BASIC COMMUTATORS

FOR

POLYNILPOTENT GROUPS

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STATEMENT

Apart from a brief historical survey appearing in the introduction, the work reported in this thesis is entirely my own with no assistance from any other person.

In this, as in any mathematical text, certain elementary and general facts are assumed in the very language used: these are described in detail in Appendix II. Apart from these only two known results are used, and these are acknowledged with references when they appear.

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<u>CONTENTS</u>

CHAPTER 0

INTRODUCTION

BACKGROUND *

The conventional theory of basic commutators may be considered to be an investigation of the properties of the lower central series $\gamma_c(F) \ : c = 1,2,\ldots$ of an absolutely free group F , or alternatively of the properties of the corresponding factor groups, which are the free nilpotent groups of various classes:
$F \,/\, \gamma_{c+1}(F) = F(\underline{\underline{N}}_c)$.

Suppose that G is a group generated by a subset $\mathcal{G}$. Then a set of "formal expressions" may be constructed using the elements of $\mathcal{G}$, the symbol 1 and the operations of inversion, multiplication and commutation. Each of these formal expressions will then represent a unique element of G in the obvious way. Certain formal expressions, known as "basic commutators" of "weight c" are defined for positive integers c and well ordered by recursion over c as follows:

* Throughout this thesis certain symbols in more or less common use will be employed without formal definition. For the reader's convenience these are collected in Appendix II, together with all terms and symbols defined in the text.

(i) The basic commutators of weight 1 are the elements of $\mathcal{G}$. They may be well-ordered in any way.

(ii) Assuming that $c \geqq 2$ and the basic commutators of weight $< c$ have been defined and ordered, the basic commutators of weight c are the expressions of the form $[x,y]$ where x and y are basic commutators of weights r and s respectively, $r + s = c$, $x > y$ and, if $x = [x_1, x_2]$ then $x_2 \leqq y$. The well-order may be extended to the basic commutators of weight c in any way so that they follow the ones of smaller weight.

The following facts have been established.

(1) A "collecting process" is defined, by means of which a formal expression for an element in $F(\underline{\underline{N}}_c)$, where $\mathcal{G}$ is a free generating set for this group, can be transformed into a particular type of expression known as a "basic product" of the form 1 or $b_1^{\beta_1} b_2^{\beta_2} \cdots b_k^{\beta_k}$ where $b_1, b_2, \ldots, b_k$ are basic commutators of weight $\leqq c$, $b_1 < b_2 < \ldots < b_k$ and $\beta_1, \beta_2, \ldots \beta_k$ are non-zero integers. The basic product represents the same element of the group as did the original expression.

(2) The representation of a particular element of $F(\underline{\underline{N}}_c)$ in this form is unique.

(3) $\gamma_c(F) / \gamma_{c+1}(F) \cong \gamma_c(F(\underline{\underline{N}}_c))$ is a free Abelian group, for

which those elements represented by basic commutators of weight c form a free basis.

(4) The upper and lower central series of $F(\underline{\underline{N}}_c)$ coincide. More specifically, provided the rank of F is > 1,
$\zeta_r(F(\underline{\underline{N}}_c)) = \gamma_{c-r}(F(\underline{\underline{N}}_c))$.

(5) The lower central series of the absolutely free group F has trivial intersection: $\bigcap_{i=1}^{\infty} \gamma_i(F) = \{1\}$. In other words, F is residually nilpotent.

(6) "Witt's formula". When the number τ of generators is finite, the number of basic commutators of weight c is also finite, and is the number $\sum_{r|c} \mu(r)\, \tau^{c/r}$, where μ is the Möbius function, defined for any positive integer r : $\mu(r) = 0$ if there exists $p > 1$ such that $p^2 \mid r$, $\mu(r) = (-1)^s$ otherwise, where s is the number of primes dividing r .

The results (1), (2) and (5) can be considered to constitute a tool for investigating these groups and the results (3), (4) and (6) as applications illustrating its power.

Parts of this theory have been applied to the study of p-groups.

The history of the subject can be covered briefly. The theory was initiated in 1934 by P. Hall [4] in a paper concerned with p-groups. Here the notion of "basic commutator" was introduced and the collecting process investigated (Result (1)), however the question of uniqueness was not treated in this paper. In 1937 E. Witt [10] showed that the whole problem could be translated into an equivalent one concerning free Lie rings, and also produced the Witt formula (Result (6)). At about the same time W. Magnus [6,7], also working in terms of free Lie rings, introduced the so-called "Magnus Ring" in terms of which the residual nilpotence of absolutely free groups (Result (5)) was proven. Finally, thirteen years later, Marshall Hall Jr. [3], using the results of all of these papers, was able to prove the "basis theorem" (Results (2) and (3)) thus rounding the theory off nicely.

Surveys of this theory may be found in P. Hall [5] and R. H. Bruck [2].

PREVIEW

The work reported in this thesis arose originally from the desire to prove the results of Chapter 5 and more generally from the feeling that it should be possible to modify the theory of basic commutators as just described to permit the properties of free polynilpotent groups to be studied in the same way.

The idea of "weight" of a commutator is well-known. This may be extended to the idea of weight of an expression (definition 1.3) and then the terms of the lower central series of a group may be defined thus: an element of G belongs to $\gamma_c(G)$ if and only if it may be written as an expression of weight $\geq c$. This is possibly not a familiar way of defining the lower central series, and is made precise in definition 1.10 and lemma 1.6.

Here the idea of weight is generalized to that of "semiweight" and "semiweight range". Then, for a given semiweight range W, subgroups $W_\alpha(G)$ of a group G, consisting of all elements which may be written as an expression of semiweight $\geq \alpha$ are defined. It is in the generalization of "weight" to "semiweight" that the crux of this thesis lies: for, just as the lower central series may be defined in terms of weight and then the conventional theory of basic commutators investigates the properties of this series, so the subgroups $W_\alpha(G)$ are defined in terms of the semiweight range W and the theory to be described here investigates the properties of these subgroups. But the semiweight range W may be chosen so that these subgroups contain among them the verbal subgroups corresponding to polynilpotent varieties.

While the semiweight range W may be chosen so that subgroups not directly connected with polynilpotent groups may be investigated, so that the early part of this thesis will be slightly more general than its title suggests, the prime consideration throughout will be the study of polynilpotent groups; in Chapters 3 and 4 it will be seen that in fact I have been able to prove some important results only for the polynilpotent case.

In Chapter 1 the idea of a semiweight range W with its associated semiweight σ is introduced. In terms of this the W-basic commutators are then defined. The most important part of this chapter consists of a proof that the number of W-basic commutators, for a given finite number of generators, is independent of the choice of W .

Chapter 2 develops the collecting process which together with the main result of Chapter 1 provides the basis theorem. In this chapter the forgoing theory is applied to Lie rings and, as might be expected, the results are pleasingly straightforward.

In Chapter 3 the "polynilpotent" semiweights are defined. A result analogous to the residual nilpotence of absolutely free groups in the conventional theory, namely "partial collectability" is proved for this case.

Chapter 4 investigates the upper and lower central series for certain semiweights (including the polynilpotent ones). It is shown that the complete inverse image of the centre of $F / W_{\alpha}(F)$, where F is an absolutely free group, is $W_{\alpha-1}(F)$ for a suitable and quite natural definition of $\alpha-1$.

By this stage results analogous to (1) - (6) of the previous section have been stated and proved.

Chapter 5 contains the solution of a problem which uses some of this theory. This chapter is in the nature of an excursion, but is included for three reasons: firstly for its intrinsic interest, secondly because it in fact caused me to embark upon this work and thirdly because it is an application, albeit not a very mysterious one, of the results of Chapters 1 - 4.

FORMAL EXPRESSIONS

The conventional theory of basic commutators concerns itself much of the time with formal expressions: not so much the elements of a group themselves as the way they are written down on paper. In the present theory it will be found that more and more emphasis is placed on this aspect and that group-theoretic results, though the primary object of this study, appear infrequently.

In order to avoid this essentially metamathematical approach in this thesis I have replaced the idea of formal expressions for the elements of a group by the elements of a free algebra. This algebra is chosen so as to be anarchic enough that we may regard (intuitively) the elements of the algebra to be in one-to-one correspondence with the possible formal expressions for elements in the group.

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CHAPTER 1

W-BASIC COMMUTATORS

THE ALGEBRA OF EXPRESSIONS

Definition 1.1

Let $\underset{\sim}{G} = \{\underset{\sim}{g}_i\}_{i<\tau}$ be some set indexed by the ordinals less than some ordinal τ, the indexing being one-to-one. Form the algebra $(\underset{\sim}{A}, \Omega, \underset{\equiv}{e})$ generated freely by the set $\underset{\sim}{G}$ with operator domain $\Omega = \{\varepsilon, \nu, \mu, \chi\}$ where

ε is a nullary operator (the "unit element"),

ν is a unary operator ("inversion"),

μ and χ are binary operators ("multiplication" and "commutation" respectively) ,

the only law being that μ is associative.

A more conventional notation will be used for the effect of the operators on $\underset{\sim}{A}$, as follows:

$$\varepsilon = \underset{\sim}{1}$$

$$\left.\begin{aligned} \underset{\sim}{x}\nu &= \underset{\sim}{x}^{-1} \\ \underset{\sim}{x}\underset{\sim}{y}\mu &= \underset{\sim}{x}\underset{\sim}{y} \\ \underset{\sim}{x}\underset{\sim}{y}\chi &= [\underset{\sim}{x},\underset{\sim}{y}] \end{aligned}\right\} \quad \text{for all } \underset{\sim}{x}, \underset{\sim}{y} \in \underset{\sim}{A} .$$

Parentheses will be used in connection with the operations of inversion and multiplication in accordance with the usual conventions. A "left-normed" convention will be used in connection with the operation of commutation, that is,

$[\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}] = [[\underset{\sim}{a}, \underset{\sim}{b}], \underset{\sim}{c}]$, and so on.

The set $\underset{\sim}{G}$ will be considered to be a subset of $\underset{\sim}{A}$ in the usual way. The elements of $\underset{\sim}{A}$ will be called "expressions" and $\underset{\sim}{A}$ itself will be called the "algebra of expressions". ◇

It should be remarked that the operators ε and ν are not bona fide unit and inversion operators with respect to μ, since the associative law of multiplication is the only law of $\underset{\sim}{A}$; for instance $\underset{\sim}{x1} \neq \underset{\sim}{x}$ and $\underset{\sim}{xx}^{-1} \neq \underset{\sim}{1}$.

We have at our disposal some well-known results concerning such an algebra, which may be summarized as follows.

<u>Lemma 1.1</u>

(A) (i) $\underset{\sim}{1} \in \underset{\sim}{A}$ and $\underset{\sim}{G} \subseteq \underset{\sim}{A}$,

(ii) $\underset{\sim}{a} \in \underset{\sim}{A} \Rightarrow \underset{\sim}{a}^{-1} \in \underset{\sim}{A}$,

(iii) $\underset{\sim}{a}, \underset{\sim}{b} \in \underset{\sim}{A} \Rightarrow \underset{\sim}{ab} \in \underset{\sim}{A}$ and $[\underset{\sim}{a}, \underset{\sim}{b}] \in \underset{\sim}{A}$.

(B) If $\underset{\sim}{a} \in \underset{\sim}{A}$ then one and only one of the following five possibilities is true:

(i) $\mathbf{a} = \mathbf{1}$.

(ii) $\mathbf{a} \in \mathbf{G}$.

(iii) There exists a unique $\mathbf{x} \in \mathbf{A}$ such that $\mathbf{a} = \mathbf{x}^{-1}$.

(iv) There exists a unique pair $\mathbf{x}$, $\mathbf{y} \in \mathbf{A}$ such that $\mathbf{a} = [x,y]$.

(v) There exists a finite sequence $\mathbf{x}_1$, $\mathbf{x}_2$, ... , $\mathbf{x}_k$ of elements of $\mathbf{A}$ ($k \geqq 2$) such that $\mathbf{a} \in \mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_k$, and none of the elements $\mathbf{x}_i$ can be written as such a product themselves, that is, each $\mathbf{x}_i$ is subject to B (i), (ii), (iii) or (iv) of this lemma, but not (v)).

(C) To each $\mathbf{a} \in \mathbf{A}$ there corresponds a uniquely determined positive integer $ht(\mathbf{a})$, called the "height" of $\mathbf{a}$, which is defined by its properties:

(i) $ht(\mathbf{1}) = 1$ and $\mathbf{g}_i \in \mathbf{G} \Rightarrow ht(\mathbf{g}_i) = 1$,

(ii) $ht(\mathbf{a}^{-1}) = ht(\mathbf{a}) + 1$,

(iii) $ht(\mathbf{ab}) = ht([\mathbf{a},\mathbf{b}]) = ht(\mathbf{a}) + ht(\mathbf{b}) + 1$.

Roughly speaking, the height of an expression is the number of symbols required to write it in terms of the generators $\mathbf{G}$ and the operators ε , ν , μ , χ using Łukasiewicz' notation. Should the reader desire it, a brief but sufficient description of this notation may be found in B.H.Neumann [8], p.26. ◇

It will be useful to have a rather artificial definition of powers of an element in $\underset{\sim}{A}$.

Definition 1.2

Let n be an integer and let $\underset{\sim}{x} \in \underset{\sim}{A}$. Then $\underset{\sim}{x}^{n}$ is defined recursively:

(i) $\underset{\sim}{x}^{0} = \underset{\sim}{1}$,

(ii) $\underset{\sim}{x}^{1} = \underset{\sim}{x}$ and if $n > 1$, $\underset{\sim}{x}^{n} = \underset{\sim}{x}^{n-1}\underset{\sim}{x}$,

(iii) $\underset{\sim}{x}^{-1}$ is as described in definition 1.1 and if $n > 1$

$\underset{\sim}{x}^{-n} = \underset{\sim}{x}^{-(n-1)}\underset{\sim}{x}^{-1}$. ◇

Since $\underset{\sim}{xx}^{-1} \neq 1$, none of the "index laws" hold good with this notation.

Definition 1.3

Let $\overline{N}$ denote the set of positive integers with an extra element ∞ adjoined. The usual addition and order on the positive integers is extended to encompass ∞ by:

$$\left.\begin{array}{c} \infty + n = n + \infty = \infty + \infty = \infty \\ n < \infty \end{array}\right\} \quad \text{for any positive integer } n .$$

The mapping $wt : \underset{\sim}{A} \to \overline{N}$ is defined recursively over the height of expressions in $\underset{\sim}{A}$ as follows:

(i) $\quad \mathrm{wt}(\mathbf{1}) = \infty \ ; \quad g_i \in \mathbf{G} \Rightarrow \mathrm{wt}(g_i) = 1$,

(ii) $\quad \mathrm{wt}(\mathbf{x}^{-1}) = \mathrm{wt}(\mathbf{x})$,

(iii) $\quad \mathrm{wt}(\mathbf{x}\mathbf{y}) = \min \{ \mathrm{wt}(\mathbf{x}) , \mathrm{wt}(\mathbf{y}) \}$,

(iv) $\quad \mathrm{wt}([\mathbf{x},\mathbf{y}]) = \mathrm{wt}(\mathbf{x}) + \mathrm{wt}(\mathbf{y})$.

For each expression $\mathbf{x}$, $\mathrm{wt}(\mathbf{x})$ is called the "weight" of $\mathbf{x}$.

◇

<u>Definition 1.4</u>

A sequence $(\mathbf{C}_n)_{n=1}^{\infty}$ of subsets of $\mathbf{A}$ is defined recursively by

(i) $\quad \mathbf{C}_1 = \mathbf{G}$,

(ii) $\quad$ For $n > 1$, $\mathbf{C}_n = \{ [\mathbf{x},\mathbf{y}] : \mathbf{x} \in \mathbf{C}_r , \mathbf{y} \in \mathbf{C}_s , r + s = n \}$.

The union $\mathbf{C} = \bigcup_{n=1}^{\infty} \mathbf{C}_n$ of this sequence is called the "set of commutators" of $\mathbf{A}$. An element of $\mathbf{C}$ is called a commutator.

◇

The simplest properties of commutators may be summarized thus:

<u>Lemma 1.2</u>

(i) $\quad \mathbf{G} \subseteq \mathbf{C} \subseteq \mathbf{A}$,

(ii) $\quad$ If $\mathbf{c} \in \mathbf{C}$ then $1 \leq \mathrm{wt}(\mathbf{c}) < \infty$ and $\mathrm{ht}(\mathbf{c}) = 2\mathrm{wt}(\mathbf{c}) - 1$,

(iii) $\quad$ If $\mathbf{c} \in \mathbf{C}$ then either

(α) $\quad \mathrm{wt}(\mathbf{c}) = 1$ and $\mathbf{c} \in \mathbf{G}$, or

(β) $\quad \mathrm{wt}(\mathbf{c}) > 1$ and there exist unique $\mathbf{x},\mathbf{y} \in \mathbf{A}$

such that $\underset{\sim}{c} = [\underset{\sim}{x},\underset{\sim}{y}]$, $wt(\underset{\sim}{c}) = wt(\underset{\sim}{x}) + wt(\underset{\sim}{y})$ and hence $wt(\underset{\sim}{x}) < wt(\underset{\sim}{c})$ and $wt(\underset{\sim}{y}) < wt(\underset{\sim}{c})$,

(iv) $\underset{\sim}{C}_n$ is the set of all commutators of weight n . ◇

It will be noticed that the weight of a commutator, as defined here, corresponds with the usual definition, but that the idea of weight is extended to arbitrary expressions. It may be felt that the process of "formalization" is being carried to extremes when commutators are defined to be expressions rather than group elements; I am of the opinion that this definition leads to a simpler treatment, especially when commutators of higher weight are being considered.

The emphasis throughout this study will be on groups, however some of the results will apply to more general algebras (see for instance theorem 2.3). These may be defined as follows:

<u>Definition 1.5</u>

A "describable algebra" (G , Ω , $\underline{\underline{e}}$) is an algebra with operator domain Ω (the same as in definition 1.1) in which multiplication is associative.

Groups will be written multiplicatively and then the operators of Ω will have the obvious effects : $\varepsilon = 1$, $x\nu = x^{-1}$, $xy\mu = xy$ and $xy\chi = x^{-1}y^{-1}xy$. $\diamond$

The method by which the elements of $\underset{\sim}{A}$ describe the elements of such an algebra, and so supplant the notion of "formal expressions" may now be made precise.

<u>Definition 1.6</u>

Let G be any describable algebra (or, more particularly, a group) and suppose G is generated by a set $\mathcal{G} = \{g_i\}_{i<\tau}$ of generators indexed by the ordinals less than τ . Let $\underset{\sim}{A}$ be as defined in definition 1.1 and let $\bar{\rho} : \underset{\sim}{G} \to \mathcal{G}$ be the mapping $g_i\bar{\rho} = g_i$ ($i < \tau$) .

Then ρ may be extended uniquely to an epimorphism $\rho : \underset{\sim}{A} \to G$. The mapping ρ is called a "description" of G ; further, if G is a relatively free algebra and $\mathcal{G}$ freely generates G , ρ is called a "free description" of G . $\diamond$

SEMIWEIGHTS

Definition 1.7

A set W is called a "semiweight range" if it has an order $\leqq$ and addition $+$ defined on it with the following properties:

(i) $\leqq$ well-orders W. W has a least element 1 and a greatest element ∞.

(ii) W is closed under addition and $W \setminus \{\infty\}$ is generated by 1 under addition.

(iii) Addition is commutative: $\alpha + \beta = \beta + \alpha$. However addition is not necessarily associative; a left-normed convention will be used: $\alpha + \beta + \gamma = (\alpha + \beta) + \gamma$ and so on.

(iv) $\alpha , \beta < \infty \Rightarrow \alpha + \beta < \infty$.

$\alpha < \infty \Rightarrow \alpha < \alpha + \beta$

$\alpha + \infty = \infty$

(v) $\alpha_1 < \alpha_2$ and $\beta < \infty \Rightarrow \alpha_1 + \beta < \alpha_2 + \beta$

(vi) $\alpha \leqq \beta \leqq \gamma \Rightarrow \gamma + \beta + \alpha = \gamma + \alpha + \beta \leqq \beta + \alpha + \gamma$.

◇

Commutativity of addition together with (iv) and (v) yield many similar results quite easily, for instance $\alpha_1 \leqq \alpha_2$ and $\beta_1 \leqq \beta_2 \Rightarrow \alpha_1 + \beta_1 \leqq \alpha_2 + \beta_2$. These will be used without further comment.

It will be noticed that at this stage the symbol $\leqq$ has appeared with two distinct meanings – the ordinary order on the integers and the order just defined for a semiweight range. Before long a third meaning will appear (definition 1.12). Since these different orders are defined on disjoint sets, no confusion should occur.

Definition 1.8

Let $\underset{\sim}{A}$ be an algebra of expressions and W a semiweight range. A mapping $\sigma : \underset{\sim}{A} \to W$ is called the "semiweight on $\underset{\sim}{A}$ associated with W" if it has the properties

(i) $\sigma(\underset{\sim}{1}) = \infty$; $\underset{\sim}{g}_i \in \underset{\sim}{G} \Rightarrow \sigma(\underset{\sim}{g}_i) = 1$.

(ii) $\sigma(\underset{\sim}{x}^{-1}) = \sigma(\underset{\sim}{x})$.

(iii) $\sigma(\underset{\sim}{x}\underset{\sim}{y}) = \min \{ \sigma(\underset{\sim}{x}) , \sigma(\underset{\sim}{y}) \}$.

(iv) $\sigma([\underset{\sim}{x},\underset{\sim}{y}]) = \sigma(\underset{\sim}{x}) + \sigma(\underset{\sim}{y})$. $\diamond$

Clearly σ is defined uniquely by W , this definition amounting to a recursive specification of $\sigma(\underset{\sim}{x})$ over the height of $\underset{\sim}{x}$.

Lemma 1.3

With the notation of definition 1.8, the mapping $\sigma : \underset{\sim}{A} \to W$ is onto, provided that the set $\underset{\sim}{G}$ of generators of $\underset{\sim}{A}$ is nonempty.

Proof

This follows easily by transfinite induction over the elements of W, using definition 1.7 (iv), definition 1.8 (iv) and the closure of $\underset{\sim}{A}$ under commutation. ◇

Lemma 1.4

With the notation of definition 1.3, $\bar{N}$ is a semiweight range and $wt : \underset{\sim}{A} \to \bar{N}$ is the associated semiweight.

Proof

This follows immediately from a comparison of definitions 1.3, 1.7 and 1.8. ◇

As remarked in the introduction, the notion of "semiweight" is a generalization of that of "weight". It will be seen throughout this thesis that if the word "semiweight" is replaced by "weight" the theorems will reduce to known ones or trivialities.

It might be worth remarking that $\bar{N}$ is the only possible semiweight range (up to isomorphism) for which addition is associative.

Lemma 1.4 provides a simple example of a semiweight. For a more interesting one we must wait until Chapter 4 when the "polynilpotent" semiweights are defined.

VERBAL SUBGROUPS

Definition 1.9

Let $\underset{\sim}{A}$ be an algebra of expressions and let $\sigma : \underset{\sim}{A} \to W$ be a semiweight. Then for each $\alpha \in W$, the subset $\underset{\sim}{W}_{\alpha}$ of $\underset{\sim}{A}$ is defined to be the set of expressions of semiweight $\geqq \alpha$,

$\underset{\sim}{W}_{\alpha} = \{ \underset{\sim}{x} : \underset{\sim}{x} \in \underset{\sim}{A} , \ \sigma(\underset{\sim}{x}) \geqq \alpha \}$. ◇

Lemma 1.5

(i) Let $\sigma : \underset{\sim}{A} \to W$ be a semiweight and θ be any endomorphism of $\underset{\sim}{A}$. Then, for any $\underset{\sim}{x} \in \underset{\sim}{A}$, $\sigma(\underset{\sim}{x}\theta) \geqq \sigma(\underset{\sim}{x})$.

(ii) With the notation of definition 1.9, $\underset{\sim}{W}_{\alpha}$ is a fully-invariant subalgebra of $\underset{\sim}{A}$.

Proof

(i) follows by an easy induction over the height of $\underset{\sim}{x}$, and (ii) is a corollary of (i). ◇

Definition 1.10

Let G be a describable algebra (or in particular a group), let $\rho : \underset{\sim}{A} \to G$ be a description of G and let $\sigma : \underset{\sim}{A} \to W$ be a semiweight. Then for each $\alpha \in W$ the subset $W_\alpha(G)$ is defined to be $W_\alpha(G) = \underset{\sim}{W_\alpha}\rho = \{ \underset{\sim}{x}\rho : \underset{\sim}{x} \in \underset{\sim}{A} , \sigma(\underset{\sim}{x}) \geqq \alpha \}$. ◇

Theorem 1.1

With the notation of the forgoing definition,

(i) $W_\alpha(G)$ is independent of the particular description of G chosen to define it; it depends only on G , W and α .

(ii) $W_\alpha(G)$ is a verbal, and hence fully-invariant, subalgebra of G .

(iii) $W_1(G) = G$ and $W_\infty(G) = \{1\}$.

(iv) $\alpha \leqq \beta \Rightarrow W_\beta(G) \subseteqq W_\alpha(G)$.

Proof

Parts (i) and (ii) follow immediately from lemma 1.5 and parts (iii) and (iv) follow easily from definitions 1.7, 1.8 and 1.10. ◇

Lemma 1.6

With the notation of lemma 1.4, $\overline{N}_c(G) = \gamma_c(G)$ for any group G and positive integer c .

Proof

Suppose $\rho : \underset{\sim}{A} \to G$ is any description of G . Then, for any $\underset{\sim}{x} \in \underset{\sim}{A}$, $wt(\underset{\sim}{x}) \geq c \Rightarrow \underset{\sim}{x}\rho \in \gamma_c(G)$. This follows by an easy induction over the height of $\underset{\sim}{x}$. Consequently, if $x \in \overline{N}_c(G)$ then there exists $\underset{\sim}{x} \in \underset{\sim}{A}$ such that $x = \underset{\sim}{x}\rho$ and $wt(\underset{\sim}{x}) \geq c$ and then $x \in \gamma_c(G)$. Thus $\overline{N}_c(G) \leq \gamma_c(G)$. The converse inclusion, $\gamma_c(G) \leq \overline{N}_c(G)$, is proved by induction over c. First $\gamma_1(G) = G = \overline{N}_1(G)$. Now suppose $c > 1$ and the result is true for all smaller values. Since $\gamma_c(G) = [\gamma_{c-1}(G), G]$, any $x \in \gamma_c(G)$ may be written in the form $x = [a_1,b_1]^{\varepsilon_1}[a_2,b_2]^{\varepsilon_2}\ldots[a_k,b_k]^{\varepsilon_k}$ where k is a finite integer and, for each i , $a_i \in \gamma_{c-1}(G)$, $b_i \in G$ and $\varepsilon_i = \pm 1$. Then by the inductive hypothesis there exists $\underset{\sim}{a}_i \in \underset{\sim}{A}$ such that $a_i = \underset{\sim}{a}_i\rho$ and $wt(\underset{\sim}{a}_i) \geq c-1$. There also exists $\underset{\sim}{b}_i \in \underset{\sim}{A}$ such that $b_i = \underset{\sim}{b}_i\rho$. Write $\underset{\sim}{x} = [\underset{\sim}{a}_1,\underset{\sim}{b}_1]^{\varepsilon_1}[\underset{\sim}{a}_2,\underset{\sim}{b}_2]^{\varepsilon_2}\ldots[\underset{\sim}{a}_k,\underset{\sim}{b}_k]^{\varepsilon_k}$. It follows that $x = \underset{\sim}{x}\rho$ and $wt(\underset{\sim}{x}) \geq c$. Hence $x \in \overline{N}_c(G)$ and the result follows. $\diamond$

As a consequence of theorem 1.1 the following definition may be made.

Definition 1.11

Let W be a semiweight range and let $\alpha \in W$. Then $\underline{\underline{W}}_\alpha$ is the variety of all groups G for which $W_\alpha(G) = \{1\}$. ◇

It follows from lemma 1.6 that $\overline{\underline{\underline{N}}}_{c+1}$ is the variety of groups G for which $\gamma_{c+1}(G) = \{1\}$, that is, the variety of groups which are nilpotent of class c. Thus $\overline{\underline{\underline{N}}}_{c+1} = \underline{\underline{N}}_c$.

W-BASIC COMMUTATORS

In this section a well-ordering of the set $\underset{\sim}{C}$ of commutators is described. This order depends on two things: the order imposed on the generators $\underset{\sim}{G}$ by the ordinals indexing them and the semiweight $\sigma : \underset{\sim}{A} \to W$. Subsequently a subset of $\underset{\sim}{C}$, the set of "W-basic commutators", is defined in terms of this. The group-theoretic results arrived at later will not depend upon the ordering of $\underset{\sim}{G}$, though they do depend very much upon W. However the intermediate steps rely heavily on this well-ordering.

The first task then is to define this well-ordering of the commutators. This must be done fairly carefully: it is defined first as a relation and subsequently proved to be a well-order. Before embarking upon the definition it should be remarked that, for any semiweight range W, the set of commutators of semiweight 1 is exactly $\underset{\sim}{G}$.

Definition 1.12

Let $\sigma : \underset{\sim}{A} \to W$ be a semiweight. The relation $<$ on $\underset{\sim}{C}$ is defined recursively over the semiweight of commutators.

(i) $\quad g_i < g_j \quad$ if $\quad i < j \quad$ (for $g_i , g_j \in \underset{\sim}{G}$) .

(ii) $\quad \sigma(\underset{\sim}{x}) < \sigma(\underset{\sim}{y}) \implies \underset{\sim}{x} < \underset{\sim}{y}$.

It remains to define the relation on pairs of commutators of the same semiweight > 1 . Suppose then that $\sigma(\underset{\sim}{a}) = \sigma(\underset{\sim}{b}) = \xi > 1$ and that the relation $<$ has been defined on the set of all commutators of semiweight $< \xi$. An intermediate definition must be made: suppose $\sigma(\underset{\sim}{x}) = \xi$. Then since $\xi > 1$ we may write $\underset{\sim}{x} = [\underset{\sim}{x}_1, \underset{\sim}{x}_2]$. Then the "leading" and "trailing" parts of $\underset{\sim}{x}$ are defined

$$\begin{aligned} \mathrm{ld}(\underset{\sim}{x}) &= \underset{\sim}{x}_1 && \text{if } \underset{\sim}{x}_2 < \underset{\sim}{x}_1 \text{ or } \underset{\sim}{x}_2 = \underset{\sim}{x}_1 \\ &= \underset{\sim}{x}_2 && \text{otherwise, and} \\ \mathrm{tr}(\underset{\sim}{x}) &= \underset{\sim}{x}_2 && \text{if } \underset{\sim}{x}_2 < \underset{\sim}{x}_1 \text{ or } \underset{\sim}{x}_2 = \underset{\sim}{x}_1 \\ &= \underset{\sim}{x}_1 && \text{otherwise.} \end{aligned}$$

then $\underset{\sim}{a} < \underset{\sim}{b}$ if

(iii) $\sigma(\underset{\sim}{a}) = \sigma(\underset{\sim}{b}) = \xi > 1$ and

(a) $ld(\underset{\sim}{a}) < ld(\underset{\sim}{b})$,

(b) $ld(\underset{\sim}{a}) = ld(\underset{\sim}{b})$ and $tr(\underset{\sim}{a}) < tr(\underset{\sim}{b})$

or (c) $ld(\underset{\sim}{a}) = ld(\underset{\sim}{b})$, $tr(\underset{\sim}{a}) = tr(\underset{\sim}{b})$ and $b_1 < a_1$

(where $\underset{\sim}{a} = [\underset{\sim}{a}_1, \underset{\sim}{a}_2]$ and $\underset{\sim}{b} = [\underset{\sim}{b}_1, \underset{\sim}{b}_2]$) .

The reversal of the relation in part (iiic) is not a misprint. With regard to this part of the definition it will be noticed that if $ld(\underset{\sim}{a}) = ld(\underset{\sim}{b})$ and $tr(\underset{\sim}{a}) = tr(\underset{\sim}{b})$ then either $\underset{\sim}{b} = \underset{\sim}{a}$ or $\underset{\sim}{b} = [\underset{\sim}{a}_2, \underset{\sim}{a}_1]$ where $\underset{\sim}{a} = [\underset{\sim}{a}_1, \underset{\sim}{a}_2]$.

(iv) The usual notations are used:

$$\underset{\sim}{a} \leqq \underset{\sim}{b} \iff \underset{\sim}{a} < \underset{\sim}{b} \text{ or } \underset{\sim}{a} = \underset{\sim}{b} ,$$

$$\underset{\sim}{a} > \underset{\sim}{b} \iff \underset{\sim}{b} < \underset{\sim}{a} ,$$

$$\underset{\sim}{a} \geqq \underset{\sim}{b} \iff \underset{\sim}{b} \leqq \underset{\sim}{a} .$$

The relation $\leqq$ is called the "W-ordering" of $\underset{\sim}{C}$. ◇

<u>Lemma 1.7</u>

With the notation of the forgoing definition, $\leqq$ is a (full) well ordering of $\underset{\sim}{C}$.

<u>Proof</u>

Notice first that

(α) $\underset{\sim}{a} \leqq \underset{\sim}{b} \Rightarrow \sigma(\underset{\sim}{a}) \leqq \sigma(\underset{\sim}{b})$,

(β) $\underset{\sim}{g}_i \leqq \underset{\sim}{g}_j \Rightarrow i \leqq j$,

(γ) $\underset{\sim}{a} \leqq \underset{\sim}{b}$ and $\sigma(\underset{\sim}{a}) = \sigma(\underset{\sim}{b}) \Rightarrow \mathrm{ld}(\underset{\sim}{a}) \leqq \mathrm{ld}(\underset{\sim}{b})$,

(δ) $\underset{\sim}{a} \leqq \underset{\sim}{b}$, $\sigma(\underset{\sim}{a}) = \sigma(\underset{\sim}{b})$ and $\mathrm{ld}(\underset{\sim}{a}) = \mathrm{ld}(\underset{\sim}{b}) \Rightarrow \mathrm{tr}(\underset{\sim}{a}) = \mathrm{tr}(\underset{\sim}{b})$,

(ε) $\underset{\sim}{a} \leqq \underset{\sim}{b}$, $\sigma(\underset{\sim}{a}) = \sigma(\underset{\sim}{b})$, $\mathrm{ld}(\underset{\sim}{a}) = \mathrm{ld}(\underset{\sim}{b})$ and $\mathrm{tr}(\underset{\sim}{a}) = \mathrm{tr}(\underset{\sim}{b})$

$\Rightarrow \underset{\sim}{b}_1 \leqq \underset{\sim}{a}_1$

(where $\underset{\sim}{a} = [\underset{\sim}{a}_1, \underset{\sim}{a}_2]$ and $\underset{\sim}{b} = [\underset{\sim}{b}_1, \underset{\sim}{b}_2]$) .

Since it is not yet known that $\leqq$ is a partial order, these statements must be proved by checking all the various possibilities listed in the definition. It is now shown that $\leqq$ is indeed a partial order.

(i) $\leqq$ is reflexive, by part (iv) of the definition.

(ii) $\leqq$ is weakly antisymmetric. Suppose $\underset{\sim}{x} \leqq \underset{\sim}{y}$ and $\underset{\sim}{y} \leqq \underset{\sim}{x}$. Then by (α), $\sigma(\underset{\sim}{x}) \leqq \sigma(\underset{\sim}{y})$ and $\sigma(\underset{\sim}{y}) \leqq \sigma(\underset{\sim}{x})$ and since the relation $\leqq$ on W is known to be an order, $\sigma(\underset{\sim}{x}) = \sigma(\underset{\sim}{y}) = \xi$ say. If $\xi = 1$, then there exist ordinals i , $j < \tau$ such that $\underset{\sim}{x} = \underset{\sim}{g}_i$ and $\underset{\sim}{y} = \underset{\sim}{g}_j$, and then $i = j$ by (β) so $\underset{\sim}{x} = \underset{\sim}{y}$. If $\xi > 1$ it may be supposed inductively that the relation is weakly antisymmetric on the set of all commutators of semiweight $< \xi$. Then, by (γ), $\mathrm{ld}(\underset{\sim}{x}) = \mathrm{ld}(\underset{\sim}{y})$, so by (δ), $\mathrm{tr}(\underset{\sim}{x}) = \mathrm{tr}(\underset{\sim}{y})$ and finally by (ε), $\underset{\sim}{x}_1 = \underset{\sim}{y}_1$ where $\underset{\sim}{x} = [\underset{\sim}{x}_1, \underset{\sim}{x}_2]$ and .

$y = [y_1, y_2]$). Hence $x = y$.

(iii) $\leq$ is transitive. Suppose $x \leq y \leq z$. Then by (α), $\sigma(x) \leq \sigma(y) \leq \sigma(z)$. If $\sigma(x) < \sigma(z)$ then $x < z$ by part (ii) of the definition. Otherwise $\sigma(x) = \sigma(y) = \sigma(z) = \xi$ say. From now on the proof follows the same pattern as that of weak antisymmetry, only now it may be assumed inductively that the relation is a partial order on the set of commutators of semiweight $< \xi$.

(iv) $\leq$ is a (full) well-ordering of C . Suppose $X \subseteq C$, $X \neq \emptyset$. It is shown that X has a least element. Let X_1 be the set of commutators of least semiweight (ξ say) in X . This is nonempty, and if it has a least element so does X . If $\xi = 1$ then X_1 is a nonempty subset of G which is well-ordered by part (i) of the definition, and so X_1 has a least element. Otherwise $\xi > 1$ and it may be assumed inductively that the set of commutators of semiweight $< \xi$ is well-ordered by $\leq$. Then the set X_2 of all commutators of least leading part (ℓ say) in X_1 is defined and non-empty and if it has a least element so does X . Further, the set X_3 of all commutators of least trailing part (t say) in X_2 is defined and non-empty and if it has a least element so does X . But $X_3 \subseteq \{ [\ell, t] , [t, \ell] \}$ and so has a least element since $[\ell, t] \leq [t, \ell]$.

◇

Definition 1.13

(A) Let $\sigma : \underset{\sim}{A} \to W$ be a semiweight. A particular type of commutator, called a "W-basic commutator", is defined recursively over its weight by

(i) Every commutator of weight 1 (that is, every member of $\underset{\sim}{G}$) is a W-basic commutator.

(ii) A commutator $\underset{\sim}{c} = [\underset{\sim}{b},\underset{\sim}{a}]$ of weight > 1 is W-basic if

(a) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are both W-basic commutators,

(b) $\underset{\sim}{a} < \underset{\sim}{b}$ (under the W-ordering of $\underset{\sim}{C}$) ,

and (c) if $\underset{\sim}{b} = [\underset{\sim}{b}_1,\underset{\sim}{b}_2]$ then $\underset{\sim}{b}_2 \leq \underset{\sim}{a}$.

(B) A "W-basic expression" is an expression of the form $\underset{\sim}{1}$ or $\underset{\sim}{b}_1^{\alpha_1}\underset{\sim}{b}_2^{\alpha_2}\cdots\underset{\sim}{b}_k^{\alpha_k}$ where

(i) k is a finite positive integer,

(ii) each $\underset{\sim}{b}_i$ is a W-basic commutator,

(iii) $\underset{\sim}{b}_1 < \underset{\sim}{b}_2 < \ldots < \underset{\sim}{b}_k$ under the W-ordering

and (iv) each α_i is a non-zero integer (positive or negative).

The set of all W-basic expressions is denoted $\underset{\sim}{B}^W$.

For any $\alpha \in W$, the set of W-basic expressions of the form above in which further

$$\text{(v)} \qquad \sigma(\underset{\sim}{b}_i) < \alpha \quad (i = 1,2,\ldots,k)$$

is denoted $\underset{\sim}{B}^W_\alpha$.

For any positive integer c , the set of W-basic expressions of the form above in which further

$$\text{(v)}' \qquad \text{wt}(\underset{\sim}{b}_i) \leqq c \quad (i = 1,2,\ldots,k)$$

is denoted $\underset{\sim}{B}^W_{(c)}$. ◇

Notice that, if $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are two W-basic expressions other than $\underset{\sim}{1}$, $\underset{\sim}{u} \in \underset{\sim}{B}^W_\alpha$ and $\sigma(\underset{\sim}{v}) \geqq \alpha$ then $\underset{\sim}{uv}$ is also a W-basic expression. This fact will be used without further comment.

THE NUMBER OF W-BASIC COMMUTATORS

This section is devoted to finding an expression for the number of W-basic commutators of a given weight when the number τ of generators is finite. The argument given here is a modified version of Witt's original one [10]; his argument does not carry over exactly, since it requires the order type of the set of basic commutators to be ω .

This section is included not only for the amusement value of proving Witt's formula: it is essential to the proof given in Chapter 2 for the basis theorem which in turn is essential to the rest of the thesis.

Definition 1.14

(A) An order $\leqq$ defined on the set $\underset{\sim}{C}$ of commutators is a "B-order" if

(i) $\leqq$ well-orders $\underset{\sim}{C}$,

(ii) $\underset{\sim}{a} < [\underset{\sim}{a},\underset{\sim}{b}]$ and $\underset{\sim}{b} < [\underset{\sim}{a},\underset{\sim}{b}]$ and

(iii) $\underset{\sim}{g}_i < [\underset{\sim}{a},\underset{\sim}{b}]$ and $i < j \Rightarrow \underset{\sim}{g}_i < \underset{\sim}{g}_j$.

(B) If $\leqq$ is any B-order, then a "($\leqq$)-basic commutator" is defined recursively over its weight:

(i) Every $\underset{\sim}{g}_i \in \underset{\sim}{G}$ is a ($\leqq$)-basic commutator,

(ii) $[\underset{\sim}{b},\underset{\sim}{a}]$ is a ($\leqq$)-basic commutator if

(a) Both $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are ($\leqq$)-basic,

(b) $\underset{\sim}{a} < \underset{\sim}{b}$

and (c) if $\underset{\sim}{b} = [\underset{\sim}{b}_1,\underset{\sim}{b}_2]$ then $\underset{\sim}{b}_2 \leqq \underset{\sim}{a}$.

(C) A B-order $\leqq$ is "good" if the order type of $\underset{\sim}{C}$ under $\leqq$ is ω (that is, if there exists an order isomorphism of the positive integers onto $\underset{\sim}{C}$).

(D) Let $\leqq_1$ and $\leqq_2$ be two B-orders. Then we write

$$\leqq_1 \sim \leqq_2 \quad (\text{nil-}c)$$

if, for any two commutators $\underset{\sim}{x}$, $\underset{\sim}{y}$ of weight $\leqq c$,

$$\underset{\sim}{x} \leqq_1 \underset{\sim}{y} \iff \underset{\sim}{x} \leqq_2 \underset{\sim}{y}$$ ◇

Two immediate consequences of this definition are

<u>Lemma 1.8</u>

(i) If $\sigma : \underset{\sim}{A} \to W$ is a semiweight and $\leqq$ is the W-ordering of $\underset{\sim}{C}$, then $\leqq$ is a B-order and a commutator is $(\leqq)$-basic if and only if it is W-basic.

(ii) If $\leqq_1$ and $\leqq_2$ are two B-orders and $\leqq_1 \sim \leqq_2$ (nil-c) , then the set of $(\leqq_1)$-basic commutators of weight $\leqq c$ is the same as the set of $(\leqq_2)$-basic commutators of weight $\leqq c$, and they are ordered in the same way. ◇

<u>Lemma 1.9</u>

Let the number τ of generators of $\underset{\sim}{A}$ be finite and let c be any positive integer. Then the number of commutators of weight $\leqq c$ is finite.

Proof

By an easy induction over c . ◇

Corollary

Suppose τ is finite and c is a positive integer. If $\leqq$ is any B-order, then there exists a good B-order $\leqq^*$ such that $\leqq \sim \leqq^*$ (nil-c) .

Proof

$\leqq^*$ may be defined to be identical with $\leqq$ on the set of commutators of weight $\leqq c$, and then extended to $\underset{\sim}{C}$ in any way which preserves weight. ◇

Definition 1.15

Suppose we are dealing with a fixed B-order $\leqq$.

(i) A commutator $\underset{\sim}{c}$ is "$\underset{\sim}{b}$-compatible", where $\underset{\sim}{b}$ is any ($\leqq$)-basic commutator, if $[\underset{\sim}{c},\underset{\sim}{b}]$ is ($\leqq$)-basic.

(ii) For each basic commutator $\underset{\sim}{b}$, write $\underset{\sim}{b}^+$ for the successor of $\underset{\sim}{b}$ under the restriction of $\leqq$ to the set of ($\leqq$)-basic commutators. ◇

Definition 1.16

For any expressions $\underset{\sim}{a}$, $\underset{\sim}{b} \in \underset{\sim}{A}$ and non-negative integer α , define $[\, \underset{\sim}{a} , \alpha \times \underset{\sim}{b} \,]$ recursively over α :

(i) $[\, \underset{\sim}{a} , 0 \times \underset{\sim}{b} \,] = \underset{\sim}{a}$,

(ii) $[\, \underset{\sim}{a} , \alpha \times \underset{\sim}{b} \,] = [[\, \underset{\sim}{a} , (\alpha-1) \times \underset{\sim}{b} \,] , \underset{\sim}{b} \,] \qquad (\alpha \geqq 1)$.

◇

This notation is just shorthand for Engel commutators of various lengths. It follows immediately that

Lemma 1.10

(i) If $\underset{\sim}{a}$, $\underset{\sim}{b} \in \underset{\sim}{C}$ then $wt([\underset{\sim}{a}, \alpha \times \underset{\sim}{b}]) = wt(\underset{\sim}{a}) + \alpha wt(\underset{\sim}{b})$.

(ii) If $\underset{\sim}{b}$ is $(\leqq)$-basic and $\underset{\sim}{c}$ is $\underset{\sim}{b}$-compatible, then $[\underset{\sim}{c}, \alpha \times \underset{\sim}{b}]$ is $(\leqq)$basic and $\underset{\sim}{b}$-compatible for each $\alpha \geqq 0$ and $\underset{\sim}{b}^{+}$-compatible for each $\alpha \geqq 1$.

◇

Lemma 1.11

Suppose $\leqq$ is a good B-order and τ is finite. Then we may may index the collection of $(\leqq)$-basic commutators by the positive integers $\{\underset{\sim}{b}_i\}_{i=1}^{\infty}$ so that $i \leqq j \iff b_i \leqq b_j$ and then, for each positive integer w , there exists an integer n_w such that

$i \geqq n_w \implies wt(\underset{\sim}{b}_i) \geqq w$. ◇

Definition 1.17

With the conditions of lemma 1.11, define a sequence $\{\underset{\sim}{X}_i\}_{i=0}^{\infty}$ of subsets of $\underset{\sim}{A}$ as follows:

(i) $\underset{\sim}{X}_0 = \underset{\sim}{G}$, and

(ii) $\underset{\sim}{X}_i$ is the set of all $\underset{\sim}{b}_i$-compatible commutators $(i \geqq 1)$. ◇

It should be remarked here that, as the indices have been defined, $\underset{\sim}{g}_0 = \underset{\sim}{b}_1$, $\underset{\sim}{g}_1 = \underset{\sim}{b}_2$ and so on.

Lemma 1.12

For each integer $r \geqq 1$,

(i) $\underset{\sim}{X}_{r-1} \setminus \underset{\sim}{X}_r = \{\underset{\sim}{b}_r\}$ and

(ii) $\underset{\sim}{X}_r \setminus \underset{\sim}{X}_{r-1} = \{ [\underset{\sim}{c}, \alpha \times \underset{\sim}{b}_r] : \underset{\sim}{c} \in \underset{\sim}{X}_r \cap \underset{\sim}{X}_{r-1}$ and $\alpha \geqq 1 \}$.

Proof

The argument is slightly different for the cases $r = 1$ and $r > 1$.

r = 1

(i) $\underset{\sim}{X}_0 \setminus \underset{\sim}{X}_1 = \{\underset{\sim}{b}_1\}$. Here $\underset{\sim}{X}_0 = \underset{\sim}{G}$, $\underset{\sim}{b}_1 = \underset{\sim}{g}_0$ and $\underset{\sim}{X}_1$ is the set of all $\underset{\sim}{g}_0$-compatible commutators. Clearly $\underset{\sim}{g}_0 \in \underset{\sim}{G}$ and is not g_0-compatible. Hence $\{\underset{\sim}{b}_1\} \subseteq \underset{\sim}{X}_0 \setminus \underset{\sim}{X}_1$. Now suppose $\underset{\sim}{c} \in \underset{\sim}{X}_0 \setminus \underset{\sim}{X}_1$. Then $\underset{\sim}{c} \in \underset{\sim}{X}_0 = \underset{\sim}{G}$ so there exists $i < \tau$ such that $\underset{\sim}{c} = \underset{\sim}{g}_i$. If $i \geqq 1$, then $[\underset{\sim}{c},\underset{\sim}{g}_0] = [\underset{\sim}{g}_i,\underset{\sim}{g}_0]$ is $(\leq)$-basic and so $\underset{\sim}{g}_i \in \underset{\sim}{X}_1$, contra hyp. Hence $\underset{\sim}{c} = \underset{\sim}{g}_0$.

(ii) $\underset{\sim}{X}_1 \setminus \underset{\sim}{X}_0 = \{ [\underset{\sim}{c},\alpha\times\underset{\sim}{b}_1] : \underset{\sim}{c} \in \underset{\sim}{X}_1 \cap \underset{\sim}{X}_0 \text{ and } \alpha \geqq 1 \}$.

It is shown first that c is g_0-compatible if and only if it is of the form $[\underset{\sim}{g}_i,\alpha\times\underset{\sim}{g}_0]$ for some $i \geqq 1$ and $\alpha \geqq 0$. A commutator of this form is $\underset{\sim}{g}_0$-compatible by lemma 1.10 . Now suppose $\underset{\sim}{c}$ is $\underset{\sim}{g}_0$-compatible. The argument proceeds by induction over the weight of $\underset{\sim}{c}$. If $wt(\underset{\sim}{c}) = 1$, then $\underset{\sim}{c} = \underset{\sim}{g}_i$ for some $i < \tau$ and is then $\underset{\sim}{g}_0$-compatible only if $i \geqq 1$. Thus $\underset{\sim}{c} = [\underset{\sim}{g}_i,0\times\underset{\sim}{g}_0]$. Now suppose $\underset{\sim}{c} = [\underset{\sim}{c}_1,\underset{\sim}{c}_2]$. Then $\underset{\sim}{c}_2 \leq \underset{\sim}{g}_0$ so $\underset{\sim}{c}_2 = \underset{\sim}{g}_0$ and then $\underset{\sim}{c}_1$ is $\underset{\sim}{g}_0$-compatible and so, by the inductive hypothesis, $\underset{\sim}{c}_1 = [\underset{\sim}{g}_i,\alpha\times\underset{\sim}{g}_0]$ and then $\underset{\sim}{c} = [\underset{\sim}{g}_i,(\alpha+1)\times\underset{\sim}{g}_0]$. It now follows that $\underset{\sim}{X}_1 \setminus \underset{\sim}{X}_0 = \{ [\underset{\sim}{g}_i,\alpha\times\underset{\sim}{g}_0] : i \geqq 1 , \alpha \geqq 1 \}$, and that $\underset{\sim}{X}_1 \cap \underset{\sim}{X}_0 = \{ \underset{\sim}{g}_i : i \geqq 1 \}$. The result follows immediately.

<u>$r > 1$</u>

(i) $\underset{\sim}{b}_r > \underset{\sim}{b}_{r-1}$. Further, if $\underset{\sim}{b}_r = [\underset{\sim}{c}_1, \underset{\sim}{c}_2]$, then $\underset{\sim}{c}_2 < \underset{\sim}{b}_r$ so that $\underset{\sim}{c}_2 \leqq \underset{\sim}{b}_{r-1}$. Hence $[\underset{\sim}{b}_r, \underset{\sim}{b}_{r-1}]$ is $(\leqq)$-basic, so $\underset{\sim}{b}_r \in \underset{\sim}{X}_{r-1}$. But $\underset{\sim}{b}_r$ is not $\underset{\sim}{b}_r$-compatible, so $\underset{\sim}{b}_r \in \underset{\sim}{X}_{r-1} \setminus \underset{\sim}{X}_r$. Conversely, suppose $\underset{\sim}{c} \in \underset{\sim}{X}_{r-1} \setminus \underset{\sim}{X}_r$. Then $[\underset{\sim}{c}, \underset{\sim}{b}_{r-1}]$ is $(\leqq)$-basic, but $[\underset{\sim}{c}, \underset{\sim}{b}_r]$ is not. Then one of the conditions of definition 1.14(B) must fail for $[\underset{\sim}{c}, \underset{\sim}{b}_r]$. But $\underset{\sim}{c}$ is $(\leqq)$-basic since $[\underset{\sim}{c}, \underset{\sim}{b}_{r-1}]$ is, and if $\underset{\sim}{c} = [\underset{\sim}{c}_1, \underset{\sim}{c}_2]$ then $\underset{\sim}{c}_2 \leqq \underset{\sim}{b}_{r-1} < \underset{\sim}{b}_r$ for the same reason. Thus the only condition that can fail is (b), that is, $\underset{\sim}{c} \leqq \underset{\sim}{b}_r$. But again, since $[\underset{\sim}{c}, \underset{\sim}{b}_{r-1}]$ is $(\leqq)$-basic, $\underset{\sim}{c} > \underset{\sim}{b}_{r-1}$ so $\underset{\sim}{c} = \underset{\sim}{b}_r$. Thus $\underset{\sim}{X}_{r-1} \setminus \underset{\sim}{X}_r = \{\underset{\sim}{b}_r\}$.

(ii) If $\underset{\sim}{c} \in \underset{\sim}{X}_{r-1} \cap \underset{\sim}{X}_r$ and $\alpha \geqq 1$, then $[\underset{\sim}{c}, \alpha \times \underset{\sim}{b}_r] \in \underset{\sim}{X}_r$ by lemma 1.10(ii). But, since $\alpha \geqq 1$, $[\underset{\sim}{c}, \alpha \times \underset{\sim}{b}_r] = [[\underset{\sim}{c}, (\alpha-1) \times \underset{\sim}{b}_r], \underset{\sim}{b}_r]$ which is not $\underset{\sim}{b}_{r-1}$-compatible. Thus $[\underset{\sim}{c}, \alpha \times \underset{\sim}{b}_r] \in \underset{\sim}{X}_r \setminus \underset{\sim}{X}_{r-1}$. Conversely, suppose $\underset{\sim}{c} \in \underset{\sim}{X}_r \setminus \underset{\sim}{X}_{r-1}$. The argument proceeds by induction over the weight of $\underset{\sim}{c}$. If $wt(\underset{\sim}{c}) = 1$, then $\underset{\sim}{c} = \underset{\sim}{g}_i$ for some $i < \tau$. But $\underset{\sim}{c} \in \underset{\sim}{X}_r$ so $[\underset{\sim}{g}_i, \underset{\sim}{b}_r]$ is $(\leqq)$-basic, which means that $\underset{\sim}{b}_r = \underset{\sim}{g}_{r-1}$ and $r-1 < i$. But then $[\underset{\sim}{g}_i, \underset{\sim}{b}_{r-1}] = [\underset{\sim}{g}_i, \underset{\sim}{g}_{r-2}]$ is also $(\leqq)$-basic, so $\underset{\sim}{g}_i \in \underset{\sim}{X}_{r-1}$ contra hyp., and so $\underset{\sim}{c}$ cannot be of weight 1 . Now write $\underset{\sim}{c} = [\underset{\sim}{c}_1, \underset{\sim}{c}_2]$. Then $[\underset{\sim}{c}, \underset{\sim}{b}_r]$ is $(\leqq)$-basic but $[\underset{\sim}{c}, \underset{\sim}{b}_{r-1}]$ is not, so one of the conditions of definition 1.14(B) must fail for the latter commutator. Now

c is $(\leqq)$-basic since $[c,b_r]$ is, and for the same reason $c > b_r > b_{r-1}$. Thus the only condition that can fail is (c), that is, $c_2 > b_{r-1}$. But again, since $[c,b_r]$ is $(\leqq)$-basic, $c_2 \leqq b_r$. Thus $c_2 = b_r$ and $c = [c_1,b_r]$. Thus c_1 is b_r-compatible. If it is also b_{r-1}-compatible, the result is true with $\alpha = 1$. If c_1 is not b_{r-1}-compatible, then $c_1 \in X_r \setminus X_{r-1}$ and by the inductive hypothesis $c_1 = [c', \alpha \times b_r]$ with $c' \in X_{r-1} \cap X_r$. Then $c = [c', (\alpha+1) \times b_r]$. ◇

It will be convenient from now on to be able to write a product xy where either x or y may be "empty", in the sense that possibly $xy = x$ or $xy = y$. This slight abuse of terminology will save much circumlocution. The same terminology may be applied to products of more than two expressions; on the other hand, the idea of a product in which all factors are empty is clearly meaningless.

<u>Definition 1.18</u>

With the conditions of lemma 1.11 and the notation of that lemma and definition 1.17,

(A) For any integer $r \geqq 0$, write P_r for the set of expressions of the form

$$\mathbf{p} = \mathbf{b}_1^{\alpha_1}\mathbf{b}_2^{\alpha_2}\cdots\mathbf{b}_r^{\alpha_r}\,\mathbf{c}_1\mathbf{c}_2\cdots\mathbf{c}_k$$

where each α_i is a non-negative integer, each $\mathbf{c}_i \in \mathbf{X}_r$ and $k \geqq 0$. The possibilities $r \geqq 0$ and $k \geqq 0$ correspond to the possibilities that the products $\mathbf{b}_1^{\alpha_1}\mathbf{b}_2^{\alpha_2}\cdots\mathbf{b}_r^{\alpha_r}$ and $\mathbf{c}_1\mathbf{c}_2\cdots\mathbf{c}_k$ may be empty. The symbol $S(\mathbf{p})$ is defined

$$S(\mathbf{p}) = \alpha_1 \mathrm{wt}(\mathbf{b}_1) + \alpha_2 \mathrm{wt}(\mathbf{b}_2) + \ldots + \alpha_r \mathrm{wt}(\mathbf{b}_r) + \\ + \mathrm{wt}(\mathbf{c}_1) + \mathrm{wt}(\mathbf{c}_2) + \ldots + \mathrm{wt}(\mathbf{c}_k) \;.$$

For each positive integer w, $\mathbf{P}_r(w)$ is the set

$$\mathbf{P}_r(w) = \{\, \mathbf{p} : \mathbf{p} \in \mathbf{P}_r \,,\; S(\mathbf{p}) = w \,\} \;.$$

(B) A mapping $\theta_r : \mathbf{P}_{r-1} \to \mathbf{P}_r$ is defined for each $r \geqq 1$. Suppose $\mathbf{p} \in \mathbf{P}_{r-1}$. Then it is of the form $\mathbf{p} = \mathbf{b}_1^{\alpha_1}\mathbf{b}_2^{\alpha_2}\cdots\mathbf{b}_{r-1}^{\alpha_{r-1}}\,\mathbf{u}$, where $\mathbf{u}$ is a (possibly empty) product of commutators from $\mathbf{X}_{r-1}$. Now $\mathbf{u}$ may or may not contain commutators of the form $\mathbf{b}_r$ as factors. In any case it may be written in the form

$$\mathbf{u} = \mathbf{b}_r^{\beta_0}\mathbf{a}_1\mathbf{b}_r^{\beta_1}\mathbf{a}_2\mathbf{b}_r^{\beta_2}\cdots\mathbf{a}_m\mathbf{b}_r^{\beta_m} \;,$$

where $m \geqq 0$, each $\beta_i \geqq 0$ and, by lemma 1.12(i), each $\mathbf{a}_i \in \mathbf{X}_{r-1} \cap \mathbf{X}_r$. Then $\mathbf{p}\theta_r$ is defined

$$\mathbf{p}\theta_r = \mathbf{b}_1^{\alpha_1}\mathbf{b}_2^{\alpha_2}\cdots\mathbf{b}_{r-1}^{\alpha_{r-1}}\mathbf{b}_r^{\beta_0} \,.\, [\mathbf{a}_1, \beta_1 \times \mathbf{b}_r][\mathbf{a}_2, \beta_2 \times \mathbf{b}_r]\cdots[\mathbf{a}_m, \beta_m \times \mathbf{b}_r] \;.$$

For $1 \leqq i \leqq m$, $[\mathbf{a}_i, \beta_i \times \mathbf{b}_r] \in \mathbf{X}_r$ by lemma 1.12(ii), and so $\mathbf{p}\theta_r \in \mathbf{X}_r$ as promised. ◇

Lemma 1.13

(i) θ_r is a 1:1 mapping of $\underset{\sim}{P}_{r-1}$ onto $\underset{\sim}{P}_r$.

(ii) If $\underset{\sim}{p} \in \underset{\sim}{P}_{r-1}$ then $S(\underset{\sim}{p}\theta_r) = S(\underset{\sim}{p})$.

Proof

It is sufficient to exhibit a function $\theta' : \underset{\sim}{P}_r \to \underset{\sim}{P}_{r-1}$ such that $\theta'\theta_r$ is the identity on $\underset{\sim}{P}_r$ and $\theta_r\theta'$ is the identity on $\underset{\sim}{P}_{r-1}$. Suppose then that $\underset{\sim}{p} \in \underset{\sim}{P}_r$. Then it is of the form $\underset{\sim}{p} = \underset{\sim}{b}_1^{\alpha_1}\underset{\sim}{b}_2^{\alpha_2}\cdots\underset{\sim}{b}_r^{\alpha_r}\,\underset{\sim}{v}$, where $\underset{\sim}{v}$ is a (possibly empty) product of commutators from $\underset{\sim}{X}_r$. By lemma 1.12(ii), $\underset{\sim}{v}$ may be written in the form $\underset{\sim}{v} = [\underset{\sim}{a}_1,\beta_1\times\underset{\sim}{b}_r][\underset{\sim}{a}_2,\beta_2\times\underset{\sim}{b}_r]\cdots[\underset{\sim}{a}_m,\beta_m\times\underset{\sim}{b}_r]$, where each $\beta_i \geq 0$ and each $\underset{\sim}{a}_i \in \underset{\sim}{X}_{r-1} \cap \underset{\sim}{X}_r$. Then defining

$$\underset{\sim}{p}\theta' = \underset{\sim}{b}_1^{\alpha_1}\underset{\sim}{b}_2^{\alpha_2}\cdots\underset{\sim}{b}_{r-1}^{\alpha_{r-1}} \cdot \underset{\sim}{b}_r^{\alpha_r}\underset{\sim}{a}_1\underset{\sim}{b}_r^{\beta_1}\underset{\sim}{a}_2\underset{\sim}{b}_r^{\beta_2}\cdots\underset{\sim}{a}_m\underset{\sim}{b}_r^{\beta_m} ,$$

it follows that $\underset{\sim}{p}\theta' \in \underset{\sim}{P}_{r-1}$, and comparing this definition with that of θ_r , $\underset{\sim}{p}\theta'\theta_r = \underset{\sim}{p}$. Similarly, if $\underset{\sim}{p}' \in \underset{\sim}{P}_{r-1}$, $\underset{\sim}{p}'\theta_r\theta' = \underset{\sim}{p}'$.

(ii) This follows by an easy calculation using the definition of $S(\underset{\sim}{p})$ and lemma 1.10(i). ◇

Corollary

With the notation of definition 1.18, for $r \geqq 0$ and $w \geqq 1$,

$| \underset{\sim}{P}_r(w) | = \tau^w$.

Proof

By the lemma, the restriction of θ_r to $\underset{\sim}{P}_{r-1}(w)$ is a 1:1 mapping of $\underset{\sim}{P}_{r-1}(w)$ onto $\underset{\sim}{P}_r(w)$. Thus $| \underset{\sim}{P}_r(w) | = | \underset{\sim}{P}_o(w) |$. But $\underset{\sim}{P}_o(w)$ is the set of all products $\underset{\sim}{u}$ of elements of $\underset{\sim}{X}_o$ for which $S(\underset{\sim}{u}) = w$. But this is just the set of all expressions of the form $g_{i_1} g_{i_2} \cdots g_{i_w}$ where $i_1, i_2, \ldots, i_w < \tau$. Hence $| \underset{\sim}{P}_o(w) | = \tau^w$. ◇

Lemma 1.14

Suppose the conditions of lemma 1.11 hold. Then the number m_w of $(\leqq)$-basic commutators of weight w is given recursively by

(i) $m_1 = \tau$,

(ii) $m_w = \tau^w - \langle m_1, m_2, \ldots, m_{w-1} \rangle \qquad (w > 1)$

where the integer $\langle m_1, m_2, \ldots, m_k \rangle$ is the number of mappings f from the set $K_k = \{ (i,j) : 1 \leqq i \leqq k , 1 \leqq j \leqq m_i \}$ into the non-negative integers satisfying $\sum_{(i,j) \in K_k} i f(i,j) = k + 1$.

Proof

By induction over w. The result $m_1 = \tau$ is already known. Now suppose the result is true for all weights up to $w - 1$.

By lemma 1.11, there exists an integer N ($= n_{w+1}$) such that $i \geqq N \Rightarrow wt(\underset{\sim}{b}_i) > w$. Consider $\underset{\sim}{P}_N(w)$. An element

$\underset{\sim}{p} \in \underset{\sim}{P}_N(w)$ is of the form $\underset{\sim}{p} = \underset{\sim}{b}_1^{\alpha_1} \underset{\sim}{b}_2^{\alpha_2} \ldots \underset{\sim}{b}_N^{\alpha_N} \underset{\sim}{u}$ where $\underset{\sim}{u}$ is a (possibly empty) product of elements of $\underset{\sim}{X}_N$. But $\underset{\sim}{u}$ is in fact empty, for otherwise $\underset{\sim}{u} = \underset{\sim}{u}'\underset{\sim}{b}_h$ where $\underset{\sim}{b}_h \in \underset{\sim}{X}_N$ and then, since $[\underset{\sim}{b}_h, \underset{\sim}{b}_N]$ is $(\leqq)$-basic, $h \geqq N$ so that $S(\underset{\sim}{p}) \geqq S(\underset{\sim}{b}_h) > w$.

Thus each element of $\underset{\sim}{P}_N(w)$ is of the form $\underset{\sim}{p} = \underset{\sim}{b}_1^{\alpha_1} \underset{\sim}{b}_2^{\alpha_2} \ldots \underset{\sim}{b}_N^{\alpha_N}$.

Now all the $(\leqq)$-basic commutators of weight $\leqq w$ appear in the $\{b_1, b_2, \ldots, b_N\}$ together possibly with some of higher weight (for, although the B-order $\leqq$ is good, it does not necessarily preserve weight). However, since $S(\underset{\sim}{p}) = w$, any commutator $\underset{\sim}{b}_i$ ($1 \leqq i \leqq N$) of weight $> w$ must have power $\alpha_i = 0$ in the given expression for $\underset{\sim}{p}$. Thus $\underset{\sim}{p}$ is defined uniquely by the powers α_i of commutators of weight $\leqq w$ in that expression.

Suppose that the $(\leqq)$-basic commutators of weight $\leqq w$ are re-indexed as follows:

$$\begin{array}{cccc} \underset{\sim}{b}_{(1,1)}, & \underset{\sim}{b}_{(1,2)}, & \cdots, & \underset{\sim}{b}_{(1,m_1)} \\ \underset{\sim}{b}_{(2,1)}, & \underset{\sim}{b}_{(2,2)}, & \cdots, & \underset{\sim}{b}_{(2,m_2)} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{b}_{(w,1)}, & \underset{\sim}{b}_{(w,2)}, & \cdots, & \underset{\sim}{b}_{(w,m_w)} \end{array}$$

where for each c ($1 \leqq c \leqq w$), $\underset{\sim}{b}_{(c,1)}, \underset{\sim}{b}_{(c,2)}, \cdots, \underset{\sim}{b}_{(c,m_c)}$ is the set of commutators of weight c (in any order) . Thus set of ($\leqq$)-basic commutators of weight $\leqq w$ has been indexed by the set K_w defined in the statement of the lemma. The order in which the commutators are written down in this array is not, of course, their order under the B-order $\leqq$.

Let us write $\underset{\sim}{Q}$ for the set of all $\underset{\sim}{p} \in \underset{\sim}{P}_N(w)$ in which some commutator of weight w has non-zero index. Then, since $S(\underset{\sim}{p}) = w$, $\underset{\sim}{Q}$ is exactly the set of all ($\leqq$)-basic commutators of weight w and so $|\underset{\sim}{Q}| = m_w$.

But now, if to each $\underset{\sim}{p} \in \underset{\sim}{P}_N(w) \setminus \underset{\sim}{Q}$ a function $f_{\underset{\sim}{p}}$ from the set K_{w-1} into the non-negative integers is defined by setting $f_{\underset{\sim}{p}}(i,j)$ to be the power of $\underset{\sim}{b}_{(i,j)}$ appearing in the expression for $\underset{\sim}{p}$, it follows that $f_{\underset{\sim}{p}}$ is uniquely determined by $\underset{\sim}{p}$

and that $\sum_{(i,j)\in K_{w-1}} if(i,j) = S(\underset{\sim}{p}) = w$, and conversely that any such function uniquely defines an element of $\underset{\sim}{P}_N(w) \setminus \underset{\sim}{Q}$. Thus $| \underset{\sim}{P}_N(w) \setminus \underset{\sim}{Q} | = \langle m_1, m_2, \ldots, m_{w-1} \rangle$ and this, together with the fact just proved that $|\underset{\sim}{Q}| = m_w$ and the corollary to lemma 1.13 proves this lemma. $\diamond$

Theorem 1.2

Let $\sigma : \underset{\sim}{A} \to W$ be any semiweight and suppose that the number τ of generators of $\underset{\sim}{A}$ is finite. Then the number m_w of W-basic commutators of weight w is given by Witt's formula:

$$m_w \equiv \frac{1}{w} \sum_{r|w} \mu(r)\ \tau^{w/r} ,$$

where μ is the Möbius function.

Proof

Let $\leqq$ be the W-ordering of $\underset{\sim}{C}$. Then by the corollary to lemma 1.9 there exists a good B-order $\leqq^*$ such that $\leqq \sim \leqq^*$ (nil-w) . By lemma 1.8(ii) the number of W-basic commutators of weight w is the same as the number of $(\leqq^*)$-basic commutators of weight w , and so by lemma 1.14 this is given by the recursive formula $m_1 = \tau$,
$m_c = \tau^c - \langle m_1, m_2, \ldots, m_{c-1} \rangle \quad (1 < c \leqq w)$. But this formula

depends only on τ and w, not on the particular semiweight range chosen. Thus it applies equally well to the semiweight range $\bar{N}$, and so the number of W-basic commutators of weight w is the same as the number of $\bar{N}$-basic commutators of weight w; but these are exactly the commutators which are basic in the conventional sense, and so m_w is given by Witt's formula.

◇

CHAPTER 2

THE COLLECTING PROCESSES

The collecting processes to be described in this chapter differ in two important respects from the process used in the conventional theory. Firstly, the processes described here are defined in terms of a particular semiweight range W, the object being to convert an arbitrary expression into a W-basic one; in this sense these processes are more general than the conventional one. Secondly, while the conventional process involves an initial expansion of the expression to be collected into a product of generators and their inverses – that is, into an expression of semiweight 1 – followed by a collection into products of basic commutators of successively higher weights, the processes described here involve no such initial expansion: they proceed through a sequence of expressions of non-decreasing semiweight, and so certain properties of commutators which may be expressed in terms of their semiweight are preserved.

It is well-known that calculations performed in the "bottom" of a nilpotent group, that is in $\gamma_c(G)$ when G is nilpotent of class c, usually have a particularly simple form. Accordingly it will be advantageous to describe first a collecting process which operates in the "bottom" of a group, however here the "bottom" may mean $W_\alpha(G)$ when $G \in W_{\alpha+1}$. This is the "special" process. Following this a "general" process will be described which can operate either anywhere in $W_\alpha(G)$ when $G \in \underline{\underline{W}}_\beta$ under certain restrictions on α and β , or else anywhere in a nilpotent group.

For the remainder of this chapter it will be assumed that we are working with a fixed algebra $\mathbf{A}$ of expressions and a fixed semiweight $\sigma : \mathbf{A} \to W$ in terms of which all definitions are made.

THE SPECIAL PROCESS

Definition 2.1

(A) Let $\mathbf{x}, \mathbf{y} \in \mathbf{A}$. Then we write $d : \mathbf{x} \to \mathbf{y}$ if $\mathbf{x}$ and $\mathbf{y}$ are of any of the following forms:

(i) $\mathbf{x} = \mathbf{ab}$, $\mathbf{y} = \mathbf{ba}$

(ii) $\mathbf{x} = (\mathbf{ab})^{-1}$, $\mathbf{y} = \mathbf{b}^{-1}\mathbf{a}^{-1}$

(iii) $\mathbf{x} = (\mathbf{a}^{-1})^{-1}$, $\mathbf{y} = \mathbf{a}$

(iv) $\mathbf{x} = \mathbf{a}^{-1}\mathbf{a}$ or $\mathbf{aa}^{-1}$, $\mathbf{y} = \mathbf{1}$

(v) $\mathbf{x} = \mathbf{a1}$ or $\mathbf{1a}$, $\mathbf{y} = \mathbf{a}$

(vi) $\mathbf{x} = \mathbf{1}^{-1}$, $\mathbf{y} = \mathbf{1}$

(vii) $\mathbf{x} = [\mathbf{a},\mathbf{a}]$, $\mathbf{y} = \mathbf{1}$

(viii) $\mathbf{x} = [\mathbf{a},\mathbf{1}]$ or $[\mathbf{1},\mathbf{a}]$, $\mathbf{y} = \mathbf{1}$

(ix) $\mathbf{x} = [\mathbf{a}^{-1},\mathbf{b}]$ or $[\mathbf{a},\mathbf{b}^{-1}]$, $\mathbf{y} = [\mathbf{a},\mathbf{b}]^{-1}$

(x) $\quad \mathbf{x} = [\mathbf{ab},\mathbf{c}]$, $\mathbf{y} = [\mathbf{a},\mathbf{c}][\mathbf{b},\mathbf{c}]$ or

$\qquad \mathbf{x} = [\mathbf{a},\mathbf{bc}]$, $\mathbf{y} = [\mathbf{a},\mathbf{c}][\mathbf{a},\mathbf{b}]$

(xi) $\quad \mathbf{x} = [\mathbf{a},\mathbf{b}]$, $\mathbf{y} = [\mathbf{b},\mathbf{a}]^{-1}$ provided $\mathbf{a}$, $\mathbf{b}$ are commutators and $\mathbf{a} < \mathbf{b}$

(xii) $\quad \mathbf{x} = [\mathbf{c},\mathbf{b},\mathbf{a}]$, $\mathbf{y} = [\mathbf{b},\mathbf{a},\mathbf{c}]^{-1}[\mathbf{c},\mathbf{a},\mathbf{b}]$ provided $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are commutators and $\mathbf{a} < \mathbf{b} < \mathbf{c}$.

The notation is extended to larger expressions by recursion over their height:

(xiii) If $d : \mathbf{a}_1 \to \mathbf{a}_2$ then $d : \mathbf{a}_1^{-1} \to \mathbf{a}_2^{-1}$ and for any $\mathbf{b} \in \mathbf{A}$,

$$d : \mathbf{a}_1\mathbf{b} \to \mathbf{a}_2\mathbf{b} , \qquad d : \mathbf{b}\mathbf{a}_1 \to \mathbf{b}\mathbf{a}_2 ,$$

$$d : [\mathbf{a}_1,\mathbf{b}] \to [\mathbf{a}_2,\mathbf{b}] , \qquad d : [\mathbf{b},\mathbf{a}_1] \to [\mathbf{b},\mathbf{a}_2] .$$

(B) Write $D : \mathbf{x} \to \mathbf{y}$ if there exists a finite sequence $(\mathbf{u}_i)_{i=0}^{k}$ $(k \geq 0)$ of expressions such that $\mathbf{u}_0 = \mathbf{x}$, $\mathbf{u}_k = \mathbf{y}$ and $d : \mathbf{u}_{i-1} \to \mathbf{u}_i$ $(1 \leq i \leq k)$. ◇

The relation $D : \mathbf{x} \to \mathbf{y}$ is clearly reflexive and transitive, that is, it is a quasi-order. It also follows from the definition that part (xiii) holds just as well for D as for d .

Definition 2.2

Suppose $\mathcal{P}$ is any property that commutators may or may not have (such as being W-basic or of weight $\leq c$ for instance). A "product of commutators with the property $\mathcal{P}$" of "length" ℓ is defined:

(i) $\underset{\sim}{1}$ is a product of commutators with the property $\mathcal{P}$ of length 1, as are $\underset{\sim}{c}$ and $\underset{\sim}{c}^{-1}$, where $\underset{\sim}{c}$ is a commutator with the property $\mathcal{P}$.

(ii) If $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ are products of commutators with the property $\mathcal{P}$ of lengths ℓ_1 and ℓ_2 respectively, then $\underset{\sim}{x}_1\underset{\sim}{x}_2$ is a product of commutators with the property $\mathcal{P}$ of length $\ell_1 + \ell_2$.

A "product of commutators" of length ℓ is defined in the same way. ◇

Definition 2.3

A relation $\leq^{\circ}$ is defined on the set $\underset{\sim}{C}$ of commutators by:

$\underset{\sim}{a} \leq^{\circ} \underset{\sim}{b}$ if and only if

(i) $\sigma(\underset{\sim}{a}) > \sigma(\underset{\sim}{b})$, or

(ii) $\sigma(\underset{\sim}{a}) = \sigma(\underset{\sim}{b})$ and $\underset{\sim}{a} \leq \underset{\sim}{b}$. ◇

Clearly $\leq^{\circ}$ is a full order but not a well-order. The relation $<^{\circ}$ is defined in the obvious way.

Lemma 2.1

If $\underset{\sim}{a}_1$, $\underset{\sim}{a}_2$ and $\underset{\sim}{b}$ are commutators and $\underset{\sim}{a}_1 < \underset{\sim}{a}_2$, then

$$[\underset{\sim}{a}_1,\underset{\sim}{b}] \text{ and } [\underset{\sim}{b},\underset{\sim}{a}_1] \quad < \quad [\underset{\sim}{a}_2,\underset{\sim}{b}] \text{ and } [\underset{\sim}{b},\underset{\sim}{a}_2] .$$

Proof

is only given that $[\underset{\sim}{a}_1,\underset{\sim}{b}] < [\underset{\sim}{a}_2,\underset{\sim}{b}]$. Proofs of the other three inequalities are similar. Since $\underset{\sim}{a}_1 < \underset{\sim}{a}_2$, $\sigma(\underset{\sim}{a}_1) \leqq \sigma(\underset{\sim}{a}_2)$. If $\sigma(\underset{\sim}{a}_1) < \sigma(\underset{\sim}{a}_2)$ then $\sigma([\underset{\sim}{a}_1,\underset{\sim}{b}]) < \sigma([\underset{\sim}{a}_2,\underset{\sim}{b}])$ so that $[\underset{\sim}{a}_1,\underset{\sim}{b}] < [\underset{\sim}{a}_2,\underset{\sim}{b}]$. Otherwise $\sigma(\underset{\sim}{a}_1) = \sigma(\underset{\sim}{a}_2)$ so that $\sigma([\underset{\sim}{a}_1,\underset{\sim}{b}]) = \sigma([\underset{\sim}{a}_2,\underset{\sim}{b}])$ and there are two possibilities:

(i) If $\underset{\sim}{b} < \underset{\sim}{a}_2$, then $\mathrm{ld}([\underset{\sim}{a}_1,\underset{\sim}{b}]) = \underset{\sim}{a}_1$ or $\underset{\sim}{b} < \underset{\sim}{a}_2 = \mathrm{ld}([\underset{\sim}{a}_2,\underset{\sim}{b}])$, and then $[\underset{\sim}{a}_1,\underset{\sim}{b}] < [\underset{\sim}{a}_2,\underset{\sim}{b}]$.

(ii) If $\underset{\sim}{b} \geqq \underset{\sim}{a}_2$, then $\mathrm{ld}([\underset{\sim}{a}_1,\underset{\sim}{b}]) = \underset{\sim}{b} = \mathrm{ld}([\underset{\sim}{a}_2,\underset{\sim}{b}])$ and $\mathrm{tr}([\underset{\sim}{a}_1,\underset{\sim}{b}]) = \underset{\sim}{a}_1 < \underset{\sim}{a}_2 = \mathrm{tr}([\underset{\sim}{a}_2,\underset{\sim}{b}])$ so again $[\underset{\sim}{a}_1,\underset{\sim}{b}] < [\underset{\sim}{a}_2,\underset{\sim}{b}]$.

◇

Corollary

If $\leqq$, $<^{o}$ or $\leqq^{o}$ are substituted for $<$ in the lemma, it is still true.

◇

<u>Lemma 2.2</u>

(i) Let $\underset{\sim}{x}$ be a product of commutators. Then there exists a product $\underset{\sim}{y}$ of commutators such that $D : \underset{\sim}{x}^{-1} \to \underset{\sim}{y}$.

(ii) Let $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ be products of commutators. Then there exists a product $\underset{\sim}{y}$ of commutators such that $D : [\underset{\sim}{x}_1, \underset{\sim}{x}_2] \to \underset{\sim}{y}$.

(iii) Let $\underset{\sim}{x}$ be any expression. Then there exists a product $\underset{\sim}{y}$ of commutators such that $D : \underset{\sim}{x} \to \underset{\sim}{y}$.

<u>Proof</u>

Parts (i) and (ii) follow by an easy induction over the lengths. Part (iii) then follows from (i) and (ii) by induction over the height of $\underset{\sim}{x}$. ◇

Part (iii) of this lemma accomplishes the first main step in the special collecting process. The next step is to show that a product of commutators can be converted into a product of W-basic commutators. The crux of this step is contained in the next lemma.

Lemma 2.3

Suppose $\underset{\sim}{a}$ is a non-W-basic commutator (that is, a commutator which is not a W-basic one). Then $D : \underset{\sim}{a} \to \underset{\sim}{b}$, where $\underset{\sim}{b}$ is of one of the forms:

(a) $\underset{\sim}{b} = \underset{\sim}{1}$

(b) $\underset{\sim}{b} = \underset{\sim}{c}^{\varepsilon}$ where $\underset{\sim}{c} \in \underset{\sim}{C}$; $\varepsilon = \pm 1$; $\underset{\sim}{c} <^{o} \underset{\sim}{a}$; $wt(\underset{\sim}{c}) = wt(\underset{\sim}{a})$.

(c) $\underset{\sim}{b} = \underset{\sim}{c}_1^{\varepsilon_1} \underset{\sim}{c}_2^{\varepsilon_2}$ where $\underset{\sim}{c}_1 , \underset{\sim}{c}_2 \in \underset{\sim}{C}$; $\varepsilon_1 , \varepsilon_2 = \pm 1$;

$\underset{\sim}{c}_1 , \underset{\sim}{c}_2 <^{o} \underset{\sim}{a}$; $wt(\underset{\sim}{c}_1) = wt(\underset{\sim}{c}_2) = wt(\underset{\sim}{a})$.

Proof

The argument proceeds by induction over the weight of $\underset{\sim}{a}$. If $wt(\underset{\sim}{a}) = 1$ then $\underset{\sim}{a}$ is W-basic and the lemma is vacuously true. Now suppose $wt(\underset{\sim}{a}) > 1$ and the lemma is true for all smaller weights. Since $\underset{\sim}{a}$ is non-W-basic, at least one of the conditions of definition 1.13(A) must fail. These are treated separately. Write $\underset{\sim}{a} = [\underset{\sim}{a}_1 , \underset{\sim}{a}_2]$.

(i) $\underset{\sim}{a}_1$ is non-W-basic. Then by the inductive hypothesis, $D : \underset{\sim}{a}_1 \to \underset{\sim}{b}_1$ and so $D : \underset{\sim}{a} \to [\underset{\sim}{b}_1 , \underset{\sim}{a}_2]$ where $\underset{\sim}{b}_1$ has one of the forms given above. If $\underset{\sim}{b}_1 = \underset{\sim}{1}$ then $D : \underset{\sim}{a} \to \underset{\sim}{1}$ by definition 2.1(viii). If $\underset{\sim}{b}_1 = \underset{\sim}{c}_1^{\varepsilon}$ where $\underset{\sim}{c}_1 \in \underset{\sim}{C}$, $\varepsilon = \pm 1$, $\underset{\sim}{c}_1 <^{o} \underset{\sim}{a}_1$ and $wt(\underset{\sim}{c}_1) = wt(\underset{\sim}{a}_1)$ then $D : \underset{\sim}{a} \to [\underset{\sim}{c}_1 , \underset{\sim}{a}_2]^{\varepsilon}$ and $[\underset{\sim}{c}_1 , \underset{\sim}{a}_2] \in \underset{\sim}{C}$, $[\underset{\sim}{c}_1 , \underset{\sim}{a}_2] <^{o} \underset{\sim}{a}$ by the corollary to lemma 2.1 and

$wt([c_1, a_2]) = wt(c_1) + wt(a_2) = wt(a_1) + wt(a_2) = wt(a)$. The argument when $b_1 = c_1^{\varepsilon_1} c_2^{\varepsilon_2}$ is similar.

(ii) a_2 is non-W-basic. The argument follows the same pattern as that for case (i).

(iii) $a_1 \leq a_2$. Then either $a_1 = a_2$ in which case $D : a \to 1$ by definition 2.1(vii) or else $a_1 < a_2$. But then $D : a \to [a_2, a_1]^{-1}$ and $\sigma(a) = \sigma([a_2, a_1])$, $ld(a) = a_2 = ld([a_2, a_1])$, $tr(a) = a_1 = tr([a_2, a_1])$ and $a_1 < a_2$. Thus, by definition 1.12(iiic), $[a_2, a_1] < a$ and so $[a_2, a_1] <^{o} a$. Clearly $wt([a_2, a_1]) = wt([a_1, a_2]) = wt(a)$.

(iv) $a_1 = [a_{11}, a_{12}]$ and $a_{12} > a_2$. By virtue of case (i) it may be assumed that $a_{11} > a_{12}$.
Then $D : a \to [a_{12}, a_2, a_{11}]^{-1}[a_{11}, a_2, a_{12}]$ by definition 2.1(xii).
Write $c_1 = [a_{12}, a_2, a_{11}]$ and $c_2 = [a_{11}, a_2, a_{12}]$.
By definition 1.7(vi), $\sigma(c_1) \geq \sigma(a)$. If $\sigma(c_1) > \sigma(a)$ then $c_1 <^{o} a$. Otherwise $\sigma(c_1) = \sigma(a)$ and it must be shown that $c_1 < a$. Now $a_{11} < [a_{11}, a_{12}] = ld(a)$. Also $a_2 < a_{11}$ so that $[a_{12}, a_2] < [a_{11}, a_{12}] = ld(a)$. But $ld(c) = a_{11}$ or $[a_{12}, a_2]$ and both of these have just been shown to be less than $ld(a)$. Hence $c_1 < a$.
Now $a_{11} > a_{12}$ so that $[a_{11}, a_2] > a_{12}$. Thus $ld(c_2) = [a_{11}, a_2]$.
Similarly $a_1 = [a_{11}, a_2] > a_2$ so $ld(a) = a_1$. But $a_{12} > a_2$

so that $[\underset{\sim}{a}_{11}, \underset{\sim}{a}_{12}] > [\underset{\sim}{a}_{11}, \underset{\sim}{a}_{2}]$, that is, $ld(\underset{\sim}{a}) > ld(\underset{\sim}{c}_2)$. Since $\sigma(\underset{\sim}{a}) = \sigma(\underset{\sim}{c}_2)$ by definition 1.7(vi), it follows that $\underset{\sim}{c}_2 < \underset{\sim}{a}$ and so $\underset{\sim}{c}_2 <^{o} \underset{\sim}{a}$.

Again it is clear that $wt(\underset{\sim}{c}_1) = wt(\underset{\sim}{c}_2) = wt(\underset{\sim}{a})$. ◇

<u>Lemma 2.4</u>

For any positive integer w , the set of commutators of weight $\leqq w$ is well-ordered by $\leqq^{o}$.

<u>Proof</u>

Suppose, for any positive integer w , $\Sigma(w)$ is the set of all possible semiweights that a commutator of weight $\leqq w$ may have, that is, $\Sigma(w) = \{ \sigma(\underset{\sim}{x}) : \underset{\sim}{x} \in \underset{\sim}{C} , wt(\underset{\sim}{x}) \leqq w \}$. It follows by an easy induction over w that $\Sigma(w)$ is finite. But for any $\alpha \in W$, the order $\leqq^{o}$ coincides with the W-ordering $\leqq$ on the set of all commutators of semiweight α , and so is a well-ordering on that set. It follows that, as far as $\leqq^{o}$ is concerned, the set of all commutators of weight $\leqq w$ is the union of a finite number of well-ordered sets. The lemma follows. ◇

Lemma 2.5

If $D : \underset{\sim}{x} \to \underset{\sim}{y}$ then $\sigma(\underset{\sim}{x}) \leq \sigma(\underset{\sim}{y})$ and $wt(\underset{\sim}{x}) \leq wt(\underset{\sim}{y})$.

Proof

This follows easily from definition 2.1 by checking its various parts separately. $\diamond$

The following theorem and its two corollaries are out of logical order in this thesis. They are placed here because they summarize the properties of the special collecting process and provide the motivation for the definitions of this section. The proof requires the results of lemmas 2.6 and 2.7 of the next section, so that the theorem strictly should be stated and proved immediately following the latter lemma.

Theorem 2.1

For any expression $\underset{\sim}{x} \in \underset{\sim}{A}$ there exists $\underset{\sim}{y} \in \underset{\sim}{A}$ such that

(i) $\underset{\sim}{y}$ is a W-basic expression,

(ii) $D : \underset{\sim}{x} \to \underset{\sim}{y}$,

(iii) $wt(\underset{\sim}{x}) \leq wt(\underset{\sim}{y})$ and $\sigma(\underset{\sim}{x}) \leq \sigma(\underset{\sim}{y})$ and

(iv) if $\rho : \underset{\sim}{A} \to G$ is any description of a group G , then $\underset{\sim}{x}\rho = (\underset{\sim}{yu})\rho$ where $\underset{\sim}{u}$ is possibly empty, but if it exists, $wt(\underset{\sim}{u}) \geq wt(\underset{\sim}{x}) + 1$ and $\sigma(\underset{\sim}{u}) \geq \sigma(\underset{\sim}{x}) + 1$.

Proof

By lemma 2.2(iii), there exists a product $\underset{\sim}{z}_1$ of commutators such that $D : \underset{\sim}{x} \rightarrow \underset{\sim}{z}_1$.

Since the length of $\underset{\sim}{z}_1$ is necessarily finite, it follows that there exists an integer w such that $\underset{\sim}{z}_1$ is a product of commutators of weight $\leqq w$. It is now shown that $D : \underset{\sim}{z}_1 \rightarrow \underset{\sim}{z}_2$ where $\underset{\sim}{z}_2$ is a product of W-basic commutators. If $\underset{\sim}{z}_1$ is itself a product of W-basic commutators then this is trivial. Otherwise, again since the length of $\underset{\sim}{z}_1$ is finite, there exists a non-W-basic commutator $\underset{\sim}{a}$, which is maximum under the ordering $\leqq^{o}$, among those factors of $\underset{\sim}{z}_1$ which are non-W-basic. This commutator may appear more than once, but in any case $\underset{\sim}{z}_1$ may be written in the form $\underset{\sim}{z}_1 = \underset{\sim}{v}_0 \underset{\sim}{a} \underset{\sim}{v}_1 \underset{\sim}{a} \underset{\sim}{v}_2 \ldots \underset{\sim}{a} \underset{\sim}{v}_k$ where $k \geqq 1$ and each $\underset{\sim}{v}_i$ is a (possibly empty) product of commutators which are either W-basic or $<^{o} \underset{\sim}{a}$. But by lemma 2.3, $D : \underset{\sim}{a} \rightarrow \underset{\sim}{b}$ where $\underset{\sim}{b}$ is a product of commutators $<^{o} \underset{\sim}{a}$ and of the same weight as $\underset{\sim}{a}$. Thus $D : \underset{\sim}{z}_1 \rightarrow \underset{\sim}{v}_0 \underset{\sim}{b} \underset{\sim}{v}_1 \underset{\sim}{b} \underset{\sim}{v}_2 \ldots \underset{\sim}{b} \underset{\sim}{v}_k$ which is a product of commutators of weight $\leqq w$ which are either W-basic or $<^{o} \underset{\sim}{a}$. Thus the maximum (under $\leqq^{o}$) non-W-basic factor of $\underset{\sim}{z}_1$ may be reduced. But then, by lemma 2.4, this can only be done a finite number of times and eventually $D : \underset{\sim}{z}_1 \rightarrow \underset{\sim}{z}_2$ where $\underset{\sim}{z}_2$ is a product of W-basic commutators.

But then $D : \underset{\sim}{z_2} \rightarrow \underset{\sim}{y}$, where $\underset{\sim}{y}$ is a W-basic expression, follows immediately from definition 2.1 A (i), (iv) and (v). Parts (i) and (ii) of the theorem are thus proved. Part (iii) follows from lemma 2.5.

For part (iv), lemmas 2.6 and 2.7 must be invoked. By lemma 2.6 $E : \underset{\sim}{x} \rightarrow \underset{\sim}{y}\underset{\sim}{u}$ where $\underset{\sim}{u}$ is possibly empty, but if it exists, $\sigma(\underset{\sim}{u}) \geqq \sigma(\underset{\sim}{x}) + 1$ and $wt(\underset{\sim}{u}) \geqq wt(\underset{\sim}{x}) + 1$, and then by lemma 2.7 $\underset{\sim}{x}\rho = (\underset{\sim}{y}\underset{\sim}{u})\rho$. ◇

<u>Corollary 1</u>

Suppose G is a group $\in \underline{\underline{W}}_{\alpha+1}$ and $x \in W_\alpha(G)$. If $\rho : \underset{\sim}{A} \rightarrow G$ is any description of G , then there exists $\underset{\sim}{x} \in \underset{\sim}{A}$ such that

(i) $\underset{\sim}{x} \in \underset{\sim}{B}^{W}_{\alpha+1}$ (see definition 1.13B) ,

(ii) $\sigma(\underset{\sim}{x}) \geqq \alpha$, and

(iii) $\underset{\sim}{x}\rho = x$.

<u>Proof</u>

Since $x \in W_\alpha(G)$, there exists $\underset{\sim}{x}' \in \underset{\sim}{A}$ such that $\underset{\sim}{x}'\rho = x$ and $\sigma(\underset{\sim}{x}') \geqq \alpha$. But then by the theorem $D : \underset{\sim}{x}' \rightarrow \underset{\sim}{x}''$ where $\underset{\sim}{x}'' \in \underset{\sim}{B}^{W}$, $\sigma(\underset{\sim}{x}'') \geqq \sigma(\underset{\sim}{x}')$ and $(\underset{\sim}{x}''\underset{\sim}{u})\rho = x$ where $\underset{\sim}{u}$ is possibly empty, but if it exists, $\sigma(\underset{\sim}{u}) \geqq \sigma(\underset{\sim}{x}) + 1 \geqq \alpha + 1$. Thus $\underset{\sim}{u}\rho = 1$ and $\underset{\sim}{x}''\rho = x$.

If $\underset{\sim}{x}'' = \underset{\sim}{1}$ or $\sigma(\underset{\sim}{x}'') \geqq \alpha + 1$ then $\underset{\sim}{x}''\rho = 1$ and the result is true with $x = 1$. Otherwise $\underset{\sim}{x}''$ is of the form

$\underset{\sim}{x}'' = \underset{\sim}{b}_1^{\alpha_1}\underset{\sim}{b}_2^{\alpha_2}\cdots\underset{\sim}{b}_k^{\alpha_k}$ where, for some $\ell \geqq 1$,

$$\alpha \leqq \sigma(\underset{\sim}{b}_i) < \alpha + 1 \qquad (1 \leqq i \leqq \ell) ,$$

$$\alpha + 1 \leqq \sigma(\underset{\sim}{b}_i) \qquad (\ell + 1 \leqq i \leqq k) .$$

The result is then true with $\underset{\sim}{x} = \underset{\sim}{b}_1^{\alpha_1}\underset{\sim}{b}_2^{\alpha_2}\cdots\underset{\sim}{b}_\ell^{\alpha_\ell}$. ◇

<u>Corollary 2</u>

Suppose G is a group, nilpotent of class c , and $\underset{\sim}{x} \in \gamma_c(G)$. If $\rho : \underset{\sim}{A} \to G$ is a description of G , then there exists $\underset{\sim}{x} \in \underset{\sim}{A}$ such that

(i) $\underset{\sim}{x} \in \underset{\sim}{B}^W_{(c)}$,

(ii) $wt(\underset{\sim}{x}) \geqq c$, and

(iii) $\underset{\sim}{x}\rho = x$.

<u>Proof</u>

The argument here is essentially the same as that for corollary 1. As before, there exists $\underset{\sim}{x}'' \in \underset{\sim}{B}^W$ such that $wt(\underset{\sim}{x}'') \geqq c$ and $\underset{\sim}{x}''\rho = x$. If $\underset{\sim}{x}'' = \underset{\sim}{1}$ or $wt(\underset{\sim}{x}'') \geqq c + 1$ then the result is true with $\underset{\sim}{x} = \underset{\sim}{1}$. Otherwise $\underset{\sim}{x}'' = \underset{\sim}{b}_1^{\alpha_1}\underset{\sim}{b}_2^{\alpha_2}\cdots\underset{\sim}{b}_k^{\alpha_k}$ and at least one of these commutators is of weight c . Then the lemma is true with $\underset{\sim}{x}$ $\underset{\sim}{b}_{i_1}^{\alpha_{i_1}}\underset{\sim}{b}_{i_2}^{\alpha_{i_2}}\cdots\underset{\sim}{b}_{i_\ell}^{\alpha_{i_\ell}}$ where $i_1, i_2, \ldots i_\ell$

is the subsequence of 1, 2, ..., k for which the corresponding commutator is of weight c . ◇

Corollaries 1 and 2 mean, in effect, that the special collecting process works in the "bottom" of a group – whether the "bottom" is taken as $W_\alpha(G)$ when $G \in \underline{\underline{W}}_{\alpha+1}$ or as $\gamma_c(G)$ when G is nilpotent of class c .

These two corollaries form a pair of similar results, the first purely in terms of the subgroups $W_\alpha(G)$ of a group G and the second in terms of the interaction between these subgroups and the lower central series. This situation is typical, and such pairs of results will appear from time to time in the sequel; each time they do it will be possible to trace the dichotomy back to that between these two corollaries.

Since the lower central series of a group G is the series $\overline{N}_c(G)$ of subgroups of G , it might be thought that corollary 2 could be generalized so that some arbitrary semiweight W' might take the place of $\overline{N}$ in it, and thus ultimately allow the investigation of the interaction of arbitrary pairs of semiweights. However if the argument leading to the corollary 2 is followed with this in view it will be seen that the property

$a \leq b \leq c$ (under the W-ordering) $\Longrightarrow$ $wt([c,b,a]) = wt([c,a,b]) \leq wt([b,a,c])$ is essential; this property need no longer hold if an arbitrary semiweight is substituted for weight.

THE GENERAL PROCESS

Definition 2.4

(A) Let $x, y \in A$. Then we write $e : x \to y$ if x and y are of any of the following forms:

(i) $x = ab$, $y = ba[a,b]$

(ii) $x = (ab)^{-1}$, $y = b^{-1}a^{-1}$

(iii) $x = (a^{-1})^{-1}$, $y = a$

(iv) $x = a^{-1}a$ or aa^{-1}, $y = 1$

(v) $x = a1$ or $1a$, $y = a$

(vi) $x = 1^{-1}$, $y = 1$

(vii) $x = [a,a]$, $y = 1$

(viii) $x = [a,1]$ or $[1,a]$, $y = 1$

(ix) $x = [a^{-1},b]$, $y = [a,b]^{-1}[b,a,a^{-1}]$ or

$x = [a,b^{-1}]$, $y = [a,b]^{-1}[b,a,b^{-1}]$

(x) $x = [ab,c]$, $y = [a,c][a,c,b][b,c]$ or

$x = [a,bc]$, $y = [a,c][a,b][a,b,c]$

(xi) $x = [a,b]$, $y = [b,a]^{-1}$ provided a and b are commutators and $a < b$

(xii) $\underset{\sim}{x} = [\underset{\sim}{c},\underset{\sim}{b},\underset{\sim}{a}]$, $\underset{\sim}{y} = \underset{\sim}{v}_1[\underset{\sim}{b},\underset{\sim}{a},\underset{\sim}{c}]^{-1}\underset{\sim}{v}_2[\underset{\sim}{c},\underset{\sim}{a},\underset{\sim}{b}]\underset{\sim}{v}_3$, provided $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ are commutators and $\underset{\sim}{a} < \underset{\sim}{b} < \underset{\sim}{c}$, where

$$\underset{\sim}{v}_1 = [\underset{\sim}{a},\underset{\sim}{c},[\underset{\sim}{c},\underset{\sim}{b}]]\ [\underset{\sim}{c},\underset{\sim}{b},[\underset{\sim}{b},\underset{\sim}{a},\underset{\sim}{c}]]$$

$$\underset{\sim}{v}_2 = [\underset{\sim}{c},\underset{\sim}{b},[\underset{\sim}{b},\underset{\sim}{a}]]\ [\underset{\sim}{b},\underset{\sim}{a},[\underset{\sim}{a},\underset{\sim}{c},\underset{\sim}{b}]]\ [\underset{\sim}{a},\underset{\sim}{c},[\underset{\sim}{c},\underset{\sim}{a},\underset{\sim}{b}]^{-1}]$$

$$\underset{\sim}{v}_3 = [\underset{\sim}{b},\underset{\sim}{a},[\underset{\sim}{a},\underset{\sim}{c}]]\ [\underset{\sim}{a},\underset{\sim}{c},[\underset{\sim}{c},\underset{\sim}{b},\underset{\sim}{a}]]$$

The notation is extended to larger expressions by recursion over their height:

(xiii) If $e : \underset{\sim}{a}_1 \to \underset{\sim}{a}_2$ then $e : \underset{\sim}{a}_1^{-1} \to \underset{\sim}{a}_2^{-1}$ and for any $\underset{\sim}{b} \in \underset{\sim}{A}$, $e : \underset{\sim}{a}_1\underset{\sim}{b} \to \underset{\sim}{a}_2\underset{\sim}{b}$, $e : [\underset{\sim}{a}_1,\underset{\sim}{b}] \to [\underset{\sim}{a}_2,\underset{\sim}{b}]$, $e : \underset{\sim}{b}\underset{\sim}{a}_1 \to \underset{\sim}{b}\underset{\sim}{a}_2$ and $e : [\underset{\sim}{b},\underset{\sim}{a}_1] \to [\underset{\sim}{b},\underset{\sim}{a}_2]$.

(B) Write $E : \underset{\sim}{x} \to \underset{\sim}{y}$ if there exists a finite sequence $(\underset{\sim}{u}_i)_{i=0}^{\infty}$ $(k \geq 0)$ of expressions such that $\underset{\sim}{x} = \underset{\sim}{u}_0$, $\underset{\sim}{y} = \underset{\sim}{u}_k$ and $e : \underset{\sim}{u}_{i-1} \to \underset{\sim}{u}_i$ $(1 \leq i \leq k)$. $\diamond$

Again the relation $E : \underset{\sim}{x} \to \underset{\sim}{y}$ is reflexive and transitive and part A(xiii) of the definition holds just as well for E as for e .

<u>Definition 2.5</u>

(i) An element $\lambda \in W$ is a "limiting value" of W if $\xi < \lambda \implies \xi + 1 < \lambda$.

(ii) If α , $\beta \in W$ then "β is much greater than α", denoted $\alpha \ll \beta$, if there exists a limiting value λ of W such that $\alpha < \lambda \leqq \beta$.

Write $\alpha \not\ll \beta$ if β is not much greater than α.

(iii) For each $\alpha \in W$ and non-negative integer n, the element $\alpha\,(+1)^n$ is defined recursively over n: $\alpha\,(+1)^0 = \alpha$, and for $n > 0$, $\alpha\,(+1)^n = \alpha\,(+1)^{n-1} + 1$. $\diamond$

<u>Lemma 2.6</u>

If $D : \underset{\sim}{x} \to \underset{\sim}{y}$ then there exists $\underset{\sim}{u} \in \underset{\sim}{A}$ (possibly empty) such that $E : \underset{\sim}{x} \to \underset{\sim}{y}\underset{\sim}{u}$, and if $\underset{\sim}{u}$ is not empty, $\sigma(\underset{\sim}{u}) \geqq \sigma(\underset{\sim}{x}) + 1$ and $wt(\underset{\sim}{u}) \geqq wt(\underset{\sim}{x}) + 1$.

<u>Proof</u>

By checking the various parts of definition 2.1 separately. For parts A(ii), (iii), (iv), (v), (vi), (vii), (viii) and (x), $E : \underset{\sim}{x} \to \underset{\sim}{y}$ by the corresponding part of definition 2.4, so that the lemma is true with $\underset{\sim}{u}$ empty.

For the remaining parts:

A(i) $\underset{\sim}{x} = \underset{\sim}{a}\underset{\sim}{b}$, $\underset{\sim}{y} = \underset{\sim}{b}\underset{\sim}{a}$. Then $E : \underset{\sim}{x} \to \underset{\sim}{y}\underset{\sim}{u}$ where $\underset{\sim}{u} = [\underset{\sim}{a},\underset{\sim}{b}]$, and $\sigma(\underset{\sim}{u}) \geqq \sigma(\underset{\sim}{x}) + 1$, $wt(\underset{\sim}{u}) = wt(\underset{\sim}{x}) + 1$.

A(ix) $\underset{\sim}{x} = [\underset{\sim}{a}^{-1},\underset{\sim}{b}]$, $\underset{\sim}{y} = [\underset{\sim}{a},\underset{\sim}{b}]^{-1}$. Then $E : \underset{\sim}{x} \to \underset{\sim}{y}\underset{\sim}{u}$ where $\underset{\sim}{u} = [\underset{\sim}{b},\underset{\sim}{a},\underset{\sim}{a}^{-1}]$. But then $\sigma(\underset{\sim}{u}) = \sigma([\underset{\sim}{b},\underset{\sim}{a}]) + \sigma(\underset{\sim}{a}) \geqq$ $\geqq \sigma(\underset{\sim}{x}) + 1$, and similarly $wt(\underset{\sim}{u}) \geqq wt(\underset{\sim}{x}) + 1$. The argument if $\mathbf{x} = [\mathbf{a},\mathbf{b}^{-1}]$ is similar .

A(x) $\quad \mathbf{x} = [\mathbf{ab},\mathbf{c}]$, $\mathbf{y} = [\mathbf{a},\mathbf{c}][\mathbf{b},\mathbf{c}]$. Then

$E : \mathbf{x} \to [\mathbf{a},\mathbf{c}][\mathbf{a},\mathbf{c},\mathbf{b}][\mathbf{b},\mathbf{c}]$ by definition 2.4A(x),

$\to \mathbf{yu}$

where $\mathbf{u} = [\mathbf{a},\mathbf{c},\mathbf{b}][\mathbf{a},\mathbf{c},\mathbf{b},[\mathbf{b},\mathbf{c}]]$ by definition 2.4A(i). But then $\sigma(\mathbf{u}) = \sigma([\mathbf{a},\mathbf{c},\mathbf{b}]) \geqq \sigma([\mathbf{a},\mathbf{c}]) + 1 \geqq \sigma(\mathbf{x}) + 1$. Similarly $wt(\mathbf{u}) \geqq wt(\mathbf{x}) + 1$. The argument if $\mathbf{x} = [\mathbf{a},\mathbf{bc}]$, $\mathbf{y} = [\mathbf{a},\mathbf{c}][\mathbf{b},\mathbf{c}]$ is similar but slightly easier.

A(xii) $\quad \mathbf{x} = [\mathbf{c},\mathbf{b},\mathbf{a}]$, $\mathbf{y} = [\mathbf{b},\mathbf{a},\mathbf{c}]^{-1}[\mathbf{c},\mathbf{a},\mathbf{b}]$ where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are commutators and $\mathbf{a} < \mathbf{b} < \mathbf{c}$. Then with the notation of definition 2.4 A (xii),

$$E : \mathbf{x} \to \mathbf{v}_1[\mathbf{b},\mathbf{a},\mathbf{c}]^{-1}\mathbf{v}_2[\mathbf{c},\mathbf{a},\mathbf{b}]\mathbf{v}_3 \to \mathbf{yu} \quad ,$$

where $\mathbf{u} = \mathbf{v}_1 \; [\mathbf{v}_1,\mathbf{y}] \; \mathbf{v}_2 \; [\mathbf{v}_2,[\mathbf{c},\mathbf{a},\mathbf{b}]] \; \mathbf{v}_3$ by several applications of definition 2.4(i). Clearly $wt(\mathbf{u}) = wt(\mathbf{x}) + 1$. to show that $\sigma(\mathbf{u}) = \sigma(\mathbf{x}) + 1$ it is sufficient to show that the semiweights of the seven commutators in the expressions for $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are all $\geqq \sigma(\mathbf{x}) + 1$.

(α) $\quad \sigma([\mathbf{c},\mathbf{b}]) \geqq \sigma(\mathbf{c}) \geqq \sigma(\mathbf{a})$ and so

$\sigma([\mathbf{a},\mathbf{c},[\mathbf{c},\mathbf{b}]]) = \sigma([\mathbf{c},\mathbf{a},[\mathbf{c},\mathbf{b}]])$ by commutativity of in W ,

$\geqq \sigma([\mathbf{c},\mathbf{b},\mathbf{a},\mathbf{c}])$ by definition 1.7(vi),

$\geqq \sigma([\mathbf{c},\mathbf{b},\mathbf{a}]) + 1 = \sigma(\mathbf{x}) + 1$.

(β) $$\sigma([\mathbf{c},\mathbf{b},[\mathbf{b},\mathbf{a},\mathbf{c}]]) \geqq \sigma([\mathbf{b},\mathbf{a},\mathbf{c}]) + 1$$
$$\geqq \sigma([\mathbf{c},\mathbf{b},\mathbf{a}]) + 1 \quad .$$

(γ) $\sigma([\mathbf{c},\mathbf{b}]) \geqq \sigma(\mathbf{b}) \geqq \sigma(\mathbf{a})$ and so
$$\sigma([\mathbf{c},\mathbf{b},[\mathbf{b},\mathbf{a}]]) = \sigma([\mathbf{b},\mathbf{a},[\mathbf{c},\mathbf{b}]])$$
$$\geqq \sigma([\mathbf{c},\mathbf{b},\mathbf{a},\mathbf{b}])$$
$$\geqq \sigma([\mathbf{c},\mathbf{b},\mathbf{a}]) + 1 \quad .$$

(δ) $$\sigma([\mathbf{b},\mathbf{a},[\mathbf{a},\mathbf{c},\mathbf{b}]]) \geqq \sigma([\mathbf{a},\mathbf{c},\mathbf{b}]) + 1$$
$$= \sigma([\mathbf{c},\mathbf{a},\mathbf{b}]) + 1$$
$$= \sigma([\mathbf{c},\mathbf{b},\mathbf{a}]) + 1 \, .$$

(ε) $$\sigma([\mathbf{a},\mathbf{c},[\mathbf{c},\mathbf{a},\mathbf{b}]]) \geqq \sigma([\mathbf{c},\mathbf{a},\mathbf{b}]) + 1$$
$$\geqq \sigma([\mathbf{c},\mathbf{b},\mathbf{a}]) + 1 \quad .$$

(ξ) $\sigma(\mathbf{c}) \geqq \sigma(\mathbf{b})$ and so $\sigma([\mathbf{c},\mathbf{a}]) \geqq \sigma(\mathbf{b}) \geqq \sigma(\mathbf{a})$. Hence
$$\sigma([\mathbf{b},\mathbf{a},[\mathbf{c},\mathbf{a}]]) \geqq \sigma([\mathbf{c},\mathbf{a},\mathbf{b},\mathbf{a}])$$
$$\geqq \sigma([\mathbf{c},\mathbf{a},\mathbf{b}]) + 1$$
$$= \sigma([\mathbf{c},\mathbf{b},\mathbf{a}]) + 1 \quad .$$

(η) $\sigma([\mathbf{a},\mathbf{c},[\mathbf{c},\mathbf{b},\mathbf{a}]]) \geqq \sigma([\mathbf{c},\mathbf{b},\mathbf{a}]) + 1$ immediately.

A(xiii) The result follows in this case by an easy induction over the height of $\mathbf{x}$.

(B) The result follows in this case by an easy induction over the length k of the sequence, using lemma 2.5 . ◇

Lemma 2.7

Suppose $E : \underset{\sim}{x} \rightarrow \underset{\sim}{y}$. Then

(i) If $\rho : \underset{\sim}{A} \rightarrow G$ is any description of a group G , then $\underset{\sim}{x}\rho = \underset{\sim}{y}\rho$.

(ii) $\sigma(\underset{\sim}{x}) \leqq \sigma(\underset{\sim}{y})$ and $wt(\underset{\sim}{x}) \leqq wt(\underset{\sim}{y})$.

Proof

(i) This follows easily from definition 2.4 by checking the various parts separately. All these parts correspond to well known group laws, except perhaps for A(xii) which can be checked by expanding into a product of $\underset{\sim}{a}\rho$, $\underset{\sim}{b}\rho$, and $\underset{\sim}{c}\rho$ and their inverses and cancelling.

(ii) This is a corollary of lemmas 2.5 and 2.6. ◇

Lemma 2.8

Let $\alpha , \beta \in W$. Then $\alpha \not\ll \beta$ if and only if there exists a non-negative integer n such that $\alpha\,(+1)^n \geqq \beta$.

Proof

Suppose that there exists a non-negative integer n such that $\alpha\,(+1)^n \geqq \beta$, then $\alpha \not\ll \beta$ for otherwise there exists a limiting value λ of W such that $\alpha < \lambda \leqq \beta$ and a least integer m

such that $\alpha\,(+1)^{m} \geqq \lambda$. But then $\alpha\,(+1)^{m-1} < \lambda$ and $\alpha\,(+1)^{m} = \alpha\,(+1)^{m-1} + 1 \geqq \lambda$ contradicting the definition of a limiting value (definition 2.5(i)).

Suppose on the contrary that $\alpha\,(+1)^{n} < \beta$ for all n . Since W is well-ordered, $\lambda = \sup\{\, \alpha\,(+1)^{n} : n \geqq 0 \,\}$ is well defined, and $\alpha < \lambda \leqq \beta$. Suppose $\xi < \lambda$. Then there exists n such that $\alpha\,(+1)^{n} > \xi$. But then $\xi + 1 < \alpha\,(+1)^{n+1} \leqq \lambda$. Thus λ is a limiting value of W and so $\alpha \ll \beta$. $\diamond$

<u>Lemma 2.9</u>

Suppose $\alpha \not\ll \beta$ and $\sigma(\underset{\sim}{x}) \geqq \alpha$. Then $E : \underset{\sim}{x} \to \underset{\sim}{y}\underset{\sim}{u}$ (either $\underset{\sim}{y}$ or $\underset{\sim}{u}$ being possibly empty) where $\underset{\sim}{y} \in \underset{\sim}{B}{}^{W}_{\beta}$ and $\sigma(\underset{\sim}{u}) \geqq \beta$ (when they exist).

<u>Proof</u>

By lemma 2.8, there exists a non-negative integer n such that $\alpha\,(+1)^{n} \geqq \beta$. Suppose that n is the least such. The argument proceeds by induction over n .

If $n = 0$ then $\alpha = \alpha\,(+1)^{0} \geqq \beta$ and so the result is true with $\underset{\sim}{y}$ empty and $\underset{\sim}{u} = \underset{\sim}{x}$.

Now suppose $n > 0$ and the result is true for smaller integers. Write $\beta' = \alpha(+1)^{n-1}$. Then, by the choice of n, $\beta' < \beta$. Also, by the inductive hypothesis, $E : \underset{\sim}{x} \to \underset{\sim}{y}'\underset{\sim}{u}'$ (either $\underset{\sim}{y}'$ or $\underset{\sim}{u}'$ possibly empty) where $\underset{\sim}{y}' \in \underset{\sim}{B}^W_{\beta'}$, and $\sigma(\underset{\sim}{u}') \geqq \beta'$. If $\underset{\sim}{u}'$ is empty, the result is true $\underset{\sim}{y} = \underset{\sim}{y}'$ and $\underset{\sim}{u}$ empty. If $\underset{\sim}{u}'$ is non-empty, then by theorem 2.1, $D : \underset{\sim}{u}' \to \underset{\sim}{v}$ where $\underset{\sim}{v} \in \underset{\sim}{B}^W$ and $\sigma(\underset{\sim}{v}) \geqq \sigma(\underset{\sim}{u}') \geqq \beta'$. By lemma 2.6 then, $E : \underset{\sim}{u}' \to \underset{\sim}{vz}$ ($\underset{\sim}{z}$ possibly empty) where $\sigma(\underset{\sim}{z}) \geqq \sigma(\underset{\sim}{u}') + 1 \geqq \beta' + 1 \geqq \beta$. Thus $E : \underset{\sim}{x} \to \underset{\sim}{y}'\underset{\sim}{vz}$. If $\underset{\sim}{v}$ is empty or $\sigma(\underset{\sim}{v}) \geqq \beta$ the result is true with $\underset{\sim}{y} = \underset{\sim}{y}'$ and $\underset{\sim}{u} = \underset{\sim}{vz}$. Otherwise

$$\underset{\sim}{v} = \underset{\sim}{b}_1^{\alpha_1}\underset{\sim}{b}_2^{\alpha_2}\cdots\underset{\sim}{b}_\ell^{\alpha_\ell}\underset{\sim}{b}_{\ell+1}^{\alpha_{\ell+1}}\underset{\sim}{b}_{\ell+2}^{\alpha_{\ell+2}}\cdots\underset{\sim}{b}_k^{\alpha_k} \quad (\ell \geqq 1) \quad \text{where}$$

$$\beta' \leqq \sigma(\underset{\sim}{b}_i) < \beta \qquad (1 \leqq i \leqq \ell),$$

$$\beta \leqq \sigma(\underset{\sim}{b}_i) \qquad (\ell < i \leqq k),$$

and the result is true with $\underset{\sim}{y} = \underset{\sim}{y}'\underset{\sim}{b}_1^{\alpha_1}\underset{\sim}{b}_2^{\alpha_2}\cdots\underset{\sim}{b}_\ell^{\alpha_\ell}$ and $\underset{\sim}{u} = \underset{\sim}{b}_{\ell+1}^{\alpha_{\ell+1}}\underset{\sim}{b}_{\ell+2}^{\alpha_{\ell+2}}\cdots\underset{\sim}{b}_k^{\alpha_k}\underset{\sim}{z}$. ◇

Lemma 2.10

Let $\underset{\sim}{x}, \underset{\sim}{y} \in \underset{\sim}{B}^W_{(c)}$ and $wt(\underset{\sim}{y}) \geqq c$ (so that $\underset{\sim}{y}$ is a product of commutators of weight c). Then $E : \underset{\sim}{xy} \to \underset{\sim}{zu}$ ($\underset{\sim}{u}$ possibly empty) where $\underset{\sim}{z} \in \underset{\sim}{B}^W_{(c)}$ and $wt(\underset{\sim}{u}) > c$.

Proof

By an easy induction over the length (as a product) of $\underset{\sim}{y}$.

◇

Lemma 2.11

Let $\underset{\sim}{x} \in \underset{\sim}{A}$ and c be any non-negative integer. Then $E : \underset{\sim}{x} \to \underset{\sim}{yu}$ (either $\underset{\sim}{y}$ or $\underset{\sim}{u}$ possibly empty) where $\underset{\sim}{y} \in \underset{\sim}{B}^W_{(c)}$ and $wt(\underset{\sim}{u}) \geqq c + 1$ (when they exist).

Proof

By induction over c . When $c = 0$ the result is true with $\underset{\sim}{y}$ empty and $\underset{\sim}{u} = \underset{\sim}{x}$.

Now suppose $c \geqq 1$ and the result is true for smaller values of c . Then $E : \underset{\sim}{x} \to \underset{\sim}{y}'\underset{\sim}{u}'$ (either $\underset{\sim}{y}'$ or $\underset{\sim}{u}'$ possibly empty) where $\underset{\sim}{y}' \in \underset{\sim}{B}^W_{(c-1)}$ and $wt(\underset{\sim}{u}) \geqq c$ where they exist. By theorem 2.1, $D : \underset{\sim}{u}' \to \underset{\sim}{v}_1$ where $\underset{\sim}{v}_1 \in \underset{\sim}{B}^W$ and $wt(\underset{\sim}{v}_1) \geqq c$. Then by lemma 2.6, $E : \underset{\sim}{u}' \to \underset{\sim}{v}_1\underset{\sim}{z}_1$ ($\underset{\sim}{z}_1$ possibly empty) where $wt(\underset{\sim}{z}_1) \geqq wt(\underset{\sim}{u}') + 1 \geqq c + 1$. Thus $E : \underset{\sim}{x} \to \underset{\sim}{y}'\underset{\sim}{v}_1\underset{\sim}{z}_1$. Now $\underset{\sim}{v}_1$ contains (possibly) some commutators of weight $\geqq c + 1$, but by definition 2.1(i), $D : \underset{\sim}{v}_1 \to \underset{\sim}{v}_2\underset{\sim}{z}_2$ (either $\underset{\sim}{v}_2$ or $\underset{\sim}{z}_2$ possibly empty) where $\underset{\sim}{v}_2 \in \underset{\sim}{B}^W_{(c)}$, $wt(\underset{\sim}{v}_2) \geqq c$ and $wt(\underset{\sim}{z}_2) \geqq c + 1$. Thus $E : \underset{\sim}{v}_1 \to \underset{\sim}{v}_2\underset{\sim}{z}_2\underset{\sim}{z}_3$ ($\underset{\sim}{z}_3$ possibly empty) where $wt(\underset{\sim}{z}_3) \geqq c + 1$

and so $E : \underset{\sim}{x} \to \underset{\sim}{y}'\underset{\sim}{v}_2\underset{\sim}{z}_2\underset{\sim}{z}_3\underset{\sim}{z}_1$. But now lemma 2.10 may be applied to $\underset{\sim}{y}'\underset{\sim}{v}_2$ so that $E : \underset{\sim}{y}'\underset{\sim}{v}_2 \to \underset{\sim}{y}\underset{\sim}{z}_4$ where $\underset{\sim}{y} \in \underset{\sim}{B}^{W}_{(c)}$ and $wt(\underset{\sim}{z}_4) \geqq c + 1$. The result is then true with $\underset{\sim}{y}$ as just defined and $\underset{\sim}{u} = \underset{\sim}{z}_4\underset{\sim}{z}_2\underset{\sim}{z}_3\underset{\sim}{z}_1$. ◇

The reader may wonder why I have taken such pains with this lemma, since it has been patently clear for some pages that any element of a group G can be described by an element of $\underset{\sim}{B}^{W}_{(c)}$ modulo $\gamma_c(G)$. The reason is that the important part of this lemma is not that such an expression exists, but that it can be arrived at by only those operations listed in definition 2.4. Group theoretic results will presently be deduced by observing what can happen to certain properties of an expression under these operations (see for instance lemma 3.11 of the next chapter).

<u>Theorem 2.2</u> <u>The basis theorem</u>

(A) Let F be an absolutely free group of **rank** τ on free generators $\mathcal{G} = \{g_i\}_{i<\tau}$ and let $\rho : \underset{\sim}{A} \to F$ be the corresponding free description. Then

(i) For any $\alpha \in W$, $W_\alpha(F) \,/\, W_{\alpha+1}(F)$ is a free Abelian group, freely generated by the set

$\{ \underset{\sim}{b}\rho : \underset{\sim}{b}$ is W-basic, $\alpha \leqq \sigma(\underset{\sim}{b}) < \alpha + 1 \}$ modulo $W_{\alpha+1}(F)$.

(ii) The restriction of the mapping ρ to the set $\underset{\sim}{B}^W$ of W-basic expressions is one-to-one.

(iii) Provided $\alpha \not< \beta$, the restriction of ρ to $\underset{\sim\alpha}{W} \cap \underset{\sim\beta}{B}^W$ (that is, to the set of all W-basic expressions involving commutators of semiweights $\geq \alpha$ and $< \beta$) is one-to-one <u>onto</u> the factor $W_\alpha(F) / W_\beta(F)$, modulo $W_\beta(F)$.

(B) Let G be a group, free with respect to being nilpotent of class c , of rank τ and freely generated by $\mathcal{G} = \{\underset{\sim}{g}_i\}_{i<\tau}$. Let $\rho : \underset{\sim}{A} \to G$ be the corresponding free description of G . Then

(i) $\gamma_c(G)$ is a free Abelian group, freely generated by the set $\{ \underset{\sim}{b}\rho : \underset{\sim}{b}$ is W-basic , $wt(\underset{\sim}{b}) = c \}$.

(ii) The restriction of ρ to $\underset{\sim}{B}^W_{(c)}$ is one-to-one <u>onto</u> G .

(iii) $W_\alpha(G) \cap \gamma_c(G)$ is a free Abelian group, freely generated by the set $\{ \underset{\sim}{b}\rho : \underset{\sim}{b}$ is W-basic, $wt(\underset{\sim}{b}) = c$ and $\sigma(\underset{\sim}{b}) \geq \alpha \}$.

<u>Proof</u>

The various parts of the theorem are proved in a different order from that in which they were stated.

(B)(i) It is clearly sufficient to prove this statement when τ is finite. Let $x \in \gamma_c(G)$. Then there exists $\underset{\sim}{x} \in \underset{\sim}{A}$ such that $\underset{\sim}{x}\rho = x$ and $wt(\underset{\sim}{x}) \geq c$. By lemma 2.11, $E : \underset{\sim}{x} \to \underset{\approx}{yu}$

($\underset{\sim}{u}$ possibly empty) where $\underset{\sim}{y} \in \underset{\sim}{B}^{W}_{(c)}$ and $wt(\underset{\sim}{u}) \geqq c + 1$. Then $\underset{\sim}{x}\rho = \underset{\sim}{y}\rho$ by lemma 2.7(i). But $wt(\underset{\sim}{y}) \geqq wt(\underset{\sim}{x}) \geqq c$ by lemma 2.7(ii). Hence $\underset{\sim}{x}\rho = (\underset{\sim}{b}_1\rho)^{\alpha_1}(\underset{\sim}{b}_2\rho)^{\alpha_2} \ldots (\underset{\sim}{b}_k\rho)^{\alpha_k}$ where the $\underset{\sim}{b}_i$ are W-basic commutators of weight c . Thus the set $\{ \underset{\sim}{b}\rho : \underset{\sim}{b}$ is W-basic, $wt(\underset{\sim}{b}) = c \}$ generates $\gamma_c(G)$. But since, by theorem 1.2, the number of W-basic commutators of weight c is the same as the number of $\bar{N}$-basic commutators of weight c , and by the conventional theory these are mapped one-to-one into G by ρ and <u>freely</u> generate $\gamma_c(G)$, it follows that the W-basic commutators of weight c are mapped one-to-one into G by ρ and freely generate $\gamma_c(G)$. This, by an easy induction over c , also proves part B(ii).

(A)(i) Let x_1 , $x_2 \in W_\alpha(F)$. Then there exist $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2 \in \underset{\sim}{A}$ such that $\underset{\sim}{x}_1\rho = x_1$, $\underset{\sim}{x}_2\rho = x_2$, $\sigma(\underset{\sim}{x}_1) \geqq \alpha$ and $\sigma(\underset{\sim}{x}_2) \geqq \alpha$. But then $E : \underset{\sim}{x}_1\underset{\sim}{x}_2 \to \underset{\sim}{x}_2\underset{\sim}{x}_1[\underset{\sim}{x}_1, \underset{\sim}{x}_2]$ and $\sigma([\underset{\sim}{x}_1, \underset{\sim}{x}_2]) \geqq \alpha + 1$. Hence $\underset{\sim}{x}_1\underset{\sim}{x}_2 = \underset{\sim}{x}_2\underset{\sim}{x}_1$ modulo $W_{\alpha+1}(F)$. Thus $W_\alpha(F) / W_{\alpha+1}(F)$ is Abelian.

Now suppose $x \in W_\alpha(F)$. Then there exists $\underset{\sim}{x} \in \underset{\sim}{A}$ such that $\underset{\sim}{x}\rho = x$ and $\sigma(\underset{\sim}{x}) \geqq \alpha$. Then, by lemma 2.9, $E : \underset{\sim}{x} \to \underset{\sim}{y}\underset{\sim}{u}$ ($\underset{\sim}{u}$ possibly empty) where $\underset{\sim}{y} \in \underset{\sim}{B}^{W}_{\alpha+1}$ and $\sigma(\underset{\sim}{u}) \geqq \alpha + 1$. Hence $\underset{\sim}{x}\rho = \underset{\sim}{y}\rho$ modulo $W_{\alpha+1}(F)$. But $\sigma(\underset{\sim}{y}) \geqq \sigma(\underset{\sim}{x}) \geqq \alpha$ so $\underset{\sim}{y}$ is a product of W-basic commutators of semiweight $< \alpha + 1$ and $\geqq \alpha$.

Thus $\{ \underset{\sim}{b}\rho : \underset{\sim}{b}$ is W-basic, $\alpha \leq \sigma(\underset{\sim}{b}) < \alpha + 1 \}$ generates $W_\alpha(F)$ modulo $W_{\alpha+1}(F)$. That the set freely generates it follows from the next section of the proof.

(A)(ii) Suppose that $\underset{\sim}{x} = \underset{\sim}{a}_1^{\alpha_1} \underset{\sim}{a}_2^{\alpha_2} \cdots \underset{\sim}{a}_k^{\alpha_k}$ and $\underset{\sim}{y} = \underset{\sim}{b}_1^{\alpha_1} \underset{\sim}{b}_2^{\alpha_2} \cdots \underset{\sim}{b}_\ell^{\alpha_\ell}$ are two W-basic expressions and $\underset{\sim}{x}\rho = \underset{\sim}{y}\rho$. Then there exists an integer c such that $\mathrm{wt}(\underset{\sim}{a}_i) \leq c \quad (1 \leq i \leq k)$ and $\mathrm{wt}(\underset{\sim}{b}_i) \leq c \quad (1 \leq i \leq \ell)$. Then $\underset{\sim}{x} , \underset{\sim}{y} \in \underset{\sim}{B}^W_{(c)}$, and $\underset{\sim}{x}\rho = \underset{\sim}{y}\rho$ modulo $\gamma_c(F)$. Hence, by part (B)(ii) of the theorem, already proved, $\underset{\sim}{x} = \underset{\sim}{y}$.

(A)(iii) Follows from (A)(ii) and lemma 2.9.

(B)(iii) Suppose $x \in W_\alpha(G) \cap \gamma_c(G)$. Since $x \in W_\alpha(G)$, there exists $\underset{\sim}{x} \in \underset{\sim}{A}$ such that $\underset{\sim}{x}\rho = x$ and $\sigma(\underset{\sim}{x}) \geq \alpha$. By the proof of (B)(i), $E : \underset{\sim}{x} \to \underset{\sim}{y}\underset{\sim}{u}$ where $\underset{\sim}{y} \in \underset{\sim}{B}^W_{(c)}$ and $\mathrm{wt}(\underset{\sim}{u}) \geq c + 1$. But then $\mathrm{wt}(\underset{\sim}{y}) \geq c$ and $\sigma(\underset{\sim}{y}) \geq \alpha$. The result follows. $\diamond$

LIE RINGS

The forgoing theory may be applied to Lie rings instead of groups in a slightly simpler form. Since several different notations occur in the literature for the operations in Lie rings, a definition is given.

Definition 2.6

A Lie ring L is a describable algebra in which the effects of the operators ε , ν , μ , χ are written

$$\varepsilon = 0$$

$$\left.\begin{aligned} x\nu &= -x \\ xy\mu &= x + y \\ xy\chi &= xy \end{aligned}\right\} \quad \text{for any } x , y \in L ,$$

and with the following laws:

(i) L is an Abelian group with respect to addition (μ) ,

(ii) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$,

(iii) $xx = 0$,

(iv) $(xy)z + (yz)x + (zx)y = 0$. ◇

Immediate consequences of these laws are

(v) $xy = -yx$,

(vi) $x(-y) = (-x)y = -(xy)$,

(vii) $x0 = 0x = 0$.

Hence the special collecting process operates anywhere in a Lie ring, in the following sense.

<u>Lemma 2.12</u>

Let L be a Lie ring and $\rho : \underset{\sim}{A} \to L$ a description of L (remembering to translate the notation for operators — that is, $(\underset{\sim}{x}\underset{\sim}{y})\rho = \underset{\sim}{x}\rho + \underset{\sim}{y}\rho$ and $[\underset{\sim}{x},\underset{\sim}{y}]\rho = (\underset{\sim}{x}\rho)(\underset{\sim}{y}\rho)$) .
If $D : \underset{\sim}{x} \to \underset{\sim}{y}$, then $\underset{\sim}{x}\rho = \underset{\sim}{y}\rho$. $\diamond$

<u>Lemma 2.13</u>

Let L be a Lie ring, W a semiweight range and $\alpha \in W$.
Then $W_\alpha(L)$ is an ideal of L .

<u>Proof</u>

(i) Let $x , y \in W_\alpha(L)$. Then there exist $\underset{\sim}{x} , \underset{\sim}{y} \in \underset{\sim}{A}$ such that $\underset{\sim}{x}\rho = x$, $\underset{\sim}{y}\rho = y$, $\sigma(\underset{\sim}{x}) \geq \alpha$ and $\sigma(\underset{\sim}{y}) \geq \alpha$. Then $\sigma(\underset{\sim}{x}\underset{\sim}{y}) \geq \alpha$ and so $x + y = (\underset{\sim}{x}\underset{\sim}{y})\rho \in W_\alpha(L)$.

(ii) Let $x \in W_\alpha(L)$ and $y \in L$. Then there exist $\underset{\sim}{x}$, $\underset{\sim}{y} \in \underset{\sim}{A}$ such that $\underset{\sim}{x}\rho = x$, $\underset{\sim}{y}\rho = y$ and $\sigma(\underset{\sim}{x}) \geqq \alpha$. Then $\sigma([\underset{\sim}{x},\underset{\sim}{y}]) \geqq \alpha$ and so $xy = [\underset{\sim}{x},\underset{\sim}{y}]\rho \in W_\alpha(L)$. Similarly $yx \in W_\alpha(L)$. ◇

Since a Lie ring is defined in terms of laws, the idea of a free Lie ring is tenable. Marshall Hall Jr. [3] has proved (restating his theorem 3.1 in the language of this thesis):

If L is a free Lie ring, freely generated by the set $\mathcal{G} = \{g_i\}_{i<\tau}$, and $\rho : \underset{\sim}{A} \to L$ is the corresponding free description of L, then the $\bar{N}$-basic commutators are mapped one-to-one into L and their images form a basis for L (that is, they generate L freely qua free Abelian group).

This makes possible

<u>Theorem 2.3</u> <u>The basis theorem for Lie rings</u>

Let L be a free Lie ring, $\rho : \underset{\sim}{A} \to L$ a free description of L and W any semiweight range. Then

(i) The restriction of the mapping ρ to $\underset{\sim}{B}^W$ maps it one-to-one <u>onto</u> L and the images of all W-basic commutators constitute a basis for L.

(ii) For any $\alpha \in W$ the set $\{ \underset{\sim}{b}\rho : \underset{\sim}{b}$ is W-basic, $\sigma(\underset{\sim}{b}) \geqq \alpha \}$ is a basis for $W_\alpha(L)$.

(iii) For any non-negative integer c , the set $\{ \underset{\sim}{b}\rho : \underset{\sim}{b} \text{ is W-basic}, \ wt(\underset{\sim}{b}) \leq c \}$ is a basis for $L / \gamma_{c+1}(L)$ modulo $\gamma_{c+1}(L)$ and the set $\{ \underset{\sim}{b}\rho : \underset{\sim}{b} \text{ is W-basic}, \ wt(\underset{\sim}{b}) \geq c + 1 \}$ is a basis for $\gamma_{c+1}(L)$.

Proof

It is sufficient to prove the theorem when the number τ of generators is finite. Let $x \in L$. Then there exists $\underset{\sim}{x} \in \underset{\sim}{A}$ such that $x \in \underset{\sim}{x}\rho$. But $D : \underset{\sim}{x} \to \underset{\sim}{y}$ where $\underset{\sim}{y} \in \underset{\sim}{B}^W$ and then $\underset{\sim}{y}\rho = x$. Hence $\underset{\sim}{B}^W \rho = L$. Thus the set of all images of W-basic commutators under ρ generates L qua Abelian group. Similarly $\{ \underset{\sim}{b}\rho : \underset{\sim}{b} \text{ is W-basic}, \ wt(\underset{\sim}{b}) \leq c + 1 \}$ generates $L / \gamma_{c+1}(L)$ modulo $\gamma_{c+1}(L)$. But then, by theorem 1.2 and the theorem of Marshall Hall Jr. just quoted, this set freely generates that factor. Since $\gamma_{c+1}(L)$ is generated by the images of W-basic commutators of weight $\geq c + 1$, it follows that $\bigcap_{c=0}^{\infty} \gamma_{c+1}(L) = \{0\}$ and hence that $\{ \underset{\sim}{b}\rho : \underset{\sim}{b} \text{ is W-basic} \}$ freely generates L . The theorem follows. ◇

PARTIAL COLLECTION

The restriction $\alpha \not\ll \beta$ appearing in lemma 2.9 and consequently in theorem 2.2 A(iii) is disquieting. It means that there is no guarantee that an arbitrary expression can be collected at all. That this restriction is real, and not just due to an inadequacy in the method of proof, is demonstrated in Appendix I, where it is shown that if $\alpha \ll \beta$ and provided the number τ of generators is at least 3, there exists an element in $W_\alpha(F_\tau)$ which cannot be described by a W-basic expression modulo $W_\beta(F_\tau)$ at all.

In default of this, a result which would be a good second-best would be: "If $\rho : \underset{\sim}{A} \to G$ is a description of a group G and if $x \in G$, $x \neq 1$, then there exists $\underset{\sim}{yu} \in \underset{\sim}{A}$ ($\underset{\sim}{u}$ possibly empty) such that $\underset{\sim}{y} \in \underset{\sim}{B}^W$ and $\sigma(\underset{\sim}{u}) > \sigma(\underset{\sim}{y})$ ". The strength of this property will be demonstrated in the next two chapters.

It seems to me to be likely that this is true for all semiweight ranges W . On the other hand I have not as yet been able to prove the result in full generality, and so I must reduce it to the status of a desirable property that a semiweight range may (or may not) have. The situation is saved however by the proof in the next chapter that the polynilpotent semiweight ranges do in fact have this property.

Definition 2.7

(i) A semiweight range W is "partially collectable mod α", where $\alpha \in W$, if for any description $\rho : \underset{\sim}{A} \to G$ of a group G and any $x \in G \setminus W_{\alpha}(G)$ there exists $\underset{\sim}{y}\underset{\sim}{u} \in \underset{\sim}{A}$ ($\underset{\sim}{u}$ possibly empty) such that $(\underset{\sim}{y}\underset{\sim}{u})\rho = x$, $\underset{\sim}{y} \in \underset{\sim}{B}^{W}$ and $\sigma(\underset{\sim}{u}) > \sigma(\underset{\sim}{y})$ (so that $\underset{\sim}{y} \neq \underset{\sim}{1}$ and $\sigma(\underset{\sim}{y}) < \alpha$).

(ii) A semiweight range W is "partially collectable" if it is partially collectable mod ∞. ◇

It will be shown in the next lemma that the "partially collected" expression $\underset{\sim}{y}\underset{\sim}{u}$ may be chosen so that $\underset{\sim}{y}$ is a product of W-basic commutators of the same semiweight which is less than $\sigma(\underset{\sim}{u})$ and less than α, and then by the basis theorem $\underset{\sim}{y}$ is uniquely determined by x (for a given description ρ). Herein lies the importance of the partial collection property, for this allows us to obtain information about elements of a group G which <u>do not</u> belong to some $W_{\alpha}(G)$. For instance, it follows immediately that if W is partially collectable mod α, then the relatively free group $F(\underline{\underline{W}}_{\alpha})$ is torsion-free. Further instances are provided by the proofs of the next lemma, theorem 4.1 and, in a slightly different context, lemma 3.17.

These remarks and especially part (iv) of the next lemma should make clear the analogy claimed in the introduction between partial collectability in this theory and residual nilpotence of absolutely free groups in the conventional one.

Lemma 2.14

Let W be a semiweight range, σ the associated semiweight and $\alpha \in W$. The following five propositions concerning α are equivalent.

(i) W is partially collectable mod α.

(ii) If F is an absolutely free group and $x \notin W_\alpha(F)$ then there exists $\beta < \alpha$ such that $x \in W_\beta(F) \setminus W_{\beta^+}(F)$, where β^+ is the successor of β under the well-ordering $\leq$ of W.

(iii) If $\rho : \underset{\sim}{A} \to F(\underline{\underline{W}}_\alpha)$ is a free description and $\underset{\sim}{x}\rho \neq 1$ then $E : \underset{\sim}{x} \to \underset{\sim}{y}\underset{\sim}{u}$ where $\sigma(\underset{\sim}{y}) = \beta$ $(\beta < \alpha)$, $\underset{\sim}{y} \in \underset{\sim}{B}^W_{\beta^+}$ (so that $\underset{\sim}{y}$ is a product of commutators of semiweight exactly β and is not $\underset{\sim}{1}$) and $\sigma(\underset{\sim}{u}) > \beta$.

(iv) For all $\beta \leq \alpha$ the relatively free group $F(\underline{\underline{W}}_\beta)$ is residually nilpotent.

(v) For all $\beta \leq \alpha$ and any prime p the relatively free group $F(\underline{\underline{W}}_\beta)$ is residually a finite p-group.

Proof

By virtue of the basis theorem, the equivalence of (i), (ii) and (iii) is obvious.

(iii) $\Rightarrow$ (iv). Let $x \in G = F(\underset{=}{W}_\beta)$, $x \neq 1$. Let $\rho : \underset{\sim}{A} \to G$ be a free description. Then there exists $\underset{\sim}{x} \in \underset{\sim}{A}$ such that $\underset{\sim}{x}\rho = x$. But then $E : \underset{\sim}{x} \to \underset{\sim}{y}\underset{\sim}{u}$ where $\sigma(\underset{\sim}{y}) = \xi$, $\underset{\sim}{y} \in \underset{\sim}{B}^W_{\xi^+}$ and $\sigma(\underset{\sim}{u}) \geq \xi^+$ and then $(\underset{\sim}{y}\underset{\sim}{u})\rho = x$. Thus $\underset{\sim}{y}$ is of the form $\underset{\sim}{y} = \underset{\sim}{b}_1^{\gamma_1}\underset{\sim}{b}_2^{\gamma_2}\ldots\underset{\sim}{b}_k^{\gamma_k}$ where $\sigma(\underset{\sim}{b}_1) = \sigma(\underset{\sim}{b}_2) = \ldots = \sigma(\underset{\sim}{b}_k) = \xi$. Since k is finite, there exists an integer c such that $wt(\underset{\sim}{b}_i) \leq c$ $(1 \leq i \leq k)$ so that $\underset{\sim}{y} \in \underset{\sim}{B}^W_{(c)}$. Collecting $\underset{\sim}{u}$ modulo $\gamma_{c+1}(G)$, $E : \underset{\sim}{u} \to \underset{\sim}{u}_1\underset{\sim}{u}_2$ where either $\underset{\sim}{u}_1$ or $\underset{\sim}{u}_2$ may be empty, but when they exist, $\underset{\sim}{u}_1 \in \underset{\sim}{B}^W_{(c)}$ and $wt(\underset{\sim}{u}_2) \geq c + 1$. But $\sigma(\underset{\sim}{u}_1) \geq \sigma(\underset{\sim}{u}) \geq \xi^+$ so $\underset{\sim}{y}\underset{\sim}{u}_1 \in \underset{\sim}{B}^W_{(c)}$ and $\underset{\sim}{y}\underset{\sim}{u} \neq \underset{\sim}{1}$. Then $x = (\underset{\sim}{y}\underset{\sim}{u}_1)\rho$ modulo $\gamma_{c+1}(G)$ so, by the basis theorem, $x \notin \gamma_{c+1}(G)$.

(iv) $\Rightarrow$ (ii). Let F be an absolutely free group and $x \notin W_\alpha(F)$. Let β' be the smallest element of W such that $x \notin W_{\beta'}(F)$. Then $\beta' \leq \alpha$ and it remains to show that there exists $\beta \in W$ such that $\beta' = \beta^+$. The factor group $F / W_{\beta'}(F)$ is residually nilpotent, so there exists an integer c such that $x \notin W_{\beta'}(F).\gamma_{c+1}(F)$. Then there exists $\underset{\sim}{x} \in \underset{\sim}{B}^W_{(c)}$, $\underset{\sim}{x} \neq \underset{\sim}{1}$, such that $\underset{\sim}{x}\rho = x$ modulo $W_{\beta'}(F).\gamma_{c+1}(F)$. Write $\underset{\sim}{x} = \underset{\sim}{b}_1^{\gamma_1}\underset{\sim}{b}_2^{\gamma_2}\ldots\underset{\sim}{b}_k^{\gamma_k}$ and $\beta = \sigma(\underset{\sim}{b}_1)$. Then $\beta < \beta'$ and

$\underset{\sim}{x}\rho \notin W_{\beta^+}(F) \cdot \gamma_{c+1}(F)$ so $x \notin W_{\beta^+}(F)$. Thus, by the choice of β' , $\beta' = \beta^+$.

(iv) $\Rightarrow$ (v). To show that $G = F(\underline{\underline{W}}_\beta)$ is residually a finite p-group it is sufficient to show that it is residually a torsion-free nilpotent group. By assumption it is residually nilpotent so that it is sufficient to show that, for each positive integer c , $G/\gamma_c(G)$ is torsion-free, and this follows immediately from the basis theorem.

(v) $\Rightarrow$ (iv) is obvious. ◇

DISTINCTNESS OF THE SUBGROUPS $W_\alpha(F)$

The question as to when the subgroups $W_\alpha(G)$ are non-trivial and when they are different from one another for an arbitrary group G is obviously a very complicated one, and one which depends very much on the special properties of G . However if F_τ is an absolutely free group of rank $\tau \geq 3$ the answer is very simple: they are all non-trivial and all different. This is proved in this section.

The proof when the number of generators is 3 depends on the property of W that any $\alpha \in W$ other than 1 or ∞ may be written in the form $\alpha = 1 + \alpha_0 + \alpha_1 + \ldots + \alpha_k$ where $k \geqq 0$, $1 = \alpha_0 \leqq \alpha_1 \leqq \ldots \leqq \alpha_k$ and for each r $(0 \leqq r < k)$, $1 + \alpha_0 + \alpha_1 + \ldots + \alpha_r \geqq \alpha_{r+1}$. The proof of this in turn involves something very much like a collecting process operating on such "formal sums" in W. This appalling prospect may be circumvented however by considering the properties of commutators in $\underset{\sim}{A}$ when the number τ of generators is infinite.

<u>Definition 2.8</u>

For any commutator $\underset{\sim}{c}$ and for any $i < \tau$, the "number of times $\underset{\sim}{c}$ mentions $\underset{\sim}{g}_i$", written $\mu_i(\underset{\sim}{c})$, is defined recursively over the weight of $\underset{\sim}{c}$.

(i) $\mu_i(\underset{\sim}{g}_j) = 1$ if $i = j$

$= 0$ if $i \neq j$,

(ii) $\mu_i([\underset{\sim}{a},\underset{\sim}{b}]) = \mu_i(\underset{\sim}{a}) + \mu_i(\underset{\sim}{b})$.

$M(\underset{\sim}{c})$ is the sequence $(\mu_i(\underset{\sim}{c}))_{i<\tau}$ ◇

Lemma 2.15

Suppose $\underset{\sim}{c}$ is a non-W-basic commutator which mentions each generator at most once (that is, $\mu_i(\underset{\sim}{c}) = 0$ or 1 for every $i < \tau$). Then there exists a commutator $\underset{\sim}{c}'$ such that $\sigma(\underset{\sim}{c}') = \sigma(\underset{\sim}{c})$, $M(\underset{\sim}{c}') = M(\underset{\sim}{c})$ and $\underset{\sim}{c}' < \underset{\sim}{c}$.

Proof

The argument proceeds by induction over the weight of $\underset{\sim}{c}$. Since $\underset{\sim}{c}$ is non-W-basic, at least one of the conditions of definition 1.13(A) must fail for this commutator.

Suppose $\underset{\sim}{c} = [\underset{\sim}{c}_1, \underset{\sim}{c}_2]$ and $\underset{\sim}{c}_1$ is non-W-basic. Then clearly $\underset{\sim}{c}_1$ mentions each generator at most once, and so there exists $\underset{\sim}{c}_1'$ such that $\sigma(\underset{\sim}{c}_1') = \sigma(\underset{\sim}{c}_1)$, $M(\underset{\sim}{c}_1') = M(\underset{\sim}{c}_1)$ and $\underset{\sim}{c}_1' < \underset{\sim}{c}_1$. Then the result is true with $\underset{\sim}{c}' = [\underset{\sim}{c}_1', \underset{\sim}{c}_2]$. The proof if $\underset{\sim}{c}_2$ is non-W-basic is similar.

Now suppose $\underset{\sim}{c} = [\underset{\sim}{c}_1, \underset{\sim}{c}_2]$ and $\underset{\sim}{c}_1 \leq \underset{\sim}{c}_2$. Then, since $\underset{\sim}{c}$ mentions each generator at most once, $\underset{\sim}{c}_1 \neq \underset{\sim}{c}_2$ so $\underset{\sim}{c}_1 < \underset{\sim}{c}_2$. Then $\underset{\sim}{c}' = [\underset{\sim}{c}_2, \underset{\sim}{c}_1]$ is the required commutator.

Finally, suppose $\underset{\sim}{c} = [\underset{\sim}{c}_{11}, \underset{\sim}{c}_{12}, \underset{\sim}{c}_2]$ and $\underset{\sim}{c}_{12} > \underset{\sim}{c}_2$. By virtue of the first possibility considered, it may be assumed that $[\underset{\sim}{c}_{11}, \underset{\sim}{c}_{12}]$ is W-basic so that $\underset{\sim}{c}_{11} > \underset{\sim}{c}_{12}$. Then $\underset{\sim}{c}' = [\underset{\sim}{c}_{11}, \underset{\sim}{c}_2, \underset{\sim}{c}_{12}]$ is the required commutator. ◇

Corollary

Let the number τ of generators of $\underset{\sim}{A}$ be infinite and let $\alpha \in W$ $(\alpha \neq \infty)$. Then there exists a W-basic commutator of s semiweight α in $\underset{\sim}{A}$.

Proof

By an easy induction over α , there exists a commutator $\underset{\sim}{c} \in \underset{\sim}{A}$ which mentions each generator at most once for which $\sigma(\underset{\sim}{c}) = \alpha$. If $\underset{\sim}{c}$ is non-W-basic, then $\underset{\sim}{c}$ may be replaced by an earlier commutator with the same properties. Thus there exists a W-basic commutator with these properties since the set of all commutators is well-ordered by $\leqq$. ◇

Lemma 2.16

If $\alpha \in W$, $\alpha \neq 1$ or ∞ , then it may be written in the form $\alpha = 1 + \alpha_0 + \alpha_1 + \ldots + \alpha_k$ $(k \geqq 0)$ where $1 = \alpha_0 \leqq \alpha_1 \leqq \ldots$ $\ldots \leqq \alpha_k$ and for each r $(0 \leqq r < k)$,

$1 + \alpha_0 + \alpha_1 + \ldots + \alpha_r \geqq \alpha_{r+1}$.

Proof

Form an algebra $\underset{\sim}{A}$ of expressions with an infinite number of generators. Then by the corollary to lemma 2.15, there exists a W-basic commutator $\underset{\sim}{b}$ of semiweight α in $\underset{\sim}{A}$. But then $\underset{\sim}{b}$ may be written in the form $\underset{\sim}{b} = [g_i, g_j, \underset{\sim}{b}_1, \underset{\sim}{b}_2, \ldots, \underset{\sim}{b}_k]$ $(k \geq 0)$ where $g_i > g_j \leq \underset{\sim}{b}_1 \leq \underset{\sim}{b}_2 \leq \ldots \leq \underset{\sim}{b}_k$ and, for $1 \leq r \leq k$, $[g_i, g_j, \underset{\sim}{b}_1, \underset{\sim}{b}_2, \ldots \underset{\sim}{b}_k] > \underset{\sim}{b}_{r+1}$. Writing $\alpha_i = \sigma(\underset{\sim}{b}_i)$ $(1 \leq i \leq k)$, the result follows. ◇

Having obtained this result the three-generator case may be considered.

Theorem 2.4

Let W be a semiweight range and $\alpha \in W$ $(\alpha \neq \infty)$.

(i) If the number τ of generators of $\underset{\sim}{A}$ is at least 3 , then there exists a W-basic commutator of semiweight α in $\underset{\sim}{A}$.

(ii) If $\alpha < \beta$ and the rank τ of the absolutely free group F is at least 3 , then $W_\beta(F)$ is a proper subgroup of $W_\alpha(F)$.

Proof

First, notice that (ii) follows immediately from (i) by the basis theorem. It remains to prove (i). When $\alpha = 1$ the result is trivial – g_o is the required commutator.

Now suppose $\alpha > 1$. It is proved by induction over k that if α may be written in the form $\alpha = 1 + \alpha_0 + \alpha_1 + \dots + \alpha_k$, where $\alpha_r \leqq \alpha_{r+1} \leqq 1 + \alpha_0 + \alpha_1 + \dots + \alpha_r$ $(0 \leqq r < k)$, then there exists at least three W-basic commutators of the form $\underset{\sim}{a} = [g_i, \underset{\sim}{a}_0, \underset{\sim}{a}_1, \dots \underset{\sim}{a}_k]$ in $\underset{\sim}{A}$ where $\sigma(\underset{\sim}{a}_r) = \alpha_r$ $(0 \leqq r \leqq k)$.

Suppose then that α may be written in the above form. If $k = 0$, then $\alpha = 1 + 1$ and the required W-basic commutators are $[g_1, g_0]$, $[g_2, g_0]$ and $[g_2, g_1]$.

Now suppose $k \geqq 1$. Then $\alpha_{k-1} \leqq \alpha_k \leqq 1 + \alpha_0 + \alpha_1 + \dots + \alpha_{k-1}$ and there exist three W-basic commutators $\underset{\sim}{a} = [g_i, \underset{\sim}{a}_0, \underset{\sim}{a}_1, \dots, \underset{\sim}{a}_{k-1}]$, $\underset{\sim}{a}' = [g_i', \underset{\sim}{a}_0', \underset{\sim}{a}_1', \dots, \underset{\sim}{a}_{k-1}']$ and $\underset{\sim}{a}'' = [g_i'', \underset{\sim}{a}_0'', \underset{\sim}{a}_1'', \dots, \underset{\sim}{a}_{k-1}'']$ where $\sigma(\underset{\sim}{a}_r) = \sigma(\underset{\sim}{a}_r') = \sigma(\underset{\sim}{a}_r'') = \alpha_r$ $(0 \leqq r < k)$. But $\alpha_{k-1} < 1 + \alpha_0 + \alpha_1 + \dots + \alpha_k$ and so three possibilities must be considered separately:

If $\alpha_{k-1} = \alpha_k < 1 + \alpha_0 + \alpha_1 + \dots + \alpha_{k-1}$, then the required commutators are $[\underset{\sim}{a}, \underset{\sim}{a}_{k-1}'']$, $[\underset{\sim}{a}', \underset{\sim}{a}_{k-1}'']$ and $[\underset{\sim}{a}'', \underset{\sim}{a}_{k-1}'']$.

If $\alpha_{k-1} < \alpha_k < 1 + \alpha_0 + \alpha_1 + \dots + \alpha_{k-1}$, let $\underset{\sim}{c}$ be any one of the three W-basic commutators of semiweight α_k , and then $[\underset{\sim}{a}, \underset{\sim}{c}]$, $[\underset{\sim}{a}', \underset{\sim}{c}]$ and $[\underset{\sim}{a}'', \underset{\sim}{c}]$ are the required commutators.

Finally, if $\alpha_{k-1} < \alpha_k = 1 + \alpha_0 + \alpha_1 + \dots + \alpha_{k-1}$, then since $\underset{\sim}{a}$, $\underset{\sim}{a}'$ and $\underset{\sim}{a}''$ are all different, it may be assumed that $\underset{\sim}{a} < \underset{\sim}{a}' < \underset{\sim}{a}''$. Then $[\underset{\sim}{a}', \underset{\sim}{a}]$, $[\underset{\sim}{a}'', \underset{\sim}{a}]$ and $[\underset{\sim}{a}'', \underset{\sim}{a}']$ are the required commutators. ◇

As regards whether this is best possible, clearly one generator is not good enough to do more than distinguish $W_1(F)$ from the rest. For some particular semiweight ranges, such as $\overline{N}$, two generators are enough. An example for which two generators are not enough must use the language and results of the next chapter. If $K = (k_i)_{i=1}^{\infty}$ is the sequence $k_i = 2$ (all i) , then $Q^K_{\delta_2}(F_2) = \delta^2(F_2)$ and $Q^K_{\delta_2+1}(F_2) = \delta^2(F_2) \cap \gamma_5(F_2)$. But it is well known that for a two-generator group these subgroups are the same.

CHAPTER 3

POLYNILPOTENT SEMIWEIGHTS

THE SEMIWEIGHT

Definition 3.1

Let $K = (k_i)_{i=1}^{\infty}$ be an infinite sequence of integers, each ≥ 2. For each non-negative integer r, let K_r be the finite sequence $K_r = (k_i)_{i=1}^{r}$; in particular K_o is the "empty" sequence. For any group G and each K_r the subgroup $P_{K_r}(G)$ is defined recursively over r by

$$P_{K_o}(G) = G$$

$$P_{K_r}(G) = \gamma_{k_r}(P_{K_{r-1}}(G)) \qquad (r \geq 1) .$$

The resulting series of subgroups of G is called the "polynilpotent series" of G of "type K".

A group G is "polynilpotent of type K_r" if $P_{K_r}(G) = \{1\}$. The class of all such groups is a variety, the "polynilpotent variety" of "type K_r", denoted $\underline{\underline{P}}_{K_r}$. ◇

For each sequence K a semiweight range Q^K will be defined which will have the property that all the varieties $\underline{\underline{P}}_{K_r}$ will be among the varieties $\underline{\underline{Q}}_{\alpha}^{K}$.

<u>Definition 3.2</u>

Let $K = (k_i)_{i=1}^{\infty}$ be a sequence of integers, each $\geqq 2$. Then

(A) Let Q^K be the set of all functions $\varphi : \omega \to \omega$ satisfying

$$\left.\begin{array}{ll}\text{(i)} & \varphi(j-1) \geqq k_j \varphi(j) \\ \text{(ii)} & \varphi(j-1) \geqq k_j \Rightarrow \varphi(j) \geqq 1\end{array}\right\} \text{ for all } j \geqq 1.$$

(iii) $\varphi(0) \geqq 1$,

together with an extra element called ∞ . The function $1 \in Q^K$ is defined $1(0) = 1$, $1(j) = 0$ $(j \geqq 1)$.

It follows from (i) and (ii), since each $k_j \geqq 2$, that for each function φ there exists an integer J_φ such that $\varphi(j) \neq 0 \Leftrightarrow j \leqq J_\varphi$.

(B) Addition is defined on Q^K as follows: if φ , ψ are functions (that is, $\neq \infty$), then

(i) If $J_\varphi = J_\psi$ ($=J$ say) and $\varphi(J) + \psi(J) \geqq k_{J+1}$ then

$$\begin{aligned}(\varphi + \psi)(j) &= \varphi(j) + \psi(j) && (j \neq J+1), \\ &= 1 && (j = J+1),\end{aligned}$$

(ii) Otherwise $(\varphi + \psi)(j) = \varphi(j) + \psi(j)$ (for all j) .

Addition is extended to encompass ∞ by

$$\infty + \infty = \varphi + \infty = \infty + \varphi = \infty \quad (\text{for all } \varphi \in Q^K) .$$

(C) The functions are ordered lexicographically from the right: if φ , ψ are functions ($\neq \infty$) , then $\varphi < \psi$ if and only if there exists $j_o \in \omega$ such that $\varphi(j_o) < \psi(j_o)$ and

$j > j_o \Rightarrow \varphi(j) = \psi(j)$.

The ordering is extended to encompass ∞ by: if φ is a function then $\varphi < \infty$.

(D) Subject to the proof in the next few lemmas that Q^K , with the definition of addition and ordering just given, is indeed a semiweight range, it is called the "polynilpotent" semiweight range of "type K". The associated semiweight is denoted π^K . ◇

<u>Lemma 3.1</u>

Q^K is closed under addition.

<u>Proof</u>

This follows easily from a comparison of parts (A) and (B) of the definition. ◇

For use in the following lemmas, some simple properties of the ordering $\leqq$ of the functions should be remarked:

If $\varphi(j) \leqq \psi(j)$ for all j, then $\varphi \leqq \psi$.

If $J_\varphi < J_\psi$ then $\varphi < \psi$.

Conversely, if $\varphi \leqq \psi$ then $J_\varphi \leqq J_\psi$.

Lemma 3.2

Q^K is well-ordered by $\leqq$. The least element is the function 1 and the greatest, the element ∞.

Proof

A non-empty subset X of Q^K either consists of the element ∞ alone, in which case it has a least element trivially, or else it contains a function $\varphi \neq \infty$. But then the set of all functions $\leqq \varphi$ in X consists of functions whose support is a subset of the (finite) support of φ. Since the order is lexicographic the result follows. ◇

Notice that, as a consequence of the definition of the ordering $\leqq$ of Q^K, the successor α^+ of any function $\alpha \in Q^K$ is $\alpha + 1$.

Lemma 3.3

If α_1 , α_2 , β are functions ($\neq \infty$) in Q^K and $\alpha_1 < \alpha_2$, then $\alpha_1 + \beta < \alpha_2 + \beta$.

Proof

Since $\alpha_1 < \alpha_2$, $J_{\alpha_1} \leqq J_{\alpha_2}$ and there exists $j_o \in \omega$ such that $\alpha_1(j_o) < \alpha_2(j_o)$ and $j > j_o \Rightarrow \alpha_1(j) = \alpha_2(j)$. Four possibilities must be considered separately.

(i) Suppose $J_{\alpha_1} = J_{\alpha_2} = J_\beta$ (= J say) . Then $j_o \leqq J$. Now $(\alpha_1 + \beta)(J + 1) = 0$ or 1 and $(\alpha_2 + \beta)(J + 1) = 0$ or 1, and if $(\alpha_1 + \beta)(J + 1) = 1$ then $\alpha_1(J) + \beta(J) \geqq k_{J+1}$. But $j_o \leqq J$, so $\alpha_1(J) \leqq \alpha_2(J)$. Thus $\alpha_2(J) + \beta(J) \geqq k_{J+1}$ and then $(\alpha_2 + \beta)(J + 1) = 1$ also, and so in any case, $(\alpha_1 + \beta)(J + 1) \leqq (\alpha_2 + \beta)(J + 1)$. Thus $j > j_o \Rightarrow (\alpha_1 + \beta)(j) \leqq (\alpha_2 + \beta)(j)$ and, since $j_o \leqq J$, $(\alpha_1 + \beta)(j_o) = \alpha_1(j_o) + \beta(j_o) < \alpha_2(j_o) + \beta(j_o) = (\alpha_2 + \beta)(j_o)$. This means that $\alpha_1 + \beta < \alpha_2 + \beta$.

(ii) Suppose $J_\beta \neq J_{\alpha_1} = J_{\alpha_2}$ (= J say) . Again $j_o \leqq J$. Then $j > j_o$ $(\alpha_1 + \beta)(j) = \alpha_1(j) + \beta(j) = \alpha_2(j) + \beta(j) =$ $= (\alpha_2 + \beta)(j)$ and

$(\alpha_1 + \beta)(j_o) = \alpha_1(j_o) + \beta(j_o) < \alpha_2(j_o) + \beta(j_o) = (\alpha_2 + \beta)(j_o)$ which means that $\alpha_1 + \beta < \alpha_2 + \beta$.

(iii) Suppose $J_\beta = J_{\alpha_1} < J_{\alpha_2}$. Let $J = J_\beta = J_{\alpha_1}$. Then $j_o = J_{\alpha_2}$. For $j > J + 1$, $(\alpha_1 + \beta)(j) = 0$ and $(\alpha_2 + \beta)(j) = \alpha_2(j) \geqq 0$; also $(\alpha_1 + \beta)(J + 1) = 0$ or 1 $(\alpha_2 + \beta)(J + 1) = \alpha_2(J + 1) \geqq 1$. Thus $j \geqq J + 1 \Longrightarrow (\alpha_1 + \beta)(j) \leqq (\alpha_2 + \beta)(j)$. But $\alpha_1(J) < k_{J+1}$ and $\alpha_2(J) \geqq k_{J+1}$ so $(\alpha_1 + \beta)(J) < (\alpha_2 + \beta)(J)$, and thus $\alpha_1 + \beta < \alpha_2 + \beta$.

(iv) Finally, suppose $J_\beta \neq J_{\alpha_1} < J_{\alpha_2}$. Then

$j > j_o \Longrightarrow (\alpha_1 + \beta)(j) = \alpha_1(j) + \beta(j) = \alpha_2(j) + \beta(j) \leqq (\alpha_2 + \beta)(j)$

and $(\alpha_1 + \beta)(j_o) = \alpha_1(j_o) + \beta(j_o) < \alpha_2(j_o) + \beta(j_o) = (\alpha_2 + \beta)(j_o)$.

◇

<u>Lemma 3.4</u>

If α , β , γ are functions ($\neq \infty$) in Q^K and $\alpha \leqq \beta \leqq \gamma$, then for all $j \in \omega$, $(\gamma + \beta + \alpha)(j) = (\gamma + \alpha + \beta)(j) \leqq (\beta + \alpha + \gamma)(j)$.

Proof

Again four possibilities must be considered separately, according to whether $J_\alpha = J_\beta$ or not and whether $J_\beta = J_\gamma$ or not.

(i) Suppose $J_\alpha = J_\beta = J_\gamma$ ($=J$ say). Then there are two subcases:

(a) If $\alpha(J) + \beta(J) + \gamma(J) < k_{J+1}$ then for all j

$(\gamma + \beta + \alpha)(j) = (\gamma + \alpha + \beta)(j) = (\beta + \alpha + \gamma)(j) = \alpha(j) + \beta(j) + \gamma(j)$.

(b) If $\alpha(J) + \beta(J) + \gamma(J) \geqq k_{J+1}$ then

$$(\gamma + \beta + \alpha)(j) = (\gamma + \alpha + \beta)(j) = (\beta + \alpha + \gamma)(j) = \alpha(j) + \beta(j) + \gamma(j) \quad (\text{for } j \neq J+1)$$

$$= 1 \quad (\text{for } j = J+1) .$$

(ii) Suppose $J_\alpha = J_\beta < J_\gamma$. Let $J = J_\alpha = J_\beta$.

Then $J_\beta \neq J_\gamma$ so $(\gamma + \beta)(j) = \beta(j) + \gamma(j)$ for all j .

Then $J_{\gamma+\beta} = J_\gamma \neq J_\alpha$ so $(\gamma + \beta + \alpha)(j) = \alpha(j) + \beta(j) + \gamma(j)$ for all j . Similarly

$(\gamma + \alpha + \beta)(j) = \alpha(j) + \beta(j) + \gamma(j)$ for all j .

Now $J_\alpha = J_\beta = J$ so $J_{\beta+\alpha} = J$ or $J + 1$. In either case

$(\beta + \alpha)(j) \geqq \alpha(j) + \beta(j)$ for all j .

Again $J_{\beta+\alpha+\gamma} = J_\gamma$ or $J_\gamma + 1$ (it is possible that $J_{\beta+\alpha} = J + 1 = J_\gamma$) . In any case

$(\beta + \alpha + \gamma)(j) \geqq (\beta + \alpha)(j) + \gamma(j) \geqq \alpha(j) + \beta(j) + \gamma(j)$ for all j .

(iii) Suppose $J_\alpha < J_\beta = J_\gamma$ ($= J$ say) . There are two subcases depending on the value of $\beta(J) + \gamma(J)$.

(a) If $\beta(J) + \gamma(J) \geq k_{J+1}$, then

$$\begin{aligned}(\gamma + \beta)(j) &= \beta(j) + \gamma(j) \qquad (j \neq J + 1) ,\\ &= 1 \qquad (j = J + 1)\end{aligned}$$

and so $J_{\gamma+\beta} = J + 1 = J_\alpha$. Thus

$$\begin{aligned}(\gamma + \beta + \alpha)(j) &= \alpha(j) + \beta(j) + \gamma(j) \qquad (j \neq J + 1) ,\\ &= 1 \qquad (j = J + 1) .\end{aligned}$$

Now $J_\gamma \neq J_\alpha$, so $(\gamma + \alpha)(j) = \alpha(j) + \gamma(j)$ (for all j). Thus $J_{\gamma+\alpha} = J_\gamma = J$. But then $(\gamma + \alpha)(J) = \gamma(J)$ so that $(\gamma + \alpha)(J) + \beta(J) \geq k_{J+1}$. Thus

$$\begin{aligned}(\gamma + \alpha + \beta)(j) &= (\gamma + \alpha)(j) + \beta(j) \qquad (j \neq J + 1) ,\\ &= 1 \qquad (j = J + 1) ;\end{aligned}$$

that is, $(\gamma + \alpha + \beta)(j) = (\gamma + \beta + \alpha)(j)$ for all j .

Similarly $(\beta + \alpha + \gamma)(j) = (\gamma + \beta + \alpha)(j)$ for all j .

(b) If $\beta(J) + \gamma(J) < k_{J+1}$, then $(\gamma + \beta)(j) = \beta(j) + \gamma(j)$ for all j . Then $J_{\gamma+\beta} = J \neq J$ so that

$(\gamma + \beta + \alpha)(j) = \alpha(j) + \beta(j) + \gamma(j)$ for all j .

Now $J_\gamma = J \neq J_\alpha$ so that $(\gamma + \alpha)(j) = \alpha(j) + \gamma(j)$ for all j .

Thus $J_{\gamma+\alpha} = J = J_\beta$. But

$(\gamma + \alpha)(J) + \beta(J) = \gamma(J) + \beta(J) < k_{J+1}$, so that

$(\gamma + \alpha + \beta)(j) = \alpha(j) + \beta(j) + \gamma(j) = (\gamma + \beta + \alpha)(j)$ for all j .

Similarly $(\beta + \alpha + \gamma)(j) = (\gamma + \beta + \alpha)(j)$ for all j .

(iv) Finally, suppose $J_\alpha < J_\beta < J_\gamma$.

Then $(\gamma + \beta)(j) = \beta(j) + \gamma(j)$ for all j , and then

$J_{\gamma+\beta} = J_\gamma \neq J_\alpha$ so that $(\gamma + \beta + \alpha)(j) = \alpha(j) + \beta(j) + \gamma(j)$

for all j .

Similarly $(\gamma + \alpha + \beta)(j) = (\beta + \alpha + \gamma)(j) = (\gamma + \beta + \alpha)(j)$

for all j . ◇

Definition 3.3

With the notation of definition 3.2, for each non-negative integer r a function $\delta_r^K \in Q^K$ is defined by

$$\begin{aligned} \delta_r^K(j) &= \prod_{i=j+1}^{r} k_i && \text{for } j < r , \\ &= 1 && \text{for } j = r \\ &= 0 && \text{for } j > r . \end{aligned}$$ ◇

Lemma 3.5

(i) $\delta_r^K(j)$ may be defined by its properties

$$\begin{aligned} \delta_r^K(j - 1) &= k_j\, \delta_r^K(j) && (j < r) , \\ &= 1 && (j = r) , \\ &= 0 && (j > r) . \end{aligned}$$

(ii) $\delta_r^K \in Q^K$.

Proof

(i) follows immediately from the definition and (ii) from (i).

◇

Lemma 3.6

$Q^K \setminus \{\infty\}$ is generated by the function 1 under addition.

Proof

By virtue of lemmas 3.2 and 3.3, it is sufficient to show that for each $\varphi \in Q^K$ other than 1 or ∞, there exist $\psi_1, \psi_2 \in Q^K$ such that $\varphi = \psi_1 + \psi_2$. Three cases must be considered, depending on the values of $\varphi(J-1)$ and $\varphi(J)$ where $J = J_\varphi$.

(i) Suppose that $\varphi(J) \geqq 2$. Let $\psi_1 = \delta_J^K$ and define ψ_2 by $\psi_2(j) = \varphi(j) - \psi_1(j)$ for all j. Then for all j,
$\psi_2(j-1) = \varphi(j-1) - \psi_1(j-1) \geqq k_j\varphi(j) - k_j\psi_1(j) = k_j\psi_2(j)$,
$\psi_2(j-1) \geqq k_j \Rightarrow j \leqq J$ (since $\varphi(J) < k_{J+1}$) $\Rightarrow \psi_2(j) \geqq 1$
and $\psi_2(0) \geqq 1$ trivially.
Hence $\psi_2 \in Q^K$. But now $J_{\psi_2} = J_{\psi_1} = J$ and
$\psi_1(J) + \psi_2(J) = \varphi(J) < k_{J+1}$ so $(\psi_1 + \psi_2)(j) = \psi_1(j) + \psi_2(j) =$
$= \varphi(j)$ for all j.

(ii) Suppose that $\varphi(J) = 1$ and $\varphi(J-1) > k_J$. Write $\psi_1 = \delta_{J-1}^K$ and define ψ_2 by $\psi_2(j) = \varphi(j) - \psi_1(j)$ for all j; the argument goes as before.

(iii) Suppose that $\varphi(J) = 1$ and $\varphi(J - 1) = k_J$. Writing $\psi_1 = \delta^K_{J-1}$ and defining ψ_2 by $\psi_2(j) = \varphi(j) - \psi_1(j)$ for all $j \neq J$, $\psi_2(J) = 0$, the argument goes as before. ◇

Lemma 3.7

With the notation of definition 3.2, Q^K is a semiweight range.

Proof

The various parts of definition 1.7 are checked separately.

(i) $\leq$ well-orders Q^K, 1 is the least element and ∞ the greatest by lemma 3.2.

(ii) Q^K is closed under addition by lemma 3.1 and Q^K $\{\infty\}$ is generated by 1 under addition by lemma 3.6.

(iii) Addition is commutative by definition 3.2(B).

(iv) Suppose $\alpha, \beta \in Q^K$, $\alpha < \infty$ and $\beta < \infty$. Then α and β are functions and then so is $\alpha + \beta$ by definition 3.2(B). Now suppose $\alpha, \beta \in Q^K$ and $\alpha < \infty$. If $\beta = \infty$ then $\alpha < \infty = \infty + \beta = \alpha + \beta$. Otherwise α and β are both functions and $(\alpha + \beta)(j) \geq \alpha(j)$ for all j. But $(\alpha + \beta)(0) \geq \alpha(0) + 1$. Hence $\alpha < \alpha + \beta$.

$\alpha + \infty = \infty$ by definition 3.2(B).

(v) Suppose $\alpha_1, \alpha_2, \beta \in Q^K$, $\alpha_1 < \alpha_2$ and $\beta < \infty$. If $\alpha_2 = \infty$ then $\alpha_1 < \infty$ so $\alpha_1 + \beta < \infty = \alpha_2 + \beta$. Otherwise they are all functions and $\alpha_1 + \beta < \alpha_2 + \beta$ by lemma 3.3.

(vi) Suppose $\alpha, \beta, \gamma \in Q^K$ and $\alpha \leqq \beta \leqq \gamma$. If any one of these is ∞ then $\gamma + \beta + \alpha = \gamma + \alpha + \beta = \beta + \alpha + \gamma = \infty$. Otherwise they are all functions and

$\gamma + \beta + \alpha = \gamma + \alpha + \beta \leqq \beta + \alpha + \gamma$ by lemma 3.4. ◇

Lemma 3.8

The limiting values of Q^K are

(i) The elements 1 and ∞ and

(ii) the functions $\varphi \in Q^K$ such that $\varphi(0) = k_1\varphi(1)$ and $\varphi(1) \geqq 2$.

Proof

The elements 1 and ∞ are limiting values trivially. Now suppose φ is a function in Q^K for which $\varphi(0) = k_1\varphi(1)$ and $\varphi(1) \geqq 2$, and suppose $\xi < \varphi$. If $J_\xi = 0$ then $(\xi + 1)(1) \leqq 1$, and $(\xi + 1)(j) = 0$ for $j > 1$. Hence $\xi + 1 < \varphi$. If $J_\xi \geqq 1$, then (since $\xi(1) = \varphi(1) \Rightarrow \xi(0) \geqq \varphi(0)$) $j_o \geqq 1$, where j_o is the integer such that $\xi(j_o) < \varphi(j_o)$ and $j > j_o \Rightarrow \xi(j) = \varphi(j)$. But then $(\xi + 1)(j) = \xi(j)$ for all $j \geqq 1$, so $(\xi + 1)(j) = \xi(j)$ for all $j \geqq 1$, so $\xi + 1 < \varphi$.

Thus φ is a limiting value.

Suppose conversely that φ is none of these elements. Then φ is a function and $\varphi(0) \geqq 2$. If $J_{\varphi} = 0$, then define ξ by: $\xi(0) = \varphi(0) - 1$ and $\xi(j) = 0$ for $j \geqq 1$. Then $\xi < \varphi$ and $\xi + 1 = \varphi$ so φ is not a limiting value.

It may now be assumed that $J_{\varphi} \geqq 1$ and that one of the conditions of (ii) break down for φ .

Suppose first that $\varphi(0) > k_1\varphi(1)$. Defining ξ by: $\xi(0) = \varphi(0) - 1$, $\xi(j) = \varphi(j)$ for $j \geqq 1$, it follows that $\xi \in Q^K$, $\xi < \varphi$ and $\xi + 1 = \varphi$.

Now suppose that $\varphi(1) = 1$. It may be assumed that $\varphi(0) = k_1$ by virtue of the forgoing case. Defining ξ by: $\xi(0) = k_1 - 1$, $\xi(j) = 0$ for $j \geqq 1$, it follows again that $\xi \in Q^K$, $\xi < \varphi$ and $\xi + 1 = \varphi$. $\diamond$

PARTIAL COLLECTABILITY

Suppose W and W' are two arbitrary semiweight ranges. Then for any $\alpha \in W$ the set $\underset{\sim}{W}_\alpha$ (definition 1.9) is a subalgebra of $\underset{\sim}{A}$ and hence a describable algebra. Consequently, for any $\beta \in W'$ the set $W'_\beta(\underset{\sim}{W}_\alpha)$ is defined (definition 1.10). Further, this is a fully invariant subalgebra of $\underset{\sim}{A}$ and defines a product variety of groups, since for any description $\rho : \underset{\sim}{A} \rightarrow G$ of a group G, $(W'_\beta(\underset{\sim}{W}_\alpha))\rho = W'_\beta(W_\alpha(G))$.

Lemma 3.9

Suppose $K = (k_i)_{i=1}^{\infty}$ is any sequence of integers, each ≥ 2 and $K' = (k'_i)_{i=1}^{\infty}$ is defined by $k'_i = k_{i+1}$ (all $i \geq 1$). Write $Q = Q^K$ and $Q' = Q^{K'}$. Then for any function $\varphi \in Q$ with the property $\varphi(0) = k_1\varphi(1)$ and any group G , $Q_\varphi(G) = Q'_{\varphi'}(\gamma_{k_1}(G))$, where φ' is the function defined by $\varphi'(j) = \varphi(j + 1)$ for all j .

Proof

First it is necessary to observe that $\varphi' \in Q'$ so that $Q'_{\varphi'}(G)$ has meaning. Let $\rho : \underset{\sim}{A} \rightarrow G$ be any description of G and write $\pi = \pi^K$, $\pi' = \pi^{K'}$. For any $\alpha \in Q$ such that $J_\alpha \geq 1$, define α' by $\alpha'(j) = \alpha(j + 1)$ for all j , and a closure

$\overline{\alpha}$ by $\overline{\alpha}(j) = \alpha(j)$ for all $j \geqq 1$, $\overline{\alpha}(0) = k_1\alpha(1)$.

Clearly $\alpha' \in Q'$ and $\overline{\alpha} \in Q$. The following properties are easily verified:

$\overline{\overline{\alpha}} = \overline{\alpha}$

$\overline{\alpha} \leqq \alpha$

$\alpha' + \beta' = (\alpha + \beta)'$. $\overline{\alpha} + \overline{\beta} = \overline{\alpha + \beta}$

$\overline{\alpha} < \overline{\beta} \Leftrightarrow \alpha' < \beta' \Rightarrow \alpha < \beta$

$\alpha \leqq \beta \Rightarrow \overline{\alpha} \leqq \overline{\beta} \Leftrightarrow \alpha' \leqq \beta'$.

(i) $Q_\varphi(G) \leqq Q'_{\varphi'}(\gamma_{k_1}(G))$. The argument proceeds by proving that $\pi(\underset{\sim}{x}) \geqq \alpha$ $(J_\alpha \geqq 1)$ $\underset{\sim}{x}\rho \in Q'_{\alpha'}(\gamma_{k_1}(G))$ by induction over the height of $\underset{\sim}{x}$. If $ht(\underset{\sim}{x}) = 1$ then either $\underset{\sim}{x} = \underset{\sim}{1}$ in which case $\underset{\sim}{x}\rho = 1 \in Q'_{\varphi'}(\gamma_{k_1}(G))$ or else $\underset{\sim}{x} = \underset{\sim}{g_i}$ in which case $\pi(\underset{\sim}{x}) = 1$ so that $J_\alpha < 1$ and the result is vacuously true.

Now suppose that $ht(\underset{\sim}{x}) > 1$ and the result is true for all smaller heights. Then there are the usual three possibilities. If $\underset{\sim}{x} = \underset{\sim}{u}^{-1}$ or $\underset{\sim}{x_1}\underset{\sim}{x_2}$ the result follows immediately from the fact that $Q'_{\varphi'}(\gamma_{k_1}(G))$ is a subgroup. Now suppose $\underset{\sim}{x} = [\underset{\sim}{x_1}, \underset{\sim}{x_2}]$. Write $\pi(\underset{\sim}{x}) = \psi$ (so that $\psi \geqq \alpha$), $\pi(\underset{\sim}{x_1}) = \psi_1$ and $\pi(\underset{\sim}{x_2}) = \psi_2$. Then $\psi = \psi_1 + \psi_2$. Several subcases must now be treated separately.

(a) $J_{\psi_1} = J_{\psi_2} = 0$. Then $\psi(1) = 1$ since $\psi > \alpha$ and $J_\alpha \geqq 1$, and so $\psi' = 1$ and $\psi(0) \geqq k_1$. Thus $wt(\underset{\sim}{x}) = \psi(0) \geqq k_1$ and so $\underset{\sim}{x}\rho \in \gamma_{k_1}(G) = Q_1'(\gamma_{k_1}(G)) = Q_{\alpha'}'(\gamma_{k_1}(G))$.

(b) $J_{\psi_1} = 0$, $J_{\psi_2} \geqq 1$. Then $\overline{\psi} = \overline{\psi}_2$ and so $\psi_2 \geqq \overline{\psi}_2 = \overline{\psi} \geqq \overline{\alpha} = \alpha$. Thus, by the inductive hypothesis, $\underset{\sim}{x}_2\rho \in Q_{\alpha'}'(\gamma_{k_1}(G))$; but this is a normal subgroup of G , so $\underset{\sim}{x}\rho = [\underset{\sim}{x}_1, \underset{\sim}{x}_2]\rho \in Q_{\alpha'}'(\gamma_{k_1}(G))$.

(c) $J_{\psi_1} \geqq 1$, $J_{\psi_2} = 0$. The argument in this case is similar to that for case (b).

(d) $J_{\psi_1} \geqq 1$, $J_{\psi_2} \geqq 1$. Then ψ_1' and ψ_2' exist and by the inductive hypothesis $\underset{\sim}{x}_1\rho \in Q_{\psi_1'}'(\gamma_{k_1}(G))$ and $\underset{\sim}{x}_2\rho \in Q_{\psi_2'}'(\gamma_{k_1}(G))$. But then $\underset{\sim}{x}\rho \in Q_{\psi_1'+\psi_2'}'(\gamma_{k_1}(G)) = Q_{\psi'}'(\gamma_{k_1}(G)) \leqq Q_{\alpha'}'(\gamma_{k_1}(G))$.

(ii) $Q_{\varphi'}'(\gamma_{k_1}(G)) \leqq Q_{\varphi}(G)$. By definitions 1.3 and 1.9, $\overline{\underset{\sim}{N}}_{k_1}$ is the set of all expressions of weight $\geqq k_1$. It has just been remarked that it is a describable algebra, so there exists a description $\rho' : \underset{\sim}{A}' \to \overline{\underset{\sim}{N}}_{k_1}$, where $\underset{\sim}{A}'$ is some algebra of expressions: $\underset{\sim}{A}'$ may be the same as $\underset{\sim}{A}$ provided the latter has enough generators. Here two free algebras of expressions are involved so a little care must be taken with the definitions of $\underset{\sim}{Q}_{\varphi}$ and $\underset{\sim}{Q}_{\varphi'}'$ (see definition 1.9, where it was assumed that only one algebra of expressions was involved). It will be assumed that $\underset{\sim}{Q}_{\varphi} \subseteqq \underset{\sim}{A}$ and $\underset{\sim}{Q}_{\varphi'}' \subseteqq \underset{\sim}{A}'$ or, more precisely,

$\underset{\sim}{Q}_{\varphi} = \{ \underset{\sim}{x} : \underset{\sim}{x} \in \underset{\sim}{A} , \pi(\underset{\sim}{x}) \geqq \varphi \}$ and

$\underset{\sim}{Q}_{\varphi'}' = \{ \underset{\sim}{x}' : \underset{\sim}{x}' \in \underset{\sim}{A}' , \pi'(\underset{\sim}{x}') \geqq \varphi' \}$.

Then $Q_{\varphi}(G) = \underset{\sim}{Q}_{\varphi}\rho$ and $Q_{\varphi'}'(\gamma_{k_1}(G)) = \underset{\sim}{Q}_{\varphi'}'\rho'\rho$. It is now sufficient to show that $\underset{\sim}{Q}'_{\varphi'}\rho' \subseteq \underset{\sim}{Q}_{\varphi}$. This is proved by showing that $\underset{\sim}{x}' \in \underset{\sim}{Q}'_{\varphi'} \implies \underset{\sim}{x}'\rho' \in \underset{\sim}{Q}_{\varphi}$ by induction over $\underset{\sim}{x}'$.

If $ht(\underset{\sim}{x}') = 1$ then either $\underset{\sim}{x}' = \underset{\sim}{1}'$ (the unit element of $\underset{\sim}{A}'$), in which case $\underset{\sim}{x}'\rho' = \underset{\sim}{1} \in \underset{\sim}{Q}_{\varphi}$, or else $\underset{\sim}{x}' = \underset{\sim}{g}'_i$ (one of the generators of $\underset{\sim}{A}'$) in which case $\pi'(\underset{\sim}{x}') = 1$, so that $\varphi(0) = k_1$ and $\varphi(1) = 1$. Thus $\underset{\sim}{Q}_{\varphi} = \overline{\underset{\sim}{N}}_{k_1}$ and so $\underset{\sim}{x}'\rho' \in \underset{\sim}{A}'\rho' = \overline{\underset{\sim}{N}}_{k_1} = \underset{\sim}{Q}_{\varphi}$.

Now suppose $ht(\underset{\sim}{x}') > 1$ and the result is true for all smaller heights. If $\underset{\sim}{x}' = \underset{\sim}{u}^{-1}$ or $\underset{\sim}{x}'_1\underset{\sim}{x}'_2$ the result follows immediately from the fact that $\underset{\sim}{Q}_{\varphi}$ is a subalgebra. If $\underset{\sim}{x}' = [\underset{\sim}{x}'_1, \underset{\sim}{x}'_2]$, write $\pi'(\underset{\sim}{x}') = \psi'$ (so that $\psi' \geqq \varphi'$), $\pi'(\underset{\sim}{x}'_1) = \psi'_1$ and $\pi'(\underset{\sim}{x}'_2) = \psi'_2$. Then $\psi' = \psi'_1 + \psi'_2$. Define $\psi \in Q$ by $\psi(j) = \psi'(j-1)$ for all $j \geqq 1$ and $\psi(0) = k_1\psi(1)$, and define ψ_1 , ψ_2 similarly . Then by the inductive hypothesis $\underset{\sim}{x}'_1\rho' \in \underset{\sim}{Q}_{\psi_1}$ and $\underset{\sim}{x}'_2\rho' \in \underset{\sim}{Q}_{\psi_2}$ so that $\underset{\sim}{x}'\rho' = [\underset{\sim}{x}'_1\rho', \underset{\sim}{x}'_2\rho'] \in \underset{\sim}{Q}_{\psi_1+\psi_2}$ and $\psi = \psi_1 + \psi_2$ so $\underset{\sim}{x}'\rho' \in \underset{\sim}{Q}_{\psi} \subseteq \underset{\sim}{Q}_{\varphi}$. ◇

<u>Corollary</u>

With the notation of the lemma, $\underline{\underline{Q}}_{\varphi} = \underline{\underline{Q}}'_{\varphi'} \cdot \underline{\underline{N}}_{k_1-1}$ ◇

The next theorem involves the notion of "partial collectability mod α " (definition 2.7). It has been seen (lemma 2.14) that this property can be expressed in terms of residual nilpotence, and for later use (lemma 3.13) it will be convenient to prove the theorem in these terms.

Theorem 3.1

Any polynilpotent semiweight range is partially collectable.

Proof

The proof will proceed by a rather strange double induction and perhaps it is advisable to describe this precisely before proceeding.

For any sequence $K = (k_i)_{i=1}^{\infty}$ of integers, each $\geqq 2$, any non-negative integer n and any function $\varphi \in Q^K$ ($\neq \infty$) such that $J_\varphi = n$, let $S(K,n,\varphi)$ be the proposition, "the group $F_\tau(\underline{\underline{Q}}^K_\varphi)$ of any rank τ is residually nilpotent". Let us write $(L,m,\psi) < (K,n,\varphi)$ if (a) $m < n$ or (b) $K = L$ and $\psi < \varphi$ (notice that $K = L$ is necessary for $\psi < \varphi$ to have meaning). Clearly this is a (partial) well-ordering of these triplets. The truth of the proposition for any K , n and φ is established by the inductive step: $S(L,m,\psi)$ is true for all $(L,m,\psi) < (K,n,\varphi)$ $\Longrightarrow$ $S(K,n,\varphi)$ is true.

Suppose then that $\varphi \in Q^K$, $J_\varphi = n$ and the inductive hypothesis is true. There are two possibilities.

(i) Suppose that φ is not a limiting value of Q^K . Then there exists $\psi \in Q^K$ such that $\varphi = \psi + 1$. Then, by the inductive hypothesis, $F_\tau(\underline{\underline{Q}}^K_\psi)$ is residually nilpotent. Suppose that $x \in G = F_\tau(\underline{\underline{Q}}^K_\varphi)$, $x \neq 1$. Then either $x \notin Q^K_\psi(G)$ or $x \in Q^K_\psi(G)$. If $x \notin Q^K_\psi(G)$ then, since $G / Q^K_\psi(G) \cong F_\tau(\underline{\underline{Q}}^K_\psi)$ is residually nilpotent, there exists an integer c such that $x \notin \gamma_c(G).Q^K_\psi(G)$ and then $x \notin \gamma_c(G)$. If on the other hand $x \in Q^K_\psi(G)$ then, since $\varphi = \psi + 1$, there exists an expression $\underset{\sim}{x}$ which is Q^K-basic, $\neq 1$, is a product of commutators of semiweight exactly ψ and such that $\underset{\sim}{x}\rho = x$. Thus $\underset{\sim}{x}$ may be written in the form $\underset{\sim}{x} = \underset{\sim}{b}_1^{\alpha_1} \underset{\sim}{b}_2^{\alpha_2} \ldots \underset{\sim}{b}_k^{\alpha_k}$ and then there exists an integer c such that $wt(\underset{\sim}{b}_i) \leq c$ $(1 \leq i \leq k)$. Then $\underset{\sim}{x} \in \underset{\sim}{B}^{Q^K}_{(c)}$ and $x = \underset{\sim}{x}\rho$ so $x \notin \gamma_{c+1}(G)$ by part B(iii) of the Basis theorem. Thus G is residually nilpotent and $S(K,n,\varphi)$ is true.

(ii) Suppose φ is a limiting value of Q^K. By lemma 3.8, either $\varphi = 1$ or ∞ or else $\varphi(0) = k_1\varphi(1)$. If $\varphi = 1$ the result is trivial since $F_\tau(\underline{\underline{Q}}^K_1) = \{1\}$ and if $\varphi = \infty$ the result is known, since $F_\tau(\underline{\underline{Q}}^K_\infty)$ is an absolutely free group. If $\varphi(0) = k_1\varphi(1)$ the corollary to lemma 3.9 is used. With the notation of that lemma, $F(\underline{\underline{Q}}^K_\varphi) = F(\underline{\underline{Q}}^{K'}_\varphi . \underline{\underline{N}}_{k_1-1})$.

But $J_{\varphi'} = J_{\varphi} - 1$ so, by the inductive hypothesis $F(\underline{\underline{Q}}^{K'}_{\varphi'})$ is residually nilpotent and then, by lemma 2.14, residually a finite p-group. But so is $F(\underline{\underline{N}}_{k_1 - 1})$ and therefore, by a theorem of Gilbert Baumslag ([1], Theorem 3), $F(\underline{\underline{Q}}^{K}_{\varphi})$ is residually a finite p-group and hence residually nilpotent.

This completes the inductive step. The theorem now follows immediately by lemma 2.14. ◇

THE SUBGROUPS Q^K_φ

First the result promised at the beginning of this chapter, that the varieties $\underline{\underline{Q}}^{K}_{\varphi}$ contain among them the varieties $\underline{\underline{P}}_{K_r}$, is proved. The remainder of this section is devoted to finding an expression for all the subgroups $Q^K_\varphi(F)$ of an absolutely free group F in group theoretical terms. The argument used to this end is outlined following definition 3.4.

Theorem 3.2

Let $K = (k_i)_{i=1}^{\infty}$ be a sequence of integers, each ≥ 2, and let $\delta_r = \delta^K_r$ be the function given in definition 3.3. Then

(i) For any group G and non-negative integer r,

$P_{K_r}(G) = Q^K_{\delta_r}(G)$,

(ii) For any non-negative integer r , $\underline{\underline{P}}_{K_r} = \underline{\underline{Q}}^{K}_{\delta_r}$.

Proof

(ii) By induction over r . The result is trivial when $r = 0$ or 1 . Suppose $r \geqq 2$ and the result is true for all smaller values. Then $\delta_r(0) = k_1 \delta_r(1)$ so by the corollary to lemma 3.9 and using the notation of that lemma,

$$\begin{aligned} \underline{\underline{Q}}^{K}_{\delta_r} &= \underline{\underline{Q}}^{K'}_{\delta'_r} \cdot \underline{\underline{N}}_{k_1 - 1} && \text{where } \delta'_r = \delta^{K'}_r , \\ &= \underline{\underline{P}}_{K'_r} \cdot \underline{\underline{N}}_{k_1 - 1} && \text{by the inductive hypothesis,} \\ &= \underline{\underline{P}}_{K_r} . \end{aligned}$$

(i) is an immediate consequence of (i). ◇

For the remainder of the chapter it will be assumed that we are working with a fixed polynilpotent semiweight range $Q = Q^K$ and its associated semiweight $\pi = \pi^K$.

Definition 3.4

(A) A new partial order $\preccurlyeq$ is defined on Q by
$\varphi \preccurlyeq \psi \iff \varphi(j) \leqq \psi(j)$ for all j , if φ and ψ are functions, and $\varphi \preccurlyeq \infty$ for all $\varphi \in Q$.

This partial order is clearly a lattice order.

(B) The "meet" $\varphi \wedge \psi$ of two functions is defined accordingly: $(\varphi \wedge \psi)(j) = \min\{\varphi(j), \psi(j)\}$ for all j if φ and ψ are functions, and $\varphi \wedge \infty = \infty \wedge \varphi = \varphi$ for all $\varphi \in Q$. Clearly if φ and $\psi \in Q$, so does $\varphi \wedge \psi$.

(C) A mapping $\hat{\pi} : \underset{\sim}{A} \to Q$ (not a semiweight) is defined recursively:

(i) $\hat{\pi}(\underset{\sim}{1}) = \infty$, $\underset{\sim}{g}_i \in \underset{\sim}{G} \Rightarrow \hat{\pi}(\underset{\sim}{g}_i) = 1$.

(ii) $\hat{\pi}(\underset{\sim}{x}^{-1}) = \hat{\pi}(\underset{\sim}{x})$

(iii) $\hat{\pi}(\underset{\sim}{x}\underset{\sim}{y}) = \hat{\pi}(\underset{\sim}{x}) \wedge \hat{\pi}(\underset{\sim}{y})$

(iv) $\hat{\pi}([\underset{\sim}{x},\underset{\sim}{y}]) = \hat{\pi}(\underset{\sim}{x}) + \hat{\pi}(\underset{\sim}{y})$.

(D) For each $\varphi \in Q$ let $\hat{\underset{\sim}{Q}}_\varphi$ be the set $\hat{\underset{\sim}{Q}}_\varphi = \{\underset{\sim}{x} : \underset{\sim}{x} \in \underset{\sim}{A}, \ \hat{\pi}(\underset{\sim}{x}) \geqslant \varphi\}$ and, for any description $\rho : \underset{\sim}{A} \to G$, let $\hat{Q}_\varphi(G)$ be the set $\hat{Q}_\varphi(G) = \hat{\underset{\sim}{Q}}_\varphi \rho$. Subject to the proof in the next lemma that $\hat{\underset{\sim}{Q}}_\varphi$ is a fully invariant subalgebra of $\underset{\sim}{A}$, let $\hat{\underline{\underline{Q}}}_\varphi$ be the corresponding variety of groups. $\diamond$

This definition deserves some explanation. In definition 1.7(i) it was postulated that the order $\leqq$ defined on an arbitrary semiweight W should be a well-order, and this property appears first in the proof of Witt's formula in Chapter 1 and second in the proof that the special collecting process can convert an arbitrary expression into a W-basic one in Chapter 2.

But there is another important consequence of the postulates of definition 1.7 and that is that the semiweight of an arbitrary expression does not become smaller under either collecting process (lemmas 2.5 and 2.7), and this fact does not require the property that $\leqq$ is a well-order.

In lemma 3.11 it will be shown that the new partial order $\precsim$ on Q satisfies properties (iv), (v) and (vi) of definition 1.7, and so this partial order is preserved by the collecting process in the same way. Thus the mapping $\hat{\pi} : \underset{\sim}{A} \to Q$ becomes quite closely analogous to a semiweight and many lemmas previously proved for $\leqq$ can now be proved for $\precsim$ with no more than a word-for-word translation, reading $\precsim$ for $\leqq$, $\hat{\pi}$ for π or σ, $\hat{Q}_\varphi(G)$ for $Q_\varphi(G)$ or $W_\varphi(G)$ and so on. Some of these lemmas will be needed in translation, and a reference to the forgoing untranslated version will be given in each case.

It is also not difficult to prove that $\precsim$ is in fact the coarsest partial order which satisfies parts (iv), (v) and (vi) of definition 1.7; this fact however is not necessary to the argument to be used in this section and is not proved here. However this means, roughly speaking, that the subalgebras $\hat{\underset{\sim}{Q}}_\varphi$ are just about the smallest ones to be closed under the collecting processes, and so it might be expected that the corresponding subgroups $\hat{Q}_\varphi(G)$ are more "natural" than the subgroups $Q_\varphi(G)$.

This turns out to be true, in the sense that they are more simply expressed in group-theoretical terms and that it is more easily proved that this expression is correct. Once this has been done it is a fairly easy matter to express a subgroup $Q_\varphi(G)$ as a product of subgroups $\hat{Q}_\varphi(G)$. This approach is used here and occupies the remainder of this section.

Lemma 3.10

With the notation of the forgoing definition, $\hat{\underset{\sim}{Q}}_\varphi$ is a fully invariant subalgebra of $\underset{\sim}{A}$ and consequently $\hat{Q}_\varphi(G)$ is a verbal subgroup of G for any group G and is independant of the particular description $\rho : \underset{\sim}{A} \to G$ chosen to define it.

Proof

Translate lemma 1.5 . ◇

Lemma 3.11

(i) The order $\preccurlyeq$ satisfies the properties

$$\alpha , \beta \prec \infty \Rightarrow \alpha + \beta \prec \infty$$

$$\alpha \prec \infty \Rightarrow \alpha \prec \alpha + \beta$$

$$\alpha_1 \prec \alpha_2 \text{ and } \beta \prec \infty \Rightarrow \alpha_1 + \beta \prec \alpha_2 + \beta$$

$$\alpha \leqq \beta \leqq \gamma \Rightarrow \gamma + \beta + \alpha = \gamma + \alpha + \beta \preccurlyeq \beta + \alpha + \gamma$$

(ii) If $D : \underset{\sim}{x} \to \underset{\sim}{y}$ or $E : \underset{\sim}{x} \to \underset{\sim}{y}$ then $\hat{\pi}(\underset{\sim}{x}) \leqslant \hat{\pi}(\underset{\sim}{y})$.

(iii) If $D : \underset{\sim}{x} \to \underset{\sim}{y}$ then there exists $\underset{\sim}{u}$ (possibly empty such that $E : \underset{\sim}{x} \to \underset{\sim}{yu}$ and $\hat{\pi}(\underset{\sim}{u}) \geqslant \hat{\pi}(\underset{\sim}{x}) + 1$.

Proof

(i) The first three of these properties follow immediately from the definitions and the last from lemma 3.4.

(ii) Translate lemmas 2.5 and 2.7.

(iii) Translate lemma 2.6. ◇

Lemma 3.12

Let $G = F(\underline{\underline{N}}_c)$ be a group, free with respect to being nilpotent of class c . Then, for any $\varphi \in Q$, $\hat{Q}_\varphi(G) \cap \gamma_c(G)$ is a free Abelian group. Further, if $\rho : \underset{\sim}{A} \to G$ is a free description of G , then the set

$\{ \underset{\sim}{b}\rho : \underset{\sim}{b}$ is a Q-basic commutator, $wt(\underset{\sim}{b}) = c$ and $\hat{\pi}(\underset{\sim}{b}) \geqslant \varphi \}$ is a free basis for $\hat{Q}_\varphi(G) \cap \gamma_c(G)$.

Proof

By virtue of the basis theorem, it is sufficient to show that the Q-basic commutators of weight c and semiweight $\geqslant \varphi$ generate $\hat{Q}_\varphi(G) \cap \gamma_c(G)$. But this follows immediately from lemma 3.11(ii). ◇

It will be noticed that this lemma amounts to a small part of the basis theorem for the subgroups $\hat{Q}_{\varphi}(G)$. The rest of the basis theorem can be proved in this context, but it will not be required here.

Lemma 3.13

The following two propositions are equivalent for any $\varphi \in Q$.

(i) $F(\hat{\underline{\underline{Q}}}_{\varphi})$ is residually nilpotent.

(ii) $F(\hat{\underline{\underline{Q}}}_{\varphi})$ is residually a finite p-group.

Proof

Translate the appropriate part of lemma 2.14. $\diamond$

Partial collectability can be proved for $\hat{Q}$ subgroups. This is expressed most conveniently in terms of residual nilpotence, using the forgoing lemma.

Lemma 3.14

The relatively free group $F(\hat{\underline{\underline{Q}}}_{\varphi}) = F / \hat{Q}_{\varphi}(F)$ is residually nilpotent.

Proof

Translate lemma 3.9 and theorem 3.1. ◇

It follows immediately from definition 3.4 that the order $\precsim$ is coarser than the order $\leq$, that is, that $\varphi \precsim \psi \Rightarrow \varphi \leq \psi$. Another immediate consequence is that, for any expression $\underset{\sim}{x}$, $\hat{\pi}(\underset{\sim}{x}) \precsim \pi(\underset{\sim}{x})$ and thus $\hat{\pi}(\underset{\sim}{x}) \leq \pi(\underset{\sim}{x})$. There exist expressions for which this order relation is strict. On the other hand, the opposite relation can be established in a weakened form as the next lemma shows.

Lemma 3.15

Let $\underset{\sim}{x}$ be an expression. Then there exist expressions $\underset{\sim}{u}_1 , \underset{\sim}{u}_2 , \ldots , \underset{\sim}{u}_k$ such that $E : \underset{\sim}{x} \to \underset{\sim}{u}_1 \underset{\sim}{u}_2 \ldots \underset{\sim}{u}_k$ and $\hat{\pi}(\underset{\sim}{u}_i) \geq \pi(\underset{\sim}{x}) \quad (1 \leq i \leq k)$.

Proof

This follows by an easy induction over the height of $\underset{\sim}{x}$. ◇

Lemma 3.16

For any group G and $\varphi \in Q$,

$$Q_{\varphi}(G) \;=\; \prod_{\psi \geqq \varphi} \hat{Q}_{\psi}(G) \quad .$$

Proof

Suppose $x \in \hat{Q}_{\psi}$, $\psi \geqq \varphi$. Then there exists $\underset{\sim}{x} \in \underset{\sim}{A}$ such that $\underset{\sim}{x}\rho = x$ and $\hat{\pi}(\underset{\sim}{x}) \succcurlyeq \psi$. Then $\pi(\underset{\sim}{x}) \geqq \hat{\pi}(\underset{\sim}{x}) \geqq \psi \geqq \varphi$ so $x \in Q_{\varphi}(G)$. Conversely, suppose $x \in Q_{\varphi}(G)$. Then, by lemma 3.15, x is a product of elements each of which belongs to some $\hat{Q}_{\psi}(G)$, $\psi \geqq \varphi$. ◇

This lemma establishes the promised connection between the Q_{φ} and the $\hat{Q}_{\varphi}$ subgroups, and in the next lemma a group theoretic expression for the $\hat{Q}_{\varphi}$ subgroups is established. It will be noticed that the property of partial collectability, given by lemma 3.14, plays an essential part in the proof; this provides the first demonstration of the usefulness of this notion.

Lemma 3.17

Let F be an absolutely free group and let $\varphi \in Q$ $(\neq \infty)$, then

$$\hat{Q}_{\varphi}(F) \;=\; \bigcap_{j=0}^{J_{\varphi}} \gamma_{\varphi(j)} \left(P_{K_j}(F) \right) \quad .$$

Proof

(i) It is shown first that $\hat{Q}_{\varphi}(F)$ is contained in the intersection displayed above. This is proved by showing that if $\hat{\pi}(\underset{\sim}{x}) \succeq \varphi$ then, for $0 \leqq j \leqq J_{\varphi}$, $\underset{\sim}{x}\rho \in \gamma_{\varphi(j)}(P_{K_j}(F))$, by induction over the height of $\underset{\sim}{x}$. If $ht(\underset{\sim}{x}) = 1$ the result is trivial. Now suppose that $ht(\underset{\sim}{x}) > 1$ and the result is true for all smaller heights: there are the usual three cases. If $\underset{\sim}{x} = \underset{\sim}{u}^{-1}$ or $\underset{\sim}{x}_1\underset{\sim}{x}_2$ the result is true since $\gamma_{\varphi(j)}(P_{K_j}(F))$ is a subgroup. If $\underset{\sim}{x} = [\underset{\sim}{x}_1, \underset{\sim}{x}_2]$, define $\varphi_1 = \hat{\pi}(\underset{\sim}{x}_1)$ and $\varphi_2 = \hat{\pi}(\underset{\sim}{x}_2)$. Then $\underset{\sim}{x}_1\rho \in \gamma_{\varphi_1(j)}(P_{K_j}(F))$ and $\underset{\sim}{x}_2\rho \in \gamma_{\varphi_2(j)}(P_{K_j}(F))$ so that $\underset{\sim}{x}\rho \in \gamma_{\varphi_1(j)+\varphi_2(j)}(P_{K_j}(F))$. But $\varphi_1(j) + \varphi_2(j) = \varphi(j)$ so that the result is true unless $j = J_{\varphi}$, $J_{\varphi_1} = J_{\varphi_2} = j - 1$ and $\varphi_1(j-1) + \varphi_2(j-1) \geqq k_j$. But in this case $\underset{\sim}{x}\rho \in \gamma_{\varphi_1(j-1)+\varphi_2(j-1)}(P_{K_{j-1}}(F)) \leqq$ $\leqq \gamma_{k_j}(P_{K_{j-1}}(F)) = P_{K_j}(F)$ and $\varphi(j) = 1$ so the result is still true.

(ii) In order to establish the reverse inclusion it must first be proved that if $x \in \gamma_c(P_{K_r}(F))$ then there exists $\underset{\sim}{x} \in \underset{\sim}{A}$ such that $\underset{\sim}{x}\rho = x$ and $\varphi(r) \geqq c$ where $\varphi = \hat{\pi}(\underset{\sim}{x})$. This is done by double induction over r and c. The result is trivial when $c = 1$ and $r = 0$. Now the result is proved for any r and c on the inductive assumption that it is true for the same r and smaller c and also for smaller r and any c . Suppose first that $c > 1$. Then, since $\gamma_c(P_{K_r}(F)) = [\gamma_{c-1}(P_{K_r}(F)) , P_{K_r}(F)]$, x may be written in the form $x = [a_1,b_1]^{\varepsilon_1}[a_2,b_2]^{\varepsilon_2}\ldots[a_k,b_k]^{\varepsilon_k}$ where, for each i , $a_i \in \gamma_{c-1}(P_{K_r}(F))$, $b_i \in P_{K_r}(F)$ and $\varepsilon_i = \pm 1$. Then there exist $\underset{\sim}{a}_i$, $\underset{\sim}{b}_i \in \underset{\sim}{A}$ such that $\underset{\sim}{a}_i\rho = a_i$, $\underset{\sim}{b}_i\rho = b_i$, $\alpha_i(r) \geq c - 1$ and $\beta_i(r) \geq 1$ where $\alpha_i = \hat{\pi}(\underset{\sim}{a}_i)$ and $\beta_i = \hat{\pi}(\underset{\sim}{b}_i)$. Then $\hat{\pi}([\underset{\sim}{a}_i,\underset{\sim}{b}_i]) = \alpha_i + \beta_i$ and $(\alpha_i + \beta_i)(r) \geq c$. Thus, writing $\underset{\sim}{x} = [\underset{\sim}{a}_1,\underset{\sim}{b}_1]^{\varepsilon_1}[\underset{\sim}{a}_2,\underset{\sim}{b}_2]^{\varepsilon_2}\ldots[\underset{\sim}{a}_k,\underset{\sim}{b}_k]^{\varepsilon_k}$, $\underset{\sim}{x}\rho = x$ and $\varphi(r) \geq c$ where $\varphi = \hat{\pi}(\underset{\sim}{x})$. Now suppose that $c = 1$ and $r > 0$. Now $x \in \gamma_1(P_{K_r}(F)) = \gamma_{k_r}(P_{K_{r-1}}(F))$ so by the inductive assumption there exists $\underset{\sim}{x} \in \underset{\sim}{A}$ such that $\underset{\sim}{x}\rho = x$ and $\varphi(r - 1) \geq k_r$ where $\varphi = \hat{\pi}(\underset{\sim}{x})$. But then $\varphi(r) \geq 1$ by definition 3.2.

It is now possible to prove the converse inclusion. Suppose $x \notin \hat{Q}_\varphi(F)$ and $x \in \bigcap_{j=0}^{J_\varphi} \gamma_{\varphi(j)}(P_{K_j}(F))$. Then by lemma 3.14 there exists an integer c such that $x \notin \hat{Q}_\varphi(F) \cdot \gamma_c(F)$. But then, by the basis theorem, there exists $\underset{\sim}{x} \in \underset{\sim}{B}^Q_{(c)}$ such that $\underset{\sim}{x}\rho = x$ modulo $\gamma_c(F)$, and then, still by the basis theorem, $\hat{\pi}(\underset{\sim}{x})$ is not $\succcurlyeq \varphi$; that is, there exists an integer r such that $\psi(r) < \varphi(r)$ where $\psi = \hat{\pi}(\underset{\sim}{x})$. Clearly then $r \leq J_\varphi$ so that $x \in \gamma_{\varphi(r)}(P_{K_r}(F))$. But it has just been proved that in this case there exists $\underset{\sim}{x}' \in \underset{\sim}{A}$ such that

$\underset{\sim}{x}'\rho = x$ and $\varphi'(r) \geqq \varphi(r)$ where $\varphi' = \hat{\pi}(\underset{\sim}{x}')$. But then $E : \underset{\sim}{x}' \to \underset{\sim}{y}\underset{\sim}{u}$ where $\underset{\sim}{y} \in \underset{\sim}{B}^{Q}_{(c)}$ and $wt(\underset{\sim}{u}) \geqq c + 1$. Then by lemma 3.12, $\underset{\sim}{x} = \underset{\sim}{y}$ so that $\hat{\pi}(\underset{\sim}{y}) = \psi$. But by lemma 3.11(ii), $\psi = \hat{\pi}(\underset{\sim}{y}) \succcurlyeq \hat{\pi}(\underset{\sim}{y}\underset{\sim}{u}) \succcurlyeq \hat{\pi}(\underset{\sim}{x}') = \varphi'$ so that $\psi(r) \geqq \varphi'(r) \geqq \varphi(r)$. But this is a contradiction: both $\psi(r) \geqq \varphi(r)$ and $\psi(r) < \varphi(r)$ have been deduced. This completes the proof of the lemma. $\diamond$

The last two lemmas contain between them a group theoretic description of the subgroups $Q_{\varphi}(F)$. This can be made a little simpler by the following lemma.

<u>Lemma 3.18</u>

Suppose, for any function $\varphi \in Q$ and any integer n $(-1 \leqq n \leqq J_{\varphi})$ the function $\varphi_{(n)} \in Q$ is defined by

$$
\begin{aligned}
\varphi_{(n)}(j) &= (\varphi(n) + 1) \prod_{i=j+1}^{n} k_i && (j < n) , \\
&= \varphi(n) + 1 && (j = n) , \\
&= \varphi(j) && (j > n ,\ j \neq J_{\varphi} + 1) , \\
&= 1 && (\text{if } j = J_{\varphi} + 1 ,\ n = J_{\varphi} \text{ and } \varphi(n) = k_{n+1} - 1) , \\
&= 0 && (\text{otherwise}) .
\end{aligned}
$$

(Clearly $\varphi_{(n)} \in Q$. Notice that $\varphi_{(-1)} = \varphi$ and $\varphi_{(0)} = \varphi + 1$). Then $\psi \geqq \varphi$ if and only if $\psi \succcurlyeq \varphi_{(n)}$ for some n $(-1 \leqq n \leqq J_{\varphi})$.

Proof

Suppose that $\psi \geqq \varphi$. Then either $\psi = \varphi$, in which case $\psi \succcurlyeq \varphi = \varphi_{(-1)}$, or $\psi \succ \varphi$, in which case there exists an integer $n \geqq 0$ (hitherto called j_o) such that $\varphi(n) < \psi(n)$ and $j > n \Rightarrow \varphi(j) = \psi(j)$. Suppose first that $n > J_\varphi$. Then $\psi(J) \geqq k_{J_\varphi+1}$, $\psi(J_\varphi - 1) \geqq k_{J_\varphi+1} . k_{J_\varphi}$ and so on. But $\varphi(J_\varphi) < k_{J_\varphi+1}$ so $\psi \succcurlyeq \varphi_{(J_\varphi)}$. Now suppose $n \leqq J$. By the same argument, $\psi \succcurlyeq \varphi_{(n)}$.
Conversly, if $\psi \succcurlyeq \varphi_{(n)}$ for some n then $\psi \geqq \varphi_{(n)}$ and clearly $\varphi_{(n)} \geqq \varphi$.

◇

All this may now be summed up in a theorem.

Theorem 3.3

Let $Q = Q^K$ be a polynilpotent semiweight range and let F be an absolutely free group. Writing $P^n = P_{K_n}(F)$ for the various terms of the polynilpotent series of type K ,

$$\hat{Q}_\varphi(F) = \bigcap_{r=0}^{J_\varphi} \gamma_{\varphi(r)}(P^r)$$

and

$$Q_\varphi(F) = \prod_{n=-1}^{J_\varphi} \hat{Q}_{\varphi_{(n)}}(F) \quad ,$$

where

$$\hat{Q}_{\varphi_{(-1)}}(F) = \hat{Q}_{\varphi}(F) \quad \text{as above, and}$$

$$\hat{Q}_{\varphi_{(n)}}(F) = \gamma_{\varphi(n)+1}(P^n) \cap \bigcap_{j=n+1}^{J_\varphi} \gamma_{\varphi(j)}(P^j)$$

for $0 \leqq n \leqq J_\varphi$. ◇

CHAPTER 4

THE UPPER CENTRAL SERIES OF FREE $\underline{\underline{W}}_{\alpha}$-GROUPS

Definition 4.1

Suppose W is an arbitrary semiweight range and σ the associated semiweight. Then

(i) For each $\alpha \in W$ the element $\alpha - 1$ of W is defined $\alpha - 1 = \min \{ \xi : \xi \in W , \xi + 1 \geqq \alpha \}$. This is well defined since W is well-ordered.

(ii) For each $\alpha \in W$ and each non-negative integer n the element $\alpha (-1)^n$ is defined recursively by $\alpha (-1)^0 = \alpha$ and $\alpha (-1)^n = \alpha (-1)^{n-1} - 1$ for $n > 0$. ◇

Clearly $\alpha (-1)^n$ could just as well be defined $\alpha (-1)^n = \min \{ \xi : \xi \in W , \xi (+1)^n \geqq \alpha \}$. Notice that, if λ is a limiting value of W, $\lambda - 1 = \lambda$.

The principal result of this chapter is, if W is partially collectable mod α and $G = F(\underline{\underline{W}}_\alpha)$ is of rank at least 2, then $\zeta(G) = W_{\alpha-1}(G)$.

Lemma 4.1

Suppose Q^K is a polynilpotent semiweight range and $\varphi \in Q^K$. Then $\varphi - 1$ may be described as follows:

(i) If $\varphi = 1$ or ∞ or is of the form $\varphi(0) = k_1 \varphi(1)$, $\varphi(1) \geqq 2$ then $\varphi - 1 = \varphi$.

(ii) If $\varphi(0) = k_1$ and $\varphi(1) = 1$ then $(\varphi - 1)(0) = k_1 - 1$ and $(\varphi - 1)(j) = 0$ for $j \geqq 1$.

(iii) If $\varphi(0) > k_1 \varphi(1)$ then $(\varphi - 1)(0) = \varphi(0) - 1$ and $(\varphi - 1)(j) = \varphi(j)$ for $j \geqq 1$. $\diamond$

For the remainder of this chapter it will be assumed that we are working with a fixed semiweight range W and its associated semiweight σ. All definitions will be made in terms of these.

<u>Definition 4.2</u>

Let $\underset{\sim}{b}$ be a commutator $> \underset{\sim}{g}_0$ (that is, other than $\underset{\sim}{g}_0$). The commutator $\underset{\sim}{b}^*$ is defined recursively over its weight.

(i) $\underset{\sim}{g}_i^* = [\underset{\sim}{g}_i, \underset{\sim}{g}_0]$,

(ii) $[\underset{\sim}{b}_1, \underset{\sim}{b}_2]^* = [\underset{\sim}{b}_1^*, \underset{\sim}{b}_2]$. $\diamond$

Suppose that x is an element of some group G which can be described by a W-basic expression $\underset{\sim}{x} = \underset{\sim}{b}_1^{\beta_1} \underset{\sim}{b}_2^{\beta_2} \dots \underset{\sim}{b}_k^{\beta_k}$ (other than $\underset{\sim}{1}$) modulo $W_\alpha(G)$, and suppose $\underset{\sim}{b}_r$ is the factor of $\underset{\sim}{x}$ maximum under the order $\leqq^0$ (definition 2.3). Then it will be proved that $[x, g_0]$ can be described by a W-basic expression modulo $W_{\alpha+1}(G)$ which contains $\underset{\sim}{b}_r^*$ as a factor and which is thus not $\underset{\sim}{1}$.

This proof occupies most of the chapter and then the main result follows immediately.

Some simple properties of the commutator $\underset{\sim}{b}^*$ are given in the next lemma.

Lemma 4.2

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are W-basic commutators $> \underset{\sim}{g}_o$, then

(i) $\sigma(\underset{\sim}{a}^*) = \sigma([\underset{\sim}{a},\underset{\sim}{g}_o]) = \sigma(\underset{\sim}{a}) + 1 > \sigma(\underset{\sim}{a})$,

(ii) $\underset{\sim}{a} < \underset{\sim}{a}^*$ and $\underset{\sim}{a}^* \overset{o}{<} \underset{\sim}{a}$,

(iii) $\underset{\sim}{a} < \underset{\sim}{b} \Rightarrow \underset{\sim}{a}^* < \underset{\sim}{b}^*$ and $\underset{\sim}{a} \overset{o}{<} \underset{\sim}{b} \Rightarrow \underset{\sim}{a}^* \overset{o}{<} \underset{\sim}{b}^*$,

(iv) $\underset{\sim}{a}^*$ is W-basic,

(v) $\underset{\sim}{a}^* = \underset{\sim}{b}^* \Rightarrow \underset{\sim}{a} = \underset{\sim}{b}$.

Proof

(i) $\sigma([\underset{\sim}{a},\underset{\sim}{g}_o]) = \sigma(\underset{\sim}{a}) + 1 > \sigma(\underset{\sim}{a})$ is trivial. It is shown that $\sigma(\underset{\sim}{a}^*) = \sigma(\underset{\sim}{a}) + 1$ by induction over the weight of a .

If $\underset{\sim}{a}$ is of weight 1 the result is again trivial. Otherwise $\underset{\sim}{a} = [\underset{\sim}{a}_1,\underset{\sim}{a}_2]$ and $\underset{\sim}{a}^* = [\underset{\sim}{a}_1^*,\underset{\sim}{a}_2]$ so, by the inductive hypothesis, $\sigma(\underset{\sim}{a}^*) = \sigma(\underset{\sim}{a}_1) + 1 + \sigma(\underset{\sim}{a}_2)$. But, since a is W-basic, $\sigma(\underset{\sim}{a}_1) \geqq \sigma(\underset{\sim}{a}_2) \geqq 1$ so $\sigma(\underset{\sim}{a}^*) = \sigma(\underset{\sim}{a}_1) + \sigma(\underset{\sim}{a}_2) + 1 = \sigma(\underset{\sim}{a}) + 1$.

(ii) follows immediately from (i).

(iii) It is shown that $\mathbf{a} < \mathbf{b} \Rightarrow \mathbf{a}^* < \mathbf{b}^*$ by induction over $\sigma(\mathbf{b})$. If $\sigma(\mathbf{b}) = 1$ then $\sigma(\mathbf{a}) = 1$ and the result is trivial. If $\sigma(\mathbf{b}) > \sigma(\mathbf{a})$ then $\sigma(\mathbf{b}^*) = \sigma(\mathbf{b}) + 1 > \sigma(\mathbf{a}) + 1 = \sigma(\mathbf{a}^*)$ so $\mathbf{b}^* > \mathbf{a}^*$. It remains to prove the result when $\sigma(\mathbf{a}) = \sigma(\mathbf{b}) > 1$. Write $\mathbf{a} = [\mathbf{a}_1, \mathbf{a}_2]$ and $\mathbf{b} = [\mathbf{b}_1, \mathbf{b}_2]$ so that $\mathbf{a}^* = [\mathbf{a}_1^*, \mathbf{a}_2]$ and $\mathbf{b}^* = [\mathbf{b}_1^*, \mathbf{b}_2]$. Since $\mathbf{a}$ and $\mathbf{b}$ are W-basic, $\mathrm{ld}(\mathbf{a}) = \mathbf{a}_1$ and $\mathrm{ld}(\mathbf{b}) = \mathbf{b}_1$. Therefore either $\mathbf{a}_1 < \mathbf{b}_1$ or $\mathbf{a}_1 = \mathbf{b}_1$ and $\mathbf{a}_2 < \mathbf{b}_2$. If $\mathbf{a}_1 < \mathbf{b}_1$ then $\mathbf{a}_1^* < \mathbf{b}_1^*$. But $\mathbf{a}_1^* > \mathbf{a}_1 > \mathbf{a}_2$ so $\mathrm{ld}(\mathbf{a}^*) = \mathbf{a}_1^*$. Similarly $\mathrm{ld}(\mathbf{b}^*) = \mathbf{b}_1$ and hence $\mathbf{a}^* < \mathbf{b}^*$. If, on the other hand, $\mathbf{a}_1 = \mathbf{b}_1$ and $\mathbf{a}_2 < \mathbf{b}_2$ then $\mathbf{a}_1^* = \mathbf{b}_1^*$ and $\mathbf{a}_2 < \mathbf{b}_2$ so again $\mathbf{a}^* < \mathbf{b}^*$.

Now suppose $\mathbf{a} \overset{\circ}{<} \mathbf{b}$. Then either $\sigma(\mathbf{a}) > \sigma(\mathbf{b})$ in which case $\sigma(\mathbf{a}^*) > \sigma(\mathbf{b}^*)$ so that $\mathbf{a}^* \overset{\circ}{<} \mathbf{b}^*$ or else $\sigma(\mathbf{a}) = \sigma(\mathbf{b})$ and $\mathbf{a} < \mathbf{b}$ in which case $\sigma(\mathbf{a}^*) = \sigma(\mathbf{b}^*)$ and $\mathbf{a}^* < \mathbf{b}^*$ and again $\mathbf{a}^* \overset{\circ}{<} \mathbf{b}^*$.

(iv) $\mathbf{a}^*$ is W-basic, by induction over the weight of $\mathbf{a}$. If $\mathrm{wt}(\mathbf{a}) = 1$ the result is trivial. Now suppose $\mathbf{a} = [\mathbf{a}_1, \mathbf{a}_2]$ and $\mathbf{a}_1^*$ is W-basic. Then $\mathbf{a}^* = [\mathbf{a}_1^*, \mathbf{a}_2]$, $\mathbf{a}_1^*$ and $\mathbf{a}_2$ are W-basic and $\mathbf{a}_1^* > \mathbf{a}_2$. Since $\mathrm{wt}(\mathbf{a}_1^*) > 1$, write $\mathbf{a}_1^* = [\mathbf{c}_1, \mathbf{c}_2]$; it remains to prove that $\mathbf{c}_2 \leqq \mathbf{a}_2$. If $\mathrm{wt}(\mathbf{a}_1) = 1$ then $\mathbf{c}_2 = \mathbf{g}_0 \leqq \mathbf{a}_2$ and if $\mathrm{wt}(\mathbf{a}_1) > 1$, writing $\mathbf{a}_1 = [\mathbf{a}_{11}, \mathbf{a}_{12}]$, $\mathbf{c}_2 = \mathbf{a}_{12} \leqq \mathbf{a}_2$ since $\mathbf{a} = [\mathbf{a}_{11}, \mathbf{a}_{12}, \mathbf{a}_2]$ is W-basic.

(v) follows immediately from (iii). ◇

Lemma 4.3

Suppose x is a product of commutators, all $<^{\circ}$ some commutator a, and b is any commutator. Then $D : [x,b] \to y$ where y is a product of commutators, all $<^{\circ} [a,b]$.

Proof

By induction over the length ℓ of x. If $\ell = 1$ then either $x = 1$, in which case the result is true with $y = 1$, or $x = c^{\pm 1}$ where c is a commutator $<^{\circ} a$, in which case $D : [x,b] \to [c,b]^{\pm 1}$ and $[c,b] <^{\circ} [a,b]$ by the corollary to lemma 2.1. Now suppose $\ell > 1$ and the result is true for shorter products. Then $x = x_1 x_2$ where x_1 and x_2 are products of commutators $<^{\circ} a$ and so $D : [x_1,b] \to y_1$ and $D : [x_2,b] \to y_2$ where y_1 and y_2 are products of commutators $<^{\circ} [a,b]$. Hence $D : x \to y_1 y_2$ which is of the required form. $\diamond$

Lemma 4.4

Suppose c is a W-basic commutator $> g_0$. Then $D : [c,g_0] \to c^{*} u$ where u is a (possibly empty) product of commutators $<^{\circ} c^{*}$.

Proof

The argument proceeds by induction over $\underset{\sim}{c}$ under the W-ordering $\leqq$ of the commutators. Suppose $wt(\underset{\sim}{c}) = 1$. Then $\underset{\sim}{c}^* = [\underset{\sim}{c}, \underset{\sim}{g}_0]$ already. Now suppose that $wt(\underset{\sim}{c}) > 1$ and the result is true for all commutators $< \underset{\sim}{c}$. Write $\underset{\sim}{c} = [\underset{\sim}{c}_1, \underset{\sim}{c}_2]$. Then $[\underset{\sim}{c}, \underset{\sim}{g}_0] = [\underset{\sim}{c}_1, \underset{\sim}{c}_2, \underset{\sim}{g}_0]$. If $\underset{\sim}{c}_2 = \underset{\sim}{g}_0$, then, since $\underset{\sim}{c}$ is W-basic, $\underset{\sim}{c}_1$ is $\underset{\sim}{g}_0$-compatible (definition 1.15) and so is of the form $[\underset{\sim}{g}_i, \alpha \times \underset{\sim}{g}_0]$ for some generator $\underset{\sim}{g}_i$ and $\alpha \geqq 0$ (lemma 1.12). Thus $[\underset{\sim}{c}, \underset{\sim}{g}_0] = [\underset{\sim}{g}_i, (\alpha+1) \times \underset{\sim}{g}_0] = \underset{\sim}{c}^*$. Otherwise $\underset{\sim}{c}_2 > \underset{\sim}{g}_0$. Then since $\underset{\sim}{c}$ is W-basic, $\underset{\sim}{g}_0 < \underset{\sim}{c}_2 < \underset{\sim}{c}_1$. Hence
$D : [\underset{\sim}{c}, \underset{\sim}{g}_0] \to [\underset{\sim}{c}_2, \underset{\sim}{g}_0, \underset{\sim}{c}_1]^{-1} [\underset{\sim}{c}_1, \underset{\sim}{g}_0, \underset{\sim}{c}_2]$. But then, by the inductive hypothesis, $D : [\underset{\sim}{c}_1, \underset{\sim}{g}_0] \to \underset{\sim}{c}_1^* \underset{\sim}{u}_1$, where $\underset{\sim}{u}_1$ is a product of commutators $\overset{\circ}{<} \underset{\sim}{c}_1^*$ and $D : [\underset{\sim}{c}_2, \underset{\sim}{g}_0] \to \underset{\sim}{c}_2^* \underset{\sim}{u}_2$ where $\underset{\sim}{u}_2$ is a product of commutators $\overset{\circ}{<} \underset{\sim}{c}_2^*$. Thus
$D : [\underset{\sim}{c}, \underset{\sim}{g}_0] \to [\underset{\sim}{c}_2^* \underset{\sim}{u}_2, \underset{\sim}{c}_1]^{-1} [\underset{\sim}{c}_1^* \underset{\sim}{u}_1, \underset{\sim}{c}_2] \to [\underset{\sim}{u}_2, \underset{\sim}{c}_1]^{-1} [\underset{\sim}{c}_2^*, \underset{\sim}{c}_1]^{-1} [\underset{\sim}{c}_1^*, \underset{\sim}{c}_2][\underset{\sim}{u}_1, \underset{\sim}{c}_2]$.

It is now shown that $[\underset{\sim}{c}_2^*, \underset{\sim}{c}_1] \overset{\circ}{<} [\underset{\sim}{c}_1^*, \underset{\sim}{c}_2]$. First $\sigma([\underset{\sim}{c}_2^*, \underset{\sim}{c}_1]) = \sigma(\underset{\sim}{c}_2) + 1 + \sigma(\underset{\sim}{c}_1)$. But $\underset{\sim}{c}$ is W-basic, so $\underset{\sim}{c}_2 < \underset{\sim}{c}_1$ and so $1 \leqq \sigma(\underset{\sim}{c}_2) \leqq \sigma(\underset{\sim}{c}_1)$; thus $\sigma([\underset{\sim}{c}_2^*, \underset{\sim}{c}_1]) \geqq \sigma(\underset{\sim}{c}_1) + 1 + \sigma(\underset{\sim}{c}_2) = \sigma([\underset{\sim}{c}_1^*, \underset{\sim}{c}_2])$. If $\sigma([\underset{\sim}{c}_2^*, \underset{\sim}{c}_1]) > \sigma([\underset{\sim}{c}_1^*, \underset{\sim}{c}_2])$ then $[\underset{\sim}{c}_2^*, \underset{\sim}{c}_1] \overset{\circ}{<} [\underset{\sim}{c}_1^*, \underset{\sim}{c}_2]$ immediately. Otherwise $\sigma([\underset{\sim}{c}_2^*, \underset{\sim}{c}_1]) = \sigma([\underset{\sim}{c}_1^*, \underset{\sim}{c}_2])$. But $\underset{\sim}{c}_1 > \underset{\sim}{c}_2$ so $\underset{\sim}{c}_1^* > \underset{\sim}{c}_2$ and thus $ld([\underset{\sim}{c}_1^*, \underset{\sim}{c}_2]) = \underset{\sim}{c}_1^*$. But $ld([\underset{\sim}{c}_2^*, \underset{\sim}{c}_1])$ is either $\underset{\sim}{c}_2^*$ or $\underset{\sim}{c}_1$ and both of these are $< \underset{\sim}{c}_1$. Hence $[\underset{\sim}{c}_2^*, \underset{\sim}{c}_1] \overset{\circ}{<} [\underset{\sim}{c}_1^*, \underset{\sim}{c}_2]$.

By lemma 4.3, since $\underset{\sim}{u}_2$ is a product of commutators $<^{\circ}$ $\underset{\sim}{c}_2^*$, $D : [\underset{\sim}{u}_2,\underset{\sim}{c}_1]^{-1} \to \underset{\sim}{v}_2$, a product of commutators $<^{\circ}$ $[\underset{\sim}{c}_2^*,\underset{\sim}{c}_1]$ and so $\underset{\sim}{v}_2[\underset{\sim}{c}_2^*,\underset{\sim}{c}_1]$ is a product of commutators $<^{\circ}$ $[\underset{\sim}{c}_1^*,\underset{\sim}{c}_2]$. Again $D : [\underset{\sim}{u}_1,\underset{\sim}{c}_2] \to \underset{\sim}{v}_1$ where $\underset{\sim}{v}_1$ is a similar product. Thus $D : [\underset{\sim}{c},g_0] \to \underset{\sim}{v}_1[\underset{\sim}{c}_2^*,\underset{\sim}{c}_1]\underset{\sim}{c}^*\underset{\sim}{v}_2 \to \underset{\sim}{c}^*\underset{\sim}{v}_1[\underset{\sim}{c}_2^*,\underset{\sim}{c}_1]\underset{\sim}{v}_2$ which is of the required form. ◇

Lemma 4.5

Suppose $\underset{\sim}{u}$ is a product of commutators $<^{\circ}$ some commutator $\underset{\sim}{c}$ and $D : \underset{\sim}{u} \to \underset{\sim}{v}$. Then so is $\underset{\sim}{v}$.

Proof

This follows immediately by checking the various parts of definition 2.1. ◇

Theorem 4.1

Suppose W is a semiweight range, $\alpha \in W$, W is partially collectable mod α and F is an absolutely free group of rank ≥ 2 .

(i) Let $G = F(\underline{\underline{W}}_\alpha)$. Then $\zeta(G) = W_{\alpha-1}(G)$.

(ii) Let c be a positive integer and $G = F(\underline{\underline{W}}_\alpha \cap \underline{\underline{N}}_c)$. Then $\zeta(G) = W_{\alpha-1}(G).\gamma_c(G)$.

Proof

Let $\rho : \underset{\sim}{A} \to G$ be a free description of G. Then the number τ of generators of $\underset{\sim}{A}$ is at least 2. Clearly the result is true when $\alpha = 1$ or $1 + 1$, for then $W_\alpha(G) = G$ or $\delta(G)$ respectively. It may now be assumed that $\alpha > 1 + 1$.

(i) First it is proved that $W_{\alpha-1}(G) \leqq \zeta(G)$. Suppose $z \in W_{\alpha-1}(G)$ and $x \in G$. Then there exist $\underset{\sim}{z}, \underset{\sim}{x} \in \underset{\sim}{A}$ such that $\underset{\sim}{z}\rho = z$, $\underset{\sim}{x}\rho = x$ and $\sigma(\underset{\sim}{z}) \geqq \alpha - 1$. Hence $\sigma([\underset{\sim}{z},\underset{\sim}{x}]) \geqq \alpha$ and so $[z,x] = 1$. But x is an arbitrary element of G, so $z \in \zeta(G)$.

Next the converse inclusion $\zeta(G) \leqq W_{\alpha-1}(G)$ is proved. Suppose $x \in G$ but $x \notin W_{\alpha-1}(G)$. Then, since W is partially collectable, there exists $\xi \in W$ such that $x \in W_\xi(G) \setminus W_{\xi^+}(G)$ and $\xi^+ \leqq \alpha - 1$. By the basis theorem there exists a W-basic expression $\underset{\sim}{x} = \underset{\sim}{b}_1^{\beta_1}\underset{\sim}{b}_2^{\beta_2}\cdots\underset{\sim}{b}_k^{\beta_k}$ (other than 1) such that $k \geqq 1$, $\underset{\sim}{x}\rho = x$ modulo $W_{\xi^+}(G)$, $\sigma(\underset{\sim}{b}_1) = \sigma(\underset{\sim}{b}_2) = \cdots = \sigma(\underset{\sim}{b}_k) = \xi$ and as usual $\underset{\sim}{b}_1 < \underset{\sim}{b}_2 < \cdots < \underset{\sim}{b}_k$ (so that $\underset{\sim}{b}_1 \overset{\circ}{<} \underset{\sim}{b}_2 \overset{\circ}{<} \cdots \overset{\circ}{<} \underset{\sim}{b}_k$) and none of the β_i are zero. But then, writing

$\underset{\sim}{x}' = \underset{\sim}{b}_k^{\beta_k}\underset{\sim}{b}_{k-1}^{\beta_{k-1}}\cdots\underset{\sim}{b}_1^{\beta_1}$, $\underset{\sim}{x}'\rho = x$ modulo $W_{\xi^+}(G)$ also.

Now $D : [\underset{\sim}{x}',\underset{\sim}{g}_0] \to [\underset{\sim}{b}_k,\underset{\sim}{g}_0]^{\beta_k}[\underset{\sim}{b}_{k-1},\underset{\sim}{g}_0]^{\beta_{k-1}}\cdots[\underset{\sim}{b}_1,\underset{\sim}{g}_0]^{\beta_1}$,

and then by lemma 4.4, for each i $(1 \leqq i \leqq k)$,

$D : [\underset{\sim}{b}_i,\underset{\sim}{g}_0] \to \underset{\sim}{b}_i^*\underset{\sim}{u}_i$ where $\underset{\sim}{u}_i$ is a product of commutators $\overset{\circ}{<} \underset{\sim}{b}_i$.

Thus $D : [\underset{\sim}{x}',\underset{\sim}{g}_o] \to (\underset{\sim}{b}_k^*)^{\beta_k} \underset{\sim}{u}_k^{\beta_k} (\underset{\sim}{b}_{k-1}^*)^{\beta_{k-1}} \underset{\sim}{u}_{k-1}^{\beta_{k-1}} \cdots (\underset{\sim}{b}_1^*)^{\beta_1} \underset{\sim}{u}_1^{\beta_1}$

$$= (\underset{\sim}{b}_k^*)^{\beta_k} \underset{\sim}{v}_1 \quad \text{say}$$

and $\underset{\sim}{v}_1$ is a product of commutators $<^o \underset{\sim}{b}_k^*$. Then $D : \underset{\sim}{v}_1 \to \underset{\sim}{v}_2 = \underset{\sim}{c}_1^{\gamma_1} \underset{\sim}{c}_2^{\gamma_2} \cdots \underset{\sim}{c}_n^{\gamma_n}$, a W-basic expression and by lemma 4.5 each $\underset{\sim}{c}_i <^o \underset{\sim}{b}_k^*$. Now $\sigma(\underset{\sim}{b}_k^*) = \xi + 1$, and so each $\underset{\sim}{c}_i$ is of semiweight at least $\xi + 1$. Thus there exists m $(0 \leqq m \leqq n)$ so that $\sigma(\underset{\sim}{c}_i) = \xi + 1$ for $1 \leqq i \leqq m$ and $\sigma(\underset{\sim}{c}_i) > \xi + 1$ for $m < i \leqq n$. Then

$$D : [\underset{\sim}{x}',\underset{\sim}{g}_o] \to \underset{\sim}{c}_1^{\gamma_1} \underset{\sim}{c}_2^{\gamma_2} \cdots \underset{\sim}{c}_m^{\gamma_m} (\underset{\sim}{b}_k^*)^{\beta_k} \underset{\sim}{c}_{m+1}^{\gamma_{m+1}} \underset{\sim}{c}_{m+2}^{\gamma_{m+2}} \cdots \underset{\sim}{c}_n^{\gamma_n} .$$

But $\sigma([\underset{\sim}{x}',\underset{\sim}{g}_o]) = \xi + 1$ so

$$(\underset{\sim}{c}_1^{\gamma_1} \underset{\sim}{c}_2^{\gamma_2} \cdots \underset{\sim}{c}_m^{\gamma_m} (\underset{\sim}{b}_k^*)^{\beta_k})\rho = [x,g_o] \quad \text{modulo} \quad W_{(\xi+1)^+}(G) .$$

Now, since $\sigma(\underset{\sim}{c}_m) = \xi + 1 = \sigma(\underset{\sim}{b}_k^*)$ and $\underset{\sim}{c}_m <^o \underset{\sim}{b}_k^*$ so that $\underset{\sim}{c}_m < \underset{\sim}{b}_k^*$, it follows that $\underset{\sim}{c}_1^{\gamma_1} \underset{\sim}{c}_2^{\gamma_2} \cdots \underset{\sim}{c}_m^{\gamma_m} (\underset{\sim}{b}_k^*)^{\beta_k} \in \underset{\sim}{B}^W_{(\xi+1)^+}$.

But, although the product $\underset{\sim}{c}_1^{\gamma_1} \underset{\sim}{c}_2^{\gamma_2} \cdots \underset{\sim}{c}_m^{\gamma_m}$ may be empty, $(\underset{\sim}{b}_k^*)^{\beta_k}$ certainly is not. Hence, by the basis theorem again, $[x,g_o] \notin W_{(\xi+1)^+}(G)$. But $\xi < \alpha - 1$ so $\xi + 1 < \alpha$ and and so $(\xi + 1)^+ \leqq \alpha$. Thus $[x,g_o] \notin W_\alpha(G) = \{1\}$ and so $x \notin \zeta(G)$.

(ii) The modifications reqired to the forgoing argument for this case follow the by now familiar form. ◇

Corollary

With the conditions of the theorem,

(i) Let $G = F(\underline{\underline{W}}_\alpha)$. Then the upper central series of G is given by $\zeta_n(G) = W_{\alpha(-1)^n}(G) \quad (n \geq 0)$. The upper central series terminates, that is, there exists an integer N such that $\zeta_{N+1}(G) = \zeta_N(G)$ and then $\alpha(-1)^N$ is a limiting value of W , in fact, the greatest limiting value $\leq \alpha$.

(ii) Let $G = F(\underline{\underline{W}}_\alpha \cap \underline{\underline{N}}_c)$. Then the upper central series of G is given by $\zeta_n(G) = W_{\alpha(-1)^n}(G) \cdot \gamma_{c+1-n}(G)$. ◇

CHAPTER 5

AN APPLICATION

It is a well-known fact (see for instance Hanna Neumann [9], section A.2.5) that if a group is nilpotent and torsion-free, then its central factor group is also torsion-free.

The following problem, which was posed by Gilbert Baumslag and L. G. Kovacs and communicated to me by the latter, could be regarded as a weakened converse: if a group G is nilpotent, relatively free and has a torsion-free central factor group, does it follow that G itself is torsion-free?

In this chapter the question will be answered, and the answer will depend upon the soluble length of G. If G has the properties listed above and is also exactly metabelian (that is, metabelian but not Abelian or, in other words, $G \in \underline{\underline{S}}_2 \setminus \underline{\underline{S}}_1$) then G is torsion-free; in fact G is a free group of the variety $\underline{\underline{S}}_2 \cap \underline{\underline{N}}_c$ for some c. On the other hand, counter-examples are given for any other soluble length; for each $\ell \neq 2$ a group is described which is soluble of length exactly ℓ, relatively free, nilpotent and has a torsion-free central factor group but which is not itself torsion-free.

It is a fairly well-known fact that if G is a free metabelian group, then $\gamma_c(G) / \gamma_{c+1}(G)$ is a free Abelian group, freely generated by the left-normed basic commutators of weight c modulo $\gamma_{c+1}(G)$. It might be desirable to state this fact and prove it in terms of the present theory, from which it emerges as an easy special case.

The derived series $\delta^n(G)$ of a group G is defined recursively by $\delta^0(G) = G$ and $\delta^{n+1}(G) = \delta(\delta^n(G)) = [\delta^n(G), \delta^n(G)]$ for $n \geq 0$. Consequently, with the notation of definition 3.1, if the sequence $K = (k_i)_{i=1}^{\infty}$ is defined by $k_i = 2$ for all i, then $P_{K_n} = \delta^n(G)$ for all n. Throughout this chapter it will be assumed that the corresponding semiweight range $Q = Q^K$ is being used. The corresponding varieties $\underline{\underline{P}}_{K_r}$ are $\underline{\underline{S}}_r$ for all $r \geq 0$. The functions $\delta_r \in Q$ (definition 3.3) are given by $\delta_r(j) = 2^{r-j}$ $(j \leq r)$ and $\delta_r(j) = 0$ $(j > r)$. Then $\underline{\underline{S}}_r = \underline{\underline{Q}}_{\delta_r}$ and for any group G, $Q_{\delta_r}(G) = \delta^r(G)$.

<u>Definition 5.1</u>

(i) A "left-normed" commutator is one of the form $[g_{i_1}, g_{i_2}, \ldots, g_{i_k}]$.

(ii) A "left-normed basic commutator" is a commutator of the form $[g_{i_1}, g_{i_2}, \ldots g_{i_k}]$ where $i_1 > i_2 \leq i_3 \leq \ldots \leq i_k$. ◇

Clearly, for any semiweight range W, a left-normed commutator is a W-basic commutator if and only if it is a "left-normed basic commutator". This justifies the terminology.

Lemma 5.1

(i) Defining a function $\lambda_c \in Q$ for each positive integer c by $\lambda_1 = 1$, $\lambda_{c+1} = \lambda_c + 1$, the left-normed basic commutators of weight c are exactly the Q-basic commutators of semiweight λ_c. Further, for $c \geqq 2$, λ_c is the function $\lambda_c(0) = c$, $\lambda_c(1) = 1$ and $\lambda_c(j) = 0$ $(j \geqq 2)$.

(ii) Suppose $G = F(\underline{\underline{S}}_2 \cap \underline{\underline{N}}_c)$. Then $\gamma_c(G)$ is a free Abelian group, freely generated by the left-normed basic commutators of weight c.

Proof

(i) is all obvious and then (ii) is the basis theorem for these semiweights. ◇

Lemma 5.2

Suppose $i < \tau$ and $\underset{\sim}{c}$ is a commutator. Let θ be the endomorphism of $\underset{\sim}{A}$ defined: $\underset{\sim}{g}_j\theta = \underset{\sim}{g}_j$ $(j \neq i)$, $\underset{\sim}{g}_i\theta = \underset{\sim}{g}_i^{\,2}$. Then $D : \underset{\sim}{c}\theta \to \underset{\sim}{c}^{2^\mu}$ where $\mu = \mu_i(\underset{\sim}{c})$ as defined in definition 2.8.

Proof

By induction over the weight of $\underset{\sim}{c}$. If $wt(\underset{\sim}{c}) = 1$ the result is trivial. If $\underset{\sim}{c} = [\underset{\sim}{c}_1, \underset{\sim}{c}_2]$ and $D : \underset{\sim}{c}_1\theta \to \underset{\sim}{c}_1^{2^{\mu_1}}$, $D : \underset{\sim}{c}_2\theta \to \underset{\sim}{c}_2^{2^{\mu_2}}$ where $\mu_1 = \mu_i(\underset{\sim}{c}_1)$ and $\mu_2 = \mu_i(\underset{\sim}{c}_2)$, then $D : \underset{\sim}{c}\theta \to [\underset{\sim}{c}_1^{2^{\mu_1}}, \underset{\sim}{c}_2^{2^{\mu_2}}] \to [\underset{\sim}{c}_1, \underset{\sim}{c}_2]^{2^{\mu_1+\mu_2}}$ by an easy induction over μ_1 and μ_2 , and $\mu = \mu_1 + \mu_2$. ◇

Lemma 5.3

Suppose $G = F(\underline{\underline{S}}_2 \cap \underline{\underline{N}}_c)$, V is a fully-invariant subgroup of G and $\rho : \underset{\sim}{A} \to G$ is any free description of G . Let $(\nu_i)_{i<\tau}$ be any sequence of integers, and suppose $\underset{\sim}{u} = \underset{\sim}{b}_1^{\alpha_1}\underset{\sim}{b}_2^{\alpha_2}\dots\underset{\sim}{b}_k^{\alpha_k}$ is a product of commutators of weight c such that $\underset{\sim}{u}\rho \in V$.

Then, writing $\underset{\sim}{v} = \underset{\sim}{b}_{p_1}^{\alpha_{p_1}}\underset{\sim}{b}_{p_2}^{\alpha_{p_2}}\dots\underset{\sim}{b}_{p_\ell}^{\alpha_{p_\ell}}$, where $p_1, p_2, \dots, p_\ell$ is the subsequence of the integers $i = 1, 2, \dots, k$ for which $M(\underset{\sim}{b}_i) = (\nu_i)_{i<\tau}$ (definition 2.8) or $\underset{\sim}{v} = \underset{\sim}{1}$ if there are no such i , there exists a non-zero integer t such that $\underset{\sim}{v}^t\rho \in V$.

Proof

By induction over the number of integers r for which $M(\underset{\sim}{b}_r) \neq (\nu_i)_{i<\tau}$. If there are no such integers, then the result is immediately true with $\underset{\sim}{v} = \underset{\sim}{u}$ and $t = 1$.

Now suppose there is some integer r for which $M(\underset{\sim}{b}_r) \neq (\nu_i)_{i<\tau}$. Then there exists some $n < \tau$ such that $\mu_n(\underset{\sim}{b}_r) \neq \nu_n$. Let θ be the endomorphism of $\underset{\sim}{A}$ defined: $\underset{\sim}{g}_n\theta = \underset{\sim}{g}_n{}^2$, $\underset{\sim}{g}_i\theta = g_i$ $(i \neq n)$. Then by lemma 5.2,

$$D : \underset{\sim}{u}\theta \to \underset{\sim}{b}_1{}^{\alpha_1\beta_1} \underset{\sim}{b}_2{}^{\alpha_2\beta_2} \cdots \underset{\sim}{b}_k{}^{\alpha_k\beta_k} , \text{ where } \beta_i = 2^{\mu_n(\underset{\sim}{b}_i)} .$$

But $\underset{\sim}{u}\rho \in V$ and V is verbal. Hence $\underset{\sim}{u}\theta\rho \in V$, that is , $(\underset{\sim}{b}_1{}^{\alpha_1\beta_1} \underset{\sim}{b}_2{}^{\alpha_2\beta_2} \cdots \underset{\sim}{b}_k{}^{\alpha_k\beta_k})\rho \in V$. Then, writing $\underset{\sim}{b}_i\rho = b_i$, $b_1{}^{\alpha_1\beta_1} b_2{}^{\alpha_2\beta_2} \cdots b_k{}^{\alpha_k\beta_k} \in V$. Then, since the commutators $\underset{\sim}{b}_i$ are all of weight c and G is nilpotent of class c , the elements b_i of G all commute. Thus

$(\underset{\sim}{u}\rho)^{-\beta_r} = b_1{}^{-\alpha_1\beta_r} b_2{}^{-\alpha_2\beta_r} \cdots b_k{}^{-\alpha_k\beta_r} \in V$ and then

$b_1{}^{\gamma_1} b_2{}^{\gamma_2} \cdots b_k{}^{\gamma_k} \in V$ where $\gamma_i = \alpha_i(\beta_i - \beta_r)$, and thus, since γ_r is zero, $(\underset{\sim}{b}_1{}^{\gamma_1} \underset{\sim}{b}_2{}^{\gamma_2} \cdots \underset{\sim}{b}_{r-1}{}^{\gamma_{r-1}} \underset{\sim}{b}_{r+1}{}^{\gamma_{r+1}} \underset{\sim}{b}_{r+2}{}^{\gamma_{r+2}} \cdots \underset{\sim}{b}_k{}^{\gamma_k})\rho \in V$. But now the inductive hypothesis applies and

$$(\underset{\sim}{b}_{p_1}{}^{\gamma_{p_1}} \underset{\sim}{b}_{p_2}{}^{\gamma_{p_2}} \cdots \underset{\sim}{b}_{p_\ell}{}^{\gamma_{p_\ell}})^{t'} \rho \in V \quad (t' \neq 0) .$$

But $\gamma_i = \alpha_{p_i}(\beta_{p_i} - \beta_r) = \alpha_{p_i}(2^{\mu_n(\underset{\sim}{b}_{p_i})} - 2^{\mu_n(\underset{\sim}{b}_r)}) =$

$= \alpha_{p_i}(2^{\nu_n} - 2^{\mu_n(\underset{\sim}{b}_r)})$. Hence

$$(\underset{\sim}{b}_{p_1}{}^{\alpha_{p_1}} \underset{\sim}{b}_{p_2}{}^{\alpha_{p_2}} \cdots \underset{\sim}{b}_{p_\ell}{}^{\alpha_{p_\ell}})^{t} \rho \in V \quad \text{where } t = t'(2^{\nu_n} - 2^{\mu_n(\underset{\sim}{b}_r)}) ;$$

but $\mu_n(\underset{\sim}{b}_r) \neq \nu_n$ and $t' \neq 0$, so $t \neq 0$. ◇

Definition 5.3

(i) The notation of definition 1.16 is extended as follows: if $\underset{\sim}{y}$, $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$, ... , $\underset{\sim}{x}_k$ are expressions and n_1 , n_2 , ... , n_k non-negative integers, then $[\underset{\sim}{y}, n_1 \times \underset{\sim}{x}_1, n_2 \times \underset{\sim}{x}_2, \ldots, n_k \times \underset{\sim}{x}_k]$ is the expression

$$[\underset{\sim}{y}, \overbrace{\underset{\sim}{x}_1, \underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_1}^{n_1 \text{ times}}, \overbrace{\underset{\sim}{x}_2, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_2}^{n_2 \text{ times}}, \ldots, \overbrace{\underset{\sim}{x}_k, \underset{\sim}{x}_k, \ldots \underset{\sim}{x}_k}^{n_k \text{ times}}] .$$

A stricter definition, by recursion, is

$$[\underset{\sim}{y}, 0 \times \underset{\sim}{x}_1] = \underset{\sim}{y}$$

$$[\underset{\sim}{y}, n_1 \times \underset{\sim}{x}_1, n_2 \times \underset{\sim}{x}_2, \ldots, n_k \times \underset{\sim}{x}_k] = [\underset{\sim}{y}, n_1 \times \underset{\sim}{x}_1, n_2 \times \underset{\sim}{x}_2, \ldots, n_{k-1} \times \underset{\sim}{x}_{k-1}] \quad \text{if } n_k = 0 ,$$

$$= [[\underset{\sim}{y}, n_1 \times \underset{\sim}{x}_1, n_2 \times \underset{\sim}{x}_2, \ldots, (n_k - 1) \times \underset{\sim}{x}_k], \underset{\sim}{x}_k] \quad \text{if } n_k > 0 .$$

If any of the n_i are 1 they may be omitted.

(ii) For any sequence $(\underset{\sim}{x}_i)$ of expressions indexed by integers, the expression $\prod_{i=m}^{n} \underset{\sim}{x}_i$ is defined recursively for $m \geqq n$ by: $\prod_{i=m}^{m} \underset{\sim}{x}_i = \underset{\sim}{x}_m$ and, for $n > m$, $\prod_{i=m}^{n} \underset{\sim}{x}_i = (\prod_{i=m}^{n-1} \underset{\sim}{x}_i)\underset{\sim}{x}_n$.

The same notations will be used in the obvious way for left-normed commutators and products of elements of an arbitrary group G . ◇

Some simple properties of commutators in the "bottom" of a nilpotent group are:

<u>Lemma 5.4</u>

Suppose $G \in \underline{\underline{S}}_2 \cap \underline{\underline{N}}_c$. Then

(i) If $u \in [z,y,x] \in \gamma_c(G)$ then $u = [y,x,z]^{-1}[z,x,y]$.

(ii) If $u \in \gamma_c(G)$ is of the form

$u = [x_1,x_2,\dots,x_\ell,a,b,y_1,y_2,\dots,y_k]$ $(\ell \geqq 2 , k \geqq 0)$

then $u = [x_1,x_2,\dots,x_\ell,b,a,y_1,y_2,\dots,y_k]$.

(iii) If $u \in \gamma_c(G)$ is of the form $u = [x_1,x_2,y_1,y_2,\dots,y_k]$ and π is any permutation of the integers $1 , 2 , \dots , k$, then $u = [x_1,x_2,y_{\pi(1)},y_{\pi(2)},\dots,y_{\pi(k)}]$.

(iv) If $u \in \gamma_c(G)$ is of the form

$u = [x_1,x_2,\dots,x_\ell,ab,y_1,y_2,\dots,y_k]$ $(\ell \geqq 0 , k \geqq 0)$ then

$u = [x_1,x_2,\dots,x_\ell,a,y_1,y_2,\dots,y_k][x_1,x_2,\dots,x_\ell,b,y_1,y_2,\dots,y_k]$.

(v) If $u \in \gamma_c(G)$ is of the form

$u = [x_1,x_2,\dots,x_\ell,n\times ab,y_1,y_2,\dots,y_k]$ $(\ell \geqq 2 , k \geqq 0 , n \geqq 0)$

then $u = \prod_{r=0}^{n} [x_1,x_2,\dots,x_\ell,r\times a,(n-r)\times b,y_1,y_2,\dots,y_k]^{\binom{n}{r}}$,

where $\binom{n}{r}$ is the usual binomial coefficient.

<u>Proof</u>

(i) follows immediately from Jacobi's identity.

(ii) follows from (i) by induction over k .

(iii) is a corollary of (ii) .

(iv) by induction over k .

(v) is a corollary of (iii) and (iv) . ◇

Theorem 5.1

Let G be a nilpotent, relatively free group which is exactly metabelian (that is, $G \in \underline{\underline{S}}_2 \setminus \underline{\underline{S}}_1$) and has a torsion-free central factor group. Then G is isomorphic with one of the groups $F_\tau(\underline{\underline{S}}_2 \cap \underline{\underline{N}}_c)$.

Proof

This theorem must be proved in two stages: first when the rank τ of G is infinite and second when it is finite.

(A) Suppose τ is infinite. Let c be the smallest integer such that $G \in \underline{\underline{S}}_2 \cap \underline{\underline{N}}_c$. Since G is not Abelian, $c \geqq 2$.

Let $F = F_\tau(\underline{\underline{S}}_2 \cap \underline{\underline{N}}_c)$. Then there exists a fully-invariant subgroup V of F such that $G \cong F/V$. It may be assumed that $G = F/V$. Let μ be the natural epimorphism $\mu : F \to G$, and let Z be the complete inverse image of $\zeta(G)$: that is, $z \in Z \iff z\mu \in \zeta(G)$. Then $F/Z \cong G/\zeta(G)$ and so is torsion-free. Also $z \in Z$ if and only if $[z,x] \in V$ for all $x \in F$.

Let $\rho : \underset{\sim}{A} \to F$ be a free description of F defined in terms of a free generating set $\mathcal{G} = \{g_i\}_{i<\tau}$ of F.

If $V = \{1\}$ then $G \cong F_\tau(\underline{\underline{S}}_2 \cap \underline{\underline{N}}_c)$ and the theorem is true. Otherwise V is non-trivial and thus it has a non-trivial intersection with the centre of F. But, by the results of the previous chapter, $\zeta(F) = \gamma_c(F)$, so $V \cap \gamma_c(F) \neq \{1\}$. Thus there exists a Q-basic expression $\underset{\sim}{u}_o = \underset{\sim}{b}_1^{\alpha_1} \underset{\sim}{b}_2^{\alpha_2} \dots \underset{\sim}{b}_k^{\alpha_k} \in \underset{\sim}{A}$ such that $\underset{\sim}{u}_o \neq \underset{\sim}{1}$, $\underset{\sim}{u}_o\rho \in V$ and $\pi(\underset{\sim}{b}_1) = \pi(\underset{\sim}{b}_2) = \dots = \pi(\underset{\sim}{b}_k) = \lambda_c$, that is, $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \dots, \underset{\sim}{b}_k$ are left-normed basic commutators of weight c. But then, by lemma 5.3, it may be assumed that $M(\underset{\sim}{b}_1) = M(\underset{\sim}{b}_2) = \dots = M(\underset{\sim}{b}_k)$.

Firstly suppose $c = 2$. Then, since the $\underset{\sim}{b}_i$ are Q-basic and of weight 2 and $M(\underset{\sim}{b}_1) = M(\underset{\sim}{b}_2) = \dots = M(\underset{\sim}{b}_k)$ it follows that $k = 1$ so that $\underset{\sim}{u}_o = \underset{\sim}{b}_1^{\alpha_1}$ $(\alpha_1 \neq 0)$. Writing $\underset{\sim}{b}_1 = [\underset{\sim}{g}_j, \underset{\sim}{g}_i]$ it follows that $[g_j, g_i]^{\alpha_1} \in V$. But then, since $[g_j, g_i]^{\alpha_1} = [g_j^{\alpha_1}, g_i]$, $[g_j^{\alpha_1}, g_i] \in V$. Now let x be any element of F and let θ_1 be the endomorphism of F defined $g_j\theta_1 = g_o$, $g_i\theta_1 = x$, $g_r\theta_1 = g_r$ $(r \neq i,j)$. This is possible since $i \neq j$. Then $[g_o^{\alpha_1}, x] = [g_j^{\alpha_1}, g_i]\theta \in V$. But x is an arbitrary element of F, so $g_o^{\alpha_1} \in Z$. But $\alpha_1 \neq 0$ and F/Z is torsion-free, so $g_o \in Z$ and thus $[g_1, g_o] \in V$. But V is verbal, so $\delta(F) \leq V$: that is, G is Abelian which contradicts the choice of c .

Now suppose $c \geqq 3$. The argument in this case is merely a more sophisticated version of the one just given. Since $\underset{\sim}{b}_1$ is of weight c , it may be written in the form $\underset{\sim}{b}_1 = [\underset{\sim}{g}_{p_1}, \underset{\sim}{g}_{p_2}, \ldots, \underset{\sim}{g}_{p_c}]$ where $p_1 > p_2 \leqq p_3 \leqq \ldots \leqq p_c$. If $\underset{\sim}{b}_r$ is one of the others $(2 \leqq r \leqq k)$, writing $\underset{\sim}{b}_r = [\underset{\sim}{g}_{p_1'}, \underset{\sim}{g}_{p_2'}, \ldots, \underset{\sim}{g}_{p_c'}]$, again $p_1' > p_2' \leqq p_3' \leqq \ldots \leqq p_c'$. It follows from these inequalities and the fact that $M(\underset{\sim}{b}_1) = M(\underset{\sim}{b}_r)$ that $p_1' \neq p_1$ and $p_2' = p_2$.

Let θ_2 be the endomorphism of F defined $g_{p_1}\theta_2 = g_1$, $g_i\theta_2 = g_o$ $(i \neq p_1)$. Then $\underset{\sim}{b}_1\rho\theta_2 = [g_1,(c-1)\times g_o]$ and $\underset{\sim}{b}_r\rho\theta_2 = 1$ $(2 \leqq r \leqq k)$. Hence $\underset{\sim}{u}_o\rho\theta_2 = [g_1,(c-1)\times g_o]^{\alpha_1}$ and so, since V is fully-invariant, $[g_1,(c-1)\times g_o]^{\alpha_1} \in V$.

Now let θ_3 be the endomorphism of F defined $g_o\theta_3 = g_o g_2$, $g_i\theta_3 = g_i$ $(i \neq 0)$. Then $u_1 = [g_1,(c-1)\times g_o g_2]^{\alpha_1} = [g_1,(c-1)\times g_o]^{\alpha_1}\theta_3$ is an element of V . But then, by lemma 5.4,

$$
\begin{aligned}
u_1 &= [g_1, g_o g_2, (c-2)\times g_o g_2]^{\alpha_1} \\
&= [g_1, g_o, (c-2)\times g_o g_2]^{\alpha_1} [g_1, g_2, (c-2)\times g_o g_2]^{\alpha_1} \\
&= \prod_{r=0}^{c-2} [g_1, (r+1)\times g_o, (c-2-r)\times g_2]^{\binom{c-2}{r}\alpha_1} \; . \\
&\qquad \cdot \prod_{s=0}^{c-2} [g_1, g_2, s\times g_o, (c-2-s)\times g_2]^{\binom{c-2}{s}\alpha_1} \; .
\end{aligned}
$$

Now apply lemma 5.3 to u_1 with $\nu_0 = c-2$, $\nu_1 = \nu_2 = 1$, $\nu_i = 0$ $(i > 2)$. Then $u_2 \in V$ where

$$u_2 = [g_1,(c-2)\times g_0,g_2]^{\binom{c-2}{c-3}\alpha_1}[g_1,g_2,(c-2)\times g_0]^{\binom{c-2}{c-2}\alpha_1}$$
$$= [g_1,(c-2)\times g_0,g_2]^{(c-2)\alpha_1}[g_1,g_2,(c-2)\times g_0]^{\alpha_1}$$
$$= [g_1,(c-2)\times g_0,g_2]^{(c-2)\alpha_1}[g_1,g_2,g_0,(c-3)\times g_0]^{\alpha_1}$$
$$= [g_1,(c-2)\times g_0,g_2]^{(c-2)\alpha_1}[g_1,g_0,g_2,(c-3)\times g_0]^{\alpha_1}[g_2,g_0,g_1,(c-3)\times g_0]^{-\alpha_1}$$
$$= [g_1,(c-2)\times g_0,g_2]^{(c-1)\alpha_1}[g_2,(c-2)\times g_0,g_1]^{-\alpha_1} .$$

Now let θ_4 be the endomorphism of F defined $g_1\theta_4 = g_2$, $g_2\theta_4 = g_1$, $g_i\theta_4 = g_i$ $(i \neq 1,2)$. Then

$$u_2\theta_4 = [g_2,(c-2)\times g_0,g_1]^{(c-1)\alpha_1}[g_1,(c-2)\times g_0,g_2]^{-\alpha_1} \in V .$$

Hence $u_3 = (u_2^{c-1})(u_2\theta_4) = [g_1,(c-2)\times g_0,g_2]^{\beta} \in V$, where $\beta = (c-1)^2\alpha_1 - \alpha_1 \neq 0$ since $c \geqq 3$. Now let x be an arbitrary element of F . Let θ_5 be the endomorphism of F defined $g_2\theta_5 = x$, $g_i\theta_5 = g_i$ $(i \neq 2)$. Then $[[g_1,(c-2)\times g_0]^{\beta},x] = [g_1(c-2)\times g_0,x]^{\beta} = u_3\theta_5 \in V$ and x is an arbitrary element of F so $[g_1,(c-2)\times g_0]^{\beta} \in Z$. But then, since $\beta \neq 0$ and F/Z is torsion-free, $[g_1,(c-2)\times g_0] \in Z$ and thus $[g_1,(c-2)\times g_0,g_{c-1}] \in V$.

It will now be proved that, for each r $(1 \leqq r \leqq c-2)$, $v_r = [g_1,(c-1-r)\times g_0,g_{c-r},g_{c-r+1},\ldots,g_{c-1}] \in V$ by induction over r . This fact has just been proved for $r = 1$. Now suppose that $v_{r+1} \in V$. Let θ_6 be the endomorphism of F defined by $g_0\theta_6 = g_0g_{c-r-1}$, $g_i\theta_6 = g_i$ $(i \neq 0)$. Then

$v_r\theta_6 = [g_1,(c-r-1)\times g_0 g_{c-r-1},g_{c-r},g_{c-r+1},\ldots,g_{c-1}] \in V$.

Since $r \leqq c - 3$, and so $c-r-1 \geqq 2$, this may be written in the form $v_r\theta_6 = [g_1,g_0 g_{c-r-1},(c-r-2)\times g_0 g_{c-r-1},g_{c-r},g_{c-r+1},\ldots,g_{c-1}] =$ $= w_1 w_2$, where $w_1 = [g_1,g_0,(c-r-2)\times g_0 g_{c-r-1},g_{c-r},g_{c-r+1},\ldots,g_{c-1}]$ and $w_2 = [g_1,g_{c-r-1},(c-r-2)\times g_0 g_{c-r-1},g_{c-r},g_{c-r+1},\ldots,g_{c-1}]$. Thus

$$w_1 = \prod_{t=0}^{c-r-2} [g_1,g_0,t\times g_0,(c-r-t-2)\times g_{c-r-1},g_{c-r},g_{c-r+1},\ldots,g_{c-1}]^{\binom{c-r-2}{t}}$$

and

$$w_2 = \prod_{t=0}^{c-r-2} [g_1,g_{c-r-1},t\times g_0,(c-r-t-2)\times g_{c-r-1},g_{c-r},\ldots,g_{c-1}]^{\binom{c-r-2}{t}} .$$

Then, applying lemma 5.3 to $w_1 w_2$ with $\nu_0 = c-r-2$, $\nu_i = 1$ $(1 \leqq i \leqq c-1)$ and $\nu_i = 0$ $(i \geqq c)$, since $w_1 w_2 \in V$, $w_1' w_2' \in V$ where

$w_1' = [g_1,(c-r-2)\times g_0,g_{c-r-1},g_{c-r},\ldots,g_{c-1}]^{c-r-2}$ and

$w_2' = [g_1,g_{c-r-1},(c-r-2)\times g_0,g_{c-r},g_{c-r+1},\ldots,g_{c-1}]$.

But $c-r-1 \geqq 1$, so

$$w_2' = [g_1,(c-r-2)\times g_0,g_{c-r-1},g_{c-r},\ldots,g_{c-1}] \cdot [g_{c-r-1},(c-r-2)\times g_0,g_1,g_{c-r},g_{c-r+1},\ldots,g_{c-1}]^{-1}$$

and thus

$$w_1' w_2' = [g_1,(c-r-2)\times g_0,g_{c-r-1},g_{c-r},\ldots g_{c-1}]^{c-r-1} \cdot [g_{c-r-1},(c-r-2)\times g_0,g_1,g_{c-r},g_{c-r+1},\ldots,g_{c-1}]^{-1} .$$

Write θ_7 for the endomorphism of F defined $g_1\theta_7 = g_{c-r-1}$, $g_{c-r-1}\theta_7 = g_1$, $g_i\theta_7 = g_i$ $(i \neq 1 , c-r-1)$: this is possible since $r \leqq c - 3$ and so $c-r-1 \geqq 2$. Then

$$(w_1' w_2')^{c-r-1}(w_1' w_2')\theta_7 = [g_1,(c-r-2)\times g_0,g_{c-r-1},g_{c-r},\ldots,g_{c-1}]^d \in V$$

where $d = (c-r-1)^2 - 1 \neq 0$. In other words $v_{r+1}{}^d \in V$.
Now let x be an arbitrary element of G and let θ_8 be the endomorphism of F defined $g_{c-1}\theta_8 = x$, $g_i\theta_8 = g_i$ $(i \neq c-1)$.
Then $[[g_1,(c-r-2)\times g_o,g_{c-r-1},g_{c-r},\dots,g_{c-2}]^d,x] = v_{r+1}{}^d\theta_8 \in V$ and, since x is an arbitrary element of F ,
$[g_1,(c-r-2)\times g_o,g_{c-r-1},g_{c-r},\dots,g_{c-2}]^d \in Z$. But $d \neq 0$ and F/Z is torsion free so $[g_1,(c-r-2)\times g_o,g_{c-r-1},g_{c-r},\dots,g_{c-2}] \in Z$.
Thus $v_{r+1} = [g_1,(c-r-2)\times g_o,g_{c-r-1},g_{c-r},\dots,g_{c-1}] \in V$. This completes the induction. Thus, in the particular case $r = c - 2$,
$v_{c-2} = [g_1,g_o,g_2,g_3,g_4,\dots,g_{c-1}] \in V$ and since V is verbal, $\gamma_c(F) \leqq V$ which contradicts the choice of c . This completes the proof of the theorem when the rank of G is infinite.

(B) Proof when the rank is finite. Suppose G_n is a relatively free group of finite rank n , nilpotent, exactly metabelian and has a torsin-free central factor group. It is proved that G_n is one of the groups $F_n(\underline{\underline{S}}_2 \cap \underline{\underline{N}}_c)$.

Let $F = F_\omega$ be an absolutely free group on free generators $\{g_i\}_{i<\omega}$ and let F_n be the subgroup of F generated by $\{g_o,g_1,\dots,g_{n-1}\}$ — an absolutely free group of rank n .
Let $\underline{\underline{V}}$ be the variety generated by G_n , let V be the corresponding verbal subgroup of F and let $V_n = F_n \cap V$. Then it may be assumed that $G_n = F_n/V_n$.

Let $G = F/V$. Then clearly G is relatively free, nilpotent, exactly metabelian and of infinite rank. It is now shown that G has a torsion-free central factor group.

First a property of V should be remarked: suppose an element u of F has the property that for any homomorphism $\theta : F \to F_n$, $u\theta \in V_n$; then it follows that $u \in V$ (for then u is a law of G_n and consequently of G).

To prove that G has a torsion-free central factor group, it is sufficient to prove the following: if r is a non-zero integer and $u \in F$ has the property that for any $x \in F$, $[u^r,x] \in V$, then, for any $y \in F$, $[u,y] \in V$.

Suppose then that r is a non-zero integer and u has the property that for any $x \in F$, $[u^r,x] \in V$ and suppose that $y \in F$.

Let θ be any homomorphism $F \to F_n$ and let x' be an arbitrary element of F_n . Then, since F is of infinite rank, there exists a homomorphism $\theta' : F \to F_n$ and an integer k such that $u\theta = u\theta'$ and $x' = g_k\theta'$. Thus $[(u\theta)^r,x'] = [u^r,g_k]\theta' \in V_n$ by the assumed property of u . But x' is an arbitrary element of F_n and G_n has a torsion-free central factor group. Thus for any $y' \in F_n$, $[u\theta,y'] \in V_n$. Thus $[u,y]\theta \in V_n$. But

θ is an arbitrary homomorphism and so $[u,y] \in V$. This completes the proof that G has a torsion-free central factor group.

Applying part (A) of this proof, there exists an integer c such that $G = F_\omega(\underline{\underline{S}}_2 \cap N_c)$ and so $G_n = F_n(\underline{\underline{S}}_2 \cap \underline{\underline{N}}_c)$. $\diamond$

The construction of counter-examples for the other soluble lengths is an easier task.

<u>Theorem 5.2</u>

For each $\ell \neq 2$ there exists a group G of soluble length exactly ℓ (that is, $G \in \underline{\underline{S}}_\ell \setminus \underline{\underline{S}}_{\ell-1}$) with the properties: G is relatively free, nilpotent, $G/\zeta(G)$ is torsion-free but G itself is not torsion-free.

<u>Proof</u>

The cyclic group of order 2 is a group with the required properties for the case $\ell = 1$.

Now suppose $\ell > 2$. It will be convenient to write $h = \ell - 1$ $h \geqq 2$.

Let $F = F_\omega(\underline{\underline{N}}_c)$ where $c = 2^h$. Then $\delta^\ell(F) \leqq \gamma_c(F)$ but $\delta^\ell(F)$ is not trivial. Thus $\delta^\ell(F)$ is a fully-invariant subgroup of F and is free Abelian. Let V be the subgroup $V = \{ x^2 : x \in \delta^\ell(F) \}$. Then V is also a fully-invariant subgroup and, writing $G = F/V$, G is nilpotent of class c, soluble of length exactly ℓ, relatively free but not torsion-free. It remains to show that $G/\zeta(G)$ is torsion-free, and this is accomplished by showing that $\zeta(G) = \gamma_c(G)$. Now $\delta^h(F) \geqq V \geqq \gamma_{c+1}(F) = \{1\}$ so the centre of F is contained in the complete inverse image of $\zeta(G) = \zeta(F/V)$ which in turn is contained in the complete inverse image of the centre of $F/\delta^h(F)$. But $F/\delta^h(F) \cong F_\omega(\underline{\underline{S}}_h \cap \underline{\underline{N}}_c)$, and so by the results of the previous chapter, the centre of this group is $\gamma_c(F/\delta^h(F))$. The result follows. ◇

APPENDIX I

NON-COLLECTABILITY

In this appendix the fact mentioned in Chapter 2 (page 78) is established: that if α , β are elements of a semiweight range W , $\alpha \ll \beta$ and provided the rank τ of the absolutely free group F_τ is large enough, there exists an element x in $W_\alpha(F_\tau)$ which cannot be described by a W-basic expression modulo $W_\beta(F_\tau)$ at all.

The proviso that τ be "large enough" here will be seen to be that τ is large enough to ensure the existence of a basic commutator of semiweight at least α but much less than β . But this is not a strong condition on τ at all, as can be seen from theorem 2.4 which tells us that if $\tau \geqq 3$ then it is large enough.

<u>Definition I.1</u>

(A) For any W-basic commutator $\underset{\sim}{c}$ other than g_0 and any non-negative integer n , the commutator $(\underset{\sim}{c},n)$ is defined recursively over n :

(i) $(\underset{\sim}{c},0) = \underset{\sim}{c}$,

(ii) $(\underset{\sim}{c},n) = (\underset{\sim}{c},n-1)$ $(n > 0)$ (see definition 4.2) .

(B) For any W-basic commutator $\underset{\sim}{c}$ other than g_0 the expression $\underset{\sim}{c}^0$ is defined recursively over the weight of $\underset{\sim}{c}$.

(i) $g_i{}^0 = [g_i, g_0{}^{-1}]$,

(ii) $[\underset{\sim}{c}_1,\underset{\sim}{c}_2]^{\circ} = [\underset{\sim}{c}_1^{\circ},\underset{\sim}{c}_2]$ if $\underset{\sim}{c}_2 \neq \underset{\sim}{g}_o$,

$= [\underset{\sim}{c}_1,\underset{\sim}{c}_2,\underset{\sim}{g}_o^{-1}]$ if $\underset{\sim}{c}_2 = \underset{\sim}{g}_o$.

(C) For any W-basic commutator other than $\underset{\sim}{g}_o$ and any non-negative integer n , the expression $(\underset{\sim}{c}/n)$ is defined

$(\underset{\sim}{c}/n) = (\underset{\sim}{c},n)^{\circ} = (\underset{\sim}{c}^{\circ},n)$. ◇

It follows immediately (see lemma 4.2(i)) that $\sigma((\underset{\sim}{c},n)) = \sigma(\underset{\sim}{c})(+1)^n$, $\sigma(\underset{\sim}{c}^{\circ}) = \sigma(\underset{\sim}{c}) + 1$ and $\sigma((\underset{\sim}{c}/n)) = \sigma(\underset{\sim}{c})(+1)^{n+1}$. Also notice that , if $[\underset{\sim}{c},\underset{\sim}{g}_o]$ is W-basic, then there exists integers $i \geqq 0$ and $k \geqq 1$ such that $\underset{\sim}{c} = [\underset{\sim}{g}_i,k\times\underset{\sim}{g}_o]$, and then

$([\underset{\sim}{c},\underset{\sim}{g}_o],n) = [\underset{\sim}{g}_i,(k+n)\times\underset{\sim}{g}_o] = (\underset{\sim}{g}_i,k+n)$,

$[\underset{\sim}{c},\underset{\sim}{g}_o]^{\circ} = [\underset{\sim}{g}_i,k\times\underset{\sim}{g}_o,\underset{\sim}{g}_o^{-1}]$ and

$([\underset{\sim}{c},\underset{\sim}{g}_o]/n) = [\underset{\sim}{g}_i,(k+n)\times\underset{\sim}{g}_o,\underset{\sim}{g}_o^{-1}] = (\underset{\sim}{g}_i/k+n)$.

It now becomes necessary to make a rather peculiar generalization of definition 2.8 to encompass arbitrary expressions.

<u>Definition I.2</u>

Let us write $\mathcal{N}^{\circ}$ for the set of sequences $N = (\nu_i)_{i<\tau}$ of non-negative integers, and define a pre-order $\leqq$ on this set by

$(\nu_i)_{i<\tau} \leqq (\nu_i')_{i<\tau} \iff$ for all i $(1 \leqq i \leqq \tau)$, $\nu_i \leqq \nu_i'$.

Notice that the values of ν_o , ν_o' are not relevant to the pre-order.

The corresponding strict relation $<$ is given by

$(\nu_i)_{i<\tau} < (\nu_i')_{i<\tau} \iff$ for all i $(1 \leqq i < \tau)$, $\nu_i \leqq \nu_i'$ and there exists r $(1 \leqq r < \tau)$ such that $\nu_r < \nu_r'$.

Addition is defined: if $N = (\nu_i)_{i<\tau}$ and $N' = (\nu_i')_{i<\tau}$, then $(N + N') = (\nu_i + \nu_i')_{i<\tau}$.

To each expression $\underset{\sim}{x}$ a subset $\mathcal{M}(\underset{\sim}{x})$ of $\mathcal{N}$ is defined recursively over the height of $\underset{\sim}{x}$.

(i) $\mathcal{M}(\underset{\sim}{1}) = \emptyset$, $\mathcal{M}(\underset{\sim}{g}_j) = \{(\nu_i)_{i<\tau}\}$ where $\nu_j = 1$, $\nu_i = 0$ $(i \neq j)$.

(ii) $\mathcal{M}(\underset{\sim}{x}^{-1}) = \mathcal{M}(\underset{\sim}{x})$.

(iii) $\mathcal{M}(\underset{\sim}{x}\underset{\sim}{y}) = \mathcal{M}(\underset{\sim}{x}) \cup \mathcal{M}(\underset{\sim}{y})$.

(iv) $\mathcal{M}([\underset{\sim}{x},\underset{\sim}{y}]) = \{ M + N : M \in \mathcal{M}(\underset{\sim}{x}) , N \in \mathcal{M}(\underset{\sim}{y}) \}$.

A sequence N is a "lower bound" for a subset $\mathcal{M}$ of $\mathcal{N}$ if $N \leqq M$ for all $M \in \mathcal{M}$; it is a "strict lower bound" if $N < M$ for all $M \in \mathcal{M}$. $\diamond$

<u>Lemma I.1</u>

Easy properties of $\mathcal{N}$.

(i) If $\underset{\sim}{c}$ is a commutator then $\mathcal{M}(\underset{\sim}{c})$ contains one and only one sequence, namely $M(\underset{\sim}{c})$ (definition 2.8).

(ii) If $\underset{\sim}{c}$ is a W-basic commutator other than $\underset{\sim}{g}_0$ and n is a non-negative integer, then $M(\underset{\sim}{c})$ is a lower bound for $\mathcal{M}(\underset{\sim}{c})$, $\mathcal{M}((\underset{\sim}{c},n))$ and $\mathcal{M}((\underset{\sim}{c}/n))$.

(iii) M is a lower bound for $\mathcal{M}(\underset{\sim}{x}_1\underset{\sim}{x}_2)$ if and only if it is a lower bound for both $\mathcal{M}(\underset{\sim}{x}_1)$ and $\mathcal{M}(\underset{\sim}{x}_2)$. Similarly M is a strict lower bound for $\mathcal{M}(\underset{\sim}{x}_1\underset{\sim}{x}_2)$ if and only if is a strict lower bound for both $\mathcal{M}(\underset{\sim}{x}_1)$ and $\mathcal{M}(\underset{\sim}{x}_2)$.

(iv) If M_1, M_2 are lower bounds for $\mathcal{M}(\underset{\sim}{x}_1)$, $\mathcal{M}(\underset{\sim}{x}_2)$ respectively, then $M_1 + M_2$ is a lower bound for $\mathcal{M}([\underset{\sim}{x}_1,\underset{\sim}{x}_2])$. If either M_1 or M_2 are strict lower bounds then so is $M_1 + M_2$.

◇

Lemma I.2

Suppose $\underset{\sim}{c}$ is a W-basic commutator other than g_o and n is a non-negative integer. Let $\rho : \underset{\sim}{A} \to G$ be any description of a group G. Then there exists a (possibly empty) expression $\underset{\sim}{z}$ such that $(\underset{\sim}{c}/n)\rho = ((\underset{\sim}{c}/n+1)^{-1}(\underset{\sim}{c},n+1)^{-1}\underset{\sim}{z})\rho$ where $M(\underset{\sim}{c})$ is a strict lower bound for $\mathcal{M}(\underset{\sim}{z})$.

Proof

The proof uses the following easily checked group identity: if a and b are elements of a group, then $[a,b^{-1}] = [a,b,b^{-1}]^{-1}[a,b]^{-1}$. If $\underset{\sim}{c} = [g_i,\underset{\sim}{k}]$ for some non-negative integer k, then $(\underset{\sim}{c}/n) = [(\underset{\sim}{c},n),g_o^{-1}]$ and so, using the group identity just mentioned,

$$\begin{aligned}(c/n)\rho &= ([(\underset{\sim}{c},n),g_o,g_o^{-1}]^{-1}[(\underset{\sim}{c},n),g_o]^{-1})\rho \\ &= ((\underset{\sim}{c}/n+1)^{-1}(\underset{\sim}{c},n+1)^{-1})\rho\end{aligned}$$

and the result is true with $\underset{\sim}{z}$ empty.

The proof may now proceed by induction over the weight of $\underset{\sim}{c}$, assuming $\underset{\sim}{c} = [\underset{\sim}{c}_1, \underset{\sim}{c}_2]$, $\underset{\sim}{c}_2 \neq \underset{\sim}{g}_0$ and the result is true for $\underset{\sim}{c}_1$. Thus $(\underset{\sim}{c}/n) = [(\underset{\sim}{c}_1/n), \underset{\sim}{c}_2]$ and so

$(\underset{\sim}{c}/n)\rho = [\,(\underset{\sim}{c}_1/n{+}1)^{-1}(\underset{\sim}{c}_1,n{+}1)^{-1}\underset{\sim}{z}_1\,,\ \underset{\sim}{c}_2\,]\rho$

where $M(\underset{\sim}{c}_1)$ is a strict lower bound for $\mathcal{M}(\underset{\sim}{z}_1)$. But now

$(\underset{\sim}{c}/n)\rho = ([\,(\underset{\sim}{c}_1/n{+}1)^{-1}(\underset{\sim}{c}_1,n{+}1)^{-1}\,,\ \underset{\sim}{c}_2\,]\,\underset{\sim}{z}_2)\rho$

where $\underset{\sim}{z}_2 = [\,(\underset{\sim}{c}_1/n{+}1)^{-1}(\underset{\sim}{c}_1,n{+}1)^{-1}\,,\ \underset{\sim}{c}_2\,,\ \underset{\sim}{z}_1\,][\,\underset{\sim}{z}_1\,,\ \underset{\sim}{c}_2\,]$.

But $M(\underset{\sim}{c}_1)$ is a strict lower bound for $\mathcal{M}(\underset{\sim}{z}_1)$ so $M(\underset{\sim}{c}) = M(\underset{\sim}{c}_1) + M(\underset{\sim}{c}_2)$
$M(\underset{\sim}{c}) = M(\underset{\sim}{c}_1) + M(\underset{\sim}{c}_2)$ is a strict lower bound for $[\underset{\sim}{z}_1, \underset{\sim}{c}_2]$ and hence for $\underset{\sim}{z}_2$. Thus

$(\underset{\sim}{c}/n)\rho = (\,[(\underset{\sim}{c}_1/n{+}1)^{-1},\underset{\sim}{c}_2]\underset{\sim}{z}_3[(\underset{\sim}{c}_1,n{+}1)^{-1},\underset{\sim}{c}_2]\underset{\sim}{z}_2\,)\,\rho$

where $\underset{\sim}{z}_3 = [(\underset{\sim}{c}_1/n{+}1)^{-1},\underset{\sim}{c}_2,(\underset{\sim}{c}_1,n{+}1)^{-1}]$. But $\mathcal{M}(\underset{\sim}{z}_3)$ contains only the sequence $N = (\nu_i)_{i<\tau}$ where $\nu_0 = 2\mu_0(\underset{\sim}{c}_1) + \mu_0(\underset{\sim}{c}_2) + 2n + 3$ and for $1 \leqq i < \tau$, $\nu_i = 2\mu_i(\underset{\sim}{c}_1) + \mu_i(\underset{\sim}{c}_2)$. But $\underset{\sim}{c}$ is W-basic, so $\underset{\sim}{c}_1 > \underset{\sim}{c}_2$ and hence $\underset{\sim}{c}_1 \neq \underset{\sim}{g}_0$. But $\underset{\sim}{c}_1$ is itself W-basic, so there exists r $(1 \leqq r < \tau)$ such that $\mu_r(\underset{\sim}{c}_1) > 0$. Then, since $M(\underset{\sim}{c})$ is the sequence $(\mu_i(\underset{\sim}{c}_1) + \mu_i(\underset{\sim}{c}_2))_{i<\tau}$, $M(\underset{\sim}{c}) < N$ and so $M(\underset{\sim}{c})$ is a strict lower bound for $M(\underset{\sim}{z}_3)$.

Then $(\underset{\sim}{c}/n)\rho = (\,[(\underset{\sim}{c}_1/n{+}1),\underset{\sim}{c}_2]^{-1}\underset{\sim}{z}_4\underset{\sim}{z}_3[(\underset{\sim}{c}_1,n{+}1),\underset{\sim}{c}_2]^{-1}\underset{\sim}{z}_5\underset{\sim}{z}_2\,)\,\rho$

where $\underset{\sim}{z}_4 = [\underset{\sim}{c}_2,(\underset{\sim}{c}_1/n{+}1),(\underset{\sim}{c}_1/n{+}1)^{-1}]$ and

$\underset{\sim}{z}_5 = [\underset{\sim}{c}_2,(\underset{\sim}{c}_1,n{+}1),(\underset{\sim}{c}_1,n{+}1)^{-1}]$ and again $M(\underset{\sim}{c})$ is a strict lower bound for $\mathcal{M}(\underset{\sim}{z}_4)$ and $\mathcal{M}(\underset{\sim}{z}_5)$. Thus

$$(c/n)\rho = ([(c_1/n+1),c_2]^{-1}[(c_1,n+1),c_2]^{-1} z_6)\,\rho$$
$$= ((c/n+1)^{-1}(c,n+1)^{-1} z_6)\,\rho$$

where $z_6 = z_4 z_3 [z_4 z_3, [(c_1,n+1),c_2]^{-1}] z_5 z_2$ and again $M(c)$ is a strict lower bound for $\mathcal{M}(z_6)$. This proves the lemma.

◇

__Lemma I.3__

If N is a strict lower bound for $\mathcal{M}(x)$ and $E : x \to y$ then N is a strict lower bound for $\mathcal{M}(y)$.

__Proof__

This follows immediately by checking the various parts of definition 2.4.

◇

__Lemma I.4__

Let c be a W-basic commutator other than g_o and let $\rho : A \to G$ be any description of a group G. Then to each non-negative integer n there exists an expression $x_n = u_n (c/n)^{(-1)^n} v_n$, either u_n or v_n possibly empty, such that

(i) $x_n \rho = c^o \rho$,

(ii) u_o is empty, and if $n \geq 1$, u_n is of the form

$u_n = b_1^{\beta_1} b_2^{\beta_2} \ldots b_k^{\beta_k} \in B^W_{\alpha_n}$ where $k \geq n$ and $\alpha_n = \sigma(c)(+1)^{n+1}$.

(iii) If v_n is nonempty, then $\sigma(v_n) \geq \alpha_n$ and $M(c)$ is a strict lower bound for $\mathcal{M}(v_n)$.

<u>Proof</u>

By induction over n . If $n = 0$, $(c/n) = (c/0) = c^0$ and the lemma is true with both u_0 and v_0 empty.

Now suppose the result is true as stated for n . The corresponding result is proved for $n + 1$. By lemma I.2, there exists an expression z_1 (possibly empty) such that
$(c/n)\rho = ((c/n+1)^{-1}(c,n+1)^{-1}z_1)\rho$ and $M(c)$ is a strict lower bound for $\mathcal{M}(z_1)$. Hence $c^0\rho = x'\rho$ where
$x' = u_n((c/n+1)^{-1}(c,n+1)^{-1}z_1)^{(-1)^n}v_n$.

Suppose n is even. Then $x' = u_n(c/n+1)^{-1}(c,n+1)^{-1}z_1v_n$
and $E : x' \to u_n(c,n+1)^{-1}(c,n+1)^{-1}z_2z_1v_n$ where
$z_2 = [(c/n+1)^{-1},(c,n+1)^{-1}]$ and so $M(c)$ is a strict lower bound for $\mathcal{M}(z_2)$ and $\sigma(z_2) \geq \alpha_{n+1}$. Thus
$E : x' \to u_n(c,n+1)^{-1}z_2z_1v_n(c/n+1)^{-1}z_3$ where
$z_3 = [(c/n+1)^{-1},z_2z_1v_n]$ and so $M(c)$ is a strict lower bound for $\mathcal{M}(z_3)$ and $\sigma(z_3) \geq \alpha_{n+1}$. But now, since $\sigma(z_2z_1v_n) \geq \alpha_n$,
$E : z_2z_1v_n \to wz_4$ where either w or z_4 may be empty, but when they are not, $w \in B^W_{\alpha_n+1}$, $\sigma(w) \geq \alpha_n$ and $\sigma(z_4) \geq \alpha_{n+1}$.

Further, $M(c)$ is a strict lower bound for $\mathcal{M}(w)$ and for $\mathcal{M}(z_4)$. Thus $E : x' \to u_n(c,n+1)^{-1} w z_4 (c/n+1)^{-1} z_3$.

It is now shown that $E : x' \to u_n w' z_5 z_4 (c/n+1)^{-1} z_3$ where $w' \in B^W_{\alpha_n+1}$ but $w' \neq 1$, $\sigma(w') \geqq \alpha_n$ and z_5 may be empty, but when it is not $\sigma(z_5) \geqq \alpha_n + 1$ and $M(c)$ is a strict lower bound for $\mathcal{M}(z_5)$. If w is empty or 1 this is true with $w' = (c,n+1)^{-1}$ and z_5 empty. Otherwise, suppose $w = a_1^{\gamma_1} a_2^{\gamma_2} \ldots a_h^{\gamma_h}$. For each i $(1 \leqq i \leqq h)$, $M(a_i) \in \mathcal{M}(w)$ and so $M(c) < M(a_i)$. Thus for each i there exists an integer r $(1 \leqq r < \tau)$ such that $\mu_r(c) < \mu_r(a_i)$. But $\mu_r((c,n+1)) = \mu_r(c)$ and so $(c,n+1) \neq a_i$. Thus $(c,n+1)$ is not the same as any of the commutators a_i $(1 \leqq i \leqq h)$. But $(c,n+1)$ is W-basic (see definition I.1(A) and lemma 4.2(iv)). Thus there exists an integer t $(0 \leqq t \leqq h)$ such that $w' = a_1^{\gamma_1} a_2^{\gamma_2} \ldots a_t^{\gamma_t} (c,n+1)^{-1} a_{t+1}^{\gamma_{t+1}} a_{t+2}^{\gamma_{t+2}} \ldots a_h^{\gamma_h}$ is a W-basic expression. But $w \in B^W_{\alpha_n+1}$ so each a_i is a commutator of semiweight $< \alpha_n + 1$. Also $\sigma((c,n+1)) = \sigma(c)(+1)^{n+1} = \alpha_n$ so $w' \in B^W_{\alpha_n+1}$. But $w' \neq 1$ since it contains the non-trivial factor $(c,n+1)$. Again, $\sigma(w) \geqq \alpha_n$ so each a_i is of semiweight $\geqq \alpha_n$ and $\sigma((c,n+1)) = \alpha_n$ so $\sigma(w') \geqq \alpha_n$. Then $E : (c,n+1)^{-1} w \to w' z_5$ where $\sigma(z_5) \geqq \sigma((c,n+1)^{-1} w) + 1 \geqq \alpha_n + 1$ and, since z_5 is a product of commutators of $(c,n+1)^{-1}$ with other W-basic commutators, $M(c)$ is a strict lower bound for $\mathcal{M}(z_5)$.

Finally, writing $u_{n+1} = u_n w'$ and $v_{n+1} = z_5 z_4 [z_5 z_4, (c/n+1)^{-1}] z_3$, $E : x' \to u_{n+1}(c/n+1)^{-1} v_{n+1}$ and u_{n+1} and v_{n+1} are of the required form.

The argument when n is odd is similar but slightly simpler.

$\diamond$

<u>Theorem I.1</u>

Suppose W is a semiweight range, α and $\beta \in W$, $\alpha \ll \beta$ and there exists a W-basic commutator $c \in A$ such that $\alpha \leqq \sigma(c) \ll \beta$. Then, if $\rho : A \to F$ is any free description of the absolutely free group F, the element $c^{\circ}\rho \in W_{\alpha}(F)$ cannot be described by a W-basic expression modulo $W_{\beta}(F)$ at all.

<u>Proof</u>

Suppose on the contrary that $c^{\circ}\rho$ can be described by a W-basic expression $w = a_1^{\gamma_1} a_2^{\gamma_2} \ldots a_n^{\gamma_n}$. Then, by lemma I.4, there exists an expression $x_{n+1} = u_{n+1}(c/n+1)^{(-1)^{n+1}} v_{n+1}$ such that $x_{n+1}\rho = c^{\circ}\rho$ where u_{n+1} is a basic expression of the form $u_{n+1} = b_1^{\beta_1} b_2^{\beta_2} \ldots b_k^{\beta_k}$ $(k \geqq n+1)$ and $\sigma(v_{n+1}) \geqq \sigma(c)(+1)^{n+2}$. Also $\sigma(c/n+1) \geqq \sigma(c)(+1)^{n+2}$ so $x_{n+1}\rho = u_{n+1}$ modulo $W_{\xi}(F)$, where $\xi = \sigma(c)(+1)^{n+2}$. But $\sigma(c) \ll \beta$, so $\xi \leqq \beta$. Now there exists h $(0 \leqq h \leqq n)$ such that

$\underset{\sim}{w}' = \underset{\sim}{a}_1^{\gamma_1} \underset{\sim}{a}_2^{\gamma_2} \cdots \underset{\sim}{a}_h^{\gamma_h} \in \underset{\sim}{B}_\xi^W$ and $\sigma(\underset{\sim}{a}_{h+1}^{\gamma_{h+1}} \underset{\sim}{a}_{h+2}^{\gamma_{h+2}} \cdots \underset{\sim}{a}_n^{\gamma_n}) \geqq \xi$. Then $\underset{\sim}{w}'\rho = \underset{\sim}{c}^{o}\rho = \underset{\sim}{u}_{n+1}\rho$ modulo $W_\xi(F)$ and both $\underset{\sim}{w}'$ and $\underset{\sim}{u}_{n+1} \in \underset{\sim}{B}_\xi^W$. Thus, by the basis theorem, $\underset{\sim}{w}' = \underset{\sim}{u}_{n+1}$ and so $h = k$. But it has already been seen that $h \leqq n$ and $k \geqq n + 1$: this is a contradiction. ◇

APPENDIX II

TERMS AND SYMBOLS
USED IN THE TEXT

SYMBOLS IN MORE OR LESS COMMON USE

Logic

$\Rightarrow$ logical implication.

$\Leftrightarrow$ logical equivalence.

Set theory

For any property $\mathcal{P}$ that the elements of a set A may have, $\{ x : x \in A , \mathcal{P}(x) \}$ is the set of all elements of A for which $\mathcal{P}(x)$ is true. When the set A is clear from the context, $\{ x : \mathcal{P}(x) \}$ may be written.

$A \cup B$ and $A \cap B$ are the union and intersection respectively of the sets A and B. The union of a family of sets is written $\bigcup_{i=1}^{\infty} A_i$, $\bigcup_{i<\tau} A_i$, $\bigcup_{i \in I} A_i$ etc., and the union of a set $\mathcal{A}$ of sets is written $\bigcup \mathcal{A}$. The symbol $\bigcap$ is used in a similar fashion for intersections.

$a \in A$ means " a is a member of A ". Occasionally $a , b \in A$ is used as shorthand for " $a \in A$ and $b \in A$ " when no confusion can arise. $a \notin A$ means " a is not a member of A ".

$\{a\}$ is the set whose only member is a , $\{a,b\}$ is the set whose only members are a and b , and so on.

$\emptyset$ is the empty set.

$A \subseteq B$ means " A is a subset of B " .

$A \setminus B$ is the complement of B in A, the set $\{ x : x \in A , x \notin B \}$.

$(x_i)_{i=1}^{n}$, $(x_i)_{i=1}^{\infty}$, $(x_i)_{i<\tau}$ etc. are finite and transfinite sequences; $\{x_i\}_{i=1}^{n}$, $\{x_i\}_{i=1}^{\infty}$, $\{x_i\}_{i<\tau}$ are the corresponding sets.

$\varphi : A \to B$ indicates that φ is a function mapping the set A into the set B . Exceptions are the notations $d : \underset{\sim}{x} \to \underset{\sim}{y}$, $D : \underset{\sim}{x} \to \underset{\sim}{y}$, $e : \underset{\sim}{x} \to \underset{\sim}{y}$ and $E : \underset{\sim}{x} \to \underset{\sim}{y}$ which are given special definitions in Chapter 2. Functions are variously written as right or left operators: the image of x under φ may be written $x\varphi$ or $\varphi(x)$; the notation used for a particular function will be made clear in the text. If X is a subset of the domain of φ , then its image under φ is written $X\varphi$ or $\varphi(X)$.

$|A|$ is the (cardinal) number of members of A .

Arithmetic

ω is the set of non-negative integers and the first infinite ordinal.

$m \mid n$, for integers m and n , means " m divides n ".

$\binom{n}{r}$ is the usual binomial coefficient: $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

Group theory

With the exception of the underlying group of a Lie ring, groups are written multiplicatively. The unit is denoted 1 .

$[x,y]$ is the element $x^{-1}y^{-1}xy$.

$A \leqq B$ means " A is a subgroup of B ". If A is a normal subgroup of B the factor group is denoted B/A .

$A \cong B$ means " A is isomorphic with B ".

AB is the subgroup generated by the normal subgroups A and B . The subgroup generated by a family of normal subgroups is denoted

$$\prod_{i=1}^{\infty} A_i \ , \quad \prod_{i<\tau} A_i \ , \quad \prod_{i\in I} A_i \quad \text{etc.}$$

$[A,B]$ is the subgroup generated by all commutators $[a,b]$ where $a \in A$ and $b \in B$.

$\gamma_c(G)$, for positive integers c , are terms of the lower central series of the group G , defined by $\gamma_1(G) = G$ and
$\gamma_c(G) = [\gamma_{c-1}(G), G]$ for $c > 1$.

$\delta^n(G)$, for non-negative integers n , are terms of the derived series of the group G , defined by $\delta^0(G) = G$ and
$\delta^n(G) = \delta(\delta^{n-1}(G)) = [\delta^{n-1}(G), \delta^{n-1}(G)]$ for $n > 0$.

$\zeta_n(G)$, for non-negative integers n , are terms of the upper central series of the group G , defined by $\zeta_0(G) = \{1\}$ and
$\zeta_n(G)$ is the complete inverse image of the centre of
$G/\zeta_{n-1}(G)$, for $n > 0$.

Varieties

The language and notation concerned with this topic will follow Hanna Neumann [9]. In particular,

F_τ , for some cardinal τ , is an absolutely free group of rank τ .
F usually denotes an absolutely free group of arbitrary rank.

Varieties themselves are distinguished by double underlining, $\underline{\underline{V}}$, $\underline{\underline{S}}_2$ and so on, to indicate German script. If $\underline{\underline{V}}$ is a variety the corresponding verbal subgroup of the absolutely free group F is written V .

$F_\tau(\underline{\underline{V}})$ is the free group of rank τ of the variety $\underline{\underline{V}}$ (a $\underline{\underline{V}}$-free group . $F_\tau(\underline{\underline{V}}) = F_\tau/V$.

Particular varieties: $\underline{\underline{N}}_c$ is the variety of groups which are nilpotent of class c and $\underline{\underline{S}}_\ell$ is the variety of groups which are soluble of length ℓ . The notation associated with polynilpotent varieties is defined precisely in definition 3.1.

Algebras

The word "algebra" in this thesis is to be construed in the sense of "universal algebra", as in B. H. Neumann [8] . Language and notation concerned with algebras follows this work.

SYMBOLS DEFINED IN THE TEXT

References, unless otherwise stated, are to definition numbers.

<u>TERMS DEFINED IN THE TEXT</u>

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