC has only one class of maximal nilpotent subgroups, namely, that of PCZ/CZ . Thus, G/ZC is a p-group. Since $G/Z \cong S_4$ {see (5.3.14)}, ZC/Z must, therefore, contain a subgroup isomorphic to the alternating group A_4 on 4 letters. In particular, the Hall q'-subgroups of ZC/Z , and hence those of $C/Z \cap C$ and C/L , cannot be nilpotent. But then the nilpotent q'-subgroup $(WL/L)_{q'}$ of C/L cannot be a Hall q'-subgroup; that is, for some r $(\neq q)$, $(WL/L)_{q'}$ cannot contain any Sylow r-subgroup $(C/L)_r$ of C/L . However, $(C/L)_r \times N/L$ is a non-primary nilpotent subgroup of G/L . Since VL/L and PL/L are primary, $(C/L)_r \times N/L$ must, therefore, be contained in some conjugate of WL/L . But now, some conjugate of $(C/L)_r$ is in $(WL/L)_{q'}$, and we have a contradiction.

Hence, $WL \nleq N$. We claim that C/L is then a q-group. Suppose this is not so, and let T/L be a non-trivial Sylow r-subgroup of C/L for some $r \neq q$. Then $(N/L) \times (T/L)$ is a non-primary nilpotent subgroup of G/L , which, as before, must be contained in some conjugate of WL/L . But then $N/L \leq WL/L$, contrary to $WL \nleq N$.

Thus, it follows that $C \leq R$ (for, G/R is a q'-group (see (5.3.15)) and $L \leq R$), and so $|N/L| \geq q^2$, since the alternative $|N/L| = q$ implies that G/C , and hence G/R is cyclic (see Theorem 1.3.10 (ii) in Gorenstein [14]), contrary to (5.3.6).

Next we show that $|N/L| = q^2$ and $WL \cap N > L$. Let Q be a Sylow q-subgroup of N/L ; Q is then elementary abelian, and so, from what has been just shown and (5.3.10), $|Q| = |N/L| = q^2$. Moreover, as the proof of Lemma 5.2.3 *(i)* shows, Q has a non-trivial subgroup Q^* such that $C_F(Q^*) > C_F(Q) \geq \{1\}$. Since V is a p'-group (see (5.3.4) and (5.3.5)), it follows then that some conjugate of $C_F(Q^*) \times Q^*$ is contained in W , whence $WL \cap N > L$.

Now, since $|N/L| = q^2$, we have that G/C is isomorphic to a subgroup of $GL(2, GF(q))$. Thus, since $C \leq R$, it follows from (5.3.14) that Z/C is the centre of G/C . But $O_q(G/C) = \{1\}$ (see Theorem 3.1.3 in Gorenstein [14]). Therefore, Z/C is a q'-group.

Hence, from (5.3.9), (5.3.12) and (5.3.14), we get that G/C is a q'-group, and so in view of (5.3.17), $V \leq C$. Thus, the maximal nilpotent subgroups of G/C are just the conjugates of PC/C and WC/C, so that $WC/C \geq Z/C$. Since Z/C is a q'-group, in fact, $W_{q'}C/C \geq Z/C$. But $W_{q'}C$ acts trivially on $(WL \cap N)/L$; so Z, too, must act trivially on $WL \cap N/L$. Consequently, $\{1\} < (WL \cap N)/L \leq C_{N/L}(Z/L) \lhd G/L$, and hence $C_{N/L}(Z/L) = N/L$ as N/L is a chief factor. It follows thus that Z acts trivially on the whole of N/L, whence $Z = C$ and *a fortiori* $Z = R$, contrary to (5.3.14). The proof of Proposition 5.3.2 is now complete. //

Let H be the binary octahedral group, that is, the group defined on the generators a, b, c by the relations

$$a^2 = b^3 = c^4 = abc .$$

As is well-known, H has just one element z of order 2, the centre Z of H is generated by z and $H/Z \cong S_4$. Let M be a vector space over $GF(3)$ which affords the representation of H induced from a non-trivial one-dimensional representation of Z, and let N be any non-zero H-invariant subspace of M. It is easy to see that, for the split extension G of N by H, $\nu(G) = 3$ and $h(G) = 4$. (In fact N can be chosen to be of order 3^4, so that G is of order $2^4 . 3^5$.) Thus, the bound obtained in Proposition 5.3.1 is the best possible.

5.4 A range for the best possible bound

Our main aim in this section is to find as small as possible a range in which the best possible bound on the Fitting length of a group G in terms of $\nu(G)$ lies. First of all, we show

(5.4.1) <u>Theorem</u>. *Let* H *be a group of order* $p^{\alpha}q^{\beta}$ *, where* $\alpha, \beta > 0$ *, and let* $C = \langle c \mid c^p = 1 \rangle$ *be a cyclic group of order* p *. Let* G = H wr C *. If the Sylow subgroups of* H *are maximal nilpotent subgroups of* H *, then the Sylow subgroups of* G *are maximal nilpotent subgroups of* G *, and*

$$\nu(G) = \{\nu(H)^p + (2p-1)\nu(H) - p\}/p \ .$$

The hard part is, of course, to perform the required count of classes of maximal nilpotent subgroups of G ; we shall not interrupt that count to observe, as can be easily done, that the claim relating to the Sylow subgroups of G is verified.

For convenience we break up the proof of the theorem into a series of lemmas, but first we fix some notation.

For any integer n , let $\underline{n}$ denote the residue class of n modulo p . The base group B of G is a direct product of p copies of H : let the corresponding canonical projections be denoted by $\pi_{\underline{i}} : B \twoheadrightarrow H$ and let the corresponding insertions be denoted by $\mu_{\underline{i}} : H \rightarrowtail B$; thus, for each b in B ,

$$b = \prod_{i=0}^{p-1} b\pi_{\underline{i}}\mu_{\underline{i}}$$

and, for each h in H ,

$$h\mu_{\underline{i}}\pi_{\underline{j}} = \begin{cases} h \ , \text{ if } \underline{i} = \underline{j} \ ; \\ 1 \ , \text{ otherwise.} \end{cases}$$

The automorphism of B induced by c is then described by

$$b^c \pi_i = b\pi_{i-1}$$

for all b in B ; consequently,

$$(h\mu_i)^c = h\mu_{i+1}$$

for all h in H . Let D denote the "diagonal subgroup" $\left\{\prod_{i=0}^{p-1} h\mu_i \mid h \in H\right\}$ of B . Notice that $D = C_B(c)$. Finally, let V denote a maximal nilpotent subgroup of G . As before, V_p, V_q will denote the Sylow p- and the Sylow q-subgroups of V , respectively.

To begin with, we show

(5.4.2) Lemma. *(i) If* $V \leq B$ *, then* $V = \prod_{i=0}^{p-1} V\pi_i\mu_i$ *, where, for each* $i = 0, 1, \ldots, p-1$ *,* $V\pi_i$ *is a maximal nilpotent subgroup of* H *and not all the* $V\pi_i$ *are Sylow* p*-subgroups of* H *. Conversely, if, for each* $i = 0, 1, \ldots, p-1$ *,* V_i *is a maximal nilpotent subgroup of* H *and not all the* V_i *are Sylow* p*-subgroups of* H *, then* $\prod_{i=0}^{p-1} V_i\mu_i$ *is a maximal nilpotent subgroup of* G *which is contained in* B *.*

(ii) Let W *be some other maximal nilpotent subgroup of* G *contained in* B *. Then* V *and* W *are conjugate in* B *if and only if, for each* $i = 0, 1, \ldots, p-1$ *,* $W\pi_i$ *and* $V\pi_i$ *are conjugate in* H *.*

Proof. That $V = \prod_{i=0}^{p-1} V\pi_i\mu_i$, where, for each

$i = 0, 1, \ldots, p-1$, $V\pi_{\underline{i}}$ is a maximal nilpotent subgroup of H , is a direct consequence of Lemma 2.5.1 *(iii)*. Since V is a maximal nilpotent subgroup of G and since $|G : B| = p$, not all the $V\pi_{\underline{i}}$ are p-subgroups of H .

Next, let U be a maximal nilpotent subgroup of G which contains $\prod_{i=0}^{p-1} V_{\underline{i}}\mu_{\underline{i}}$ and consider the Sylow q-subgroup Q of the latter. By Lemma 2.5.1 *(iii)*, $Q = \prod_{i=0}^{p-1} Q\pi_{\underline{i}}\mu_{\underline{i}}$, and therefore, by Lemma 2.5.2, $C_G(Q) = C_B(Q)$. In particular, $U_p \leq B$, and so $U \leq B$. Also, for each $i = 0, 1, \ldots, p-1$, $U\pi_{\underline{i}} \geq V_{\underline{i}}$, and hence, by the maximality of $V_{\underline{i}}$, $U\pi_{\underline{i}} = V_{\underline{i}}$. Thus, finally, in view of the first part, $U = \prod_{i=0}^{p-1} U\pi_{\underline{i}}\mu_{\underline{i}} = \prod_{i=0}^{p-1} V_{\underline{i}}\mu_{\underline{i}}$ and *(i)* is proved.

(ii) If $V = W^b$ for some $b \in B$, then, by *(i)*,
$\prod_{i=0}^{p-1} V\pi_{\underline{i}}\mu_{\underline{i}} = \prod_{i=0}^{p-1} (W\pi_{\underline{i}}\mu_{\underline{i}})^{b\pi_{\underline{i}}\mu_{\underline{i}}}$, and so $V\pi_{\underline{i}} = (W\pi_{\underline{i}})^{b\pi_{\underline{i}}}$ for $i = 0, 1, \ldots, p-1$. Conversely, if there exist elements $h_0, h_1, \ldots, h_{p-1}$ of H such that $V\pi_{\underline{0}} = (W\pi_{\underline{0}})^{h_0}$, $V\pi_{\underline{1}} = (W\pi_{\underline{1}})^{h_1}, \ldots, V\pi_{\underline{p-1}} = (W\pi_{\underline{p-1}})^{h_{p-1}}$, then $V = W^b$ with $b = \prod_{i=0}^{p-1} h_i\mu_{\underline{i}}$. //

Next we show

(5.4.3) Lemma. *Assume* $V \leq B$ *and let* $\underline{\underline{C}}$ *be the class of conjugates of* V *in* B *and* $\tilde{\underline{\underline{C}}}$ *the class of conjugates of* V *in* G. *Then*

(i) for each $W \in \tilde{\underline{\underline{C}}}$, *there is a positive integer* $\ell \leq p$ *such that* W *is conjugate in* B *to* $V^{c^{\ell}}$;

(ii) $\underline{\underline{C}} = \tilde{\underline{\underline{C}}}$ *if and only if* $V\pi_{\underline{0}}, V\pi_{\underline{1}}, \ldots, V\pi_{\underline{p-1}}$ *are pairwise conjugate in* H.

If $\underline{\underline{C}} \subset \tilde{\underline{\underline{C}}}$, then $\tilde{\underline{\underline{C}}}$ is the (disjoint) union of the p classes of conjugates of $V, V^c, \ldots, V^{c^{p-1}}$, respectively, in B.

Proof. *(i)* Trivial.

(ii) Assume first of all that $\underline{\underline{C}} = \tilde{\underline{\underline{C}}}$ and consider V^c. By our assumption, $V^c = V^b$ for some $b \in B$. Then

$$V\pi_{\underline{i-1}} = (V^c)\pi_{\underline{i}} = (V^b)\pi_{\underline{i}} = (V\pi_{\underline{i}})^{b\pi_{\underline{i}}}$$

for each $i = 0, 1, \ldots, p-1$, and hence $V\pi_{\underline{0}}, \ldots, V\pi_{\underline{p-1}}$ are pairwise conjugate in H, as required.

Assume next that $V\pi_{\underline{0}}, \ldots, V\pi_{\underline{p-1}}$ are pairwise conjugate in H and consider V^g for any $g \in G$. Clearly $g = c^j b$ for some $b \in B$ and some j with $0 \leq j \leq p-1$. Thus, by Lemma 5.4.2 *(i)*, $V^g = V^{c^j b} = \left(\prod_{i=0}^{p-1} V\pi_{\underline{i}}\mu_{\underline{i+j}}\right)^b$. Now, in view of our assumption, there exist elements $h_0, h_1, \ldots, h_{p-1}$ such that $V\pi_{\underline{0}} = (V\pi_{\underline{j}})^{h_0}$,

$V\pi_{\underline{1}} = \left(V\pi_{\underline{1+j}}\right)^{h_1}, \ldots, V\pi_{\underline{p-1}} = \left(V\pi_{\underline{p+j-1}}\right)^{h_{p-1}}$. Let $d = \prod_{i=0}^{p-1} h_i\mu_{\underline{i+j}}$, so that $d\pi_{\underline{i+j}} = h_i$. Then, clearly

$$V^g = \left(\prod_{i=0}^{p-1} V\pi_{\underline{i}}\mu_{\underline{i+j}}\right)^b = \left(\prod_{i=0}^{p-1} \left[\left(V\pi_{\underline{i+j}}\right)^{h_i}\right]\mu_{\underline{i+j}}\right)^b =$$

$$\left(\prod_{i=0}^{p-1} (V^d)\pi_{\underline{i+j}}\mu_{\underline{i+j}}\right)^b = V^{db}$$

since, firstly, as i runs from 0 to $p-1$, $\underline{i+j}$ runs from $\underline{0}$ to $\underline{p-1}$, and, secondly, Lemma 5.4.2 *(i)* applies also to V^d in place of V . Since $db \in B$ and g was an arbitrary element of G , it follows thus that $\tilde{\underline{\underline{C}}} = \underline{\underline{C}}$, and so *(ii)* is proved.

Finally, assume that $\underline{\underline{C}} \subset \tilde{\underline{\underline{C}}}$ and consider V^{c^ℓ} and V^{c^k} , where $0 \le \ell$, $k \le p-1$ and $\ell \ne k$. We have to show that V^{c^ℓ} and V^{c^k} are not conjugate in B . Suppose to the contrary that $V^{c^\ell} = V^{c^k b}$ for some $b \in B$. Then, by Lemma 5.4.2 *(i)*,

$$\prod_{i=0}^{p-1} V\pi_{\underline{i}}\mu_{\underline{i+\ell}} = \left(\prod_{i=0}^{p-1} \left(V\pi_{\underline{i}}\mu_{\underline{i+k}}\right)\right)^b ,$$

and so $V\pi_{\underline{j-\ell}} = \left(V\pi_{\underline{j-k}}\right)^{b\pi_{\underline{j}}}$ for each $j = 0, 1, \ldots, p-1$. Thus, since $\ell \ne k$, it follows that $V\pi_{\underline{0}}, \ldots, V\pi_{\underline{p-1}}$ are pairwise conjugate in H , and hence, by *(ii)*, $\underline{\underline{C}} = \tilde{\underline{\underline{C}}}$, contrary to our assumption. With this contradiction the proof of the lemma is now complete. //

An application of the preceding lemma yields

(5.4.4) Lemma. *The number of conjugacy classes of maximal nilpotent subgroups of* G *which are contained in* B *is*

$$\{\nu(H)^p+\nu(H)(p-1)-p\}/p \ .$$

Proof. By Lemma 5.4.2 and the hypothesis of Theorem 5.4.1, the maximal nilpotent subgroups of G contained in B constitute $\nu(H)^p - 1$ conjugacy classes of maximal nilpotent subgroups of B , and hence, by Lemma 5.4.3, they constitute $\frac{(\nu(H)^p-1)-(\nu(H)-1)}{p} + \nu(H) - 1$ conjugacy classes of maximal nilpotent subgroups of G , as required. //

It now remains to show that the number of conjugacy classes of maximal nilpotent subgroups V of G which are not contained in B is $\nu(H)$. Throughout the rest of this section we assume that $V_p \nleq B$.

(5.4.5) Lemma. $V_p \cap B = \prod_{i=0}^{p-1} (V_p \cap B)\pi_{\underline{i}}\mu_{\underline{i}}$.

Proof. Let $P = \prod_{i=0}^{p-1} (V_p \cap B)\pi_{\underline{i}}\mu_{\underline{i}}$. Since, by Lemma 2.5.1 *(i)*, $C_B(V_q) = \prod_{i=0}^{p-1} \left[C_H(V_q\pi_{\underline{i}})\right]\mu_{\underline{i}}$, and since $V_p \cap B \leq C_B(V_q)$, we conclude that $P \leq C_B(V_q)$, and hence $\langle g, P\rangle \leq C_G(V_q)$ where $g \in V_p \backslash V_p \cap B$. However, by Lemma 2.5.1 *(ii)*, $g \in N_G(P)$ since $g \in N_G(V_p \cap B)$, and so $\langle g, P\rangle$ is a p-group. Also, $V_p \leq \langle g, P\rangle$. Since, by the maximality of V , V_p is a Sylow p-subgroup of $C_G(V_q)$, it follows finally that $\langle g, P\rangle = V_p$, and therefore $V_p \cap B = P$, as required. //

(5.4.6) Lemma. V_p *contains a complement of* B *in* G .

Proof. Since $V_p B = G$ and since G/B is of prime order and is generated by cB , we must have $V_p \cap cB \neq \emptyset$. Let $g \in V_p \cap cB$. Then $g = cb$ for some b in B . Now

$$g^p = b^{c^{(p-1)}+c^{(p-2)}+\ldots+c+1} \in V_p \cap B ,$$

and hence

$$g^p \pi_{\underline{0}} = b\pi_{\underline{1}} b\pi_{\underline{2}} \ldots b\pi_{\underline{p-1}} b\pi_{\underline{0}}$$

for each $i = 0, 1, \ldots, p-1$. Let $g_1 = g\left(g^{-p}\pi_{\underline{0}}\mu_{\underline{0}}\right)$. Then, on account of $g^p \in V_p \cap B$ and Lemma 5.4.5, $g_1 \in V_p$; also, $g_1 = cd$ where $d = b\left(g^{-p}\pi_{\underline{0}}\mu_{\underline{0}}\right) \in B$. Thus

$$\begin{aligned}\left(g_1^p\right)\pi_{\underline{0}} &= \left(d^{c^{(p-1)}+c^{(p-2)}+\ldots+c+1}\right)\pi_{\underline{0}} \\ &= d\pi_{\underline{1}} d\pi_{\underline{2}} \ldots d\pi_{\underline{p-1}} d\pi_{\underline{0}} \\ &= b\pi_{\underline{1}} b\pi_{\underline{2}} \ldots b\pi_{\underline{p-1}} b\pi_{\underline{0}} \left(b\pi_{\underline{1}} b\pi_{\underline{2}} \ldots b\pi_{\underline{p-1}} b\pi_{\underline{0}}\right)^{-1} \\ &= 1 ,\end{aligned}$$

since

$$d\pi_{\underline{i}} = \begin{cases} b\pi_{\underline{i}} , & \text{if } \underline{i} \neq \underline{0} , \\ b\pi_{\underline{0}}\left(g^p \pi_{\underline{0}}\right)^{-1} , & \text{if } \underline{i} = \underline{0} . \end{cases}$$

On the other hand, if $0 < i < p$, then $\left(g_1^p\right)^{c^{p-i}} = ag_1^pa^{-1}$, where $a = d^{c^{p-i-1}+c^{p-i-2}+\ldots+c+1} \in B$, so that

$$g_1^p\pi_{\underline{i}} = \left(\left(g_1^p\right)^{c^{p-i}}\right)\pi_{\underline{0}} = \left(ag_1^pa^{-1}\right)\pi_{\underline{0}} = \left(a\pi_{\underline{0}}\right)\left(g_1^p\pi_{\underline{0}}\right)\left(a\pi_{\underline{0}}\right)^{-1}$$
$$= 1$$

since $g_1^p\pi_{\underline{0}} = 1$. Therefore, $g_1^p = 1$. Since $g_1 \in V_p \backslash V_p \cap B$, the lemma is clearly proved. //

Since, by a result of C.H. Houghton (see the proof of Theorem 3.3 of Houghton [24], and also Theorem 10.1 of Neumann [31]) any two complements of B in G are conjugate in G, it follows, in view of Lemma 5.4.6, that

(5.4.7) Lemma. *Some conjugate of* V *in* G *contains* c. //

Now, for a maximal nilpotent subgroup W of G which contains c, define $W^* = (W \cap B)\pi_{\underline{0}}$. Note that $W_p^* = \left(W_p \cap B\right)\pi_{\underline{0}}$ and $W_q^* = W_q\pi_{\underline{0}}$.

(5.4.8) Lemma. *If* W *is a maximal nilpotent subgroup of* G *which contains* c, *then* $W_p \cap B = \prod_{i=0}^{p-1} W_p^*\mu_{\underline{i}}$.

Proof. Clearly $W_p \cap B \triangleleft W_p$. Thus, for each $i = 0, 1, \ldots, p-1$, $\left(W_p \cap B\right)^{c^i} = W_p \cap B$, and so

$$\left(W_p \cap B\right)\pi_{\underline{i}} = \left(\left(W_p \cap B\right)^{c^{p-i}}\right)\pi_{\underline{0}} = \left(W_p \cap B\right)\pi_{\underline{0}} = W_p^* .$$

An application of Lemma 5.4.5 then completes the proof. //

(5.4.9) Lemma. *Let* W *be a maximal nilpotent subgroup of* G *which contains* c . *Then* $W_q = D \cap \prod_{i=0}^{p-1} W_q^*\mu_{\underline{i}}$.

Proof. If $b \in W_q$, then, for each $i = 0, 1, \ldots, p-1$, $b^{c^i} = b$, and so $b\pi_{\underline{i}} = b^{c^{k-i}}\pi_{\underline{0}} = b\pi_{\underline{0}} \in W_q^*$. Thus clearly $W_q \leq D \cap \prod_{i=0}^{p-1} W_q^*\mu_{\underline{i}}$. On the other hand, it follows from Lemma 5.4.8, that $D \cap \prod_{i=0}^{p-1} W_q^*\mu_{\underline{i}}$ centralizes $\left(W_p \cap B\right)$, and hence also $\langle c, W_p \cap B\rangle = W_p$. Thus, by the maximality of W , $W_q = D \cap \prod_{i=0}^{p-1} W_q^*\mu_{\underline{i}}$, as required. //

Since $W_p = \langle c, W_p \cap B\rangle$, it is now immediate from Lemmas 5.4.8 and 5.4.9 that

(5.4.10) Lemma. *If* W *is a maximal nilpotent subgroup of* G *and* $c \in W$

$$W = \left\langle c, \prod_{i=0}^{p-1} W_p^*\mu_{\underline{i}}, D \cap \prod_{i=0}^{p-1} W_q^*\mu_{\underline{i}}\right\rangle . \quad //$$

(5.4.11) Lemma. *Let* U *be a maximal nilpotent subgroup of* H . *Then* $U = W^*$ *for some maximal nilpotent subgroup* W *of* G *which contains* c .

Proof. Let $P = \prod_{i=0}^{p-1} U_p\mu_{\underline{i}}$ and $Q = D \cap \prod_{i=0}^{p-1} U_q\mu_{\underline{i}}$. Clearly c normalizes P, so that $\langle c, P\rangle$ is a p-subgroup of G. On the other hand, it is clear that $\langle c, P\rangle$ centralizes Q, so that $\langle c, P, Q\rangle$ is a nilpotent subgroup of G.

Now, by the maximality of U, U_p is a Sylow p-subgroup of $C_H(U_q)$. Thus, by Lemma 2.5.1 *(i)* and *(iii)*, P is a Sylow p-subgroup of $C_B(Q)$. Hence, if W is a maximal nilpotent subgroup of G which contains $\langle c, P, Q\rangle$, then, since $P \le V_p \cap B \le C_B(Q)$, $P = V_p \cap B$. Since, moreover, $|W_p/W_p \cap B| = |W_pB/B| = p$ and $c \in W_p \backslash P$, it follows then that $W_p = \langle P, c\rangle$.

Next, consider $\prod_{i=0}^{p-1} U_q\mu_{\underline{i}}$. By Lemma 2.5.1 *(i)* and *(iii)* it is clearly a Sylow q-subgroup of $C_B(P)$. Thus, since W_q centralizes P, $W_q \le \left(\prod_{i=0}^{p-1} U_q\mu_{\underline{i}}\right)^b$ for some $b \in B$. Since W_q also centralizes c, we have, moreover, that $W_q \le D = C_B(c)$, and so

$W_q \le D \cap \left(\prod_{i=0}^{p-1} U_q\mu_{\underline{i}}\right)^b$. But, as $D \cong H$ and $\pi_{\underline{0}}$ is onto, it is clear that

$$\left|D \cap \left(\prod_{i=0}^{p-1} U_q\mu_{\underline{i}}\right)^b\right| = \left|\left(D \cap \left(\prod_{i=0}^{p-1} U_q\mu_{\underline{i}}\right)^b\right)\pi_{\underline{0}}\right| \le \left|U_q^{b\pi_{\underline{0}}}\right| = |U_q| = \left|D \cap \prod_{i=0}^{p-1} U_q\mu_{\underline{i}}\right| = |Q| .$$

Hence, since $Q \le W_q$, it follows now that $W_q = Q$, whence

$\langle c, P, Q\rangle = W$ is a maximal nilpotent subgroup of G which contains c. Since, as it can be easily checked, $W^* = U$, we are finally done. //

(5.4.12) <u>Lemma</u>. *If* W *is a maximal nilpotent subgroup of* G *which contains* c*, then* W^* *is a maximal nilpotent subgroup of* H.

Proof. Let U be a maximal nilpotent subgroup of H which contains W^*. By Lemma 5.4.11, there is a maximal nilpotent subgroup X of G which contains c and for which $U = X^*$. Since both X and W contain c, and since $W^*_p \leq U_p = X^*_p$ and $W^*_q \leq U_q = X^*_q$, it follows, by Lemma 5.4.10, that

$$W = \left\langle c, \prod_{i=0}^{p-1} W^*_p \mu_{\underline{i}}, D \cap \prod_{i=0}^{p-1} W^*_q \mu_{\underline{i}} \right\rangle$$

$$\leq \left\langle c, \prod_{i=0}^{p-1} U_p \mu_{\underline{i}}, D \cap \prod_{i=0}^{p-1} U_q \mu_{\underline{i}} \right\rangle$$

$$= X .$$

Hence $W = X$, and so $W^* = X^* = U$, as required. //

Finally, we show

(5.4.13) <u>Lemma</u>. *Let* U *and* W *be two maximal nilpotent subgroups of* G *both of which contain* c. *Then* U *and* W *are conjugate in* G *if and only if* U^* *and* W^* *are conjugate in* H.

Proof. Suppose first that $U^* = (W^*)^h$ for some $h \in H$ and let $d = \prod_{i=0}^{p-1} h\mu_{\underline{i}}$. Clearly $d \in D = C_B(c)$. Thus, using Lemma 5.4.10, we have

$$U = \left\langle c, \prod_{i=0}^{p-1} U_p^* \mu_{\underline{i}}, D \cap \prod_{i=0}^{p-1} U_q^* \mu_{\underline{i}} \right\rangle$$

$$= \left\langle c^d, \prod_{i=0}^{p-1} \left[(W_p^*)^h \right] \mu_{\underline{i}}, D \cap \prod_{i=0}^{p-1} \left[(W_q^*)^h \right] \mu_{\underline{i}} \right\rangle$$

$$= \left\langle c, \prod_{i=0}^{p-1} W_p^* \mu_{\underline{i}}, D \cap \prod_{i=0}^{p-1} W_q^* \mu_{\underline{i}} \right\rangle^d$$

$$= W^d .$$

Conversely, if $U = W^g$ for some $g \in G$, then, since $g = c^j b$ for some $b \in B$ and some j with $0 \leq j \leq p-1$, and since $c \in W$, we have $U = W^b$, and so

$$U^* = (U \cap B)\pi_{\underline{0}} = \left(B \cap W^b\right)\pi_{\underline{0}} = \left((B \cap W)^b\right)\pi_{\underline{0}}$$

$$= \left((B \cap W)\pi_{\underline{0}}\right)^{b\pi_{\underline{0}}}$$

$$= (W^*)^{b\pi_{\underline{0}}} .$$

The proof is complete. //

The remaining claim towards the proof of Theorem 5.4.1, namely, that the number of conjugacy classes of maximal nilpotent subgroups of G which are not contained in B is $\nu(H)$, follows now from Lemmas 5.4.7, 5.4.11, 5.4.12 and 5.4.13. //

Having established Theorem 5.4.1, we proceed to construct groups G for which $h(G) \geq \log_3 \left(\log_3 \nu(G)\right)$.

Let $G_1 = S_3$, the symmetric group on 3 letters, and, for

$i = 1, 2, \ldots$, let $G_{2i} = G_{2i-1} \text{ wr } C_3$ and $G_{2i+1} = G_{2i} \text{ wr } C_2$, where C_3 and C_2 are cyclic groups of order 3 and 2, respectively. Then, as it can be easily checked, the groups G_j, $j = 1, 2, \ldots$ satisfy the hypotheses of Theorem 5.4.1. Hence, by the same theorem, for each $i = 1, 2, \ldots$,

$$\nu(G_{2i}) = \left[\nu(G_{2i-1})^3 + 5\nu(G_{2i-1}) - 3\right]/3$$

and

$$\nu(G_{2i+1}) = \left[\nu(G_{2i})^2 + 3\nu(G_{2i}) - 2\right]/2 .$$

In particular, since $\nu(G_1) = 2$ and the above expressions for $\nu(G_{2i})$ and $\nu(G_{2i+1})$ are monotone in the relevant range, it is easy to see that

$$\nu(G_{2i}) \le \nu(G_{2i-1})^3$$

and

$$\nu(G_{2i+1}) \le \nu(G_{2i})^2 .$$

Therefore,

$$\nu(G_{2i}) \le \nu(G_{2(i-1)})^6 \le \nu(G_{2(i-1)})^{6^2} \le \ldots \le \nu(G_2)^{6^{i-1}} .$$

But $\nu(G_2) = 5$. Thus, $\nu(G_{2i}) \le 5^{6^{i-1}} \le 5^{6^i}$, and so

$$\log_3\left[\log_3 \nu(G_{2i})\right] \le \log_3\left[6^{i-1}\log_3 5\right] = i\log_3 6 + \log_3\left[\log_3 5\right]$$

$$\le 2i + \log_3\left[\log_3 5\right] .$$

In particular,

$$i \ge \frac{1}{2}\left\{\log_3\left[\log_3 \nu(G_{2i})\right] - \log_3\left[\log_3 5\right]\right\} .$$

But $2i + 1 = h(G_{2i})$. Hence,

$$h(G_{2i}) \ge \log_3\left[\log_3 \nu(G_{2i})\right] - \log_3\left[\log_3 5\right] + 1 \ge \log_3\left[\log_3 \nu(G_{2i})\right] ,$$

for each $i = 1, 2, \ldots$.

It is now clear in view of the preceding examples and Theorem 5.2.8 that, if G is a group in which the best possible upper bound is attained, then

$$\log_3\left[\log_3 \nu(G)\right] \le h(G) \le 2\left[1 + \log_3 \frac{18\nu(G)-19}{2}\right] .$$

CHAPTER 6

THE FITTING LENGTH OF A GROUP AND THE NUMBER OF CONJUGACY CLASSES OF ITS MAXIMAL METANILPOTENT SUBGROUPS

In this brief chapter we show that an upper bound on the Fitting length of a group can be obtained in terms of the number of conjugacy classes of its maximal metanilpotent subgroups. In fact, our result is rather more general. Let $\underline{\underline{F}}$ be any saturated formation of characteristic π, say, which is also a Fischer class, and let $\underline{\underline{F}}_\pi = \underline{\underline{S}}_{\pi'}\underline{\underline{F}}$, the class of all finite soluble groups G with a normal Hall π'-subgroup $G_{\pi'}$ such that $G/G_{\pi'} \in \underline{\underline{F}}$. If $\underline{\underline{X}}$ denotes the class $\underline{\underline{F}}_\pi^k$, where $k > 1$, then we show that

(6.1) <u>Theorem</u>. *The* $\underline{\underline{F}}$*-length,* $h_{\underline{\underline{F}}}(G)$, *of a group* G *is at most* $\nu_{\underline{\underline{X}}}(G) + k - 1$.

We will deduce Theorem 6.1 from a series of lemmas.

(6.2) <u>Lemma</u>. *Let* Σ *be a Sylow system of a group* $H \notin \underline{\underline{F}}_\pi$, *let* D *be the* $\underline{\underline{F}}_\pi$*-normalizer of* H *corresponding to* Σ *and let* V *be the* $\underline{\underline{F}}_\pi$*-injector of* H *into which* Σ *reduces. If* $H = DV$ *and* V *is an* $\underline{\underline{F}}_\pi$*-projector of every proper subgroup of* H *which contains* V, *then* $H \in \underline{\underline{F}}_\pi^2$.

Proof. Let $\{\underline{\underline{F}}_\pi(p)\}$ be the full, integrated family of formations which defines $\underline{\underline{F}}_\pi$ locally, and let, for each q dividing $|H|$, $J_q/V_q^H = O_{q'}(H/V_q^H)$ and $I_q/V_q^H = O_{q'q}(H/V_q^H)$. Since $H = DV$, every chief factor of H is, by Theorem 4.1.6, either $\underline{\underline{F}}_\pi$-central in H or $\underline{\underline{F}}_\pi$-covered in H or both. Thus, in particular, every q-chief factor of G/V_q^H is $\underline{\underline{F}}_\pi$-central in G/V_q^H, and hence, since I_q/V_q^H is the intersection of the centralizers of the q-chief factors of G/V_q^H, $H/I_q \in \underline{\underline{F}}_\pi(q)$. In fact, since the family $\{\underline{\underline{F}}_\pi(q)\}$ is full and integrated, $H/J_q \in \underline{\underline{F}}_\pi(q) \leq \underline{\underline{F}}_\pi$.

Now, let $K = \bigcap_{q \mid |H|} J_q$. Since $H/J_q \in \underline{\underline{F}}_\pi$ for each q dividing $|H|$ and since $\underline{\underline{F}}_\pi$ is a formation, $H/K \in \underline{\underline{F}}_\pi$. Thus, it remains to be shown that $K \in \underline{\underline{F}}_\pi$, that is to say, that $K \leq V$. In order to show this, we observe that by our hypothesis, Theorem 2.3.7 and the definition of J_q, $\Sigma_q \cap V$ is a Sylow q-subgroup of J_q. Therefore, since $K \trianglelefteq G$ and $K \leq J_q$, $\Sigma_q \cap V \cap K$ is a Sylow q-subgroup of K. Since this holds for each q dividing $|H|$, it clearly follows that $K \leq V$, and the lemma is proved. //

As a consequence of Lemma 6.2, we have

(6.3) <u>Lemma</u>. *Let* H *be a group and* V *an* $\underline{\underline{F}}_\pi$*-injector of* H. *If* V *is* $\underline{\underline{F}}_\pi^2$*-maximal in* H, *then* V *is also an* $\underline{\underline{F}}_\pi$*-projector of* H.

Proof. The result is trivially true if $V = H$. Hence we assume that $V < H$, that is to say, that $H \notin \underline{\underline{F}}_{\pi}$, and proceed by induction on $|H|$. Then it is clear that V is an $\underline{\underline{F}}_{\pi}$-projector of K whenever $V \leq K < H$; for, V is certainly $\underline{\underline{F}}_{\pi}^2$-maximal in K and, by Theorem 2.3.4 *(iv)*, it is also an $\underline{\underline{F}}_{\pi}$-injector of K, so that, by the induction hypotheses, V is an $\underline{\underline{F}}_{\pi}$-projector of K.

Let Σ be a Sylow system of H which reduces into V and let D be the $\underline{\underline{F}}_{\pi}$-normalizer of H corresponding to Σ. By Theorem 4.1.5, $DV = VD$.

Suppose that $DV = H$. Then, from what has been observed above and Lemma 6.2, $H \in \underline{\underline{F}}_{\pi}^2$. But then $V = H$ since V is, by hypothesis $\underline{\underline{F}}_{\pi}^2$-maximal in H and we have a contradiction.

Hence $DV \neq H$. Let $\overline{M}$ be a maximal subgroup of H which contains DV and let $\overline{\Sigma}$ be a Sylow system of H which reduces into $\overline{M}$. By Theorem 2.1.4, $\overline{\Sigma}^g = \Sigma$, for some $g \in G$. Thus, Σ reduces into $\overline{M}^g$, and hence, since $\overline{M}^g$ contains an $\underline{\underline{F}}_{\pi}$-injector of H, namely V^g, $V \leq \overline{M}^g$ by Lemma 2.3.5. Let $M = \overline{M}^g$ and let $\overline{D}$ be the $\underline{\underline{F}}_{\pi}$-normalizer of M corresponding to $\Sigma \cap M$. Since $D^g \leq M$, it follows, by Theorem 2.2.6 *(ii)*, that M is $\underline{\underline{F}}_{\pi}$-abnormal in H. Thus, by Theorem 2.2.6 *(iii)*, $D \leq \overline{D} \leq M$. But then, since V is, by the induction hypothesis, an $\underline{\underline{F}}_{\pi}$-projector of M and since, by our assumption, $\Sigma \cap M$ reduces into V, we have, by Theorem 2.2.10 *(i)*, that $\overline{D} \leq V$, and so $D \leq V$.

Now, let $H^{\underline{\underline{F}}_\pi}/K$ be a chief factor of H . Since $\underline{\underline{F}}_\pi$ has characteristic the whole prime set, it follows, by Lemma 2.2.2, that $H/H^{\underline{\underline{N}}} \in \underline{\underline{F}}_\pi$, and so $H^{\underline{\underline{F}}_\pi} \leq H^{\underline{\underline{N}}} < H$. Moreover, since $H \notin \underline{\underline{F}}_\pi$, $H^{\underline{\underline{F}}_\pi} \neq \{1\}$. Also, by Theorem 2.2.6 *(vi)*, D , and hence V , covers $H/H^{\underline{\underline{F}}_\pi}$. Clearly $VK \neq H$, since otherwise $H/K \cong V/V \cap K \in \underline{\underline{F}}_\pi$, whence $H^{\underline{\underline{F}}_\pi} \leq K$, a contradiction. Thus VK complements $H^{\underline{\underline{F}}_\pi}/K$ in H . Hence, since $VK/K = DK/K$, it follows by Theorems 2.2.6 *(v)* and 2.2.10 *(ii)*, that VK/K is an $\underline{\underline{F}}_\pi$-projector of H/K . But, by the induction hypothesis, V is an $\underline{\underline{F}}_\pi$-projector of $VK < H$. Hence, finally, by Lemma 2.2.9, V is an $\underline{\underline{F}}_\pi$-projector of H , and the proof is complete. //

Throughout the rest of this chapter $\underline{\underline{Y}}$ will denote the class $\underline{\underline{F}}_\pi^{k-1}$.

(6.4) <u>Lemma</u>. *If* V *is an* $\underline{\underline{X}}$*-injector of* G *, then* $G_{\underline{\underline{Y}}} = V_{\underline{\underline{Y}}}$ *; moreover,* $V/G_{\underline{\underline{Y}}}$ *is an* $\underline{\underline{F}}_\pi$*-injector of* $G/G_{\underline{\underline{Y}}}$.

Proof. Clearly $\underline{\underline{X}} = \bigcap_{\lambda \in \Lambda} \underline{\underline{X}}_\lambda \underline{\underline{S}}_{\pi(\lambda)'} \overline{\underline{\underline{X}}}_\lambda$, where $\underline{\underline{X}}_\lambda = \underline{\underline{Y}}$, $\pi(\lambda) = \pi$ and $\overline{\underline{\underline{X}}}_\lambda = \underline{\underline{F}}$, for each $\lambda \in \Lambda$. Thus, the first part of the lemma, namely, $G_{\underline{\underline{Y}}} = V_{\underline{\underline{Y}}}$, is a consequence of Lemma 10 of Hartley [20].

Next, let $(N/G_{\underline{\underline{Y}}}) \lhd\lhd (G/G_{\underline{\underline{Y}}})$ and consider $N \cap V/G_{\underline{\underline{Y}}}$. Since V

is an $\underline{\underline{X}}$-injector of G , $N \cap V$ is an $\underline{\underline{X}}$-injector of N , and so, by the first part, $(N \cap V)_{\underline{\underline{Y}}} = N_{\underline{\underline{Y}}}$. Thus, since $N \cap V$ is $\underline{\underline{X}}$-maximal in in N , it follows that $(N \cap V)/N_{\underline{\underline{Y}}}$ is $\underline{\underline{F}}_\pi$-maximal in $N/N_{\underline{\underline{Y}}}$. But it can be easily checked that $N_{\underline{\underline{Y}}} = G_{\underline{\underline{Y}}}$. Hence, $N \cap V/G_{\underline{\underline{Y}}}$ is $\underline{\underline{F}}_\pi$-maximal in $N/G_{\underline{\underline{Y}}}$. Since $N/G_{\underline{\underline{Y}}}$ was an arbitrary subnormal subgroup of $G/G_{\underline{\underline{Y}}}$, this shows that $V/G_{\underline{\underline{Y}}}$ has the defining properties of $\underline{\underline{F}}_\pi$-injectors, and so $V/G_{\underline{\underline{Y}}}$ is an $\underline{\underline{F}}_\pi$-injector of $G/G_{\underline{\underline{Y}}}$, as required. //

(6.5) <u>Lemma</u>. *Let* V *be as in Lemma* 6.4. *If* $V/G_{\underline{\underline{F}}_\pi}$ *is* $\underline{\underline{X}}$*-maximal in* $G/G_{\underline{\underline{F}}_\pi}$ *, then* $V/G_{\underline{\underline{X}}}$ *is an* $\underline{\underline{F}}_\pi$*-projector of* $G/G_{\underline{\underline{X}}}$.

Proof. Consider $V/G_{\underline{\underline{Y}}}$. By Lemma 6.4, it is $\underline{\underline{F}}_\pi$-injector of $G/G_{\underline{\underline{Y}}}$. Moreover, since $h_{\underline{\underline{F}}_\pi}\left(G_{\underline{\underline{Y}}}/G_{\underline{\underline{F}}_\pi}\right) \le k - 2$, it follows by our assumption, that $V/G_{\underline{\underline{Y}}}$ is $\underline{\underline{F}}_\pi^2$-maximal in $G/G_{\underline{\underline{Y}}}$. Thus, by Lemma 6.3, $V/G_{\underline{\underline{Y}}}$ is an $\underline{\underline{F}}_\pi$-projector of $G/G_{\underline{\underline{Y}}}$. Since, by Lemma 2.2.8 *(ii)*, $\underline{\underline{F}}_\pi$-projectors of G are homomorphism invariant, we have finally that $V/G_{\underline{\underline{X}}}$ is an $\underline{\underline{F}}_\pi$-projector of $G/G_{\underline{\underline{X}}}$, as required. //

We will now prove the main result of this chapter, namely, Theorem 6.1.

Proof of Theorem 6.1. We proceed by induction on $|G|$. Thus we can assume that $\nu_{\underline{\underline{X}}}\left(G/G_{\underline{\underline{F}}_\pi}\right) = \nu_{\underline{\underline{X}}}(G)$. Also we can assume that $h_{\underline{\underline{F}}}(G) > k$ since otherwise the result is trivially true. Now let V be an

$\underline{\underline{X}}$-injector of G. Then, since $\nu_{\underline{\underline{X}}}\left(G/G_{\underline{\underline{F}}_\pi}\right) = \nu_{\underline{\underline{X}}}(G)$, $V/G_{\underline{\underline{F}}_\pi}$ is, by Lemma 5.1.1, $\underline{\underline{X}}$-maximal in $G/G_{\underline{\underline{F}}_\pi}$. Hence, by Lemma 6.5, $V/G_{\underline{\underline{X}}}$ is an $\underline{\underline{F}}_\pi$-projector of $G/G_{\underline{\underline{X}}}$. Next, let $W/G_{\underline{\underline{F}}_\pi}$ be an $\underline{\underline{X}}$-injector of $G/G_{\underline{\underline{F}}_\pi}$. Then, since $V \nleq G_{\underline{\underline{F}}_\pi^{k+1}}$, $V/G_{\underline{\underline{F}}_\pi}$ and $W/G_{\underline{\underline{F}}_\pi}$ belong to two distinct conjugacy classes of $\underline{\underline{X}}$-maximal subgroups of $G/G_{\underline{\underline{F}}_\pi}$. Hence, $W/G_{\underline{\underline{F}}_\pi^2}$ is not $\underline{\underline{X}}$-maximal in $G/G_{\underline{\underline{F}}_\pi^2}$. For otherwise, by Lemma 6.5, $W/G_{\underline{\underline{X}}}$ is an $\underline{\underline{F}}_\pi$-projector of $G/G_{\underline{\underline{X}}}$ and hence conjugate to $V/G_{\underline{\underline{X}}}$. But then V and W are conjugate in G and we have a contradiction. Thus we have shown that $\nu_{\underline{\underline{X}}}\left(G/G_{\underline{\underline{F}}_\pi^2}\right) < \nu_{\underline{\underline{X}}}(G)$. In fact, $\nu_{\underline{\underline{X}}}\left(G/G_{\underline{\underline{F}}_\pi^2}\right) \leq \nu_{\underline{\underline{X}}}(G) - 2$, since $V/G_{\underline{\underline{F}}_\pi^2} < VG_{\underline{\underline{F}}_\pi^{k+1}}/G_{\underline{\underline{F}}_\pi^2} \in \underline{\underline{X}}$. Hence, by induction,

$h_{\underline{\underline{F}}}\left(G/G_{\underline{\underline{F}}_\pi^2}\right) \leq k + \left(\nu_{\underline{\underline{X}}}(G)-2\right) - 1$. Since $h_{\underline{\underline{F}}}\left(G_{\underline{\underline{F}}_\pi^2}\right) = 2$, it follows now that $h_{\underline{\underline{F}}}(G) \leq k + \nu_{\underline{\underline{X}}}(G) - 1$ and we are done. //

We end this chapter with a remark that the above result is no more true for $k = 1$ as the case when $\underline{\underline{F}} = \underline{\underline{N}}$, the class of all finite nilpotent groups, and $\nu_{\underline{\underline{F}}}(G) = \nu(G) = 2$ shows (see the remark following the proof of Proposition 5.3.1).

REFERENCES

[1] Donald W. Barnes and Otto H. Kegel, "Gaschütz functors on finite soluble groups", *Math. Z.* 94 (1966), 134-142.

[2] R.W. Carter, "On a class of finite soluble groups", *Proc. London Math. Soc.* (3) 9 (1959), 623-640.

[3] Roger W. Carter, "Nilpotent self-normalizing subgroups of soluble groups", *Math. Z.* 75 (1961), 136-139.

[4] Roger Carter and Trevor Hawkes, "The $\underline{\underline{F}}$-normalizers of a finite soluble group", *J. Algebra* 5 (1967), 175-202.

[5] Roger Carter, Bernd Fischer, Trevor Hawkes, "Extreme classes of finite soluble groups", *J. Algebra* 9 (1968), 285-313.

[6] Walter Feit and John G. Thompson, "Solvability of groups of odd order", *Pacific J. Math.* 13 (1963), 775-1029.

[7] B. Fischer, Habilitationsschrift, Universität Frankfurt. (196)

[8] B. Fischer, "Classes of conjugate subgroups in finite solvable groups", (Yale University lecture notes, 1966).

[9] B. Fischer, W. Gaschütz und B. Hartley, "Injektoren endlicher auflösbarer Gruppen", *Math. Z.* 102 (1967), 337-339.

[10] Wolfgang Gaschütz, "Praefrattinigruppen", *Arch. Math.* 13 (1962), 418-426.

[11] Wolfgang Gaschütz, "Zur Theorie der endlichen auflösbaren Gruppen", *Math. Z.* 80 (1963), 300-305.

[12] Wolfgang Gaschütz und Ursula Lubeseder, "Kennzeichnung gesättigter Formationen", *Math. Z.* 82 (1963), 198-199.

[13] Wolfgang Gaschütz, "Selected topics in the theory of soluble groups", (Lectures given at the Ninth Summer Research Institute of the Australian Mathematical Society in Canberra, 1969. Notes by J. Looker).

[14] Daniel Gorenstein, *Finite groups* (Harper and Row, New York, Evanston, London, 1968).

[15] P. Hall, "A note on soluble groups", *J. London Math. Soc.* 3 (1928), 98-105.

[16] P. Hall, "A characteristic property of soluble groups", *J. London Math. Soc.* 12 (1937), 198-200.

[17] P. Hall, "On the Sylow systems of a soluble group", *Proc. London Math. Soc.* (2) 43 (1937), 316-323.

[18] P. Hall, "On the system normalizers of a soluble group", *Proc. London Math. Soc.* (2) 43 (1937), 507-528.

[19] P. Hall, "On non-strictly simple groups", *Proc. Cambridge Philos. Soc.* 59 (1963), 531-553.

[20] B. Hartley, "On Fischer's dualization of formation theory", *Proc. London Math. Soc.* (3) 19 (1969), 193-207.

[21] Trevor Hawkes, "Analogues of prefrattini subgroups", *Proc. Internat. Conf. Theory of groups, Austral. Nat. Univ., Canberra,* 1965, 145-150 (Gordon and Breach, New York, London, Paris, 1967).

[22] Trevor Hawkes, "On formation subgroups of a finite soluble group", *J. London Math. Soc.* 44 (1969), 243-250.

[23] Trevor Hawkes, "On Fitting formations", *Math. Z.* 117 (1970), 177-182.

[24] C.H. Houghton, "On the automorphism groups of certain wreath products", *Publ. Math. Debrecen* 9 (1962), 307-313.

[25] Bertram Huppert, *Endliche Gruppen I* (Die Grundlehren der mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[26] Bertram Huppert, "Zur Theorie der Formationen", *Arch. Math.* 19 (1968), 561-574.

[27] H. Lausch and A. Makan, "On a relation between the Fitting length of a soluble group and the number of conjugacy classes of its maximal nilpotent subgroups", *Bull. Austral. Math. Soc.* 1 (1969), 3-10.

[28] U. Lubeseder, "Construction of formations in finite soluble groups" (German), (Thesis, University of Kiel, 1963).

[29] A. Makan, "Another characteristic conjugacy class of subgroups of finite soluble groups", *J. Austral. Math. Soc.* 11 (1970), 395-400.

[30] A. Mann, "A criterion for pronormality", *J. London Math. Soc.* 44 (1969), 175-176.

[31] Peter M. Neumann, "On the structure of standard wreath products of groups", *Math. Z.* 84 (1964), 343-373.

[32] John S. Rose, "Finite soluble groups with pronormal system normalizers", *Proc. London Math. Soc.* (3) 17 (1967), 447-469.

[33] John S. Rose, "Nilpotent subgroups of finite soluble groups", *Math. Z.* 106 (1968), 97-112.

[34] Eugene Schenkman, *Group theory* (Van Nostrand, Princeton, New Jersey; Toronto; New York; London; 1965).

[35] Hermann Schunck, "H-Untergruppen in endlichen auflösbaren Gruppen", *Math. Z.* 97 (1967), 326-330.

[36] Jack Shamash, "On the Carter subgroup of a solvable group", *Math. Z.* 109 (1969), 288-310.

[37] John G. Thompson, "Nonsolvable finite groups all of whose local subgroups are solvable", *Bull. Amer. Math. Soc.* 74 (1968), 383-437.

[38] Charles R.B. Wright, "On screens and L-Izers of finite solvable groups", *Math. Z.* 115 (1970), 273-282.