Accepted Manuscript

Title: Uncertainty, rationality and complexity in a multi-sectoral dynamic model: the Dynamic Stochastic Generalized Aggregation approach

Author: Michele Catalano Corrado Di Guilmi

PII: S0167-2681(17)30280-9
DOI: https://doi.org/doi:10.1016/j.jebo.2017.10.006
Reference: JEBO 4166

To appear in: *Journal of Economic Behavior & Organization*

Received date: 29-3-2016
Revised date: 29-7-2017
Accepted date: 5-10-2017

Please cite this article as: Michele Catalano, Corrado Di Guilmi, Uncertainty, rationality and complexity in a multi-sectoral dynamic model: the Dynamic Stochastic Generalized Aggregation approach, *<![CDATA[Journal of Economic Behavior and Organization]]>* (2017), https://doi.org/10.1016/j.jebo.2017.10.006

- New approach for the analytical solution of agent-based models (named DSG-A).
- DSG-A incorporates infinite horizon optimization in a complex economic system.
- Identification of two types of equilibrium: rational and uncertainty equilibrium.
- Applying DSGA a multi-sectoral agent-based model solved through a dynamical system.

Title: Uncertainty, rationality and complexity in a multi-sectoral dynamic model: the Dynamic Stochastic Generalized Aggregation approach.

Author1: Michele Catalano,
Prometeia - Bologna, Italy

Author 2: Corrado Di Guilmi,
Economics Discipline Group - University of Technology Sydney AND Centre for Applied Macroeconomic Analysis - Australian National University.

Abstract

The paper proposes an innovative approach for the analytical solution of agent-based models.

The approach is termed Dynamic Stochastic Generalized Aggregation (DSGA) and is tested on a macroeconomic model articulated in a job and in a goods markets with a large number of heterogeneous and interacting agents (namely firms and workers).

The agents heuristically adapt their expectations by interpreting the signals from the market and give rise to macroeconomic regularities.

The model is analytically solved in two different scenarios.

In the first, the emergent properties of the system are determined uniquely by the myopic behavior of the agents while, in the second, a social planner quantifies the optimal number of agents adopting a particular strategy. The integration of the DSGA approach with intertemporal optimal control allows the identification of multiple equilibria and their qualitative classification.

Keywords: aggregation, uncertainty, opinion dynamics, master equation, optimal control.

1. Introduction

The standard assumption of perfect rationality rules out the possibility of agents making mutually inconsistent decisions which might lead to situations of aggregate disequilibrium, multiple equilibria or indeterminacy. These outcomes can emerge in models that feature some sort of bounded rationality or heterogeneous beliefs (not confined to a predefined distribution) and which allow for agents' interaction and learning. Agent-based models (ABMs) represent a suitable and well-known example of this modeling strategy. This approach has been proven capable of replicating a wide range of stylized facts (see Delli Gatti et al., 2005; Dosi et al., 2013; Lengnick, 2013, among many others) and providing original policy indications (Journal of Economic Behavior & Organization, 2008).

Therefore Dynamic Stochastic General Equilibrium (DSGE) and ABM approaches appear at odds with their conceptual pillars and the range of possible implications. While it is possible to find examples of DSGE models which incorporate some of the insights and modeling strategies of ABMs[1], to the best of our knowledge the literature has not yet provided an original theoretical framework sufficiently flexible to include the defining features of both approaches simultaneously.

Recent applications of statistical mechanics tools in macroeconomics can open new perspectives for a closer integration of the two approaches.[2] In this literature, the macroeconomic system is structured as a continuum of states, each corresponding to a discrete value of a state quantity such as production or price level. Microeconomic agents are classified in a grid of states (such as production levels, leverage ratio, etc...), and can switch stochastically across them according to probability laws defined by the transition rates. Each macroeconomic state can be associated to different configurations at the micro-level. Since the evolution of the aggregate quantities depends on how the agents are distributed across micro-states, it is possible to quantify a probability for the macro-states. This method is feasible with any amount of information on the possible microeconomic configurations.

This analytical representation has proved capable of replicating the results of agent-based models with higher degrees of heterogeneity (Chiarella and Di Guilmi, 2011; Di Guilmi et al., 2012; Di Guilmi and Carvalho, 2017) but has not been applied or extended to more comprehensive multi-sectoral frameworks to provide a general methodology comparable to that adopted by general equilibrium models.

[1]See Gobbi and Grazzini (2015) and the papers reviewed by Dilaver et al. (2016). From this perspective the works of Per Krusell and co-authors are also relevant (see in particular Krusell et al., 2012).

[2]See Alfarano et al. (2008); Aoki (1996, 2002); Aoki and Yoshikawa (2006); Foley (1994); Lux (1995, 1998); Smith and Foley (2008); Weidlich (2000, 2008). In this paper we will draw in particular on the seminal contributions by Aoki (2002) and Aoki and Yoshikawa (2006) and further developments by Di Guilmi (2008) and Landini and Uberti (2008).

This paper is a first attempt in this direction and provides three main contributions. First, it introduces a general methodology for an aggregate representation of ABMs by means of an original use of the master equation. This methodology adopts a bottom-up modeling approach to build a representation of the economy based on a dynamical system, as in standard DSGE models but without reducing the complexity of the microeconomic interactions, and allowing for out-of-equilibrium dynamics. The dynamics of the macroeconomic variables depend on the evolution of the proportions of agents adopting the different available strategies, which is modeled as a Markov process and quantified by the master equation. The aggregate model preserves the behavioral assumptions of the ABM by embodying them in the transition rates. Consequently, the aggregate system inherits the disequilibrium dynamics from the ABM. For this reason, a side result is that our methodology can be regarded as an original alternative to model dynamical disequilibrium. We define this innovative approach as *Dynamic Stochastic Generalized Aggregation (DSGA)*.

Second, this original use of the master equation is a relevant contribution to the vast and growing literature on opinion dynamics (see Brock and Hommes, 1997; Lux, 2009, among many others) from a dual perspective: first, the model is explicitly microfounded, and second, a closed form solution is identified, allowing for a full analytical representation of the evolution of the macro-variables as dependent on agents' choices.

The third and perhaps most relevant contribution is the stochastic analysis of a scenario in which a social planner identifies the optimal proportions of the agents in the different groups in order to maximize the expected streams of profit and utility. In that scenario, complexity and rationality coexist, and it is possible to assess the role of uncertainty and interaction in determining macroeconomic outcomes. Indeed, the DSGA approach incorporates infinite horizon optimization in a complex economic system, allowing for comparison with the heuristic behavior of the standard ABM treatment. This allows us to model and discriminate between: a) rational incentives driving microeconomic behavior and via aggregation the macroeconomic dynamics, b) uncertainty arising from irreducible complexity due to agents' interactions. This representation involves interaction at the microeconomic level, and it cannot be reproduced using standard stochastic processes, nor included in a standard decision process under uncertainty. This allows us not only to identify a multiple-equilibria system but also to distinguish qualitatively different types of equilibria. Two kinds of equilibria emerge: *rational equilibrium* in which the system is in a quasi-steady state defined by rationality principles, and *uncertainty equilibrium* which is the result of system complexity and agents' interactions. In the rational equilibrium, agents receive signals which lead the macroeconomic system to stabilize according to an optimization rule identified by a social planner. The uncertainty equilibrium corresponds to a sub-optimal configuration determined by the uncoordinated response of agents to changes in the economic environment.

Other techniques have been proposed with the aim of providing an analytical counterpart to ABMs. In particular Assenza and Delli Gatti (2013) and other related works model the macroeconomic dynamics as depended on the moments of the distribution of a particular microeconomic variable. The DSGA is distinguished from this approach by three main aspects. First, it focuses on the (open-ended) state-space of the macroeconomy rather on the moments of the distribution of a particular variable. Consequently its application does not depend on the particular distribution of the micro-variable, which might raise issues (for example, if the distribution is Pareto the second moment may not be finite). Second, it preserves and embodies interaction as the source of the emergent properties of the system. Third, it allows the joint analysis of multiple state variables.

Anticipating some results, our analysis shows that full employment equilibria is attainable only in a context with perfect rationality. In both the heuristic and the optimizing treatments, the model exhibits structural imbalances that lead to periodic crashes when agents are boundedly rational, or affect the long-run trend in the optimizing case. Even in the case of perfect rationality, the economy can be caught in an *uncertainty trap* (Aoki and Yoshikawa, 2006, chap. 4).

The remainder of the paper is structured as follows. Section 2 briefly presents the ABM, which is an adaptation of Russo et al. (2007). Section 3 introduces the aggregation method. Section 3.1 describes the specific application of the master equation and its solution, comprising an ordinary differential equation plus a stochastic component. Both solution components (the parameters of the differential equation and the moments of the distribution of fluctuations) depend on the transition rates. In turn, the transition rates are defined as the product of the probability of transition and the probability to be in the state from which the transition occurs. Sections 3.2 and 3.3 presents the functional identification of the latter. Section 3.2 uses maximum entropy while 3.3 presents the derivation in the equilibrium condition of the master equation. This second derivation provides the two different treatments of the stochastic evolution of the agents: the heuristic (or zero-intelligence) case, and the full rationality case.

Section 4 implements the methodology in the ABM described in section 2 by configuring the analytical solution. Section 4.1 derives the equations for the relevant macroeconomic variables (output, price, wage and consumption), which are dependent on the share of agents in the different states. Section 4.2 details the procedure for the determining the transition probabilities according to the model's behavioral assumptions. It provides the calculation of the relevant quantities for defining the probability of being in one of the states for, respectively, the heuristic and perfect rationality treatments. Section 5 describes and contrasts the results of the numerical simulations for the heuristic case (section 5.1), and contrasts them with the numerical solution of the ABM (section 5.2) and with the full rationality case (section 5.3). Section 6 deepens the analysis of the different types of equilibria by proposing an analytical criteria for classifying them quantitatively and by identifying the transition paths between them in section 6.1. Section 6.2 exploits these analytical results to conduct a simple policy experiment. Section 7 offers some concluding remarks.

2. The Agent-Based Model

The model on which we test the aggregation technique borrows heavily from Russo et al. (2007). This model provides a suitable environment for testing the DSGA because it includes different markets in a rather simple setting with basic heuristic behavioral rules for agents.

Our model is composed of large and close populations of firms and workers, and the economic system consists of a circular flow of goods and money. Firms operate in a monopolistically competitive market: they produce goods that are close substitutes but, as a result of market imperfections, can be sold at different prices. Firms can be heterogeneous in their production and price levels. Workers have identical skills but, given labor market frictions, they can be characterized by different reservation wages. They also can have different levels of wealth. Firms set the production quantity, the selling prices and the labor demand. Information is incomplete and limited in the model, and agents adaptively revise their expectations each period according to a set of simple rules. The timeline of events is detailed in Appendix A.

At the beginning of each period, firms heuristically determine the quantity to produce depending on whether they sold the whole of their production in the previous unit of time, or have unsold goods in stock s_i. Namely, the i^{th} firm decides to adjust the desired produced quantity $y_i(t)$ based on a simple dichotomous strategy:

$$y_i(t) = y_i(t-dt) \times \begin{cases} (1+\delta) \text{ if } s_i(t-dt) = 0 \\ (1-\delta) \text{ if } s_i(t-dt) > 0, \end{cases} \quad (1)$$

where $0 < \delta < 1$ and dt is an arbitrarily small time interval. Firms adopt a linear technology which employs only labor l with unitary labor productivity constant across firm and through time, expressed by

$$y_i(t) = l_i(t) \quad (2)$$

which also quantifies the demand for labor.

Workers set their satisficing wages w according to their previous occupation status, which is modeled as a dichotomous variable *occ* equal to 1 if the worker was employed and equal to 0 otherwise. Thus the satisficing wage of worker j is equal to

$$w_j(t) = w_j(t-dt) \times \begin{cases} (1+\delta) \text{ if } occ_j(t-dt) = 1 \\ (1-\delta) \text{ if } occ_j(t-dt) = 0. \end{cases} \quad (3)$$

As the labor market opens, firms set vacancies according to the difference between labor demand and their current stock of labor force. If the difference is positive new vacancies must be filled, otherwise labor is destroyed. Then the j^{th} worker sends applications to a subset h^w_{ji} of randomly chosen firms, indicating her satisficing wage. Firms collect the workers' curricula, sort them according to the satisficing wage, and hire the cheapest workers. As a consequence, a firm may not receive enough applications to satisfy its demand for new labor, and then the actual increase in output will be smaller than the planned δy_i. In the case of a firm reducing its output, the most expensive employees will be laid off.

We follow Russo et al. (2007) and assume workers to have complete market power: a worker earns her satisficing wage if hired, otherwise she remains unemployed. Accordingly, wages are never below the satificing levels and workers are able to extract all the surplus in the labor market bargaining.

Once the quantity of labor is set and the wage costs are known, firms set the goods prices p for the current period. Their decision is dependent on whether they were able or not to sell all their production in the previous unit of time. If a firm did not sell all its output and has accumulated stocks, it will revise its price downward. It will make the opposite decision if its stocks are equal to zero. Accordingly, the desired price $p^*_i(t)$ for firm i is

$$p^*_i(t) = \begin{cases} p_i(t-dt)(1+\delta) & \text{if } s_i(t-dt) = 0 \\ p_i(t-dt)(1-\delta) & \text{if } s_i(t-dt) > 0. \end{cases} \tag{4}$$

A firm will actually apply the variation calculated as in (4) if the resulting price is at least equal to the production costs. Let us define the average cost of the subset h^l_{ij} of employed workers as

$$p^1_i = \frac{\sum_{j \in h^l_{ij}} w_j(t)}{y_i(t)} \tag{5}$$

The actual price for the firm i at time t will be equal to

$$p_i(t) = \begin{cases} p^*_i(t) & \text{if } p^*_i(t) \geq p^1_i(t) \\ p^1_i(t) & \text{if } p^*_i(t) < p^1_i(t). \end{cases} \tag{6}$$

The j^{th} consumer wants to consume all her wealth z_j. The consumer searches for the cheapest goods by collecting a subset of the posted prices from h^c_{ji} randomly chosen firms. Then she sorts these prices and buys the maximum quantity of goods allowed by her stock of wealth (given the prices) or by her suppliers' availability. As such, consumers can be supply-constrained, if the suppliers' availability is below the desired quantity, with the consequence that $c_j < z_j$.

Worker's wealth is increased in each period by the amount of labor income plus a share of firms' profits (equally allocated among all households) less consumption. If the consumer is supply-constrained, the wealth in excess of consumption is remunerated at the constant interest rate r in each period. Accordingly, the nominal wealth z of each consumer j evolves according to

$$z_j(t) = (1+r)z_j(t-dt) + occ_j(t)w_j(t) + \frac{\Pi(t)}{N_c} - c_j(t), \tag{7}$$

where Π is the total amount of profits, and N_c is the number of consumers. Therefore when $z_j(t) > 0$, the consumer is supply-constrained.

In the following period she will try again to spend all her wealth.

Finally, stocks for firm i are calculated as the difference between production and the actual consumption of the consumers in its pool h_{ij}^{y}

$$s_i(t) = y_i(t) - \sum_{h_{ij}^{y}} \frac{c_j(t)}{p_i(t)}. \tag{8}$$

3. Methodology

Figure 1 provides a graphical representation of the structure of the DSGA approach. ABMs are represented as Markovian processes in which agents are endowed with a particular interaction protocol and a set of behavioral rules and strategies which represent the basis and starting point of our approach. Agents may change their strategies as a consequence of the interaction, according to the behavioral rules of the ABM. Consequently, the functional form of the transition rates of the Markovian process can be identified from the explicit behavioral assumptions of the ABM. It is worth noting that this functional representation takes account of both the complex interactions among agents which drive the agents' choice of a particular strategy, and the macroeconomic conditions. The transition rates are arguments of the master equation, which is employed to describe the evolution of agents. The DSGA representation (the vertex of the pyramid) exploits the solutions of the master equation to quantify, through a dynamical system, the evolution of the aggregate variables as dependent on the distribution of agents. In turn, the macroeconomic variables determine the transition rates (the base of the pyramid) in a circular fashion. More precisely, given the distribution function of the variable of interest at the agent level (for example the levels of excess demand for each firm's product), its aggregate value (aggregate excess demand) will represent the reference point for calculating the likelihood for the agents to change strategy (firms' decision about increase or decrease in production). In a nutshell, both the algorithmic structure of the ABM (strategies and interaction protocols) and the set of macroeconomic variables determine the distribution of agents across states or strategies.

This section presents the main tools used in the DSGA approach. The master equation is introduced in section 3.1 together with its solution and a discussion of the particular application proposed in the present paper. The solution of the master equation depends on the transition rates of the underlying Markov process. The next two subsections illustrate the procedure for deriving the transition rates. Section 3.2 introduces the concept of statistical entropy, which is central to the stochastic aggregation process, and solves a maximum entropy problem in order to derive the quantities to be used in the remainder of the analysis, and in particular, for the definition of the transition rates. Section

3.3 shows the functional definitions of the transition rates. It describes also how the different degrees of rationality and uncertainty are incorporated in the transition rates in order to determine the stochastic evolution of the system.

3.1. Master Equation and dynamics

Consider a population of $i = [1, ..., N_k]$ agents, where k identifies the subgroup, for example firms and workers. At any point in time, an agent is in a state $s_k = [1, ..., S_k]$ and adopts the associated strategy f_{s_k} in order to set the evolution of its control variables $\mathbf{x}_k \in R^{M_k}$, where M_k is the number of control variables for each type of agent. Thus, the control variables of the individual i of type k evolve according to $\dot{x}_{i,k} = f_{s_k}(x_{i,k})$.

For the whole economy, it is possible to identify a $S \times m_k$ functional matrix F and a m_k-dimensional control variable vector $\dot{\mathbf{x}}$ such that

$$\dot{\mathbf{x}}(t) = F(\mathbf{x}(t)). \tag{9}$$

considering $dt \to 0$ in a continuous time setting.

In order to reduce the dimensionality of the problem and aggregate the system, we indicate the probability of an agent to be in state s_k as $P(s_k)$. The resulting dynamical system is:

$$\dot{X}_k = \sum_{s_k} f_{s_k}(X_k)P(s_k), \tag{10}$$

where X_k is the aggregate control variable. The above system is the aggregate stochastic macro-rule originating from the set of micro-rules (9).

Assuming that the stochastic process governing the switching of a specific j-type agent is Markovian[3] allows us to use the master equation to model the evolution of the probabilities $P(s)$. The master equation has been already applied in macroeconomic models with a large state-space for a single agent-level state variable (Di Guilmi et al., 2012). Here, we focus on the case of binary options for agents, which provides a more immediate analytical representation of the new methodology with multiple state variables and is appropriate for the ABM described in section 2. Let us consider two states $s = [1, 2]$ and use as a reference the state $s = 1$, such that n_k denotes the relative density of agents of type k in state 1. Accordingly, the master equation for the density n_k is given by

$$\frac{dP(n_k,t)}{dt} = \lambda_k(t)P(n_k-1/N_k,t)+\gamma_k(t)P(n_k+1/N_k,t)-[\lambda_k(t)+\gamma_k(t)]\,P(n_k,t), \tag{11}$$

[3]It is worth noting that the transition rates of the process are time varying and, as a consequence, the assumption of Markovianity does not imply the memory-less property at the agent level. In fact, the probability of transition of an agent depends on its current endowment, which is the result of its previous history, and its current micro-state, which is updated every period. See Izquierdo et al. (2009) for a discussion on the use of Markov processes to represent the dynamics of ABMs.

where λ_k and γ_k are the respective transition rates in and out of state $s = 1$. Equation (11) is a balance flow equation between the probability of observing a density equal to n_k starting from a different density, and the probability of already having a proportion n_k of agents in state $s = 1$ and observing any transition.

The transition rates are given by

$$\lambda_k(t) = (1 - \eta_k)\zeta_k(t), \tag{12}$$
$$\gamma_k(t) = \eta_k \iota_k(t). \tag{13}$$

where η_k is the probability to be in state $s = 1$, which is considered as exogenous for the moment, while ζ_k and ι_k are the probabilities of a single agent entering and exiting state $s = 1$, respectively. The transition probabilities are dependent on the model's behavioral assumptions, and the agent's condition at any given time.[4]

In order to asymptotically solve (11), we apply the approximation method introduced by Aoki (2002), which splits the fraction of agents in state s_k into a drift term m_k and an additive spread term u_k (divided by $N_k^{1/2}$ to normalize its standard deviation) as follows:

$$n_k(t) = m_k(t) + u_k(t)N_k^{-1/2}. \tag{14}$$

Di Guilmi (2008) derives a system of coupled equations for the generic terms m and u:

$$\frac{dm_k}{dt} = \lambda_k(t)m_k(t) - [\lambda_k(t) + \gamma_k(t)]m_k(t)^2, \tag{15}$$
$$du_k = -a_1(m_k)u_k dt + a_2(m_k)d\mathcal{W}, \tag{16}$$

where $d\mathcal{W}$ is a Wiener process.[5]

Accordingly, equation (10) can be re-formulated as follows

$$\dot{X}_k(t) = \sum_{s_k} f_{s_k}(X_k(t))n_{s_k}(t). \tag{17}$$

Finally it is possible to build a dynamical system to describe the model. This system is composed of two subsystems: the subsystem of macro-equations (17) which is nothing more than a weighted average of the rules of the system at the micro level, and the subsystem of the master equations' solutions which

[4] The DSGA makes use of endogenous and microfounded transition rates as opposed to the standard approach in DSGE with heterogeneous agents, where heterogeneity is modeled as an idiosyncratic exogenous stochastic process.

[5] Di Guilmi (2008) derives a solution of the master equation (11) composed of equation (15) and a Fokker-Planck equation whose stationary solution is $P(u_k) \sim \mathcal{N}(0, \frac{\lambda_k \gamma_k}{(\lambda_k+\gamma_k)^2})$. As demonstrated by Gardiner (2002) and van Kampen (1992), the stochastic process of the noise can be expressed as the Ito stochastic differential equation (16).

determine the number of agents in each state according to equations (14), (15) and (16). The first subsystem uses the proportions of agents in each state to describe the time evolution of the macroeconomic variables; the second subsystem uses the transition rates, which are updated according to the response of agents to changes in the macroeconomy, to provide the proportions of agents in the different states.

3.2. Entropy and inference

In order to provide a functional specification of the transition rates according to (12-13), we need to identify the transition probabilities ζ and ι and the probability η. The transition probabilities depend on the specific assumptions in the underlying ABM and are derived in section 4.2. We present two possible formulations for the probability η. The first is introduced in this section, and makes use of the maximum entropy method, without involving the master equation. For our purposes, the main reason for using this method is to derive the uncertainty variable, identified below as β, which will feature also in the second formulation of η introduced in section 3.3

Maximization of the system entropy, which is known in information theory as MaxEnt (Jaynes, 1957), provides the most likely probability function for the number of agents following a particular behavioral rule s, subject to the normalization constraint, and to additional constraints representing the available information.

As demonstrated in Appendix B, in the two-state case $s = [1, 2]$, the resulting functional form for the share of agents n_k in state $s = 1$ is given by

$$P(n_k) = \eta_k = \frac{exp(-\beta_k V_1)}{exp(-\beta_k V_1) + exp(-\beta_k V_2)}, \tag{18}$$

where V_1, V_2 are the returns associated to the two strategies. The quantity β_k is the intensity of switching, and measures the degree of uncertainty in the system: for $\beta \to \infty$ there is no uncertainty and all agents adopt the best-performing strategy; for $\beta \to 0$ the return plays no role and all the strategies have an equal probability. This functional form has become extremely popular in discrete choice models since the pioneering work by Brock and Hommes (1997). Chiarella and Di Guilmi (2015) show that (18) can be endogenously derived by maximizing the statistical entropy. In this approach, β_k is the Lagrange multiplier of the constraint to the entropy maximization concerning the returns coming from the different states, rather than an exogenous parameter. As shown in Appendix B, in the case of two available strategies 1 and 2, β can be expressed as

$$\beta_k = (V_1 - V_2)^{-1} log\left(\frac{n_k}{1 - n_k}\right). \tag{19}$$

where n_k is the proportion of agents in state 1 as above. It is easy to see that, provided that $V_1 - V_2 \neq 0$ and is finite, $\beta_k \to \pm\infty$ when n_k goes to 0 or 1 (minimum uncertainty: all agents adopt the same strategy), while $\beta_k \approx 0$ when $n_k = 1/2$ (maximum uncertainty: the two strategies are equally likely).

3.3. Stochastic equilibrium

In solving the model, we use a different formulation of the probability η, identifying it as the stationary distribution of the Markovian process of agents' switching which is introduced in section 3.3.1. As shown in section 3.3.2, derivation of the stationary distribution differs depending whether it is determined only by the heuristic behavior of economic agents or is the result of an optimization process.

3.3.1. Equilibrium distribution

The master equation is in equilibrium if the probability inflows are equal to the outflows. Given the dependence of macroeconomic variables on the proportions of agents in the two states illustrated in section 3.1, the equilibrium condition for the master equation will correspond to the statistical representation of a deterministic economic steady state.

Following Aoki (2002, 46), in the case of two possible states $s = [1, 2]$, under this condition the probability function for the proportion of agents of type k in state 1, indicated by n_1, is of the Gibbs type with the following functional form[6]

$$P(n_1) = \eta_k = \frac{exp[\beta_k g_k(n_1)]}{exp[\beta_k g_k(n_1)] + exp[-\beta_k g_k(n_2)]}. \tag{20}$$

where β_k is the same as in (19). This formulation for the stationary probability allows us to incorporate additional information with respect to the result in (18). In (20), economic behavior is not measured simply by the return of the strategy but is modeled by $g(n_s)$, which is a function evaluating the difference in the utility between the different strategies.

In the standard opinion formation literature, the function $g(n_s)$ in (20) represents an assumed fitness function which quantifies the returns associated to the particular strategy s. Alternatively, the suitable functional form for g can be identified endogenously using the economic *potential* (Smith and Foley, 2008). The potential is a functional which quantifies the likelihood of a state of the system as a consequence of the states of its parts. In this perspective, it provides a measure of uncertainty since it depends on how many combinations of agents' choices are compatible with a given macroeconomic state. In particular, large values of the potential signal that a particular macroeconomic state can be generated by a great number of different configurations at the micro-level, therefore the degree of uncertainty in the system is large.[7] Consequently, the

[6]This result stems from the Markov-Gibbs equivalence demonstrated by the Hammersley and Clifford theorem (Clifford, 1990).

[7]In statistical physics, the minimum of the potential are the points where the free energy of the system, and consequently the uncertainty, reaches a minimum. In an economy, this

minima of the potential represent possible absorbing states where agents have no incentive to change their strategy, and discontinuities may prevent the transition from one equilibrium to another. Following Aoki and Yoshikawa (2006) and Di Guilmi (2008), for systems with two micro-states, the potential is defined as

$$U_k = -2\int_0^{n_1} g_k(x)dx - \frac{H(n_1)}{\beta_k}, \tag{21}$$

where H is the statistical entropy, which in the Shannon formulation is given by

$$H = -n_1 log(n_1) - n_2 log(n_2). \tag{22}$$

As equation (21) shows, a negative relationship exists between the level of the potential and the uncertainty, quantified by β_k. When the uncertainty is at its maximum ($\beta_k \to 0$) the potential is not defined, and tends to infinity.

3.3.2. Agents' behavior and stochastic equilibrium

The g function plays a relevant role in our story. It factors the degree of rationality in the determination of the stochastic equilibrium. We distinguish two cases. In the first scenario, the heuristic case, the aggregate properties are the results of the uncoordinated choices of agents who behave like atoms. In the second case, we identify the proportions of agents in each group that maximize some measure of social welfare by applying standard maximization tools.

Derivation of g: the heuristic case.
To identify the minima of the potential, we must find the critical points of the equation (21). Taking the derivative with respect n_1, the first order condition is given by

$$\frac{\partial U_k}{\partial n_1} = -2g_k(n_1) - \frac{dH}{dn_1}\beta_k^{-1} = 0. \tag{23}$$

Considering that $n_2 = 1 - n_1$ and substituting (19) and (22) into (23), we obtain

$$g_k(n_1) = \frac{V_1 - V_2}{2}. \tag{24}$$

At the point of minimum uncertainty, g_k is equal to the relative value of the returns, with an equal probability for the two alternatives. In this situation the probability functions (18) and (20) are identical, meaning that, in the absence of any further specification about agents' behavior being included in g_k, the inference process considers unsophisticated agents (as atoms) and the macroeconomic results are not impacted by their behaviors.

corresponds to a situation where incentives to take opposite actions offset one another and the system reaches a statistical equilibrium.

Substituting (19) and (24) into (20), we obtain

$$P(n_1) = \frac{exp[\beta_k g_k(n_1)]}{\mathcal{Z}} = \frac{exp[\frac{V_1 - V_2}{2} log\left(\frac{n_1}{1-n_1}\right)(V_1 - V_2)^{-1}]}{\mathcal{Z}} = \frac{n_1}{1-n_1}\frac{1}{\mathcal{Z}},$$

where $\mathcal{Z} = \sum_s P(n_s)$. Under the assumption of heuristic behavior, the product of the uncertainty variable β_k and the g_k function vanishes: if economic behavior is quasi-random, the economic incentives are indistinguishable due to uncertainty.

Derivation of g: full rationality and optimization.
Rationality is introduced in the model to preserve the probabilistic nature of the aggregation method, allowing for a comparison with the heuristic case. The decision rules for agents are the same as in the heuristic setting but we assume also that a social planner maximizes an objective function in which the arguments are the fractions of agents adopting a particular rule, obtaining standard Euler equations. In a treatment similar to Brock and Durlauf (2001), the social decision maker optimizes an intertemporal objective function controlling for the densities of agents in each group.[8] In this perspective, the g_k function becomes the outcome of a dynamic intertemporal control problem specified as marginal net utility from available alternatives. As a consequence, the transition rates incorporate (through the g_k function) non-trivial behavior: a coordination process in which agents behave optimally in transitioning from one heuristic rule to the other.

To the best of our knowledge, this is the first attempt to link the master equation approach to the standard dynamic optimization, following the suggestion in Aoki and Yoshikawa (2006, 68): "*The apparently very simple binary choice model actually accommodates sophisticated dynamic optimization under uncertainty*". Our approach differs from other works where the social planner's optimization subject to a market clearing condition involves the distribution of agents as in Nuño and Moll (2015), in order to compare the outcomes of centralized versus decentralized optimization processes. Specifically, we propose an investigation of the outcomes of rational optimization in a complex system characterized by random interactions among agents and not bounded by an imposed market-clearing equilibrium.

The social planner considers equation (17) for the macro-state Y as the state equation of a standard intertemporal control problem. The control instruments are given by the fractions of agents choosing a particular strategy.

[8] This is a standard modeling strategy in macroeconomics, for example in the dynamic programming applied to search and matching labor market (Trigari, 2006; Gertler and Trigari, 2009). In these models, the labor participation decision is modeled as the intertemporal problem of a planner who chooses the optimal fraction of the household's working members. Optimality conditions provide dynamic labor supply equations which depends on the expected net labor income. Similarly, we use the densities of agents in each group as control variables, to be set by the social planner given some dynamic constraints, in order to maximize the intertemporal utility/profits.

Formally, the vector of the occupation numbers $\mathbf{n}$ quantifies the weights for a collection of available strategies $s = 1, ..., S$ which determine the macroeconomic variable Y. The intertemporal optimization problem is defined by the following infinite horizon return stream

$$g_k\left(Y, n_s\right) = \max_{\mathbf{n}(t)} \int_{t_0}^{\infty} exp(-\theta t)\psi(Y(t), \mathbf{n}(t))dt, \tag{25}$$

subject to the macroeconomic rule:

$$\dot{Y}(t) = \sum_s f_s(Y(t))n_s, \tag{26}$$

where ψ and θ respectively are the instant payoff function and the discount factor. The steps required to obtain a closed form solution for the above problem are:

1. to represent the optimization problem under the Hamilton-Jacobi-Belmann equation which provides a dynamic forward looking ordinary differential equation for the discounted payoff function (25);
2. to calculate the first order condition with respect to $\mathbf{n}$, subject to the dynamics of the state variable given by eq. (26);
3. finally, to find a closed form solution for g_k, *i.e.* the solution in time for the dynamics of the value function found in step 2.

In economic terms, the problem boils down to a decision about the fractions of individuals playing each of the available strategies. If applied at the individual level, such a model would describe an agent playing a mixed-strategy. From a frequentist or aggregate perspective, the solution for the g_k function is used to describe a world where agents are guided by rationality principles while adopting simple behavioral rules. This approach to rationality does not need a reduction in the degree of system complexity, and it is able to describe rationality in a context of disequilibrium.

4. Stochastic aggregate model

In this section we apply the aggregation method introduced in section 3 to the model presented in section 2. Firms and workers are classified according to their strategy or condition. The densities of agents in each condition determines the evolution of the relevant macro-variables in our economy, namely production, price, average wage and consumption. The dynamics of these densities are identified by a set of four master equations.

Consequently, the dynamical system describing the model is composed of four aggregate equations such as (17), and the solutions of the master equations, given by (14), (15) and (16) for each of the macroeconomic variables. The master equations' solutions feature the transition rates. These rates are specified in accordance with the ABM's behavioral rules, and therefore constitute the link between the microeconomic level and the macroeconomy. The equations comprising the analytical solution are summarized in Appendix C.

The steps required to perform the aggregation are:

1. identification of the states over the agents' sub-groups. Namely, we have $k = \{y, c, w, p\}$ for, respectively, firms clustered according to production change, consumers, workers and firms clustered according to price change. Table 1 shows the states representing agents' choices;
2. the definition of the equations for the agents' state variables according to equation (9);
3. specification of the transition rates. In order to define the transition rates we need: first, to identify the transition probabilities and, second, to quantify the generic probability η of an agent being in one the two states. This latter requires us to specify the uncertainty variable β and the value functions g for the heuristic case and the rational case respectively.

The remainder of this section presents the derivation of the results of steps 2 and 3.

4.1. Aggregate equations

In order to complete step 2, we need to exploit the law of motion of the variables at the micro-level defined in the previous section, and aggregate by calculating a suitable weighted average where the weights are given by the proportion of agents in each the two states, determined by the master equation. The aim is to apply the generic equation (17) to this model in order to obtain the dynamics of the aggregate variables. To obtain an explicit functional relationship between the number of agents in each state and the macroeconomic variable consistent with the assumptions and the structure of the ABM requires analytical approximations of agents' behavior.[9] While this solution process potentially could be applied to any ABM, identification of the best specific approximation for each model depends on its particular assumptions and structure. As we show below, numerical simulations of the ABM can provide guidance for the identification of a suitable analytical representation. The same remarks apply to the derivation of the transition rates in section 4.2 below.

[9]Di Guilmi (2008) and subsequent papers adopt the mean-field approximation to reduce the vector of observables within a cluster of agents to a single value to be used in the agents' behavioral equations. Here we propose a different strategy using aggregate functions to approximate the random behaviors and matching of agents. Although the procedure adopted in Di Guilmi (2008) has been proven to be able to replicate the numerical results of the ABM, the alternative proposed here is more consistent with the general approach in the present paper and the application in a more standard modeling technique.

4.1.1. Output

The equation governing the dynamics of aggregate production is the weighted mean of the variation in production for firms that reduce production and firms that increase production to the extent allowed by the imperfect matching in the labor market.

The symbols $u_n(t)$ and $y_i(t)\delta$ represent respectively the unemployed workers and the new vacancies, and δ is the upward (downward) adjustment in output targeted by firms in case of null (positive) inventories as specified in section 2. The imperfect matching generated by the random procedure of the ABM can be approximated analytically by the matching function $f_l(u_n(t), y_i(t)\delta) = u_n(t)^a(y_i(t)\delta)^b$, with a, b as positive parameters, and the associated per-vacancy matching probability $q(t) = \frac{f_l(t)}{y_i(t)\delta}$ (Mortensen and Pissarides, 1994). Thus, for the analytical treatment to consider the matching frictions, equation (1) must be re-expressed as

$$y_i(t) = y_i(t-dt) \times \begin{cases} (1 + f_l(t)) & \text{if } s_i(t-dt) = 0 \\ (1 - y_i(t)\delta) & \text{if } s_i(t-dt) > 0. \end{cases} \tag{27}$$

According to the model's assumptions, a fraction n_y of firms revise their output upwards by a fraction δ, while the rest of the firm population decreases its output by the same factor. However, only a fraction q of the planned increase in production can be realized due to matching frictions in the job market. Consequently, the aggregate production evolves as

$$\dot{Y}(t) = Y(t)\left[n_y(t)\delta q(t) - (1 - n_y(t))\delta\right] = Y(t)\delta\left[1 + q(t)\right]\left[n_y(t) - \frac{1}{1+q(t)}\right]. \tag{28}$$

4.1.2. Price

With reference to the price determination, firms can be in two possible situations: a fraction of n_p firms decide according to the rule (4); the remaining $(1 - n_p)$ firms are cost-constrained, and therefore adopt the price p_i^1, defined in (5).

Considering that the sign of the price variation is the same as the variation in production, we can approximate (4) as

$$\dot{p}_i(t) = \left\{ \begin{array}{ll} p_i(t)\delta & \text{if } s_i(t) = 0 \\ \\ -p_i(t)\delta & \text{if } s_i(t) > 0 \end{array} \right\} = sign(\dot{y}_i)p_i(t)\delta. \tag{29}$$

Further, we can write $w_i = p_i^1$ to denote the average wage paid by firm i. Subtracting from both sides of (5) the expression $-w_i(t-dt) + 1 - p_i(t-dt)$, and rearranging we obtain

$$p_i(t) - p_i(t-dt) - w_i(t-dt) = w_i(t) - w_i(t-dt) - p_i(t-dt).$$

Using the continuous time approximations $\dot{p}_i(t) \approx p_i(t) - p_i(t-dt)$ and $\dot{w}_i(t) \approx w_i(t) - w_i(t-dt)$, we can rewrite it as

$$\dot{p}_i(t) = \dot{w}_i(t) + w_i(t) - p_i(t). \tag{30}$$

A fraction n_p of firms sets the price according to the rule (4); the remaining firms are cost constrained and adopt the price p_i^1. Thus, in aggregate, the change in price will be equal to

$$\dot{P}(t) = n_p(t)\left[sign(\dot{Y})P(t)\delta + P(t) - W(t) - \dot{W}(t)\right] + \dot{W}(t) + W(t) - P(t). \tag{31}$$

From (31), if $n_p \to 1$ the price adjustment follows the same procedure as quantity, if $n_p \to 0$ the aggregate price grows as much as wages.

4.1.3. Wage

At each point in time, a single worker can be in one of the two states: "employed" ($occ(t) = 1$) or "unemployed" ($occ(t) = 0$). We are interested in specifying the jump-process transition probabilities of the worker for the state "employed". Workers set their satisficing wages w_j^s according to their occupational status:

$$\dot{w}_j^s(t) = \begin{cases} w_j^s(t)\delta & \text{if } occ_j(t) = 1 \\ -w_j^s(t)\delta & \text{if } occ_j(t) = 0 \end{cases} \tag{32}$$

Given (32), the aggregate variation of wage is quantified by

$$\dot{W}(t) = W(t)\delta\left(n_w(t) - \frac{1}{2}\right). \tag{33}$$

where n_w is the proportion of workers revising their satisficing wage upward.

4.1.4. Consumption

Finally, consumers can be classified into two states according to their demand level: for a fraction n_c of consumers, demand is set simply as the same level of real wealth; for the remaining consumers their demand is equal to their firms' supply because they are constrained. Accordingly, the aggregate equation is:

$$C(t) = n_c(t)Z(t) + (1 - n_c(t))P(t)Y(t), \tag{34}$$

where Z is the aggregate wealth and C is the aggregate consumption. Equation (34) states that aggregate consumption is the weighted mean between aggregate wealth and aggregate supply. If n_c goes to zero we have a fully supply-constrained market.

The model in aggregate is closed for the equation governing the evolution of consumer wealth. Given that the productivity of labor is fixed at 1, the number of employed worker is equal to the output Y, and the number of unemployed is quantified by $N_w - Y(t)$, where N_w is the total number of workers. Accordingly the law of motion of the aggregate wealth is

$$\dot{Z}(t) = [rZ(t) + W(t) - C(t)]\, Y(t) + [rZ(t) - C(t)]\, (N_w - Y(t)) \,. \quad (35)$$

The first term in equation (35) is the change in wealth for employed workers: interest on previous money deposits plus total wages less consumption. The second term refers to unemployed workers, for whom wage earning is null.

4.2. Transition rates

The transition rates are needed to define the four master equations (one for each alternative for firms, workers and consumers), according to (11). As discussed in section 3 we are looking for a specification that can incorporate the algorithm structure of the ABM, taking into consideration interaction and agents' behavior.

From (12-13), the transition rates are the product of two factors: the transition probability and the probability of being in the state from which the transition occurs. The transition probabilities are developed according to the underlying behavioral assumptions and the emergent properties produced by the numerical simulations of the ABM. For simplicity, in this first implementation of the DSGA the transitions are considered independent among the different master equations. In addition to allowing a neater presentation of the method, this is justified by the fact that the analysis considers strategies at the meso-level of aggregation and abstracts from the evolution of each single agent.

In order to find a functional form for the generic probability of being in a given state η_k as per equation (20), we need to the determine the uncertainty variable β_k and the value function g_k for the heuristic case, and for the rational optimization case.

The transition probabilities and the coefficient β_k are the same in both the heuristic and the optimizing treatments, the distinctive feature being determination of the value function g_k. Below, we present the derivation of the transition rates and β_k in the part devoted to the heuristic approach, while the description of the rational approach focuses on the determination of g_k in that particular case.

4.2.1. Heuristic behavior

Here, we provide the derivation of the transition rates, β_k and g_k for each macroeconomic variable in the model described in section 2. The quantity β_k is computed according to equation (19), while g_k, in the heuristic treatment, is formulated according to equation (24).

Output

Firms have two available strategies: increasing or decreasing output according to the rule (1). We know that a firm will increase output if (8) is equal to zero which implies $\frac{y_i(t)p_i(t)}{\sum_{h_{ij}^y} c_j} = 1$. At the level of the aggregate economy, we can define $V_y(t) = \frac{Y(t)P(t)}{C(t)}$ as the aggregate excess supply ratio, and $v_y(t) = \frac{y(t)p(t)}{c(t)}$ as the corresponding quantity at the firm level.

When at aggregate level $V_y < 1$, individual firms are likely to sell all their output and, consequently, to increase production in the following period jumping to state 1. In the opposite case, firms are more likely to decrease ouput, as the goods market experiences supply excess. As explained in section 3, we take the aggregate value of the variable of interest as a likelihood measure for the single agents, given the agent-level distribution as it emerges from the simulations. Numerical simulations of the ABM reveal that v_y is uniformly distributed within the interval $[0.75, 1.25]$ across firms. Accordingly, we can define the probability respectively to enter and exit from the state 1, respectively, as

$$\zeta_y(t) = P(v_y \leq V_y(t)) = F(V_y(t)) = \frac{V_y(t) - 0.75}{0.5}, \tag{36}$$

and

$$\iota_y(t) = 1 - F(V_y(t)) = 1 - \frac{V_y(t) - 0.75}{0.5}. \tag{37}$$

We now need to derive β_y, according to (19) and g_y for the heuristic treatment, according to (24). The returns for the alternatives V_1 and V_2 are, respectively, $V_1 = u_n^a(Y\delta)^b$ in the case of the increasing strategy, and $V_2 = -\delta Y$ for the decreasing strategy. The return V_1 is simply represented by the matching function since, given (2) and (27), the increase in production is bounded upward by the number of workers that the firm can recruit in the job market. Accordingly

$$\beta_y(t) = -[u_n(t)^a(Y(t)\delta)^b + Y(t)\delta]^{-1} log\left(\frac{n_y(t)}{1 - n_y(t)}\right), \tag{38}$$

$$g_y(t) = \frac{1}{2}\left[u_n(t)^a(Y(t)\delta)^b + Y(t)\delta\right]. \tag{39}$$

Price

Let us define the variable $V_p(t) = P(t) - W(t)$ as the difference between the aggregate price index and the average wage in the economy and $v_p(t) = p(t) - w(t)$ as the corresponding quantity at the firm level. The simulations reveal that at the firm level $v_p \sim N(0, 1)$ across firms. Let us use $s = 1$ to denote the state in which firms can set the price according to (4) and $s = 2$ to denote the state in which firms are cost-constrained, and consider $s = 1$ as the reference state, such that n_p denotes the share of firms that can freely adjust the price. As in the case of output, when aggregate price grows at a higher pace than the wage, the likelihood that a single firm is able to freely set price without being cost constrained is higher.

Accordingly, the probability for a firm to access state $s = 1$ is given by

$$\zeta_p(t) = F(v_p < V_p(t)) = \frac{1}{2}\left[1 + erf\left(\frac{V_p(t)}{\sqrt{2}}\right)\right], \tag{40}$$

The probability of the opposite event is quantified simply by

$$\iota_p(t) = 1 - \zeta_p(t). \tag{41}$$

Considering (6), the returns for the two conditions of price increasing and the cost-constrained price are, respectively $V_1 = sign(\dot{Y})P\delta$ and $V_2 = \dot{W} + W - P$. Accordingly

$$\beta_p(t) = -[sign(\dot{Y}(t))P(t)\delta + P(t) - W(t) - \dot{W}(t)]^{-1} log\left(\frac{n_p(t)}{1 - n_p(t)}\right), \tag{42}$$

$$g_p(t) = \frac{1}{2}\left[\dot{W}(t) + W(t) - P(t) - sign(\dot{Y}(t))P(t)\delta\right]. \tag{43}$$

Wage

Workers have two possible strategies for setting their reservation wage: to revise it upward if employed or revise it downward if unemployed. The probability of being employed is conditioned on the probability of firms increasing their production and hence their demand for labor. Accordingly, once a worker has been matched with a firm the possible combined events are four: a) if the firm adjusts upward and the worker is already employed the probability to be employed in next period is 1; b) if the firm adjusts downward and the worker is employed there is a positive probability to be fired; c) if the firm adjusts upward and the worker is unemployed there is a positive probability to be hired; d) if the firm adjusts downward and the worker is unemployed the probability is zero. Recalling that the binary variable $occ(t)$ is equal to 1 (0) if the worker is employed (unemployed), the different four events are:

$$\begin{array}{l} M_1 = dY(t-dt) \geq 0 \ \cap \ occ(t-dt) = 1 \\ M_2 = dY(t-dt) < 0 \ \cap \ occ(t-dt) = 1 \\ M_3 = dY(t-dt) \geq 0 \ \cap \ occ(t-dt) = 0 \\ M_4 = dY(t-dt) < 0 \ \cap \ occ(t-dt) = 0 \end{array}. \tag{44}$$

We define $P(occ(t) = 1)$ as the probability to access the "employed" state. This probability can be quantified as the conditioned probability over the four distinct events defined by (44):

$$P(occ(t) = 1) = \sum_{h=1}^{4} P(occ(t) = 1|M_h)P(M_h). \tag{45}$$

Let us define w^s as a cut-off wage level. Specifically, if the economy is creating (destroying) jobs, w^s will be the wage of the most (least) expensive employee that is hired (fired). We indicate with $M(w_j > w^s)$ the probability that worker j has a satisficing wage higher than the cutoff level. Using $M(w_j > w^s)$ in equation (45) and expanding, we have

$$\begin{aligned} P(occ(t) = 1) = \ & P(dY(t-dt) \geq 0)P(occ(t-dt) = 1) + \\ & +(1 - M(w_j > w^s))P(dY(t-dt) < 0)P(occ(t-dt) = 1) + \\ & +(1 - M(w_j > w^s))P(dY(t-dt) \geq 0)P(occ(t-dt) = 0). \end{aligned} \tag{46}$$

Considering that $P(occ(t-dt)=0) = 1-P(occ(t-dt)=1)$ and $P(dY(t-dt) < 0) = 1 - P(dY(t-dt) \geq 0)$, equation (46) can be rewritten as

$$\begin{aligned} P(occ(t)=1) \quad &= P(occ(t-dt)=1)+ \\ &+M(w_j > w^s)\left[P(dY(t-dt) \geq 0) - P(occ(t-dt)=1)\right]. \end{aligned} \tag{47}$$

Since we know that $P(dY(t) > 0) = \eta_y(t)$, the only quantity left to be determined in (47) is $M(w_j > w^s)$. An intuitive way to simplify the dynamics of the ABM is to consider the employed marginal worker, that is the one with the highest wage among the employed workers. According to the ABM's assumptions, in case of a small decrease in production this worker will be the first to be laid off, while in case of a small increase she will maintain her job. This situation can be captured by using a cumulative probability $F(x) = 1/2 - 1/2erf(x/\sqrt{2})$ with $x = -\alpha/(\dot{Y}/Y100)$ and α as a small positive constant. Since w^s is larger the bigger is the positive variation in production, we can write $M(w_j > w^s) = 1 - F(x)$.

Consequently, using the approximation $\dot{P}(occ(t) = 1) \approx P(occ(t) = 1) - P(occ(t-dt) = 1)$ in (47) we can express the variation in probability to be employed as

$$\dot{P}(occ(t)=1) = \left[\eta_y(t-dt) - P(occ(t-dt)=1)\right]\left[1-F(x)\right]. \tag{48}$$

Considering as a reference the state $s = 1$ in which the satisficing wage is increased because the worker is employed, equation (48) can be used to define the transition probabilities as

$$\zeta_w(t) = P(occ(t)=1), \tag{49}$$

$$\iota_w(t) = 1 - P(occ(t)=1). \tag{50}$$

According to (3), the returns associated to the wage-increasing and wage-decreasing strategies are, respectively, $V_1 = W\delta$ and $V_2 = -W\delta$.

$$\beta_w(t) = -(2W(t)\delta)^{-1} log\left(\frac{n_w(t)}{1-n_w(t)}\right), \tag{51}$$

$$g_w(t) = W(t)\delta. \tag{52}$$

Consumption

Consumers can be supply-constrained or not depending on whether they are able to spend all their wealth on consumption. Let us define the two variables $V_c(t) = \frac{P(t)Y(t)}{Z(t)} - 1$ and $v_c(t) = \frac{p(t)y(t)}{\sum_{h^y_{ij}} z_j(t)} - 1$ at aggregate level and general firm level, respectively. The two variables are negative if consumers are supply-constrained and positive if they are not. Accordingly, the larger is $V_c(t)$ the higher will be the probability for a consumer to be able to spend all her wealth in consumption. We can use the fact that $v_c \sim N(0,1)$ across consumers, as shown by the simulations. Thus, considering $s = 1$ the state of non-constrained consumers as a reference, the transition probabilities are

$$\zeta_c(t) = F(v_c < V_c(t)) = \frac{1}{2}\left[1 + erf\left(\frac{V_c(t)}{\sqrt{2}}\right)\right], \tag{53}$$

$$\iota_c(t) = 1 - \zeta_c(t). \tag{54}$$

For consumers who are not supply-constrained the return is given simply by the consumption $V_1 = PY$, while for those who accumulate involuntary savings the return is given by the wealth: $V_2 = Z$. It follows that

$$\beta_c(t) = (P(t)Y(t) - Z(t))^{-1} \, log\left(\frac{n_c(t)}{1 - n_c(t)}\right), \tag{55}$$

$$g_c(t) = \frac{1}{2}\left(P(t)Y(t) - Z(t)\right). \tag{56}$$

It is easy to see that β_c tends to infinity (minimum uncertainty) when supply and demand are in equilibrium, that is $PY - Z = 0$. As shown below, a particularly interesting situation arises when, in a situation of goods market equilibrium, $n_c \approx 1/2$ pushing β_c to 0 (maximum uncertainty). In this case numerical simulations are needed to determine what is the final effect on β_c.

4.2.2. Rational optimization

Optimal control is adopted for the real variables (firms' supply and households' consumption decisions), while price and wage setting are determined by agents' heuristic behavior. The choice to limit the optimization to the real variables allows comparison between the different consequences of nominal frictions in the two treatments and how they impact on the interactions among agents.

As mentioned above, transition probabilities and β_ks are the same in the two cases while the determination of g_k differs.

Since the stationary fraction of agents choosing one strategy is determined by the stationary probability (20), the social planner must solve a suitable $g(n_1)$ function in order to maximize the expected future streams of profits and utility. The problem involves the allocation of agents over two different populations (firms and workers). Consequently, the goal of the social planner is to maximize the stream of firms' expected profits and households' expected life-time utility subject to equation (28) and (35), respectively. In the firms' problem, a proportion n_y of firms wants to increase their production, and as a consequence will incur search costs, denoted κ. Recalling that we consider a unitary labor productivity, $1 - \frac{W(t)}{P(t)}$ represents the unitary profit. Thus the remaining $(1 - n_y)$ firms will suffer a profit loss, quantified by $\frac{W(t)}{P(t)} - 1$ due to the decrease in production. Considering (25) and using θ to indicate the household's discount parameter, the firms' problem can be formulated as

$$g_y = J(Y) = \max_{n_y} \int_0^\infty exp(-\theta t) \left[n_y(t)(-\kappa) + (1 - n_y(t)) \left(\frac{W(t)}{P(t)} - 1 \right) \right] dt, \tag{57}$$

s.t.

$$\dot{Y}(t) = Y(t)\delta \left[1 + q(t)\right] \left[n_y(t) - \frac{1}{1 + q(t)} \right]. \tag{28}$$

For the household problem, let us use $u(Z)$ and $u(Y)$ respectively to denote the current utility deriving from the available wealth and consumption of firms' production, respectively. Consumers who are not supply-constrained will enjoy the utility generated by their total wealth Z, while supply-constrained consumers will derive their utility only from the level of consumption they can actually access. Accordingly, using (26) we can write the dynamic optimization problem for the proportion of non-supply-constrained consumers as

$$g_c = J(Z) = \max_{n_c} \int_0^\infty exp(-\theta t) \left[n_c(t)u(Z(t)) + (1 - n_c(t))u(Y(t))\right] dt, \tag{58}$$

s.t.

$$\dot{Z}(t) = \left[rZ(t) + W(t) - C(t)\right] Y(t) + \left[rZ(t) - C(t)\right] (N_w - Y(t)). \tag{35}$$

We then apply dynamic programming to equations (57) and (58). The corresponding Hamilton-Jacobi-Bellman equations are given by

$$-J(Y,t) = \max_{n_y} \quad exp(-\theta t) \left[n_y(t)(-\kappa) + (1 - n_y(t)) \left(\frac{W(t)}{P(t)} - 1 \right) \right] + J_y(Y,t)\dot{Y}(t), \tag{59}$$

$$-J(Z,t) = \max_{n_c} \quad exp(-\theta t) \left[n_c(t)u(Z(t)) + (1 - n_c(t))u(Y(t))\right] + J_z(Z,t)\dot{Z}(t). \tag{60}$$

We guess the solutions $J(Y,t) = exp(-\theta t)V(Y)$ and $J(Z,t) = exp(-\theta t)V(Z)$. Observing that $J_Y(Y,t) = V_Y(Y)$ and $J_Z(Z,t) = V_Z(Z)$, the conditions become[10]

[10]To verify this result, consider the dynamic programming equation in discrete time

$$J(x_t, t) = max_{y_t} U(x_t, y_t, t) + J(x_{t+1}, t+1),$$

subject to

$$x_{t+1} = h(x_t, y_t, t),$$

$$-V(Y)exp(-\theta t)exp(-\theta t) = \max_{n_y} \left[n_y(t)(-\kappa) + (1-n_y(t))\left(\frac{W(t)}{P(t)} - 1\right)\right] + \\ + V_y(Y)exp(\theta t)exp(-\theta t)\dot{Y}(t), \tag{61}$$

$$-V(Z)exp(\theta t)exp(-\theta t) = \max_{n_c} exp(-\theta t)\left[n_c u(Z(t)) + (1-n_c)u(Y(t))\right] + \\ + V_Z(Z)exp(-\theta t)exp(-\theta t)\dot{Z}. \tag{62}$$

The result is an ordinary differential equation in the state variable. Substituting the dynamical constraints (28) and (35) into respectively (61) and (62), and taking the partial derivatives for n_y and n_c, the first order conditions can be expressed as

$$\left[1 - \frac{W(t)}{P(t)} - \kappa\right] + J_Y(Y)Y(t)\delta = 0, \tag{63}$$

$$u(Z(t)) - u(Y(t)) + J_Z(Z(t) - Y(t)) = 0. \tag{64}$$

Substituting equations (63) and (64) into (59) and (60) for J_Y and J_Z, and assuming $u(X) = X$, we get two ordinary differential equations in the macro variables Y and Z. Applying the standard solution method, we get the closed form solutions:

$$g_y = V(Y) = \pi(t)\frac{q(t)}{1+q(t)}, \tag{65}$$

$$g_c = V(Z) = \frac{\theta\frac{Z(t)}{P(t)} - Y(t)(\frac{W(t)}{P(t)} - 2n_c(t))}{\theta(\theta + 2(n_c(t) - \frac{1-r}{2}))}. \tag{66}$$

where $\pi(t) = 1 - \frac{W(t)}{P(t)} - \kappa$ denotes the unitary profits as labor productivity is equal to 1. The probability $q(t) = (1-Y(t))^a(Y(t)\delta)^{b-1}$ is equivalent to the matching probability in the labor market, introduced in section 4.1. Equations (65) and (66) are the solutions to the intertemporal Euler equation for firms and households.

While in the zero intelligent behavior the product of β_k and g_k function vanishes, in case of rational behavior, the equilibrium depends on both the g_k

given x_0. The expansion of the value function in Δt gives

$$J(x_t,t) = max_{y_t}U(x_t,y_t,t)\delta t + J(x_t,t) + \frac{\partial J(x_t,t)}{\partial x_t}\frac{x_{t+\Delta t} - x_t}{\Delta t}\Delta t + \frac{\partial J(x_t,t)}{\partial t}\Delta t + o(\Delta t).$$

Taking the limit $\Delta_t \to 0$ gives equations (59) and (60) function which embodies the rational behavior, and the uncertainty variable β_k. Thus, uncertainty and the state of the macroeconomy affect the macroeconomic equilibrium and simultaneously determine the behavior of the economic agents. Therefore we can identify two types of equilibrium solutions of the dynamical system, depending on whether the effects of g_k and β_k offset each other or not. In the first case, since g_k vanishes, agents' incentives do not affect the resulting equilibrium which can be defined as *uncertainty equilibrium*. This equilibrium always arises in the bounded rationality setting but occurs only if $\beta \approx 0$ under the full-rationality assumption. Under this equilibrium, the system may be lock in an uncertainty trap (Aoki and Yoshikawa, 2006) as shown by the simulations below.

In the second case, the product βg does not vanish in (20): rational incentives affect the distribution of firms across states. This equilibrium corresponds to the optimal situation in which no further arbitrage would be profitable. This situation is consistent with a perfectly rational behavior, in which all the profit opportunities are exhausted and full employment is achieved: therefore it can be qualified as *rational equilibrium*.

Looking at the firms' decision about production levels, equation (65) states that the macroeconomy reaches the equilibrium ($g_y = 0$) in two cases: null profits or, more interestingly, full employment. The latter condition is represented analytically by the no-match situation $q(t) = 0$, not considering the uninteresting no-match case in which no worker supplies labor.

For households the result is more complex. Households' behavior is in equilibrium if equation (66) is null. This leads to the macroeconomic condition:

$$\frac{Z(t)}{P(t)} = \theta^{-1} Y(t) \left(\frac{W(t)}{P(t)} - 2n_c(t) \right), \tag{67}$$

which can be interpreted as the demand function in terms of real wealth for each level of n_c. More precisely, demand depends negatively on n_c since when $n_c > 0$ part of the wealth in the system is not spent on consumption.

When consumers are fully supply-constrained ($n_c = 0$), the equilibrium condition becomes $\frac{Z(t)}{P(t)} = \theta^{-1} Y(t) \frac{W(t)}{P(t)}$. This condition implies the general equilibrium since all labor income $Y(t)\frac{W(t)}{P(t)}$ is consumed, there is no accumulation of wealth and the circular flow in the goods market is fully closed.

Interestingly, if $n_c = 1/2$ and the real wage is equal to productivity ($W/P = 1$), the (67) becomes $Z/P = 0$: if consumers are wealth-constrained and supply-constrained in the same proportion, there are no savings at the macroeconomic level.

5. Simulations

We run different series of simulations in order to study the dynamical system, to compare the outcomes of the agent based model and the stochastic dynamic aggregate model, and to contrast the heuristic scenario with the optimizing case.[11]

Simulation codes are written for Matlab and are available upon request. The parameter setting in the benchmark scenario is shown in table 3. For the stochastic dynamic aggregation, the complete simulated system is composed of equations (28) for production setting behavior, (31) for price setting behavior, (33) for wage, (34) and (35) for consumers. The system is completed by (14) and the coupled equations (15) and (16) for each of the agents' densities n_w, n_y, n_p, n_c.

The properties of the system are investigated in the baseline simulation presented in section 5.1. Then the stochastic aggregation is tested by comparing the outcomes of the aggregate model with the ABM's results in section 5.2 which presents the results of a Monte Carlo experiment in the form of impulse/response functions, represented as deviations from the baseline simulation. Section 5.3 presents the results when g_k is optimally set.

The initial conditions for the aggregate system are set close to the macroeconomic equilibrium with the real variables (consumption, real wages and production) equal to 1 (see Table 2). The initial condition for the stochastic spread is set to 0. For the ABM, wages and prices are set equal to 1, and the initial worker allocation for each firm is a uniform distribution over a population of 100 firms and 500 workers.

5.1. Heuristic behavior

Figures 2-3 show the outcome of a single simulation including in the same charts production, consumption and wealth for the aggregate system (figure 2) and the dynamics of occupation numbers for each sub-sector (figure 3).

Fluctuations and crises are due to the fact that the labor and goods markets incorporate search frictions at the micro-level. Note that the macroeconomic effects of these real frictions originate from the interaction between firms and workers. Labor market frictions determine a real wage above productivity (which is constant and equal to 1). Since the real wage is always above 1, the circular flow of the system is not in equilibrium, and periodic macroeconomic crises are needed in order to re-balance the real-valued resources.

[11] In order to approximate continuous time, we apply the Euler-Maruyama procedure in the interval $t \in (T_0, T_{max})$. Given the number of steps N_{step} we can determine numerically the solution with the discretization of time

$$h = \frac{(T_0 - T_{max})}{N_{step}}. \tag{68}$$

Thus for example, we can approximate equation (28) in the following way:

$$y(i+1) = y(i) + h\left[y(i)\delta\left[1+q(i)\right]\left(n_y(i) - \frac{1}{1+q(i)}\right)\right]. \tag{69}$$

We generate the Wiener process with $d\mathcal{W} = \sqrt{(h)}u$ where u is distributed as a standardized Gaussian distribution.

The dynamics of real variables display the quasi-stable disequilibrium in figure 2: the aggregate production level stabilizes close to the full employment equilibrium but without reaching it. Agents over-save because the average real wealth is higher than 1. Moreover, real consumption always lies between supply and aggregate wealth. This is due simply to the fact that consumption is a weighted mean of supply (production) and demand (wealth). The system oscillates around the equilibrium displaying endogenous crises, heterogeneous in duration and depth.

Three different phases can be detected in the system dynamics. The first phase (first box on the left in figure 2) is characterized by a deep recession in which consumption, wealth and production display a sudden drop, causing a fall in employment. The recession could be prolonged since the fall in employment leads to a reduction in wealth. The second phase (second box from left) is the subsequent recovery when aggregate production, consumption, and wealth grow to an almost full-employment equilibrium. The real quantities grow steadily maintaining consumption below production levels. This means that consumers are not supply-constrained and, at the same time, they have excess income to rebuild their stock of wealth. When quasi-full-employment equilibrium is reached, the economy enters the third phase (third box from left): wealth, consumption, and production oscillate just below the full-employment equilibrium. When consumers accumulate enough wealth, aggregate demand increases, pushing firms to increase prices. Higher inflation leads to wealth devaluation followed by a fall in consumption level. This phase cannot be characterized as a recession since the fall stops when consumption and wealth hit the production level, which seems to act as a floor.

Figure 3 displays the dynamics of the occupation numbers in order to assess their evolution during the different phases. The shaded areas mark phases in which the production level is less than 95% of the full-employment equilibrium level. During these phases, a growing proportion of firms reduces production. The lower aggregate supply in the goods market leads to an increase in the proportion of supply-constrained consumers n_c. Finally, due to the accumulation of inventories, more firms decrease prices causing an increase in the proportion of firms that are cost-constrained n_p.

5.2. Monte Carlo experiments

In order to compare the outcomes of the ABM and the stochastic aggregation, and in particular to contrast the different responses to shocks in the parameters, we perform Monte Carlo simulations.

We follow a different approach from existing works (such as for example Chiarella and Di Guilmi, 2011), comparing the results of the two solutions by testing their reaction to an exogenous shock. This approach is consistent with the fact that we run two different sets of simulations for the ABM and the aggregate model, and as a consequence the results of single simulations are not directly comparable.

Shocks are imposed on the initial conditions defined in table 2 for: a) the aggregate production level $Y(0)$ (-1%), b) the aggregate price level $p(0)$ $(+1\%)$, c) labor productivity $(+1\%)$, and d) the interest rate r $(+1\%)$. The plotted data are obtained by subtracting the series generated in the baseline scenario to the series with the shock in the parameters.[12]

In all experiments but more clearly in the case of experiments a) and b) (figures 4-5), the aggregate model correctly predicts the sign of the reaction to the shock, while the magnitude of the reaction and the speed of adjustment in some cases are different. In particular, in the case of the shock on price the ABM displays slower convergence.[13] In order to verify the similarity of the series generated by the two simulations, we run an Engle-Granger test with the null hypothesis of no cointegration. For all the experiments, the null hypothesis is rejected at the 95% level of confidence.

Figure 4 reports the results for experiment a). When the initial condition for employment is below the full employment level, the aggregate production and consumption also start below their equilibrium level and converge to it in the long run in both sets of simulations. When unemployment increases, workers adjust the nominal wage while the real wage remain approximately constant. This is because firms lower their prices due to the accumulation of inventories.

Figure 5 refers to experiment b). If the initial price is above its equilibrium level, in both simulations the aggregate production drops due to a decrease in real aggregate demand. The drop in real wages leads to a larger fall in all the variables in the ABM case, which for wealth prevents the adjustment to the equilibrium level within the time-frame of the simulation.

In the case of a shock to the structural parameters (technology and interest rate) the change in the ABM is introduced at a later stage and not at time 0 in order to better visualize the response of the system. In the case of a technology shock in experiment c) (figure 6), the ABM and the DSGA display comparable dynamics. Production raises as expected, while real wage and wealth decrease due to an increase in prices and a decrease in the number of employed workers. This result is in line with the New-Keynesian DSGE literature where a negative impact on employment due to a technology shock is generated by price rigidities in goods or labor markets (see Liu and Phaneuf, 2013, for a survey). Aggregate consumption increases in both systems because a fraction n_c of consumers are constrained and the increase in supply allows for a partial relaxation of the market constraint.

Finally, figure 7 shows that the monetary shock has a positive impact on the economy due mainly to the implicit adaptive expectations included in this model

[12]The different time scale accounts for the conversion from continuous to discrete time as specified in footnote 11.

[13]The difference in the speed of convergence can be reduced by a formal calibration of the parameters aimed to achieve a perfect match between the results of the two treatments. Given the scope and the length of the present paper, this aspect is left to future research.

and the fact that, in the absence of financial liabilities, monetary frictions create a positive wealth effect leading to an increase in the aggregate available income.

5.3. Rational optimization

This subsection presents the results for the model introduced in section 2 and 4, using equations (65) and (66). The initial conditions are the same as in table 2.

Figure 8 provides a zoom of a limited time span within the simulation while figure 9 shows a longer time span. Figure 10 visualizes the dynamics of g_c and β_c while figure 11 reports the evolution of the proportion n_c of households who are not supply constrained. Examining of all the figures 8-11 shows that the system can generate two possible equilibria. Figure 8 shows that, after the phasing-in period (until around period 90), the system is in a *rational* equilibrium condition until around period 650. In fact, over this time-span g_c is equal to 0, implying that there is no relevant difference between the returns of the two consumption strategies, and $b_c \neq 0$ (figure 10) which indicates a low degree of uncertainty in the system. The economic interpretation is straightforward: the economic system is in a quasi-stable equilibrium with low uncertainty and maximized consumers' utility and firms' profits. Thus, when the system is in the rational-equilibrium, the β_c variable, following the dynamics of the macro-variables, is as expected far from zero. The rational equilibrium is associated with two emergent facts: there is full employment as production reaches the maximum level where employment is equal to $N_c = 500$ and 1 units of consumption per capita, and real wealth is near to production (figure 8). However, the endowment of real wealth is higher than unity due to goods market frictions, allowing for a level of consumption higher than aggregate production. In relation to households' behavior, n_c (the proportion of wealth-constrained households) fluctuates around 0.5 during the same period (figure 11). The stability of this quantity (net of stochastic noise) is due to there being no incentives for agents to change their strategy.

Since we introduce rationality only in production and consumption decisions, prices and wages are still set sub-optimally. This setting, together with the interaction in the goods and labor market, allows for real inefficiencies: the average real salary is always higher than productivity. This affects the long run behavior of the economy but in a different way to the heuristic case. Figures 8 and 9 show that around period 650 the rational equilibrium no longer holds due to dynamic inefficiencies in the labor and goods markets. Instead of periodic crises, the system is caught in what we can describe as an *uncertainty trap* (Aoki and Yoshikawa, 2006): a scenario in which increasing uncertainty locks the macroeconomy out from the optimal equilibrium.[14]

[14]In a different treatment, Fajgelbaum et al. (2014) define uncertainty traps "*as the coexistence of multiple stationary points in the dynamics of uncertainty and economic activity*". Fajgelbaum et al. (2014) in their paper refer to the uncertainty that agents face in their opti-

Figures 8 and 9 show that households reduce their consumption and increase their saving. This creates a growing gap between wealth and the other variables. Soon after period 650, wealth increases rapidly, diverging from the rational equilibrium level. As the dynamics of g_c and β_c demonstrates, the system transitions to the uncertainty equilibrium in which economic behavior is not rational ($g_c \neq 0$), uncertainty is high, as shown by $\beta_c \to 0$ (figure 10) and consumers become fully supply-rationed ($n_c \to 0$). The inefficiencies determine growing uncertainty, as β_c approaches 0 until period 650 when it peaks because a large fraction of consumers switch to the supply-constrained state. This determines a discontinuity in g_c (capped at 100 in the simulations), which subsequently takes high negative values due to the negative denominator when $n_c \to 0$ in equation (66). From around period 700, β_c remains at 0 signaling high uncertainty.

With reference to equation (55), the simulations demonstrate that the effects of the (dis)equilibrium in the goods market dominates the effects of agents' choices. In fact, in the rational equilibrium $\beta \neq 0$ even when $n_c \approx 1/2$ because $P(t)Y(t) \approx Z(t)$, while we have an uncertainty equilibrium even with $n_c = 0$ due to the disequilibrium in the goods market.

In the heuristic treatment, since the behavior of agents is quasi-zero intelligent, small idiosyncratic shocks can trigger feedback effects at agent level which produce the cyclical crises. In the optimization case, the accumulation of wealth is self-sustaining as shown in equation (66): the perceived relative pay-offs from consumption behaviors change, and in the absence of re-equilibrating mechanisms for the nominal variables, this leads to a permanent difference between wealth and consumption.

Further insights are provided by the phase diagram in figure 12. Since the model is multidimensional, a complete geometric representation of the equilibria is impossible but exploration of the consumption-wealth plane (C, Z) is sufficient to represent the dynamics of adjustment towards the two equilibria.

We tested a wide range of initial values (only a few of them are shown in the graph to ensure readability) and in almost all cases the model converges to a sink: a local rational equilibrium $C \cong Z$ in the point $(1.02, 1.05)$. The exceptions are given by the starting points located in the high-wealth region ($Z \gg 1$) from which the system is attracted to the uncertainty equilibrium in point $(1, 1.4)$. In this case, consistently with figures 8 and 9, the system is in a situation which we define as uncertainty trap, which is characterized by an over-accumulation of wealth.

In order to obtain further insights into the transition between different equilibria, in the next section this standard phase-space investigation is enriched from a combinatorial perspective by the analysis of the potential.

---

mization process, and rule out the existence of multiple equilibria, while Aoki and Yoshikawa (2006) consider the uncertainty generated by the complex interaction of agents, regardless of their objective functions and behavioral rules, which generate possible multiple equilibria.

6. Analytical identification of the equilibria

This section proposes an analytical criterion to identify quantitatively the different equilibria. In particular, we determine the critical levels of the proportion of supply-constrained consumers which determine the transition from one equilibrium to another.[15]

The depiction in figure 11 is completed by the analysis of the potential, introduced in section 3. in addition to identifying the critical points, this study defines the possible transition paths between different attractors, and the conditions under which our artificial economy becomes locked into the uncertainty trap. We focus on the households subsystem since it allows us to study the evolution of wealth and consumption demand at the same time.

6.1. Identification of the equilibrium type and transitions

The potential function (21) for the number of not-rationed consumers is

$$U = -2\int_0^{n_c} g_c(t)dZ - \frac{H(n_c,t)}{\beta(n_c,t)}, \tag{70}$$

with g_c defined by (66). The critical points of (70) are the equilibria for the occupation number n_c, and therefore for aggregate wealth.

Considering equation (19) and (70), the first order condition is

$$\frac{\partial U}{\partial n_c} = -2g_c(t) - \frac{H^{'}(n_c(t))}{\beta(n_c(t))} + \frac{\beta^{'}(n_c(t))}{\beta(n_c(t))^2} = 0. \tag{71}$$

With respect to the formula (23) presented in section 3.2, equation (71) introduces the term $\frac{\beta^{'}(nc)}{\beta(nc)^2}$. This term is not present in the heuristic treatment, and represents the additional knowledge available to the social planner which is not usable by boundedly rational agents.

Since a closed form solution cannot be derived analytically, we evaluate it numerically. Figures 13-15 display the results of the numerical analysis. Due to the discontinuity around $n_c = 0.001$, in order to make critical points and discontinuities visible we need to split the plot in tree sub-regions: one near $n_c = 0$ and two around $n_c = 0.5$. Figure 13 plots the whole function with zoom on the areas around the critical points for an overall assessment. Figure 14 plots the neighborhoods of the critical points. The potential has two local minima at $n_{c,1} = 0$ and $n_{c,2} = 0.48$, and the two points are separated by a discontinuity. Consequently, transition between the two equilibria is not possible. The first panel in figure 14 also shows a local maximum which splits the region on the

[15] The relevance for ABMs of the "tipping points" where phase transitions occur is stressed by Gualdi et al. (2015), in particular with reference to the parameter space.

left. In the local minimum $n_{c,1} = 0$, the simulations allow us to verify that $\beta_c = 0$ and $g_c \neq 0$ (figures 10 and 11): uncertainty is maximal and the equilibrium is not rational. The middle panel in figure 14 shows the neighborhood of $n_{c,2} = 0.48$, which is associated with $g(n_c) = 0$ (figures 10 and 11) and therefore can be identified as a rational equilibrium. Also, since $\lim_{n_c \to n_{c,2}} \beta(n_c) = -\infty$, as the economy approaches the social optimum, uncertainty reduces. Thus, in the absence of nominal frictions, the rational equilibrium is stable. The macroeconomic signal received by agents drives them not to change their behavior. Finally, we have a third extreme point located in $n_c = 0.51$ which is a maximum and therefore represents an unstable equilibrium. The critical points are shown clearly in figure 15 where the $log(U^{'})$ is displayed.

6.2. Policy Implications

The model presented above is parsimonious and serves the purpose of introducing and applying the aggregation method. Nevertheless, it is possible to run a simple policy experiment to assess how the policy maker can influence the type of equilibrium that the system can achieve. Within the present framework, the policy maker's task is to prevent the system from being locked into the uncertainty equilibrium or at least to reduce the likelihood of its occurrence. In fact, in the uncertainty equilibrium agents can have contrasting incentives and this coordination failure can jeopardize the success of the policy measures. The following analysis focuses on the impact of interest rate changes on the equilibrium condition in the household sector since the interest rate is the only policy variable in the model.

The goal is to investigate the effect of a zero lower bound on the number and the characteristics of the equilibria and to assess whether the rational and uncertainty equilibria are affected by the level of the interest rate.

We run three different simulations of the potential function for $r = \{-0.005, 0, 0.005\}$. The respective equilibrium values for per-capita production and real wage are equal to 1 and 1.005 as obtained in the simulations, while $\delta = 0.005$. Figure 16 replicates figure 15 for the different values of the interest rate, which in the baseline scenario is set to 0.001. Clearly, the number and the type of equilibria change as r moves from positive to negative values. In the case of a positive interest rate the results do not differ substantially from the baseline simulations: the two possible outcomes are a rational and an uncertainty equilibrium. For $r = -0.005$ the system can avoid the uncertainty equilibrium and the only attractors are the two rational equilibria in the neighborhood of $n_c = 0.5$. Also the discontinuity near the uncertainty equilibrium disappears, allowing the system to adjust to the rational equilibrium in the long run. This is possible because the negative interest rate prevents the accumulation of wealth in the long run. The uncertainty equilibrium disappears also in the case of a zero interest rate but indeterminacy arises since no critical point can be identified.

To conclude, the zero lower bound can be a source of indeterminacy and can lead the economy into the *uncertainty trap*. If the real factors of instability originating from the interactions of agents are not eliminated, negative nominal interest rates may be needed to curb the effects of uncertainty. While this result is well known in macroeconomic theory, the DSGA approach reveals the conditions under which such a situation occurs, and determine the structure of the equilibria through analysis of the potential.

7. Conclusions

The paper introduces and tests the Dynamic Stochastic Generalized Aggregation approach (DSGA) as an original methodology for analytical aggregation of ABMs which considers different levels of rationality while taking account also of the complexity arising from agents' interactions. We apply the DSGA approach to a medium-scale ABM, providing three main contributions. First, we estimate a set of macro equations starting from the micro behaviors of different sets of agents (namely firms, workers and consumers) to represent analytically the ABM. Building on simple micro-behavioral rules, we elaborate a macro-model that is able to qualitatively replicate the results of the ABM. This kind of methodology could be helpful for evaluating analytically the properties of complex models by obtaining reduced forms to validate and estimate ABMs.

Second, the application of the method to different sets of strategies provides an alternative way to link opinion dynamics and microeconomic behavioral rules.

The third contribution is that, by assuming that the proportions of agents adopting one or the other strategy are set optimally, it is possible to isolate the effects of rational incentives and uncertainty. In particular, the numerical and analytical treatments identify two types of equilibria: the rational equilibrium where all the opportunities for a welfare increase are exhausted, and the uncertainty equilibrium, in which the complex chain of interactions and feedback among agents generates a suboptimal outcome.

To summarize the results, in the presence of market inefficiencies, the accumulation of wealth determines over-saving and over-consumption leading to periodic crises in the heuristic setting, and to the uncertainty trap in the optimizing setting. The analytical investigation and the simulations show that, in the optimizing case for consumption, the system has two attractors which are associated respectively to an uncertainty equilibrium and a rational equilibrium. The analysis of the potential function is also applied in a simple policy exercise. In a world where agents are rational but interact in a complex economy, multiple equilibria are possible and monetary policy can affect their number and quality. In particular, the zero lower bound limits the capacity of the policy maker to avoid the uncertainty trap.

The paper aims to provide a general methodology for the analytical treatment of ABMs and the inclusion of standard optimization processes in a complex system structure. The procedure for constructing the dynamical system can be applied to any ABM, although the specific approximation for the functional identification of the transition probabilities will be different for and specific to each model. Admittedly, the aggregation procedure presented here suffers from two limitations. The first is that it restricts the choices of agents to a binary option. As we mention in the paper, although the solution for the master equation for a number of states larger than two is already available, it is not used here since a two-state master equation is appropriate for the application to this specific ABM, and we want to keep the analytical complexity to a minimum. In any case, the two-state limit is still at the moment binding for the analytical tractability of the optimization case. The second limitation is that for simplicity the transition rates are derived under the assumption of independence among the different master equations. This can be overcome either by a multivariate master equation, or a derivation of the transition probabilities conditioned on other firm-level variables. Both procedures involve further analytical complications which we prefer to avoid for this first implementation of the aggregation technique.

This promising methodology could be employed to build more general analytical models including in principle standard structural macroeconomic models (DSGE) as special cases. A full exploration of this possibility is the next item of our research agenda. Comparison with DSGE models could also involve the treatment of the transition rates. The method presented here endogenously generates the transition rates and could be used for a comparison with the empirically estimated rates presented in DSGE with Markov switching. The approach could be extended also by estimating the actual probability of the uncertainty trap in a real economy using a model calibrated with empirical data.

Appendix A. Timeline of events

1. At time t, firms determine the output to produce for the current period based on the level of stocks according to (1) and, consequently, quantify the number of workers to hire;
2. Workers update their satisficing wage on the basis of their occupational status according to (3);
3. The labor market opens. All workers send applications to a random sample of h^w_{ji} firms. Each firm collects the applications, sorts them in ascending order of w_j and hires the cheapest workers until either its demand for labor is satisfied or the list of applications is exhausted. Given that $h^y_{ij} \ll N$, at the end of this process some workers can be unemployed, and some firms can have unfilled vacancies;
4. Production takes place;
5. The market for consumption goods opens. Each consumer has access to a random sample of h^c_{ji} suppliers. The consumer sorts the prices in ascending order and purchases goods from the cheapest firms until either her wealth is entirely spent, or the supplying firms in her sample run out of goods. Given that $h^c_{ji} \ll N$, at the end of this process some consumers

can be supply-constrained and accumulate positive savings while some firms can be left with unsold goods which are stocked as inventories;

6. Firms' profits are determined as the difference between revenue and the wage bill and distributed to households.

Appendix B. Entropy maximization

This appendix presents the solution of the system (20). The goal is to maximize Entropy:

$$H = -n_1 log(n_1) - n_2 log(n_2), \tag{B.1}$$

subject to the constraints

$$\begin{cases} n_1 + n_2 = 1, \\ V_1 n_1 + V_2 n_2 = \dot{Y}. \end{cases} \tag{B.2}$$

The first constraint is a simple normalization constraint, while the second can be considered an accounting identity. For example, if V_1 and V_2 are the change in production for the firms of respectively, type 1 and type 2, the variation in total output must be equal to $\dot{y}$. The Lagrangian for this problem is

$$-n_1 log(n_1) - n_2 log(n_2) + \delta_1(n_1 + n_2 - 1) + \delta_2(V_1 n_1 + V_2 n_2 - \dot{Y}). \tag{B.3}$$

The first order conditions are given by

$$\frac{dL}{dn_1} \Rightarrow -log(n_1) - 1 + \delta_1 + \delta_2 V_1 = 0, \tag{B.4}$$

$$\frac{dL}{dn_2} \Rightarrow -log(n_2) - 1 + \delta_1 + \delta_2 f_2 = 0, \tag{B.5}$$

$$\frac{dL}{d\delta_1} \Rightarrow n_1 + n_2 = 1, \tag{B.6}$$

$$\frac{dL}{d\delta_2} \Rightarrow \dot{Y} = V_1 n_1 + V_2 n_2. \tag{B.7}$$

Imposing the following identities

$$\begin{aligned} \delta_1 &= 1 - \alpha, \\ \delta_2 &= -\beta, \end{aligned} \tag{B.8}$$

the first two equations in (B.7) become

$$n_1 = exp[-(\alpha + \beta V_1)], \tag{B.9}$$

$$n_2 = exp[-(\alpha + \beta V_2)]. \tag{B.10}$$

Substituting the above equations into the third in (B.7) we obtain

$$exp(-\alpha) = \frac{1}{exp(-\beta V_1) + exp(-\beta V_2)}, \tag{B.11}$$

which, substituted in the last of (B.7) gives

$$(V_1 - \dot{Y})exp(-\beta V_1) + (V_2 - \dot{Y})exp(-\beta V_2) = 0. \tag{B.12}$$

Rearranging we obtain an expression for β

$$\beta = (V_2 - V_1)^{-1} log\left(\frac{n_1}{1 - n_1}\right), \tag{B.13}$$

Since

$$n_1 = exp[-\alpha(t)]e^{-\beta(t)V_1(t)}, \tag{B.14}$$

$$n_2 = exp[-\alpha(t)]e^{-\beta(t)V_2(t)}. \tag{B.15}$$

Then

$$p_1 = \frac{exp[-\beta(t)V_1(t)]}{Z}, \tag{B.16}$$

$$p_2 = \frac{exp[-\beta(t)]V_2(t)}{Z}, \tag{B.17}$$

where Z represents the partition function:

$$Z = exp[-V_1(t)\beta(t)] + exp[-V_2(t)\beta(t)].$$

Appendix C. Full dynamical system

To help the reader this appendix summarizes the different equations included in the dynamical system for each macroeconomic variable, and the functional links among them.

The dynamical system is given by

$$\dot{X}_k(t) = \sum_{s_k} f_{s_k}(X_k(t))n_{s_k}(t) \tag{17}$$

$$n_k(t) = m_k(t) + u_k(t)N_k^{-1/2}, \tag{14}$$

$$\frac{dm_k}{dt} = \lambda_k(t)m_k(t) - [\lambda_k(t) + \gamma_k(t)]m_k(t)^2, \tag{15}$$

$$du_k = -a_1(m_k)u_k dt + a_2(m_k)dW, \tag{16}$$

with $P(u) \sim \mathcal{N}(0, \frac{\lambda_k \gamma_k}{(\lambda_k+\gamma_k)^2})$. Equation (17) expresses the macro-variables X_k as a function of the proportion of agents in each state n_{s_k}. The evolution of this proportion is determined by the solution of the master equation in (15-16) and depends on the two transition rates, which are defined as

$$\lambda_k(t) = (1 - \eta_k)\zeta_k(t),, \tag{12}$$

$$\gamma_k(t) = \eta_k \iota_k(t). \tag{13}$$

The transition probabilities ζ, ι depend on the behavioral assumptions of the model and require the further specifications and approximations presented in section 4.2. The probability η is defined as

$$P(n_1) = \eta_k = \frac{exp[\beta_k g_k(n_1)]}{exp[\beta_k g_k(n_1)] + exp[-\beta_k g_k(n_2)]}, \tag{20}$$

with

$$\beta_k = (V_1 - V_2)^{-1} log\left(\frac{n_k}{1 - n_k}\right). \tag{19}$$

The definition of g_k differs between the heuristic and the full rationality cases. In the former it is defined as

$$g_k(n_1) = \frac{V_1 - V_2}{2}. \tag{24}$$

In the full rationality case g_k is obtained from an intertemporal optimization problem depending on the returns of the different strategies

$$g\left(Y, n_s\right) = \max_{\mathbf{n}(t)} \int_{t_0}^{\infty} e^{-\theta t} \psi(Y(t), \mathbf{n}(t)) dt. \tag{25}$$

Tables and figures

Agent	States	Fraction	Description
Firms' quantity	y_1	n_y	Upward adjustment
	y_0	$1 - n_y$	Downward adjustment
Firms' price	p_1	n_p	Free adjustment
	p_0	$1 - n_p$	Unitary cost constrained
Workers' wage w	$occ = 1$	n_w	Upward adjustment
	$occ = 0$	$1 - n_w$	Downward adjustment
Workers' consumption	c_1	n_c	Not rationed
	c_0	$1 - n_c$	Rationed

Table 1: Agents and States

Variable and initial value	Description
$y = 480$	Aggregate Production
$p = 1$	Price
$w = 1$	Wage
$c = 1$	Consumption
$z = 1$	Wealth
$m_y = 0.5$	Drift for production
$s_y = 0$	Spread for production
$m_p = 0.5$	Drift for price
$s_p = 0$	Spread for price
$m_w = 0.5$	Drift for wage
$s_w = 0.1$	Spread for wage
$m_c = 0.01$	Drift for supply constrained consumers
$s_c = 0$	Spread for supply constrained consumers

Table 2: Initial conditions.

Parameter and value	Description
$\delta = 0.01$	Price and wage adjustment rate
$r = 0.1\%$	Interest rate
$a = 0.5$	Matching function parameter
$b = 0.5$	Matching function parameter
$h^w_{ji} = h^c_{ji} = 3$	Subset of firms for each workers/consumers in labor and goods market
$\kappa = 0$	Searching cost in goods market
$N_y = 100$	Number of firms
$N_c = 500$	Number of consumers

Table 3: Parameters

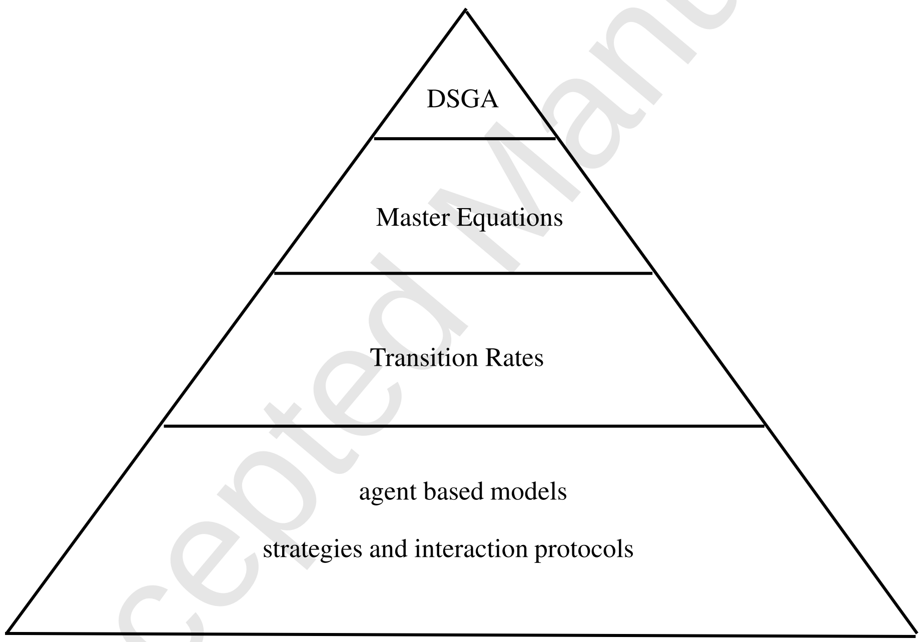

Figure 1: Conceptual representation of the methodology.

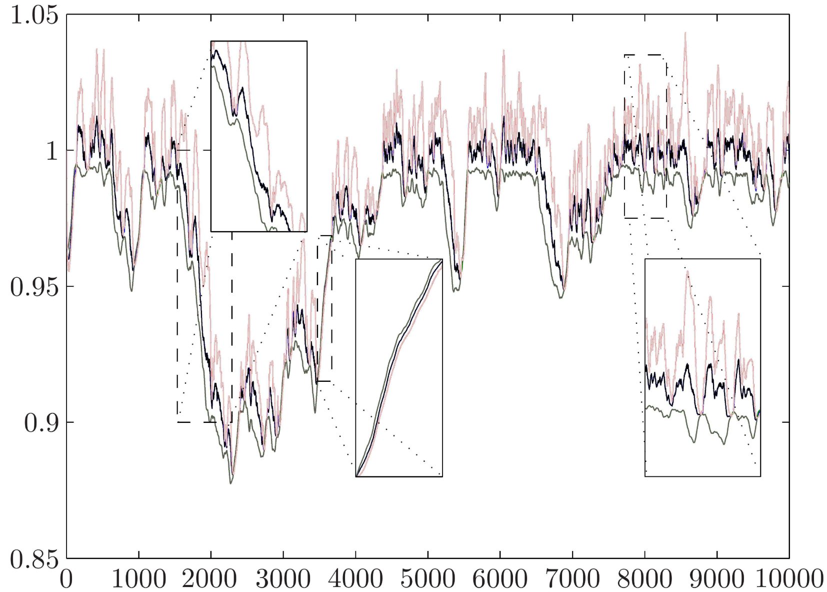

Figure 2: Simulation of the system for the aggregate model. Black line: per-capita production, light gray line: per-capita wealth, dark gray line: per-capita consumption.

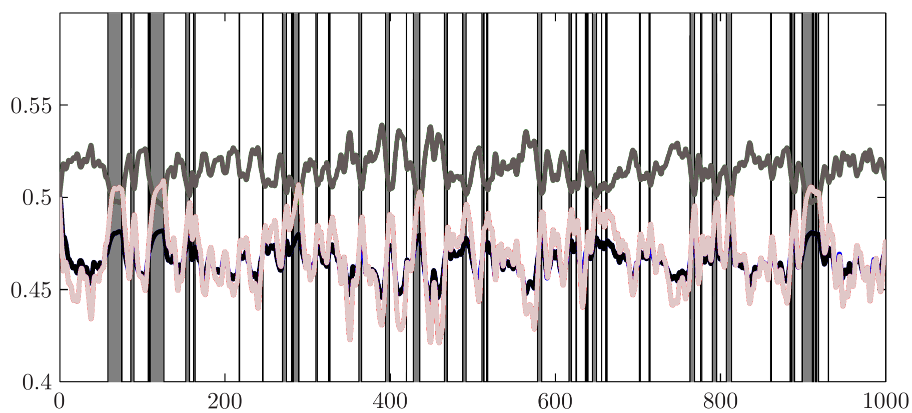

Figure 3: Simulation of the system for the aggregate model: occupation numbers. Light gray line: increasing-production firms (n_y), dark gray line: not-rationed consumers (n_c), black line: not cost-constrained firms (n_p). Shaded areas: periods in which the production level is less than 95% of the full-employment equilibrium level.

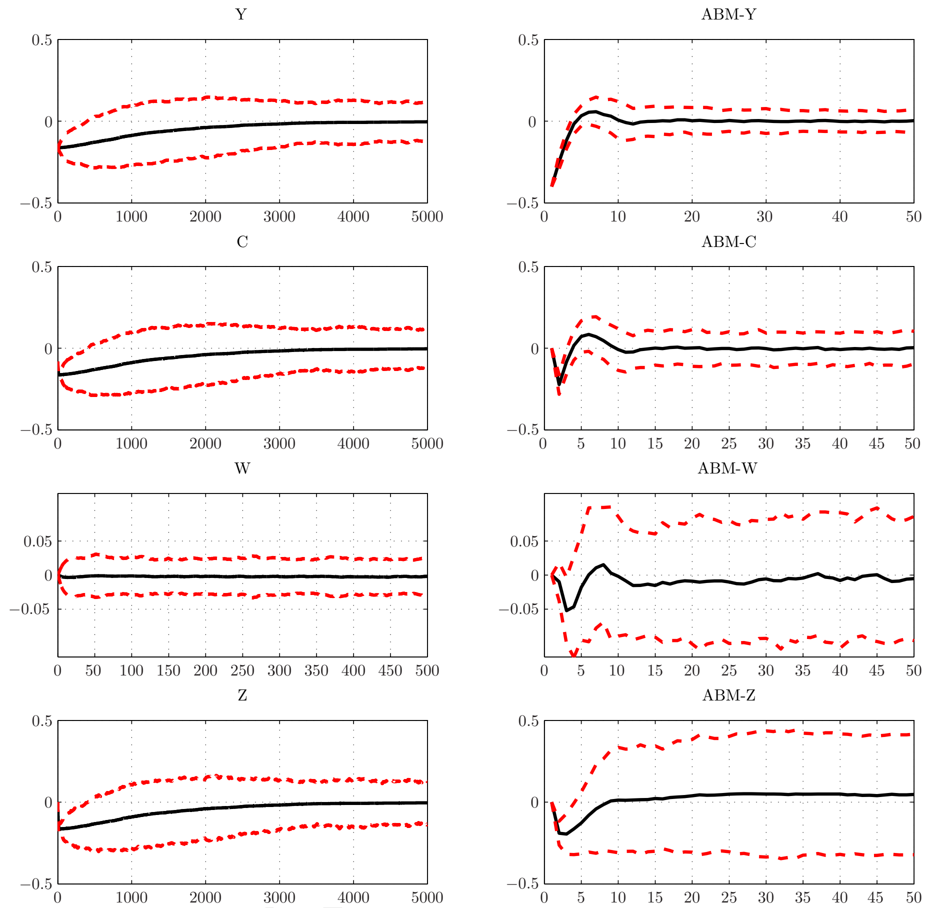

Figure 4: Shock in Production (−1%). Monte Carlo simulation with 1000 replications. Percentage deviation from baseline simulation and confidence intervals. Left panels: DSGA system; right panels: ABM.

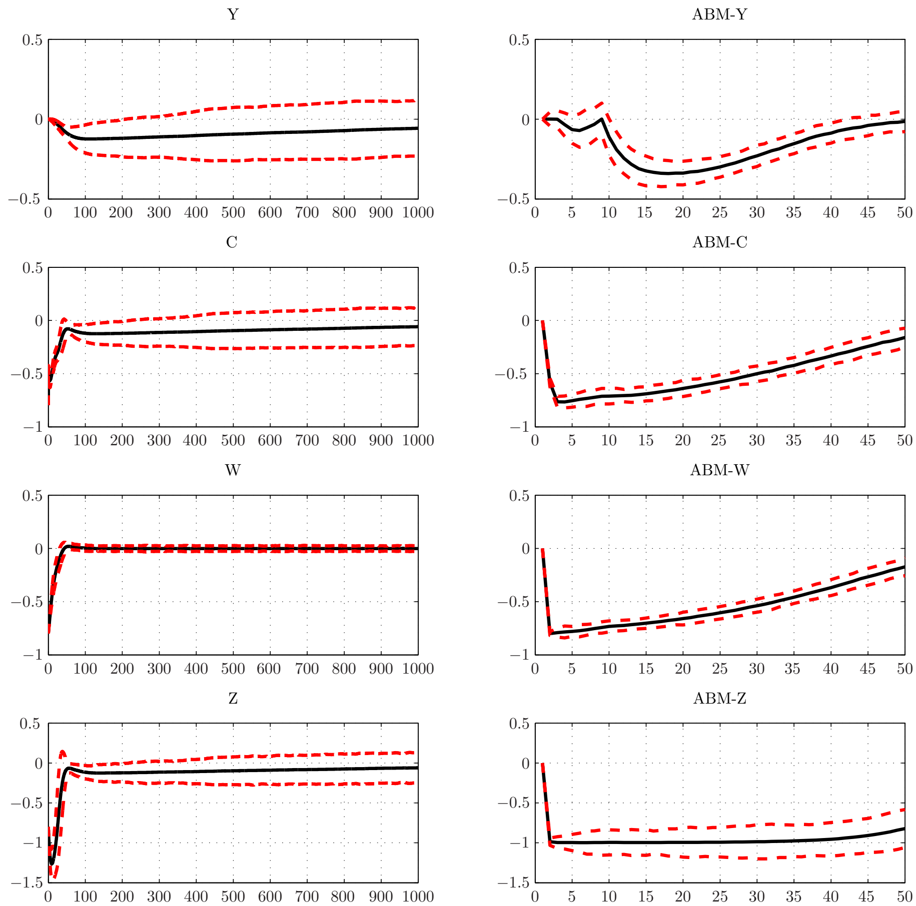

Figure 5: Shock in Prices (+1%). Monte Carlo simulation with 1000 replications. Percentage deviation from baseline simulation and confidence intervals. Left panels: DSGA system; right panels: ABM.

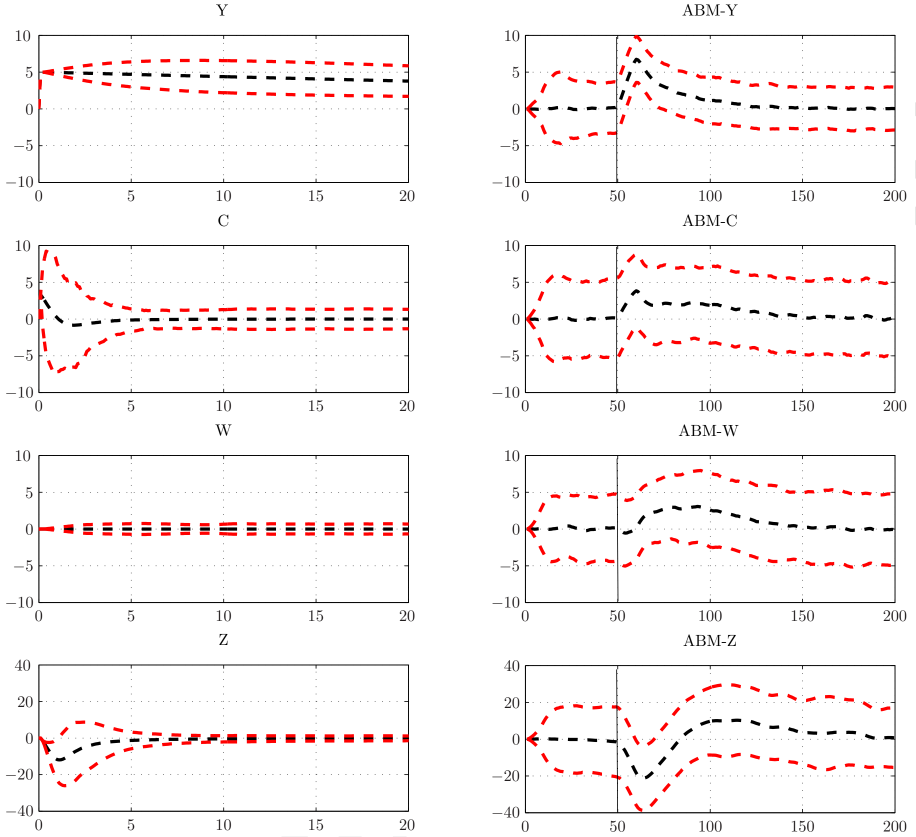

Figure 6: Shock in labor productivity (+1%). Monte Carlo simulation with 1000 replications. Percentage deviation from baseline simulation and confidence intervals. Left panels: DSGA system; right panels: ABM.

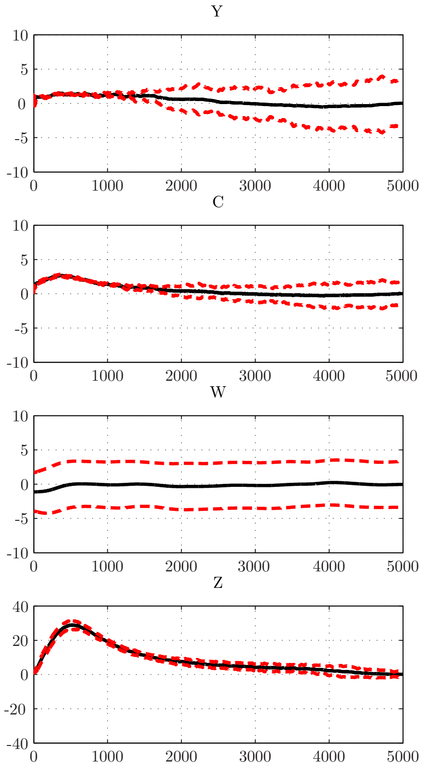

Figure 7: Shock in interest rate (+1%). Monte Carlo simulation with 1000 replications. Percentage deviation from baseline simulation and confidence intervals. Left panels: DSGA system; right panels: ABM.

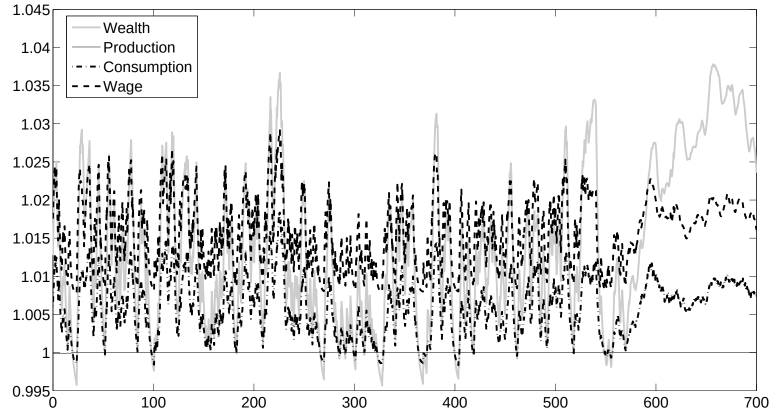

Figure 8: Simulation of the system for the aggregate model with optimization: per-capita real variables (zoom on periods 0-700).

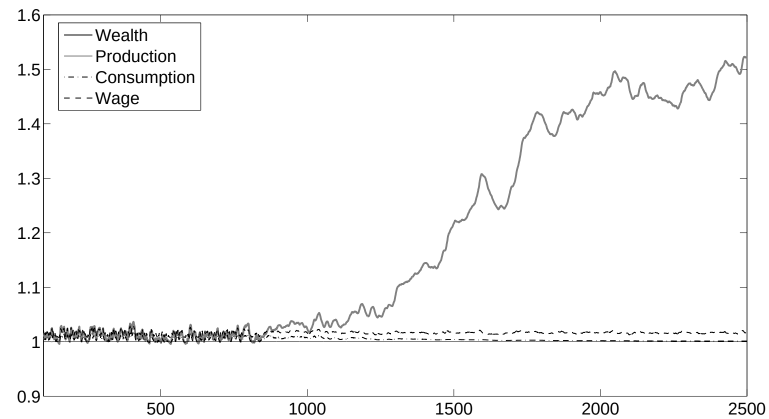

Figure 9: Simulation of the system for the aggregate model with optimization: per-capita real variables (burn-in period 0-100 omitted).

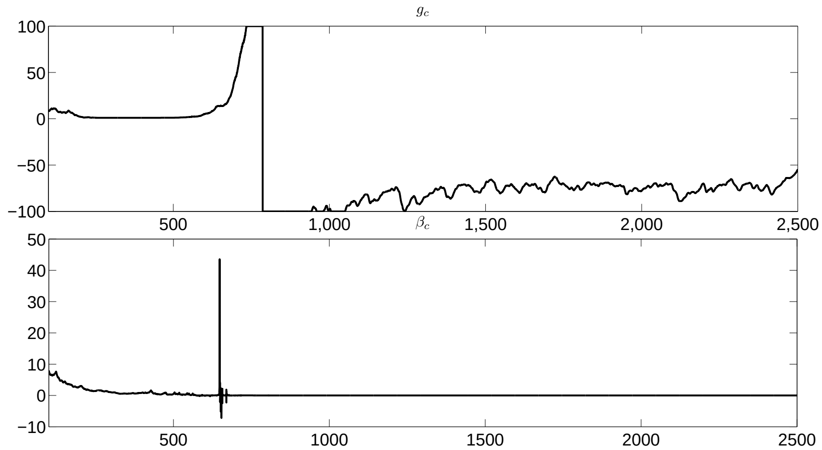

Figure 10: Simulation of the system for the aggregate model with optimization: value function g_c (upper panel) and uncertainty index β_c (bottom panel) for supply-constrained consumers (burn-in period 0-100 omitted).

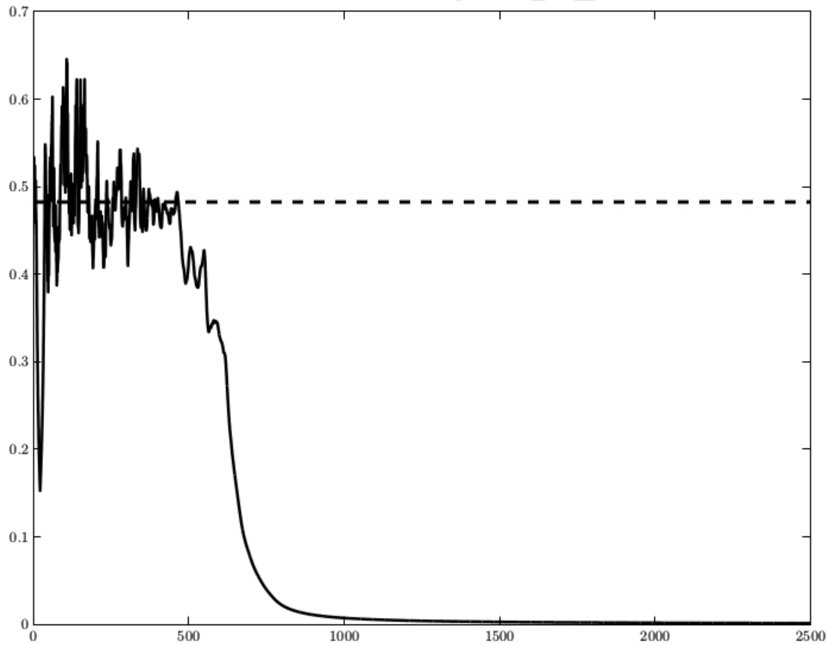

Figure 11: Simulation of the system for the aggregate model with optimization: proportion of households not supply constrained. Dashed line for $n_c = 0.48$.

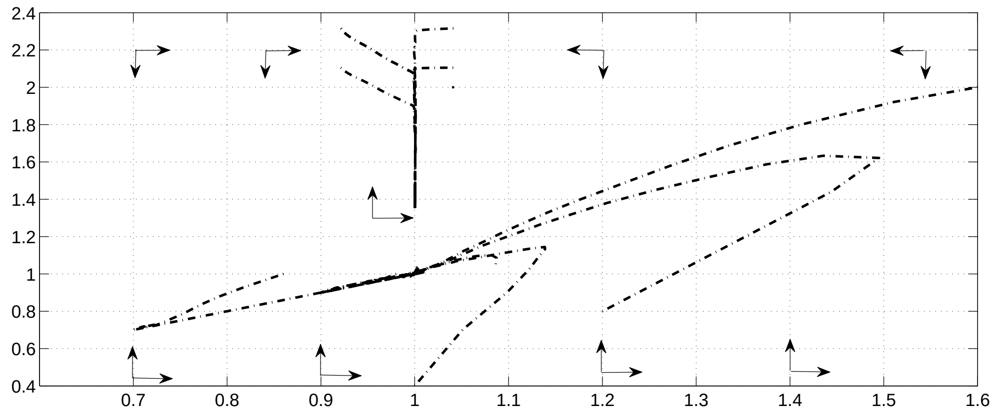

Figure 12: Numerical Exploration of system phase-space. Consumption vs Wealth for different initial conditions.

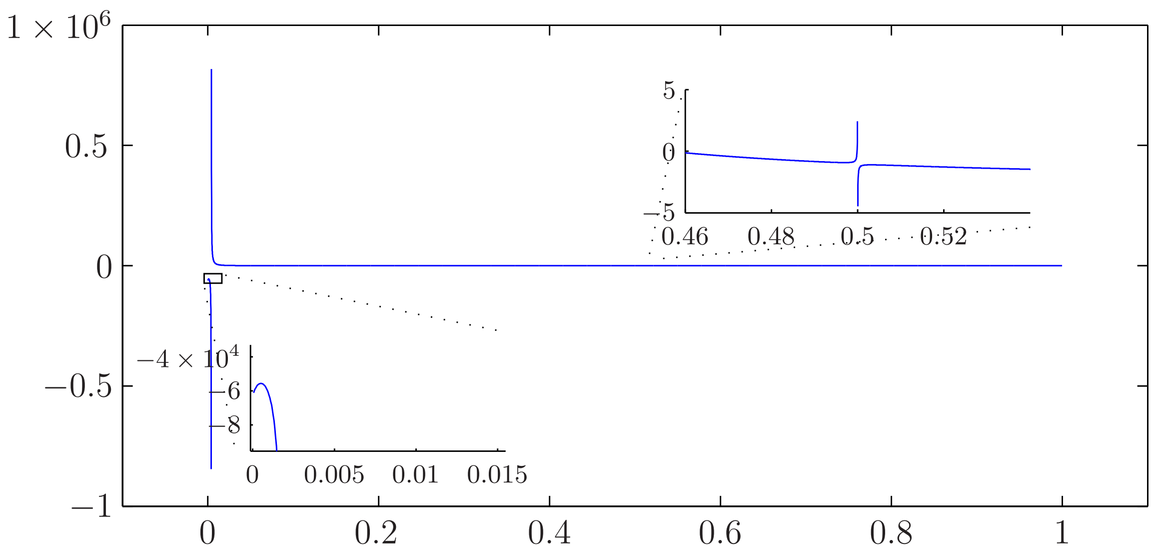

Figure 13: Simulation of the system for the aggregate model with optimization: potential function U (vertical axis) vs. n_c (horizontal axis) with zoom around the critical points.

46

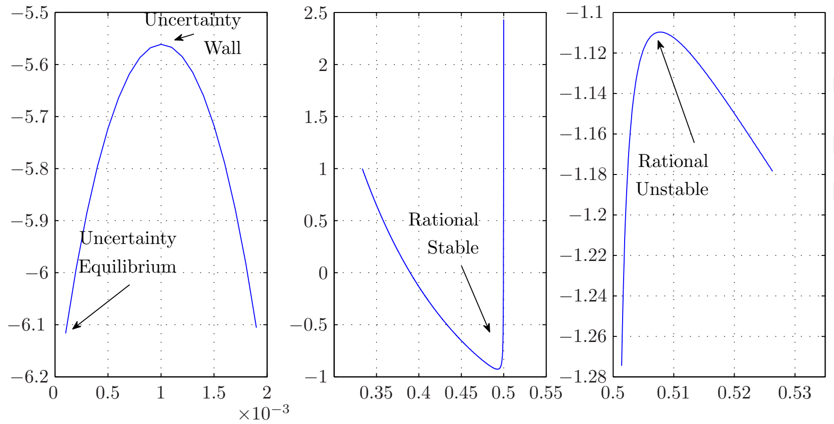

Figure 14: Simulation of the system for the aggregate model with optimization: potential function U (vertical axis) vs. n_c (horizontal axis). Detail around the critical points.

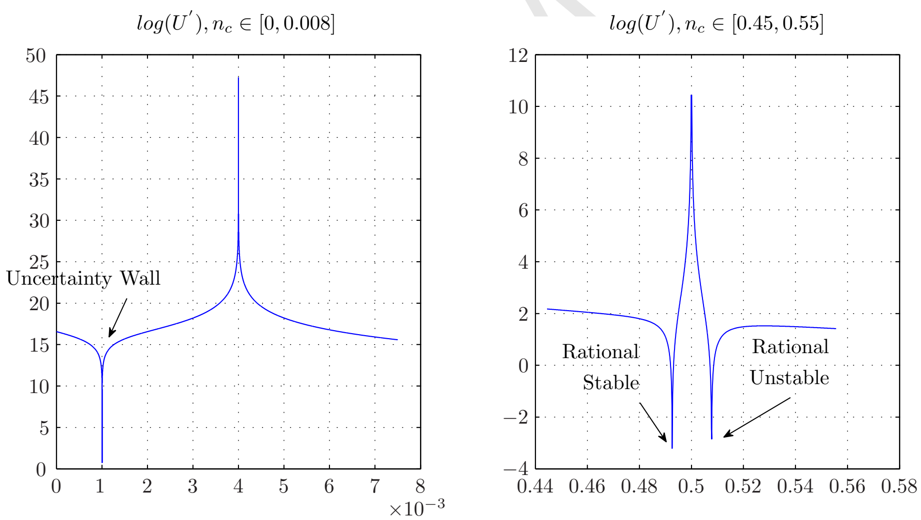

Figure 15: Simulation of the system for the aggregate model with optimization: logarithm of the potential function U (vertical axis) vs. logarithm of n_c (horizontal axis). Detail around the critical points.

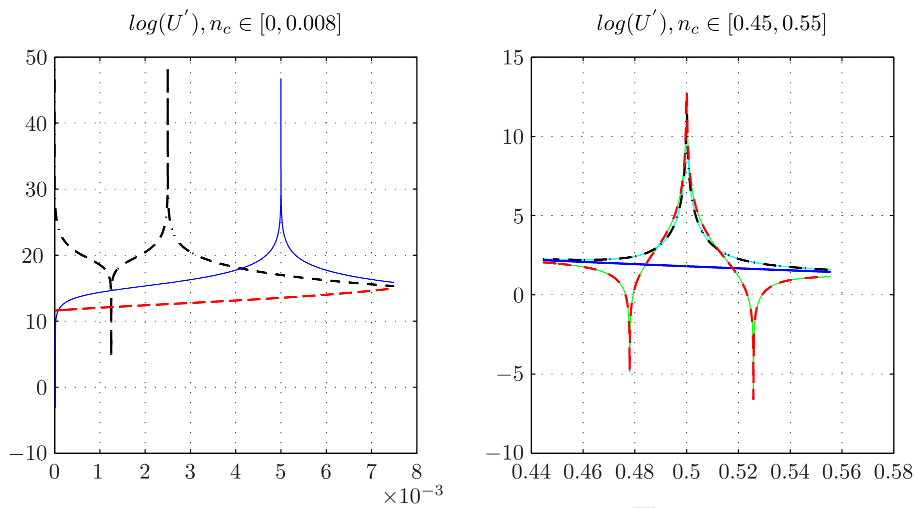

Figure 16: Simulation of the system for the aggregate model with optimization: logarithm of the potential function U (vertical axis) vs. logarithm of n_c (horizontal axis). Detail around the critical points for different interest rates. Red line: $r = -0.5$ per cent, black line $r = 0.5$ per cent, blue line $r = 0$ per cent.

Alfarano, S., Lux, T., Wagner, F., 2008. Time variation of higher moments in a financial market with heterogeneous agents: An analytical approach. Journal of Economic Dynamics and Control 32, 101 – 136.

Aoki, M., 1996. New approaches to macroeconomic modeling. Cambridge University Press.

Aoki, M., 2002. Modeling aggregate behavior and fluctuations in economics : stochastic views of interacting agents. Cambridge Univ. Press.

Aoki, M., Yoshikawa, H., 2006. Reconstructing Macroeconomics. Cambridge University Press.

Assenza, T., Delli Gatti, D., 2013. E pluribus unum: Macroeconomic modelling for multi-agent economies. Journal of Economic Dynamics and Control 37, 1659–1682.

Brock, W.A., Durlauf, S.N., 2001. Discrete Choice with Social Interactions. Review of Economic Studies 68, 235–60.

Brock, W.A., Hommes, C.H., 1997. A rational route to randomness. Econometrica 65, 1059–1096.

Chiarella, C., Di Guilmi, C., 2011. The financial instability hypothesis: A stochastic microfoundation framework. Journal of Economic Dynamics and Control 35, 1151 – 1171.

Chiarella, C., Di Guilmi, C., 2015. The limit distribution of evolving strategies in financial markets. Studies in Nonlinear Dynamics & Econometrics 19, 137–159.

Clifford, P., 1990. Markov random fields in statistics, in: Grimmett, G.R., Welsh, D.J.A. (Eds.), Disorder in Physical Systems. A Volume in Honour of John M. Hammersley, Clarendon Press, Oxford. pp. 19–32.

Delli Gatti, D., Di Guilmi, C., Gaffeo, E., Giulioni, G., Gallegati, M., Palestrini, A., 2005. A new approach to business fluctuations: Heterogeneous interacting agents, scaling laws and financial fragility. Journal of Economic Behavior & Organization 56, 489–512.

Di Guilmi, C., 2008. The Generation of Business Fluctuations: Financial Fragility and Mean-field Interactions. Peter Lang Publishing.

Di Guilmi, C., Carvalho, L., 2017. The dynamics of leverage in a demand-driven model with heterogeneous firms. Journal of Economic Behavior & Organization 140, 70–90.

Di Guilmi, C., Gallegati, M., Landini, S., Stiglitz, J.E., 2012. Dynamic Aggregation of Heterogeneous Interacting Agents and Network: An Analytical Solution for Agent Based Models. Technical Report. mimeo. URL: http://www2.gsb.columbia.edu/faculty/jstiglitz/topics.cfm.

Dilaver, O., Jump, R., Levine, P., 2016. Agent-based Macroeconomics and Dynamic Stochastic General Equilibrium Models: Where do we go from here? Working Paper DP 01/16. University of Surrey, School of Economics.

Dosi, G., Fagiolo, G., Napoletano, M., Roventini, A., 2013. Income distribution, credit and fiscal policies in an agent-based Keynesian model. Journal of Economic Dynamics and Control 37, 1598–1625.

Fajgelbaum, P., Schaal, E., Taschereau-Dumouchel, M., 2014. Uncertainty Traps. NBER Working Papers 19973. National Bureau of Economic Research, Inc. URL: https://ideas.repec.org/p/nbr/nberwo/19973.html.

Foley, D.K., 1994. A statistical equilibrium theory of markets. Journal of Economic Theory 62, 321–345.

Gardiner, C.W., 2002. Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences. Springer, Berlin.

Gertler, M., Trigari, A., 2009. Unemployment Fluctuations with Staggered Nash Wage Bargaining. Journal of Political Economy 117, 38–86.

Gobbi, A., Grazzini, J., 2015. Learning and heterogeneity in DSGE models: an agent-based approach. Technical Report. mimeo.

Gualdi, S., Tarzia, M., Zamponi, F., Bouchaud, J.P., 2015. Tipping points in macroeconomic agent-based models. Journal of Economic Dynamics and Control 50, 29–61.

Izquierdo, L.R., Izquierdo, S.S., Galı, J.M., Santos, J.I., 2009. Techniques to understand computer simulations: Markov chain analysis. Journal of Artificial Societies & Social Simulation 12, 8.

Jaynes, E.T., 1957. Information theory and statistical mechanics. Phys. Rev. 106, 620–630. doi:10.1103/PhysRev.106.620.

Journal of Economic Behavior & Organization, 2008. Special issue: Agent-based models for economic policy design. volume 67,2. Editors: Dawid, H. and Fagiolo, G.

van Kampen, N.G., 1992. Stochastic processes in physics and chemistry. North-Holland, Amsterdam.

Krusell, P., Mukoyama, T., Rogerson, R., Sahin, A., 2012. Is Labor Supply Important for Business Cycles? Technical Report 17779. NBER.

Landini, S., Uberti, M., 2008. A statistical mechanics view of macro-dynamics in economics. Computational Economics 32, 121–146.

Lengnick, M., 2013. Agent-based macroeconomics: A baseline model. Journal of Economic Behavior and Organization 86, 102–120.

Liu, Z., Phaneuf, L., 2013. The Transmission of Productivity Shocks: What Do We Learn About DSGE Modeling? Annals of Economics and Statistics , 283–304.

Lux, T., 1995. Herd behaviour, bubbles and crashes. Economic Journal 105, 881–96.

Lux, T., 1998. The socio-economic dynamics of speculative markets: interacting agents, chaos, and the fat tails of return distributions. Journal of Economic Behavior & Organization 33, 143–165.

Lux, T., 2009. Rational forecasts or social opinion dynamics? identification of interaction effects in a business climate survey. Journal of Economic Behavior & Organization 72, 638 – 655.

Mortensen, D.T., Pissarides, C.A., 1994. Job Creation and Job Destruction in the Theory of Unemployment. Review of Economic Studies 61, 397–415.

Nuño, G., Moll, B., 2015. Controlling a distribution of heterogeneous agents. Working Papers 1533. Banco de España.

Russo, A., Catalano, M., Gaffeo, E., Gallegati, M., Napoletano, M., 2007. Industrial dynamics, fiscal policy and R&D: Evidence from a computational experiment. Journal of Economic Behavior & Organization 64, 426–447.

Smith, E., Foley, D.K., 2008. Classical thermodynamics and economic general equilibrium theory. Journal of Economic Dynamics and Control 32, 7–65.

Trigari, A., 2006. The Role of Search Frictions and Bargaining for Inflation Dynamics. Working Papers 304. IGIER (Innocenzo Gasparini Institute for Economic Research), Bocconi University.

Weidlich, W., 2000. Sociodynamics: A Systematic Approach to Mathematical Modelling in the Social Sciences. Gordon and Breach: London.

Weidlich, W., 2008. Dynamics of political opinion formation including catastrophe theory. Journal of Economic Behavior & Organization 67, 1 – 26.