PERMUTATIONAL PRODUCTS OF GROUPS AND EMBEDDING THEORY OF GROUP AMALGAMS

by

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Statement.

Except where it is indicated, this thesis is my own work.

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INDEX

Notations.

Most of the notations used in the thesis are standard in the modern publications in Group Theory. However, for purposes of clarity and convenience we list these here. A and B shall denote groups unless stated otherwise.

$gp\{x,y,\ldots;\ r_i=1, i=1,2,\ldots\}$	the group generated by $x,y,\ldots;$ satisfying the relations $r_i=1, i=1,2,\ldots$
$\{x,y,\ldots;\ \ldots\}$	the set consisting of $x,y,\ldots,$ such that ...
$H \subseteq A$	H is a subgroup of A.
$C_A(S)$	the centraliser of S in A.
$[x,y]$	$x^{-1}y^{-1}xy;\ x,y \in A$
$[A,B]$	$gp\{[a,b];\ a \in A, b \in B\}$
$K = S \times T \times H$	$\{(s,t,h);\ s \in S, t \in T, h \in H\}$
$A \times B$	the direct product of A,B.
$\underline{\underline{A}} = am(A,B;\ H)$	the amalgam of A and B amalgamating H
$(A \times B;\ H)$	the direct product of $\underline{\underline{A}}$.
$(A * B;\ H)$	the free product of $\underline{\underline{A}}$.
$P(\underline{\underline{A}};\ S,T)$	the permutational product of $\underline{\underline{A}}$ using S and T as transversals.

CHAPTER 0 (Introduction)

The concept of 'permutational products' of groups was introduced by B.H. Neumann [1]. This group theoretic construction is based on a method given by him in his famous essay [3] for the embeddability of an amalgam with a single group amalgamated, in a permutation group. Use of this construction was made to answer various questions about the embedding theory of group amalgams (cf. [1], [2], [3]). We begin by defining the notion of an amalgam[1] and some related concepts.

An 'amalgam' $\underline{\underline{A}}$ of (for convenience only) two groups A and B with a common subgroup H is an 'incomplete group' whose elements are those of A and B with the elements of H thought of as identified in the two groups. The product of two elements of $\underline{\underline{A}}$ is defined if and only if they both belong to A or both belong to B, and its value is as in that group. If there is a group G containing A and B as subgroups such that in G the intersection of A and B is the prescribed group H, then we speak of an 'embedding' of the amalgam $\underline{\underline{A}} = \mathrm{am}(A,B;\ H)$ in G. A and B are called 'constituents' of $\underline{\underline{A}}$ and H the 'amalgamated subgroup'.

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1. This term was first introduced by Baer [15]. A deeper account of results concerning group amalgams and their embeddability can be found in [3], [4], [5] and [11].

By a 'transversal' of a subgroup H of a group A we shall mean a set $S \subseteq A$ such that every element a of A is uniquely representable in the form

$$a = sh \quad s \in S,\ h \in H,$$

We now come to the definition of a permutational product (cf. [1]). Let $\underline{\underline{A}} = \text{am}(A,B;\ H)$ be an amalgam of the groups A and B. We choose transversals S of H in A and T of H in B. Form the set product $K = S \times T \times H$. The elements of K are ordered triplets (s,t,h), $s \in S$, $t \in T$, $h \in H$. For each $a \in A$, we define a mapping $\rho(a) : K \longrightarrow K$ by

$$(s,t,h)^{\rho(a)} = (s',t,h')$$

whene $s' \in S$, $h' \in H$ are determined by the equation

$$sha = s'h'$$

Similarly for b in B we define a mapping $\rho(b) : K \longrightarrow K$ by

$$(s,t,h)^{\rho(b)} = (s,t'',h'')$$

where

$$thb = t''h''.$$

It is easy to verify that for $a = b \in H$ no ambiguity arises in the definition of ρ. Moreover, the mapping $\rho : A \longrightarrow \rho(A)$ is a homomorphism; for if a,a' are two elements of A, then

$$(s,t,h)^{\rho(a)\rho(a')} = (s',t,h')^{\rho(a')}$$
$$= (s'',t,h'')$$

where $sha = s'h'$, $s'h'a' = s''h''$ so that $shaa' = s''h''$ which means that

$$(s,t,h)^{\rho(aa')} = (s'',t,h'') .$$

Thus $\rho(a).\rho(a') = \rho(aa')$. The proof for $\rho(b)\rho(b') = \rho(bb')$ is similar. It follows, therefore, that $\rho(A) = \{\rho(a);\ a \in A\}$ and $\rho(B) = \{\rho(b);\ b \in B\}$ are groups. However the homomorphism $A \longrightarrow \rho(A)$ turns out to be an isomorphism, for if $\rho(a) = i_k$, the identity mapping of K, then

$$(s,t,h)^{\rho(a)} = (s,t,h)$$

for all $(s,t,h) \in K$ means that $sha = sh$ for all $s \in S$, $h \in H$ and therefore $a = 1$.

The above remarks show that the mappings $\rho(a)$, $\rho(b)$; $a \in A$, $b = B$ are infact permutations of K. Furthermore, the intersection of $\rho(A)$ and $\rho(B)$ is $\rho(H)$, because if $\rho(a) \in \rho(B)$ then $\rho(a)$ leaves the first component of each triplet (s,t,h) fixed and so

$$(s,t,h)^{\rho(a)} = (s,t,ha),$$

therefore $ha \in H$, that is $a \in H$.

The permutation group P of K generated by $\rho(A)$ and $\rho(B)$ contains isomorphic copies of A and B with $\rho(A) \cap \rho(B) = \rho(H)$ isomorphic to H and, therefore, embeds the amalgam $\underline{\underline{A}}$. P <u>is called a permutational product of</u> $\underline{\underline{A}} = am(A,B;\ H)$. We use here the indefinite article because P depends not only on $\underline{\underline{A}}$ but also on the choice of transversals S,T of H in A,B respectively (for details see [1]). By $P(\underline{\underline{A}};\ S,T)$ we shall denote the permutational product of $\underline{\underline{A}}$ corresponding to the transversals S,T of H in A and B respectively.

Next we define the free product of groups as follows: Let $\{G_\alpha\}$ be a family of groups indexed by a set I of finite or infinite cardinality. A group G, which we shall write as $\prod\limits_{\alpha\in I}{}^{*} G_\alpha$, is said to be the "ordinary free product" of its subgroups G_α, $(\alpha \in I)$ if

(i) the subgroups G_α generate G, that is, if every element $g \neq 1$ of G is expressible as a product of a finite number of elements from the G_α:

$$g = g_1 g_2 \ldots g_n\ ;\quad g_i \in G_{\alpha_i} \quad i = 1,2,\ldots,n \qquad \text{(A)}$$

where

$$g_i \neq 1,\ \alpha_i \neq \alpha_j \quad \text{for} \quad j \neq i \pm 1\ ; \qquad \text{and}$$

(ii) The expression (A) is unique for every $g \neq 1$ in G.

If the set I is finite, we shall use the notation

$$G = G_1 * G_2 * \ldots * G_n .$$

The subgroups G_α of G are called the 'free factors' of G while the expression (A) is called the 'normal form' of an element g of G.

If each of the free factors G_α contains a subgroup $H_{\alpha\beta}$ isomorphic to a $H_{\beta\alpha}$ of G_β, then let G^* be the group obtained from the free product G by introducing all relations $h_{\alpha\beta} = h_{\beta\alpha}$ $\alpha \neq \beta$, identifying pairs of elements of $H_{\alpha\beta}$ and $H_{\beta\alpha}$ which correspond under some fixed isomorphism between these two subgroups G_α and G_β respectively. This makes G^* a homomorphic image of G in a natural way. If in G^* the images of the subgroups G_α of G still are isomorphic to G_α for each α and their intersections in pairs are precisely (the images of) the subgroups $H_{\alpha\beta} = H_{\beta\alpha}$ then G^* will be called 'the free product of G_α with amalgamated $H_{\alpha\beta}$'. We shall write G^* as:

$$G^* = (\Pi^* G_\alpha;\ H_{\alpha\beta} = H_{\beta\alpha}, \alpha,\beta \in I,\ \alpha \neq \beta).$$

Let $\underline{\underline{A}}$ be an amalgam of the groups G_α with amalgamated $H_{\alpha\beta}$, $(\alpha,\beta \in I)$. Let H_α be the group generated by all $H_{\alpha\beta}(\alpha,\beta \in I, \alpha$ fixed). The amalgam $\underline{\underline{A}}'$ formed by the groups H_α amalgamating $H_{\alpha\beta}$ is called the 'reduced amalgam' of the groups G_α. A necessary and sufficient condition for the embeddability of $\underline{\underline{A}}$ is that the reduced amalgam $\underline{\underline{A}}'$ is embeddable (Hanna Neumann [4]). Apart from that no necessary and sufficient condition for the embeddability of an amalgam is known, not even in the case of an amalgam of three groups.

This thesis comprises of three chapters. Chapter I deals with a discussion on Permutational products and their properties. A generalisation of a theorem of B.H. Neumann concerning changes in the transversals of the amalgamated subgroup is given by Theorem I.2.1.

In Chapter II are given two necessary and sufficient criteria for the existance of the generalised free product of the reduced amalgam of three groups. A special amalgam of more than three groups and its embeddability is discussed in this chapter. The result obtained here generalises Theorem 9.0 of [4].

A problem of B.H. Neumann and Hanna Neumann about the embeddability of an embeddable finite amalgam in a finite group is considered with particular reference to those amalgams whose embeddability has been proved in Chapter II. Theorem III.2.2 generalises some known results. (cf. Theorem 9.1. [4] and [14]). It is mentioned, however, that the question of embeddability in a finite group of a finite amalgam of type S is still open.

Lastly a counter example is given to answer various questions about different embeddings of an amalgam of three groups.

CHAPTER I

PERMUTATIONAL PRODUCTS OF GROUPS.

I.0 Introduction: Let $\underline{\underline{A}} = am(A,B;H)$ be an amalgam of two groups A and B with the subgroup H amalgamated. Let $P(\underline{\underline{A}},S,T)$ be the permutational product of the amalgam $\underline{\underline{A}}$ using transversals S and T in A and B respectively. It has been shown by B.H. Neumann that a change in the transversals of H in the constituents greatly alters the nature and character of a permutational product. In I.1 we study the effect of some special changes in the transversals of H which do not change the isomorphism type of the permutational product of A and B.

B.H. Neumann has shown that if H is central in one of the constituents, say A, then the isomorphism type of the permutational product is independent of the change of transversals in the other constituents, that is in B, (cf. Theorem 4.2 [1]). We prove (Theorem I.2.1) that such is also the case under a less restrictive condition namely that the amalgamated subgroup possesses in one of the constituents a transversal which it centralises. It then follows that if H has, in both A and B, transversals that it centralises, then all permutational products of the amalgam formed with transversals of which at least one centralises H, are isomorphic.

In I.3, we examine the structure of a permutational product of an amalgam in which the amalgamated subgroup possesses in both the constituents transversals that it centralises, and show that for this kind of amalgam the permutational product corresponding to transversals centralised by the amalgamated subgroup belongs to the least 'variety'[1] containing both the constituents.

Some analogies between the generalised free product and permutational products of groups with an amalgamated subgroup are discussed in I.4. A result similar to theorem 1.1[3] is proved for permutational products.

Following B.H. Neumann, let F* denote one of the following properties of groups: being locally finite (LF), of finite exponent (FE), or periodic (P). It is known that a soluble amalgam (that is, an amalgam of soluble groups) or an amalgam of groups having the property F*, need not be embeddable in a soluble or F* group, respectively, (cf. [1], [2]). However, under some sufficient conditions on the constituents, B.H. Neumann has shown it to be possible. For the case of soluble amalgams, theorem 5.4 of [1] is shown to hold under the weaker assumption that the amalgamated subgroup is contained in the centraliser of one of its transversals. Some other sufficient conditions for the embeddability of a soluble, nilpotent or F* group amalgam in a soluble, nilpotent or F* group respectively are also discussed.

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1. A 'variety' is a class of groups closed under the operations of taking subgroups, epimorphic images and cartesian products [cf. [12]. [16]).

I.1. Some very special changes of the transversals are possible in any amalgam without changing the isomorphism type of the permutational product. In this paragraph, we make a study of these changes. Let S be a transversal of the amalgamalid subgroup in one of the constituents. By $S^* = Sh^*$, we shall denote the transversal of H obtained from S by multiplying on the right every element s of S by a fixed element $h^* \in H$. We now prove:

I.1.1. The permutational products $P = P(\underline{\underline{A}};S,T)$ and $P^* = P^*(\underline{\underline{A}};S^*,T^*)$ of the groups A and B amalgamating H, corresponding to the transversals S,T and S^*,T^* respectively where $S^* = Sh^*$, $T^* = Th^*$, $h^* \in H$, are isomorphic.

Proof: Let $K = S \times T \times H$, $K^* = S^* \times T^* \times H$. The permutation representations of $\underline{\underline{A}}$ using K and K^* will be denoted by ρ and ρ^* respectively. Define a mapping $\varphi : K \longrightarrow K^*$ as follows: For any $(s,t,h) \in K$, $s \in S$, $t \in T$, $h \in H$,

$$(s,t,h)^{\varphi} = (sh^*,th^*,h_1)$$

with $h = h^*h_1$. Then φ is clearly one to one and so φ^{-1} exists. For any element a of A,

$$\begin{aligned}(sh^*,th^*,h_1)^{\varphi^{-1}\rho(a)\varphi} &= (s,t,h)^{\rho(a)\varphi} \\ &= (s',t,h')^{\varphi} \\ &= (s'h^*,th^*,h_1')\end{aligned}$$

where $sha = s'h' = s'h^*h_1'$ (i)

Also

$$(sh^*,th^*,h_1)^{\rho^*(a)} = (s''h^*,th^*,h'')$$

where,

$$sh^*h_1a = sha = s''h^*h''$$

$$= s'h^*h_1' \qquad \text{by (i)}$$

Consequently $s'' = s'$, $h'' = h_1'$ and therefore,

$$\varphi^{-1}\rho(a)\varphi = \rho^*(a) \qquad \text{for all } a \in A$$

Similarly,

$$\varphi^{-1}\rho(b)\varphi = \rho^*(b) \qquad \text{for all } b \in B.$$

so that

$$\varphi^{-1}P\varphi = P^*$$

Hence P and P^* are isomorphic.

In the following proposition, we show that conjugation of elements of both the transversals by a fixed element of the amalgamated subgroup does not alter the isomorphism type of the permutational product, that is we prove:

I.1.2 If $P = P(\underline{\underline{A}};S,T)$ and $\overline{P}^* = \overline{P}^*(\underline{\underline{A}};\overline{S}^*,\overline{T}^*)$ are permutational products of $\underline{\underline{A}} = am(A,B;\ H)$ for transversals S,T and $\overline{S}^*,\overline{T}^*$ respectively, where $\overline{S}^* = h^{*-1}\overline{S}h^*$, $\overline{T}^* = h^{*-1}\overline{T}h^*$, then P and $\overline{P}^*$ are isomorphic.

Proof: We denote by K and $\bar{K}^*$ the set products $S \times T \times H$ and $\bar{S}^* \times \bar{T}^* \times H$ respectively and define a mapping $\varphi : K \to \bar{K}^*$ such that, for $(s,t,h) \in K$,

$$(s,t,h)^{\varphi} = (h^{*-1}sh^*, h^{*-1}th^*, h_1)$$

with $h = h^*h_1$, $h_1 \in H$. Then again, φ is one-one. Also for $a \in A$,

$$\begin{aligned}(h^{*-1}sh^*, h^{*-1}th^*, h_1)^{\varphi^{-1}\rho(a)\varphi} &= (s,t,h)^{\rho(a)\varphi} \\ &= (s',t,h')^{\varphi} \\ &= (h^{*-1}s'h^*, h^{*-1}th^*, h_1')\end{aligned}$$

where $sha = s'h' = s'h^*h_1'$; (i)

and

$$(h^{*-1}sh^*, h^{*-1}th^*, h_1)^{\bar{\rho}^*(a)} = (h^{*-1}s''h^*, h^{*-1}th^*, h_1'')$$

with

$$h^{*-1}sh^*h_1a = h^{*-1}s''h^*h_1'', \quad \text{i.e.} \quad sha = s''h^*h_1'' \qquad \text{(ii)}$$

From (i) and (ii) we get

$$s' = s'', \; h_1'' = h_1'$$

$$\varphi^{-1}\rho(a)\varphi = \bar{\rho}^*(a) \qquad \text{for all } a \in A.$$

On account of the symmetrical nature of the transformation, we also have

$$\varphi^{-1}\rho(b)\varphi = \bar{\rho}^*(b) \qquad \text{for all } b \in B.$$

Since P and P* are generated by $\rho(A),\rho(B)$ and $\bar{\rho}*(A)$, $\bar{\rho}*(B)$ respectively, they are isomorphic as before.

If we have transversals S,T and S**, T** such that $S** = h*Sh**^{-1}$ and $T** = h*Th**^{-1}$, then writting these as $S** = h*Sh*^{-1}h*h**^{-1} = \bar{S}*h'**$, and $T** = \bar{T}*h'**$ and using first I.1.2, then I.1.1 we obtain

I.1.3. <u>Corollary</u>: The permutational product $P(\underline{\underline{A}};S,T)$ and $P**(\underline{\underline{A}};S**,T**)$ where $S** = h*Sh**^{-1}$, $T** = h*Th**^{-1}$, are isomorphic.

We discuss now the changes in those transversals of H in $\underline{\underline{A}}$ which consist of powers of one single element. The class of group amalgams such that one group possesses such a transversal, is, of course, very restricted. In such an amalgam, a particular transversal of H in one of the constituents, say A, is written as

$$S = \left\{s^0, s^1, \ldots s^{m-1}\right\},\ s^m = 1$$

Thus the first element of S is taken as the unit representative of H in A. S will be called a "cyclic transversal" of H and at least one of the constituents in $\underline{\underline{A}}$ would be supposed to have one such transversal.

By $S_{(i,h)}$, h∈H, we denote the transversal obtained from S by multiplying the ith element of S by h on the right. Then we prove:

I.1.4. If one of the constituents in $\underline{\underline{A}}$, say A has a cyclic transversal and two other transversals S', S'' of H in A are choosen such that $S' = S_{(1,h*)}$, $S'' = S_{(2,h*)}$, and moreover, there is a mapping

φ from $K' = S' \times T \times H$ to $K'' = S'' \times T \times H$, defined by

$$(s^0 h^*, t, h')^{\varphi} = (s^1 h^*, t, h')$$

and

$$(s^k, t, h')^{\varphi} = (s^{k+1}, t, h') \quad , \quad 1 \leq k \leq m-1$$

then for all $h \in H$, $\varphi^{-1}\rho'(h)\varphi = \rho''(h)$, where ρ' and ρ'' are permutation mappings of K' and K'' respectively.

__Proof:__ φ is clearly one to one and so has an inverse. For an $h \in H$, we have

(i) $$(s^1 h^*, t, h')^{\varphi^{-1}\rho'(h)\varphi} = (s^0 h^*, t, h')^{\rho'(h)\varphi}$$
$$= (s^0 h^*, t, h'h)^{\varphi}$$
$$= (s^1 h^*, t, h'h)$$

and

$$(s^1 h^*, t, h')^{\rho''(h)} = (s^1 h^*, t, h'h)$$

while

(ii) $$(s^{k+1}, t, h')^{\varphi^{-1}\rho'(h)\varphi} = (s^k, t, h')^{\rho'(h)\varphi}$$
$$= (s^k, t, h'h)^{\varphi}$$
$$= (s^{k+1}, t, h'h)$$
$$= (s^{k+1}, t, h')^{\rho''(h)}$$

Thus in both cases $\varphi^{-1}\rho'(h)\varphi = \rho''(h)$.

The above result holds also when H has cyclic transversals S,T in both A and B respectively and S',S'' and T',T'' are chosen such that $S' = S_{(1,h^*)}$, $S'' = S_{(2,h^*)}$, $T' = T_{(1,h^*)}$, $T'' = T_{(2,h^*)}$. But in such a case, one can show more, and we prove, by defining the mapping φ somewhat more carefully, the following

I.1.5. Theorem. If H has cyclic transversals S,T where $|S| = m$, $|T| = n$ in both A,B and transversals S',S'' and T',T'' are chosen such that

$$S' = S_{(1,h^*)},\ S'' = S_{(2,h^*)},\ T' = T_{(1,h^*)},\ T'' = T_{(2,h^*)}$$

then $P'(\underline{\underline{A}};S',T')$ and $P''(\underline{\underline{A}};S'',T'')$ are isomorphic.

Proof: The mapping $\varphi : K' \longrightarrow K''$ is defined as

(i) $$(s^0h^*,t^0h^*,h')^\varphi = (s^1h^*,t^1h^*,h')$$

(ii) $$(s^0h^*,t^\ell,h')^\varphi = (s^1h^*,t^{\ell+1},h')$$

(iii) $$(s^k,t^0h^*,h')^\varphi = (s^{k+1},t^1h^*,h')$$

(iv) $$(s^k,t^\ell,h')^\varphi = (s^{k+1},t^{\ell+1},h')$$

Where k,ℓ are both nonzero such that $1 \le k \le m-1$, $1 \le \ell \le n-1$.

Again φ is one to one and φ^{-1} exists. Also for $a \in A$,

$$\begin{aligned}(s^1h^*,t^1h^*,h')^{\varphi^{-1}\rho'(a)\varphi} &= (s^0h^*,t^0h^*,h')^{\rho'(a)\varphi} \\ &= (s^{k'},t^0h^*,h'_1)^\varphi \\ &= (s^{k'+1},t^1h^*,h'_1)\end{aligned}$$

where $s^0h^*h'a = s^{k'}h'_1$ i.e. $s^1h^*h'a = s^{k'+1}h'_1$ (1)

and

$$(s^1h^*, t^1h^*, h')^{\rho''(a)} = (s^{k''}, t^1h^*, h_2')$$

where

$$s^1h^*h'a = s^{k''}h_2' = s^{k'+1}h_1' \qquad \text{from (1)}$$

$\therefore$ $k'' = k'+1$, $h_1' = h_2'$ and so $\varphi^{-1}\rho(a)\varphi = \rho'(a)$ in this case.

Also

$$\begin{aligned}(s^{k+1}, t^{\ell+1}, h')^{\varphi^{-1}\rho'(a)\varphi} &= (s^k, t^\ell, h')^{\rho'(a)\varphi} \\ &= (s^{k_1}, t^\ell, h_3')^{\varphi} \\ &= (s^{k_1+1}, t^\ell, h_3')\end{aligned}$$

with $s^kh'a = s^{k_1}h_3'$ i.e. $s^{k+1}h'a = s^{k_1+1}h_3'$ $\qquad$ (1′)

and

$$(s^{k+1}, t^{\ell+1}, h')^{\rho''(a)} = (s^{k_2}, t^{\ell+1}, h_3'')$$

where

$$s^{k+1}h'a = s^{k_2}h_3'' = s^{k_1+1}h_3' \qquad \text{from (1′)}$$

Therefore

$k_1+1 = k_2$, $h_3' = h_3''$ and consequently again

$\varphi^{-1}\rho'(a)\varphi = \rho''(a)$.

Also for $b \in B$,

$$\begin{aligned}(s^1h^*, t^{\ell+1}, h')^{\varphi^{-1}\rho(b)\varphi} &= (s^0h^*, t^\ell, h')^{\rho(b)\varphi} \\ &= (s^0h^*, t^{\ell'}, h_1)^{\varphi} \\ &= (s^1h^*, t^{\ell'+1}, h_1)\end{aligned}$$

where

$$t^{\ell}h'b = t^{\ell'}h_1 \quad \text{i.e.} \quad t^{\ell+1}h'b = t^{\ell'+1}h_1 \qquad (2)$$

and

$$(s^1h^*, t^{\ell+1}, h')^{\rho''(b)} = (s^1h^*, t^{\ell''}h_2)$$

where

$$t^{\ell+1}h'b = t^{\ell''}h_2 = t^{\ell'+1}h_1 \qquad \text{from (2)}$$

$$\therefore \qquad \ell'' = \ell'+1,\ h_2 = h_1 \quad \text{and so} \quad \varphi^{-1}\rho'(b)\varphi = \rho(b)$$

Moreover,

$$(s^{k+1}, t^1h^*, h')^{\varphi^{-1}\rho'(b)\varphi} = (s^k, t^0h^*, h')^{\rho'(b)\varphi}$$

$$= (s^{k+1}, t^{\ell'}, h_3)$$

with

$$t^0h^*h'b = t^{\ell'}h_3 \quad \text{i.e.} \quad t^1h^*h'b = t^{\ell'+1}h_3 \qquad (2')$$

and

$$(s^{k+1}, t^1h^*, h')^{\rho''(b)} = (s^{k+1}, t^{\ell''}, h_4)$$

where

$$t^1h^*h'b = t^{\ell''}h_4 = t^{\ell'+1}h_3 \qquad \text{from (2')}$$

$$\therefore \qquad \ell'+1 = \ell'',\ h_3 = h_4 \quad \text{and again} \quad \varphi^{-1}\rho'(b)\varphi = \rho''(b).$$

The other two cases for both $a \in A$ and $b \in B$ follow similarly, thus, once again, we have

$$\varphi^{-1}\rho'(A)\varphi = \rho''(A),\ \varphi^{-1}\rho'(B)\varphi = \rho''(B) \quad \text{so that}$$

$\varphi^{-1}P'\varphi = P''$ and P', P'' are isomorphic as before.

From the above result we see that if S is a cyclic transversal and $S' = S_{(1,h)}$, $S'' = S_{(2,h)}$, then $P_1'(\underline{\underline{A}};S',T)$ and $P_1''(\underline{\underline{A}};S'',T)$ are isomorphic. Corresponding results for the changed transversals T' and T'' also hold.

However, in all the above cases $P(\underline{\underline{A}};S,T)$ is not, in general, isomorphic to $P_1'(\underline{\underline{A}};S',T)$ as the following example shows:

I.1.6. Example: We take A and B both isomorphic to the symmetric group of degree three, i.e.

$$A = gp\{a,b;\ \ a^3 = b^2 = (ab)^2 = 1\}$$

$$B = gp\{c,d;\ \ c^3 = d^2 = (cd)^2 = 1\}$$

and let

$$H = gp\{h;\ \ h^2 = 1,\ h = b = d\}$$

H possesses cyclic transversals $S = (1,a,a^2)$, $T = (1,c,c^2)$ in both A and B respectively. However, the permutational products $P(\underline{\underline{A}};S,T)$ and $P_1'(\underline{\underline{A}};S',T)$ where $S' = (h,a,a^2)$ or $P(\underline{\underline{A}},S,T)$ and $P_2'(\underline{\underline{A}};S,T')$ where $T' = (h,c,c^2)$ are not isomorphic because P is of order 18 and $P_i'(i = 1,2)$ is of order 162. (cf. [1]. B.H. Neumann).

But if A is a cyclic group of finite or infinite order, then we have a positive answer, namely, we have the following:

I.1.7. Theorem: Let A be a cyclic group and $H = gp\{h;\ \ h = s^m\}$ where A is generated by s, then $P(\underline{\underline{A}};S,T)$ and $P_1'(\underline{\underline{A}};S',T)$ are isomorphic. $(S' = S_{(1,h)})$.

<u>Proof</u>: Define a mapping $\varphi : K \rightarrow K'$ such that for $(s^k, t, h^j) \in K = S \times T \times H$

$$(s^k, t, h^j)^\varphi = (s^{k+1}, t, h^j) \in K'.$$

φ is one-one and therefore possesses an inverse. Also for $a \in A$, we have,

$$(s^{k+1}, t, h^j)^{\varphi^{-1}\rho(a)\varphi} = (s^k, t, h^j)^{\rho(a)\varphi}$$

$$= (s^{k'}, t, h^{j'})^\varphi$$

$$= (s^{k'+1}, t, h^{j'})$$

and

$$(s^{k+1}, t, h^j)^{\rho'(a)} = (s^{k''}, t, h^{j''})$$

where in both cases

$$s^{k+1}h^j a = s^{k''}h^{j''} = s^{k'+1}h^{j'}$$

Hence

$$k'' = k'+1,\ j'' = j' \text{ and therefore } \varphi^{-1}\rho(a)\varphi = \rho'(a)$$

for all $a \in A$. For $b \in B$,

$$(s^{k+1}, t, h^j)^{\varphi^{-1}\rho(b)\varphi} = (s^k, t, h^j)^{\rho(b)\varphi}$$

$$= (s^k, t', h^{j'})^\varphi$$

$$= (s^{k+1}, t', h^{j'})$$

$$= (s^{k+1}, t, h^j)^{\rho'(b)}$$

So that $\varphi^{-1}\rho(b)\varphi = \rho'(b)$ for all $b \in B$. Hence $\varphi^{-1}P\varphi = P'$ and the isomorphism of P and P' follows because φ is one-one. [If $S' = S_{(1,h)}$, $S'' = S_{(1,2,h)}$, ..., $S^{(m)} = S_{(1,2,\ldots,m,h)}$ where $S = (a^0, a^1, \ldots, a^{m-1})$, and $S_{(1,2,\ldots,i,h)} = (a^0h, a^1h, \ldots, a^{i-1}h, a^i, \ldots, a^{m-1})$, then we have,

I.1.71. <u>Corollary</u>: The permutational products $P(\underline{\underline{A}},S,T)$, $P_1'(\underline{\underline{A}};S',T)$, ..., $P^{(m)}(\underline{\underline{A}};S^{(m)},T)$ are all isomorphic.

We have discussed only some particular changes in the transversals of the amalgamated subgroup. General results of some kind or other in this direction seem to be very difficult to prove. It is known that if H is central in one of the constituents, the isomorphism type of the permutational product is independent of the change of transversals in the other constitutents. In the case of theorem I.1.7; therefore, we need examine the changes in the transversals of H in A only where A is a cyclic group. It is true that amongst the permutational products obtained by changing the transversals of H in the cyclic constituent A, a large number are isomorphic but I have not been able to prove that they are all isomorphic to each other although this seems a possibility.

I.2. It is known that given two groups A and B with an amalgamated subgroup H where H is central in one of the constituents say A, the isomorphism type of permutational products of A and B amalgamating H is independent of the change of transversal in the other constituent i.e. in B. (cf. [1]. Theorem 2). We generalise this result and prove that this is also true if the amalgamated subgroup possesses in one of the constituents a transversal which it centralises, that is, we show:

I.2.1. <u>Theorem</u>: Given two groups A and B with an amalgamated subgroup H, let S be a transversal of H in A which is centralised by H, then the isomorphism type of the permutational product $P(\underline{\underline{A}};\ S,T)$ independent of the change of transversals T in the other constituent, i.e. in B.

<u>Proof</u>: Let T and T' be two distinct transversals of H in B and $P(\underline{\underline{A}};S,T)$, $P'(\underline{\underline{A}};S,T')$ permutational products of A,B corresponding to the transversals S,T and S,T' of H in A and B respectively.
We define a one to one mapping φ from $K = S \times T \times H$ to $K' = S \times T' \times H$ in the following manner:

If $(s,t,h) \in K$, then

$$(s,t,h)^{\varphi} = (s,t',h')$$

where $(s,t',h') \in K'$ and $th = t'\,h'$.

Let $a \in A$, then, since φ^{-1} exists,

$$(s,t',h')^{\varphi^{-1}\rho(a)\varphi} = (s,t,h)^{\rho(a)\varphi}$$

$$= (s_1,t,h_1)^{\varphi}$$

$$= (s_1,t',h_2h_1)$$

where $sha = s_1h_1$, $th = t'h'$, $th_1 = t'h_2h_1$ (1)

Also

$$(s,t',h')^{\rho'(a)} = (s_2,t',h_2')$$

where

$$sh'a = s_2h_2' \qquad (2)$$

Now from $th = t'h'$, we have $t' = thh'^{-1}$ and putting it in $t = t'h_2$, we get

$$t = thh'^{-1}h_2$$

so that $hh'^{-1}h_2 = 1$ i.e. $h' = h_2h$.

Therefore

$$sh'a = sh_2ha = h_2sha, \text{ by the assumption that } [s,h] = 1 \text{ for all } h \in H,\ s \in S.$$

$$= h_2s_1h_1 \quad \text{from (1),}$$

$$= s_1h_2h_1 \quad \text{by assumption}$$

$$= s_2h_2' \quad \text{from (2).}$$

Therefore $s_1 = s_2$, $h_2h_1 = h_2'$, and $\varphi^{-1}\rho(a)\varphi = \rho'(a)$ for all $a \in A$.

For $b \in B$, we have

$$(s,t',h')^{\varphi^{-1}\rho(b)\varphi} = (s,t,h)^{\rho(b)\varphi}$$

$$= (s,t_1,h_1)^{\varphi}$$

$$= (s,t_1',h_1')$$

and

$$(s,t',h')^{\rho'(b)} = (s,t_2',h_2)$$

where in the first case

$$thb = t_1h_1 = t_1'h_1'$$

and from the second equation,

$$t'h'b = t_2'h_2'.$$

Since $th = t'h'$, therefore,

$$thb = t'h'b = t_1'h_1' = t_2'h_2' .$$

Hence $t_1' = t_2'$, $h_1' = h_2'$ so that $\varphi^{-1}\rho(b)\varphi = \rho'(b)$ for all $b \in B$, i.e. $\varphi^{-1}P\varphi = P'$ and therefore P and P' are isomorphic.

This gives us

I.2.11. Corollary: If H is a direct factor in A, then the isomorphism type of the permutational product is independent of the change of transversals in B.

Proof: This is immediate: choose a complementary direct factor as a transversal.

I.2.12. Corollary. If H possesses in each, A and B, at least one transversal which it centralises, then all permutational products of the amalgam formed with transversals of which at least one centralises H, are isomorphic.

Proof: Let S_1 and T_1 be transversals of H in A and B respectively which are centralised by H, and S and T any arbitrary transversals. Then by theorem 2.1

$$\rho(\underline{\underline{A}};S,T_1) \cong \rho(\underline{\underline{A}};S_1,T_1) \cong \rho(\underline{\underline{A}};S_1,T)$$

as required.

One may, quite naturally expect that when the amalgamated subgroup has transversals which it centralises in both the constituents, there is only one isomorphism type of permutational product. However, the following example shows that this is hoping too much.

I.2.13. <u>Example</u>: The groups A and B are taken as isomorphic to the dihedral group of order 12 which can be considered as the direct product of the dihedral group of order 6 by a cyclic group of order 2. Thus

$$A = gp\{a,b,c;\ a^3 = b^2 = c^2 = (ab)^2 = [a,c] = [b,c] = 1\}$$
$$B = gp\{a',b',d;\ a'^3 = b'^2 = d^2 = (a'b')^2 = [a',d] = [b',d] = 1\}$$

We take H as

$$H = gp\{g,h;\ g^3 = h^2 = (gh)^2 = 1,\ g = a = a', h = b = b'\}$$

The transversals $S = (1,c)$, $T = (1,d)$ are centralised by H. The permutational product $P(\underline{\underline{A}};S,T)$ is the direct product of the four group by a group isomorphic to H and so has order 24. However, if we choose the transversals as $S' = (h,c)$, $T' = (g,d)$, the permutational product $P'(\underline{\underline{A}};S',T')$ turns out to be of order 72. P and P' are obviously non-isomorphic.

As already mentioned, when H is central in both A and B and so centralises all its transversals in the constituents, then the isomorphism type of the permutational product is unique. In fact it is then the generalised direct product of A and B amalgamating H (cf. [1]). The examples given by B.H. Neumann [1] also show how drastic the effect of a change in the transversals can be if the amalgamated subgroup is not central in both the constituents. It may therefore be asked whether, in all other cases, excepting the one above (i.e. of H being central in both A and B) permutational products of A and B always depend on the choice of transversals of the amalgamated subgroup. This, however, is not the case, as the following example constructed by B.H. Neumann in a different context (cf. [2]) shows:

I.2.2. Example: Let H be the restricted direct product of an infinite number of cyclic groups of order 2, that is

$$H = gp\{h_0, h_1, h_2, \ldots;\ h_i^2 = [h_i, h_j] = 1,_{j,i=0,1,2,\ldots}\}$$

Let α, β be the automorphisms of H defined by

$$h_{2i}^{\alpha} = h_{2i+1},\ h_{2i+1}^{\alpha} = h_{2i},\ (i = 0,1,2,\ldots)$$

and

$$h_0^{\beta} = h_0,\ h_{2i+1}^{\beta} = h_{2i+2},\ h_{2i+2}^{\beta} = h_{2i+1},\ (i = 0,1,2,\ldots)$$

We now extend H by cyclic groups $C_1 = gp\{a;\ a^2 = 1\}$ and $C_2 = gp\{b;\ b^2 = 1\}$ corresponding to these automorphisms to get two groups A and B respectively. Thus

$$A = gp\{a, H;\ a^2 = h_i^2 = [h_i, h_j] = 1, h_{2i}^a = h_{2i+1}, h_{2i+1}^a = h_{2i}\}$$

$$B = gp\{b, H;\ b^2 = h_i^2 = [h_i, h_j] = 1,\ h_0^b = h_0, h_{2i+1}^b = h_{2i+2},$$
$$h_{2i+2}^b = h_{2i+1}\}$$

Let P be a permutational product of A and B amalgamating H, then in P, ab is an element of infinite order, because for any non-zero integer n

$$h_0^{(ab)^n} = h_0^{ab.(ab)^{n-1}} = h_1^{b(ab)^{n-1}} = h_2^{(ab)^{n-1}} = \ldots = h_{2n} \neq h_0$$

If F is the free product of A and B amalgamating H then, by the definition of the free product, there is a homomorphism of F onto P. To show that F and P are isomorphic, it is, therefore, enough to prove that it is impossible to add an additional relation in F different from those already implied by the relations of A and B, without making any of the groups collapse.

Now it follows from the general theory of free products with one amalgamated subgroup that a general element of F can be written uniquely in the form

$$r = ha^{\varepsilon_1}bab \ldots ab^{\varepsilon_2}$$

$\varepsilon_i = 0$ or 1 $i = 1,2$, and $h \in H$. Hence a relation $r = 1$ gives

$$h = b^{\varepsilon_2}abab \ldots a^{\varepsilon_1} .$$

If the right hand side is equal to 1, then this is a relation in H; hence we may assume it to be different from 1. Then $\varepsilon_1, \varepsilon_2$ cannot simultaneously be 1 or 0, for the right hand side in such a situation becomes $(ba)^{m+1}$ or $(ab)^m$ for some integer m according as $\varepsilon_1 = \varepsilon_2 = 1$ or $\varepsilon_1 = \varepsilon_2 = 0$. Therefore, because ab and ba are of infinite order in any group embedding the amalgam, they are of infinite order in F whereas h is of order 2. Thus either ε_1 or ε_2 is zero. Without any loss of generality, suppose that $\varepsilon_2 = 0$, then $\varepsilon_1 = 1$ and

$$h = ababa \ldots ba .$$

The right hand side has an odd number of factors, each of order 2, hence it is a conjugate of the central factor, 'a' in our case, by a power of ba:

$$h = a^{(ba)^k}$$

for some integer k. But $H = gp\{h_i\}$ is normal in each of the constituents A and B, hence transforming h by $(ba)^{-k}$ gives $a \in H$, which is impossible because this leads to the collapse of A. Therefore no proper homomorphic image of F embeds the amalgam of A and B. The kernel of this homomorphism being trivial, an isomorphism between F and P is established. As the free product of an amalgam is unique to within isomorphism, this amalgam possesses only one permutational product but for isomorphism.

Thus the case of the amalgamated subgroup being central in one of the constituents is, by no means, the only one for which we get a unique permutational product of an amalgam of two groups.

The above example also proves another interesting fact; that in some cases the free product of two groups with amalgamation may coincide with their permutational product.

However, the free product of two groups with trivial amalgamation can never coincide with their permutational product because the permutational product of such an amalgam degenerates into their direct product and since by a theorem of Baer and Levi [10], a group which is decomposable into the free product of its subgroups cannot be decomposed into their direct product, we conclude that in such a case (amalgam with trivial amalgamation), the permutational product of A,B is always different from their free product.

Finally we mention that there exist amalgams of two groups such that their permutational product is different from their free product, the amalgamated subgroup is central in none of the constituents and still we have only one isomorphism type of permutational product. This is shown by the following example.

I.2.3. <u>Example</u>: Let

$$A = gp\{a,b;\ a^3 = b^2 = (ab)^2 = 1\}$$

$$B = gp\{c,d;\ c^3 = d^2 = (cd)^2 = 1\}$$

and

$$H = gp\{h;\ h^3 = 1,\ h = a = c\}$$

All the permutational products of A and B are different from their free product (this is due to the fact that a permutational product of a proper amalgam of two finite groups being subgroups of the permutation group on a finite set $K = S \times T \times H$ where S and T are coset representatives of H in A and B, is finite whereas their free product is always infinite.) Moreover, the amalgamated subgroup is central in none of the constituents. But the only different looking permutational products of this amalgam corresponding to distinct transversals, are given by

$$P_1 = gp\{\alpha,\beta,\gamma;\ \beta^2 = \gamma^2 = (\beta\gamma)^6 = 1,\ (\beta\gamma)^2 = \alpha\}$$

$$P_2 = gp\{a',b',c';\ a'^3 = b'^2 = c'^2 = (b'c')^2 = (c'a')^2 = (a'b')^2 = 1\}$$

P_1 can also be generated by β and γ alone. The mapping

$$\beta \longrightarrow b'a', \ \gamma \longrightarrow c'$$

is an isomorphism between P_1 and P_2.

I.3. We now examine the structure of the permutational product $P(\underline{\underline{A}};S,T)$ of $\underline{\underline{A}}$ corresponding to the transversals S and T which are centralised by H. It will be shown that this particular permutational product possesses the properties of the generalised direct product of groups. In fact, we shall prove that this permutational product is itself the generalised direct product of some groups isomorphic to subgroups of the constituents in $\underline{\underline{A}}$.

Let us denote by $C_A(S)$ the centraliser of S in A, then we have:

I.3.1. <u>Lemma</u>: Let $H \subseteq C_A(S) \cap C_B(T)$, then in the permutational product $P(\underline{\underline{A}};S,T)$ of $\underline{\underline{A}}$, $\rho(S)$ and $\rho(T)$ commute elementwise.

<u>Proof</u>: Let $(s_1,t_1,h_1) \in K = S \times T \times H$, then for any $s \in S$ and $t \in T$, we have,

$$(s_1,t_1,h_1)^{\rho(s)\rho(t)} = (s_2,t_1,h_2)^{\rho(t)} = (s_2,t_2,h_3)$$

where $\quad s_1h_1s = s_2h_2, \quad t_1h_2t = t_2h_3 \qquad (1)$

Also

$$(s_1,t_1,h_1)^{\rho(t)\rho(s)} = (s_1,t_2',h_2')^{\rho(s)} = (s_2',t_2',h_3')$$

where $\quad s_1h_2's = s_2'h_3', \quad t_1h_1t = t_2'h_2' \qquad (2)$

From (1), and (2), we have,

$$s_1 s = s_2 h_2 h_1^{-1} = s_2' h_3' h_2'^{-1} \quad \text{i.e.} \quad s_2 = s_2', \quad h_2 h_1^{-1} = h_3' h_2'^{-1} \qquad (3)$$

and

$$t_1 t = t_2 h_2^{-1} h_3 = t_2' h_1^{-1} h_2'$$

$$\therefore \quad t_2 = t_2', \; h_2^{-1} h_3 = h_1^{-1} h_2' \qquad (4).$$

From (3) and (4), $h_3 = h_2 h_1^{-1} h_2' = h_3'$. Therefore $\rho(s)\rho(t) = \rho(t)\rho(s)$. This being true for all $(s_1, t_1, h_1) \in K$, and $s \in S$, $t \in T$, we have

$$[\rho(S), \rho(T)] = 1$$

~~$[\rho(s), \rho(t)] = 1$~~.

I.3.2. <u>Theorem</u>. Let H possess in A and B transversals S and T respectively which it centralises, then the permutational product $P(\underline{\underline{A}}; S, T)$ of $\underline{\underline{A}} = am(A,B;\ H)$ can be represented as the generalised direct product of any of the following three sets of groups.

(i) The groups K, L, and $\rho(H)$ where $K = gp\{\rho(S)\}$, $L = gp\{\rho(T)\}$; or

(ii) $\rho(A)$ and L; or

(iii) $\rho(B)$ and K.

<u>Proof</u>: To prove (i) we first note that since $\rho(S)$ and $\rho(T)$ commute elementwise (lemma 3.4), so do also the groups $K = gp\{\rho(S)\}$ and $L = gp\{\rho(T)\}$. Let $K \cap L = R$, then R is central in K because R is a subgroup of L which centralises K; so also it is in L because then we consider it as a subgroup of K, and L and K commute elementwise.

Moreover, R being a subgroup of both K and L and so also of $\rho(A)$ and $\rho(B)$, R is contained in $\rho(H)$. Furthermore, R is central in $\rho(H)$ because it is a subgroup of K and L which are centralised by $\rho(H)$. Since K,L and $\rho(H)$ contain R as a central subgroup and P is generated by these groups it is their generalised direct product amalgamating R.

We now proceed to the proof of (ii). The group P which is generated by $\rho(S)$, $\rho(T)$, $\rho(H)$, can also be generated by $\rho(A)$ and $\rho(T)$ i.e. by $\rho(A)$ and L only, where $L = gp\{\rho(T)\}$. Let now $\rho(A) \cap L = R_1$: we show that R_1 is central in L as well as in $\rho(A)$. Since R_1 is a subgroup of $\rho(A)$ and L, R_1 is contained in $\rho(H)$ (the meet of A and B being only H.). As a subgroup of $\rho(H)$, R_1 and L commute elementwise because H centralises T and so also L. Also R_1 is central in $\rho(H)$ because R_1 is in L. Since $\rho(A)$ is generated by $\rho(S)$ and $\rho(H)$, and L centralises $\rho(S)$, therefore R_1 centralises $\rho(S)$, that is R_1 is central also in $\rho(A)$. P being generated by $\rho(A)$ and L with their meet central in both is their generalised direct product amalgamating R_1.

The proof of (iii) is exactly the same as that of (ii).

The theorem is now completely proved.

The following theorem gives the nature of subgroups generated by elements of A and B contained in the centraliser of H in A and B respectively, in a permutational product P of A and B amalgamating H.

I.3.3. Theorem. Let $\{a_i\}$ and $\{b_j\}$ be two sets, finite or infinite, of elements of A and B respectively such that each $a_i \in C_A(H)$ $b_i \in C_B(H)$ and H abelian, then in any permutational product P of A and B, $A_m = gp\{\rho(a_1), \ldots, \rho(a_m)\}$, $B_n = gp\{\rho(b_1), \ldots, \rho(b_n)\}$ generate their generalised direct product.

Proof: Let $(s,t,h) \in K = S \times T \times H$, then we have to show that $\rho(a_i)\rho(b_j) = \rho(b_j)\rho(a_i)$ for $i = 1,2,\ldots,m$ $j = 1,2,\ldots,n$.

Now

$$(s,t,h)^{\rho(a_i)\rho(b_j)} = (s',t,h')^{\rho(b_j)} = (s',t',h'')$$

where

$$sha_i = sa_ih = s'h' \text{ and } th'b_j = tb_jh' = t'h'' \quad (1)$$

Also

$$(s,t,h)^{\rho(b_j)\rho(a_i)} = (s,t_1,h_1)^{\rho(a_i)} = (s_1,t_1,h_2)$$

where

$$sh_1a_i = sa_ih_1 = s_1h_2, \text{ and } thb_j = tb_jh = t_1h_1 \quad (2)$$

From (1) and (2), we have,

$$sa_i = s_1h_2h_1^{-1} = s'h'h^{-1} \text{ and } tb_j = t_1h_1h^{-1} = t'h''h'$$

Therefore

$$s_1 = s', \quad h_2h_1^{-1} = h'h^{-1}, \text{ and } t_1 = t', \; h_1h^{-1} = h''h'$$

that is,

$$h_2 = h'h^{-1}h_1 = h_1h^{-1}h' = h'' \qquad \text{(because H is abelian).}$$

Thus

$\rho(a_i)\rho(b_j) = \rho(b_j)\rho(a_i)$ and consequently, A_m and B_n generate their generalised direct product in P.

Remark 1. If $a_i = s_i^* h_i^*$, $b_j = t_j^* h_j^*$, then for all $h \in H$,

$$1 = [a_i,h] = [s_i^* h_i^*,h] = [s_i^*,h]^{h_i^*}[h_i^*,h] = [s_i^*,h]^{h_i^*}$$

gives,

$$[s_i^*,h] = 1$$

Therefore $s_i^* \in C_A(H)$ for all $i = 1,2,\ldots,m$

similarly: $t_j^* \in C_B(H)$ for all $j = 1,2,\ldots,n$.

Thus if $S^* = \{s^*; s^* \in C_A(H)\}$, $T^* = \{t^*; t^* \in C_B(H)\}$ then since $[\rho(s^*), \rho(t^*)] = 1$ for all $s^* \in S^*$, $t^* \in T^*$, if $K^* = gp\{\rho(S^*)\}$, $L^* = gp\{\rho(T^*)\}$, then K^* and L^* generate their direct product in any permutational product of A and B amalgamating an abelian subgroup H.

Remark 2. The condition on H about its being abelian is necessary. Example I.2.13 would suffice to show this.

I.4. Although the concept and nature of the generalised free product with amalgamations is entirely different from that of a permutational product of a given family of groups still there are some results which exhibit certain analogies in the behaviour of free products and permutational products. For example, it is known that if $\{G_\alpha\}$ and $\{G'_\alpha\}$, (α belonging to an index set I) are two families of groups each having a common subgroup H and H' respectively, and a system of homomorphisms φ_α of G_α onto G'_α where any two φ_α and φ_β agree on H, is given, and further if F and F' are the free products of $\{G_\alpha\}$ and $\{G'_\alpha\}$

amalgamating H and H' respectively then there exists a homomorphism φ of F onto F' which extends all the φ_α. The proof of the above theorem is due to Hanna Neumann (cf. for example [3] Theorem 1.1). We prove a corresponding result giving a relationship between the permutational products of two families of groups $\{G_\alpha\}$ and $\{G'_\alpha\}$ amalgamating H and H' respectively. We have, of course, to choose the transversals in a particular manner. It is sufficient to show this for the permutational products of only two groups because the proof for more than two groups is not at all different.

Let A,B and A',B' be the given groups having common subgroups H and H' respectively. Let S and S' be coset representatives of H and H' in A and A' respectively. Let φ_A be a homomorphism of A onto A'. If S' is the set of distinct ~~elements~~ coset representatives in the image $S\varphi_A$ of S under this mapping, then we say that S and S' are "equivalent transversals" of H and H' in A and A' respectively. We similarly choose a pair of equivalent transversals T and T' of H and H' in B and B' respectively corresponding to a homomorphism $\varphi_B : B \rightarrow B'$ which coincides with φ_A on H.

We now prove the following:

I.4.1. <u>Theorem</u>. Let $\varphi_A : A \rightarrow A'$, $\varphi_B : B \rightarrow B'$ be homomorphisms of A onto A' and of B onto B' such that $\varphi_A|H = \varphi_B|H$, and further A = SH, B = TH. If a pair of transversals S', T' of H' in A',B' equivalent to S and T respectively, is chosen then there exists a homomorphism φ of the permutational product $P(\underline{\underline{A}};S,T)$ of $\underline{\underline{A}} = am(A,B;H)$ onto

$P'(\underline{\underline{A}}';S',T')$ of

$\underline{\underline{A}}' = \mathrm{am}(A',B';\ H')$, which extends both φ_A and φ_B.

<u>Proof</u>: First we show that in the permutational products $P(\underline{\underline{A}};S,T)$ and $P'(\underline{\underline{A}};S',T')$, the mappings

$$\rho(a) \longrightarrow \rho'(a') \quad \text{and} \quad \rho(b) \longrightarrow \rho'(b')$$

where $a \xrightarrow{\varphi_A} a'$, $b \xrightarrow{\varphi_B} b'$, $a \in A$, $b \in B$, $a' \in A'$, $b' \in B'$, are homomorphisms of $\rho(A)$ onto $\rho'(A')$ and of $\rho(B)$ onto $\rho'(B')$ respectively.

Denote by ρ^{-1} the inverse of the isomorphic mapping $a \longrightarrow \rho(a)$ of A onto $\rho(A)$ for $a \in A$. Then if

$$a_1 \xrightarrow{\varphi_A} a_1', \quad a_2 \xrightarrow{\varphi_A} a_2'$$

we have

$$\rho(a_1)\ \rho^{-1}\varphi_A\ \rho' = (a_1)\varphi_A\ \rho' = (a_1')\rho' = \rho'(a_1')$$

$$\rho(a_2)\ \rho^{-1}\varphi_A\ \rho' = \rho'(a_2')$$

and

$$\begin{aligned} \rho(a_1)\rho(a_2)\ \rho^{-1}\varphi_A\ \rho' &= \rho(a_1a_2)\ \rho^{-1}\varphi_A\rho' \\ &= (a_1a_2)\varphi_A\rho' \\ &= (a_1'a_2')\rho' \\ &= \rho'(a_1'a_2') \\ &= \rho'(a_1')\rho'(a_2')\ . \end{aligned}$$

Thus $\rho(a) \longrightarrow \rho'(a')$ is a homomorphism of $\rho(A)$ onto $\rho'(A')$, $a \in A$, $a' \in A'$. Similarly $\rho(b) \longrightarrow \rho'(b)$ gives a homomorphism of $\rho(B)$ onto $\rho'(B')$.

To prove that P' is a homomorphic image of P, we have to show that any law which hold in P also holds in P'. Let $(s,t,h) \in \mathfrak{K} = S \times T \times H$ and let w be a word in elements from $\rho(A)$ and $\rho(B)$, that is

$$w = \rho(a_1)^{\delta_1}\rho(b_1) \ldots \rho(a_n)\rho(b_n)^{\delta_2}$$

$$= \rho(a_1^{\delta_1})\rho(b_1) \ldots \rho(a_n)\rho(b_n^{\delta_2})$$

where δ_1 and δ_2 are 0 or 1, and $a_i \in A$, $b_j \in B$. Then

$$(s,t,h)^w = (s,t,h)^{\rho(a_i^{\delta_1})\rho(b_1) \ldots \rho(a_n)\rho(b_n^{\delta_2})}$$

$$= (s_1,t_1,h_1^*)^{\rho(a_2) \ldots \rho(a_n)\rho(b_n^{\delta_2})}$$

$$\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots$$

$$= (s_n,t_n,h_n^*) .$$

Here

$$sha_1^{\delta_1} = s_1h_1 \qquad \text{and} \qquad t\,h_1b_1 = t_1h_1^*$$

$$s_1h_1^*a_2 = s_2h_2 \qquad\qquad t_1h_2b_2 = t_2h_2^*$$

$$\ldots\ldots\ldots\ldots \qquad\qquad \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots \qquad\qquad \ldots\ldots\ldots\ldots$$

$$s_{n-1}h_{n-1}^*a_n = s_nh_n \qquad\qquad t_{n-1}h_nb_n^{\delta_2} = t_nh_n^* \quad .$$

Also if the elements s',t' of S',T' correspond to the elements s,t of S,T under the homomorphisms φ_A, φ_B respectively and $a_i' = a_i\varphi_A$, $b_i' = b_i\varphi_B$, then

$$(s',t',h')^{\rho'(a_1'^{\delta_1})\rho'(b_1') \ldots \rho'(a_n')\rho'(b_n'^{\delta_2})}$$

$$= (s\varphi_A, t\varphi_B, h\varphi_A)^{\rho'(a_1^{\delta_1}\varphi_A)\rho'(b_1\varphi_B) \ldots \rho'(a_n\varphi_A)\rho'(b_n^{\delta_2}\varphi_B)}$$

$$= (s_1\varphi_A, t_1\varphi_B, h_1^*\varphi_A)^{\rho'(a_2\varphi_A)\rho'(b_2\varphi_B) \ldots \rho'(a_n\varphi_A)\rho'(b_n^{\delta_2}\varphi_B)}$$

....

....

$$= (s_n\varphi_A, t_n\varphi_B, h_n^*\varphi_A)$$

because

$$s\varphi_A h\varphi_A a_1^{\delta_1}\varphi_A = (sha_1^{\delta_1})\varphi_A = (s_1h_1)\varphi_A = s_1\varphi_A h_1\varphi_A$$

and

$$s_i\varphi_A h_i^*\varphi_A a_{i+1}\varphi_A = (s_i h_i^* a_{i+1})\varphi_A = (s_{i+1}h_{i+1})\varphi_A = s_{i+1}\varphi_A h_{i+1}\varphi_A .$$

Similarly

$$t\varphi_B h_1\varphi_B b_1\varphi_B = (th_1b_1)\varphi_B = (t_1h_1^*)\varphi_B = t_1\varphi_B h_1^*\varphi_B$$

and

$$t_i\varphi_B h_{i+1}\varphi_B b_{i+1}\varphi_B = (t_i h_{i+1} b_{i+1})\varphi_B = (t_{i+1}h_{i+1}^*)\varphi_B$$

$$= t_{i+1}\varphi_B h_{i+1}^*\varphi_B .$$

Thus if

$$\gamma(\ldots,\rho(a),\rho(b),\ldots) = 1$$

is a relation in $P(\underline{\underline{A}};S,T)$, then

$$\gamma(\ldots,\rho'(a'),\rho'(b'),\ldots) = 1$$

is also a relation in $P'(\underline{\underline{A}}';S',T')$. By von Dyck's theorem, there exists a homomorphism φ of P onto P' which extends both φ_A and φ_B. This completes the proof of the theorem.

We shall make use of the above theorem in the special case when φ_A and φ_B are isomorphisms in a different context in Chapter III.

B.H. Neumann, in his famous "Essay on the generalised free product of groups with amalgamations" has proved the following theorem, which deals with a very restricted class of subgroups of the generalised free product.

I.4.2. <u>Theorem of B.H. Neumann</u> [3]. Let P be the generalised free product of two groups with an amalgamated subgroup H, that is $A \cap B = H$.

Let a set of elements t_α (where α ranges over an index set I) be given with the following two properties:

(i) Every t_α belongs to the normaliser of H in B, i.e. $t_\alpha \in B$ and $t_\alpha H = Ht_\alpha$.

(ii) No two different elements t_α, t_β, $\alpha \neq \beta$ lie in the same left coset (or what amounts to the same thing because of (i), right coset) modulo H, i.e.

$$t_\alpha H \neq t_\beta H \quad \text{for } \alpha \neq \beta .$$

Then the subgroup Q of P generated by the groups

$$G_\alpha = t_\alpha^{-1} A t_\alpha$$

is the generalised free product of these G_α with

$$H = G_\alpha \cap G_\beta \quad (\alpha \neq \beta)$$

amalgamated.

We show by an example that in the corresponding situation, the subgroups G_α of a particular permutational product of A and B, generate in this permutational product a subgroup which is no permutational product of these G_α.

4.21. Example:

Let

$$A = gp\{a;\ a^9 = 1\}$$
$$B = gp\{b,c,d;\ b^3=c^3=d^2=(bd)^2=(cd)^2=[b,c]=1\}$$

and

$$H = gp\{h;\ h^3=1, h=a^3=b\}$$

B is a splitting extension of an elementary abelian group of order 9 by a cyclic group of order 2.

As H is central in A, the isomorphism type of permutational product of A and B is independent of the choice of transversals of H in B. Also since H has index 3 in A and since H and so every coset of H has order 3 there are $3^3 = 27$ choices of transversals of H in A, and consequently 27 permutational products of A and B. However, in all these permutational products the corresponding groups

$$A_c = \rho(c)^{-1}\rho(A)\rho(c) \quad \text{and} \quad A_d = \rho(d)^{-1}\rho(A)\rho(d)$$

generate a nonabelian group. But as these groups are cyclic and hence abelian, the only permutational product obtainable from them is their generalised direct product which must be abelian, and therefore not isomorphic to the group generated by A_c and A_d in any of the corresponding permutational products of A and B amalgamating H:

If we are given an amalgam of two groups A and B amalgamating H and if

$$A = A_1 \times A_2 \times \ldots \times A_n, \quad B = B_1 \times B_2 \times \ldots, B_n$$

in such a way that

$$A \cap B = H = H_1 \times H_2 \times \ldots \times H_n$$

where $H_i = A_i \cap B_i$ for all $i = 1,2,\ldots,n$, one will expect the permutational product of A and B to be the direct product of the permutational products of the amalgams $am(A_i, B_i;\ H_i)$ provided the transversals are chosen in the natural way. The following theorem for which we suppose the groups to have, without any loss of generality, only two factors, confirms this guess.

I.4.3. <u>Theorem</u>: Let $P_1(\underline{\underline{A}}_1;S_1,T_1)$ and $P_2(\underline{\underline{A}};S_2,T_2)$ be permutational products of A_1,B_1, amalgamating H_1 and of A_2,B_2 amalgamating H_2. Let further, $A = A_1 \times A_2$, $B = B_1 \times B_2$ such that $H = H_1 \times H_2$. Then the permutational product $P(\underline{\underline{A}};S,T)$ of A and B amalgamating H is the direct product of P_1 and P_2 if $S = S_1 \times S_2$ and $T = T_1 \times T_2$.

<u>Proof</u>: We first prove that in P, the group P_1' generated by $\rho'(A_1)$, $\rho'(B_1)$ is a direct factor of P. For this we have to show that $\rho'(A_1)$, $\rho'(B_2)$ and $\rho'(A_2)$, $\rho'(B_1)$ commute elementwise.

Since the transversals of H in A and B are taken as $S = S_1 \times S_2$, $T = T_1 \times T_2$, therefore, every $(s,t,h) \in K = S \times T \times H$, can be written as:

$$(s_2s_1,t_1t_2,h_1h_2)$$

where $s_i \in S_i$, $t_i \in T_i$, $h_i \in H_i$, $i = 1,2$.

Let $a_1 \in A_1$, $b_2 \in B_2$, then

$$\begin{aligned}(s_2s_1,t_1t_2,h_1h_2)^{\rho'(a_1)\rho'(b_2)} &= (s_2s_1',t_1t_2,h_2h_1')^{\rho'(b_2)}\\ &= (s_2s_1',t_1t_2',h_2'h_1')\end{aligned}$$

where

$$s_2s_1h_2h_1a_1 = s_2s_1h_1a_1h_2 = s_2s_1'h_2h_1' = s_2s_1'h_1'h_2$$

i.e.

$$s_1h_1a_1 = s_1'h_1'$$

and

$$t_1t_2h_2h_1'b_2 = t_1t_2h_2b_2h_1' = t_1t_2'h_2'h_1'$$

hence

$$t_2h_2b_2 = t_2'h_2'$$

Also if

$$\rho(a_{11}^{\varepsilon_1})\ \rho(b_{11})\ \ldots\ \rho(a_{1n})\ \rho(b_{1n}^{\delta_1})$$

$$= \rho(a_{21}^{\varepsilon_2})\ \rho(b_{21})\ \ldots\ \rho(a_{2m})\ \rho(b_{2m}^{\delta_2})$$

$$\neq$$

then for any $(s_2s_1,\ t_2t_1,\ h_2h_1) \in K$, we have,

$$(s_2s_1,\ t_2t_1,\ h_2h_1)^{\rho(a_{11}^{\varepsilon_1})\ \rho(b_{11})\ \ldots\ \rho(a_{1n})\ \rho(b_{1n}^{\delta_1})} = (s_2s_1',\ t_2t_1',\ h_2h_1')$$

$$= (s_2s_1,\ t_2t_1,\ h_2h_1)^{\rho(a_{21}^{\varepsilon_2})\ \rho(b_{21})\ \ldots\ \rho(a_{2m})\ \rho(b_{2m}^{\delta_2})}$$

$$= (s_2's_1,\ t_2't_1,\ h_2'h_1)$$

then

$$s_2's_1 = s_2s_1',\quad t_2't_1 = t_2t_1',\quad h_2'h_1 = h_2h_1'$$

that is

$$s_2^{-1}s_2' = s_1's_1^{-1} \in A_1 \cap A_2,\quad t_2^{-1}t_2' = t_1'\,t_1^{-1} \in B_1 \cap B_2$$

$$h_2^{-1}h_2' = h_1'h_1^{-1} \in H_1 \cap H_2$$

and since ...

However, also

$$(s_2 s_1, t_1 t_2, h_2 h_1)^{\rho'(b_2)\rho'(a_1)} = (s_2 s_1', t_1 t_2', h_2' h_1') \ .$$

Therefore, $[\rho(a_1),\rho(b_2)] = 1$ and consequently in P $\rho'(A_1)$, $\rho'(B_2)$ commute elementwise. By symmetry $\rho'(A_2)$ and $\rho'(B_1)$ also commute elementwise. Hence if

$$P_1' = gp\{\rho(A_1),\rho(B_1)\} \quad \text{and} \quad P_2' = gp\{\rho'(A_2), \rho'(B_2)\}$$

then P_1' and P_2' commute element by element. Also, since A_1,A_2; B_1,B_2; A_1,B_2; A_2,B_1 all intersect trivially therefore $P_1' \cap P_2' = \{1\}$. Since P is generated by P_1' and P_2' together, P_1' and so also P_2' is a direct factor in P. Thus $P = P_1' \times P_2'$. We now show that $P_i' \cong P_i$, where P_i is the permutational product of A_i,B_i amalgamating H_i corresponding to the transversals S_i,T_i of H_i in A_i and B_i respectively.

We take a word

$$w = \rho(a_{11})\rho(b_{11}) \ldots \rho(a_{1n})\rho(b_{1n}), \ a_{1i} \in A_1, \ b_{1i} \in B_1,$$

in P_1. Then if $(s_1,t_1,h_1) \in K_1 = S_1 \times T_1 \times H_1$, we have

$$(s_1,t_1,h_1)^{\rho(a_{11})\rho(b_{11}) \ldots \rho(a_{1n})\rho(b_{1n})}$$

$$= (s_2,t_1,h_2')^{\rho(b_{11}) \ldots \rho(a_{1n})\rho(b_{1n})}$$

$$= (s_2 t_2, h_2)^{\rho(a_{12})\rho(b_{12}) \ldots \rho(a_{1n})\rho(b_{1n})}.$$

....

....

$$= (s_{n+1}, t_{n+1}, h_{n+1})$$

where

$$s_1h_1a_{11} = s_2h'_2 \quad \text{and} \quad t_1h'_2b_{11} = t_2h_2$$

$$s_2h_2a_{12} = s_3h'_3 \quad \text{"} \quad t_2h'_3b_{12} = t_3h_3$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$s_nh_na_{1n} = s_{n+1}h'_{n+1} \qquad t_nh'_{n+1}b_{1n} = t_{n+1}h_{n+1} .$$

Also if $(s_2s_1,t_2t_1,h_2h_1) \in K = S \times T \times H$, then

$$(s_2s_1,t_2t_1,h_2h_1)^{\rho'(a_{11})\rho'(b_{11}) \ldots \rho'(a_{1n})\rho'(b_{1n})}$$

$$= (s_2s_{n+1},t_2t_{n+1},h_2h_{n+1}) .$$

Therefore if $w = 1$ is a relation in P_1, it is also a relation in P'_1, but the converse also holds. Since the relation of P_1 and P'_1 are in one to one correspondence therefore they are isomorphic. Similarly $P_2 \cong P'_2$. Thus

$$P \cong P_1 \times P_2$$

as required.

That the choice of transversals of H in A and B in this particular way is necessary, is shown by the following example.

I.4.31. <u>Example</u>: Let A_i and B_i be symmetric groups of degree three, that is

$$A_i = gp\{a_i, c_i;\ a_i^3 = c_i^2 = (a_i c_i)^2 = 1\}$$

$$B_i = gp\{b_i, c_i;\ b_i^3 = c_i^2 = (b_i c_i)^2 = 1\}$$

and

$$A = A_1 \times A_2,\ B = B_1 \times B_2 \text{ and } H = H_1 \times H_2$$

where $H_i = gp\{c_i;\ c_i^2 = 1\}$. Let P_i be the permutational product of A_i, B_i amalgamating H_i corresponding to the transversals $S_i = \{c_i, a_i, a_i^2\}$ and $T_i = \{1, b_i, b_i^2\}$. Then each of the P_i's $i = 1,2$, has order 162 (c.f. B.H. Neumann [1]) and therefore the order of $P_1 \times P_2$ is $162 \times 162 = 26244$. However, the permutational product P of A and B corresponding to the transversals $S = \{1, a_1, a_1^2\} \times \{1, a_2, a_2^2\}$ and $T = \{1, b_1, b_1^2\} \times \{1, b_2, b_2^2\}$, being an extension of an elementary abelian group of exponent 3 and order 81 by the four group is of order 324. P is, therefore, not isomorphic to $P_1 \times P_2$.

$P(\underline{\underline{A}}, S, T)$ is, of course, isomorphic to $P_1' \times P_2'$ where $P_i' = P_i'(\underline{\underline{A}}_i; S_i', T_i')$ with $S_i' = \{1, a_i, a_i^2\}$, $T_i' = \{1, b_i, b_i^2\}$.

I.5. Let P be a property satisfied by the groups of a certain amalgam $\underline{\underline{A}}$ (e.g, the property of being finite, soluble, etc.). As is shown in [1] and [2], an amalgam with a property P may not always be embeddable in a group having the same property. Sufficient conditions of one kind or another on the amalgam are, therefore essential. A condition which is fairly close to the hypothesis that the amalgamated subgroup be central in both the constituents is the existence of transversals, one in each of the constituents, which are centralised by the amalgamated subgroup.

We have seen in theorem I.3.2 that when the amalgamated subgroup H has, in the constituents A and B, transversals S and T respectively which it centralises, the permutational product $P(\underline{\underline{A}};S,T)$ of A and B amalgamating H is the generalised direct product of K,L and $\mathbf{P}(H)$ where $K = gp\{\mathbf{P}(S)\}$, $L = gp\{\mathbf{P}(T)\}$, amalgamating $K \cap L = R$. But then in such a case $P(\underline{\underline{A}};S,T)$ belongs to the least variety containing both A and B, so that if A and B are in a variety $\underline{\underline{V}}$, say, then so is also P in $\underline{\underline{V}}$, we have therefore:

I.5.0. <u>Theorem</u>: Let $\underline{\underline{A}} = am(A,B;\ H)$ be an amalgam of two groups belonging to a variety $\underline{\underline{V}}$, then $\underline{\underline{A}}$ is embeddable in a group belonging to $\underline{\underline{V}}$ provided that H possesses transversals S and T which it centralises in both A and B.

B.H. Neumann (cf. [1]) has shown that an amalgam of two soluble groups is embeddable in a soluble group if the amalgamated subgroup is central in one of the constituents. The above remark slightly varies this result. However, going further, we prove that, as suggested by theorem I.2.1, the condition that the amalgamated subgroup is central in one of the constituents can be replaced by the requirement that it possesses in one of the constituents a transversal which it centralises.

Some more results concerning the embeddability of a soluble amalgam (that is, an amalgam of soluble groups.) in a soluble group using a different sufficient condition will also be obtained.

We first repeat some of the definitions in [1]. Given two soluble groups A and B with a common subgroup H, by S and T we shall denote arbitrary but then fixed transversals of H in A and B respectively. By B^S, we mean the set of all functions on S with values in B. This is turned into a group by defining the multiplication of any two functions $f, g \in B^S$ as

$$fg(s) = f(s)g(s) \qquad \text{for all } s \in S$$

<u>Definition</u>: A mapping γ of the set K of all triplets (s,t,h), $s \in S$, $t \in T$, $h \in H$, into itself is a quasi-multiplication (or more precisely a quasi B-S multiplication) if there is a function f on S to B such that

$$(s,t,h)^{\gamma} = (s,t',h')$$

with

$$t'h' = thf(s).$$

The mapping γ associated with f is denoted by $\gamma(f)$. The set of all such functions ~~are~~ is known to form a group Γ isomorphic to B^S (cf. lemma 5.1 [1]).

To see that the results in [1] and [2] hold under this weaker condition, namely the existence of a transversal centralised by the amalgamated subgroup in one of the constituents, it is enough to show that the fundamental Lemma 5.2 [1] holds.

We restate the lemma making use of the new hypothesis.

I.5.10. Lemma (Compare with Lemma 5.2 [1]). Let H possess in one of the constituents, say A, a transversal S which it centralises. Then $\rho(A)$ normalises Γ. More precisely, for $a \in A$, $\gamma = \gamma(f) \in \Gamma$ there is an element $\gamma' = \gamma(f')$ of Γ such that

$$\rho^{-1}(a)\gamma(f)\rho(a) = \gamma(f'),$$

for $f' \in B^S$, and ρ a permutation of $K = S \times T \times H$.

Proof: The proof is essentially the same as in [1] except that here we just use the condition that H is contained in the centraliser of one of its transversals, say S, in a constituent A.

We compute

$$\gamma' = \rho(a^{-1})\gamma(f)\rho(a)$$

for $a \in A$, $\gamma(f) \in \Gamma$, $f \in B^S$. Let $(s,t,h) \in K = S \times T \times H$, then

$$(s,t,h)^{\rho(a^{-1})\gamma(f)\rho(a)} = (s_1,t,h_1)^{\gamma(f)\rho(a)}$$

$$= (s_1,t_1,h_2)^{\rho(a)}$$

$$= (s_2,t_1,h_3)$$

where

$$sha^{-1} = s_1h_1,\ th_1f(s_1) = t_1h_2,\ s_1h_2a = s_2h_3 \qquad \text{(i)}$$

Also

$$(s,t,h)^{\gamma(f')} = (s,t',h')$$

with

$$thf'(s) = t'h' \qquad \text{(ii)}$$

We have to show that $s_2 = s$, $t' = t_1$, $h' = h_3$ to prove that $\gamma(f') = \rho(a^{-1})\ \gamma(f)\rho(a)$.

Now from (i) we have

$$sh = s_1h_1a = s_1h_1h_2^{-1}s_1^{-1}s_2h_3 = s_2h_1h_2^{-1}h_3$$

($\because$ $[s_1,h_1] = [s_2,h_1h_2^{-1}] = 1$). Therefore $s = s_2$ and $h = h_1h_2^{-1}h_3$, that is

$$h_3 = h_2h_1^{-1}h \qquad \text{(i')}$$

Also

$$t_1h_3 = t_1h_2h_1^{-1}h \qquad \text{from (i')}$$

$$= th_1f(s_1)h_1^{-1}h \qquad \text{from (i)}$$

$$= thh^{-1}h_1f(s_1)h_1^{-1}h$$

$$= th(h_1^{-1}h)^{-1}f(s_1)h_1^{-1}h \quad .$$

Thus $$t_1h_3 = thc$$

where

$$c = (h_1^{-1}h)f(s_1)h_1^{-1}h$$

which depends only on s_1 and $h_1^{-1}h$. If we write

$$(sha^{-1})^{\sigma} = (sa)^{\sigma} = s_1, \ (sha)^{-\sigma+1} = h_1$$

we see that s_1 is independent of h. Also from $sha^{-1} = s_1h_1$ that is, $h_1^{-1}hsa^{-1} = s_1$, we have,

$$h_1^{-1}h = s_1as^{-1} = (sa^{-1})^{\sigma}as^{-1}$$

which depends only on a and not on h. Thus if we define the elements f_1, g, f' of B^S by

$$(1) \qquad f_1(s) = f(s_1) = f(sa^{-1})^{\sigma}$$

$$(2) \qquad g(s) = h_1^{-1}h = (sa^{-1})^{\sigma}as^{-1}$$

then

$$f'(s) = (h_1^{-1}h)^{-1}f(s_1)h_1^{-1}h.$$

$$= (g(s))^{-1}f_1(s)g(s)$$

for all $s \in S$ and we have

$$f' = g^{-1}f_1g$$

and

$$\gamma' = \gamma(f') = \gamma(g^{-1}f_1g)$$

where $f_1 \in B^S$, $g \in H^S$. This completes the proof of the lemma.

This gives us

I.5.11. Corollary (Compare with corollary 5.2 [1]). If $H \subseteq C_A(S)$, then

$$[\rho(A),\Gamma] \subseteq \Gamma .$$

Here $[K,L]$ means the group generated by all commutators $[k,\ell]$, $k \in L$, $\ell \in L$. The proof of the above corollary follows from the fact that $\rho(A)$ normalises Γ.

I.5.12. Corollary (Compare with corollary 5.3 [1]). If $H \subseteq C_A(S)$, then

$$[\rho(A),\Gamma'] \subseteq \Gamma'$$

where Γ' denotes the derived group of Γ.

I.5.13. Corollary (Compare with corollary 5.3 [1]). If $H \subseteq C_A(S)$, then

$$[\rho(A),\rho(B)] \subseteq \Gamma' .$$

Consequently, we have,

I.5.2. Theorem (Compare with theorem 5.4 [1]). If, in one of the constituents, say A, the amalgamated subgroup H possesses a transversal S which is centralised by H, and if further, A and B are soluble of length ℓ and m respectively, then the permutational product $P(\underline{A};S,T)$ of A and B is soluble of length n where n satisfies the relation

$$n \leq \ell + m - 1 .$$

We further remark, without going into the details that the results proved in [2] based on lemma 5.2 of [1] still hold under this weaker assumption.

Let F* denote one of the following properties of a group; being locally finite (LF), of finite exponent (FE), or being periodic (P). We discuss here the embeddability of a soluble or F* amalgam in a soluble or F* group respectively, making use of a sufficient condition of somewhat different nature. The following lemma plays a key role in the discussion that follows.

I.5.3. Lemma: Let the groups A and B be extensions of a normal subgroup S of A by H and of a normal subgroup T of B by H respectively. Then S and T serve as transversals and the permutational product $P(\underline{\underline{A}};\ S,T)$ of the amalgam $\underline{\underline{A}} = am(A,B;\ H)$ belongs to the least variety containing both A and B.

Proof: We first show that in P, $\rho(S)$ and $\rho(T)$ commute elementwise. Let $(s_1,t_1,h_1) \in K = S \times T \times H$, then for $s \in S$, $t \in T$, we have

$$(s_1,t_1,h_1)^{\rho(s)\rho(t)} = (s_1s',t_1,h_1)^{\rho(t)}$$
$$= (s_1s',t_1t',h_1)$$

and

$$(s_1,t_1,h_1)^{\rho(t)\rho(s)} = (s_1s',t_1t',h_1)$$

where

$$s_1h_1s = s_1s'h_1,\ t_1h_1t = t_1t'h_1$$

in both cases. Therefore $[\rho(s),\rho(t)] = 1$ for all $s \in S$, $t \in T$.

Before going into further details of the proof of the above lemma we remark that for a slightly more general situation when $S \cap T = Z \neq \{1\}$ is central in both S and T and S_1, T_1, given by $S_1 Z = S$, $T_1 Z = T$ are taken as transversals, $\rho(S)$ and $\rho(T)$ still commute elementwise in the permutational product $P(\underline{\underline{A}};\ S_1, T_1)$ of the amalgam $\underline{\underline{A}} = am(A,B : \{Z,H\})$.

Since P is generated by $\rho(S)$, $\rho(T)$ and $\rho(H)$ and moreover $\rho(H)$ normalises both $\rho(S)$ and $\rho(T)$ and hence also $\rho(S) \times \rho(T)$, P is an extension of $\rho(S) \times \rho(T)$ by $\rho(H)$.

Next we look at the amalgam of the groups A and B rather differently. We regard these groups as generated by S,H and T,H_1 respectively and suppose that there is a fixed isomorphism φ between H and H_1 so that the amalgam of A and B consists of quintuplets $(A,B,H,H_1,\varphi;\ H\varphi = H_1)$. We take the direct product G of A and B. Since $A = SH$, $B = TH_1$ and S,T are normal subgroups of A and B respectively, $S \times T$ is normal in G. Take the 'diagonal'

$$H' = \{(h,h_1)=(h,h\varphi);\ h \in H,\ h_1 \in H_1,\ h_1 = h\varphi\}$$

of the direct product $H \times H_1$ in G. H' is clearly isomorphic to H. Also the groups $A' = \{(sh,h\varphi);\ s \in S,\ h \in H\}$ and $B' = \{(h,th\varphi);\ h \in H,\ t \in T\}$ are isomorphic to A and B respectively under the isomorphisms $a = sh \longrightarrow a' = (sh,h\varphi)$ $b = th\varphi \longrightarrow b' = (h,th\varphi)$ and since $(sh,h\varphi) = (h',th'\varphi)$ implies $sh = h'$, $h\varphi = th'\varphi$ which give $s = 1 = t$, $h = h'$, the intersection of A' and B' is precisely H'.

The groups A' and B' can also be taken as generated by $S' = \{(s,1);\ s \in S\} \cong S$ and H' and by $T' = \{(1,t);\ t \in T\} \cong T$ ~~by~~ and H' respectively. However, since

$$h'^{-1}s'h' = (h,h\varphi)^{-1}(s,1)(h,h\varphi)$$
$$= (s^h,1),$$

and

$$h'^{-1}t'h' = (1,t^{h\varphi}).$$

$h' \in H'$, $s' \in S'$, $t' \in T'$; H' induces the same automorphisms in S' and T' as H and H_1 do in S and T respectively. The group P' generated by $S' \times T'$ and H' in $A \times B$, therefore, contains isomorphic copies A', B' of A and B, intersect in a common subgroup H' isomorphic to H and H_1 and is an extension of $S' \times T'$ by H' corresponding to the above automomorphisms.

As shown above P also is an extension of $\rho(S) \times \rho(T) \cong S' \times T'$ by $\rho(H) \cong H'$. Further, these two extensions correspond to the 'same' groups of automorphisms as induced by H in S and T and are, therefore, 'equivalent'. (cf. Kurosh, [18]).

Thus P' is isomorphic to P. Since P is a subgroup of $A \times B$ and belongs to the least variety containing both A and B, $P(\underline{\underline{A}};\ S,T)$ also has this property. This completes the proof of the lemma.

As a consequence of the above remarks, we have:

I.5.4. <u>Corollary</u>: A soluble or nilpotent amalgam of two groups A and B which are extensions of their normal subgroups S and T respectively by a group H, is embeddable in a soluble or nilpotent group.

I.5.5. Corollary: If the groups A and B of lemma I.5.3 have the property F*, then their amalgam is embeddable in an F* group.

I.5.6. Corollary: If the groups A and B of lemma I.5.3 are p-groups for the same p, that is, every element has order a power of p, then their amalgam is embeddable in a p-group.

In the case of finite groups this is a very special case of a result of Graham Higman [17].

CHAPTER II

EXISTENCE THEOREMS FOR GENERALISED FREE PRODUCTS OF GROUPS.

II.0 Introduction: Since, by a theorem of Hanna Neumann (cf. [4] theorem 5.0) the question of embeddability of an amalgam of an arbitrary collections of groups with amalgamations is reduced to that of their reduced amalgam, in this chapter, therefore, we shall consider only the reduced amalgams of groups as regards their embeddability. While any amalgam of two groups, by Schreier's theorem, is embeddable in a group F, the 'generalised free product of the amalgam', a corresponding statement for an amalgam of more than two groups does not hold in general. Even for an amalgam of three groups, not much is known about its embeddability. Some sufficient conditions for the existence of the generalised free product of amalgams of three groups are given in [4]. We establish here (II.1) two necessary and sufficient eriteria for the embeddability of such amalgams.

We also prove the existence of the generalised free product of an amalgam of three dihedral groups represented in a special form.

In II.2 we consider an amalgam of n groups having a 'limited' number of intersections, briefly called 'an amalgam of type S'. We show that while for $n = 3$ some kind of additional restriction is essential, for $n > 3$ no further condition is required for embeddability.

II.1. We start by stating, without proof, some of the known facts about the existence of the generalised free product of a given family of groups G_α amalgamating $H_{\alpha\beta}$, (α,β belong to an index set I).

It is known that the generalised free product of arbitrarily given groups with amalgamations need not exist at all. Some, fairly obvious, necessary conditions have, therefore, to be satisfied. The following theorem (due to Hanna Neumann [4]) gives some necessary conditions for the existence of the generalised free product of groups G_α amalgamating $H_{\alpha\beta}$.

II.1.10. <u>Theorem</u>: For the existence of the generalised free product of the groups G_α amalgamating $H_{\alpha\beta}$, it is necessary that for any three different suffixes α,β,γ, the three groups

$$H_{\alpha\beta} \cap H_{\alpha\gamma} = H_{\alpha\beta\gamma}$$

$$H_{\beta\gamma} \cap H_{\beta\alpha} = H_{\beta\gamma\alpha}$$

$$H_{\gamma\alpha} \cap H_{\gamma\beta} = H_{\gamma\alpha\beta}$$

are isomorphic and moreover, the isomorphisms $I_{\alpha\beta}$, $I_{\beta\gamma}$, $I_{\gamma\alpha}$ of $H_{\alpha\beta}, H_{\beta\gamma}, H_{\gamma\alpha}$ onto $H_{\beta\alpha}, H_{\gamma\beta}, H_{\alpha\gamma}$ respectively are such that for each $h_{\alpha\beta\gamma} \in H_{\alpha\beta\gamma}$

$$I_{\beta\gamma}(I_{\alpha\beta}\, h_{\alpha\beta\gamma}) = I_{\gamma\alpha}(h_{\alpha\beta\gamma}) .$$

However the above conditions for the existence of the generalised free product of G_α are, by no means, sufficient, because there are amalgams for which the conditions of the above theorem are satisfied while their generalised free products do not exist. (cf. [3], [4]).

Let G_α be the given groups with $H_{\alpha\beta}$ amalgamated $(\alpha,\beta \in I)$ $\alpha \neq \beta$. Denote by H_α the group generated by all $H_{\alpha\beta}$, $(\alpha,\beta \in I$, α fixed). The groups H_α then form an amalgam (called the reduced amalgam of the amalgam of G_α), with the same amalgamations. A necessary and sufficient condition for the existence of the generalised free product of G_α in terms of that of H_α is given by the following theorem, first proved by Hanna Neumann [4], (cf. also [13]).

II.1.11 Theorem: The generalised free product G of the groups G_α amalgamating $H_{\alpha\beta}$ exists if and only if the generalised free product H of H_α amalgamating $H_{\alpha\beta}$ exists.

The above theorem allows us to consider only the reduced amalgams. In the subsequent paragraphs, therefore, we shall consider, without mentioning, only the reduced amalgams of groups.

We shall also need the following concept. Let G be the generalised free product of G_α with amalgamated $H_{\alpha\beta}$. Following Hanna Neumann, we call a normal subgroup N of G 'tidy' with respect to G_α if it has the following two properties:

(i) $G_\alpha \cap N = 1$ for all α

(ii) N does not contain any element of the form $g_\alpha g_\beta$ where g_α, g_β are non-trivial elements of G_α and G_β respectively, $(\alpha \neq \beta)$.

Thus if N is a normal subgroup of G and is tidy with respect to G_α, then the factor group G/N contains groups G'_α, G'_β isomorphic to G_α and G_β

respectively with their intersection in G/N isomorphic to $H_{\alpha\beta}.G/N$, therefore, also embeds the amalgam of G_α.

We shall also require the following theorem in the proof of some of the results in this chapter.

II.1.12 Theorem (Hanna Neumann [4].). Let P be the free product of groups $G_\alpha (\alpha \in I)$ with one amalgamated subgroup H. In every G_α, let there be given a subgroup A_α which intersects H in a fixed subgroup B,

$$A_\alpha \subseteq G, \quad A_\alpha \cap H = B.$$

Then the subgroup Q of P generated by the A_α is their generalised free product with amalgamated subgroup B. If, in particular, the subgroups A_α have trivial intersection with H, then they generate their ordinary free product.

With these preliminaries in mind, we shall now examine the following problem:

"Given a reduced amalgam of groups G_α with amalgamated subgroups $H_{\alpha\beta}$, under what conditions does their generalised free product exist?". We shall consider this problem for some special cases only.

We begin with the consideration of an amalgam of three groups. Some of the examples of non-embeddable amalgams mentioned in [3], [4] consist of only three groups which is the least possible number since by Schreiers' theorem [9] an amalgam of two groups is always embeddable.

Even in this case no necessary and sufficient conditions are known, only some sufficient ones (cf. [4], Hanna Neumann). We prove here a different sufficient condition. Since we need take only the reduced amalgam, we write these groups as:

$$A = gp\{K,L\},\ B = gp\{K,M\},\ C = gp\{L,M\}.$$

The intersections $K \cap L$; $L \cap M$; $M \cap K$ are assumed to be all isomorphic, inview of the theorem II.1.10. We now prove:

II.1.2. <u>Theorem</u>: Let $A = K \times L$, $B = gp\{K,M\}$, $C = gp\{L,M\}$, such that K and L are normal subgroups of B and C respectively. Then the generalised free product of A,B,C exists.

<u>Proof</u>: We have only to show that there is a group embedding the amalgam of A,B,C. Since K is normal in B, L is normal in C, by Lemma I.5.3 K and L serve as transversals of M in these groups, and in the permutational product $P(\underline{\underline{A}};K,L)$ of the amalgam $\underline{\underline{A}} = am(B,C;\ M)$, $\rho(K)$ and $\rho(L)$ centralise each other and so generate a group isomorphic to A. P, therefore, embeds A also and hence the amalgam. Thus the generalised free product of A,B,C exists.

However, we can show a little more, namely, that the permutational product $P(\underline{\underline{A}};K,L)$ is isomorphic to the generalised free product F of A,B,C. Since F is ~~freely~~ generated by K,L,M, and M normalises both K and L the group generated by these, that is the group A is normal in F. The factor group F/A must then be isomorphic to M.

Thus F is an extension of A by M. Since P also is an extension of $\rho(A)$ by $\rho(M)$ and a homomorphic image of F of the same order as F, P is isomorphic to F. Since a permutational product of two finite groups is always finite, F itself is finite. We have, as a consequence, the following:

II.1.21 <u>Corollary</u>: The generalised free product of any three finite groups satisfying the assumptions of theorem II.1.2 is finite.

The conditions of theorem II.1.2 can be relaxed a little: The intersections $K \cap L$, $L \cap M$, $M \cap K$ need not be trivial for the existence of the generalised free product of A,B,C. Thus it suffices to assume that A is the generalised direct product of K and L amalgamating a central subgroup Z. The meet of L,M and K,M must then also, by theorem II.1.10, be Z. Moreover Z is a normal subgroup of M because if $z \in Z$, then since $z \in K$, $z^m = z' \in K$ and since $z \in L$, $z^m = z' \in L$ for any $m \in M$. Thus $z' \in K \cap L = Z$. By the remark in the proof of lemma I.5.3 we have, because of the centralising property of $\rho(K)$ and $\rho(L)$:

II.1.22 <u>Corollary</u>: If A is the generalised direct product of K and L amalgamating a central subgroup Z and in $B = gp\{K,M;\ K \cap M = Z\}$, $C = gp\{L,M;\ L \cap M = Z\}$, K and L are respectively normal, then the generalised free product of A,B,C exists.

That it is not sufficient by itself that K and L are normalised by M in B and C is shown by the following:

II.1.23 Example. We take K and L as the elementary abelian groups of order 4 and M as a cyclic group of order 3. A,B,C are then taken as splitting extensions of L by K; K by M and L by M respectively, so that

$$A = gp\{a,b,c,d;\ a^2=b^2=c^2=d^2=(ab)^2=(cd)^2 = 1\ c^a=d, d^a=c,\ c^b=d, d^b = c\}$$

$$B = gp\{a,b,f;\ a^2=b^2=(ab)^2=f^3=1, a^f=ab, b^f=a\}$$

$$C = gp\{c,d,f;\ c^2=d^2=f^3=(cd)^2=1, c^f=cd, d^f=c\}$$

If

$$F = gp\{a,b,c,d,f;\ R_1 \cup R_2 \cup R_3\}$$

where R_1, R_2, R_3 are the sets of relations of A,B and C respectively, and the amalgam is embeddable, then this group, being the group freely generated by it, must embed it. But in F, we have,

$$c^a = d \quad \text{or} \quad acad = 1$$

Therefore,

$$1 = (acad)^f = a^f c^f a^f d^f$$

$$= ab.cd.ab.c$$

However,

$$c^a=d,\ d^a=c \quad \text{and} \quad c^b=d,\ d^b=c \quad \text{gives}$$

$$(cd)^{ab} = (c^a d^a)^b = (dc)^b = d^b c^b = cd\ .$$

Therefore,

$$1 = (ab)^2.cd.c = d$$

From $d^f = c$, we have $c = 1$, so that in F the group C has collapsed, and consequently, F does not embed the amalgam, that is, the generalised free product does not exist.

Similarly if, instead of K and L being normal in B and C respectively, we have M normal in both B and C while A remains the direct product of K and L, then, in general, the free product of A,B,C does not exist, as the following example shows

II.1.24 Example. M here is taken as the elementary abelian 2-group of order 8 and K and L cyclic groups each of order 3. Then

$$A = gp\{a,b;\ a^3=b^3=[a,b]=1\}$$

$$B = gp\{a,h_i;\ a^3=h_i^2=[h_i,h_j]=1,\ h_1^a=h_1h_2, h_2^a=h_3, h_3^a=h_2h_3\}$$

$$C = gp\{b,h_i;\ b^3=h_i^2=[h_i,h_j]=1,\ h_1^b=h_1h_2, h_2^b=h_1, h_3^b=h_1h_3\}$$

B and C are split extensions of M by K and of M by L respectively. Let G be any group embedding the amalgam of A,B,C: Then in G, we have,

$$h_1 = h_1^{[a,b]} = h_1^{a^{-1}b^{-1}ab} = h_1h_2h_3^{b^{-1}ab} = h_1h_2h_3^{ab} = h_1^b = h_1h_2$$

Thus $h_2 = 1$, but then, from the relations

$$h_2^a = h_3,\ h_3^b = h_1h_3$$

we also get $h_3 = h_1 = 1$, so that in G, B and C have collapsed, and G consists of just the group A, and therefore does not embed the amalgam, that is, the generalised free product of A,B,C does not exist.

However, if we add the requirement that the automorphisms induced by B and C in M respectively commute, then we obtain a necessary and sufficient criterion for the existence of the generalised free product of this special type of amalgam.

II.1.3 <u>Theorem</u>: Let A be the direct product of K and L and $B = gp\{K,M\}$, $C = gp\{L,M\}$ with M normal in both B and C. Let K' and L' be the groups generated by the automorphisms induced by K and L in M respectively. The generalised free product of A,B,C exists if and only if K' and L' commute elementwise.

<u>Proof</u>: Since M is normal in both B and C and $K \cap M = L \cap M = \{1\}$ we can regard B and C as split extensions of M by K and of M by L respectively, so that there exist homomorphisms $\varphi : K \rightarrow K'$ and $\psi : L \rightarrow L'$. We first show that the condition is necessary, that is if the generalised free product exists then $[k',\ell'] = 1$ for all $k' \in K'$, $\ell' \in L'$.

Since M is normalised by K and L and F is generated by K,L and M, M is normal in F. Moreover, the factor group F/M is isomorphic to the group generated by K and L, that is, to A. We can, therefore, regard F as an extension of M by A. Let A' be the group generated by the automorphisms induced by A in M, then

$A' = gp\{K', L'\}$ and there is a homomorphism $\varphi^* : A \rightarrow A'$ which coincides with φ on K and with ψ on L. Since in A, $[k,\ell] = 1$, therefore,

$$[k,\ell]\varphi = [k\varphi, \ell\varphi] = [k', \ell'] = 1 \quad \text{for all} \quad k' \in K', \ \ell' \in L'.$$

so that K' and L' commute elementwise.

Next we prove the sufficiency of the condition. For this we first observe that since every embedding of the amalgam A,B,C automatically embeds the amalgam of B and C, the desired group - if it exists - must be isomorphic to a factor group of F^A where F^A is the generalised free product of B and C, using the maximality of the free embedding. In this factor group F^A/N^*, say, K and L must become elementwise permutable. Now in F^A, K and L generate their free product $\bar{F}^A$ by II.1.12. If N is the normal closure of the set of all commutators $[k,\ell]$, $k \in K$, $\ell \in L$ in $\bar{F}^A$, then $\bar{F}^A/N \cong A$. Therefore N^* must contain N and if N^* is the least normal subgroup of F^A containing N then F^A/N^* will be the free embedding of the given amalgam provided that it is an embedding at all. Thus the free product of the amalgam exists if and only if

(i) $N^* \cap \bar{F}^A = N$ so that $\bar{F}^A N^*/N^* \cong \bar{F}^A/\bar{F}^A \cap N^* \cong \bar{F}^A/N \cong A$,

(ii) N^* is tidy with respect to K and L in F^A.

(iii) N^* contains no element of the form fb or fc, $b \neq 1 \in B$, $c \neq 1 \in C$, $f \neq 1 \in \bar{F}^A$, so that in F^A/N^* the image A of $\bar{F}^A$ has the correct intersection with B and with C.

Let m be an element of M, since N is generated by $[k,\ell]$, $k \in k$, $\ell \in L$ and normal in $\bar{F}^A$ (by Golovin [6]), the elements $m^{-1}[k,\ell]m$ generate N^*. We show that these are elements of N.

Let α_k, β_ℓ be the automorphisms induced by k^{-1} and ℓ^{-1} on M, that is, for all $m \in M$,

$$m\alpha_k = kmk^{-1} \in M$$

and

$$m\beta_\ell = \ell m\ell^{-1} \in M.$$

Then
$$\begin{aligned} m^{-1}[k,\ell]m &= m^{-1}k^{-1}\ell^{-1}k\ell m \\ &= m^{-1}k^{-1}\ell^{-1}k(m\beta_\ell)\ell \\ &= m^{-1}k^{-1}\ell^{-1}(m\beta_\ell\alpha_k)k\ell \\ &= m^{-1}(m\beta_\ell\alpha_k\beta_{\ell^{-1}}\alpha_{k^{-1}})[k,\ell] \end{aligned}$$

But $\beta_{\ell^{-1}} = \beta_\ell^{-1}$, $\alpha_{k^{-1}} = \alpha_k^{-1}$ and $[\alpha_k,\beta_\ell] = 1$, therefore $m^{-1}(k,\ell]m = [k,\ell] \in N$ and so $N^* = N$ proving (i).

To prove (ii) we note that each element of B is uniquely of the form km, $k \in K$, $m \in M$, since M is normal in B and $K \cap M = \{1\}$. Similarly each element of C is uniquely of the form ℓm, $\ell \in L$, $m \in M$. Now assume N^* contains an element of the form $bc = km_1\ell m_2 = k\ell m_1' m_2$. As $N^* \subset \bar{F}^A$, $bc \in \bar{F}^A$, $k\ell \in \bar{F}^A$, hence $m_1'm_2 \in \bar{F}^A$. But as K and L both intersect M trivially, it follows by II.1.12 that $K*L = \bar{F}^A$ also intersects M trivially; hence $m_1'm_2 = 1$ and so $bc = k\ell \in N^* = N$. But N is tidy in $\bar{F}^A$ being the kernel of the mapping $\bar{F}^A = K*L \longrightarrow K \times L$. Hence $k = \ell = 1$, that is, $bc = 1$. This proves (ii).

To see that (iii) holds, assume $n = b\bar{f}$ for some $n \in N^*$, $b \in B$, $\bar{f} \in \bar{F}^A$, then since n and $\bar{f}$ are both in $\bar{F}^A$, b also lies in $\bar{F}^A$ that is $b \in K$. Similarly $n = c\bar{f}$, $c \in C$, $\bar{f} \in \bar{F}^A$ implies that $c \in L$, so that no element of N^* can be written in the above form. This completes the proof of the theorem.

We mention, without proof, that the above result still holds under the weaker assumption: when the intersection of K,L; L,M; M,K is a non trivial fixed group Z so that in this case A becomes the generalised direct product of K and L amalgamating the central subgroup Z.

Furthermore, if the groups A,B,C of theorem II.1.3 are finite, then their generalised free product is also finite.

However, there exist embeddable amalgams of three groups of the form described in theorem II.1.3 but with a different (seemingly weaker) restriction on A, namely we suppose that, instead of A being the direct product of K and L, there is a homomorphism of A onto the group generated by the automorphisms induced by K and L in M. It is interesting to note that this condition again turns out to be both necessary and sufficient for the existance of the generalised free product of such groups. Thus we prove:

II.1.4. <u>Theorem</u>. Let $A = gp\{K,L\}$, $B = gp\{K,M\}$, $C = gp\{L,M\}$ with M normal in both B and C. The generalised free product of A,B,C exists if, and only if, there is a homomorphism φ of A onto the group generated by the automorphisms induced by K and L in M.

<u>Proof</u>: Let K', L' be the groups generated by the automorphisms induced by K and L in M respectively. Then there exist homomorphisms $\varphi_1 : K \to K'$ and $\varphi_2 : L \to L'$. Let φ be the given homomorphism of A onto $A^* = gp\{K', L'\}$ which coincides with φ_1 on K and with φ_2 on L.

We form the extension G of M by A determined by the homomorphism φ i.e., the group of pairs (a,m), $a \in A$, $m \in M$ where

$$(a_1,m_1)(a_2,m_2) = (a_1a_2, m_1^{a_2\varphi} m_2)$$

$a_1, a_2 \in A$, $m_1, m_2 \in M$, and show that this group embeds the amalgam. Since $\varphi|K = \varphi_1$, $\varphi|L = \varphi_2$, in G the groups B_1, C_1 consisting of the pairs (k,m), $k \in K$, $m \in M$ and of (ℓ,m), $\ell \in L$, $m \in M$ are isomorphic to B and C and intersect precisely in the group of pairs $(1,m)$, $m \in M$ isomorphic to M. That these groups have the right intersection also with the group A_1 consisting of the pairs $(a,1)$, $a \in A$, isomorphic to A is obvious. Thus G embeds the amalgam of A, B, C.

Conversely, suppose the generalised free product F of A, B and C exists. Since M is normalised by K and L and F is generated by K, L and M, therefore M is normal in F and moreover $F/M \cong gp\{K,L\} = A$ so that F is an extension of M by A. Hence there is a homomorphism of A onto the group generated by the automorphisms induced by K and L in M. The proof of the theorem is now complete.

It is worth mentioning that here again, the generalised free product of groups obeying conditions as those in the above theorem is finite if the groups themselves are finite.

We now turn to investigate the embeddability of an amalgam of three groups of a different kind. This amalgam consists of three dihedral groups A,B,C of order 2ℓ, $2m$, and $2n$ respectively and taken in the special form as:

$$\begin{aligned} A &= gp\{a,b;\ a^2 = b^2 = (ab)^\ell = 1\} \\ B &= gp\{b,c;\ b^2 = c^2 = (bc)^m = 1\} \\ C &= gp\{c,a;\ c^2 = a^2 = (ca)^n = 1\} \end{aligned} \qquad (1)$$

The amalgam $\underline{\underline{A}}$ formed by these groups is their reduced amalgam. If any two of the ℓ,m,n are equal to 2 then this amalgam falls under the category of those considered in theorem II.1.4. The following theorem shows that the above amalgam is embeddable for all values of ℓ,m,n:

II 1.5. Theorem: The generalised free product of the amalgam $\underline{\underline{A}}$ of groups of type A,B,C in (1) exists.

Proof: In view of the above remark, we need discuss the existance of the generalised free product for m and n not both equal to 2. Following the scheme of proof of theorem II.1.3, we first take the generalised free product F^A of B and C and knowing that the subgroup $\bar{F}^A$ of F^A, generated by K and L, and in this case by a and b, is their free product, we take the normal closure N of the group generated by the left hand sides of the relations of A and their conjugates. If N^* is the normal closure of N in F^A and satisfies the conditions (i), (ii) and (iii) in the proof of theorem II.1.3, the factor group F^A/N^* will then be isomorphic to the generalised free product of A,B,C. Thus the problem of existence of the generalised free product of $\underline{\underline{A}}$ reduces to that of showing that N^* possesses the properties (i), (ii), and (iii) mentioned above.

The generalised free product of B and C is

$$F^A = gp\{a,b,c;\ a^2=b^2=c^2=(bc)^m=(ca)^n=1\} .$$

If we put $bc = g$, $ca = h$, then F^A has an alternative representation

$$F^A = gp\{g,h,c;\ g^m=h^n=(gh)^\ell=c^2=(gc)^2=(ch)^2=1\}$$

as the free products of $B' = gp\{g,c;\ g^m=c^2=(gc)^2=1\} = B$ and $C' = gp\{h,c;\ h^n=c^2=(hc)^2=1\} = C$ amalgamating $\{c\}$. In F^A, a and b generate their free product which is the infinite dihedral group and has, as its homomorphic images, all the finite dihedral groups. In particular, since,

$$a^{-1}baa = ab = (ba)^{-1},\ b^{-1}bab = ab = (ba)^{-1}$$

the group generated by $(ba)^\ell$, $\ell \geq 2$ is a normal subgroup of $\bar{F}^A$. Taking this as N we have $\bar{F}^A/N \cong A$. Let N^* be the normal closure of N in F^A. N^* is then generated by $(ba)^\ell$ and $c(ba)^\ell c$. We first show that N^* is a free group of rank 2. Writting $ba = gh$, $c(ba)c = c(gh)c = g^{-1}h^{-1} = (hg)^{-1}$, we see that N^* is generated by $(gh)^\ell$ and $(hg)^{-\ell}$ These two are independent generators of N^*, for if

$$(gh)^{\ell} = (hg)^{-p\ell}$$

for some $p > 1$, ($p \neq 1$ because $(cghc)^{\ell} = c(gh)^{\ell}c = (g^{-1}h^{-1})^{\ell} \neq (gh)^{\ell}$).
Then

$$(gh)^{\ell} = (hg)^{-p\ell} = (g^{-1}h^{-1})^{p\ell} = g^{-1}(h^{-1}g^{-1})^{p\ell}g = g^{-1}(gh)^{-p\ell}g$$

i.e. $g(gh)^{\ell}g^{-1} = (gh)^{-p\ell}$. Hence $g^2(gh)^{\ell}g^{-2} = (gh)^{p^2\ell}$.
Continuing in this way, we have,

$$g^m(gh)^{\ell}g^{-m} = (gh)^{\ell} = (gh)^{(-1)^m p^m \ell}$$

i.e. $(gh)^{((-1)^m p^m - 1)\ell} = 1$. Since $p \neq 1$, this relation implies that gh is of finite order, a contradiction.

Further, in the second representation of F^A, $\{g\}$ and $\{h\}$ generate their free product. A word w of the form

$$(gh)^{\varepsilon_1 \ell}(hg)^{\delta_1 \ell} \ldots (gh)^{\varepsilon_n \ell}(hg)^{\delta_n \ell}$$

cannot reduce to identity because in w there are no cancellations but only amalgamations of the form $g.g$, $h.h$, $g^{-1}.g^{-1}$, $h^{-1}.h^{-1}$ and since not both m and n are equal to 2, these amalgamations cannot reduce the above word to identity. Thus there is no non trivial relation of the form $w = 1$. The group N^* generated by $(gh)^{\ell}$ and $c(gh)^{\ell}c$ is, therefore, free of rank 2.

Now

$$c(gh)^{\ell}c = c(ba)^{\ell}c$$

does not belong to $\bar{F}^A$ and moreover, $N^* \cap F^A$ as a subgroup of a free group, is free, by Schreier's theorem; as a subgroup of the infinite dihedral group $\bar{F}^A$ it can be free only if it is an infinite cycle; as it contains N, this cycle must be generated by $(gh)^{\ell_1}$ with $\ell_1 | \ell$. But the free group with free generator $(gh)^{\ell}$ contains no proper root of this. Hence $\ell_1 = \ell$, and $N^* \cap \bar{F}^A = N$, so that condition (i) is satisfied.

Also N^* is tidy with respect to B and C, for every element of N^* is of the form

$$n^* = (gh)^{\varepsilon_1 \ell}(hg)^{\delta_1 \ell} \ldots (gh)^{\varepsilon_n \ell}(hg)^{\delta_n \ell} .$$

In n^*, c occurs an even number of times and since every element of B is of the form cg^k, $k = 0,1, \ldots, m-1$, $n^* \neq cg^k$ because then n^* like cg^k will be of finite order. Similarly $N^* \cap C = \{1\}$. This proves (ii).

Also no element of N^* can be written in the form $f \in \bar{F}^A$, $b^* \in B$, because any element f of $\bar{F}^A$ is of the form

$$f = a^{\varepsilon} bab \ldots ab^{\delta}$$

and every $b^* \in B$ has a representation as $b^* = c(bc)^k$, $k = 0,1,2, \ldots, m-1$ and in fb^*, between no two consecutive c the element $a \in A$ occurs whereas in any n^* it does. Similarly $n^* \neq fc^*$, $c^* \in C$. This proves (iii) and also completes the proof of the theorem.

The discussion of this particular amalgam has been inserted at this point because the argument is close to the earlier arguments in this paragraph. It arose out of my endeavour to show that the group

$$F = gp\{a,b,c;\ a^2=b^2=c^2=(ab)^\ell=(bc)^m=(ca)^n=1\}$$

described in Coxeter and Moser (cf. [8], p.37/39) as the group of reflections in the sides of a spherical triangle with angles π/ℓ, π/m, π/n does in fact embed the amalgam of three dihedral groups and therefore is its free product. F is known to be finite if $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} > 1$ and otherwise infinite. If we write F as

$$F = gp\{g,h,c;\ g^m=h^n=(gh)^\ell=c^2=(gc)^2=(ch)^2=1\}$$

then it is easy to see that F is a split extension of

$$P = P(m,n,\ell) = gp\{g,h;\ g^m=h^n=(gh)^\ell = 1\}$$

by a cyclic group of order 2. P belongs to the well known family of groups called the 'polyhedral groups'. It possesses, among its factor groups, many of the known finite simple groups. (cf; [7].)

These remarks are wanted in the last chapter when we use these and other groups given by Coxeter and Moser to give examples of the range of different embeddings an amalgam may possess.

II.2. Let $\underline{\underline{A}}$ be the reduced amalgam of three groups H_1, H_2, H_3 such that $H_i \cap H_j = H_{ij} = H_{ji}$ for all $i,j = 1,2,3,\ i \neq j$. We can take these groups as

$$H_1 = gp\{H_{12},H_{13}\},\ H_2 = gp\{H_{12},H_{23}\},\ H_3 = gp\{H_{13},H_{23}\}.$$

It is known that the amalgam $\underline{\underline{A}}$ is embeddable in a group if any one of the H_i's, H_3 say, is the generalised free product of H_{i3}, $i = 1,2$, (cf. Hanna Neumann Theorem 9.0 [4]). Hanna Neumann has also shown that, in general, this theorem cannot be generalised to the case of more than three groups. The examples she has constructed deal with the most general type of group amalgams. However, if we have an amalgam of more than three groups with a 'limited' number of intersections, then the Theorem 9.0 [4] of Hanna Neumann can be generalised.

The kind of amalgam we consider involve n groups $G_1, G_2, \ldots, G_n$, where for every $i = 1,2, \ldots, n$ and with $G_{n+1} = G_1$, we have

$$G_i \cap G_{i+1} = H_i \quad \text{and} \quad G_i \cap G_j = H \quad \text{for} \quad j \neq i \pm 1.$$

For $n = 3$ this is the same as the amalgam considered above. Any amalgam of the above form shall be called 'of a special type' or more briefly 'of type S'. While for $n = 3$ such an amalgam may not be embeddable without one or other additional restriction, we show that for $n > 3$, no further condition is required.

<u>Theorem</u>: Let n given groups $G_1, G_2, \ldots, G_n$; $n > 3$ form an amalgam $\underline{\underline{A}}$ of type S. Then $\underline{\underline{A}}$ is embeddable in a group, that is, the generalised free product of $G_1, G_2, \ldots, G_n$ exists.

<u>Proof</u>: Since we need consider only the reduced amalgam, it can be supposed that the amalgam $\underline{\underline{A}}$ formed by $G_1, G_2, \ldots, G_n$ is already reduced. In

that event each G_{i+1} is generated by its subgroups H_i, H_{i+1}, $i = 1,2, \ldots, n$ where we assume again that $G_{n+1} = G_1$, $H_{n+1} = H_1$. We arrange these groups in pairs as follows:

When n is even, $n = 2m$, as:

$$(G_1, G_{2m}), (G_2, G_{2m-1}), \ldots, (G_i, G_{2m-i+1}), \ldots, (G_m, G_{m+1}), \qquad (1)$$

and when $n = 2m+1$, as:

$$(G, G_{2m+1}), \ldots, (G_i, G_{2m-i+2}), \ldots, (G_{m-1}, G_{m+3}) \qquad (2)$$

together with the triplet (G_m, G_{m+1}, G_{m+2}). Then the free product of each pair exists by Schreier's theorem; we postpone consideration of the triplet (G_m, G_{m+1}, G_{m+2}).

In the case of (1) the free products F_i of (G_i, G_{2m-i+1}) amalgamating H for $i = 2,3,\ldots,m-1$ and H_{2m} or H_m for $i = 1$ or m respectively, are such that in each of these:

(a) two consecutive groups H_i, H_{i+1} generate the group G_{i+1}, and

(b) any pair H_i, H_j; $j \neq i \pm 1$ generate their free product amalgamating H.

To substantiate this claim, it is sufficient to prove it for F_1 only. Since F_1 is the generalised free product of $G_1 = \mathrm{gp}\{H_{2m}, H_1\}$, $G_{2m} = \mathrm{gp}\{H_{2m-1}, H_{2m}\}$, (a) is obviously satisfied. Also by the properties of the free product, as $H_1 \subset G_1$, $H_{2m-1} \subset G_{2m}$, and $H_1 \cap H_{2m-1} \subseteq G_1 \cap G_{2m} = H_{2m}$, we have

$$H_1 \cap H_{2m-1} = H_1 \cap H_{2m} \cap H_{2m-1} = (H_1 \cap H_{2m}) \cap (H_{2m} \cap H_{2m-1})$$

$$= H \cap H = H$$

Hence by II.1.12 H_1, H_{2m-1}, generate their free product amalgamating H, so that F_1 has (b) also.

We now take the first two of the free products F_i, $(i = 1,2, \ldots,m)$:

$$F_1 = gp\{H_1, H_{2m-1}, H_{2m}\}, \; F_2 = gp\{H_1, H_2, H_{2m-2}, H_{2m-1}\}$$

The groups H_1 and H_{2m-1} are contained in F_1 and F_2 and generate in both their free products amalgamating H. The free product $F^{(1)}$ of F_1 and F_2 amalgamating $\{H_1 * H_{2m-1}\}$ can, therefore, be formed. In $F^{(1)}$, since $H_{2m} \subset G_{2m}$, $H_2 \subset G_2$ and $H_2 \cap H_{2m} \subseteq G_2 \cap G_{2m} = H$, we have,

$$\begin{aligned} H_2 \cap H_{2m} &= H_2 \cap H \cap H_{2m} \\ &= (H_2 \cap H) \cap (H \cap H_{2m}) \\ &= H \end{aligned}$$

Similarly, $H_{2m-2} \cap H_{2m} = H$. Thus for $F^{(1)}$ also the conditions (a) and (b) are satisfied. We then form the generalised free product $F^{(2)}$ of $F^{(1)}$ and F_3 amalgamating $\{H_2 * H_{2m-2}; \; H\}$ and continuing in this way, lastly, the free product $F^{(m-2)}$ of $F^{(m-3)}$ and F_{m-1} amalgamating $\{H_{m-2} * H_{m+2}; \; H\}$. As it has already been shown that $F^{(1)}$ satisfies the requirements (a) and (b), we have a basis for induction. Suppose that we have already proved that $F^{(m-2)}$ has the properties (a) and (b). Then, since $\{H_{m-1} * H_{m+1}; \; H\} = H*$

is contained in both $F^{(m-2)}$ and F_m, the free product $F^{(m-1)}$ of the amalgam $\underline{\underline{A}}' = am(F^{(m-1)}, F_m;\ H^*)$ may be formed. Now for $i \neq m$, $m-1$, or $m+1$,

$$H_i \cap H_m \subset F^{(m-2)} \cap F_m = H^*$$

However, by assumption, $H_i \cap H_{m-1} = H_i \cap H_{m+1} = H$; and also in F_m, $H_m \cap H_{m-1} = H_m \cap H_{m+1} = H$, therefore, $H_i \cap H^* = H^* \cap H_m = H$. Thus $H_i \cap H_m = H_i \cap H^* \cap H_m = (H_i \cap H^*) \cap (H^* \cap H_m) = H \cap H = H$. Hence $F^{(m-1)}$ also satisfies the conditions (a) and (b). But then $F^{(m-1)}$ contains all the G_i's with their precise intersections and therefore embeds their amalgam.

For the odd case $n = 2m+1$, we show that the generalised free product of the triplet (G_m, G_{m+1}, G_{m+2}) exists provided that $n > 3$, i.e., $m > 1$. For this, we have, in view of II.1.11 only to show that the reduced amalgam of G_m, G_{m+1}, G_{m+2} is embeddable. But since $G_m \cap G_{m+1} = H_m$, $G_{m+1} \cap G_{m+2} = H_{m+1}$, $G_{m+2} \cap G_m = H$ because $m > 1$, the reduced amalgam of these groups consists of just the group G_{m+1} and hence is embeddable, so that F_m exists. In fact if $X = \{G_m * G_{m+1};\ H_m\}$ then $F_m = \{X * G_{m+2};\ H_{m+1}\}$ and in F_m the group H_{m-1} and H_{m+2} generate their free product by II.1.12. That the free product $F^{(m-1)}$ of $F^{(m-1)}$ and F_m amalgamating $\{H_{m-1} * H_{m+2};\ H\}$ and so also of G_1, G_2, ..., G_n exists follows in exactly the same way as above. This completes the proof of the theorem.

CHAPTER III.

EMBEDDABILITY IN A FINITE GROUP OF AN EMBEDDABLE FINITE AMALGAM

§III.0

Introduction.

In this chapter we discuss a problem of B.H. Neumann and Hanna Neumann about the embeddability in a finite group of a finite amalgam of three or more groups. Most of the amalgams considered here are those whose embeddability has already been shown in the preceeding chapter. In the case of some amalgams of three groups, a permutational product of some two of the groups of an amalgam turns out to be isomorphic to the generalised free product of the amalgam. However, not for all amalgams of three groups does this situation occur.

Theorem III.2.1 mentions a sufficient condition for embedding of a finite amalgam of n groups of type S in a finite group. Since embeddability of an amalgam and the existence of its generalised free product are synonymous terms, this result generalises theorem 9.1 of [4]. Theorem III.2.2 gives a different set of sufficient conditions for such an amalgam to have a finite embedding. In theorem III.2.3 we make use of a known property of permutational products to give a sufficient criterian for finit embeddability of an amalgam of 4 groups. We also mention that the question of embeddability of a finite amalgam of type S in a finite group is still open.

Lastly we give an example to answer some questions regarding non-free embeddings of an amalgam.

§III.1.

B.H. Neumann and Hanna Neumann in their well known paper, "A contribution to the embedding theory of group amalgams", (cf [11]) mention the following problem:

<u>If a finite amalgam, that is, an amalgam of a finite number of finite groups is embeddable in a group, is it also embeddable in a finite group</u> ?

The answer to this question is not known in general, not even in the case of an amalgam of only three groups. For a finite amalgam with one amalgamated subgroup, this is, however, always possible because in such a situation a permutational product of these groups embeds the amalgam and being a subgroup of the permutation group of a certain finite set, is itself finite (cf. [1], [3]). In particular,

III.1.1 An amalgam of two finite groups is embeddable in a finite group.

Let $\underline{\underline{A}}$ be an amalgam of groups $G_1, G_2, \ldots, G_n$ with amalgamated subgroups H_{ij} $i,j = 1,2,\ldots,n$ and let $\underline{\underline{A}}'$ be their reduced amalgam, that is, the amalgam formed by the groups H_i generated by all the H_{ij}, $j \neq i$, i fixed,

It is known that the generalised free product of $\underline{\underline{A}}$ exists if and only if that of $\underline{\underline{A}}'$ does (cf. Theorem 5.0 [4], [13]). We prove here a similar 'reduction theorem' for the existence of a finite embedding of a finite amalgam.

III.1.2. <u>Theorem</u>. A finite amalgam $\underline{\underline{F}}$ of the groups $G_1, G_2, \ldots, G_n$ amalgamating H_{ij} is embeddable in a finite group if and only if their reduced amalgam $\underline{\underline{F}}'$ is so embeddable.

Proof: The necessity of the condition is immediate. To prove the sufficiency, let H be a finite embedding of the reduced amalgam $\underline{\underline{F}}'$. Then for any integer i, $i = 1,2,\ldots,n$ the groups G_i and H both contain a subgroup H_i generated by all H_{ij}, $j \neq i$, i fixed, and their amalgan is embeddable in a finite group K_i say, by III.1.1. Each of the groups K_i so obtained contains as a subgroup the finite group H. We, therefore have a finite amalgam of finite groups $K_1,\ldots,K_n$ amalgamating a single group H. A permutational product of these groups amalgamating H embeds the amalgam and is finite, as required.

In view of the above theorem, we need, therefore, examine the embeddability of only the reduced amalgam in a finite group. A finite embedding of the reduced amalgam will ensure that of the whole amalgam as well.

We first consider amalgams of only three groups. To my knowledge, little is known as regards the finite embeddings of such amalgams. Because of the difficulties involved in knowing whether such an amalgam is embeddable or not, the problem of finding a finite embedding becomes even more complicated. However, we have been able to find a few results concerning the embeddability in a finite group for some special classes of such group amalgams.

The reduced amalgam of three groups A, B, C, consists of the groups K, L and M which in pairs K, L; K, M; and L, M say, generate these groups respectively. We have already shown in Chapter II that such kind of amalgams need not even be embeddable, hence the imposition of sufficient conditions of some form or other is essential to making the embedding possible. Theorems II.1.2, II.1.3 and II.1.4 of the previous chapter give sufficient conditions for the existence of the generalised free products of such group amalgams and hence also for their embeddability. We have also seen that in the above three cases there is a unique finite embedding namely their generalised free product.

It is, however, sometimes possible that in a permutational product of some two of the groups of the amalgam, B and C, say, the groups K and L generate a subgroup isomorphic to A, thus making a finite embedding sure. That this is not true in general is shown by the following example.

III.1.3 <u>Example</u>: We take K, L and M as cyclic groups of order 3, 3 and 2 respectively and put

$$A = gp\{a,b; a^3 = b^3 = (ab)^2 = 1\} \cong A_4$$

$$B = gp\{a,c; a^3 = c^2 = (ac)^2 = 1\} \cong S_3$$

$$C = gp\{b,c; b^3 = c^2 = (bc)^2 = 1\} \cong S_3 .$$

B.H. Neumann [1] has shown that there are only three permutational products of B and C namely of order 18, 162 and 9!. However, the only group embedding the amalgam of A,B,C is their free product,

$$F = gp\{a,b,c; a^3 = b^3 = c^2 = (ab)^2 = (bc)^2 = (ac)^2 = 1\}$$

F is an extension of A_4 by M and is obviously different from all the permutational products of B and C.

Let us have three finite groups

$$A = gp\{K,L\} , \quad B = gp\{K,M\}, \quad C = gp\{L,M\} ;$$

$$K \cap L = L \cap M = M \cap K = \{1\}$$

with K and L normal in B and C respectively. An amalgam of three such groups need not be embeddable as was shown by example II.1.23. But whenever such an amalgam is embeddable, it is also embeddable in a finite group. This is shown by the following theorem:

III.1.4. <u>Theorem</u>: If A , B , C are three groups generated by K , L ; K , M ; and L , M respectively such that K is normal in B , L is normal in C and the amalgam is embeddable in a group G say, then G is finite.

<u>Proof</u>: Since the amalgam of A,B,C is embeddable in a group G , we can take G as generated by K , L and M . In G, the groups K and L must generate A and since K and L are normalised by M , A is a normal subgroup of G . The factor group G/A is then isomorphic to M , so that G is an extension of A by M . Since A and M are finite, G is also finite, as required.

§III.2.

Let $\underline{\underline{A}}$ be an amalgam of type S (see p. 66) of n groups $G_1,G_2,\ldots,G_n$. Since we need consider only the reduced amalgam, we can take each G_i as generated by H_{i-1} and H_i , the intersection of these groups being a fixed group H . That the amalgam $\underline{\underline{A}}$ of $G_1,G_2,\ldots,G_n$, $n \neq 3$

is embeddable in a group which was proved in the theorem of § 2 Chapter II. It is not known whether, in general, such an amalgam has also a finite embedding. In the case of $n = 3$ some sufficient conditions for the embeddability in a finite group of this finite amalgam were given by theorems II.1.3, II.1.4, II.1.5 and III.1.5. Here we find some sufficient conditions for such a finite amalgam of more than three groups to have a finite embedding. We prove:

III.2.1 <u>Theorem</u>: Let $\underline{\underline{A}}$ be an amalgam of n groups of type S. If for even n, $n/2$ and for odd n, $(n+1)/2$ of the groups G_i, $G_{m+1}, G_{m+2}, \ldots G_n$ say, where $m = [n/2]$, have the property that in each G_i, H_{i-1} and H_i permute elementwise and further $H_i \cap H_j = H$ is central in G_i, G_j for all $i,j = 1,2,\ldots,n$; then $\underline{\underline{A}}$ is embeddable in a finite group.

<u>Proof</u>: When n is even, we take the n groups in pairs as follows:

$$(G_1, G_{2m}), (G_2, G_{2m-1}), \ldots, (G_m, G_{m+1}) ;$$

when n is odd, $n = 2m+1$, we consider only the pairs

$$(G_1, G_{2m+1}), (G_2, G_{2m}), \ldots, (G_{m-1}, G_{m+3}) ;$$

and then the triplet

$$(G_m, G_{m+1}, G_{m+2}).$$

When $n = 2m$, we form the generalised direct products F_1 of G_1 and H_{2m-1}, F_i of G_i and G_{2m-i+1}, for $i = 2,3,\ldots,m-1$ and F_m of G_m, H_{m+1} amalgamating H in all these cases. Each of the F_i's exists because H is central in all the G_i's $i = 1,2,\ldots,n$. Now in F_1, H_{2m-1} and H_{2m} commute element by element and therefore generate a group isomorphic to G_{2m}. Similarly in F_m, H_m and H_{m+1} generate G_{m+1}. Since

$$F_1 = gp\{H_1, H_{2m-1}, H_{2m}\}$$

$$F_2 = gp\{H_1, H_2, H_{2m-2}, H_{2m}\}$$

and in both, H_1, H_{m-1} commute elementwise and generate their generalised direct product $(H_1 \times H_{2m-1}; H)$ and since F_1, F_2 are finite groups, we can take a finite embedding $F^{(1)}$ of F_1 and F_2 amalgamating $(H_1 \times H_{2m-1}; H)$. Then $F^{(1)}$ contains the groups G_1, G_2, G_{2m-1}, G_{2m}. Also since $F_3 = gp\{H_2, H_3, H_{2m-3}, H_{2m-2}\}$ and both $F^{(1)}$ and F_3 contain $(H_2 \times H_{2m-2}; H)$, there is a finite embedding $F^{(2)}$ of $F^{(1)}$ and F_3 amalgamating $(H_2 \times H_{2m-2}; H)$. Continuing in this way we obtain a finite embedding $F^{(m-1)}$ of $F^{(m-2)}$ and F_m amalgamating $(H_{m-1} \times H_{m+1}; H)$.

When $n = 2m+1$, we follow the same procedure except that in this case F_m is the generalised direct product of G_m and G_{m+2} amalgamating H. F_m automatically contains G_{m+1} because in F_m, H_m and H_{m+1} generate their generalised direct product and hence the group G_{m+1}.

That the finite group $F^{(m-1)}$ so constructed contains $G_1, \ldots, G_n$ is obvious. That in $F^{(m-1)}$ these groups have their right intersections follows by reasoning exactly the same way as in the theorem of §II.2; using the direct product with one amalgamation in place of the free product with one amalgamation. This completes the proof of the theorem.

Hanna Neumann has shown (cf. [4]pp. 623) that in the case of the most general type of an amalgam of n groups, $n > 3$ if $n - 1$ of the groups G_i have the property that their subgroups H_{in} (i fixed) commute elementwise, the generalised free product of this amalgam may not exist. However for this particular amalgam of type S which reduces to that in Theorem 9.1 [4] for $n = 3$ such a result and even a slightly more general version of it is seen to be true.

Also it is known (cf [14]) that if each of the groups G_i has the property that H_{i-1} and H_i generate in G_i their generalised direct product then a finite embedding of the amalgam is possible. In such a case even a permutational product type construction can be made, as was shown by R.A. Bryce in [14]. The above theorem generalises his result in the sense that here we assume only $n/2$ or $(n+1)/2$ of the groups to have that property, according as n is even or odd.

Another sufficient condition for the embeddability of an amalgam of type S in a finite group is given by the following theorem:

III.2.2 Theorem: Let the finite groups $G_1, G_2, \ldots, G_n$ form a reduced amalgam $\underline{\underline{A}}$ of type S . Let further H_1 and H_{n-1} be normal subgroups of G_1 and G_n respectively and $H = G_i \cap G_j$ $j \neq i \pm 1$ be central in each G_i , $i = 1,2,\ldots,n$. Then A is embeddable in a finite group if in the even case $n = 2m$, H_{m-1} and H_{m+1} are normal in G_m and G_{m+1} respectively and in the odd case $n = 2m + 1$, G_{m+1} is such that its subgroups H_m and H_{m+1} are permutable element by element.

Proof: To begin with we note that for $n = 3$ the generalised free product of the amalgam exists and is finite by Theorem II.1.2. Therefore we have only to prove that $\underline{\underline{A}}$ has a finite embedding even for $n > 3$ provided that the conditions in the theorem are satisfied. For this we, once again follow, the procedure in the proof of theorem of §2 Chapter II, and write $G_1, G_2, \ldots, G_n$ in pairs as (G_i, G_{2m-i+1}) or (G_i, G_{2m-i+2}) according as $n = 2m$ or $n = 2m + 1$, leaving G_{m+1} alone in the second case.

We consider first the case when $n = 2m$. Here H_1 is normal in G_1 and H_{n-1} normal in G_n, and G_1, G_n are generated by H_n, H_1 and H_{n-1}, H_n. The remark in the proof of Lemma I.5.3 shows that the permutational product P_1 of G_1, G_n amalgamating H_n (formed with transversals H_1', H_{n-1}' where $H_1'H = H_1$, $H_{n-1}'H = H_{n-1}$) is such that the groups $H_1 = \rho(H_1)$ and $H_{n-1} = \rho(H_{n-1})$ in P_1 generate their generalised direct product amalgamating a central subgroup H. Similarly, because of the normality of H_{m-1} in G_m and of H_{m+1} in G_{m+1}, the permutational product $P_m(\underline{\underline{A}}_m; H_{m-1}', H_{m+1}')$ of $\underline{\underline{A}}_m = am(G_m, G_{m-1}; H_m)$ with $H_{m-1}'H = H_{m-1}$, $H_{m+1}'H = H_{m+1}$ is such that in it, the groups H_{m-1} and H_{m+1} generate their generalised direct product amalgamating H. Moreover, since H is central in every G_i and $G_i \cap G_{2m-i+1} = H$, the generalised direct product P_i of G_i, G_{2m-i+1} amalgamating H (which is also their permutational product, (cf [1]) exists for every $i = 2,3,\ldots,m-1$. Each P_i is then finite and has the following property:

Whenever H_i and H_{i+1} belong to such a permutational product, then they generate in it a group isomorphic to G_{i+1} , and whenever H_i , H_j with $j \neq i \pm 1$ belong to such a permutational product, then they generate in it their generalised direct product amalgamating H .

Now the groups $P_1 = gp\{H_1, H_{2m-1}, H_{2m}\}$ and $P_2 = gp\{H_1, H_2, H_{2m-2}, H_{2m-1}\}$ have a common subgroup $(H_1 \times H_{2m-1}; H)$. There exists a finite embedding $P^{(1)}$ say of P_1 and P_2 amalgamating this subgroup, $P^{(1)}$ then contains the groups G_1 , G_2 , G_{2m-1} , G_{2m} . Since $P_3 = gp\{H_2, H_3, H_{2m-3}, H_{2m-2}\}$ contains the group $(H_2 \times H_{2m-2}; H)$ and this is also a subgroup of P_2 and hence of $P^{(1)}$ a finite embedding $P^{(2)}$ of $P^{(1)}$ and P_3 amalgamating $(H_2 \times H_{2m-2}; H)$ can be found. Continuing in this way we see that there is a finite embedding $P^{(m-2)}$ of $P^{(m-3)}$ and P_{m-1} amalgamating $(H_{m-2} \times H_{m+2}; H)$. In P , the groups H_{m-1} and H_{m+1} commute elementwise because they do so in P_{m-1} . However, in P_m also, these groups are elementwise permutable as shown above, therefore a permutational product $P^{(m-1)}$ of $P^{(m-2)}$ and P_m amalgamating $(H_{m-1} \times H_{m+1}; H)$ embeds the amalgam of $P^{(m-2)}$ and P_m . $P^{(m-1)}$ obviously contains the groups $G_1, \ldots, G_n$.

When $n = 2m + 1$, then form the generalised direct products P_i of G_i , G_{2m-i+2} amalgamating a central subgroup H , $i = 2, 3, \ldots, m$ and also the permutational product P_1 of G_1 , G_n amalgamating H_n using the transversals H_1' and H_{n-1}' where $H_{n-1}'H = H_{n-1}$ and $H_1'H = H_1$. In P_1 the groups H_1 and H_{n-1} generate their generalised direct product amalgamating H . We now construct similarly, step by step, a finite group $P^{(m-1)}$ which is an embedding of the finite groups $P^{(m-2)}$ and P_m amalgamating $(H_{m-2} \times H_{m+2}; H)$. However $m > 1$, G_m and G_{m+2} have

If there exist isomorphisms φ of G_1 onto G_2 and ψ of G_4 onto G_3 such that φ is identity on $H_1 = G_1 \cap G_2$ and ψ is identity on $H_3 = G_4 \cap G_3$ and further $\varphi|H_4 = \psi|H_4$ with $H_4\varphi = H_4\psi = H_2$, then $\underline{\underline{A}}$ ...

only H in common and P_m is the generalised direct product of G_m, G_{m+2} amalgamating H. Hence the group $G_{m+1} = (H_m \times H_{m+1} ; H)$ is contained in P_m and so also in $P^{(m-1)}$. Thus in this case again, $P^{(m-1)}$ contains all the G_i's $i = 1,2,\ldots,n$. That they have their right intersections in $P^{(m-1)}$ follows by an argument similar to that in the theorem of §2 Chapter II.

The proof of the theorem is now complete.

This is as far as one could hope to go in the general case. Things look more obscure when we do not have any restriction on the constituents. Even for the case of four groups we have not been able to prove a general result without any restrictions on the groups forming the amalgam. However the following theorem gives a sufficient condition in this case.

III.2.3 <u>Theorem</u>: Let the groups G_1, G_2, G_3 and G_4 form an amalgam $\underline{\underline{A}}$ of type S. ~~If G_1, G_2 and G_3, G_4 are isomorphic and also H_2 isomorphic to H_4~~, then $\underline{\underline{A}}$ is embeddable in a finite group.

<u>Proof</u>: It has already been shown that the generalised free product of G_i's exist. In the free products of G_1, G_4 and G_2, G_3, the groups H_1, H_3 generate their free products amalgamating H. Therefore, if there exist two finite groups embedding the amalgams of G_1, G_4 and of G_2, G_3 such that the group generated by H_1, H_3 in both is the same then we can take a permutational product of these two groups amalgamating the group generated

by H_1 and H_3, and get a finite embedding of the amalgam of G_1, G_2, G_3, G_4. We show that this is possible provided that the conditions in the theorem are satisfied.

We here use the notion of 'equivalent transversals' as defined in the paragraph preceding theorem 4.1 of Chapter I. Since G_1 and G_2 are isomorphic and H_1 is a common subgroup of both G_1 and G_2, if φ is an isomorphism of G_1 onto G_2 and S a transversal of H_1 in G_1, then since $G_1\varphi = (SH_1)\varphi = S\varphi H_1\varphi = S\varphi H_1 = G_2$ we take $S\varphi = S'$ as a transversal of H_1 in G_2. Similarly a transversal T' of H_3 in G_4, $T' = T\psi^{-1}$ where $G_4\psi = G_3$, is chosen corresponding to a transversal T of H_3 in G_3. Since $H_2 \cong H_3$, the isomorphisms φ and ψ of G_1 onto G_2 and of G_4 onto G_3 are such that $\varphi|H_4 = \psi|H_4$. We now have the groups G_1, G_4 and G_2, G_3 with amalgamated subgroups H_4 and H_2 respectively and the isomorphisms φ and ψ of G_1 onto G_2 and of G_4 onto G_3 respectively. Therefore by Theorem 4.1 Chapter I, there is an isomorphism χ of the permutational product $P(S,T;H_4)$ onto $P'(S',T';H_2)$ which extends both φ and ψ. In P and P', the group H_1, H_3 generate isomorphic subgroups and in fact the same groups. Any permutational product of P and P' amalgamating the group generated by H_1, H_3 will then embed the amalgam $\underline{\underline{A}}$ of G_1, G_2, G_3 and G_4 and be finite.

This completes the proof of the theorem.

The above results give only a partial answer to the question of finite embeddability of an amalgam of type S. <u>Whether every such amalgam is embeddable in a finite group is still an open question</u>.

§III.3

In this section we construct an example to answer the following questions:

(i) Is there an infinite embedding of three finite groups A , B , C other than the free one? If so, is there also a finite embedding of A , B , C ?

(ii) Let P and P' be two finite embeddings of the groups A , B and C such that the order of P is larger than that of P' . Does there always exist a homomorphism of P onto P' ?

(iii) Is there always a unique minimal embedding of an amalgam of three finite groups ?

Many of these questions may be answered by interpreting groups described by Coxeter and Moser as generalised free products.

III.3.1 <u>Example</u>: Let $(\ell,m,n;p)$ denote the group generated by S and T having the defining relations

$$S^{\ell} = T^{m} = (ST)^{n} = (ST^{-1}S^{-1}T)^{p} = 1 .$$

When $n = 2$, so that

$$ST = (ST)^{-1} = T^{-1}S^{-1} ,$$

we have

$$ST^{-1}S^{-1}T = SSTT = S^2T^2$$

Hence $(\ell,m,2;p)$ is also defined by

$$S^{\ell} = T^{m} = (ST)^{2} = (S^2T^2)^{p} = 1$$

If $\ell = 5$, $m = 5$, $p = 3$, then the group (5,5,2;3) has a representation as

$$S^5 = T^5 = (ST)^2 = (S^2T^2)^3 = 1$$

The infiniteness of (5,5,2;3) follows by Coxeter [7] page 93.

The amalgam we construct is such that its generalised free product shall have as a proper homomorphic image a group containing a subgroup isomorphic to (5,5,2;3) . The three groups we take are the dihedral groups of orders 4 , 10 and 10 such that the amalgamated subgroup of each pair is a cyclic group of order 2 . Thus

$$A = gp\{a,b;a^2 = b^2 = (ab)^2 = 1\}$$

$$B = gp\{b,c;b^2 = c^2 = (b\ c)^5 = 1\}$$

$$C = gp\{c,a;c^2 = a^2 = (ca)^5 = 1\}$$

The free product of A , B , C is

$$F = gp\{a,b,c;a^2 = b^2 = c^2 = (bc)^5 = (ca)^5 = (ab)^2 = 1\}$$

which can, following Coxeter and Moser [8], also be considered as a group generated by reflections 'a' , 'b' , 'c' in the sides of a spherical triangle with angles $\pi/5$, $\pi/5$ and $\pi/2$ respectively. F can also be represented as

$$F = gp\{g,h,c;g^5 = h^5 = (gh)^2 = (gc)^2 = (ch)^2 = c^2 = 1\}$$

where $bc = g$, $ca = h$ so that $ba = gh$. Since

$$g^c = c(bc)c = cb = (bc)^{-1} = g^{-1}$$

and

$$h^c = c(ca)c = ac = (ca)^{-1} = h^{-1} ,$$

the subgroup

$$G_1 = gp\{g,h; g^5 = h^5 = (gh)^2 = 1\} \tag{1}$$

of F is normal in F. Moreover, since $G_1 \cap \{c\} = \{1\}$, F is a split extension of G_1 by a cyclic group of order 2. Further, G_1 is an infinite group by Coxeter [8] p. 54.

Let N be the normal closure of $\{(g^2h^2)^3\}$ in G_1, we show that the normal closure N^* of N in F coincides with N. For this we have only to prove that

$$c^{-1}(g^2h^2)^3c$$

is an element of N. Now,

$$c^{-1}(g^2h^2)^3c = (g^{-2}h^{-2})^3 = (h^2g^2)^{-3}$$

and since,

$$g^{-2}(g^2h^2)^3g^2 = g^{-2}g^2h^2g^2h^2g^2h^2g^2 = (h^2g^2)^3$$

belongs to N, therefore $c^{-1}(g^2h^2)^3c$ being the inverse of $(h^2g^2)^3$ is also in N, showing that N is also normal in F. If we denote the factor group F/N by F_1 then $\mathbf{F_1} = gp\{g',h',c'; g'^5 = h'^5 = (g'h')^2 = (g'^2h'^2)^3 = (g'c')^2 = (c'h')^2 = c'^2 = 1\}$ or, on identification of g', h', c' with g, h, c respectively,

$$F_1 = gp\{g,h,c; g^5 = h^5 = (gh)^2 = (g^2h^2)^3 = (gc)^2 = (ch)^2 = c^2 = 1\} .$$

However, the group

$$G_2 = gp\{g,h; g^5 = h^5 = (gh)^2 = (g^2h^2)^3 = 1\} \qquad (2)$$

is a normal subgroup of F_1 and since $G_2 \cap \{c\} = 1$, F_1 is a splitting extension of G_2 by a two-cycle. G_2 being the same as the group $(5,5,2;3)$ is infinite and so also is, therefore, F_1 . In terms of a , b , c , F_1 has the representation

$$F_1 = gp\{a,b,c; a^2 = b^2 = c^2 = (bc)^5 = (ca)^5 = (ab)^2 = (bca)^6 = 1\} .$$

Since every element of B and C is of the form

$$(bc)^k b \quad \text{and} \quad (ca)^{k'} c$$

respectively, $k,k' = 0,1,2,3,4$, and for no values of k and k' , the relation $(bc)^k b = (ca)^{k'} c$ or $(bc)^{k+1} = (ca)^{k'}$ is implied by the additional relation $(bca)^6 = 1$, therefore there are no additional amalgamations of B and C and similarly not of C and A or of A and B . The groups A , B , C have, therefore, their precise intersections in F_1 and F_1 consequently embeds their amalgam.

However, F_1 is different from the free embedding F of the amalgam of A , B , C and moreover, is infinite, answering the first part of question (i).

We now show that there is also a finite embedding of the amalgam of A , B , C .

Consider the group

$$G_3 = gp\{g',g'; g'^5 = h'^5 = (g'h')^2 = (g^{-1}.h')^3 = 1\} \tag{3}$$

G_3 is isomorphic to the smallest non-abelian simple group A_5 of even permutations on five letters (cf Coxeter, [7]). However, a factor group of G_1 determined by the normal closure N' of $\{(g^{-1}h)^3\}$ in G_1 is also isomorphic to G_3. We prove that N' is a normal subgroup of F.

Since $g = bc$, $h = ca$, and

$$a^{-1}(g^{-1}h)a = acbcaa = acbc = h^{-1}g = (g^{-1}h)^{-1}$$

therefore $\quad a^{-1}(g^{-1}h)^3a = (g^{-1}h)^{-3} \in N'$

Further, since

$$b^{-1}g^{-1}hb = bcbcab = g^2.(gh)^{-1} = g^2h^{-1}g^{-1}$$

hence,

$$\begin{aligned} b^{-1}(g^{-1}h)^3b &= g^2h^{-1}g^{-1}g^2h^{-1}g^{-1}g^2h^{-1}g^{-1} \\ &= g^2h^{-1}g.h^{-1}g.h^{-1}g.g^{-2} \\ &= g^2(h^{-1}g)^3g^{-2} \\ &= g^2(g^{-1}h)^{-3}g^{-2} \in N' \end{aligned}$$

Also

$$c^{-1}g^{-1}hc = ccbcac = bc.ac = gh^{-1}$$

Thus

$$c^{-1}(g^{-1}h)^3c = gh^{-1}gh^{-1}gh^{-1} = g.h^{-1}gh^{-1}gh^{-1}g.g^{-1}$$

$$= g(h^{-1}g)^3g^{-1}$$

$$= g(g^{-1}h)^{-3}g^{-1} \in N'$$

showing that N' is a normal subgroup of F. Therefore

$$F/N' = gp\{G_1/N', c'\} = gp\{G_3, c'\} = F_2 \text{ say.}$$

Again identifying g', h', c' with g, h, c respectively we have

$$F_2 = gp\{g,h,c; g^5 = h^5 = (gh)^2 = (g^{-1}h)^3 = (gc)^2 = (ch)^2 = c^2 = 1\}$$

and in terms of a, b, c,

$$F_2 = gp\{a,b,c; a^2 = b^2 = c^2 = (bc)^5 = (ca)^5 = (ab)^2 = (cbca)^3 = 1\}$$

By a reasoning similar to that above, we see that also in F_2, A, B, C have their precise intersections. F_2, therefore, embeds the amalgam of A, B, C. The finiteness of F_2 follows from the fact that it is an extension of a finite group G_3 by a cyclic group of order 2. This completes the answer to the first question.

We now show that there exist two finite embeddings with the property that the smaller embedding is not a homomorphic image of the larger one. we consider the group

$$G_4 = gp\{g'', h''; g''^5 = h''^5 = (g''h'')^2 = (g''^{-1}h'')^4 = 1\} \qquad (4)$$

which is isomorphic to the simple group A_6 of even permutations on six letters (Coxeter, [7] p. 79), and is a homomorphic image of G_1 where G_1 is the same as in (1). The kernel N'' of this homomorphism is the normal closure of $\{(g^{-1}h)^4\}$ in G_1. We show again that N'' is a normal subgroup of F. It is enough to show that the transforms of $(g^{-1}h)^4$ by a, b, c are in N''. Now

$$a^{-1}g^{-1}ha = (g^{-1}h)^{-1}$$
$$b^{-1}(g^{-1}h)b = g^2h^{-1}g^{-1}$$
$$c(g^{-1}h)c = gh^{-1}$$

and therefore, since, as before, $a(g^{-1}h)^4a = (g^{-1}h)^{-4}$, $b(g^{-1}h)^4b = g^2(g^{-1}h)^{-4}g^{-2}$ and $c(g^{-1}h)^4c = g(g^{-1}h)^{-4}g^{-1}$ are elements of N'', N'' is a normal subgroup of F and the factor group

$$F_3 = F/N'' = gp\{G_4, c''\}$$

being a split extension of G_4 by $\{c''\}$, is finite.

In terms of a, b, c, F_3 is given by

$$F_3 = gp\{a,b,c; a^2 = b^2 = c^2 = (bc)^5 = (ca)^5 = (ab)^2 = (cbca)^4 = 1\}$$

That F_3 embeds the amalgam of A, B and C follows as before. However, considering

$$F_3 = gp\{G_4, c''\} \quad \text{and} \quad F_2 = gp\{G_3, c'\} ,$$

one sees that F_2 cannot be a homomorphic image of F_3, because of the simplicity of G_3 and G_4, answering (ii).

Moreover F_2 and F_3 are minimal finite embeddings of A , B , C in the sense that there are no proper factor groups of F_2 and F_3 which embed the amalgam of A , B , C giving a negative answer to question (iii).

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