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Tue Gørgens, Dean Hyslop

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*Highlights (for review)

Highlights:

- Discrete time binary data may be modeled using state occupancy or transitions.
- Dynamic binary response and multi-spell duration models are commonly used models.
- There is a one-to-one correspondence between the representations required for each.
- There is a one-to-one correspondence between the sets of conditional probabilities.
- First and second-order DBR models are nested in simple MSD models.

Equivalent representations of discrete-time two-state panel data models

Tue Gørgens* Dean Hyslop†

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Abstract: There are two common approaches to analyzing discrete-time two-state panel data: one focuses on modeling the determinants of state occupancy, the other on modeling the determinants of transition between states. This note shows that there are one-to-one correspondences between the two representations, between the two probability distributions in an unrestricted context, and between low-order Markov models of state occupancy and semi-Markov models of transition between states with strictly limited duration dependence.

Keywords: Panel data, binary response, dynamic models.

JEL classification codes: C33, C35, C41.

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*Address: Research School Economics, The Australian National University, Acton ACT 2601, Australia. E-mail: tue.gorgens@anu.edu.au.

†Address: Motu Economic and Public Policy Research, PO Box 24390, Wellington 6142, New Zealand. E-mail: dean.hyslop@motu.org.nz.

1 Introduction

There are two distinct approaches to analyzing discrete-time two-state panel data in the applied econometrics literature. One approach focuses on modeling the determinants of state occupancy using autoregressive dynamic binary response models and low-order Markov assumptions (e.g. Hyslop, 1999). The other approach focuses on modeling the determinants of transition between states using multi-spell duration models and semi-Markov assumptions (e.g. Stevens, 1999). Overwhelmingly, the applied literature prefers to model occupancy rather than transition. With a few exceptions (e.g. Barmby, 1998; Cappellari et al., 2007; Bhuller et al., 2016), there appears to be little awareness of the connection between the two.

This note presents three equivalence results. First, the representations of the outcomes in terms of state occupancy or transition between states are equivalent in the sense that there is a one-to-one correspondence between them. Second, the unrestricted probability distributions for the occupancy and transition outcomes are also equivalent. Third, models of state occupancy with a first- or second-order Markov assumption are equivalent to models of transition between states with a semi-Markov assumption and no duration dependence after one or two periods.

Equivalence implies that, in principle, a data analyst may choose to model occupancy probabilities and will be able to infer the corresponding transition probabilities, and vice versa. Equivalence also implies that any differences between the approaches arise from auxiliary assumptions such as parametric specifications. Importantly, equivalence means that auxiliary assumptions about probabilities of state occupancy have implications for transition probabilities, and vice versa. As we show in related research (see Gørgens and Hyslop, 2016), the restrictions embodied in typical models of state occupancy can be unacceptably unrealistic.

2 Three equivalence results

Suppose time is divided into periods of equal length, an entity occupies one of two states during each period, and transitions between states occur between periods. Entities are indexed by i and times are indexed by t. Let Y_{it} be the indicator of the state occupied by entity i at time t, and let C_{it} be the indicator of whether or not the entity makes a transition between times $t-1$ and t, with $Y_{it} \in \{0,1\}$ and $C_{it} \in \{0,1\}$ for $t = 1, \dots, T$.

To keep the note concise, we make several simplifications. First, we assume the process begins at time 1 for all entities and lasts at least until period T. Second, we focus on the outcomes experienced by the entities, whether observed or not. In practice, it is necessary to distinguish between the underlying process and the observational scheme, and issues of sampling and incomplete data are important. Third, we abstract from covariates. Our results hold more generally, but the simple setup is sufficient for our purpose in this note. For example, the results continue to hold if all probabilities concerning Y_{it} and C_{it} are conditional on a covariate vector, X_{it}.

The following theorem shows that there is a one-to-one correspondence between the two outcome representations.

Theorem 1. *The outcome representations* $\{Y_{i1}, Y_{i2}, \dots, Y_{iT}\}$ *and* $\{Y_{i1}, C_{i2}, \dots, C_{iT}\}$ *are equivalent.*

Proof. The conclusion follows from

$$C_{it} = 1(Y_{it-1} \neq Y_{it}), \quad t = 2, \dots, T, \tag{1}$$

and

$$Y_{it} = \left(Y_{i1} + \sum_{k=2}^{t} C_{ik} \right) \bmod 2, \quad t = 2, \dots, T, \tag{2}$$

where "x mod y" yields the remainder after dividing x by y. □

The aim of most empirical studies is to understand the determinants of either probabilities of state occupancy or probabilities of transition between states, and it is usually important to characterize the influence of the past. Let H_{it} denote an entity's history at time t. From the theorem above, we know the history has several equivalent representations. For concreteness, let $H_{it} = \{Y_{i1}, \ldots, Y_{it}\}$. Let $\mathcal{H}_t = \{0,1\}^t$ denote the support of H_{it}, and let h_t denote a generic element of $\mathcal{H}_t$.

In analyses that focus on state occupancy, the interest is in the conditional probability distribution of Y_{it}, the state occupied by an entity in period t, given the history prior to that time. That is,

$$\begin{aligned} &\chi = \mathsf{P}(Y_{i1} = 1), \\ &\zeta(h_{t-1}) = \mathsf{P}(Y_{it} = 1 | H_{it-1} = h_{t-1}), \quad h_{t-1} \in \mathcal{H}_{t-1}, \quad t = 2, \ldots, T. \end{aligned} \tag{3}$$

There are 2^{t-1} elements in $\mathcal{H}_{t-1}$, so the total number of probabilities in (3) is $2^T - 1$.

In analyses that focus on transition between states, the interest is in the conditional probability distribution of C_{it}, the change in state by an entity between periods $t-1$ and t, given the history prior to that time. Formally,

$$\begin{aligned} &\chi = \mathsf{P}(Y_{i1} = 1), \\ &\xi(h_{t-1}) = \mathsf{P}(C_{it} = 1 | H_{it-1} = h_{t-1}), \quad h_{t-1} \in \mathcal{H}_{t-1}, \quad t = 2, \ldots, T. \end{aligned} \tag{4}$$

Obviously, there are also $2^T - 1$ distinct probabilities in (4).

The following theorem shows that there is a one-to-one correspondence between the set of conditional probabilities of state occupancy and the set of conditional probabilities of transition between states.

Theorem 2. *Given $t \geq 2$ and $h_{t-1} \in \mathcal{H}_{t-1}$, let y' denote the state occupied in period $t-1$ according to h_{t-1}. The probability distributions in (3) and (4) are equivalent in the sense that $\zeta(h_{t-1}) = 1 - \xi(h_{t-1})^{y'}(1-\xi(h_{t-1}))^{1-y'}$ and $\xi(h_{t-1}) = 1 - \zeta(h_{t-1})^{y'}(1-\zeta(h_{t-1}))^{1-y'}$.*

Proof. The conditioning event is the same in each set of probability distributions given in (3) and (4). The conclusion therefore follows from the facts that $Y_{it} = 1$ happens if and only if either $(Y_{it-1} = 1, C_{it} = 0)$ or $(Y_{it-1} = 0, C_{it} = 1)$, and that $C_{it} = 1$ happens if and only if either $(Y_{it-1} = 1, Y_{it} = 0)$ or $(Y_{it-1} = 0, Y_{it} = 1)$. $\square$

Data analysts who focus on state occupancy typically impose a Markov assumption of order r, where r is quite low, either one or two. If the conditional probabilities depend on history only through the r most recent previous periods, then

$$\mathsf{P}(Y_{it} = 1 | H_{it-1} = h_{t-1}) = \mathsf{P}(Y_{it} = 1 | Y_{it-r} = y'_{-r}, \ldots, Y_{it-1} = y'_{-1}),$$
$$h_{t-1} \in \mathcal{H}_{t-1}, \quad t = r+1, \ldots, T, \quad (5)$$

where $y'_{-r}, \ldots, y'_{-1}$ are the most recent r elements of h_{t-1}. For simplicity, we suppress the dependence and do not write e.g. $y'_{-r}(h_{t-1})$.

The set of distinct probabilities of interest under the Markov assumption can be represented by those at the beginning of the process (i.e. the first $r+1$ periods), namely

$$\begin{aligned} &\chi = \mathsf{P}(Y_{i1} = 1), \\ &\zeta(h_{t-1}) = \mathsf{P}(Y_{it} = 1 | H_{it-1} = h_{t-1}), \quad h_{t-1} \in \mathcal{H}_{t-1}, \quad t = 2, \ldots, r, \\ &\zeta^M(y_{-r}, \ldots, y_{-1}) = \mathsf{P}(Y_{ir+1} = 1 | Y_{i1} = y_{-r}, \ldots, Y_{ir} = y_{-1}), \\ &\qquad\qquad\qquad\qquad\qquad\qquad (y_{-r}, \ldots, y_{-1}) \in \{0,1\}^r, \end{aligned} \quad (6)$$

where $\zeta(h_r)$ in (3) is recast as $\zeta^M(y'_{-r}, \ldots, y'_{-1})$ for $h_r \in \mathcal{H}_r$. The total number of probabilities in (6) is $2^{r+1} - 1$, which is usually small compared to the $2^T - 1$ probabilities in (3).

Data analysts who focus on transition between states are usually interested in how the transition probabilities depend on elapsed time in the ongoing spell, and therefore often work with a semi-Markov assumption. Let D_{it} denote the elapsed time in the ongoing spell by the end of period t. If the history prior to entering the ongoing spell does not influence the conditional probabilities, then

$$\mathsf{P}(C_{it} = 1 | H_{it-1} = h_{t-1}) = \mathsf{P}(C_{it} = 1 | Y_{it-1} = y', D_{it-1} = d')$$
$$h_{t-1} \in \mathcal{H}_{t-1}, \quad t = 2, \ldots, T, \quad (7)$$

where y' and d' are constructed from h_{t-1} by letting y' be the most recent element and letting d' be the elapsed time in the most recent spell.

The set of distinct probabilities of interest under the semi-Markov assumption can also be represented by those at the beginning of the process (i.e. the first spells in the two states); that is,

$$\begin{aligned} &\chi = \mathsf{P}(Y_{i1} = 1), \\ &\xi^S(y,d) = \mathsf{P}(C_{id+1} = 1 | Y_{id} = y, D_{id} = d), \quad (y,d) \in \{0,1\} \times \{1,2,\ldots,T-1\}, \end{aligned} \tag{8}$$

where $\xi(h_{t-1})$ in (4) is replaced by $\xi^S(y',d')$ for $h_{t-1} \in \mathcal{H}_{t-1}$. There are $1+T(T-1)$ total probabilities in (8). This is usually much smaller than the $2^T - 1$ probabilities in (4).

The following theorem shows that the widely used first- and second-order autoregressive dynamic binary response models are equivalent to particularly simple multi-spell duration models, where the transition probabilities are constant after one or two periods. Given $r \leq 2$, define the indicator $G_{it} = 1(D_{it-1} \geq r)$, then restrictive semi-Markov assumption is

$$\begin{aligned} \mathsf{P}(C_{it} = 1 | H_{it-1} = h_{t-1}) = \mathsf{P}(C_{it} = 1 | Y_{it-1} = y', G_{it-1} = g') \\ h_{t-1} \in \mathcal{H}_{t-1}, \quad t = 2,\ldots,T, \end{aligned} \tag{9}$$

where g' is also constructed from h_{t-1} by letting $g' = 1$ if the elapsed time in the most recent spell is at least r (i.e. $d' \geq r$) and $g' = 0$ otherwise.

Theorem 3. *Suppose $r \leq 2$. Then the Markov property, (5), holds if and only if the restrictive semi-Markov property, (9), holds.*

Proof. First, by previous results Y_{it} and C_{it} can be inferred from each other given $H_{it-1} = h_{t-1}$. Therefore, (5) implies

$$\begin{aligned} \mathsf{P}(C_{it} = 1 | H_{it-1} = h_{t-1}) = \mathsf{P}(C_{it} = 1 | Y_{it-r} = y'_{-r}, \ldots, Y_{it-1} = y'_{-1}), \\ h_{t-1} \in \mathcal{H}_{t-1}, \quad t = r+1,\ldots,T. \end{aligned} \tag{10}$$

We begin with the "only if" part of the theorem. If $r = 1$, then (10) implies that the history affects the transition probability only through Y_{it-1}, and it follows that (9) holds and that G_{it-1} does not matter. If $r = 2$, note that Y_{it-2} can be inferred from (Y_{it-1}, G_{it-1}). If $G_{it-1} = 0$, then $D_{it-1} = 1$ which means there was a transition at time $t-1$, so $Y_{it-2} = 1 - Y_{it-1}$. If $G_{it-1} = 1$, then $D_{it-1} \geq 2$ which means there was no transition at time $t-1$, so $Y_{it-2} = Y_{it-1}$. Consequently, conditioning on (Y_{it-1}, G_{it-1}) is the same as conditioning on $(Y_{it-2}, Y_{it-1}, G_{it-1})$, so (9) is equivalent to

$$\mathsf{P}(C_{it} = 1 | H_{it-1} = h_{t-1}) = \mathsf{P}(C_{it} = 1 | Y_{it-2} = y'_{-2}, Y_{it-1} = y'_{-1}, G_{it-1} = g'),$$
$$h_{t-1} \in \mathcal{H}_{t-1}, \quad t = 2, \ldots, T. \quad (11)$$

Therefore, if $r = 2$, then (10) implies that the history affects the transition probability only through (Y_{it-2}, Y_{it-1}), and it follows that (11), and hence (9), hold and that G_{it-1} does not matter.

We next show the "if" part. If $r = 1$, then $G_{it-1} = 1$ always, so conditioning on $G_{it-1} = g'$ is redundant, and hence (11) implies (10). If $r = 2$, note that $Y_{it-2} \neq Y_{it-1}$ implies $D_{it-1} = 1$ and $G_{it-1} = 0$, and $Y_{it-2} = Y_{it-1}$ implies $D_{it-1} \geq 2$ and $G_{it-1} = 1$. Consequently, conditioning on $(Y_{it-2}, Y_{it-1}, G_{it-1})$ is the same as conditioning on (Y_{it-2}, Y_{it-1}), so (11) implies (10). □

3 Concluding remarks

In related work (see Gørgens and Hyslop, 2016), we compare parametric specifications of these models, including both observed and unobserved heterogeneity. In that context, we also find that prototypical Markov models are special cases of prototypical semi-Markov models. In an empirical case study, we find that the Markov assumption is strongly rejected.

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