CONSTRUCTIONS FOR FITTING FORMATIONS

by

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The work presented in this thesis is my own except where otherwise stated.

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ABSTRACT

Fitting formations, namely Fitting classes that are also formations, made their first appearance in the work of Trevor Hawkes in connection with skeletal classes of finite soluble groups and primitive saturated formations.

In 1970 Hawkes showed that each metanilpotent Fitting formation is saturated and can be characterised by a local definition consisting of formations of nilpotent groups. However the situation in the general case seems considerably more complex. We are motivated by the search for a classification result.

Since Bryce and Cossey have shown that saturated Fitting formations can be defined locally by Fitting formations it follows that the non-saturated situations are of particular interest.

All groups in this thesis are finite and soluble.

We supply a general method for constructing Fitting formations by examination of properties of chief factors. Let k be a field of characteristic q. For each group G suppose $M(G)$ is a given class of irreducible kG-modules such that the family $\{M(G)\}$ satisfies certain closure conditions. Then, loosely speaking, the class of groups G where q-chief factors are contained in $M(G)$ is a Fitting formation.

Extensive use is made of this method.

In our first example we generalise the already known non-saturated classes of Hawkes and Berger-Cossey. The class we define is called $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ where π is an arbitrary set of primes and $\underline{\underline{X}}$ any Fitting formation subject to a condition dependent on π and q.

When $\pi = \{q\}$, $\underline{\underline{X}} = \underline{\underline{S}}$ we reduce to Hawkes' case; when $\pi = \{p\}$, $\underline{\underline{X}} = \underline{\underline{S}}_{p'}\underline{\underline{S}}_p$ we have the Berger-Cossey example.

Further we give criteria under which members of this family are non-saturated. Our main theorem will be that $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ is non-saturated if there exists primes r,s such that $r \in \pi'$, $r \neq q$, $s \in \pi$ with $\underline{\underline{S}}_r\underline{\underline{S}}_s \subseteq \underline{\underline{X}}$. We have not been able to find exact conditions for non-saturation.

In our second main example we define a new family of Fitting formations $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$, with π and q as above and $\underline{\underline{X}} \subseteq \underline{\underline{S}}_{\pi}\underline{\underline{S}}_{\pi'}$.

For each group G we let $H_q^{\pi}(G)$ be the class of irreducible kG-modules V which on restriction to the π-radical of G are homogeneous and has $G/\ker V \in \underline{\underline{X}}$. The Fitting formation is then defined by the general method described above. Non-saturation occurs in similar situations as for $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$.

Finally we investigate the possibility that all Fitting formations can be constructed by our methods. We define a closure operation P_{π} on the class of Fitting formations, $\mathcal{F}$, where π is an arbitrary set of primes. We put $P_{\mathbb{P}} = P$. Essentially P_{π} is defined locally at each prime $q \in \pi$. For $\underline{\underline{X}} \in \mathcal{F}$, $P_q(\underline{\underline{X}})$ is the class of groups G such that each q-chief factor of G extended by its automiser is contained in $\underline{\underline{X}}$.

Since $\underline{\underline{S}}_{\pi'} \subseteq P_{\pi}(\underline{\underline{X}})$ it is clear that $\underline{\underline{X}}$ cannot be P_{π}-closed. However, we conjecture $\underline{\underline{X}}$ is P-closed. This proves to be a difficult problem to decide in general although we shall show all saturated cases are P-closed and all known non-saturated instances are also.

CONTENTS

CHAPTER 1

INTRODUCTION

Let G be a finite soluble group and m be a co-prime divisor of $|G|$. Then an old theorem of Philip Hall [17] states that there exists a subgroup of G of order m, and that all such subgroups, the Hall subgroups, form a unique characteristic conjugacy class in G. In the soluble case this provides a generalisation of the first Sylow theorem.

In 1961 R.W. Carter [7] showed that for each finite soluble group there exists a self-normalising nilpotent subgroup, the Carter subgroup, and that Carter subgroups form a unique characteristic conjugacy class of subgroups in the full group.

These two facts prompted interest in developing a uniform account of when and how characteristic conjugacy classes of subgroups arise.

In 1963 Wolfgang Gaschütz [13] presented the first such description. He introduced the seminal concept of a *saturated formation*, a class of finite soluble groups (as are all groups in this thesis) closed under taking quotients and subdirect products and saturated. Roughly speaking, he showed that one class of conjugacy subgroups is obtained for each saturated formation $\underline{\underline{X}}$; the subgroups in the class being called $\underline{\underline{X}}$-covering subgroups (to use the notation of Carter and Hawkes [8]). For example $\underline{\underline{S}}_\pi$-covering subgroups are exactly the Hall π-subgroups (for π a set of primes $\underline{\underline{S}}_\pi$ is the class of π-groups) and the $\underline{\underline{N}}$-covering subgroups are just the Carter subgroups (for $\underline{\underline{N}}$ the class of nilpotent groups).

Following Gaschütz, U. Lubeseder [28] showed that every saturated formation arises via a local definition from a family of formations (Chapter 2 §2.4). This showed that the theory of formations lay at the heart of Gaschütz's methods. Much of early formation theory was worked out by Gaschütz [13], Carter and Hawkes [8] and B. Huppert [23].

In 1967 Herman Schunck [32] was able to significantly generalise Gaschütz's work. He replaced Gaschütz's saturated formations with a more general class - the Schunck class - and showed that for each group G there exist a set of subgroups, the $\underline{\underline{X}}$-projectors of G, which are conjugate in G.

Saturated formations are Schunck classes although the converse is false. In the cases when $\underline{\underline{X}} = \underline{\underline{S}}_\pi$ or $\underline{\underline{N}}$ say, the $\underline{\underline{X}}$-projectors of a group coincide with the $\underline{\underline{X}}$-covering subgroups.

Meanwhile in 1966 B. Fischer [11] dualized the concept of a formation to define the *Fitting class* - a class closed under subnormal subgroups and products of normal subgroups. Fischer showed that if a Fitting class $\underline{\underline{X}}$ enjoys a certain weak closure property (similar to subgroup closure) then each group G possesses a unique conjugacy class of maximal $\underline{\underline{X}}$-subgroups containing the $\underline{\underline{X}}$-radical - the Fischer $\underline{\underline{X}}$-subgroups, as Hartley [19] names them.

However in 1967 B. Fischer, W. Gaschütz and B. Hartley [12] obtained a simpler dualization of formation theory. They showed that there exists a unique characteristic conjugacy class of subgroups, called $\underline{\underline{X}}$-injectors, for each Fitting class $\underline{\underline{X}}$, without additional closure properties on $\underline{\underline{X}}$ being necessary. For example the $\underline{\underline{S}}_\pi$-injectors of a group G coincide with the $\underline{\underline{S}}_\pi$-projectors and are the Hall π-subgroups of G. Indeed, given a Fitting class which is also a Schunck class, $\underline{\underline{X}}$ say, such that $\underline{\underline{X}}$-injectors and $\underline{\underline{X}}$-projectors coincide for all soluble groups, then $\underline{\underline{X}} = \underline{\underline{S}}_\pi$ for some set of primes π. The $\underline{\underline{N}}$-injectors of a group G are the Fischer $\underline{\underline{N}}$-subgroups.

By the late 1960s attention had shifted to a more formal study of the specific classes that give rise to conjugacy classes of subgroups.

One line of research has been to explore the interrelationship of closure properties on a class. For instance it was recently proved by R.A. Bryce and John Cossey [4] that subgroup closed Fitting classes are formations. It seems to have been a contribution of Hartley in [19] and [12] to express the standard closure properties as closure operations on a class in the terminology introduced by P. Hall [18] in 1963.

The first important result of this new kind had in fact appeared quite early. This was the Lubeseder result [28] on saturated formations as mentioned above. It provides a characterisation of saturated formations in terms of formations. Another early result was by Peter Neumann [30] who showed that a formation of nilpotent groups is subgroup closed.

However it was work of Trevor Hawkes that gave direct encouragement to these new sorts of questions. In [21] he pointed out that primitive saturated formations are subgroup closed Fitting classes and suggested that the converse might also be true. This question first attracted attention to the study of Fitting classes that are also formations. Hawkes called such classes, the subject of this thesis, *Fitting formations*.

In 1970 Hawkes [20] presented a classification of metanilpotent Fitting formations showing that they are defined locally by nilpotent groups. It follows that such classes are both saturated and subgroup closed. However these results do not carry over to the case of nilpotent length 3 ; Hawkes is able to construct an example which is neither saturated nor subgroup closed. This seems to indicate, as he notes, that the classification problem may be a difficult one in the general case. We describe this important example in Chapter 2, §3.7.

Inspired by Hawkes' results Bryce and Cossey [3] showed that a subgroup closed Fitting formation is saturated; with the converse failing in general, however holding in the case of nilpotent length 3 . The methods of [3] also enabled the authors to give an affirmative answer to Hawkes' original question in [21]: Every subgroup closed Fitting formation is a primitive saturated formation. This means that subgroup closed Fitting formations are completely classifiable by local definitions.

This thesis is motivated by the problem of extending this classification to the general case.

In [3] Bryce and Cossey showed that all saturated Fitting formations can be defined locally by Fitting formations, so it seems likely that a treatment of the non-saturated case will be a major step in the general classification.

However, work on the non-saturated cases has been impeded by a lack of examples. Indeed, apart from Hawkes' example mentioned above, no new examples appeared until 1978 when Tom Berger and John Cossey [1] were able to construct an example using a variation of Hawkes' methods. We describe this example in Chapter 2 §3.8.

Accordingly this thesis is a consideration of constructions for Fitting formations. We place particular emphasis on the non-saturated cases.

All examples under our examination have a common feature : they are defined by collecting those groups whose chief factors are subject to certain specified conditions. The thesis ends by considering whether all Fitting formations are of this type.

The thesis is divided into six chapters. Chapter 1 is the introduction. Chapter 2 settles preliminary questions of notation and convention. We briefly summarise the elementary properties of a Fitting formation and collect and arrange other results, particularly from representation theory, which we will need.

In Chapter 3 we formalise the method of constructing a Fitting formation used implicitly by Hawkes and Berger-Cossey in the construction of their examples which we shall call $\underline{\underline{Y}}_q$ and $\underline{\underline{Y}}_q^{\,p}$ respectively (where p,q are primes). An axiomatic method emerges. Loosely speaking, choose a set of irreducible modules $M(G)$ for each group G for which certain specified closure conditions hold. Let q be the characteristic of the underlying field. Then a Fitting formation is defined by collecting those groups which, loosely speaking, have all q-chief factors in $M(G)$.

We make extensive use of this method.

In Chapter 4 we show that Hawkes' class $\underline{\underline{Y}}_q$ and Berger-Cossey's class $\underline{\underline{Y}}_q^{\,p}$ are members of one family of Fitting formations we call $\underline{\underline{Y}}_q^{\,\pi}(\underline{\underline{X}})$, for π a set of primes and $\underline{\underline{X}}$ a Fitting formation subject to certain requirements.

In particular we have $\underline{\underline{Y}}_q$ when $\pi = \{q\}$, $\underline{\underline{X}} = \underline{\underline{S}}$ and $\underline{\underline{Y}}_q^{\,p}$ when $\pi = \{p\}$, $\underline{\underline{X}} = \underline{\underline{S}}_{p'}\underline{\underline{S}}_p$.

$\underline{\underline{Y}}_q^{\,\pi}(\underline{\underline{X}})$ is constructed using the method of Chapter 3. Accordingly, for each G we select a set of irreducible modules called $Y_q^{\,\pi}(G)$ subject to a set of conditions, generalisations of those which appear in the Berger-Cossey construction.

Next we give sufficient conditions on the choice of $\underline{\underline{X}}$ subject to π under which $\underline{\underline{Y}}_q^{\,\pi}(\underline{\underline{X}})$ is non-saturated. We are not able to give necessary and sufficient conditions. Finally we consider when two such classes may be equal and remark on further elementary properties.

In Chapter 5 we construct a new example of a non-saturated Fitting formation, called $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$ where once again π is a set of primes and $\underline{\underline{X}}$ a Fitting formation subject to specific conditions. The construction is by the method of Chapter 3. Roughly speaking, for each group we select those modules which are homogeneous under restriction to the appropriate π-subgroup.

The conditions we provide under which $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$ is non-saturated are similar to the case for $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$.

All Fitting formations dealt with to this point have been constructed by the method of Chapter 3, via consideration of chief factors. In Chapter 6 we try to answer a very important question : Can all Fitting formations be so constructed?

Although we cannot provide a definite answer to this question all suggestions seem to indicate the answer might be affirmative.

We define a new closure operation P on the class of Fitting formations, $\mathcal{F}$. If $\underline{\underline{X}}$ is any Fitting formation $P(\underline{\underline{X}})$ is constructed by the method of Chapter 3. Basically $P(\underline{\underline{X}})$ is a class with the property that certain well chosen primitive $P(\underline{\underline{X}})$-groups are in $\underline{\underline{X}}$. The primitive groups under consideration occur naturally as suitable extensions of chief factors.

The important question is whether each Fitting formation is P-closed or not. If not then P may prove useful in providing new examples of non-saturated Fitting formations. If so then we will have an answer to a basic question : Are Fitting formations characterised by properties of their chief factors?

All saturated Fitting formations are P-closed, as are all known non-saturated cases, $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ and $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$. The general case however seems a difficult problem.

The chapter closes with a partially successful consideration of how P behaves under products of classes.

CHAPTER 2

PRELIMINARIES

§1. Introduction

Most of the notation and terminology of this thesis can be found in Wolfgang Gaschütz's book [14]. The aim of this Chapter is to settle remaining details of notation and convention. Foremost amongst these is the following reaffirmation : All groups in this thesis, unless explicitly stated otherwise, are both finite and soluble.

We assume familiarity with the following texts :

- For Fitting class and formation theory Gaschütz [14] Chapters 0, I, VI, VII, IX and X.
- For general group theory Gorenstein [15] Chapters 1, 2, 5.
- For representation theory Curtis and Reiner [10] Chapters I, II, VII and Huppert [24] Chapter V.

Most of the preliminary results quoted in this chapter fall outside these references; overlapping occurs for the sake of emphasis or re-expression in a more suitable form.

In general those proofs which are either elementary or for which an adequate reference exists have been omitted.

The material we present is divided into three sections.

Since no unified description of the basic properties of a Fitting formation already exists in the literature, §2 is devoted to that aim. Our account ends with Hawkes' characterisation of all metanilpotent examples. We will be careful to say for each result which of the closure operations are involved.

In §3 our attention is focussed on modules with a group acting on them. Indeed, such modules will be our main way of studying Fitting formations. The fact which enables this is that a chief factor of a finite soluble group may be viewed as an irreducible module for the group itself over a field of prime order. We record several elementary consequences and establish certain conventions and notations. The section culminates with a description of the two classes that have stimulated this thesis : Hawkes' Fitting formation and the Berger-Cossey class.

The final section, §4 , is a brief review of the representation theory we shall need. First we state the collection of results henceforth referred to as Clifford's theorem. We then go on to examine a situation not fully explored by this theorem : roughly when an irreducible module restricted to a normal subgroup is rendered homogeneous. The section concludes by providing conditions under which an irreducible module for a normal subgroup can be expressed as the restriction of an irreducible module for the full group. The application of these conditions will turn out to be crucial in Chapter 5.

§2. Fitting formations

Recall that a Fitting formation is a class of groups which is both a Fitting class and a formation.

Let $\mathcal{F}$ be the class of all Fitting formations.

According to standard definitions a Fitting class is a class subnormal subgroup closed (S_n-closed) and normal product closed (N_0-closed); and a formation is a class quotient group closed (Q-closed) and subdirect product closed (R_0-closed).

Since there are a number of equivalent ways of expressing closure under these operations it is appropriate to state which version we shall use.

Accordingly let $\underline{\underline{X}}$ be a class. Then $\underline{\underline{X}}$ is

- Q-closed whenever $G \in \underline{\underline{X}}$ implies $G/N \in \underline{\underline{X}}$ for each N normal in G.
- R_0-closed whenever $G/N_i \in \underline{\underline{X}}$, $i = 1,2$ implies $G/(N_1 \cap N_2) \in \underline{\underline{X}}$ for each N_i normal in G .
- S_n-closed whenever $G \in \underline{\underline{X}}$ implies all maximal normal subgroups of G are in $\underline{\underline{X}}$.
- N_0-closed whenever $G = N_1N_2$ and N_i are maximal normal subgroups of G in $\underline{\underline{X}}$ implies $G \in \underline{\underline{X}}$.

In addition $\underline{\underline{X}}$ is saturated (E_Φ-closed) whenever $G/\Phi(G) \in \underline{\underline{X}}$ implies $G \in \underline{\underline{X}}$ for each G .

$\mathbb{P}$ is the set of all primes and π an arbitrary set of primes.

Some elementary examples of Fitting formations are :

$\underline{\underline{1}}$ the class consisting of only $\langle 1 \rangle$

$\underline{\underline{S}}$ the class of all finite soluble groups

$\underline{\underline{S}}_\pi$ the class of all finite soluble π-groups

$\underline{\underline{N}}$ the class of all nilpotent groups

$\underline{\underline{N}}_\pi$ the class of all nilpotent π-groups.

Amongst the non-elementary examples there are two of particular interest to us. These are Hawkes' class, $\underline{\underline{Y}}_q$, defined in [20] and the Berger-Cossey class $\underline{\underline{Y}}_q^p$ defined in [1] , where p,q are primes. The notation we use for Hawkes' class brings it into line with Berger-Cossey, and foreshadows the chief result of Chapter 4 which shows that these examples belong to a single family of Fitting formations. A description of $\underline{\underline{Y}}_q$ and $\underline{\underline{Y}}_q^p$ is best reserved till §3 .

Let $\underline{\underline{X}}$ be a formation and G a group. The $\underline{\underline{X}}$-*residual* of G , written $G^{\underline{\underline{X}}}$, is the intersection of all normal subgroups of G whose factor group belongs to $\underline{\underline{X}}$. Since $\underline{\underline{X}}$ is R_0-closed $G^{\underline{\underline{X}}}$ is the uniquely determined smallest normal subgroup of G whose factor group is in $\underline{\underline{X}}$.

If φ is an epimorphism of G then

$$\left(G^{\underline{\underline{X}}}\right)^{\varphi} = \left(G^{\varphi}\right)^{\underline{\underline{X}}}$$

(Gaschütz [14] V.13).

Now let $\underline{\underline{X}}$ be a Fitting class. The $\underline{\underline{X}}$-*radical* of G , written $G_{\underline{\underline{X}}}$, is the product of all normal subgroups of G which belong to $\underline{\underline{X}}$. Since $\underline{\underline{X}}$ is N_0-closed $G_{\underline{\underline{X}}}$ is the uniquely determined largest normal subgroup of G in $\underline{\underline{X}}$.

If H is subnormal in G then

$$H_{\underline{\underline{X}}} = H \cap G_{\underline{\underline{X}}}$$

(Gaschütz [14] X. 3d)

If $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are both Fitting formations then the formation class product of $\underline{\underline{X}}$ and $\underline{\underline{Y}}$, namely

$$\{G \mid G^{\underline{\underline{Y}}} \in \underline{\underline{X}}\}$$

determines the same class as the Fitting class product

$$\{G \mid G/G_{\underline{\underline{X}}} \in \underline{\underline{Y}}\} .$$

Thus the expression $\underline{\underline{XY}}$ is unambiguous and is just the usual class product of $\underline{\underline{X}}$ and $\underline{\underline{Y}}$. For a proof of this see Gaschütz [14] X.6 . It is an elementary result that $\underline{\underline{XY}}$ is again a Fitting formation. To see this combine Gaschütz [14] VII.6 and X.7.

Furthermore it is easy to see that both $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are contained in $\underline{\underline{XY}}$.

Now suppose we wish to show $\underline{\underline{X}} \subseteq \underline{\underline{Y}}$ where $\underline{\underline{X}},\underline{\underline{Y}}$ are Fitting formations. Typically we take a counter example of minimal order, G say, and refine the structure of G leading to a contradiction. The first step of this refinement can be completed in general.

2.1 <u>LEMMA</u> *Let* $\underline{\underline{X}}$ *and* $\underline{\underline{Y}}$ *be Fitting formations and* G *a group of minimal order in* $\underline{\underline{X}}$ *but not in* $\underline{\underline{Y}}$. *Then* G *has a unique maximal normal subgroup and a unique minimal normal subgroup.*

The proof of this result is simple combining the minimality of G with the R_0,N_0-closure of $\underline{\underline{Y}}$.

By the *characteristic* of a group G , written $\gamma(G)$, we shall mean the set of primes which divide the order of G .

Likewise by the characteristic of a class of groups $\underline{\underline{X}}$, written $\gamma(\underline{\underline{X}})$, we shall mean the set of primes which divide the order of some group in $\underline{\underline{X}}$ i.e. $\gamma(\underline{\underline{X}}) = \bigcup_{G \in \underline{\underline{X}}} \gamma(G)$.

The following is a well known result.

2.2 LEMMA *Let* $\underline{\underline{X}}$ *be a Fitting class. Then*

$$\underline{\underline{X}} \cap \underline{\underline{N}} = \underline{\underline{N}}_{\gamma(\underline{\underline{X}})}$$

The proof of this is an immediate consequence of Gaschütz [14] X.4c. Loosely speaking Lemma 2.2 shows that a Fitting class is amply supplied with nilpotent groups.

The following is a similar kind of result.

Let N be a normal subgroup of G . Then G acts on N by conjugation. In particular if $M \trianglelefteq G$ and $M \subseteq C_G(N)$ then G/M acts on N . Now under this action form the semidirect product N.G/M . The following lemma shows that a formation is rich enough to contain all extensions of this kind when $G \in \underline{\underline{X}}$.

2.3 LEMMA *Let* $\underline{\underline{X}}$ *be a formation,* G *a group in* $\underline{\underline{X}}$ *,* N *a normal subgroup of* G *and* M *be a normal subgroup of* G *contained in* $C_G(N)$ *. Then*

$$N.G/M \in \underline{\underline{X}}$$

This is proved by Bryant, Bryce, Hartley [2] Lemma 1.8.

Let $\underline{\underline{X}}$ be a formation. Suppose there exists a formation $\underline{\underline{X}}(p)$ for each $p \in \gamma(\underline{\underline{X}})$ such that

$$\underline{\underline{X}} = \Big(\bigcap_{p \in \gamma(\underline{\underline{X}})} \underline{\underline{S}}_{p'} \underline{\underline{S}}_{p} \underline{\underline{X}}(p) \Big) \cap \underline{\underline{S}}_{\gamma(X)}$$

Then we say $\underline{\underline{X}}$ is *locally defined* by $\underline{\underline{X}}(p)$.

The following is a well known result of Gaschütz and Lubeseder.

2.4 THEOREM *A formation has a local definition if and only if it is a saturated formation.* (Huppert [24] VI Hauptsatz 7.5 and Satz 7.25).

Next we quote Hawkes' characterisation of metanilpotent Fitting formations, Theorem 1 of [20].

2.5 THEOREM *Let* $\underline{\underline{X}}$ *be a Fitting formation contained in* $\underline{\underline{N}}^2$. *For* $p \in \gamma(\underline{\underline{X}})$ *take*

$$\tau(p) = \{r \in \gamma(\underline{\underline{X}}) \mid r \in \gamma(G/O_{p'p}(G)) \text{ for some } G \in \underline{\underline{X}}\}$$

and put

$$\underline{\underline{X}}(p) = \begin{cases} \underline{\underline{1}} & \text{if } \tau(p) = \phi \\ \underline{\underline{N}}_{\tau(p)} & \text{otherwise} \end{cases} .$$

Then $\underline{\underline{X}}$ *is defined locally by* $\underline{\underline{X}}(p)$, *that is*

$$\underline{\underline{X}} = \{G \in \underline{\underline{S}}_{\gamma(\underline{\underline{X}})} \mid G/O_{p'p}(G) \in \underline{\underline{X}}(p) \text{ for each } p \in \gamma(\underline{\underline{X}})\} .$$

If G and H are groups let $G \text{ wr } H$ denote the standard wreath product of G and H, and let G^H be the base group, $\prod_{h \in H} G_h$ (cf Neumann [29] §2). The next result interrelates the $\underline{\underline{X}}$=radical of $G \text{ wr } H$ with that of G. $\underline{\underline{X}}$ is assumed to be a Fitting formation.

2.6 LEMMA *In terms of the above when* $G \notin \underline{\underline{X}}$ *then*

$$(G \text{ wr } H)_{\underline{\underline{X}}} = (G_{\underline{\underline{X}}})^H$$

Originally Cossey expressed this result for $\underline{\underline{X}}$ a Lockett class (Lemma 2.2 [9]) however as is well known (Lockett [27] Theorem 2.2(d)) a Q-closed Fitting class, in particular a Fitting formation is a Lockett class.

To close this section we quote a purely group theoretic result concerning properties of the Fitting and Frattini subgroups, denoted $F(G)$ and $\Phi(G)$ respectively. These properties are culled from Huppert.

2.7 LEMMA *For a group* G *if* $\Phi(G) = 1$ *then* $F(G)$ *is complemented in* G *. Moreover* $F(G)$ *is the direct product of minimal normal subgroups of* G *, say* M_i , $i = 1,\ldots,n$, *and* $F(G) \subseteq C_G(M_i)$.

Proof All references are to Huppert [24] III.

The complement for $F(G)$ is assured by Hilfssatz 4.4, and the direct product property by Satz 4.5. The centralising property is contained in Satz 4.2e, as required.

Henceforth all classes of groups are assumed to be non-trivial.

§3. G-modules and Fitting formations

Our first remarks in this section concern the situations in which we may change the group which acts on a given module. Frequent use will be made of such changes. Our practice will be to leave it to the context to settle how the modules are being considered.

Let V be a G-module for a group G . If N is normal and contained in $\ker_G V$ then under the appropriate action V may be considered a G/N-module.

Further, if ϕ is a homomorphism of a group X into G , then V may be considered an X-module under the obvious action for $x \in X$, $v \in V$:

$$xv = \phi(x)v \ .$$

In this case we say that V is an X-module given by *inflation* from G. It will be usual to leave the exact homomorphism understood.

If U,V are isomorphic as G-modules we write $U \cong_G V$ to signify module isomorphism with respect to the group G.

If V is a G-module then there is an implied homomorphism of G into $\mathrm{Aut}\, V$. Under this homomorphism we are able to construct a semidirect product, always written $V.G$. Note that $U \cong_G V$ implies $U.G \cong V.G$ but the converse fails.

Moreover if V is an irreducible G-module then $G/\ker_G V$ acts faithfully and irreducibly on V. In line with Bryce, Cossey [6] we shall call $G/\ker_G V$ the *automiser* of V. The following semidirect product will be of great significance in Chapter 6.

3.1 <u>DEFINITION</u> *If* V *is any* finite *G-module then we put*

$$P(V,G) = V.G/\ker_G V \ .$$

3.2 <u>THEOREM</u> *For* V *an irreducible* G*-module* $P(V,G)$ *is a primitive group and* V *is a unique minimal normal subgroup.*

<u>Proof</u>. For convenience put $P = P(V,G)$. Then because V is irreducible under G it follows V is a minimal normal subgroup of P. Moreover because $G/\ker_G V$ acts faithfully on V it follows that $C_P(V) = V$.

Thus Theorem I.8 of Gaschütz [24] applies showing that P is primitive. Uniqueness follows immediately from I.4 of Gaschütz again. This completes the proof.

Now in a finite soluble group chief factors are elementary abelian q-groups, q a prime, and so can be regarded as vector spaces over $\mathbb{Z}_q$. Moreover if H/K is a q-chief factor of G then G acts on H/K by conjugation in the usual way. Under this action H/K becomes an irreducible $\mathbb{Z}_q G$-module. The kernel of H/K under this action is the centraliser of H/K in G and is denoted by $C_G(H/K)$.

Further, if N is normal in G then we say H/K is above N if $K \supseteq N$ and below N if $H \subseteq N$.

Note in passing that if $\phi : G \to H$ is an epimorphism and U is a chief factor of H then there exists a chief factor of G, V say, such that $V \cong_G U$.

Choose a definite chief series for G and let $\{U_i\}_{i=1}^{r}$ be the set of all q-chief factors contained in it. Then by Huppert [24] III 4.3 we have

(3.3) $$O_{q'q}(G) = \bigcap_{i=1}^{n} C_G(U_i) .$$

Since $F(G) = \bigcap_{q \in \mathbb{P}} O_{q'q}(G)$ it follows that $F(G)$ is just the intersection of centralisers belonging to the chief factors of chief series for G.

A useful corollary is that when V is any q-chief factor and N is normal in G and $N \in \underline{\underline{S}}_{q'}\underline{\underline{S}}_q$ then

(3.4) $$N \subseteq C_G(V) .$$

We can now extend Lemma 2.3 to a consideration of chief factors. The following is a most useful result relevant to Chapter 6.

3.5 LEMMA *Let* V *be any chief factor of* G *and let* $G \in \underline{\underline{X}}$ *where* $\underline{\underline{X}}$ *is a formation. Then*

$$P(V,G) \in \underline{\underline{X}} .$$

Proof Suppose $V = H/K$.

That $P(V,G) \cong P(V,G/K)$ is elementary.

But then (2.3) applies, yielding the required result.

Now let k be an arbitrary field of characteristic q . We know that any q-chief factor, V say, may be extended to a kG-module of the form $k \otimes_{\mathbb{Z}_q} V$. Composition factors of chief factors extended in this way constitute the basic material for the several constructions of Fitting formations we consider in this thesis.

3.6 DEFINITION *Let* k *be any field of characteristic* q *and let* G *be a group.*

Denote by $\Gamma_k(G)$ *the class of irreducible* kG-*modules which appear as composition factors of* kG-*modules of the form* $k \otimes_{\mathbb{Z}_q} V$, *where* V *is a* q-*chief factor of* G .

We are now in a position to describe Hawkes' class $\underline{\underline{Y}}_q$, and the Berger-Cossey class $\underline{\underline{Y}}_q^p$.

3.7 EXAMPLE (Hawkes)

Let k be an algebraically closed field of characteristic q . Define the absolute arithmetic q-rank of a group G to be the least common multiple of the k-dimensions of the modules in $\Gamma_k(G)$.

Then the class of groups which have absolute arithmetic q-rank a q'-number is a Fitting formation. Hawkes [20] Theorem 2.

3.8 EXAMPLE (Berger-Cossey)

Let k be an algebraically closed field of characteristic q, and p be a prime. For each group G define $Y_q^p(G)$ to be the class of irreducible kG-modules V which satisfy the following conditions

BC1 p does not divide the dimension of V.

BC2 If θ is the representation of G afforded by V, then for $g \in G$, $\det(\theta(g))$ lies in the p'-roots of unity in k.

BC3 $G/\ker_G V$ is p-nilpotent.

Then the class of groups

$$\underline{\underline{Y}}_q^p = \{G \mid \Gamma_k(G) \subseteq Y_q^p(G)\}$$

is a non-saturated Fitting formation. Berger-Cossey [1] Theorem 3.1.

§4. Representation theory

In this section we collect the representation theory needed.

First we state what we shall mean by Cliffords Theorem (cf [31] Theorem 2.2A).

4.1 THEOREM *Let* k *be an arbitrary field, and* G *be a group with* N *a normal subgroup of* G. *If* V *is an irreducible* kG*-module then*

(1) V_N *is completely reducible with all irreducible components conjugate under* G.

(2) Let W *be an homogeneous component of* V_N *and* S *the stabiliser of* W *in* G *. Then* $N \subseteq S \subseteq G$ *and* W *is an irreducible* kS*-module with* $V \cong_G W^G$.

(3) The dimensions of the homogeneous components are all the same, and there are exactly $|G:S|$ *of them.*

The next theorem also due to Clifford is co-extensive with the last. The most valuable feature of Theorem 4.1 is that we are able to write V as being induced from an irreducible module of smaller dimension than V . However this only applies in the case $|G:S| \neq 1$. When $|G:S| = 1$ (4.1) reveals nothing about the structure of V as a G-module. It is this case, when V_N is homogeneous, that the following result covers.

4.2 <u>THEOREM</u> *Let* G *and* N *be as before, however this time take* k *to be an algebraically closed field. Let* V *be an irreducible* kG-*module and* V_N *homogeneous. Let* D *be the representation of* G *afforded by* V *and let* R *be the representation afforded by an irreducible constituent of* V_N . *Suppose the multiplicity of* R *in* D *is* t .

Then there exist irreducible projective representations ([10], §51) *of* G , P_1 *and* P_2 *say, such that*

(1) $D(g) = P_2(g) \otimes P_1(g)$ *for* $g \in G$

(2) $P_{2N} = R$ (or $P_2(g) = R(g)$ *for* $g \in N$) , *and*

(3) P_1 *has degree* t *and has* N *in its kernel.*

Furthermore if there exists an ordinary representation R^* *of* G *such that* $R^*_N = R$ *then we may take* $P_2 = R$ *and in this case* P_1 *will also be an ordinary representation.*

For a proof of this result see Curtis and Reiner [10] Theorem 51.7.

Note that when there exists a kG-module W with the property $W_N = U$ we say that U *extends* or *lifts* to G .

As a corollary we have the following

4.3 THEOREM *Let* G,N,k *and* V *be as in Theorem 4.2. Suppose* $|G:N| = q$, *where* q *is any prime. Then* V_N *is irreducible.*

Proof Let U be an irreducible component of V_N . Applying Theorem 4.2 and using the same terms we have

$$V = R \otimes S ,$$

where R affords an irreducible projective representation for G and S one for G/N .

But G/N is cyclic and so $\dim S = 1$. This is because by Huppert [24] V 25.3 the Schur multiplier of G/N is trivial and so the factor set of S is trivial. Thus S is really an ordinary irreducible, and for this we know $\dim S = 1$.

Thus $\dim V = \dim R$.

But by Theorem 4.2 (2) $\dim R = \dim U$ and so $\dim V = \dim U$.

Hence V_N is irreducible, as required.

Theorems 4.2 and 4.3 are used in Chapters 4 and 5.

The most useful version of 4.2 is when U lifts so that V can be written as a tensor product of ordinary irreducibles.

The following results which provide sufficient conditions for such lifting are therefore of considerable value.

4.4 THEOREM *Let* k *be an arbitrary field and* N *a normal Hall subgroup in a group* G *with complement* K . *Let* V *be an irreducible* kN-*module such that* V *is invariant in* G *i.e. module isomorphic with its* G-*conjugates. Moreover let* θ *be the* N-*representation afforded by* V . *Then*

(1) V *lifts to a* kG-*module,* V^* *say.*

If in addition k is algebraically closed then

(2) V^* *may be chosen so that the* G-*representation afforded by* V^* , *say* θ , *has the property*

$$\det \theta^*(g) = \begin{cases} 1 & g \in K \\ \det \theta(g) & g \in N \end{cases}$$

Proof For (1) we have Isaacs [26] Theorem A.

For (2) we know that such a complement K exists by the Schur-Zassenhaus Theorem, now apply Hawkes [22] (2.6). This completes the proof.

In the situation where N is no longer a Hall subgroup Isaacs [26] (4.5) has the following.

4.5 <u>THEOREM</u> *Let* k *be an arbitrary field. Let* G *be a group with* N *a normal subgroup. Let* V *be an irreducible* kN-*module such that* V *is invariant in* G . *If for each Sylow subgroup* P/N *of* G/N *we have that* V *lifts to* P , *then* V *lifts to* G .

CHAPTER 3

A GENERAL CONSTRUCTION VIA CHIEF FACTORS

§1. Introduction

In 1970 Hawkes [20] took the first step towards characterising Fitting formations by showing that all examples of nilpotent length 2 can be defined locally by nilpotent groups (Theorem 2.5) . At the same time however, as we have noted, he constructed an example of nilpotent length 3 which is neither saturated nor subgroup closed (essentially Example 3.7).

Later, in 1978, Tom Berger and John Cossey [1] were able to construct further cases of non-saturated Fitting formations by applying a variation of Hawkes' technique.

Since then no new such classes have appeared.

Nevertheless the methods of Hawkes, Berger and Cossey seem to contain fruitful ideas about how new examples might arise. Foremost with both sets of authors is the central role chief factors are given in the construction.

Now it is a well known fact that the chief factors of a soluble group may be considered as irreducible modules for the group over a suitable field of prime order. The approach Hawkes, Berger and Cossey take is to define their Fitting formations by collecting those groups for which certain modules, essentially the chief factors, are restricted in some way. Specifically, for a field k of characteristic q and a group G , restrictions are placed on the modules in $\dot{\Gamma}_k(G)$, the class of composition factors of the q-chief factors of G extended to k , as defined in 2.3.6.

In this chapter our aim is to systematise the procedure of placing restrictions on $\Gamma_k(G)$.

In later chapters we shall see how this method enables the construction of new examples and also facilitates a more systematic study of old ones.

For instance, in Chapter 4 we show that Hawkes' class $\underline{\underline{Y}}_q$ and Berger-Cossey's class $\underline{\underline{Y}}_q^{\,p}$ belong to one family of examples and in Chapter 5 we construct a new family of Fitting formations for which some members are of non-saturated type. Finally in Chapter 6 we review the question of whether it is possible to characterise all Fitting formations as occurring in this way, by a construction through properties of chief factors.

§2. The General Construction

By way of introducing our result we point out that Hawkes and Berger-Cossey proceed in two stages : *Step 1* and *Step 2*. It will be our aim to emphasise that the substantial work is done in *Step 1* ; whereas *Step 2* may be replaced once and for all by a general argument.

For Hawkes and Berger-Cossey then we have the following format :

Step 1

For each group there is specified a class of irreducible modules over a field k *and* the family of *these classes* is *shown to satisfy certain closure properties.*

Step 2

The Fitting formation is defined as the class of groups G *for which* $\Gamma_k(G)$ *is contained in the class of modules specified in Step 1.*

Closure on this class of groups now follows from the closure properties on the classes of modules determined in Step 1.

Note that in the case of $\underline{\underline{Y}}_q^p$ *Step 1* is fairly explicit. In particular Berger-Cossey define a class of irreducible kG-modules, $Y_q^p(G)$, over an algebraically closed field of characteristic q (Example 2.3.8) and then establish two closure results, Theorem 2.1 (i) and (ii) [1].

Further, note that chief factors only enter the construction in *Step 2* where $\Gamma_k(G)$ comes under consideration.

Our construction is modelled on this format.

We begin by ascertaining the closure properties, denoted M1 to M5 , which are satisfied (perhaps implicitly) by the modules in *Step 1*. For instance M4 and M5 are suggested by Berger-Cossey [1] Theorem 2.1 (i) and (ii). Now we suppose that $M(G)$ is a class of irreducible kG-modules (for k an *arbitrary* field of characteristic q) which satisfies this list. This completes the equivalent of *Step 1* in our general construction.

The equivalent of *Step 2* is now satisfied by Theorem 2.1 below. This result states (in the terms we have just set up) that the class of groups G which satisfy the relation

$$\Gamma_k(G) \subseteq M(G)$$

defines a Fitting formation. In this way *Step 2* is permanently replaced and therefore constructions of this kind are reduced to a one step process.

2.1 <u>THEOREM</u> *Let* k *be a field of characteristic* q *not necessarily algebraically closed. Suppose for each soluble group* G *we have a class of irreducible* kG*-modules denoted by* $M(G)$. *Suppose further that for each* G , $M(G)$ *satisfies the following closure conditions.*

M1 *The trivial irreducible* kG*-module is in* $M(G)$.

M2 *If* $V \in M(G)$ *and* $N \subseteq \ker_G(V)$ *where* N *is normal in* G , *then* $V \in M(G/N)$.

M3 *If* $V \in M(N)$ *and there is an epimorphism* $\phi : G \to N$, *then* $V \in M(G)$.

M4 *If* $V \in M(G)$ *and* N *is a maximal normal subgroup of* G *and* U *is a composition factor of* V_N , *then* $U \in M(N)$.

M5 *If* V *is an irreducible* kG*-module where* $G = N_1N_2$ *for* N_1, N_2 *maximal normal subgroups of* G *and if each composition factor of* V_{N_i} *is in* $M(N_i)$ *for* $i = 1,2$, *then* $V \in M(G)$.

Then the class of groups

$$\underline{\underline{M}} = \{G \mid \Gamma_k(G) \subseteq M(G)\} \tag{2.2}$$

is a Fitting formation.

For an explanation of how we are thinking of the modules in M2 and M3 see Chapter 2, 3.

<u>Proof</u> First we must establish that $\underline{\underline{M}}$ is isomorphism closed.

Let G , H be groups with $G \cong H$, $G \in \underline{\underline{M}}$ and $F \in \Gamma_k(H)$ say. Our aim is to show $F \in M(H)$, for then $H \in \underline{\underline{M}}$, as required.

But using the isomorphism it follows that $F \in \Gamma_k(G)$ and so then by M3 we have $F \in M(H)$, as desired.

For Q-closure let $G \in \underline{\underline{M}}$ and N be normal in G. We must show $\Gamma_k(G/N) \subseteq M(G/N)$.

Let $F \in \Gamma_k(G/N)$, and let U be the q-chief factor of G/N associated with F i.e. F is taken to be a composition factor of $k \otimes U$.

Now by inflation F is an irreducible kG-module with $N \subseteq \ker_G(F)$. It is easy to see that F must now be in $\Gamma(G)$ and so by hypothesis in $M(G)$.

Thus M2 applies. Hence $F \in M(G/N)$, as required.

For R_0-closure let $G/N_i \in \underline{\underline{M}}$ for $i = 1,2$ and $N_1 \cap N_2 = 1$. We must show $\Gamma_k(G) \subseteq M(G)$.

Let $F \in \Gamma_k(G)$ and U be the associated q-chief factor of G.

It is an elementary result (Bryce-Cossey [6] Remark (iii) §2.5) that U is isomorphic as a G-module to a chief factor of G/N_i for $i = 1$ or 2. Say $i = 1$.

It follows $F \in \Gamma_k(G/N_1)$ and so, by hypothesis, $F \in M(G/N_1)$. M3 now applies to F. Hence $F \in M(G)$, as required.

For S_n-closure take $G \in \underline{\underline{M}}$ and let N be a maximal normal subgroup of G. We must show $\Gamma_k(N) \subseteq M(N)$.

As usual let $F \in \Gamma_k(N)$ and U be a chief factor associated with F; by the Jordan-Hölder theorem, we can assume U is chosen as follows.

Now take a chief series of G passing through N and refine it to a chief series for N. Let V be the chief factor of G which contains U. U is an irreducible constituent of V_N, and is therefore, by Clifford's theorem, (module) isomorphic to a direct summand of V_N. F is thus a composition factor of $(k \otimes V)_N$.

Now let F^* be the composition factor of $k \otimes V$ which has F as an irreducible constituent of F^*_N. Since $G \in \underline{\underline{M}}$, we have $F^* \in M(G)$.

Condition M4 now applies giving $F \in M(N)$, as required.

Finally, for N_0-closure let $G = N_1 N_2$ where N_i are maximal normal subgroups of G and $N_i \in \underline{\underline{M}}$ for $i = 1,2$. We must show $\Gamma_k(G) \subseteq M(G)$.

Let $F \in \Gamma_k(G)$ and U the associated chief factor of F.

The argument divides into two cases depending on whether U is above $N_1 \cap N_2$ or below.

Suppose first U is above $N_1 \cap N_2$.

Then by maximality of N_1, N_2 it follows that U is (isomorphic to) either G/N_1 or G/N_2.

If $U = G/N_1$ then as a G-module U is trivial as well as being irreducible. Thus $F = k \otimes U$. It follows that F_{N_i} is trivial and irreducible $i = 1,2$. Thus by M1 $F_{N_i} \in M(N_i)$ and hence by M5 $F \in M(G)$, as we require in this case.

In the case $U = G/N_2$ a similar argument, making only the necessary changes of notation, applies.

In the final case suppose U is below $N_1 \cap N_2$.

By Clifford's theorem we have that

$$U_{N_i} = \bigoplus_{j=1}^{r_i} W_{ij}$$

where the W_{ij} are irreducible kN_i-modules for $i = 1,2$, $j = 1,\ldots,r_i$.
Hence

$$(k\otimes U)_{N_i} = \bigoplus_{j=1}^{r_i} k \otimes W_{ij}$$

Thus each composition factor of F_{N_i} is also one for $k \otimes W_{ij}$ for some j .

But note that the W_{ij} are module isomorphic to chief factors of N_i .

It follows that each composition factor of F_{N_i} must, by hypothesis, be in $M(N_i)$.

M5 now applies and gives $F \in M(G)$, in this case as well.

Combining the parts it follows that $F \in M(G)$ in each case, as required for N_0-closure.

This completes the proof of the theorem.

Following this theorem we are now in possession of the following technique for constructing a Fitting formation via consideration of chief factors :

2.3 <u>CONSTRUCTION</u> *Take a field* k *of characteristic* q *and choose a set of irreducible modules* $M(G)$ *for each group* G *for which M1 to M5 are all satisfied. Then the class*

$$\underline{\underline{M}} = \{G \mid \Gamma_k(G) \subseteq M(G)\}$$

forms a Fitting formation.

We mention two trivial examples.

1) If for each G we have $M(G) = \underline{\underline{T}}^*$ then it follows

$\underline{\underline{M}} = \underline{\underline{S}}_{q'}\underline{\underline{S}}_q$ (given that the characteristic of k is q).

2) If for each G, $M(G)$ is all irreducible kG-modules then $\underline{\underline{M}} = \underline{\underline{S}}$.

It is worth noting that if $G \in \underline{\underline{S}}_{q'}\underline{\underline{S}}_q$ then because the characteristic of k is q, $\Gamma_k(G)$ equals $\underline{\underline{T}}$. Hence $\underline{\underline{S}}_{q'}\underline{\underline{S}}_q \subseteq \underline{\underline{M}}$, and so Fitting formations determined in this way exhibit a very rich structure.

It would be interesting to know what are the exact conditions which must hold on $M(G)$ in order to produce a Fitting formation in the way Theorem 2.1 has described. Indeed there seems to be no reason to expect that the conditions M1 to M5 are at all necessary. Moreover, we have been unable to find criteria useful for the selection of suitable conditions.

$\underline{\underline{T}}^*$ consists of the trivial kG-module alone.

CHAPTER 4

A GENERALISATION OF $\underline{\underline{Y}}_q$ AND $\underline{\underline{Y}}_q^{\,p}$

§1. Introduction

The purpose of this chapter is to extend knowledge of the non-saturated Fitting formations already known : Hawkes' class, $\underline{\underline{Y}}_q$ (Example 2.3.7) and Berger-Cossey's class, $\underline{\underline{Y}}_q^{\,p}$ (Example 2.3.8).

In §2 we show that these classes are members of a more general family of Fitting formations we shall call $\underline{\underline{Y}}_q^{\,\pi}(X)$. Here π is a set of primes and $\underline{\underline{X}}$ any Fitting formation contained in $\underline{\underline{S}}_{\pi_0}\underline{\underline{S}}_{\pi_0}$, for $\pi_0 = \pi \setminus \{q\}$.

Throughout this chapter q is the characteristic of an underlying algebraically closed field, k .

$\underline{\underline{Y}}_q^{\,\pi}(\underline{\underline{X}})$ reduces to $\underline{\underline{Y}}_q$ when $\pi = \{q\}$ and $\underline{\underline{X}} = \underline{\underline{S}}$; and reduces to $\underline{\underline{Y}}_q^{\,p}$ when $\pi = \{p\}$ and $\underline{\underline{X}} = \underline{\underline{S}}_p\underline{\underline{S}}_p$.

In §3 we give sufficient conditions for this new family to be non-saturated. Typically these conditions will be on the choice of $\underline{\underline{X}}$ in relation to π and q . Unfortunately we are not able to provide necessary and sufficient conditions for $\underline{\underline{Y}}_q^{\,\pi}(\underline{\underline{X}})$ to be non-saturated.

A number of the key results in this section are used again to determine non-saturation in the example of Chapter 5.

Finally, in §4 elementary properties of $\underline{\underline{Y}}_q^{\,\pi}(\underline{\underline{X}})$ are discussed. Most importantly we provide necessary and sufficient conditions for two specified classes $\underline{\underline{Y}}_q^{\,\pi}(\underline{\underline{X}})$ and $\underline{\underline{Y}}_q^{\,\sigma}(\underline{\underline{X}})$ say to be equal, and discuss further results of a similar kind.

§2. The Construction of $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$

The generalisation we propose to construct in this section is motivated by a number of quite simple observations.

1. Suppose G is a group for which $\Gamma_k(G)$ satisfies BC1, then using Hawkes' terms (as in Example 2.3.7) this is equivalent to saying that the absolute arithmetic q-rank of G is a p'-number.

2. For $\underline{\underline{Y}}_q^p$ when $p = q$, BC2 holds automatically. This is because of the elementary fact that in a field of characteristic q there are no elements of order a power of q.

3. Berger-Cossey make only one substantive use of BC3 in the whole of their proof; this is to guarantee that BC2 holds for the modules which occur in the consideration of M4 (essentially Berger-Cossey's Theorem 2.1(i)[1]).

4. In the Berger-Cossey construction (Example 2.3.8) their proof still holds if p is replaced by a set of primes π.

Prompted by the suggestions contained in these observations we will use the method described in Chapter 3 to construct classes more general than $\underline{\underline{Y}}_q$ or $\underline{\underline{Y}}_q^p$.

In the first step, for each G we collect the irreducible kG-modules which satisfy new conditions Y1, Y2, Y3 and call this class of modules $Y_q^{\pi}(G)$. In each case Y1, Y2 and Y3 are generalisations of BC1, BC2 and BC3 respectively.

In the second step our theorem below shows that M1 to M5 each hold on $Y_q^{\pi}(G)$.

Thus, by Theorem 3.2.1 it will follow that the class

$$\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}}) = \{G \in \underline{\underline{S}} \mid \Gamma_k(G) \subseteq Y_q^{\pi}(G)\}$$

is a Fitting formation. Here $\underline{\underline{X}}$ is a Fitting formation which appears in condition Y3. It is chosen subject to π and q ; details are provided below.

Y1 and Y2 are straight forward generalisations of BC1 and BC2 and follow the suggestion of 4 above, replacing the single prime p by a set of primes π .

Y3 is a little harder to motivate. Let V be an irreducible kG-module. It is convenient to have G/ker V in some Fitting formation, $\underline{\underline{X}}$ say. However, in order to satisfy the co-prime relationships which seem essential to the proof, we have $\underline{\underline{X}}$ contained in $\underline{\underline{S}}_{\pi_0'}\underline{\underline{S}}_{\pi_0}$ for $\pi_0 = \pi \setminus \{q\}$.

Note that π_0 is used instead of π so that when $\pi = \{q\}$, $\underline{\underline{X}}$ may be set to equal $\underline{\underline{S}}$. It follows that Y3 is automatically satisfied for all relevant modules. Moreover from 2 we have that Y2 is also satisfied. Hence we are left with only one substantial condition on $Y_q^q(G)$, namely Y1 . But by 1 we know that this is just Hawkes' original condition, thus $\underline{\underline{Y}}_q^q(\underline{\underline{S}})$ is Hawkes' class $\underline{\underline{Y}}_q$.

Before proceeding to the theorem we will need to explain what we shall mean by the determinantal order of a representation.

If V is any kG-module for a field k and θ is the representation afforded by V then

$$\det \theta : G \to k^{*}$$

is a linear representation of G defined in the obvious way :

(2.1) $$(\det\theta)(g) := \det\theta(g)$$

Define $(\det\theta)^n(g) := (\det\theta(g))^n$. Then following Isaacs' notation ([25] Chapter 6 p.88) (where in fact he is talking about characters afforded by V not representations) the *determinantal order* of θ , denoted $o(\det\theta)$ is the smallest n such that $\det\theta^n(g) = 1$ for all $g \in G$.

In particular $o(\det\theta(g)) \mid o(\det\theta)$ for all $g \in G$.

Further it is easy to check that

(2.2) $$|G : \ker\det\theta| = o(\det\theta)$$

and so $o(\det\theta) \mid |G|$.

2.3 <u>THEOREM</u> *Let* k *be an algebraically closed field of characteristic* q *and* π *be an arbitrary set of primes. Let* $\underline{\underline{X}}$ *be any Fitting formation with the constraint that* $\underline{\underline{X}} \subseteq \underline{\underline{S}}_{\pi_0'}\underline{\underline{S}}_{\pi_0}$ *where* $\pi_0 = \pi \setminus \{q\}$.
Further, for each soluble group G , *let* $\underline{\underline{Y}}_q^{\pi}(G)$ *be the class of all irreducible* kG-*modules* V *satisfying the following conditions :*

Y1 dim V *is a* π'*-number*

Y2 if V *affords the representation* θ, *then* $o(\det\theta)$ *is a* π'*-number*

Y3 $G/\ker V \in \underline{\underline{X}}$.

Then the modules $Y_q^{\pi}(G)$ *so determined satisfy the module conditions M1 to M5 of Theorem 3.2.1 and so the class*

(2.4) $$Y_q^{\pi}(\underline{\underline{X}}) = \{G \in \underline{\underline{S}} \mid \Gamma_k(G) \subseteq Y_q^{\pi}(G)\}$$

is a Fitting formation.

Proof For convenience we drop all notation where ambiguity will not arise : thus $Y_q^{\pi}(G) = Y(G)$, and so on.

M1 is satisfied trivially.

For M2 we are given $V \in Y(G)$ and N normal in G with $N \subseteq \ker_G V$ and must show $V \in Y(G/N)$. Clearly Y1 holds, and Y2 holds because the determinantal order of the representation afforded by V as a kG/N-module is just that for V as a kG-module. For Y3 we have $\ker_{G/N}(V) = \ker V/N$ so

$$G/N \Big/ \ker_{G/N}(V) \cong G/\ker V$$
$$\in \underline{\underline{X}}$$

For M3 we are given $V \in Y(N)$ and an epimorphism $\phi : G \to N$ and must show $V \in Y(G)$. Once again Y1 and Y2 follow automatically. To show V satisfies Y3 we must show $G/\ker V \in \underline{\underline{X}}$. But $\phi : G \to N$ defines an epimorphism

$$G/\ker \phi \to N/\ker V$$

whose kernel is $\ker_G V/\ker \phi$. Thus

$$G/\ker_G V \cong N/\ker V$$
$$\in \underline{\underline{X}} .$$

For M4 let N be a maximal normal subgroup of G which must thus be of prime index, t say. Let V be an irreducible kG-module where

$V \in Y(G)$, and U an irreducible component of V_N . We must show $U \in Y(N)$.

Now suppose $\ker V \nsubseteq N$. Then by maximality of N , $G = N \ker V$. So G acts on U and thus U is a kG-module. Hence $V_N = U$ since V is irreducible. Thus U satisfies Y1 and Y2 immediately. Observing that

$$\begin{aligned} N/\ker U &\cong N \ker V/\ker V \\ &\cong G/\ker V \\ &\in \underline{\underline{X}} \end{aligned}$$

it follows that Y3 is also satisfied. Therefore $V_N \in Y(N)$ and M4 is satisfied in this case.

We now work modulo $\ker V$. But since $\ker V$ is contained in N and $\ker U$ this is equivalent to assuming $\ker V = 1$.

U is any irreducible component of V_N . Now by Clifford's Theorem V_N is either homogeneous or else it breaks up into a direct sum of t non-isomorphic irreducible components.

In the first case if V_N is homogeneous then by Theorem 2.4.3 it is also irreducible. Now by the argument we used above for the case $V \subsetneq N$ it follows that $V_N \in Y(N)$.

Hence M4 is satisfied in this case.

In the second case V_N breaks up. We have

$$V_N = V_1 \oplus \dots \oplus V_t$$

where the V_i are non-isomorphic irreducible kN-modules $i = 1,\dots,t$. We must show $V_i \in Y(N)$ for each i .

Now $\dim V = t \dim V_i$ and $\dim V$ is a π'-number because $V \in Y(G)$. Thus $\dim V_i$ is a π'-number, and so Y1 holds. Note for later use that t is also a π'-number. It remains to show that V_i satisfies the conditions Y2 and Y3.

Now for Y2 let θ_i be the representations afforded by V_i $i = 1,\ldots,t$ and let θ be the representation afforded by V. We must show $o(\det \theta_i)$ is a π'-number, that is $o(\det \theta_i(n))$ is a π'-number for all $n \in N$ and for each $i = 1,\ldots,t$. Over a field of characteristic q (as k is) this is equivalent to showing $\det \theta_i(n) = 1$ for all $n \in N$ where n is of order a π_0-number and $\pi_0 = \pi \setminus \{q\}$.

It is this last claim we shall prove.

By Clifford's theorem there exists $z_1 = 1, z_2, \ldots, z_t$ elements of G such that $V_i \cong V_1^{z_i}$. Put $O^{\pi}(G) \cap N = L$. Since the V_i are non-isomorphic and $O^{\pi}(G) \nleq N$ we have for each z_i, $i \neq 1$ that there exists $x_i \in O^{\pi}(G) \setminus L$ at $y_i \in N$ such that $z_i = x_i y_i \ell_i$ for some $x_i \in L$. Thus

$$V_i \cong V_1^{z_i}$$

$$= V_1^{x_i}$$

for $x_i \in O^{\pi}(G) \setminus L$, $i = 2,\ldots,t$ and $x_1 = 1$.

Translated into terms of representations it follows that

$$\theta_i(n) = \theta_1\left(n^{x_i}\right)$$

$$= \theta_1(n[n,x_i])$$

and thus

$$\det \theta_i(n) = \det \theta_1(n) \det \theta_1[n,x_i]$$

$n \in N$.

Our aim now is to show the following :

2.5 *For all* $n \in N$ *of order a* π_0*-number* $\det \theta_i(n) = \det \theta_1(n)$.

In order to do this we take such an n and show $\det \theta_1[n,x_i] = 1$.

Let α be the order of n , a π_0-number. Thus $(\det \theta_i(n))^{\alpha} = 1$ and since $\det \theta_1[n,x_i] = \det \theta_i(n) \det \theta_1(n)^{-1}$ it follows

$$(\det \theta_1[n,x_i])^{\alpha} = 1 . \tag{A}$$

On the other hand since $x_i \in O^{\pi}(G)$ and $O^{\pi}(G)$ and N are both normal in G it follows $[n,x_i] \in L$ and so

$$[n,x_i] \ker V_i \in L \ker V_i/\ker V_i .$$

Choose β to be the order of $[n,x_i] \ker V_i$ in this group. Hence

$$(\det \theta_1[n,x_i])^{\beta} = 1 . \tag{B}$$

Now $(\alpha,\beta) = 1$, for put $L_0 = O^{\pi_0}(G) \cap N$ and observe

$$L_0 \ker V_i/\ker V_i \cong L_0/L_0 \cap \ker V_i$$

where the last term is in $\underline{\underline{S}}_{\pi_0'}$ because $G \in \underline{\underline{X}}$ implies $O^{\pi_0}(G) \in \underline{\underline{S}}_{\pi_0'}$ and so $L_0 \in \underline{\underline{S}}_{\pi_0'}$. Now L is normal in L_0 thus $L \ker V_i/\ker V_i \in \underline{\underline{S}}_{\pi_0'}$

and hence β is a π_0'-number. But we already have that α is a π_0-number, thus $(\alpha,\beta) = 1$.

The only way (A) and (B) can now hold is if $\det \theta_1[n,x_i] = 1$. This proves (2.5).

We are now in a position to see how Y2 holds. Since $V_N = V_1 \oplus \ldots \oplus V_t$ we have

$$\det \theta(n) = \prod_{i=1}^{t} \det \theta_i(n) .$$

where recall t is a prime in π' . However by hypothesis $\det \theta(n) = 1$ hence, applying (2.5)

$$(\det \theta_1(n))^t = 1$$

But n and hence $\det \theta_1(n)$ is of order a π_0-number, and this is coprime to t .

Therefore $\det \theta_1(n) = 1$ for all $n \in N$, where n is of order a π_0-number. This is what we set out to prove. Y2 is now satisfied.

Lastly Y3 follows immediately on the assumption $\ker V = 1$.

Therefore $V_i \in Y(N)$ for each i and so M4 is satisfied.

To show M5 is satisfied let N_i be a maximal normal subgroup of G where $|G:N_i| = t_i$ a prime for $i = 1,2$.

Now by Clifford's theorem either V_{N_i} is homogeneous or else it breaks up into a direct sum of t_i non-isomorphic irreducibles. If it is homogeneous then by Theorem 2.4.3 it follows that it is irreducible.

We may best express these alternatives by the following compact expression.

(2.6) $$V_{N_i} = V_{i1} \oplus \ldots \oplus V_{ij} \oplus \ldots \oplus V_{is_i}$$

where V_{ij} is an irreducible kN_i-module for $j = 1,\ldots,s_i$ and $s_i = 1$ or t_i, $i = 1,2$. Note that by hypothesis, $V_{ij} \in Y(N_i)$.

We must show $V \in Y(G)$, that is that conditions Y1, Y2, Y3 apply for V. We do this in the order : Y2, Y3, Y1.

For Y2 let θ be the representation afforded by V and θ_{ij} the representation afforded by V_{ij}. Then for each $g \in G$ it is possible to write $g = n_1 n_2$ with $n_1 \in N_1$ and $n_2 \in N_2$. Thus

$$\det \theta(g) = \det \theta(n_1) \det \theta(n_2)$$

and so from (2.6)

$$\det \theta(g) = \prod_{i,j} \det \theta_{ij}(n_i) .$$

But $o(\det \theta_{ij})$ is a π'-number so $o(\det \theta)$ is a π'-number and this completes Y2.

For Y3 we must show $G/\ker V \in \underline{\underline{X}}$. Now $N_i/\ker V_{ij} \in \underline{\underline{X}}$ and because $N_i \cap \ker V = \cap \ker V_{ij}$ it follows by R_0-closure of $\underline{\underline{X}}$ that $N_i/N_i \cap \ker V \in \underline{\underline{X}}$. Thus $G/\ker V = \prod_i N_i \ker V/\ker V \in \underline{\underline{X}}$. This satisfies Y3.

For Y1 the argument becomes more involved. We must show $\dim V$ is a π'-number. Assume the contrary, that $\dim V$ is not a π'-number. But by Clifford's theorem we have $\dim V = s_i \dim V_{ij}$ and by hypothesis we have that $\dim V_{ij}$ is a π'-number. Hence the assumption $\dim V$ is not a π'-number leads to the conclusion $s_i \neq 1$ and $s_i = t_i \in \pi$, for each $i = 1,2$.

The plan for the proof is to show first that $V_{N_1 \cap N_2}$ is homogeneous and then that this implies V_{N_1} is homogeneous.

By Theorem 2.4.3 V_{N_1} is irreducible thus $\dim V = \dim V_{11}$ and this contradicts the assumption $\dim V = t_1 \dim V_{11}$, $t_1 \neq 1$.

To show $V_{N_1 \cap N_2}$ is homogeneous note first that by the maximality of N_1, N_2 it follows $|N_i : N_1 \cap N_2| \in \pi$ for $i = 1,2$. Thus applying Clifford's Theorem $V_{ijN_1 \cap N_2}$ must be homogeneous or else t_{3-i} a π-number will divide $\dim V_{ij}$, a contradiction. Hence, by Theorem 2.4.3 , $V_{ijN_1 \cap N_2}$ is irreducible for all i,j . Now suppose $V_{1j} = V_{2\ell}$ for certain j and ℓ . Then because $G = N_1 N_2$ it follows that V_{ij} is a proper kG-submodule of V , a contradiction for V is irreducible. This shows that the two decompositions of $V_{N_1 \cap N_2}$ are distinct. An elementary argument now shows that $V_{N_1 \cap N_2}$ must then be homogeneous, as asserted. (for otherwise, since all the irreducibles are distinct the decomposition is unique.)

But V_{N_1} is also homogeneous. For since $V_{ijN_1 \cap N_2}$ is irreducible it is certainly homogeneous, in fact $V_{ijN_1 \cap N_2} \cong V_{11}$ as $V_{N_1 \cap N_2}$ is homogeneous and V_{11} extends to N_1 . Thus, applying Theorem 2.4.2, there exists an ordinary irreducible $N_1/N_1 \cap N_2$-module, S say, such that

(2.7) $$V_{1j} \cong V_{11} \otimes S .$$

Since $|N_1 : N_1 \cap N_2| = t_2$ is prime $\dim S = 1$. Moreover, either $\ker S = N_1$ or $\ker S = N_1 \cap N_2$.

If $\ker S = N_1$ then S is trivial so $V_{ij} \cong V_{11}$; and so V_{N_1} is homogeneous.

If $\ker S = N_1 \cap N_2$, let $x \in N_1$ and σ be the representation afforded by S and $\ell = \dim V_{11}$. Then from (2.7)

$$\det \theta_{ij}(x) = \det \theta_{11}(x) \ (\det \sigma(x))^{\ell}$$

So $o(\det \sigma)$ is a π'-number.

On the other hand $(\det \sigma(x))^{t_2} = (\det \sigma(x\ N_1 \cap N_2))^{t_2} = 1$ because $N_1/N_1 \cap N_2 \cong C_{t_2}$ and so $o(\det \sigma)$ divides t_2 which lies in π.

Thus $\det \sigma(x) = 1$ for each $x \in N_1$.

But since $\dim S = 1$ we conclude that S is trivial. Hence by (2.7) again $V_{1j} \cong V_{11}$ and so V_{N_1} is homogeneous as we asserted.

This completes the proof that V satisfies Y1. Therefore M5 is satisfied.

This completes the proof of theorem.

We now have the following.

2.8 <u>COROLLARY</u> *Using the same notation as the theorem. If*

(1) $\pi = \{p\}$, *then* $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{S}}_p, \underline{\underline{S}}_p)$ *is the Berger-Cossey class,* $\underline{\underline{Y}}_q^{p}$;

(2) $\pi = \{q\}$, *then* $\underline{\underline{Y}}_q^{\pi'}(\underline{\underline{S}})$ *is the Hawkes class,* $\underline{\underline{Y}}_q$.

§3. $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ and non-saturation

The main results of this section give conditions on $\underline{\underline{X}}$ so that $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ is non-saturated. Our method will be to construct certain groups in $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ and show that a contradiction arises if saturation is assumed.

We need a number of lemmas.

The first of these is a fact concerning an arbitrary Fitting formation, $\underline{\underline{X}}$ say.

Let p,q be primes and let E_{q^n} be an elementary abelian group of order q^n. Now in Hawkes [20](1.5) we have that if $\underline{\underline{X}}$ contains a copy of the unique non-nilpotent extension of E_{q^n}* by E_p, called D_q^p, then $\underline{\underline{X}}$ contains all extensions of q-groups by cyclic p-groups. Our first lemma extends this result.

Roughly speaking, just as Lemma 2.2.2 shows that formations come equipped with all relevant nilpotent groups, we show when a Fitting formation contains all relevant metanilpotent groups.

3.1 <u>LEMMA</u> *Let* $\underline{\underline{X}}$ *be a Fitting formation and* p,q *primes* .

If $\underline{\underline{X}}$ *contains a non-nilpotent* $\underline{\underline{S}}_p\underline{\underline{S}}_q$*-group, then* $\underline{\underline{S}}_p\underline{\underline{S}}_q \subseteq \underline{\underline{X}}$.

<u>Proof</u> For convenience put $\underline{\underline{R}} = \underline{\underline{X}} \cap \underline{\underline{S}}_p\underline{\underline{S}}_q$. The proof is divided into two steps. The first step characterises the elements of $\underline{\underline{R}}$ in terms of automisers for the chief factors. Since $\underline{\underline{R}} \subseteq \underline{\underline{N}}^2$ we can do this by applying the Hawkes characterisation theorem, Theorem 2.2.5. The second step takes $G \in \underline{\underline{S}}_p\underline{\underline{S}}_q$ and shows that it is in $\underline{\underline{R}}$, thus $G \in \underline{\underline{X}}$ i.e. $\underline{\underline{S}}_p\underline{\underline{S}}_q \subseteq \underline{\underline{X}}$, as required.

* n is the smallest positive integer such that $p | q^n - 1$.

We use the notation of Theorem 2.2.5.

Firstly $\gamma(\underline{\underline{R}}) = \{p,q\}$ since $\underline{\underline{R}} \subseteq \underline{\underline{S}}_p\underline{\underline{S}}_q$ implies $\gamma(\underline{\underline{R}}) \subseteq \{p,q\}$ and the fact $\underline{\underline{R}}$ contains a non-nilpotent group implies $\gamma(\underline{\underline{R}})$ cannot be a singleton. Put $\pi = \{p,q\}$. Note $G \in \underline{\underline{S}}_p\underline{\underline{S}}_q$ implies $O_q(G) = O_{p'}(G)$.

We have $q \in \tau(p)$. For suppose not and let G be a non-nilpotent group in $\underline{\underline{R}}$.

Now since $G \in \underline{\underline{R}}$ and $\underline{\underline{R}} \subseteq \underline{\underline{S}}_p\underline{\underline{S}}_q$ the Sylow p-subgroup of G is just $O_p(G)$. On the other hand since $q \notin \tau(p)$ it follows $q \notin \gamma(G/O_{p'}(G))$ because $O_{p'p}(G)/O_{p'}(G)$ is a p-group. But $O_{p'}(G) = O_q(G)$. Thus $O_q(G)$ is a Sylow q-subgroup of G. Therefore all Sylow subgroups of G are normal and so G is nilpotent, a contradiction. Hence $q \in \tau(p)$.

Further $\tau(q) = \phi$ since for every $G \in \underline{\underline{R}}$, $G/O_{q'}(G) = G/O_p(G) \in \underline{\underline{S}}_q$ and so $O_{q'q}(G) = G$ and thus $\gamma(G/O_{q'q}(G)) = \phi$.

Hence applying Theorem 2.2.5 $G \in \underline{\underline{R}}$ if and only if $G/O_{p'p}(G) \in \underline{\underline{N}}_{\tau(p)}$ and $G/O_{q'q}(G) \in \underline{\underline{1}}$.

Now for a given chief series of G let $\{U_i\}$ $i = 1,\ldots,r$ be the complete set of p-chief factors of G and $\{W_j\}$ $i = 1,\ldots,s$ be the complete set of q-chief factors of G. Then from (2.3.3) we know $O_{p'p}(G) = \bigcap_{i=1}^{r} C_G(U_i)$ and $O_{q'q}(G) = \bigcap_{j=1}^{s} C(W_j)$. Therefore, using R_0-closure where necessary, it follows that $G \in \underline{\underline{R}}$ if and only if $G/C_G(U_i) \in \underline{\underline{N}}_{\tau(p)}$ and $G/C_G(W_j) \in \underline{\underline{1}}$ for all i,j.

This completes the first step.

For the second let $G \in \underline{\underline{S}}_p\underline{\underline{S}}_q$ and take U to be any p-chief factor of G. Then by (2.3.4) it follows that $O_p(G) \subseteq C_G(U)$. So

$$G/C_G(U) \cong G/O_p(G) \Big/ C_G(U)/O_p(G)$$

$$\in \underline{\underline{S}}_q .$$

But $q \in \tau(p)$; thus $\underline{\underline{S}}_q \subseteq \underline{\underline{N}}_{\tau(p)}$, so $G/C_G(U) \in \underline{\underline{N}}_{\tau(p)}$.

Moreover if W is a q-chief factor of G then by (2.3.4) again $G \subseteq C_G(W)$, since G is q-nilpotent. Thus $G = C_G(U)$ hence $G/C_G(W) = 1$.

Therefore $G \in \underline{\underline{R}}$ from the characterisation provided in the first step.

Thus $\underline{\underline{S}}_p\underline{\underline{S}}_q \subseteq \underline{\underline{X}}$, as required.

The next lemma we will need is of a similar kind.

3.2 LEMMA *If* $\underline{\underline{X}} \not\subseteq \underline{\underline{S}}_\sigma\underline{\underline{S}}_{\sigma'}$, *where* σ *is a set of primes and* $\underline{\underline{X}}$ *is a Fitting formation, then there exists* $r \in \sigma'$ *and* $s \in \sigma$ *such that* $\underline{\underline{S}}_r\underline{\underline{S}}_s \subseteq \underline{\underline{X}}$.

Proof Let G be a group of minimal order in $\underline{\underline{X}}$ but not in $\underline{\underline{S}}_\sigma\underline{\underline{S}}_{\sigma'}$.

Then $O_\sigma(G) = 1$. For suppose the contrary. By the minimality of G it now follows $G/O_\sigma(G) \in \underline{\underline{S}}_\sigma\underline{\underline{S}}_{\sigma'}$, and so $G \in \underline{\underline{S}}_\sigma\underline{\underline{S}}_{\sigma'}$, a contradiction. Thus $O_\sigma(G) = 1$.

Now let L be a minimal normal subgroup of G. Then L must be of order a power of r, for some $r \in \sigma'$.

We will show that the group

$$G_0 = L \, . \, G/L$$

is a non-nilpotent $\underline{\underline{S}}_r\underline{\underline{S}}_s$-group for some s a prime in σ .

Put $O_{\sigma'}(G) = M$ and take N/M to be a minimal normal subgroup of G/M . It is clear that N/M must be a σ-group.

Moreover $O_\sigma(N) = N \cap O_\sigma(G) = 1$. Hence $N \notin \underline{\underline{S}}_\sigma\underline{\underline{S}}_{\sigma'}$. So by the minimality of G , $N = G$. Therefore $G \in \underline{\underline{S}}_{\sigma'}\underline{\underline{S}}_\sigma$ and G/M is cyclic of order a prime, s say, with $s \in \sigma$.

Now consider L . By the minimality of G it follows $G/L \in \underline{\underline{S}}_\sigma\underline{\underline{S}}_{\sigma'}$. Moreover $G \in \underline{\underline{S}}_{\sigma'}\underline{\underline{S}}_\sigma$ implies $G/L \in \underline{\underline{S}}_{\sigma'}\underline{\underline{S}}_\sigma$. Combining these $G/L \in \underline{\underline{S}}_\sigma\underline{\underline{S}}_{\sigma'} \cap \underline{\underline{S}}_{\sigma'}\underline{\underline{S}}_\sigma$.

Thus we write

$$G/L = H/L \times K/L$$

for some $H/L \in \underline{\underline{S}}_{\sigma'}$ and $K/L \in \underline{\underline{S}}_\sigma$.

But we have $O_\sigma(K) = 1$ so $K \notin \underline{\underline{S}}_\sigma\underline{\underline{S}}_{\sigma'}$ and from minimality of G again, $K = G$. Thus $L = M$ and so G/L is a cyclic group of order a prime s , $s \in \sigma$.

It remains to show that G_0 is non-nilpotent. If it were nilpotent then $G_0 \in \underline{\underline{S}}_\sigma\underline{\underline{S}}_{\sigma'}$.

By the Schur-Zassenhaus theorem G splits over L so $G \cong G_0$, thus $G \in \underline{\underline{S}}_\sigma\underline{\underline{S}}_{\sigma'}$, a contradiction.

Hence G_0 is a non-nilpotent $\underline{\underline{S}}_r\underline{\underline{S}}_s$-group contained in $\underline{\underline{X}}$.

Now by Lemma 3.1 it follows $\underline{\underline{S}}_r\underline{\underline{S}}_s \subseteq \underline{\underline{X}}$, as required.

The following lemmas contain the heaviest part of the construction we shall use.

The first of these is a generalisation of Berger-Cossey [1] Lemma 2.2.

3.3 <u>LEMMA</u> *Let* r, s *be different primes and* k *an algebraically closed field of characteristic* q , $q \neq r$.

Let H *be a group with the following properties*

- F(H) *is an extraspecial group of order* r^{2m+1} , $m \in \mathbb{N}$

- $|H:F(H)| = s^n$, $n \in \mathbb{N}$

- Z(H) = Z(F(H))

Then there exists a faithful irreducible $\mathbb{Z}_q$H*-module,* W *say* .

Moreover for the group $H_0 = W.H$ *we have :*

(1) If $E \in \Gamma_k(H_0)$ *then* dim E *is an* r*-number,*

and

(2) If E *affords the irreducible representation* ξ *then* o(det ξ) *is an* r*-number.*

<u>Proof Lemma 3.3</u>

First we prove the following.

(3.4) *There exists a faithful irreducible* kH*-module,* V *say, such that*

(1)* $\dim V = r^m$

(2) if* V *affords the representation* θ , *then* o(det θ) *is an* r*-number.*

This fact is proved by applying a well known theorem about the representation theory of extraspecial groups.

According to Huppert [24] V 16.14 there are $r - 1$ faithful and irreducible $kF(H)$-modules $\mathcal{U}$ with the following properties.

(α) $\dim \mathcal{U} = r^m$

(β) $\mathcal{U}$ is determined by its restriction to $Z(F(H))$.

Moreover if $\mathcal{U}$ affords the representation ω then because $F(H)$ is an r-group it follows

(γ) $o(\det \omega)$ is an r-number.

Now $\mathcal{U}$ is invariant in H . For by (β) we need only consider $x \in Z(F(H))$. Thus taking $h \in H$ we have

$$\omega^g(x) := \omega(x^{g^{-1}})$$

$$= \omega(x) ,$$

since $Z(H) = Z(F(H))$.

Moreover $F(H)$ is a Hall subgroup in H .

Theorem 2.4.4 (1) now applies. This shows $\mathcal{U}$ lifts to H , that is, there exists an irreducible kH-module V such that $V_{F(H)} = \mathcal{U}$.

Hence (1^*) holds.

(2^*) follows from Theorem 2.4.4 (2) and (γ) above.

Lastly V is faithful for if not $\ker V$ must be a power of s , since $F(H)$ acts faithfully on V . But then $\ker V \subseteq F(H)$, a contradiction.

This completes the proof of (3.4).

Now choose W to be an irreducible $\mathbb{Z}_q H$-module such that V as in (3.4) is a composition factor of $k \otimes W$.

It is well known that $k \otimes W$ is completely reducible and breaks up into a direct sum of Galois conjugates of V (Curtis and Reiner [10] 70.15 and Isaacs [26] 9.21). The Galois conjugates of V will each have the same dimension, kernel and determinantal order as V.

Therefore, since V is a faithful module for H it follows that W is also.

(1) and (2) now follow by application of (3.4).

This completes the proof of the lemma.

We know from a note at the end of Chapter 3 that $\underline{\underline{S}}_q, \underline{\underline{S}}_q \subseteq \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$. As a corollary to Lemma 3.3 we can now give examples of groups in $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ which are not in $\underline{\underline{S}}_{q'}\underline{\underline{S}}_q$. These will be important in our enquiry into non-saturation later.

3.5 COROLLARY *Let* π *be a set of primes such that* $r \in \pi'$ *and* $s \in \pi$ *(note* $r \neq q$, *as in Lemma 3.3) and let* $\underline{\underline{X}}$ *be any Fitting formation contained in* $\underline{\underline{S}}_{\pi_0'}\underline{\underline{S}}_{\pi_0}$ *(recall* $\pi_0 = \pi \setminus \{q\}$*).*

Now with H *and* H_0 *as in Lemma 3.3 assume* $H \in \underline{\underline{X}}$, *then* $H_0 \in \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$.

Proof We must show $\Gamma(H_0) \subseteq Y_q^{\pi}(H_0)$.

Let $E \in \Gamma(H_0)$.

Y1 and Y2 are satisfied by (1) and (2) of the lemma respectively.

For Y3, since $W \subseteq \ker_{H_0} E$ and $H \in \underline{\underline{X}}$ it follows $H_0/\ker_{H_0} E \in \underline{\underline{X}}$, as required.

In §4 we will use an important application of this corollary.

In the same notation if $r \in \gamma(\underline{\underline{X}})$, then by Lemma 2.2.2 the extraspecial group of order r^{2m+1} , R say, is in $\underline{\underline{X}}$.

Thus putting $H = R$ in Lemma 3.3 there is a group $W.R \in \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$.

There exists such a group for each $r \in \gamma(\underline{\underline{X}})$.

Next we show that such groups as in the data of Lemma 3.3 in fact arise for every choice of primes r,s , $r \neq s$.

3.6 <u>LEMMA</u> *For any primes* r,s , $r \neq s$ *there exists an* $\underline{\underline{S}}_r\underline{\underline{S}}_s$*-group* H *say with the following properties.*

(1) $F(H) = O_r(H)$ *and is an extraspecial group of order* r^{2m+1} , *for some* m .

(2) $|H:F(H)| = s$

(3) $Z(H) = Z(F(H))$

<u>Proof</u> Let S be a cyclic group of order s , and V be a faithful irreducible $\mathbb{Z}_r S$-module of dimension m say. Take V^* to be the contragredient $\mathbb{Z}_r S$-module of V .

Since $r \nmid |S|$, $V \otimes V^*$ is completely reducible.

By Theorem 43.14 of Curtis and Reiner [10], $i(V,V) \geq 1$ and so $V \otimes V^*$ has an irreducible component, T say, which is trivial of dimension 1 .

We may write

$$V \otimes V^* = T \oplus W$$

for some W a $\mathbb{Z}_r S$-module.

Consider the set

$$B = \{(x,y) \mid x \in V \oplus V^* , y \in V \otimes V^*/W\} .$$

Define a multiplication on B by putting

$$(u+u^*,t_1+W)(v+v^*,t_2+W) = (u+v+u^*+v^*,u\otimes v^*+t_1+t_2+W)$$

with $u,v \in V$, $u^*, v^* \in V$ and $t_1, t_2 \in V \otimes V^*$.

Then, under this operation B is a group of order r^{2m+1} (since $\dim(V\otimes V^*/W) = 1)$.

In fact B is extraspecial with

$$B' = Z(B)$$

$$= \{(0,y) \mid y \in V \otimes V^*/W\} .$$

(cf. Huppert [24] VI 7.22).

Now form the semidirect product $H = B.S$ via the module action of S on V and V^*, as explained in Huppert [24] VI 7.22.

Because S is faithful on B it follows $Z(H) \subseteq B$ and thus $Z(H) \subseteq Z(B)$.

Conversely, because $Z(B)$ is centralised by S , $Z(B) \subseteq Z(H)$.

Thus $Z(H) = Z(B)$, and so (3) is satisfied.

Moreover it is clear that $F(H) = B$, and that $B = O_r(H)$ and $|H:F(H)| = s$.

This satisfies (1) and (2), and completes the proof of the lemma.

This completes the preparation. We now come to the main part of the section.

3.7 THEOREM *Let* k *be an algebraically closed field of characteristic* q . *Let* π *be a set of primes, and set* $\pi_0 = \pi \setminus \{q\}$. *Let* $\underline{\underline{X}}$ *be a Fitting formation contained in* $\underline{\underline{S}}_{\pi_0'}\underline{\underline{S}}_{\pi_0}$.

If there exists primes r,s *such that* $r \in \pi'$, $r \neq q$, $s \in \pi$ *and* $\underline{\underline{S}}_r\underline{\underline{S}}_s \subseteq \underline{\underline{X}}$, *then* $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ *is non-saturated.*

Proof Use r,s as provided in the data to construct a group H , as in Lemma 3.6.

In particular $H \in \underline{\underline{S}}_r\underline{\underline{S}}_s$ and so $H \in \underline{\underline{X}}$.

Now using the same terms and notation of Corollary we have that $H_0 = W.H \in \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$.

Suppose on the contrary that $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ is saturated. The characterisation of a saturated formation Theorem 2.2.4 now applies. Accordingly there exists a formation $\underline{\underline{X}}(p)$ where $H_0/O_{p'p}(H_0) \in \underline{\underline{X}}(p)$ for each prime p in the characteristic of $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$.

Thus $H \in \underline{\underline{X}}(q)$ and so $H/Z(H) \in \underline{\underline{X}}(q)$. For convenience put $\bar{H} := H/Z(H)$. Now if X is any irreducible $\mathbb{Z}_q\bar{H}$-module then from (2.2.5) again $X.\bar{H} \in \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$.

Our aim is to obtain a contradiction by showing that there is such an X for which $X.\bar{H} \notin \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$.

First observe that any irreducible $k\bar{H}$-module M has either dimension 1 or s. This is because F(H) an extraspecial group implies F(H)/Z(F(H)) is abelian and so, by Dade [33],

dim M divides $|\bar{H} : F(H)/Z(F(H))| = s$.

Now let $\{X_i : i = 1,\ldots,s\}$ be a complete set of irreducible $\mathbb{Z}_q\bar{H}$-modules. Thus $\bigcap_{i=1}^{\ell} \ker X_i = O_q(\bar{H}) = 1$ (see Huppert [24] V 5.17 and VI 7.20). Furthermore $k \otimes X_i$ breaks up into a direct sum of irreducible $k\bar{H}$-modules of equal dimension. Suppose this dimension is 1 for each $i = 1,\ldots,\ell$. It then follows by an elementary argument that $\bar{H}/\ker X_i$ is abelian for each i. But $\bigcap_{i=1}^{\ell} \ker X_i = 1$, hence by the subdirect product closure of abelian groups $\bar{H}$ is abelian, a contradiction. Therefore there exists an X_i such that $k \otimes X_i$ breaks up into a direct sum of irreducibles of dimension s. Put $X = X_i$.

Now $X.\bar{H} \in \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$.

However this is impossible, since the modules in $\Gamma_k(X.\bar{H})$ obtained from X, being of dimension s fail to satisfy condition Y1.

This completes the contradiction and finishes the proof of the theorem.

As an example of how this theorem can be used we show that $\underline{\underline{Y}}_q^{p}$ is non-saturated.

By Corollary 2.8 we have $\pi = \{p\}$ and $\underline{\underline{X}} = \underline{\underline{S}}_{p'}\underline{\underline{S}}_p$. So let r be a prime not equal to q and contained in p'. Then $\underline{\underline{S}}_r\underline{\underline{S}}_p \subseteq \underline{\underline{S}}_{p'}\underline{\underline{S}}_p = \underline{\underline{X}}$, and thus (3.7) shows $\underline{\underline{Y}}_q^{p}$ is non-saturated.

The crucial condition of Theorem 3.7, $\underline{\underline{S}}_r\underline{\underline{S}}_s \subseteq \underline{\underline{X}}$ is not as difficult to satisfy as might at first be thought.

We shall mention two general situations that will suffice :

(3.8) *If there exists primes* r,s *with* $r \in \pi'$, $r \neq q$ *and* $s \in \pi$ *such that there exists a non-nilpotent* $\underline{\underline{S}}_r\underline{\underline{S}}_s$*-group in* $\underline{\underline{X}}$, *then* $\underline{\underline{S}}_r\underline{\underline{S}}_s \subseteq \underline{\underline{X}}$.

(3.9) If $q \in \pi$ and $\underline{\underline{X}} \not\subseteq \underline{\underline{S}}_{\pi}\underline{\underline{S}}_{\pi'}$, *then there are primes* r, s *with* $r \in \pi'$, $r \neq q$, $s \in \pi$ *such that* $\underline{\underline{S}}_r\underline{\underline{S}}_s \subseteq \underline{\underline{X}}$.

(3.8) is established by Lemma 3.1 and (3.9) by Lemma 3.2 with $\sigma = \pi$.

By Theorem 3.7 $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ is non-saturated when either of (3.8) or (3.9) hold.

Unfortunately the condition $q \in \pi$ has to be inserted in (3.9). This guarantees that $r \neq q$.

§4. Elementary Properties of $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$

The results in this section explore the relationships between classes of the type $Y_q^{\pi}(\underline{\underline{X}})$ for varying π and $\underline{\underline{X}}$. The most important of these are obtained as a corollary to the main theorem which gives necessary and sufficient conditions for equality to hold between the classes $Y_q^{\pi}(\underline{\underline{X}})$ and $Y_q^{\sigma}(\underline{\underline{X}})$ for π, σ arbitrary sets of primes.

4.1 THEOREM *Let* π, σ *be arbitrary sets of primes. Then* $\underline{\underline{Y}}_q^{\sigma}(\underline{\underline{X}}) \subseteq \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ *if and only if* $\pi \cap \gamma(\underline{\underline{X}}) \subseteq \sigma \cap \gamma(\underline{\underline{X}})$.

Proof For convenience we drop all notation where ambiguity will not arise : thus $\gamma(\underline{\underline{X}}) = \gamma$, $\underline{\underline{Y}}_q^{\sigma}(\underline{\underline{X}}) = \underline{\underline{Y}}^{\sigma}$ and so on.

First, given $\gamma \cap \pi \subseteq \gamma \cap \sigma$, we show that $\underline{\underline{Y}}^{\sigma} \subseteq \underline{\underline{Y}}^{\pi}$.

Let $G \in \underline{\underline{Y}}^{\sigma}$. Take $V \in \Gamma_k(G)$, where by hypothesis

(1) dim V is a σ'-number,

(2) if V affords θ, $o(\det \theta)$ is a σ'-number,

and

(3) $G/\ker V \in \underline{\underline{X}}$.

Now because V is an absolutely irreducible faithful $k(G/\ker V)$-module it follows from Dade [33] that $\dim V \mid |G/\ker V|$. Thus $\dim V$ is a γ-number by (3). Let r be a prime divisor of $\dim V$, so $r \in \gamma$. Now from the data $\gamma' \cup \sigma' \subseteq \gamma' \cup \pi'$ and $r \in \sigma'$ (from (1)) thus $r \in \gamma' \cup \pi'$. But $r \in \gamma$ so $r \notin \gamma'$ hence $r \in \pi'$. Thus $\dim V$ is a π'-number, as desired.

Now let r be a prime which divides $o(\det \theta)$. It is obvious that $\ker V \subseteq \ker(\det \theta)$, thus $|G/\ker \det \theta|$ divides $|G/\ker V|$. But by (2.2) $o(\det \theta) = |G/\ker \det \theta|$, hence $o(\det \theta)$ is a γ-number and $r \in \gamma$. Then by the same argument as for the dimension of V it follows that $r \in \pi'$, so $o(\det \theta)$ is an π'-number.

By (3) it follows $G/\ker V \in \underline{\underline{X}}$.

Hence $G \in Y^\pi$ as desired.

Conversely, we have $\underline{\underline{Y}}^\sigma \subseteq \underline{\underline{Y}}^\pi$ and wish to show $\gamma \cap \pi \subseteq \gamma \cap \sigma$.

Suppose the contrary holds and let $r \in \gamma \cap \pi \setminus \gamma \cap \sigma$. Our aim will be to construct a $G \in \underline{\underline{Y}}^\sigma \setminus \underline{\underline{Y}}^\pi$, thus obtaining a contradiction.

Let R be an extraspecial group of order r^3 . Then by the remarks which follow Corollary 3.5 we have that there is an irreducible kR-module W such that $W.R \in \underline{\underline{Y}}^\sigma$. Moreover $W.R \notin \underline{\underline{Y}}^\pi$ for by Lemma 3.3 (1) we have that all modules in $\Gamma_k(W.R)$ are of dimension r , and by hypothesis we have $r \notin \pi'$.

Thus $W.R \in \underline{\underline{Y}}^\sigma \setminus \underline{\underline{Y}}^\pi$, a contradiction of the hypothesis.

Therefore $\gamma \cap \pi \subseteq \gamma \cap \sigma$, as desired.

4.2 <u>COROLLARY</u> *For terms as in Theorem 4.1* $\underline{\underline{Y}}_q^\sigma(\underline{\underline{X}}) = Y_q^\pi(\underline{\underline{X}})$ *if and only if* $\gamma(\underline{\underline{X}}) \cap \pi = \gamma(\underline{\underline{X}}) \cap \sigma$.

4.3 <u>COROLLARY</u> *For terms as in Theorem 4.1* $\underline{\underline{Y}}_q^{\sigma}(\underline{\underline{X}}) \subset \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ *if and only if* $\gamma(\underline{\underline{X}}) \cap \pi \subset \gamma(\underline{\underline{X}}) \cap \sigma$.

<u>Proof</u> Assuming $\underline{\underline{Y}}^{\sigma} \subset \underline{\underline{Y}}^{\pi}$ we show $\gamma \cap \pi \subset \gamma \cap \sigma$. By Theorem 4.1 we have $\gamma \cap \pi \subseteq \gamma \cap \sigma$, where it suffices to show $\gamma \cap \pi \neq \gamma \cap \sigma$. Suppose the contrary. The Corollary 4.2 applies giving $\underline{\underline{Y}}^{\sigma} = \underline{\underline{Y}}^{\pi}$, a contradiction. The converse is proved in a similar way.

4.4 <u>COROLLARY</u> *If* $\pi \cap \gamma(\underline{\underline{X}}) \subseteq \sigma \cap \gamma(\underline{\underline{X}})$ *and* $\underline{\underline{X}} \subseteq \underline{\underline{Y}}$ *then* $\underline{\underline{Y}}^{\sigma}(\underline{\underline{X}}) \subseteq \underline{\underline{Y}}^{\pi}(\underline{\underline{Y}})$ *whenever these terms are defined.*

4.5 <u>THEOREM</u> *For any set of primes* π *for which* $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ *is defined we have*

$$\underline{\underline{S}}_{q'}\underline{\underline{S}}_q(\underline{\underline{S}}_{\pi'} \cap \underline{\underline{X}}) \subseteq \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}}) \subseteq \underline{\underline{S}}_{q'}\underline{\underline{S}}_q\underline{\underline{X}} .$$

and

$$\underline{\underline{Y}}_q(\underline{\underline{X}}) = \begin{cases} \underline{\underline{S}}_{q'}\underline{\underline{S}}_q & \text{for } \gamma(\underline{\underline{X}}) \subseteq \pi \\ \underline{\underline{S}}_{q'}\underline{\underline{S}}_q\underline{\underline{X}} & \text{for } \gamma(\underline{\underline{X}}) \subseteq \pi' \end{cases}$$

<u>Proof</u> Let $V \in \Gamma_k(G)$, then it is easy to see that $G \in \underline{\underline{S}}_{q'}\underline{\underline{S}}_q\underline{\underline{X}}$ if and only if $G/\ker V \in \underline{\underline{X}}$.

First suppose $G \in \underline{\underline{S}}_{q'}\underline{\underline{S}}_q(\underline{\underline{S}}_{\pi'} \cap \underline{\underline{X}})$. Now for each $V \in \Gamma_k(G)$, Y1 follows by Dade [33], Y2 by (2.2) and Y3 by our assertion above. Thus $G \in \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$.

That $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}}) \subseteq \underline{\underline{S}}_{q'}\underline{\underline{S}}_q\underline{\underline{X}}$ is immediate from the converse of the above.

If $\gamma(\underline{\underline{X}}) \subseteq \pi$, take $G \in \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$. Thus $\Gamma_k(G)$ contains the trivial kG-module alone since $G/\ker V \in \underline{\underline{S}}_{\pi}$. Hence $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}}) = \underline{\underline{S}}_{q'}\underline{\underline{S}}_q$.

If $\gamma(\underline{\underline{X}}) \subseteq \pi'$, take $G \in \underline{\underline{S}}_{q'}\underline{\underline{S}}_q\underline{\underline{X}}$. Then $G \in \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ since $G/\ker V \in \underline{\underline{S}}_{\pi'}$. Hence $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}}) = \underline{\underline{S}}_{q'}\underline{\underline{S}}_q\underline{\underline{X}}$, as required.

The intersection property is straight forward.

4.6 THEOREM *If* $\underline{\underline{X}}$ *and* $\underline{\underline{Y}}$ *are Fitting formations and* π,σ *are sets of primes then*

$$\underline{\underline{Y}}_q^{\sigma}(\underline{\underline{X}}) \cap \underline{\underline{Y}}_q^{\pi}(\underline{\underline{Y}}) = Y_q^{\sigma\cup\pi}(\underline{\underline{X}}\cap\underline{\underline{Y}}) .$$

Proof Let $G \in \underline{\underline{Y}}^{\sigma}(\underline{\underline{X}}) \cap \underline{\underline{Y}}^{\pi}(\underline{\underline{Y}})$.

Then, for all $V \in \Gamma_k(G)$ if r is a prime dividing $\dim V$ then $r \in \sigma' \cap \pi' = (\sigma\cup\pi)'$, and so $\dim V$ is a $(\sigma\cup\pi)'$-number.

Similarly $o(\det\theta)$ is a $(\sigma\cup\pi)'$-number.

Also $G/\ker V \in \underline{\underline{X}} \cap \underline{\underline{Y}}$.

Moreover

$$\underline{\underline{X}} \cap \underline{\underline{Y}} \subseteq \underline{\underline{S}}_{\sigma_0'}\underline{\underline{S}}_{\sigma_0} \cap \underline{\underline{S}}_{\pi_0'}\underline{\underline{S}}_{\pi_0}$$

$$\subseteq S_{(\sigma_0\cup\pi_0)'}S_{\sigma_0\cup\pi_0} .$$

Hence

$$\underline{\underline{Y}}^{\sigma}(\underline{\underline{X}}) \cap \underline{\underline{Y}}^{\pi}(\underline{\underline{Y}}) \subseteq \underline{\underline{Y}}^{\sigma\cup\pi}(\underline{\underline{X}}\cap\underline{\underline{Y}}) .$$

The converse is established by a similar elementary argument.

A trivial example of Theorem 4.6 is that

$$\underline{\underline{Y}}_q^{\pi}\left(\bigcap_{r\in\pi_0} \underline{\underline{S}}_{r'}\underline{\underline{S}}_r\right) = \bigcap_{r\in\pi_0} \underline{\underline{Y}}_q^{r}(\underline{\underline{S}}_{r'}\underline{\underline{S}}_r) .$$

What this shows is that when $\underline{\underline{X}} = \bigcap_{r\in\pi_0} \underline{\underline{S}}_{r'}\underline{\underline{S}}_r$, the class $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ is the intersection of the Berger-Cossey classes $\underline{\underline{Y}}_q^{r}$, $r \in \pi \setminus \{q\}$.

CHAPTER 5

MORE NON-SATURATED FITTING FORMATIONS

§1. Introduction

In this Chapter we give new examples of non-saturated Fitting formations.

The construction takes place in §2 where we define $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$, a family of Fitting formations. Once again q is the characteristic of an underlying field, π is a set of primes and $\underline{\underline{X}}$ a Fitting formation subject to a co-prime condition dependent on π . We proceed by the module method of Chapter 3.

In §3 we give necessary (though not necessary and sufficient) conditions on the choice of $\underline{\underline{X}}$ under which non-saturation occurs. These conditions resemble those we found for non-saturation in the case $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$.

§2. The Construction of $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$

Before we discuss any details of the construction the following lemma, which sorts out the consequences of basic co-prime relationships, needs to be stated.

2.1 LEMMA *Let* π *be a set of primes and* K *a group in* $\underline{\underline{S}}_{\pi}\underline{\underline{S}}_{\pi'}$. *Then* $O_{\pi}(K) = O^{\pi'}(K)$.

Moreover if M *is any* K*-module then*

$$O_{\pi}(K/\ker M) = O_{\pi}(K)\ker M/\ker M$$

and so any $O_{\pi}(K)$*-module may also be considered to be an* $O_{\pi}(K/\ker M)$*-module.*

<u>Proof</u> We have $K \in \underline{\underline{S}}_{\pi}\underline{\underline{S}}_{\pi'}$. Now

$$O_{\pi}(K)O^{\pi'}(K)/O_{\pi'}(K) \cong O_{\pi}(K)/O_{\pi}(K) \cap O^{\pi'}(K)$$

$$\in \underline{\underline{S}}_{\pi}$$

But this is impossible unless $O^{\pi'}(K) \supseteq O_{\pi}(K)$.

However $K \in \underline{\underline{S}}_{\pi}\underline{\underline{S}}_{\pi'}$ says $O^{\pi'}(K) \in \underline{\underline{S}}_{\pi}$, and so $O^{\pi'}(K) \subseteq O_{\pi}(K)$.

Thus $O^{\pi'}(K) = O_{\pi}(K)$.

It follows that

$$\begin{aligned} O_{\pi}(K/\ker M) &= O^{\pi'}(K/\ker M) \\ &= O^{\pi'}(K) \ker M/\ker M \\ &= O_{\pi}(K) \ker M/\ker M \end{aligned}$$

In particular the action of $O_{\pi}(K)$ on M is just the action of $O_{\pi}(K/\ker M)$ on M .

This completes the proof.

We now state the main theorem :

2.2 <u>THEOREM</u> *Let* π *be a set of primes and* k *an algebraically closed field of characteristic* q *. Let* $\underline{\underline{X}}$ *be a Fitting formation such that* $\underline{\underline{X}} \subseteq \underline{\underline{S}}_{\pi}\underline{\underline{S}}_{\pi'}$ *. For each* G *define* $H_q^{\pi}(G)$ *to be the class of irreducible* kG*-modules* V *satisfying :*

H1 $V_{O_{\pi}(G/\ker V)}$ *is homogeneous.*

H2 $G/\ker V \in \underline{\underline{X}}$.

Then the class

$$\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}}) = \{G \mid \Gamma_k(G) \subseteq H_q^{\pi}(G)\}$$

is a Fitting formation.

By Theorem 3.2.1 it is enough to show that the module M1,...,M5 apply for $H_q^{\pi}(G)$. Application of Theorem 3.2.1 then implies $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$ is a Fitting formation as we require.

Just as in the example of Chapter 4, $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$, the conditions M1, M2, M3 are easily checked and only use the condition that G/ker V is contained in a class $\underline{\underline{X}}$ without regard to any of the finer structure available. The cases M4 and M5 (which show that $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$ is a Fitting class) are the difficult situations. It is here the condition $\underline{\underline{X}} \subseteq \underline{\underline{S}}_{\pi}\underline{\underline{S}}_{\pi'}$ is indispensible. The principal reason for this is that crucial parts of the argument use Isaac's lifting theorem for invariant irreducible representations of normal Hall subgroups. Unlike the role the same condition Y3 plays in $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$, H2 is used extensively in establishing this result.

It is worthwhile to draw attention to an often repeated situation in the proof and simultaneously show why lifting results are so significant. We have an irreducible kG-module V say and want to find out something about it. All we know is that V_N is homogeneous for N normal in G. The way we proceed is to use Theorem 2.4.2 and write V as a tensor product of two irreducible projective representations. At this point the lifting property becomes crucial. Let U be an irreducible component of V_N. Then if U lifts to G, Theorem 2.4.2 tells us that V can be written as the tensor product of two ordinary irreducible kG-modules : U and some other T say, with N in the kernel. Thus

$$V = U \otimes T .$$

Proof Instead of $H_q^{\pi}(G)$ we simply write $H(G)$.

M1 is satisfied trivially.

For M2 let $V \in H(G)$ and $N \subseteq \ker_G V$. We must show $V \in H(G/N)$. Put $\bar{G} = G/N$.

Now $\ker_{\bar{G}} V = \ker V/N$, so $\bar{G}/\ker_{\bar{G}} V \cong G/\ker V$ thus it follows that $V \in H(\bar{G})$.

For M3 let $V \in H(N)$ and let $\phi : G \to N$ be an epimorphism. We must show $V \in H(G)$.

Now it is easy to see that $\psi : g \ker \phi \to \phi(g) \ker_N V$ defines an epimorphism

$$\psi : G/\ker \phi \to N/\ker_N V ,$$

which has $\ker_G V/\ker \phi$ as kernel.

Thus $G/\ker_G V \cong N/\ker_N V$, hence $V_{O_\pi(G/\ker_G V)}$ is homogeneous and $G/\ker_G V \in \underline{\underline{X}}$.

Therefore $V \in H(G)$, as we require.

To show M4 is satisfied let N be a maximal normal subgroup of G and $V \in H(G)$. Take U to be an irreducible component of V_N , then we must show $U \in H(N)$, that is $U_{O_\pi(N/\ker U)}$ is homogeneous.

If $\ker V \not\subseteq N$ then by maximality of N , $G = N \ker V$, and so U is a G-module. Thus $U = V$.

This has two consequences : firstly $U \in H(G)$ so that $U_{O_\pi(G/\ker U)}$ is homogeneous; secondly, $\ker_N U = \ker V \cap N$ so that $O_\pi(G/\ker U) \cong O_\pi(N/\ker_N(U))$.

Combining these it follows that $U_{O_\pi(N/\ker_N U)}$ is homogeneous, and moreover $N/\ker U \in \underline{\underline{X}}$.

Hence $U \in H(N)$ as required.

Therefore consider the case $\ker V \subseteq N$. We will work modulo $\ker V$. Indeed, by M2, we may as well assume that $\ker V = 1$, that is $G \in \underline{\underline{S}}_\pi \underline{\underline{S}}_{\pi'}$.

That U satisfies H2 is now immediate since $N/\ker U$ is an element of $QS_n\underline{\underline{X}}$.

What remains - H1 - is to show that $U_{O_\pi(N/\ker_N U)}$ is homogeneous. By lemma 2.1 this is equivalent to showing $U_{O_\pi(N)}$ is homogeneous on the assumption $V_{O_\pi(G)}$ is homogeneous. For convenience put $Q = O_\pi(G)$ and $Q_2 = O_\pi(N)$. By Gaschütz [14] X.3d we have $Q_2 = N \cap Q$.

Two situations arise.

(1) $|G:N| \in \pi'$

and

(2) $|G:N| = p$, for $p \in \pi$

In situation (1), because $Q = O_\pi(G) = O^{\pi'}(G)$ by Lemma 2.1 it follows $Q \subseteq N$. Moreover $Q_2 = N \cap Q = Q$. Thus if U is any irreducible component of V_N then because $V_Q = (V_N)_{Q_2}$ is homogeneous so is U_Q .

Therefore U_{Q_2} is homogeneous, as required.

This dispenses with situation (1).

In situation (2), $|G:N| = p$.

The first thing we do is build up our knowledge of how V_N breaks up.

Let W be an irreducible component of V_Q. Then from Clifford's Theorem we have that W is invariant in G. Moreover Q is a normal Hall subgroup in G. Hence by Theorem 2.4.4(1) W extends to $\bar{W}$, an irreducible kG-module with $\bar{W}_Q = W$.

Now using this extendibility we apply Theorem 2.4.2 to write $V = \bar{W} \otimes T$ where T is some irreducible kG-module with Q in the kernel. Thus

$$V_N = \bar{W}_N \otimes T_N .$$

Examining these new terms we observe that T_N is irreducible. This is because any submodule of T_N must also be a module for Q and so a module for $G = NQ$ contradicting the irreducibility of T.

As for $\bar{W}_N$ there are only two alternatives. Either $\bar{W}_N$ is homogeneous or else it breaks up into $|G:N| = p$ non-isomorphic irreducibles.

In the homogeneous case let X be the irreducible constituent and put $\bar{W}_N \cong (X\oplus\ldots\oplus X)_n$ (i.e. X appears with multiplicity n in $\bar{W}_N$). Then we have

$$\begin{aligned}\bar{W}_{Q_2} &= (\bar{W}_N)_{Q_2} \\ &\cong \left(X_{Q_2}\oplus\ldots\oplus X_{Q_2}\right)_n . \qquad (\alpha)\end{aligned}$$

Moreover since $|N:Q_2|$ is a π'-number we have by Clifford's Theorem that the number of homogeneous components in (α) is a π'-number. But again by Clifford's Theorem since $|Q:Q_2| = p$, W_{Q_2} must either be homogeneous or the direct sum of p non-isomorphic modules.

We assume first the latter, namely,

$$W_{Q_2} = W_1 \oplus \ldots \oplus W_p$$

where W_i is an irreducible kQ_2-module for $i = 1,\ldots,p$. Since $\bar{W}_Q = W$ we have the immediate consequence

$$\bar{W}_{Q_2} = W_1 \oplus \ldots \oplus W_p \tag{β}$$

Now comparing the two decompositions of $\bar{W}_Q$ found in (α) and (β) we obtain a contradiction : (α) shows $\bar{W}_{Q_2}$ has π' homogeneous components whereas (β) shows $\bar{W}_{Q_2}$ has precisely p and $p \in \pi$.

Thus W_{Q_2} must be homogeneous.

But then $V_{Q_2} = (V_Q)_{Q_2}$ must also be homogeneous, hence U_{Q_2} is homogeneous and thus M4 is satisfied in this case.

Therefore we assume that $\bar{W}_N$ is not homogeneous. So it must break up into a set of p non-isomorphic irreducible kN-modules S_i, $i = 1,\ldots,p$, with

$$\bar{W}_N = S_1 \oplus \ldots \oplus S_p .$$

Combining this decomposition with the fact $V_N = \bar{W}_N \otimes T_N$ we get

$$V_N = S_1 \otimes T_N \oplus \ldots \oplus S_p \otimes T_N .$$

Thus U, an irreducible component of V_N, is an irreducible component of $S_\ell \otimes T_N$ for some $\ell \in \{1,\ldots,p\}$. Moreover we have

$$\bar{W}_{Q_2} = S_{1Q_2} \oplus \ldots \oplus S_{pQ_2} ,$$

and comparing this with (β) it follows that $\dim W_j = \dim S_{iQ_2}$ for all i,j .

Hence by the irreducibility of W_i we have that S_{iQ_2} is irreducible for each $i = 1,\ldots,p$. This implies

$$(S_\ell \otimes T_N)_{Q_2} = S_{\ell Q_2} \otimes T_{Q_2}$$

$$(S_{\ell Q_2} + \ldots + S_{\ell Q_2}) \dim T$$

is homogeneous.

Therefore U_{Q_2} is homogeneous, as we require.

So M4 is satisfied.

For M5 let N_i be a maximal normal subgroup of G for $i = 1,2$ and let $G = N_1 N_2$. Further assume V is an irreducible kG-module such that all irreducible components of V_{N_i} are in $H(N_i)$ for each $i = 1,2$. Then we must show $V \in H(G)$, that is that $V_{O_\pi(G/\ker V)}$ is homogeneous.

Firstly we show that V satisfied H2, viz $G/\ker V \in \underline{\underline{X}}$.

Consider V_{N_j} and let U_{ij} be an irreducible component for $i = 1,\ldots,t_j$, $j = 1,2$. Then by hypothesis $N_j/\ker U_{ij} \in \underline{\underline{X}}$ for each i . Thus by subdirect product closure of $\underline{\underline{X}}$, $N_j/\ker_{N_j} V \in \underline{\underline{X}}$.

So $N_j \ker V/\ker V \in \underline{\underline{X}}$.

Thus by normal product closure of $\underline{\underline{X}}$, $G/\ker V \in \underline{\underline{X}}$.

Hence V satisfies H2.

Suppose $\ker V \not\subseteq N_i$ for $i = 1$ or $i = 2$, say $i = 1$. Then by the maximality of N_1, $G = N_1 \ker V$. Let U be an irreducible component of V_{N_1}. Then U an N_1-module is also of course a module for $\ker V$. But $G = N_1 \ker V$ thus U is an kG-module and so $V_{N_1} = U$.

Thus $\ker U = \ker V \cap N_1 := \ker_{N_1} V$ and $V_{N_1} \in H(N_1)$.

Hence $G/\ker V \cong N_1/\ker U$ and this implies $V_{O_\pi(G/\ker V)}$ is homogeneous, so H1 holds.

Therefore consider the case $\ker V \subseteq N_i$, $i = 1,2$. Just as in the proof of M4 we now work modulo $\ker V$ so that by M2, we may assume $\ker V = 1$.

As before put $Q = O_\pi(G)$.

Two situations arise :

(1) $|G:N_i| \in \pi'$ for $i = 1,2$

and

(2) $|G:N_i| = p$ say, $p \in \pi$ for either $i = 1$ or 2.

In the first situation assume $|G:N_1| = t_1$ and $|G:N_2| = t_2$ for t_1, t_2 primes in π'. Since $O^{\pi'}(G) = O_\pi(G) := Q$ it is clear that $Q \subseteq N_i$; $i = 1,2$ and thus

$$O_\pi(N_i) = N_i \cap Q$$
$$= Q.$$

We must show V_Q is homogeneous.

If either V_{N_1} or V_{N_2} is homogeneous then it follows immediately that $V_Q = (V_{N_i})_Q$ is homogeneous where $i = 1$ or 2. Therefore assume the contrary. Thus by Clifford's Theorem we may write

$$V_{N_i} = V_{i1} \oplus \ldots \oplus V_{it_i}$$

for $i = 1,2$, where each V_{ij} is an irreducible kN_i-module.

By hypothesis $V_{ij} \in H(N_i)$ and so V_{ijQ} is homogeneous and a homogeneous component of V_Q for each i,j. But this implies $(V_{N_1})_Q$ must have t_1 homogeneous components whilst $(V_{N_2})_Q$ has t_2 homogeneous components hence $t_1 = t_2$ and each V_{1k} equals a $V_{2\ell}$ for some ℓ.

In particular this means that V_{11}, an irreducible N_1-module is also a N_2-module. But then since $G = N_1N_2$ this implies V_{11} is a kG-module of V.

Hence $V_{11} = V$.

Thus $V_Q = V_{11Q}$ is homogeneous, as required.

In case (2) put $|G:N_2| = p$, $p \in \pi$.

Then by Clifford's Theorem there are only two possibilities to consider : either V_{N_2} is homogeneous or else it breaks up into p non-isomorphic irreducibles. Put $Q_2 := O_\pi(N_2)$.

When V_{N_2} is homogeneous we have by Theorem 2.4.3 that V_{N_2} is irreducible. Thus by hypothesis $V_{N_2} \in H(N_2)$ and so by lemma 2.1 V_{Q_2} is homogeneous.

We must show V_Q is homogeneous.

The way we do this is to use the homogeneity of V_{Q_2} in order to apply Theorem 2.4.2 and by this determine a decomposition for V_Q.

Let W be an irreducible component of V_{Q_2}.

Now in order to use the full force of Theorem 2.4.2 it is necessary to show that W may be extended to an irreducible kG-module, $\bar{W}$ say, such that $\bar{W}_{Q_2} = W$. We use Theorem 2.4.5 to show this extension exists. The conditions we need are that if L/Q_2 is any Sylow subgroup of G/Q_2 then W extends to L. We show that this holds by examining as separate cases the Sylow p-subgroups and the Sylow p'-subgroups.

Since $Q = O_{\pi}(G) = O^{\pi'}(G)$, Q/Q_2 is a Sylow p-subgroup of G/Q_2. Now let E be the irreducible of V_Q which has W as an irreducible component of E_{Q_2}. But V_{Q_2} homogeneous implies that E_{Q_2} is homogeneous and thus by Theorem 2.4.3 $E_{Q_2} = W$. Hence W extends to Q.

Now let R/Q_2 be a Sylow r-subgroup of G/Q_2 for $r \neq p$. Then $r \in \pi'$ and it follows $R \leq N_2$. Moreover Q_2 is a normal Hall subgroup of N_2 and so Theorem 2.4.4 (1) applies. This shows that there is an irreducible kN_2-module, F say, such that $F_{Q_2} = W$. It follows that F_R is an irreducible kR-module and $(F_R)_{Q_2} = W$, hence W extends to R.

Therefore Theorem 2.4.5 applies as we have foreshadowed and there exists an irreducible kG-module, $\bar{W}$ such that $\bar{W}_{Q_2} = W$.

Now consider V and apply Theorem 2.4.2 in the familiar way to write $V = \bar{W} \otimes T$ where T is some irreducible kG-module with Q_2 in the kernel.

Thus

$$V_Q = \bar{W}_Q \otimes T_Q .$$

Now we know that $\bar{W}_Q$ is irreducible. Moreover T_Q is homogeneous. For by Clifford's Theorem the irreducible components are conjugates of one another by elements of $G \setminus Q$. Since G/Q_2 is the direct product of Q/Q_2 and N/Q_2 there is no effect to conjugation.

But since $|Q:Q_2| = p$ it must break up into a direct sum of isomorphic one dimensionals irreducible modules. Applying Theorem 2.4.2 again it follows that $V_Q = \bar{W}_Q \otimes T_Q$ must be homogeneous, as we desire.

This completes the argument for the case when V_{N_2} is homogeneous.

Now suppose V_{N_2} breaks up into p non-isomorphic irreducibles V_i , $i = 1,\ldots,p$, so that we may write

$$V_{N_2} = V_1 \oplus \ldots \oplus V_p .$$

By hypothesis $V_i \in H(N_2)$ for each i . This says V_{iQ_2} is homogeneous for each i . Now consider V_1 and let W be an irreducible component of V_{1Q_2} . Thus

$$V_{1Q} \cong (W\oplus\ldots\oplus W)_n .$$

By Clifford's Theorem $V = V_1^G$ and then by Mackey's theorem $(V_1^G)_Q = (V_{1Q_2})^Q$. Hence

$$V_Q = (V_{1Q_2})^Q$$

$$\cong (W^Q\oplus\ldots\oplus W^Q)_n$$

Thus V_Q is homogeneous if W^Q is irreducible.

We shall show that W^Q is irreducible.

What we do know about W is that it is either invariant in Q or it is not.

Let us suppose W is Q-invariant; our aim will be to use the homogeneity of V_{iQ_2} to arrive at a contradiction.

Because Q_2 is a normal Hall subgroup of N_2 and W is invariant in N_2 by Theorem 2.4.4 (1) we have that W extends to an irreducible kN_2-module $\bar{W}$, such that $\bar{W}_{Q_2} = W$. So using Theorem 2.4.2 in the familiar way we may write

$$V_1 = \bar{W} \otimes X$$

for X a certain kN_2-module with Q_2 in the kernel.

Now we know from Clifford's theorem that there exists a set of elements in G, $x_1 = 1,\ldots,x_p$ such that $V_i = V_1^{x_i}$, $i = 1,\ldots,p$. Moreover because G/Q_2 is the direct product of Q/Q_2 and N_2/Q_2 it is possible to choose these x_i such that they are all in Q.

It follows from this that $X^{x_i} = X$, for each $i = 1,\ldots,p$. Thus

$$\begin{aligned} V_1^{x_i} &= (\bar{W}\otimes X)^{x_i} \\ &= \bar{W}^{x_i} \otimes X^{x_i} \\ &= \bar{W}^{x_i} \otimes X . \end{aligned}$$

Now using the assumption that W is invariant in Q we have that $W^{x_i} \cong W$. Thus

$$\bar{W}_{Q_2}^{x_i} = W^{x_i}$$

$$\cong W \ .$$

Hence, considering $\bar{W}^{x_i}$, Theorem 2.4.2 applies again so that we may write

$$\bar{W}^{x_i} \cong \bar{W} \otimes T$$

for some T an irreducible kN_2-module with Q_2 in the kernel. Moreover by iteration it follows that

$$\bar{W}^{x_i^s} = \bar{W} \otimes T^s$$

for $s \in \mathbb{N}$.

However we know that x_i , $i \neq 1$, has order a π-number, say m. Thus $\bar{W} \cong \bar{W} \otimes T^m$. This implies $T^m = 1_{N_2}$.

But T is a linear representation of N_2/Q_2 a π'-group and so there is an $n \in \mathbb{N}$, a π'-number such that $T^n = 1_{N_2}$.

Finally, $(m,n) = 1$ and so T must be trivial. But this shows $\bar{W}^{x_i} \cong \bar{W}$ and thus $V_i \cong V_1$, a contradiction.

Therefore W cannot be Q-invariant.

Let U be a proper irreducible submodule of W^Q .

Now look at U_{Q_2} . It breaks up into a direct sum of irreducibles. But each one of these irreducibles is an irreducible constituent of $(W^Q)_{Q_2}$ and thus of V_{Q_2} and so also of $V_{1Q}^{x_i}$ for some i , $i = 1,\ldots,p$ and so is a Q-conjugate of W .

But W is not Q-invariant and this means that U_{Q_2} cannot be homogeneous.

Thus by Clifford's theorem U_{Q_2} breaks up into p non-isomorphic irreducibles, say U_i, $i = 1,\ldots,p$.

But then

$$\dim(U) = p \dim U_i$$

$$= p \dim W$$

$$= \dim(W^Q)$$

and so W^Q must be irreducible.

Thus V_Q is homogeneous.

This completes the proof of M5.

The theorem is now proved.

§3. $H_q^{\pi}(\underline{\underline{X}})$ and Non-saturation

In this section the objective is to describe conditions on the choice of $\underline{\underline{X}}$ which relate to the set of primes π and guarantee that $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$ is in fact a non-saturated Fitting formation.

As a preliminary to these results we record the following lemma, a counterpart to Corollary 4.3.5, which supplies a range of useful examples of groups in $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$ and contained in $\underline{\underline{S}}_q, \underline{\underline{S}}_{q'}$.

3.1 LEMMA *Let* π *be a set of primes and* $\underline{\underline{X}}$ *a Fitting formation such that* $\underline{\underline{X}} \subseteq \underline{\underline{S}}_{\pi}\underline{\underline{S}}_{\pi'}$. *Suppose* r,s *are primes with* $r \in \pi$ *and* $s \in \pi'$ *and let* H *and* H_0 *be groups satisfying the same conditions as in Lemma 4.3.3.*

Further suppose $H \in \underline{\underline{X}}$.

Then $H_0 \in \underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$.

<u>Proof</u> What we need to show is that if $E \in \Gamma_k(H_0)$ then $E_{O_\pi(H_0/\ker E)}$ is homogeneous. Now by Lemma 4.3.3

$$H_0/\ker E \cong H$$

$$\in \underline{\underline{X}}$$

so H2 is satisfied. Moreover $O_\pi(H) = F(H)$.

Now by Lemma 4.3.3 (1) we know dim E is an r-number. Suppose $E_{F(H)}$ is not homogeneous. Then by Clifford's theorem the number of homogeneous components must be an s-number.

But this implies $s \mid \dim E$, a contradiction, so $E_{F(H)}$ is homogeneous.

Therefore $H_0 \in \underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$, as asserted.

3.2 <u>COROLLARY</u> *Let* $r \in \gamma(\underline{\underline{X}}) \cap \pi$. *Take* R *to be an extraspecial group of order* r^{2m+1} *so that by Lemma 2.2.2,* $R \in \underline{\underline{X}}$. *Then there exists an irreducible* $\mathbb{Z}_q R$*-module,* W, *such that*

$$W.R \in \underline{\underline{H}}_q^{\pi}(\underline{\underline{X}}) .$$

Corollary 3.2 gives simple examples of groups in $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$ not in $\underline{\underline{S}}_q$, $\underline{\underline{S}}_q$.

The main non-saturation theorem follows.

3.3 THEOREM *Let* π *be a set of primes and put* $\pi_0 = \pi \setminus \{q\}$. *Let* $\underline{\underline{X}}$ *be any Fitting formation contained in* $\underline{\underline{S}}_{\pi}\underline{\underline{S}}_{\pi'}$.

If there exists primes r, s *such that* $r \in \pi_0$, $s \in \pi'$ *and* $\underline{\underline{S}}_r\underline{\underline{S}}_s \subseteq \underline{\underline{X}}$, *then* $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$ *is non-saturated.*

This condition is basically the same as the one for non-saturation in the case $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$. We refer the reader to the remarks we made following the statement and proof of Theorem 4.3.7 as they also apply here. Just as in those remarks, as a special case of this theorem, it follows that the class $\underline{\underline{H}}_q^{p}(\underline{\underline{S}}_p\underline{\underline{S}}_{p'})$ is non-saturated.

The proof of Theorem 3.3 follows the same pattern as the proof of Theorem 4.3.7.

<u>Proof of Theorem 3.3</u> Taking the primes r, s as in this theorem we use Lemma 4.3.6 to construct a group $H = B.S$ (where $F(H) = B$ is an extraspecial group of order r^{2m+1} and S is a cyclic group of order s) with the property that H satisfies the properties (1), (2) and (3) of Lemma 4.3.6.

Now lemma 3.1 applies, so that $H_0 \in \underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$.

We now assume that $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$ is saturated, and so the characterisation of saturated formations, Theorem 2.2.5, must apply. Using the notation of this theorem we have that $H \in \underline{\underline{X}}(q)$ and thus $H/Z(H) \in \underline{\underline{X}}(q)$. Put $\bar{H} = H/Z(H)$.

Just as in the proof of Theorem 4.3.7 we can now choose X to be a faithful irreducible $\mathbb{Z}_q\bar{H}$-module such that if U is any irreducible component of $k \otimes X$ then $\dim U \neq 1$.

But by Theorem 2.2.5 again $X.\bar{H} \in \underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$, thus $U_{O_\pi(\bar{H})}$ is homogeneous. But $|\bar{H}:O_\pi(\bar{H})| = s$, thus by Theorem 2.4.3 $U_{O_\pi(\bar{H})}$ is irreducible. However since B is extraspecial $O_\pi(\bar{H})$ is abelian and therefore $\dim U = 1$.

But this contradicts our assumption $\dim U \neq 1$.

Hence $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$ cannot be saturated.

This completes the proof of 3.3.

In the special case when $q \notin \pi$ Theorem 3.3 can be reexpressed in a more practically useful way.

3.4 <u>THEOREM</u> *Let* π *be a set of primes* $q \notin \pi$. *If* $\underline{\underline{X}} \not\subseteq \underline{\underline{S}}_\pi \cup \underline{\underline{S}}_{\pi'}$, *then* $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$ *is non-saturated.*

<u>Proof</u> Since $\underline{\underline{X}} \subseteq \underline{\underline{S}}_\pi \underline{\underline{S}}_{\pi'}$, the stated condition is equivalent by definition to $\underline{\underline{X}} \notin \underline{\underline{S}}_{\pi'} \underline{\underline{S}}_\pi$.

Now we use Lemma 4.3.2 to find primes r, s , $r \in \pi$, $s \in \pi'$, and use these in exactly the same was as in Theorem 3.3.

From this we conclude $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$ is non-saturated.

CHAPTER 6

MORE ON CONSTRUCTIONS BY CHIEF FACTORS

§1. Introduction

In Chapter 3 we defined a method for constructing Fitting formations which depended on the representation properties of chief factors. In Chapters 4 and 5 we used this method to construct interesting examples.

In this chapter we try to show that all Fitting formations can be characterised by such a method.

In §2 we start by defining a closure operation P_π on F, the class of all Fitting formations (where π is an arbitrary set of primes, as usual). The definition of P_π involves the construction technique of Chapter 3. An important role will be given to certain naturally occurring primitive groups. The section culminates with a theorem which shows that the case $\pi = \mathbb{P}$ is essential to understanding the general case $\pi \subset \mathbb{P}$. We put $P_{\mathbb{P}} = P$.

In §3 we conjecture that all Fitting formations are P-closed. If the conjecture is true then this provides a characterisation of an arbitrary Fitting formation. We show that all saturated examples are P-closed and that all known non-saturated cases (those of Chapters 4 and 5) are as well. The general case however seems a difficult problem.

In §4 we close by considering the behaviour of P_π (for an arbitrary π) under products of classes. The problems here too seem considerable. For instance we are unable to show that P preserves products of classes in the general case, although this would be consistent with the truth of the conjecture in §3.

§2. The closure operation P_π on F

The objective of this section is to define a closure operation P_π on the class of Fitting formations, F, where π is an arbitrary set of primes. The essential features of P_π are determined by the case of a single prime, q say.

As indicated in the introduction the construction of $P_q(\underline{\underline{X}})$ where $\underline{\underline{X}}$ is any Fitting formation will be by examination of chief factors using the module technique of Chapter 3. An important role will be given to certain naturally occurring primitive groups.

More precisely, if V is any G-module for a group G then we have from Definition 2.3.1 that

$$P(V,G) = V.G/\ker V .$$

Moreover, if V is any chief factor of G then Lemma 2.3.2 shows that P(V,G) is a primitive group, with V the unique minimal normal subgroup of P(V,G) .

Now, expressed as simply as possible, $P_q(\underline{\underline{X}})$ will be the class of groups G which have P(W,G) contained in $\underline{\underline{X}}$ for each q-chief factor W of G .

Before proceeding with the details of the construction we note the following elementary results.

Let W be any G-module for a group G . Then

- If N is normal in G and also contained in ker W then $P(W,G/N) \cong P(W,G)$

- W can be thought of as a module for $P(W,G)$ on which W acts trivially and so considered we have $P(W,P(W,G)) \cong P(W,G)$.

Moreover if $\underline{\underline{X}}$ is any Fitting formation and W is a chief factor of G then recalling Lemma 2.3.5 we have

$$P(W,G) \in \underline{\underline{X}} .$$

We shall use the foregoing facts without explicit reference.

We now construct the Fitting formation $P_q(\underline{\underline{X}})$.

Note that $\Gamma_{\mathbb{Z}_q}(G)$ is just the class of q-chief factors of G considered in the natural way as irreducible $\mathbb{Z}_q G$-modules.

2.1 <u>THEOREM</u> *Let* $\underline{\underline{X}}$ *be a Fitting formation and let* q *be any prime. For each group* G *let* $P_q(G)$ *be the class of irreducible* $\mathbb{Z}_q G$*-modules* V *, such that*

$$P(V,G) \in \underline{\underline{X}}$$

Then the modules in $P_q(G)$ *satisfy the conditions M1 to M5 of Theorem 3.2.1 and therefore the class of groups*

$$P_q(\underline{\underline{X}}) = \{G \mid \Gamma_{\mathbb{Z}_q}(G) \subseteq P_q(G)\}$$

is a Fitting formation.

<u>Proof</u>

M1 is satisfied trivially.

M2 is immediate.

For M3 let V be an irreducible $\mathbb{Z}_q N$-module in $P_q(N)$ and $\phi : G \to N$ an epimorphism. We must show $V \in P_q(G)$.

Now the epimorphism $\phi : G \to N$ induces an epimorphism $G \to N/\ker_N(V)$ in the usual way with kernel $\ker_G(V)$.

Thus $G/\ker_G(V) \cong N/\ker_N(V)$ and so since the action of these two groups (as defined by inflation) is the same,

$$\begin{aligned} P(V,G) &\cong P(V,N) \\ &\in \underline{\underline{X}} \end{aligned}$$

as required.

For M4 let N be a maximal normal subgroup of G and take V to be an irreducible $\mathbb{Z}_q G$-module in $P_q(G)$. We must show that if U is an irreducible component of V_N then $U \in P_q(N)$, that is $P(U,N) \in \underline{\underline{X}}$.

Now $P(V,N)$ is normal in $P(V,G) \in \underline{\underline{X}}$ so by S_n-closure of $\underline{\underline{X}}$, $P(V,N) \in \underline{\underline{X}}$.

Since by Clifford's theorem V_N is completely reducible there is a $\mathbb{Z}_q N$-module W say such that $U \cong_N V/W$.

Thus $U.N/\ker_N V \cong P(V,N)/W \in \underline{\underline{X}}$.

By Q-closure again the desired result follows, $P(U,N) \in \underline{\underline{X}}$.

For M5 let $G = N_1 N_2$ where $N_i \trianglelefteq G$, $i = 1,2$, and take an irreducible $\mathbb{Z}_q G$-module V for which each irreducible component of the restriction V_{N_i} is contained in $P_q(N_i)$, for each $i = 1,2$. What we must show is that $V \in P_q(G)$, that is $P(V,G) \in \underline{\underline{X}}$.

As $G = N_1N_2$ it follows that

$$G/\ker V = (N_1 \ker V/\ker V)(N_2 \ker V/\ker V) .$$

This means we can rewrite $P(V,G)$ as a product of normal subgroups

$$P(V,G) = P(V,N_1 \ker V)P(V,N_2 \ker V)$$

That $P(V,G) \in \underline{\underline{X}}$ will now follow from the N_0-closure of $\underline{\underline{X}}$ provided we can show $P(V,N_i \ker V) \in \underline{\underline{X}}$, $i = 1,2$. We do this by writing $P(V,N_i)$ as a subdirect product of terms in $\underline{\underline{X}}$. Note that $P(V,N_i) \cong P(V,N_i \ker V)$. Without loss suppose $i = 1$.

Now by Clifford's theorem

$$V_{N_1} = U_1 \oplus \dots \oplus U_k$$

for U_j , an irreducible $\mathbb{Z}_q N_1$-module $j = 1,\dots,k$.

It is now possible to define a set of k epimorphisms,

$$\phi_j : P(V,N_1) \to P(U_j,N_1)$$

$j = 1,\dots,k$ by the projection of V onto U_j in the usual way

Moreover

$$\ker \phi_j = \{(x,y) \mid x \in U_1 \oplus \dots \oplus \hat{U}_j \oplus \dots \oplus U_k , y \in \ker_{N_1}(U_j)/\ker_{N_1}(V)\}$$

Thus

$$\bigcap_{j=1}^{k} \ker \phi_j = \{(0,\ker_{N_1} V)\}$$

$$= 1 ,$$

since $\bigcap_{j=1}^{k} \ker_{N_1}(U_j) = \ker_{N_1}(V)$.

Therefore $P(V,N_1)$ is a subdirect product of the $P(U_j,N_1)$.

But by hypothesis each of these is in $\underline{\underline{X}}$.

It follows that $P(V,N_i) \in \underline{\underline{X}}$, $i = 1,2$, and hence that $P(V,G) \in \underline{\underline{X}}$, as we required.

This completes the proof of the theorem.

We can now extend this construction to an arbitrary set of primes π . With $P_q(\underline{\underline{X}})$ as above put

$$P_\pi(\underline{\underline{X}}) = \bigcap_{q\in\pi} P_q(\underline{\underline{X}})$$

It is an elementary result that $P_\pi(\underline{\underline{X}})$ so defined is also a Fitting formation.

We now have the following result.

2.2 <u>THEOREM</u> *For any set of primes* π , P_π *is an operation on* F *defined by*

$$P_\pi : \underline{\underline{X}} \to P_\pi(\underline{\underline{X}})$$

Moreover P_π *is a closure operation. In particular for* $\underline{\underline{X}},\underline{\underline{Y}}$ *in* F

(1) If $\underline{\underline{X}} \subseteq \underline{\underline{Y}}$, *then* $P_\pi(\underline{\underline{X}}) \subseteq P_\pi(\underline{\underline{Y}})$

(2) $\underline{\underline{X}} \subseteq P_\pi(\underline{\underline{X}})$

and

(3) $P_\pi^{\,2}(\underline{\underline{X}}) = P_\pi(\underline{\underline{X}})$.

Proof

(1) is obvious

(2) is an immediate corollary of Lemma 2.3.5.

(3) uses (2) and the identity $P(W,P(W,G)) \cong P(W,G)$ for W and G .

This completes the proof.

Now we know from a remark made at the end of Chapter 3 that $\underline{\underline{S}}_{\pi'} \subseteq P_\pi(\underline{\underline{X}})$.

When $\pi = \mathbb{P}$ we put $P_{\mathbb{P}} = P$ and in §3 we conjecture that all Fitting formations are P-closed.

In the remainder of this section though we develop properties of P_π for an arbitrary π .

We need the following technical lemma.

2.3 LEMMA *Let* π *be an arbitrary set of primes,* $\underline{\underline{X}}$ *be any Fitting formation and* G *a group in* $P_\pi(\underline{\underline{X}})$. *For convenience put* $\bar{G} = G/O_{\pi'}(G)$. *Then*

$$\bar{G}^{\underline{\underline{X}}} \subseteq \Phi(\bar{G}) .$$

Proof The important case is when π contains just a single prime, q say. We do this first.

Let $G \in P_q(\underline{\underline{X}})$ and put $\bar{G} = G/O_{q'}(G)$. It will suffice to show $\bar{G}/\Phi(\bar{G}) \in \underline{\underline{X}}$, for this implies $\bar{G}^{\underline{\underline{X}}} \subseteq \Phi(\bar{G})$, as we require.

For convenience put $S = \bar{G}/\Phi(\bar{G})$.

By an elementary result (Huppert [24] III Hilfssatz 3.4.(b)) $\Phi(S) = 1$, and therefore Lemma 2.2.7 applies. From this we get that

$$F(S) = M_1 \times \ldots \times M_\ell$$

for some ℓ where M_j is a minimal normal subgroup of S, $j = 1,\ldots,\ell$.

Choose a complement, K say, for $F(S)$ in S (this exists by Huppert [24] III 4.5 and 4.4). Now define a complement in S for each M_i in the following way,

$$K_i = (M_1 \times \ldots \times \hat{M}_i \times \ldots \times M_\ell).K .$$

Further, put $L_i = C_{K_i}(M_i)$ and note L_i is normal in S, being normalised by K_i and centralised by M_i while $S = K_i M_i$.

Now since $G \in P_q(\underline{\underline{X}})$, $S \in P_q(\underline{\underline{X}})$ and so $P(M_j, S) \in \underline{\underline{X}}$ for each j.

Thus

$$\begin{aligned} P(M_j,S) &\cong M_j.K_j/L_j \\ &\cong S/L_j \\ &\in \underline{\underline{X}} . \end{aligned}$$

Noting that

$$L_j \cap F(S) = M_1 \times \ldots \times \hat{M}_j \times \ldots \times M_\ell$$

we can show by a standard argument that $\bigcap_{j=1}^{\ell} L_j = 1$.

Hence by the subdirect product closure of $\underline{\underline{X}}$ we have $S \in \underline{\underline{X}}$.

This completes the case for $\pi = \{q\}$.

In the last part of the proof we generalise the situation to an arbitrary set of primes, π .

Since $G \in P_\pi(\underline{\underline{X}}) = \bigcap_{q \in \pi} P_q(\underline{\underline{X}})$ we have

$$G/O_{q'}(G) \Big/ \Phi(G/O_{q'}(G)) \in \underline{\underline{X}}$$

for each $q \in \pi$.

Now define H_q to be the normal subgroup of G such that

$$H_q/O_{q'}(G) = \Phi(G/O_{q'}(G))$$

Then by R_0-closure of $\underline{\underline{X}}$

$$G/\cap H_q \in \underline{\underline{X}}$$

But it is easy to see

$$\cap H_q/O_{\pi'}(G) = \Phi(G/O_{\pi'}(G)) .$$

Hence

$$G/O_{\pi'}(G) \Big/ \Phi(G/O_{\pi'}(G)) \in \underline{\underline{X}}$$

as required.

This completes the proof of the lemma.

As a first approximation to the range of relevant groups the following result sandwiches $P_\pi(\mathfrak{X})$ between two classes which are only dependent on the choice of $\mathfrak{X}$ and π.

For instance we know that both $\mathfrak{X}$ and $\mathfrak{S}_{\pi'}$ are contained in $P_\pi(\mathfrak{X})$. What we now show is that $P_\pi(\mathfrak{X})$ must at least contain the class $\mathfrak{S}_{\pi'}\mathfrak{X}$.

2.4 <u>THEOREM</u> *Let* $\mathfrak{X}$ *be any Fitting formation and* π *an arbitrary set of primes. Then*

(2.5) $$\bigcap_{q \in \pi} \mathfrak{S}_{q'}\mathfrak{S}_q\mathfrak{X} \supseteq P_\pi(\mathfrak{X}) \supseteq \mathfrak{S}_{\pi'}\mathfrak{X}$$

In particular this means

(2.6) $$\mathfrak{N}\mathfrak{X} \supseteq P(\mathfrak{X}) \supseteq \mathfrak{X} .$$

<u>Proof</u> It suffices to consider the case when $\pi = \{q\}$.

Let $G \in \mathfrak{S}_{q'}\mathfrak{X}$ and put $\bar{G} = G/O_{q'}(G)$ so that $\bar{G} \in \mathfrak{X}$, thus $\bar{G} \in P_q(\mathfrak{X})$.

But now all q-chief factors of G can be thought of as q-chief factors of $\bar{G}$, and this implies by M3 that $G \in P_q(\mathfrak{X})$. Thus $P_q(\mathfrak{X}) \supseteq \mathfrak{S}_{q'}\mathfrak{X}$.

Now consider $G \in P_q(\mathfrak{X})$ and take $\{U_i\}_{i=1}^{r}$ to be a complete set of q-chief factors belonging to a chief series of G. Then because $P(U_i, G) \in \mathfrak{X}$ it follows $G/C_G(U_i) \in \mathfrak{X}$ for each i and thus $G/\bigcap_{i=1}^{r} C_G(U_i) \in \mathfrak{X}$.

But by (2.3.3) this implies $G/O_{q'q}(G) \in \mathfrak{X}$, therefore $G \in \mathfrak{S}_{q'}\mathfrak{S}_q\mathfrak{X}$ and $\mathfrak{S}_{q'}\mathfrak{S}_q\mathfrak{X} \supseteq P_q(\mathfrak{X})$.

Thus (2.5) is established, as required.

For (2.6) put $\pi = \mathbb{P}$ in (2.5).

Now

$$\bigcap_{q \in \mathbb{P}} G_{\underline{\underline{S}}_{q'}\underline{\underline{S}}_q} = G_{\bigcap_{q \in \mathbb{P}} \underline{\underline{S}}_{q'}\underline{\underline{S}}_q}$$

is easy to see.

Therefore, since $N = \bigcap_{q \in \mathbb{P}} \underline{\underline{S}}_{q'}\underline{\underline{S}}_q$, (2.6) follows as required.

This completes the proof of the theorem.

Now let π be an arbitrary set of primes and take the product of each term in (2.6) on the left by $\underline{\underline{S}}_{\pi'}$. Thus

$$\underline{\underline{S}}_{\pi'}\underline{\underline{NX}} \supseteq \underline{\underline{S}}_{\pi'}P(\underline{\underline{X}}) \supseteq \underline{\underline{S}}_{\pi'}\underline{\underline{X}} \quad . \qquad (\alpha)$$

Now compare this with the case in (2.5) where we have

$$\bigcap_{q \in \pi} \underline{\underline{S}}_{q'}\underline{\underline{S}}_q\underline{\underline{X}} \supseteq P_\pi(\underline{\underline{X}}) \supseteq \underline{\underline{S}}_{\pi'}\underline{\underline{X}} \quad . \qquad (\beta)$$

Observe that $\underline{\underline{S}}_{q'}\underline{\underline{NX}} = \underline{\underline{S}}_{q'}\underline{\underline{S}}_q\underline{\underline{X}}$.

This is because if $G/O_{q'}(G) \in \underline{\underline{NX}} = \bigcap \underline{\underline{S}}_{q'}\underline{\underline{S}}_q\underline{\underline{X}}$ then

$$G \in \underline{\underline{S}}_{q'}{}^2\underline{\underline{S}}_q\underline{\underline{X}}$$

$$= \underline{\underline{S}}_{q'}\underline{\underline{S}}_q\underline{\underline{X}}$$

for each $q \in \mathbb{P}$, and conversely.

Thus $\underline{\underline{S}}_{\pi'}\underline{\underline{NX}} = \bigcap_{q \in \pi} \underline{\underline{S}}_{q'}\underline{\underline{S}}_q\underline{\underline{X}}$.

Comparing (α) and (β) the natural question to ask now is whether or not $P_\pi(\underline{\underline{X}}) = \underline{\underline{S}}_{\pi'}P(\underline{\underline{X}})$.

We provide an affirmative answer to this in the following theorem.

2.7 THEOREM *Let* π *be an arbitrary set of primes and* $\underline{\underline{X}}$ *any Fitting formation. Then*

$$P_\pi(\underline{\underline{X}}) = \underline{\underline{S}}_{\pi'}P(\underline{\underline{X}}) .$$

Proof It is enough to consider the case $\pi = \{q\}$.

Firstly we demonstrate that $P_q(\underline{\underline{X}}) \subseteq \underline{\underline{S}}_{q'}P(\underline{\underline{X}})$.

Let $G \in P_q(\underline{\underline{X}})$ and put $\bar{G} = G/O_{q'}(G)$. Then the aim is to show $\bar{G} \in P(\underline{\underline{X}})$.

Now from lemma 2.3 it follows $\bar{G}^{\underline{\underline{X}}} \subseteq \Phi(\bar{G}) \subseteq F(\bar{G})$. But by the way $\bar{G}$ is chosen we have $F(\bar{G}) = O_q(\bar{G})$.

Hence $\bar{G}/O_q(\bar{G}) \in \underline{\underline{X}}$.

If $\{U_i\}_{i=1}^r$ is the set of chief factors for $\bar{G}$ then we know $F(\bar{G}) = \bigcap_i C_{\bar{G}}(U_i)$ so $O_q(G) \subseteq C_{\bar{G}}(U_i)$ for each i . Thus

$$P(U_i,\bar{G}) \cong P(U_i,\bar{G}/O_q(\bar{G}))$$

$$\in \underline{\underline{X}}$$

for each $i = 1,\ldots,r$ (since $\bar{G}/O_q(\bar{G}) \in \underline{\underline{X}}$ and then by lemma 2.3.5).

Therefore $\bar{G} \in P(\underline{\underline{X}})$, and the containment is proved.

Finally we show the converse : $\underline{\underline{S}}_{q'}P(\underline{\underline{X}}) \subseteq P_q(\underline{\underline{X}})$.

This is simple for we already know that $\underline{\underline{S}}_{q'}P(\underline{\underline{X}}) \subseteq \underline{\underline{S}}_{q'}P_q(\underline{\underline{X}})$. But now observe that $\underline{\underline{S}}_{q'}P_q(\underline{\underline{X}}) = P_q(\underline{\underline{X}})$.

The converse now follows.

This completes the proof of the theorem..

Theorem 2.7 provides a crucial insight into the structure of the class $P_\pi(\underline{\underline{X}})$, $\pi \neq \mathbb{P}$. What we can see is that $P_\pi(\underline{\underline{X}})$ is essentially determined by $P(\underline{\underline{X}})$. In other words the over all effect of the set of primes π is not very interesting, whereas deeper, more substantial questions involve only the representation properties of the chief factors.

Accordingly in the next section we study $P(\underline{\underline{X}})$.

§3. Is each Fitting formation P-closed?

In this section we concentrate on the possibility of characterising an arbitrary Fitting formation, $\underline{\underline{X}}$ say, using the methods developed in §2 .

We make the following conjecture.

(3.1) *Each Fitting formation is* P*-closed.*

Using Theorem 2.7 we may reword (3.1) in the following way :

Let $\underline{\underline{X}}$ *be any Fitting formation, then* $P_\pi(\underline{\underline{X}}) = \underline{\underline{S}}_{\pi'}\underline{\underline{X}}$ *for each* $\pi \subseteq \mathbb{P}$.

If (3.1) is true then every Fitting formation can be characterised by a construction via the representation properties of chief factors.

Even if (3.1) is false the situation would still be of interest.

Our first result is a simple corollary to Lemma 2.3.

3.2 THEOREM *If $\underline{\underline{X}}$ is any saturated Fitting formation then $\underline{\underline{X}}$ is P-closed.*

Equivalently, when $\underline{\underline{X}}$ is saturated then

$$P_\pi(\underline{\underline{X}}) = \underline{\underline{S}}_{\pi'}\underline{\underline{X}}$$

for any set of primes, π .

Proof By (2.5) we only need to show $P_\pi(\underline{\underline{X}}) \subseteq S_{\pi'}\underline{\underline{X}}$.

But this is an immediate corollary of Lemma 2.3, for by that result $\bar{G}/\Phi(\bar{G}) \in \underline{\underline{X}}$ and so by saturation of $\underline{\underline{X}}$, $\bar{G} \in \underline{\underline{X}}$.

Thus $G \in \underline{\underline{S}}_{\pi'}\underline{\underline{X}}$, as required.

Thus (3.1) holds in the saturated case.

We also have the following.

3.3 COROLLARY *The containment $P_\pi(\underline{\underline{X}}) \supseteq \underline{\underline{S}}_{\pi'}\underline{\underline{X}}$ in Theorem 2.4 is in general the best possible one.*

Proof By the theorem, equality holds in the saturated case.

3.4 COROLLARY *If $\underline{\underline{X}}$ is saturated, then so is $P_\pi(\underline{\underline{X}})$.*

Proof By the theorem we have $P_\pi(\underline{\underline{X}}) = \underline{\underline{S}}_{\pi'}\underline{\underline{X}}$.

It is well known that this is saturated.

We point out that it is not known whether the other containment in Theorem 2.4, namely $\bigcap_{q\in\pi} \underline{\underline{S}}_{q'}\underline{\underline{S}}_{q}\underline{\underline{X}} \supseteq P_\pi(\underline{\underline{X}})$, is best possible (obviously if it is then our conjecture is false).

Corollary 3.4 is included to tidy up a dead end : one cannot construct a non-saturated Fitting formation by applying P_{π} to a saturated example. However, what happens when P_{π} is applied to a non-saturated case is unknown; again the question seems to be related to the possible P-closure of F .

It is easy to check that all Fitting formations in the classes $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ and $\underline{\underline{H}}_q^{\pi}(\underline{\underline{X}})$, specifically the non-saturated cases, are P-closed.

For instance consider $\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$.

Let $G \in P(\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}}))$ and $U \in \Gamma_k(G)$. We will show $U \in Y_q^{\pi}(G)$.

Since $U \in \Gamma_k(G)$ we know by definition there exists a q-chief factor V of G with U a direct summand of V . So, by definition of P , $P(V,G) \in \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$.

Thus $U \in Y_q^{\pi}(P(V,G))$.

But then U as a G-module immediately satisfies Y1 and Y2.

Noting that the kernel of U as a P(V,G)-module is just V.ker U/ker V it follows that $G/\ker U \in \underline{\underline{X}}$.

Hence Y3 is also satisfied.

Thus $P(\underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})) = \underline{\underline{Y}}_q^{\pi}(\underline{\underline{X}})$ and $Y_q^{\pi}(\underline{\underline{X}})$ is P-closed.

Similarly $U \in H_q^{\pi}(P(V,G))$ implies $U \in H_q^{\pi}(G)$ and so $H_q^{\pi}(\underline{\underline{X}})$ is also P-closed.

Thus all known Fitting formations are P-closed.

The section closes with the following theorem which indicates how difficult it might be to find a counter example to (3.1).

3.5 THEOREM *If* $\underline{\underline{X}}$ *and* $\underline{\underline{Y}}$ *are* P-*closed Fitting formations, then* $\underline{\underline{XY}}$ *is also* P-*closed.*

Proof We must show $P(\underline{\underline{XY}}) \subseteq \underline{\underline{XY}}$.

Assume on the contrary that this containment fails, and let G be a counter example of minimal order.

Thus G possesses a unique minimal normal subgroup. Call it N . Let N be a p-group, say.

Take U to be any chief factor of $G^{\underline{\underline{Y}}}$. Our aim will be to show that $P(U,G^{\underline{\underline{Y}}}) \in \underline{\underline{X}}$, for then $G \in P(\underline{\underline{X}})\underline{\underline{Y}}$ and so by hypothesis $G \in \underline{\underline{XY}}$, a contradiction.

It suffices to take U to be a minimal normal subgroup of $G^{\underline{\underline{Y}}}$ contained in N . For if U = H/K say with $K \neq 1$, then by minimality $G/K \in \underline{\underline{XY}}$, so

$$G^{\underline{\underline{Y}}}/K = (G/K)^{\underline{\underline{Y}}} \in \underline{\underline{X}}$$

and

$$P(H/K,G^{\underline{\underline{Y}}}) = P(H/K,(G/K)^{\underline{\underline{Y}}}) \in \underline{\underline{X}}$$

as we require.

We call this U , N_0 . To reiterate : the aim is to show $P(N_0,G^{\underline{\underline{Y}}}) \in \underline{\underline{X}}$.

Now $P(N,G) \in \underline{\underline{XY}}$ so $P(N,G)^{\underline{\underline{Y}}} \in \underline{\underline{X}}$.

When $P(N,G) \notin \underline{\underline{Y}}$ this provides us with crucial information. For convenience put $C = C_G(N)$ and $\bar{G}^{\underline{\underline{Y}}} = \bar{G}^{\underline{\underline{Y}}} / \bar{G}^{\underline{\underline{Y}}} \cap C$. Then it follows that

$$N.\bar{G}^{\underline{\underline{Y}}} \cong P(N,G)^{\underline{\underline{Y}}}$$

$$\in \underline{\underline{X}} .$$

But by Clifford's theorem $\mathbb{N}_{\bar{G}^{\underline{\underline{Y}}}}$ is completely irreducible with N_0 isomorphic to a direct summand.

Hence $N_0.\bar{G}^{\underline{\underline{Y}}}$ is a quotient of $N.\bar{G}^{\underline{\underline{Y}}}$ (cf. the proof of S_n-closure in Theorem 2.1 , and thus $N_0.\bar{G}^{\underline{\underline{Y}}} \in \underline{\underline{X}}$.

Hence $P(N_0,G^{\underline{\underline{Y}}}) \in \underline{\underline{X}}$. as we require.

Now consider the case where $P(N,G) \in \underline{\underline{Y}}$. Here $C \supseteq G^{\underline{\underline{Y}}}$ since $G/C \in \underline{\underline{Y}}$ and so we have the containments

$$N_0 \subseteq N \subseteq Z(G^{\underline{\underline{Y}}}) .$$

This implies $P(N_0,G^{\underline{\underline{Y}}}) \in \underline{\underline{S}}_p$.

But p is in the characteristic of $\underline{\underline{X}}$. So by Lemma 2.2.2 , $P(N_0,G^{\underline{\underline{Y}}}) \in \underline{\underline{X}}$, as we require in this case also.

We complete the proof by showing p is in the characteristic of $\underline{\underline{X}}$.

Suppose the contrary holds.

Then $(|N|,|G^{\underline{\underline{Y}}}/N|) = 1$ unless $G/N \in \underline{\underline{Y}}$. But if $G/N \in \underline{\underline{Y}}$ then since we have assumed in this case $P(N,G) \in \underline{\underline{Y}}$, it follows $G \in P(\underline{\underline{Y}}) = \underline{\underline{Y}}$, a contradiction.

Thus $G^{\underline{\underline{Y}}}$ splits over N with complement K .

However because N centralises the whole of $G^{\mathfrak{Y}}$, this is a direct product. Thus K is normal in G yet $N \cap K = 1$, contradicting the uniqueness of N.

Hence p is in the characteristic of $\mathfrak{X}$.

This completes proof of the theorem.

§4. The behaviour of P_π under products of classes

Let $\mathfrak{X},\mathfrak{Y}$ be Fitting formations and π an arbitrary set of primes. Then it is easy to see that the equation

$$(4.1) \qquad P_\pi(\mathfrak{X}\mathfrak{Y}) = P_\pi(\mathfrak{X})P_\pi(\mathfrak{Y})$$

is in general false.

For instance put $\mathfrak{X} = \mathfrak{Y} = \mathfrak{S}_\pi$. Then because $\mathfrak{S}_\pi$ is saturated we have $P_\pi(\mathfrak{X}\mathfrak{Y}) = \mathfrak{S}_{\pi'}\mathfrak{S}_\pi$ whereas $P_\pi(\mathfrak{X})P_\pi(\mathfrak{Y}) = (\mathfrak{S}_{\pi'}\mathfrak{S}_\pi)^2$ and so equality fails.

In this section we look for necessary and sufficient condition under which (4.1) holds. Unfortunately the best we are able to do is provide rather weak results. In particular we have little knowledge of the truth or otherwise of (4.1) in the most important special case $\pi = \mathbb{P}$.

Our first result leads to a sufficiency condition.

4.2 <u>THEOREM</u> *Let* $\mathfrak{X},\mathfrak{Y}$ *be Fitting formations and* π *an arbitrary set of primes. Then*

(1) $P_\pi(\mathfrak{X})\mathfrak{Y} \subseteq P_\pi(\mathfrak{X}\mathfrak{Y})$

(2) $P_\pi(\mathfrak{X}\mathfrak{Y}) \subseteq P_\pi(\mathfrak{X} \cup \mathfrak{Y})\mathfrak{Y}$.

Proof We prove (1) first.

Assume on the contrary that G is a counterexample of minimal order i.e. a group of smallest order in $P_\pi(\underline{\underline{X}})\underline{\underline{Y}}$ but not in $P_\pi(\underline{\underline{XY}})$. Then G has a unique minimal normal subgroup, N say.

Now let H/K be any chief factor of G with $K \neq 1$ and H/K a π-group.

Then consider G/K.

By the minimality of G it is in $P_\pi(\underline{\underline{XY}})$ and so

$$P(H/K,G) \cong P(H/K,G/K)$$

$$\in \underline{\underline{XY}}$$

If N is a π'-group then it follows $G \in P_\pi(\underline{\underline{XY}})$ and this provides a contradiction, completing the proof in this case.

If, on the other hand, N is a π-group suppose $N \neq C_G(N)$. Then the minimality of G implies $P(N,G) \in P_\pi(\underline{\underline{XY}})$ and so

$$P(N,G) \cong P(N,P(N,G))$$

$$\in \underline{\underline{XY}}$$

Hence $G \in P_\pi(\underline{\underline{XY}})$, a contradiction. So $N = C_G(N)$.

We now consider $P(N,G)$ and show that it is in $\underline{\underline{XY}}$ in every case. This will provide the final contradiction. To reduce notation put $\bar{N} := P(N,G)$.

Since $G \in P_\pi(\underline{\underline{X}})\underline{\underline{Y}}$ it follows that $\bar{N} \in P_\pi(\underline{\underline{X}})\underline{\underline{Y}}$. Thus if U is a chief factor of $\bar{N}^{\underline{\underline{Y}}}$ then

$$P_\pi(U,\bar{N}^{\underline{\underline{Y}}}) \in \underline{\underline{X}} .$$

However by Clifford's theorem, since N is a chief factor of $\bar{N}$ it follows that

$$N_{\bar{N}^{\underline{\underline{Y}}}} = U_1 \oplus \ldots \oplus U_r$$

where the U_i , $i = 1,\ldots,r$ are (isomorphic to) chief factors of $\bar{N}^{\underline{\underline{Y}}}$ contained in N . Hence, by the same argument in Theorem 2.1 used to show $P_\pi(\underline{\underline{X}})$ is N_0-closed, we have that $P_\pi(N,\bar{N}^{\underline{\underline{Y}}})$ is the subdirect product of the $P_\pi(U_i,\bar{N}^{\underline{\underline{Y}}})$ for $i = 1,\ldots,r$. This implies

$$P(N,\bar{N}^{\underline{\underline{Y}}}) \in \underline{\underline{X}} .$$

Using elementary properties it follows that $\bar{N}^{\underline{\underline{Y}}} \in \underline{\underline{X}}$.

Thus $\bar{N} \in \underline{\underline{XY}}$, as we needed to show.

Therefore $G \in P_\pi(\underline{\underline{XY}})$ and so $P_\pi(\underline{\underline{X}})\underline{\underline{Y}} \subseteq P_\pi(\underline{\underline{XY}})$, as asserted in part (1).

In part (2) follow the proof of Theorem 3.5 making all appropriate changes up to the point where we consider the case $P(N,G) \in \underline{\underline{Y}}$. In this case it follows that $P(N_0,G^{\underline{\underline{Y}}}) \in \underline{\underline{Y}}$ and so $G \in P_\pi(\underline{\underline{X}} \cup \underline{\underline{Y}})\underline{\underline{Y}}$, a contradiction.

This completes the proof of the theorem.

Theorem 4.2 has an immediate corollary.

4.3 <u>COROLLARY</u> *Retain the terms of the theorem whenever* $\underline{\underline{X}} \supseteq \underline{\underline{Y}}$ *we have*

$$P_\pi(\underline{\underline{XY}}) = P_\pi(\underline{\underline{X}})\underline{\underline{Y}}$$

If in addition $\underline{\underline{P}}_\pi(\underline{\underline{X}}) = \underline{\underline{Y}}$ *then*

$$P_\pi(\underline{\underline{XY}}) = P_\pi(\underline{\underline{X}})P_\pi(\underline{\underline{Y}}) \ .$$

In the next two results we try to turn the situation around.

4.4 <u>THEOREM</u> *For* $\underline{\underline{X}}, \underline{\underline{Y}}, \pi$ *as above.*

If

$$P_\pi(\underline{\underline{XY}}) = P_\pi(\underline{\underline{X}})P_\pi(\underline{\underline{Y}})$$

and if

$$P_\pi(\underline{\underline{X}})P_\pi(\underline{\underline{Y}}) \subseteq P_\pi(\underline{\underline{X}})\underline{\underline{Y}} \quad \text{and} \quad P_\pi(\underline{\underline{X}}) \neq P_\pi(\underline{\underline{X}})\underline{\underline{S}}_{\pi'},$$

then

$$P_\pi(\underline{\underline{Y}}) = \underline{\underline{Y}} \ .$$

<u>Proof</u> The method of this proof is to assume the contrary and then derive a contradiction by constructing a group which is in $P_\pi(\underline{\underline{X}})P_\pi(\underline{\underline{Y}})$ but not in $P_\pi(\underline{\underline{XY}})$.

In order to effect this construction we use the wreath product. Hence use of Lemma 2.2.6, which interrelates the radical and the wreath product is crucial.

Let G be a counterexample of minimal order in $P_\pi(\underline{\underline{Y}})$ but not $\underline{\underline{Y}}$. Further take H to be a group of minimal order in $P_\pi(\underline{\underline{X}})\underline{\underline{S}}_{\pi'}$, but not in $P_\pi(\underline{\underline{X}})$. It is well known that H must have a unique maximal normal subgroup, call this M say. Moreover by the minimality of H it is clear that M is the $P_\pi(\underline{\underline{X}})$-radical of H . Thus $H/M \in \underline{\underline{S}}_{\pi'}$.

Now consider H wr G . Our aim will be to show $H \text{ wr } G \in P_\pi(\underline{\underline{X}})P_\pi(\underline{\underline{Y}})$ whereas $H \text{ wr } G \notin P_\pi(\underline{\underline{XY}})$, providing the desired contradiction.

Since $H \in P_\pi(\underline{\underline{X}})$ we may use Lemma 2.2.6 to calculate the $P_\pi(\underline{\underline{X}})$-radical of $H \text{ wr } G$.

Thus

$$(H \text{ wr } G)_{P_\pi(\underline{\underline{X}})} = (H_{P_\pi(\underline{\underline{X}})})^G$$

$$= M^G ,$$

and hence

$$H \text{ wr } G/(H \text{ wr } G)_{P_\pi(\underline{\underline{X}})} \cong (H/M)^G.G .$$

For convenience put $W = (H/M)^G.G$.

Now because $(H/M)^G$ has characteristic a π'-number all q-chief factors, $q \in \pi$ of W occur above $(H/M)^G$. Moreover each chief factor above $(H/M)^G$ may be thought of as an inflation of a chief factor in G .

But $G \in P_\pi(\underline{\underline{Y}})$, so $W \in P_\pi(\underline{\underline{Y}})$. Hence $H \text{ wr } G \in P_\pi(\underline{\underline{X}})P_\pi(\underline{\underline{Y}})$.

However assume $H \text{ wr } G \in P_\pi(\underline{\underline{XY}})$. Then by hypothesis $H \text{ wr } G \in P_\pi(\underline{\underline{X}})\underline{\underline{Y}}$ and so $(H/M)^G.G \in \underline{\underline{Y}}$. This implies $G \in \underline{\underline{Y}}$.

But this contradicts the minimality of G .

It therefore follows that $P_\pi(\underline{\underline{Y}}) = \underline{\underline{Y}}$.

This completes the proof of the theorem.

Observe in (4.2) that the hypothesis $P_\pi(\underline{\underline{X}}) \neq P_\pi(\underline{\underline{X}})\underline{\underline{S}}_\pi$, specifically excludes the possibility $\pi = \mathbb{P}$. Our next result aims to include this crucial case.

4.5 <u>THEOREM</u> *Let* $\underline{\underline{X}}, \underline{\underline{Y}}, \pi$ *be as before.*

If

$$P_q(\underline{\underline{X}})P_q(\underline{\underline{Y}}) \subseteq P_q(\underline{\underline{X}})\underline{\underline{Y}} \quad \text{and} \quad P_q(\underline{\underline{X}}) = P_q(\underline{\underline{X}})\underline{\underline{S}}_{q'}$$

for each $q \in \pi$ *, then*

$$P_\pi(\underline{\underline{Y}}) = \underline{\underline{S}}_{\pi'}\underline{\underline{Y}} \, .$$

<u>Proof</u> Suppose the conclusion is false. Let G be a group of minimal order in $P_q(\underline{\underline{Y}})$ not in $\underline{\underline{S}}_{q'}\underline{\underline{Y}}$. Further take H to be any soluble group of minimal order not in $P_q(\underline{\underline{X}})$.

The first step is to refine the structures of H and G .

First we work on H .

By minimality it follows that the $P_q(\underline{\underline{X}})$-radical is the unique minimal normal subgroup, M say. Moreover $P_q(\underline{\underline{X}}) = P_q(\underline{\underline{X}})\underline{\underline{S}}_{q'}$ implies $H/M \notin \underline{\underline{S}}_{q'}$. Thus $|H/M| = q$.

Now we refine the structure of G .

It is well known that there is a unique minimal normal subgroup, L say.

Now suppose L is a q'-group.

Then writing $O_{q'}(G/L) = K/L$ we have that K is a q'-group. But

$$G/K \cong G/L \, / \, O_{p'}(G/L)$$

$$\in \underline{\underline{Y}}$$

by minimality of G with respect to G/L .

Thus $G \in \underline{\underline{S}}_q, \underline{\underline{Y}}$, a contradiction.

Hence L is a q-group.

Next we consider $M \text{ wr } C_q$. We shall show $M \text{ wr } C_q \notin P_q(\underline{\underline{X}})$. This will provide the basis for the penultimate contradiction. Remember this.

Assume that $M \text{ wr } C_q \in P_q(\underline{\underline{X}})$.

It is clear that

$$(H \text{ wr } C_q)/M^{C_q} \cong (H/M) \text{ wr } C_q$$

and that this is a q-group.

Thus $(M \text{ wr } C_q)/M^{C_q}$ is subnormal in $(H \text{ wr } C_q)/M^{C_q}$ and so $M \text{ wr } C_q$ is subnormal in $H \text{ wr } C_q$. We deduce from this that

$$(M \text{ wr } C_q)_{P_q(\underline{\underline{X}})} \subseteq (H \text{ wr } C_q)_{P_q(\underline{\underline{X}})}$$

But now by assumption

$$(M \text{ wr } C_q)_{P_q(\underline{\underline{X}})} = M \text{ wr } C_q$$

and moreover by Lemma 2.2.6

$$(M \text{ wr } C_q)_{P_q(\underline{\underline{X}})} = M \text{ wr } C_q .$$

Thus $M^{C_q}.C_q \subseteq M^{C_q}$, a contradiction.

So $M \text{ wr } C_q \notin P_q(\underline{\underline{X}})$.

In the next step we look at $M \text{ wr } G$ and show that the $P_q(\underline{\underline{X}})$-radical is just M^G. This will lead straight to the final contradiction.

For convenience put $R = (M \text{ wr } G)_{P_q(\underline{\underline{X}})}$.

The first thing to notice is that $M^G \subseteq R$. This is for the obvious reason that $M \in P_q(\underline{\underline{X}})$ by the minimality of H.

Now suppose $R/M^G \neq 1$.

Recall that L is the unique minimal normal subgroup of G. Thus $M^G.L/M^G$ is the unique minimal normal subgroup of $M^G.G/M^G$.

Thus $M^G.L \subseteq R$ and by this it follows $M^G.L \in P_q(\underline{\underline{X}})$.

Moreover since L is a q-group let C be a subgroup isomorphic to C_q. Then $M^G.C$ is normal in $M^G.L$ and so

$$M^G.C \in P_q(\underline{\underline{X}})$$

Now by definition

$$M^G = \prod_{g \in G} M_g$$

and so putting

$$N = \prod_{y \notin C} M_y$$

it follows that

$$M^G = M^C \times N .$$

Now since C acts trivially on N it follows that N is normal in $M^G.C$.

Hence

$$
\begin{aligned}
M^G.C/N &= M^C \times N.C/N \\
&\cong M^{C_q}.C_q \\
&= M \text{ wr } C_q \\
&\in P_q(\underline{\underline{X}})
\end{aligned}
$$

But this is a contradiction.

So $R = M^G$ as claimed.

The final contradiction follows immediately. Since

$$
\begin{aligned}
M \text{ wr } G/M^G &\cong G \\
&\in P_q(\underline{\underline{Y}})
\end{aligned}
$$

it follows $M \text{ wr } G \in P_q(\underline{\underline{X}})\ P_q(\underline{\underline{Y}})$.

Thus by hypothesis $M \text{ wr } G \in P_q(\underline{\underline{X}})\underline{\underline{Y}}$.

We deduce from this $G \in \underline{\underline{Y}}$, a final contradiction of the minimality of G .

This completes the proof of the theorem.

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