VARIETIES AND SECTION CLOSED CLASSES OF GROUPS

by

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STATEMENT

The work contained in this thesis is my own except where otherwise stated.

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ABSTRACT

The central concept of this thesis is the relationship between a locally finite variety and the section closed classes of groups which generate it. R.M. Bryant and L.G. Kovács defined the *skeleton* $S(\underline{\underline{V}})$ *of a variety* $\underline{\underline{V}}$ of groups to be the intersection of the section closed classes of groups which generate $\underline{\underline{V}}$. Of particular interest are those varieties generated by their skeletons, for they are generated by a unique minimal section closed class of groups. Since a locally finite variety $\underline{\underline{V}}$ is generated by its finite monolithic groups, $S(\underline{\underline{V}})$ is always contained in $\text{Q}\text{s}M(\underline{\underline{V}})$, the section closure of the class $M(\underline{\underline{V}})$ of finite monolithic groups in $\underline{\underline{V}}$. For a positive integer m, let $\underline{\underline{A}}_m$ denote the variety of all abelian groups of exponent dividing m. Bryant and Kovács showed that, for $m > 1$ and a locally finite variety $\underline{\underline{V}}$, $S(\underline{\underline{A}}_m\underline{\underline{V}})$ is equal to $\text{Q}\text{s}M(\underline{\underline{A}}_m\underline{\underline{V}})$. Earlier Cossey showed that the skeleton $S(\underline{\underline{U}})$ of a variety $\underline{\underline{U}}$ of A-groups is $\text{Q}\text{s}M(\underline{\underline{U}})$.

These results are generalized here by showing that for a nontrivial variety $\underline{\underline{U}}$ of A-groups and a locally finite variety $\underline{\underline{V}}$, the skeleton $S(\underline{\underline{U}}\underline{\underline{V}})$ is $\text{Q}\text{s}M(\underline{\underline{U}}\underline{\underline{V}})$. As a corollary necessary and sufficient conditions are given for $S(\underline{\underline{U}}\underline{\underline{V}})$ to consist of all finite groups in $\underline{\underline{U}}\underline{\underline{V}}$. Examples are given to show that a product of two nontrivial locally finite varieties need not be generated by its skeleton, or, even if it is, the skeleton need not contain all the critical groups in the variety.

In proving the main theorem above, we are led to consider a variety which, for some prime p, is generated by finite monolithic groups each of which is an extension of a nontrivial abelian p-group by a p'-group. In the appendix, knowledge of the skeleton of such a variety is applied to show that if $\underline{\underline{U}}$ is a variety of A-groups, $\underline{\underline{V}}$ a locally finite variety whose lattice of subvarieties is distributive and the exponents of $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are coprime then the lattice of subvarieties of $\underline{\underline{UV}}$ is distributive.

The consideration of such extensions of abelian p-groups by p'-groups leads to an interesting question. When is such a group in a locally finite variety $\underline{\underline{V}}$ only if it is in $S(\underline{\underline{V}})$? R.M. Bryant and L.G. Kovács have shown the answer to be always, provided the p-group is cyclic or elementary abelian. If the p-group is not cyclic and has sufficiently large exponent then, it is shown here, there is a locally finite variety $\underline{\underline{V}}$ containing the group, but the group is not in $S(\underline{\underline{V}})$. In particular if the p-group has exponent at least p^3 and the p'-group is cyclic this is true. Further special cases of the problem are considered.

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CHAPTER ONE

INTRODUCTION

1. The Problem and Its History

The central concept of this thesis is the relationship between a locally finite variety of groups and the section closed classes of groups which generate it. For a class $\mathcal{B}$ of varieties and a variety $\underline{\underline{V}}$ in $\mathcal{B}$, the *spine of* $\underline{\underline{V}}$ *relative to* $\mathcal{B}$ is defined to be the intersection of the section closed classes of groups which generate varieties in $\mathcal{B}$ containing $\underline{\underline{V}}$. A number of results can be rephrased in this language. Cossey [9] showed that a monolithic A-group is in the spine of the variety it generates relative to the class of varieties of A-groups. For a positive integer m let $\underline{\underline{A}}_m$ denote the variety of abelian groups of exponent dividing m. Brisley and Kovács [2] showed that, for a prime p, any finite group in the product variety $\underline{\underline{A}}_p\underline{\underline{A}}_p$ is in the spine of $\underline{\underline{A}}_p\underline{\underline{A}}_p$ relative to the class of soluble locally finite varieties.

Two special cases of the relative spine are of interest. The *skeleton* $S(\underline{\underline{V}})$ *of a variety* $\underline{\underline{V}}$ is defined to be the spine of $\underline{\underline{V}}$ relative to the class which consists of $\underline{\underline{V}}$ alone. The *spine* $T(\underline{\underline{V}})$ *of a locally finite variety* $\underline{\underline{V}}$ is the spine of $\underline{\underline{V}}$ relative to the class of all locally finite varieties. A finite group is said to be *hypocritical* if it is in the spine of the variety it generates.

Bryant and Kovács [5] have shown that for $m > 1$ and any locally finite variety $\underline{\underline{V}}$, the skeleton $S(\underline{\underline{A}}_m\underline{\underline{V}})$ of $\underline{\underline{A}}_m\underline{\underline{V}}$ is the section closure of the class of monolithic groups in $\underline{\underline{A}}_m\underline{\underline{V}}$ and have given a more precise description of the groups in the skeleton. In unpublished work (given in Lemma 9.4 below) they have shown that for a prime p and a locally finite variety $\underline{\underline{V}}$ of p'-exponent, the product variety $\underline{\underline{A}}_p\underline{\underline{V}}$ is generated by its spine; in fact it is generated by hypocritical groups.

Much of this thesis is devoted to generalizing the results of Cossey and Bryant and Kovács mentioned above. In this chapter a language and some elementary results are established, and in the next chapter some familiarity with skeletons is developed. In particular, necessary and sufficient conditions are given for a Cross variety to be generated by its skeleton.

It is shown in Chapter Three that the skeleton $S(\underline{\underline{UV}})$ of a product variety $\underline{\underline{UV}}$ of a nontrivial variety $\underline{\underline{U}}$ of A-groups with a locally finite variety $\underline{\underline{V}}$ is the section closure of the finite monolithic groups in $\underline{\underline{UV}}$, and a more precise description of the groups in the skeleton is given. To prove this a technical theorem is needed which deals with varieties generated by monolithic groups which, for a prime p, are an extension of a nontrivial abelian group of p-power order by a p'-group. That such varieties are of interest in other contexts is shown in the appendix to Chapter Three. The appendix is a paper which applies the technical lemma to show that if $\underline{\underline{U}}$ is a variety of A-groups and $\underline{\underline{V}}$ is a locally finite variety whose lattice of subvarieties is distributive and the exponents of $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are coprime then the lattice of subvarieties of $\underline{\underline{UV}}$ is distributive.

In Chapter Four the study of the monolithic groups described above is continued and attention is focused on deciding which of them are hypocritical. In particular if the normal p-subgroup is cyclic or elementary abelian the group is hypocritical. Otherwise, if the normal p-subgroup has large enough exponent then the group is not hypocritical, in which case it is said to be *sincere*. The problem of which of these groups are hypocritical and which are sincere is not resolved here, but a number of partial answers are given which indicate the complexity of the problem and offer scope for further research.

Apologia

The freedom with which we talk about sets of varieties when in fact they are classes is an accepted abuse of terminology. It is done in the faith that, with less elegance but a clearer set theoretic foundation, one could discuss equivalent results in the language of subgroups of free groups. In this thesis the precedent is followed and if language is abused perhaps even further it is done merely to avoid cumbersome and pedantic statements about normal subgroups of free groups.

2. Groups and Varieties

In this section we establish notation and definitions relating to groups and varieties and gather facts which will be needed later. Notation and definitions which are not given here are as in Hanna Neumann [20]. *Group* will mean finite group unless otherwise stated or unless this restriction is repugnant to the context. A *class of groups* is a union of isomorphism classes of groups and may contain both finite and infinite groups. A group G is said to be *monolithic* if the intersection of the nontrivial normal subgroups is nontrivial, and when nontrivial this intersection is called the *monolith*. If $\mathcal{G}$ is a set or class of groups we denote the class of groups isomorphic to

cartesian products of groups in $\mathcal{G}$ by $\mathrm{C}\mathcal{G}$,

subgroups of groups in $\mathcal{G}$ by $\mathrm{S}\mathcal{G}$,

factor groups of groups in $\mathcal{G}$ by $\mathrm{Q}\mathcal{G}$,

finite groups in $\mathcal{G}$ by $F(\mathcal{G})$, and

monolithic groups in $\mathcal{G}$ by $M(\mathcal{G})$.

If $\mathcal{G}$ consists of a single group G we write $\mathrm{C}G$, $\mathrm{S}G$, $\mathrm{Q}G$ respectively for $\mathrm{C}\mathcal{G}$, $\mathrm{S}\mathcal{G}$, $\mathrm{Q}\mathcal{G}$.

If $\mathcal{G}$ is contained in $\mathcal{H}$ we write $\mathcal{G} \subseteq \mathcal{H}$ and reserve $\mathcal{G} \subset \mathcal{H}$ for proper containment. A class $\mathcal{G}$ of groups is said to be *section closed* if $\mathrm{Q}\mathcal{G} \subseteq \mathcal{G}$ and $\mathrm{S}\mathcal{G} \subseteq \mathcal{G}$. For any class $\mathcal{G}$ of groups it is easy to see that $\mathrm{QS}\mathcal{G}$ is section closed. A *section* of a group G is an element of $\mathrm{QS}G$. A section closed class $\mathcal{G}$ of groups is called a *variety* if $\mathrm{C}\mathcal{G} \subseteq \mathcal{G}$. Birkhoff [20, 15.23] showed that if $\mathcal{G}$ is a class of groups then $\mathrm{QSC}\mathcal{G}$ is a variety. It is called the variety *generated* by $\mathcal{G}$ and denoted by var $\mathcal{G}$. If $\mathcal{G}$ is a section closed class of groups which generates $\underline{\underline{V}}$ we write $\mathcal{G}\ \mathrm{scg}\ \underline{\underline{V}}$. A monolithic group is said to be *critical* if it is not in the variety generated by its proper subgroups. We write $\mathcal{C}(G)$ for the class of critical groups in G. A *Cross variety* is a variety generated by a finite group. A variety $\underline{\underline{V}}$ is said to be *locally finite* if every group in it is locally finite. An *A-group* is a locally finite group whose nilpotent subgroups are abelian. A *variety of A-groups* is a variety which consists of A-groups.

The *exponent* of a locally finite variety is the order of the free group on one generator of the variety. The *exponent* of a group is the least common multiple of the orders of the elements of the group. For a prime p a group G or a variety $\underline{\underline{V}}$ is said to have *p-prime* (p') *exponent* if p does not divide the (finite) exponent of G or $\underline{\underline{V}}$. The *socle* σG of a group G is the product of the minimal normal subgroups of G. If an action of G is defined on H (for example H may be a section of G or a G-module) then the *centralizer* $C_G(H)$ *in* G *of* H is defined to be the set of elements of G which act trivially on H; it is always a subgroup of G. We write σ^*G for $C_G(\sigma G)$. If N is a normal subgroup of G we write $N \triangleleft G$ and if N is characteristic, N char G. If H is a subgroup of G we write $H \leq G$; if H is proper, $H < G$; and if H is isomorphic to a subgroup of G then $H \lesssim G$. If $H \leq G$ and T is a set of (right) coset representatives for H in G we say T is a (*right*) *transversal* for H in G.

Suppose G is a group and $a, b, a_1, \ldots, a_n, a_{n+1}$ are elements of G. We write $a^{-1}b^{-1}ab = [a, b] = [a, 1b]$ and inductively for $n > 1$,

$$[a_1, \ldots, a_n, a_{n+1}] = [[a_1, \ldots, a_n], a_{n+1}]$$

and

$$[a, nb] = [[a, (n-1)b], b] .$$

We denote $b^{-1}ab$ by a^b. The *derived group* G' is the subgroup of G generated by $[a, b]$ for all a, b in G. If G is generated by $a_1, a_2, \ldots$ we write $G = \mathrm{gp}(a_1, a_2, \ldots)$. If $H, K \leq G$ then $[H, K] = \mathrm{gp}([h, k] \mid h \in H, k \in K)$. Let $G_{(1)} = G$ and for $c > 1$, $G_{(c)} = [G_{(c-1)}, G]$.

For a prime p, $S_p(G)$ denotes the set of Sylow p-subgroups of G. The Frattini subgroup of G is denoted by ΦG and the center of G by $Z(G)$. The automorphism group of G is denoted by Aut G.

For groups G and H a homomorphism from G to H is denoted $G \to H$, from G onto H by $G \twoheadrightarrow H$, an embedding by $G \rightarrowtail H$ and an isomorphism by $G \rightarrowtail\!\!\!\!\twoheadrightarrow H$ or $G \cong H$. If φ is a homomorphism of a multiplicatively written group G then exponential notation is used, G^φ or a^φ for $a \in G$, unless this becomes too cumbersome typographically in which case circle notation is used, $a \circ \varphi$. If A is an additively written group then multiplicative notation is used for a homomorphism, $A\varphi$ or $a\varphi$. If $\varphi : G \to \mathrm{Aut}\, H$ is a (fixed) homomorphism then GH denotes the split extension of H by G where h^g, $h \in H$ and $g \in G$, denotes the image of h under g^φ. In particular if $H = A$ is a G-module, written additively then GA is written multiplicatively and we will switch without comment from additive to multiplicative notation and *vice versa* as seems appropriate. If φ is a homomorphism of G and $H \leq G$ then $\varphi|_H$ denotes the restriction of φ to H.

For varieties $\underline{\underline{U}}$ and $\underline{\underline{V}}$, $\underline{\underline{U}} \vee \underline{\underline{V}}$ denotes the variety generated by the set theoretic union $\underline{\underline{U}} \cup \underline{\underline{V}}$, and $\underline{\underline{U}} \wedge \underline{\underline{V}}$ the variety of groups in the set theoretic intersection $\underline{\underline{U}} \cap \underline{\underline{V}}$. The product variety of $\underline{\underline{U}}$ by $\underline{\underline{V}}$ is denoted by $\underline{\underline{UV}}$.

The following results are well known.

2.1 LEMMA. *If G is a finite A-group then σ^*G is abelian.*

Proof. Suppose by way of contradiction that $(\sigma^*G)' \neq 1$. Then $(\sigma^*G)' \lhd G$ so there is a minimal normal subgroup N of G contained in $(\sigma^*G)'$. Notice $N \leq Z(\sigma^*G)$. Since σ^*G is an A-group we can apply [15, VI 14.3 (b)] to get

$$N \leq Z(\sigma^*G) \cap (\sigma^*G)' = 1$$

which is the desired contradiction. //

2.2 COROLLARY (Cossey [9]). *If G is a monolithic A-group with σG a p-group for some prime p then $\sigma^*G \in S_p(G)$.*

Proof. Since σG is a normal p-subgroup of G , if $S \in S_p(G)$ then $\sigma G \leq S$. Since G is an A-group, $S \leq \sigma^*G$. By 2.1, σ^*G is abelian and since G is monolithic, $S = \sigma^*G$. //

2.3 LEMMA. *If G is a monolithic group with a nontrivial normal abelian Sylow p-subgroup S then $S = \sigma^*G$.*

Proof. Clearly $S \leq \sigma^*G$. If $(\sigma^*G)' \neq 1$ it contains a minimal normal subgroup N of G . Then $N = \sigma G \leq S$ and $N \leq Z(\sigma^*G)$. Since S is abelian we can apply [15, VI 14.3 (a)] to get

$$N \leq Z(\sigma^*G) \cap (\sigma^*G)' \cap S = 1$$

which is a contradiction. Thus σ^*G is abelian so $S = \sigma^*G$. //

The following theorem is proved in [15, I 18.1 and 18.3]; in this generality the proof relies on the Feit Thompson Theorem.

2.4 SCHUR ZASSENHAUS THEOREM. *If G is a group, $N \lhd G$ and the order $|G/N|$ of G/N is coprime to the order $|N|$ of N then there is a complement for N in G and all complements of N in G are conjugate.* //

2.5 LEMMA. *If* S *is a normal Sylow* p*-subgroup of* G *then* $\Phi S = S \cap \Phi G$.

Proof. Using some elementary results about Frattini subgroups [15, III §3] the problem may be reduced to the case $\Phi S = 1$. By the Schur Zassenhaus Theorem S has a complement H in G . If $S \cap \Phi G > 1$ it contains an irreducible H-module which has a complement, A say, in S (by Maschke's Theorem). But the split extension HA is a maximal subgroup of G avoiding nontrivial elements of $S \cap \Phi G$ which is a contradiction. //

For groups G and H , $G \text{ wr } H$ denotes the (restricted) wreath product of G and H , and G^H denotes the set of functions from H to G . Under pointwise multiplication G^H is a group, called the *base group* of $G \text{ wr } H$. We identify G with the subgroup of G^H of functions trivial everywhere except possibly at $1 \in H$.

2.6 LEMMA. *If* G *is a monolithic group and* $G > \sigma^*G$ *and* H *is a group then* $G \text{ wr } H$ *is monolithic,*

$$\sigma(G \text{ wr } H) = (\sigma G)^H$$

and

$$\sigma^*(G \text{ wr } H) = (\sigma^*G)^H .$$

Proof. We first show $(\sigma G)^H$ is a minimal normal subgroup of $G \text{ wr } H$. Let φ be a nontrivial element of $(\sigma G)^H$. Then there is an $a \in H$ such that $1 \neq \varphi(a) \in \sigma G$. Since $G > \sigma^*G$ there is a $b \in G$ such that $[b, \varphi(a)] \neq 1$. Let $\psi \in (\sigma G)^H$ be defined by $\psi(a) = b$, $\psi(a') = 1$: here and below the range of a' is $H\backslash\{a\}$. Then $\chi = [\psi, \varphi]$ satisfies $\chi(a) \neq 1$ and $\chi(a') = 1$, and χ is in the normal closure of φ in $G \text{ wr } H$. The normal closure of χ in

G^H is

$$\{\mu \mid \mu \in G^H,\ \mu(a) \in \sigma G,\ \mu(a') = 1\}$$

and the normal closure of this in $G \text{ wr } H$ is $(\sigma G)^H$. Thus $(\sigma G)^H$ is a minimal normal subgroup of $G \text{ wr } H$.

Let $c\xi$ centralize $(\sigma G)^H$ in $G \text{ wr } H$. Then $[c\xi, \chi] = 1$ so $c = 1$. Let $d \in H$. Since $\left[(\sigma G)^H, \xi\right] = 1$, $\xi(d) \in \sigma^* G$ and since this is true for all $d \in H$, $\xi \in (\sigma^* G)^H$. It follows that any nontrivial normal subgroup of $G \text{ wr } H$ contains $(\sigma G)^H$. Thus $G \text{ wr } H$ is monolithic with monolith $(\sigma G)^H$ and monolith centralizer $(\sigma^* G)^H$. //

2.7 THEOREM. *Suppose p is a prime and P is a relatively free p-group. Let $N \leq \Phi P$, $N \lhd P$ and let G^* be a p'-subgroup of* Aut P/N. *Then there is a group $G \leq$ Aut P such that the map $P \twoheadrightarrow P/N$ induces an isomorphism of G and G^* as abstract groups. If $G_1 \leq$ Aut P and G_1 has the same properties as G then G_1 and G are conjugate in* Aut P.

Proof. Let $a_1, \ldots, a_k$ be free generators of P and let $\varphi \in$ Aut P/N. Let $b_i \in (a_i N)^\varphi$. Since P is relatively free the map $a_i \mapsto b_i$ induces an endomorphism ψ of P. Since $N \leq \Phi P$, ψ is an automorphism. Thus the map $P \twoheadrightarrow P/N$ induces a homomorphism π of the stabilizer of N in Aut P onto Aut P/N. Let H in Aut P be the complete inverse image of G^* under π.

By a theorem of P. Hall [15, III 3.18], ker π is a p-group. By assumption $G^* \cong H/\ker \pi$ and G^* is a p'-group. Thus by the Schur Zassenhaus Theorem there is a complement G for $\ker \pi$ in H and all complements are conjugate in H , and $H \leq \text{Aut } P$. //

The following lemma is well known and is proved in Brady [1, 2.3.7].

2.8 LEMMA. *Suppose* G *and* H *are groups,* $\theta : G \rightarrowtail\!\!\!\!\rightarrow H$, $K \leq \text{Aut } G$, $L \leq \text{Aut } H$ *and* $\theta^{-1}K\theta$ *and* L *are conjugate in* $\text{Aut } H$. *Then the split extensions* KG *and* HL *are isomorphic.* //

The concept of a minimal representation due to Kovács and Newman [17] is used repeatedly in this thesis. Suppose that $\mathcal{G}$ is a section closed class of groups such that $\text{var } \mathcal{G}$ is locally finite, and G is a group in $\text{var } \mathcal{G}$. Then G is a section of a finite direct product of groups in $\mathcal{G}$, generally in many ways by [20, 51.1]. (The argument offered in [20] in support of 51.1 appears to require a further idea which can be adapted from the proof of [20, 15.74].) Each such direct product determines a finite non-increasing sequence of integers, each integer the order of a direct factor. Order these sequences lexicographically, that is by putting one sequence before another when its entry in the first place where they differ is the smaller. In this ordering there is a unique first sequence. An isomorphism

$$G \cong H/K \ , \quad H \leq H_1 \times \ldots \times H_t$$

corresponding to this first sequence and such that no proper subgroup of H has a factor group isomorphic to G is called a *minimal representation* of G on $\mathcal{G}$. The assumption that H be as small as possible is not usually made in writings about minimal representations, but is made here because it has as a consequence that $K \leq \Phi H$.

To describe a frequently used fact about minimal representations we need another definition. If G and H are groups, $M \triangleleft G$, $N \triangleleft H$ and there exist isomorphisms $\theta : M \rightarrowtail\!\!\!\twoheadrightarrow N$ and $\mu : G/C_G(M) \rightarrowtail\!\!\!\twoheadrightarrow H/C_H(N)$ such that

$$(m^a) \circ \theta = (m \circ \theta)^{a \circ \mu} \quad \text{for all } m \in M,\ a \in G/C_G(M)$$

then we say M *is similar in* G *to* N *in* H. (Here m^a is defined to be the common value of m^x for x in the coset a of $C_G(M)$ in G.)

2.9 LEMMA ([20, 53.25]). *Suppose* G *is a group and*

$$G \cong H/K\,,\quad H \leq H_1 \times \ldots \times H_t\,,\quad H_i \in \mathrm{H} \ \text{for}\ i = 1, \ldots, t$$

is a minimal representation of G *on a section closed class* H *of groups. Then for each* i, H_i *is critical and* G *has a minimal normal subgroup* N_i *which is similar in* G *to* σH_i *in* H_i. //

2.10 LEMMA. *If* G *is a monolithic group with a nontrivial normal Sylow* p*-subgroup* S *and*

$$G \cong H/K\,,\quad H \leq H_1 \times \ldots \times H_t$$

is a minimal representation of G *then a Sylow* p*-subgroup* T *of* H *is normal in* H *and*

$$K \leq \Phi T\,.$$

Proof. As noted earlier $K \leq \Phi H$ so K is nilpotent and the Sylow subgroups of K are normal in H. Since G is monolithic and $1 \neq S \triangleleft G$, $S \geq \sigma G$ so σG is a p-group. Since $\sigma H_i \cong \sigma G$ for each i, σH_i is a p-group. As H is a subdirect product of the H_i, σH is a p-group. Because the Sylow subgroups of K are normal in H, K is a p-group.

The Sylow p-subgroup of G is normal in G, so the same is true for H/K, and since K is a normal p-subgroup of H, a Sylow p-subgroup T of H is normal in H. By 2.5,

$$\Phi T = T \cap \Phi H \geq K . \quad //$$

In the following well known formula μ denotes the Möbius function.

2.11 WITT'S FORMULA ([19, 5.11]). *Suppose F is the infinite absolutely free group on k generators, $k > 1$. The rank of $F_{(c)}/F_{(c+1)}$ as a free abelian group is*

$$n(c) = \frac{1}{c} \sum_{d|c} \mu(d) k^{c/d} . \quad //$$

For a positive integer m, $\underline{\underline{A}}_m$, $\underline{\underline{N}}_m$ and $\underline{\underline{B}}_m$ denote respectively the variety of all abelian groups of exponent dividing m, the variety of all nilpotent groups of class at most m, and the variety of all groups of exponent dividing m.

For a variety $\underline{\underline{V}}$ the *lattice of* $\underline{\underline{V}}$ means the lattice of subvarieties of $\underline{\underline{V}}$ using $\vee$ and $\wedge$ defined earlier. It is modular. In a modular nondistributive lattice there are always three elements whose pairwise joins and meets are respectively equal [22, Theorems 32 and 33]. Higman [14] gave the first example of a variety with a nondistributive lattice and showed that for each prime $p > 5$ the lattice of $\underline{\underline{B}}_p \wedge \underline{\underline{N}}_{p-1}$ is not distributive. Kovács and Newman, in unpublished work, showed $\underline{\underline{A}}_2\underline{\underline{A}}_8$ has a nondistributive lattice. Bryce [7, 4.4.8] showed for any prime p, $\underline{\underline{A}}_{p^2}\underline{\underline{A}}_{p^2} \wedge \underline{\underline{N}}_{p+2}$ has a nondistributive lattice. Brooks [3] showed $\underline{\underline{A}}_3\underline{\underline{A}}_9$ has a nondistributive lattice. Thus we get the following result.

2.12 THEOREM. *For each prime p there exist three distinct locally finite varieties of p-power exponent whose pairwise joins and meets are respectively equal.* //

3. Representation Theory

In this section much of the representation theory needed later is developed. Most of it is well known. Notation and terminology not here defined are as in Curtis and Reiner [10] though here *module* shall mean finitely generated right module except where otherwise stated. Throughout this section let G be a group, p a prime, α a positive integer and R_α the ring of integers modulo p^α. On occasion the ring of integers modulo p will be denoted by Z_p. By [10, 70.24] there exists a finite splitting field Λ for G obtained by adjoining a primitive mth root of unity to Z_p, where m is the exponent of G. Both R_α and Λ are quasi-Frobenius rings (as defined in [10, 58.5]). Let R be a commutative quasi-Frobenius ring of p-power characteristic. Then the group ring RG is also quasi-Frobenius [10, 2(d) p. 402]. The regular RG-module will also be denoted by RG.

If C is a direct sum of A and B, $C = A \oplus B$, then A is said to be a *direct summand* of C. For a positive integer r, $A^{\oplus r}$ denotes the direct sum of r copies of A. An RG-module A is said to be *injective* if, whenever it is a submodule of a module C then it is a direct summand of C. An RG-module A is said to be *projective* if whenever there is a homomorphism of C onto A then A is isomorphic to a direct summand of C. A module isomorphic to an indecomposable direct summand of RG is called a *principal indecomposable module*.

3.1 LEMMA ([10, 56.6 and 58.14]). *An RG-module is injective if and only if it is projective if and only if it is a direct sum of principal indecomposable modules.* //

An RG-module is said to be *completely reducible* if every submodule is a direct summand. Recall that R is of p-power characteristic.

3.2 MASCHKE'S THEOREM ([11, 3.2.2]). *Suppose G is a p'-group and the submodule A of the RG-module C is a direct factor of C as abelian group. Then A is a direct summand of C. In particular if R is a field then C is completely reducible.* //

3.3 KRULL SCHMIDT THEOREM ([10, 14.5]). *If*

$$A = A_1 \oplus \ldots \oplus A_r = B_1 \oplus \ldots \oplus B_s$$

are two decompositions of an RG-module A into direct sums of nonzero indecomposable submodules then $r = s$ and there is a permutation π of $\{1, \ldots, r\}$ such that $A_i \cong B_{i\pi}$ for each i. //

If A is a right R-module and B is a left R-module then $A \otimes_R B$ or $A \otimes B$ will denote the tensor product of A and B over R. For a positive integer r, $A^{\otimes r}$ denotes the tensor product of r copies of (the two sided module) A. If $H \leq G$ and A is an RH-module then $A^G = A \otimes_{RH} RG$ is the RG-module *induced* from A.

3.4 LEMMA. *If $N \lhd G$ and A is an injective RN-module then A^G is an injective RG-module.*

Proof. By the definition of A^G and [10, 12.14],

$$(RN)^G = RN \otimes_{RN} RG \cong RG .$$

If A is an injective RN-module then there is an RN-module B and a positive integer r such that $A \oplus B = (RN)^{\oplus r}$ by 3.1. By [10, 12.12] the direct sum distributes over tensor products so

$$A^G \oplus B^G \cong (A \oplus B)^G \cong \left((RN)^{\oplus r}\right)^G \cong (RG)^{\oplus r} .$$

By 3.1 the lemma follows. //

If A is an RG-module and $H \leq G$ the restriction of A to H gives an RH-module denoted by A_H. If A is isomorphic to a submodule of B we write $A \lesssim B$. For an RG-module A, as noted earlier GA denotes the (multiplicatively written) split extension of A by G where the action of G on A by conjugation is the module action.

The first paragraph of the proof of [10, 63.2] can be adapted to prove the following lemma.

≤ 3.5 LEMMA. *If* $N \not\triangleleft G$ *and* A *is an* RG*-module then* $A \lesssim (A_N)^G$. //

For an RG-module A, $\ker A$ is by definition the centralizer in G of A and G is said to *act faithfully* on A if $\ker A = 1$. If $\ker A = G$ then G is said to *act trivially* on A.

3.6 LEMMA. *If* A *and* B *are* RG*-modules and* $A \lesssim B$ *then* $GA/\ker A$ *is a section of* $GB/\ker B$.

Proof. Since $A \lesssim B$, $\ker B \leq \ker A$. Now $GA/\ker A$ is a factor group of $GA/\ker B$ which is isomorphic to a subgroup of $GB/\ker B$. //

Suppose $g \in G$ and g centralizes every irreducible Z_pG-module. Then $g - 1$ is in the Jacobson radical (defined in [15, V 2.1]) of the group ring Z_pG which is nilpotent by [15, V 2.4]. Thus there is an r such that $(g-1)^r = 0$. Let n be such that $p^n \geq r$. Then $(g-1)^{p^n} = 0$ and since Z_pG has characteristic p, $g^{p^n} = 1$. It follows that if the maximal normal p-subgroup of G is 1 then G has a faithful completely reducible module A. If G is also monolithic then G must act faithfully on some irreducible direct summand of A.

3.7 LEMMA. *If G is monolithic, p is a prime, and σG is not a p-group then there is an irreducible $Z_p G$-module on which G acts faithfully.* //

3.8 REMARK. If $N \triangleleft G$, A is an RN-module and $g \in G$ then $(A^G)_N \geq A \otimes g$. The set $A \otimes g$ is an RN-module since for any $n \in N$ and $a \in A$,

$$(a \otimes g)n = an^{g^{-1}} \otimes g .$$

$A \otimes g$ is called a *conjugate module*. Identifying $A \otimes 1$ with A we have $(NA)^g = N(A \otimes g)$ so that

$$NA \cong N(A \otimes g) .$$

If B is an RG-module and $D \leq C \leq B_N$, then the subset Cg of B is in fact a submodule of B_N. With $(C/D)g$ defined as Cg/Dg we have that

$$(C/D)^G \geq (C/D) \otimes g \cong (C/D)g ,$$

the obvious isomorphism being such that $(c+D) \otimes g \mapsto cg + Dg$ for all c in C.

3.9 LEMMA. *If N is a normal p'-subgroup of G and A is a homocyclic $R_\alpha G$-module of exponent p^α then $(A_N)^G$ is an injective $R_\alpha G$-module.*

Proof. By Maschke's Theorem A_N is an injective $R_\alpha N$-module. By 3.4, $(A_N)^G$ is injective. //

If $\alpha \geq \beta$ there is a natural homomorphism $R_\alpha G \twoheadrightarrow R_\beta G$. Under it an $R_\beta G$-module of exponent dividing p^β can be considered as an

$R_\beta G$-module. In particular if G is a p'-group and A is an indecomposable $R_\alpha G$-module of exponent p^β then A is homocyclic by [11, 5.2.2] and, considered as $R_\beta G$-module, it is injective by Maschke's Theorem. Applying 3.1 gives the following lemma.

3.10 LEMMA. *If G is a p'-group then an indecomposable $R_\alpha G$-module of exponent p^β, $\beta \leq \alpha$, considered as $R_\beta G$-module, is a principal indecomposable $R_\beta G$-module.* //

If A is a module then for any positive integer n, nA denotes the submodule whose underlying set is $\{na \mid a \in A\}$.

3.11 LEMMA. *If G is a p'-group and A is a principal indecomposable $R_\alpha G$-module then*

$$A \supset pA \supset \dots \supset p^{\alpha-1}A \supset 0$$

is the unique composition series for A and all the factors are isomorphic.

Proof. Since A is ~~monolithic~~ it is indecomposable ~~and~~ therefore homocyclic by [11, 5.2.2]. By Maschke's Theorem $p^{\alpha-1}A$ is irreducible. For $\beta < \alpha$ the map $p^\beta A/p^{\beta+1}A \to p^{\alpha-1}A$ defined by

$$p^\beta a + p^{\beta+1}A \longmapsto p^{\alpha-1}a$$

for any $a \in A$ is an isomorphism since A is homocyclic. Thus $p^{\beta+1}A$ is a maximal submodule of $p^\beta A$. If B is any maximal submodule of $p^\beta A$ then $p^\beta A/B$ is of exponent p and so $p^{\beta+1}A \leq B$. Thus $p^{\beta+1}A$ is the unique maximal submodule of $p^\beta A$. //

The join of the minimal submodules of a module A is called the *socle* σA of the module.

3.12 THEOREM. *Suppose G is a p'-group and A is an $R_\alpha G$-*

module. The module B is isomorphic to a submodule of A if and only if it is isomorphic to a factor module of A .

Proof. First suppose A_1 and A_2 are indecomposable $R_\alpha G$-modules of exponent p^β with a common composition factor. Consider A_1 and A_2 as $R_\beta G$-modules. By 3.10, they are now principal indecomposables, so 3.11 gives that $\sigma A_1 \cong \sigma A_2$; thus A_1 and A_2 are $R_\beta G$-injective hulls of isomorphic irreducibles and hence they are isomorphic. This shows that an indecomposable $R_\alpha G$-module is determined up to isomorphism by its exponent and a composition factor.

Suppose C is an irreducible $R_\alpha G$-module. If A is an $R_\alpha G$-module then the join of all the indecomposable submodules of A whose socle is isomorphic to C is called the *C-component of* A . Suppose that in an unrefinable direct decomposition of A there are n indecomposable direct summands in the C-component of A and they have exponer $p^{c(1)}, \ldots, p^{c(n)}$ with $c(i) \geq c(i+1)$ for all i . The C-component of A is characterized by a sequence

$$\big(a(1),\ a(2),\ \ldots\big)$$

where $a(i) = p^{c(i)}$ for $i \leq n$ and $a(i) = 1$ for $i > n$, and this is called the *C-sequence of* A .

It will be shown that B is isomorphic to a submodule or factor module of A if and only if for each irreducible $R_\alpha G$-module C , the C-sequence of B ,

$$\big(b(1),\ b(2),\ \ldots\big)$$

is such that $a(i) \geq b(i)$ for all $i = 1, 2, \ldots$.

The sufficiency is clear. For the necessity we may suppose by way of contradiction that there is a smallest k such that $a(k) < b(k)$. Then $(b(k)p^{-1})B$ has at least $k + 1$ indecomposable direct summands with socle C in an unrefinable direct decomposition while $(b(k)p^{-1})A$ has only k. This contradiction establishes the condition for submodules. The condition for factor modules is established using a similar argument considering $B/(b(k)p^{-1})B$ and $A/(b(k)p^{-1})A$.

The classes of modules isomorphic to submodules and factor modules are defined by the same conditions and so must be the same class of modules. //

CHAPTER TWO

SOME REMARKS ON SKELETONS

In Section Four, the first section of this chapter, some lemmas are proved which have some interest in their own right and which are used in the proof of the main theorems of the next chapter. In Section Five necessary and sufficient conditions for a Cross variety to be generated by its skeleton are given.

4. Lemmas and an Example

The definition of the skeleton given in Section 1 is equivalent to the statement that the skeleton $S(\underline{\underline{V}})$ of a variety $\underline{\underline{V}}$ is the intersection of the section closed classes of groups which generate $\underline{\underline{V}}$. The first lemma helps reduce the problem of finding the skeleton of $\underline{\underline{V}}$ to that of finding skeletons of subvarieties of $\underline{\underline{V}}$.

4.1 LEMMA. *If* Λ *is an index set and for each* $\lambda \in \Lambda$, $\underline{\underline{V}}_\lambda$ *is a variety then*

$$S\left(\bigvee_{\lambda\in\Lambda} \underline{\underline{V}}_\lambda\right) \subseteq \bigcup_{\lambda\in\Lambda} S\left(\underline{\underline{V}}_\lambda\right) .$$

Proof. Let F be the (infinite) absolutely free group of countably infinite rank. For each $\lambda \in \Lambda$ let

$$N_\lambda = \left\{N \mid N \lhd F \text{ and } F/N \in \bigvee_\lambda \underline{\underline{V}}_\lambda \setminus S\left(\underline{\underline{V}}_\lambda\right)\right\}$$

and for each $N \in N_\lambda$ let $G_{\lambda N} \mathrm{scg} \underline{\underline{V}}_\lambda$ be such that $F/N \notin G_{\lambda N}$. It is easy to see that $S\left(\underline{\underline{V}}_\lambda\right) = \bigcap_{N\in N_\lambda} G_{\lambda N}$. Let $\prod N_\lambda$ denote the cartesian product of the N_λ and for $\mu \in \prod N_\lambda$, $\mu(\lambda)$ denotes the N_λ component of μ . Then for any $\mu \in \prod N_\lambda$,

$$S\left(\bigvee_{\lambda} \underline{\underline{V}}_{\lambda}\right) \subseteq \bigcup_{\lambda} G_{\lambda,\mu(\lambda)} .$$

Since

$$\bigcap_{\mu} \left(\bigcup_{\lambda} G_{\lambda,\mu(\lambda)}\right) = \bigcup_{\lambda} \left(\bigcap_{N} G_{\lambda,N}\right)$$

the lemma follows. //

To show that equality need not always hold in 4.1 and that a product of two non-trivial locally finite varieties need not be generated by its skeleton, we give an example.

4.2 EXAMPLE. Let q be a prime. By 2.12 there are three distinct locally finite varieties $\underline{\underline{U}}_1, \underline{\underline{U}}_2, \underline{\underline{U}}_3$ of q-power exponent whose pairwise joins and meets are respectively equal. Let $\underline{\underline{U}} = \underline{\underline{U}}_1 \vee \underline{\underline{U}}_2$ and let $\underline{\underline{V}}$ be a locally finite variety.

By 4.1 and since $\underline{\underline{U}} = \underline{\underline{U}}_1 \vee \underline{\underline{U}}_2$,

$$S(\underline{\underline{UV}}) \subseteq S(\underline{\underline{U}}_1\underline{\underline{V}}) \cup S(\underline{\underline{U}}_2\underline{\underline{V}}) .$$

Since $\underline{\underline{U}} = \underline{\underline{U}}_1 \vee \underline{\underline{U}}_3$,

$$S(\underline{\underline{UV}}) \subseteq S(\underline{\underline{U}}_1\underline{\underline{V}}) \cup S(\underline{\underline{U}}_3\underline{\underline{V}}) .$$

Because the lattice of sets is distributive,

$$S(\underline{\underline{UV}}) \subseteq S(\underline{\underline{U}}_1\underline{\underline{V}}) \cup \left(S(\underline{\underline{U}}_2\underline{\underline{V}}) \cap S(\underline{\underline{U}}_3\underline{\underline{V}})\right) .$$

By [20, 21.23],

$$\underline{\underline{U}}_2\underline{\underline{V}} \wedge \underline{\underline{U}}_3\underline{\underline{V}} = (\underline{\underline{U}}_2 \wedge \underline{\underline{U}}_3)\underline{\underline{V}} \subseteq \underline{\underline{U}}_1\underline{\underline{V}} ,$$

so $S(\underline{\underline{UV}}) \subseteq \underline{\underline{U}}_1\underline{\underline{V}}$. Similarly $S(\underline{\underline{UV}}) \subseteq \underline{\underline{U}}_2\underline{\underline{V}}$ so $S(\underline{\underline{UV}}) \subseteq (\underline{\underline{U}}_1 \wedge \underline{\underline{U}}_2)\underline{\underline{V}}$.

21.21 By [20, ~~23.32~~], $(\underline{\underline{U}}_1 \wedge \underline{\underline{U}}_2)\underline{\underline{V}} \subset \underline{\underline{U}}_1\underline{\underline{V}} \subset \underline{\underline{UV}}$ so $S(\underline{\underline{UV}})$ cannot generate $\underline{\underline{UV}}$. //

The above is perhaps the simplest example of a locally finite product variety not generated by its skeleton. Some results of

Woeppel [23] can be used to show that there is a locally finite product variety $\underline{\underline{UV}}$ not generated by its skeleton in which the lattice of $\underline{\underline{U}}$ is distributive. The next lemma is a presumably well known variant of [20, 22.43] which will be useful later.

4.3 LEMMA. *If* $\mathcal{G}$ *and* $\mathcal{H}$ *are nonempty classes of groups generating* $\underline{\underline{U}}$ *and discriminating* $\underline{\underline{V}}$ *respectively then*

$$\{G \text{ wr } H \mid G \in \mathcal{G},\ H \in \mathcal{H}\}$$

generates $\underline{\underline{UV}}$.

Proof. Clearly $\underline{\underline{U}} = \bigvee_{G \in \mathcal{G}} \text{var } G$ and $\underline{\underline{UV}} = \bigvee_{G \in \mathcal{G}} (\text{var } G \cdot \underline{\underline{V}})$ by [20, 21.23]. The set $\{G \text{ wr } H \mid H \in \mathcal{H}\}$ generates $(\text{var } G)\underline{\underline{V}}$ by [20, 22.43] so

$$\{G \text{ wr } H \mid G \in \mathcal{G},\ H \in \mathcal{H}\} = \bigcup_{G \in \mathcal{G}} \{G \text{ wr } H \mid H \in \mathcal{H}\}$$

generates $\bigvee_{G \in \mathcal{G}} (\text{var } G \cdot \underline{\underline{V}})$, which completes the proof. //

Recall that the spine $T(\underline{\underline{V}})$ of a locally finite variety $\underline{\underline{V}}$ is the intersection of the skeletons of the locally finite varieties containing $\underline{\underline{V}}$, so that $T(\underline{\underline{V}}) \subseteq S(\underline{\underline{V}})$. The next lemma shows that equality may sometimes hold. Recall that $M(\underline{\underline{V}})$ denotes the class of monolithic groups in $\underline{\underline{V}}$.

4.4 LEMMA. *If* $\underline{\underline{U}}$ *is a nontrivial locally finite variety generated by monolithic groups with nonabelian monoliths, and* $\underline{\underline{V}}$ *is a locally finite variety then*

$$S(\underline{\underline{UV}}) = T(\underline{\underline{UV}}) = Qs\{G \mid G \in M(\underline{\underline{UV}}) \text{ and } \sigma G \text{ is not abelian}\}\ .$$

Proof. Let

$$\mathcal{G} = \{G \text{ wr } H \mid G \in M(\underline{\underline{U}}),\ \sigma G \notin \underline{\underline{A}} \text{ and } H \in F(\underline{\underline{V}})\}\ .$$

Let $\mathcal{H} = \{H \mid H \in M(\underline{\underline{UV}}) \text{ and } \sigma H \notin \underline{\underline{A}}\}$. By 4.3, $\mathcal{G}$ generates $\underline{\underline{UV}}$. By 2.6, $\mathcal{G} \subseteq \mathcal{H}$. If we show $\mathcal{H} \subseteq T(\underline{\underline{UV}})$ we shall have

$$S(\underline{\underline{UV}}) \subseteq Qs\mathcal{G} \subseteq Qs\mathcal{H} \subseteq T(\underline{\underline{UV}}) \subseteq S(\underline{\underline{UV}})$$

proving the lemma.

Suppose K is a section closed class of groups generating a locally finite variety containing $\underline{\underline{UV}}$. Let $H \in H$ and take a minimal representation of H on K:

$$H \cong K/L \ , \quad K \le K_1 \times \dots \times K_t \ , \quad K_i \in K \ \text{ for all } \ i = 1, \dots, t \ .$$

By 2.9, $H = H/\sigma^*H \cong K_1/\sigma^*K_1 \in K$. Thus $H \subseteq K$ so $H \subseteq T(\underline{\underline{UV}})$. //

4.5 LEMMA. *If $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are nontrivial locally finite varieties and either $\underline{\underline{U}}$ is abelian or not of prime power exponent then* $M(\underline{\underline{V}}) \subseteq T(\underline{\underline{UV}})$.

Proof. If $\underline{\underline{U}}$ is abelian the lemma follows from [5, 1.2]. Suppose $\underline{\underline{U}}$ is not of prime power exponent. Let $G \in M(\underline{\underline{V}})$ and p be a prime divisor of the exponent of $\underline{\underline{U}}$ such that σG is not a p-group. By 3.7 there is a faithful irreducible $Z_p G$-module A, and it is easy to see that A is self-centralizing in the split extension GA. Now $GA \in \underline{\underline{UV}}$ and if

$$GA \cong H/K \ , \quad H \le H_1 \times \dots \times H_t$$

is a minimal representation of GA on a section closed class G of groups generating a locally finite variety containing $\underline{\underline{UV}}$, then $G \cong GA/A = GA/\sigma^*(GA) \cong H_1/\sigma^*H_1 \in G$ by 2.9. //

4.6 LEMMA. *If a variety $\underline{\underline{V}}$ is generated by its skeleton and*

$$\underline{\underline{V}} = \bigvee_{i=1}^{n} \underline{\underline{V}}_i = \bigvee_{j=1}^{m} \underline{\underline{W}}_j$$

then $\underline{\underline{V}} = \bigvee_{i,j} \operatorname{var}\left(S(\underline{\underline{V}}) \cap \underline{\underline{V}}_i \cap \underline{\underline{W}}_j\right)$.

Proof. By 4.1, $S(\underline{\underline{V}}) = \bigcup_i \left(S(\underline{\underline{V}}) \cap \underline{\underline{V}}_i\right)$ and $S(\underline{\underline{V}}) = \bigcup_j \left(S(\underline{\underline{V}}) \cap \underline{\underline{W}}_j\right)$ so

$$S(\underline{\underline{V}}) = \bigcup_{i,j} \left(S(\underline{\underline{V}}) \cap \underline{\underline{V}}_i \cap \underline{\underline{W}}_j\right) .$$

It follows that $\underline{\underline{V}} = \bigvee_{i,j} \underline{\underline{V}}(i, j)$ where $\underline{\underline{V}}(i, j) = \text{var}\big(S(\underline{\underline{V}}) \cap \underline{\underline{V}}_i \cap \underline{\underline{W}}_j\big)$. //

A variety is said to be *join irreducible* if it cannot be written as the join of two proper subvarieties. The next lemma concerns a join irreducible variety of A-groups and will be useful in the next chapter.

4.7 LEMMA. *If* $\underline{\underline{U}}$ *is a join irreducible variety of* A*-groups then there are critical groups* $G_1, G_2, \ldots$ *with* $\text{var}\, G_1 \subseteq \text{var}\, G_2 \subseteq \ldots$ *and* $\underline{\underline{U}} = \bigvee_{k=1}^{\infty} \text{var}\, G_k$.

Proof. Let $H_1, H_2, \ldots$ be the critical groups in $\underline{\underline{U}}$ and let $G_0 = 1$ and $G_1 = H_1$. Suppose there exist critical groups $G_1, G_2, \ldots, G_n$ with G_{i-1} , $H_i \in \text{var}\, G_i$ for $i = 1, \ldots, n$. We show there is a G_{n+1} extending this sequence. Let

$$S_1 = \{G \mid G \in C(\underline{\underline{U}}) \text{ and } H_{n+1} \notin \text{var}\, G\} ,$$

$$S_2 = \{G \mid G \in C(\underline{\underline{U}}) \text{ and } G_n \notin \text{var}\, G\} ,$$

and

$$S_3 = \{G \mid G \in C(\underline{\underline{U}}) \text{ and } G_n, H_{n+1} \in \text{var}\, G\} .$$

Since $C(\underline{\underline{U}}) = \bigcup_{i=1}^{3} S_i$, $\underline{\underline{U}} = \bigvee_i \text{var}\, S_i$. Because $\underline{\underline{U}}$ is join irreducible $\underline{\underline{U}} = \text{var}\, S_i$ for some i . Since $\underline{\underline{U}}$ is a variety of A-groups, $G_n, H_{n+1} \in QsS_i$ by Cossey [9]. Thus $i = 3$ and S_3 is not empty. Let $G_{n+1} \in S_3$. Now $G_n, H_{n+1} \in \text{var}\, G_{n+1}$. Continuing in this way we see $\bigvee_{k=1}^{\infty} \text{var}\, G_k$ contains $C(\underline{\underline{U}})$ so $\underline{\underline{U}} = \bigvee_{k=1}^{\infty} \text{var}\, G_k$. //

With notation as in 4.7, if $G \in \underline{\underline{U}}$ then G is a section of a finite direct product of the G_k and so is in the variety generated by one of them. Since a critical A-group generates a join irreducible Cross variety by Cossey [9], we have the following corollary.

4.8 COROLLARY. *A finite group in a join irreducible variety of A-groups is in a join irreducible Cross subvariety.* //

5. The Skeleton of a Cross Variety

A *Cross variety* is ~~defined to be~~ a variety generated by a finite group. Let G be a group and

$$G \cong H/K\ ,\quad H \leq H_1 \times \dots \times H_t\ ,\quad H_i \in \operatorname{var} G \quad \text{for}\quad i = 1, \dots, t$$

be a minimal representation of G on $\operatorname{var} G$. The class $\mathrm{Qs}\{H_1, \dots, H_t\}$ is called a *critical class for* G.

5.1 THEOREM. *Let* $\underline{\underline{V}}$ *be a Cross variety. The skeleton* $S(\underline{\underline{V}})$ *of* $\underline{\underline{V}}$ *generates* $\underline{\underline{V}}$ *if and only if each finite group generating* $\underline{\underline{V}}$ *has a unique critical class. If* $S(\underline{\underline{V}})$ *and* G *each generate* $\underline{\underline{V}}$ *and* $\mathcal{G}$ *is a critical class for* G *then* $\mathcal{G} = S(\underline{\underline{V}})$.

Proof. Assume first that each finite group generating $\underline{\underline{V}}$ has a unique critical class. Let G generate $\underline{\underline{V}}$ and $\mathcal{G}$ be its critical class. Let $\mathcal{H}$ be a section closed class of groups generating $\underline{\underline{V}}$ and let

$$G \cong H/K\ ,\quad H \leq H_1 \times \dots \times H_t\ ,\quad H_i \in \mathcal{H} \quad \text{for}\quad i = 1, \dots, t$$

be a minimal representation of G on $\mathcal{H}$. Since G and $\mathcal{H}$ generate $\underline{\underline{V}}$, $\mathrm{Qs}\{H_1, \dots, H_t\}$ is a critical class for G and by assumption is equal to $\mathcal{G}$. Thus $\mathcal{G} \subseteq \mathcal{H}$, so $\mathcal{G} \subseteq S(\underline{\underline{V}})$. Since $\mathcal{G}$ generates $\underline{\underline{V}}$, $S(\underline{\underline{V}}) \subseteq \mathcal{G}$ so $S(\underline{\underline{V}}) = \mathcal{G}$ and $S(\underline{\underline{V}})$ generates $\underline{\underline{V}}$.

On the other hand suppose G generates $\underline{\underline{V}}$ and has two distinct critical classes $\mathcal{G}_1$ and $\mathcal{G}_2$. Then there is a $G_1 \in \mathcal{G}_1$ such that $G_1 \notin \mathcal{G}_2$ so

$$S(\underline{\underline{V}}) \subseteq \mathcal{G}_1 \cap \mathcal{G}_2 \subset \mathcal{G}_1 .$$

By definition of critical class, $S(\underline{\underline{V}})$ cannot generate $\underline{\underline{V}}$. //

In fact for a Cross variety $\underline{\underline{V}}$ generated by its skeleton there is an explicit construction for $S(\underline{\underline{V}})$. Following Bryant [4] we call a set $G = \{G_1, \ldots, G_t\}$ of groups *critical* if, for each i, G is not in the variety generated by $(G \cup (\mathrm{Qs}{-}1)G_i) \setminus \{G_i\}$. Any set G can be refined to a critical set: if G is in the variety generated by $G_1 = (G \cup (\mathrm{Qs}{-}1)G_i) \setminus \{G_i\}$ then in G_1, G_i has been replaced by groups of smaller order. Continuing with this process we arrive at a critical set H with $\mathrm{var}\, H = \mathrm{var}\, G$. We call H a *critical refinement of* G. If G consists of a single group G we call H a *critical refinement of* G.

5.2 THEOREM. *If a Cross variety* $\underline{\underline{V}}$ *is generated by a finite group* G, H *is a critical refinement of* G *and* $S(\underline{\underline{V}})$ *generates* $\underline{\underline{V}}$ *then* $S(\underline{\underline{V}}) = \mathrm{Qs}H$.

Proof. Because $G \in \mathrm{var}\, H$ there is a minimal representation

$$G \cong H/K , \quad H \leq H_1 \times \ldots \times H_t , \quad H_i \in \mathrm{Qs}H \text{ for } i = 1, \ldots, t$$

of G on $\mathrm{Qs}H$. By 5.1, $\mathcal{G} = \mathrm{Qs}\{H_1, \ldots, H_t\}$ is the unique critical class for G. If $\mathcal{G} \subset \mathrm{Qs}H$ then $\mathrm{var}\, \mathcal{G} = \mathrm{var}\, H$ contradicts the definition of critical refinement. Thus $\mathcal{G} = \mathrm{Qs}H$ and by 5.1 we are done. //

If G is critical then $\mathrm{Qs}\{G\}$ is a critical class for G so 5.1 has a corollary.

5.3 COROLLARY. *If* G *is critical and* $\mathrm{var}\, G$ *is the join of two proper subvarieties then* $S(\mathrm{var}\, G)$ *does not generate* $\mathrm{var}\, G$. //

5.4 EXAMPLE. Suppose p is an odd prime and G is a nonabelian group of order p^3 and exponent p^2. Let H be a nonabelian group of order p^3 and exponent p. Then by [20, 54.22],

$$\text{var } G = \underline{\underline{A}}_{p^2} \vee \text{var } H .$$

Since G is critical, 5.3 implies $S(\text{var } G)$ does not generate var G. //

From 5.1 we have another corollary.

5.5 COROLLARY. *A join irreducible Cross variety is generated by its skeleton if and only if it is generated by a unique critical group.* //

Lemma 4.6 shows that if a Cross variety is generated by its skeleton then it has a unique decomposition in terms of join irreducible subvarieties. It is easy to see that each of these subvarieties must be generated by its skeleton. Now 5.5 gives the following result.

5.6 THEOREM. *If a Cross variety is generated by its skeleton then it has a unique decomposition as an irredundant join of join irreducible subvarieties each of which is generated by a unique critical group.* //

As example 5.4 shows, the converse of 5.6 is not true.

CHAPTER THREE

THE SKELETON OF A PRODUCT VARIETY

In this chapter the skeleton of the product of a nontrivial variety of A-groups and a locally finite variety is characterized in two ways, Theorems 6.1 and 6.3 below. In Section 6 it is shown that these characterizations follow from a description of the skeleton of a certain product variety, given in Theorem 6.4. In Section 7 a discussion of $R_\alpha G$-modules in a varietal setting lays the foundation for a proof of Theorem 6.4 which follows in Section 8.

6. The Theorems

As a locally finite variety $\underline{V}$ is always generated by its critical groups, $S(\underline{V}) \subseteq QsC(\underline{V})$. In fact equality may sometimes occur.

6.1 THEOREM. *If* $\underline{U}$ *is a nontrivial variety of* A*-groups and* $\underline{V}$ *is a locally finite variety then*

$$S(\underline{UV}) = QsM(\underline{UV}) = QsC(\underline{UV})$$

and therefore $S(\underline{UV})$ *generates* $\underline{UV}$.

For a (nontrivial) variety $\underline{V}$, $F_\infty(\underline{V})$ denotes the (infinite) relatively free group of countably infinite rank.

An interesting corollary can be derived from Theorem 6.1 and [5, 1.5].

6.2 COROLLARY. *Suppose* $\underline{U}$ *is a nontrivial variety of* A*-groups and* $\underline{V}$ *is a nontrivial locally finite variety. The skeleton* $S(\underline{UV}) = F(\underline{UV})$ *if and only if*

(a) $\underline{U}$ *is abelian of exponent a power of a prime* p *and*

$Z(F_\infty(\underline{\underline{V}}))$ *is a p-group, or*

(b) $\underline{\underline{U}}$ *is nonabelian and join irreducible.*

Proof. For abelian $\underline{\underline{U}}$ the result is given by [5, 1.5]. Suppose $\underline{\underline{U}}$ is not abelian. If $\underline{\underline{U}}$ is the join of two proper subvarieties $\underline{\underline{U}}_1$ and $\underline{\underline{U}}_2$ then by [20, 24.34], $\underline{\underline{U}}_i\underline{\underline{V}} \neq \underline{\underline{UV}}$ for $i = 1, 2$. Thus there is an $n(i)$ such that $F_{n(i)}(\underline{\underline{UV}}) \notin \underline{\underline{U}}_i\underline{\underline{V}}$. Let n be the larger of $n(1)$ and $n(2)$. Then $F_n(\underline{\underline{UV}}) \notin \underline{\underline{U}}_1\underline{\underline{V}} \cup \underline{\underline{U}}_2\underline{\underline{V}}$. However $S(\underline{\underline{UV}}) \subseteq \underline{\underline{U}}_1\underline{\underline{V}} \cup \underline{\underline{U}}_2\underline{\underline{V}}$ by definition so $F_n(\underline{\underline{UV}}) \notin S(\underline{\underline{UV}})$.

Suppose on the other hand that $\underline{\underline{U}}$ is join irreducible. By 4.7 there are critical groups $G_1, G_2, \ldots$ with $\operatorname{var} G_1 \subseteq \operatorname{var} G_2 \subseteq \ldots$ and $\underline{\underline{U}} = \bigvee_{k=1}^{\infty} \operatorname{var} G_k$. If G is a finite group in $\underline{\underline{UV}}$ then $\underline{\underline{V}}(G) \in \operatorname{var} G_k$ for some k (as in the proof of 4.8). Thus $G \in (\operatorname{var} G_k)\underline{\underline{V}}$. As $\underline{\underline{U}}$ is not abelian there is an $l \geq k$ such that G_l is not abelian and $G \in (\operatorname{var} G_l)\underline{\underline{V}}$.

Let $\underline{\underline{V}}(G) \cong H/K$, $H \leq G_l^n$, the direct power of G_l. Since $\underline{\underline{V}}$ is nontrivial there is an $L \in \underline{\underline{V}}$ with $|L| \geq n$. Let $\hat{G} = G_l \operatorname{wr} ((G/\underline{\underline{V}}(G))\times L)$. Since G_l is not abelian $G_l > \sigma^* G_l$. Thus $\hat{G} \in M(\underline{\underline{UV}})$ by 2.6. By 6.1, $\hat{G} \in S(\underline{\underline{UV}})$. We show $G \in \mathrm{QS}\hat{G}$. By [20, 22.14 and 22.12],

$$H \operatorname{wr} G/\underline{\underline{V}}(G) \leq G_l^n \operatorname{wr} G/\underline{\underline{V}}(G) \lesssim \hat{G} .$$

By [20, 22.21 and 22.11],

$$G \in \mathrm{QS}(H/K \operatorname{wr} G/\underline{\underline{V}}(G)) \subseteq \mathrm{QS}(H \operatorname{wr} G/\underline{\underline{V}}(G)) .$$

This proves the corollary. //

6.3 THEOREM. *Let* $\underline{\underline{U}}$ *be a nontrivial variety of A-groups and*

$\underline{\underline{V}}$ *be a nontrivial locally finite variety. Let* $\{\underline{\underline{U}}_\lambda \mid \lambda \in \Lambda\}$ *be the set of nonabelian join irreducible Cross subvarieties of* $\underline{\underline{U}}$ *. Then*

$$S(\underline{\underline{U}}\underline{\underline{V}}) = \bigcup_{\lambda\in\Lambda} F(\underline{\underline{U}}_\lambda\underline{\underline{V}}) \cup S((\underline{\underline{U}}\wedge\underline{\underline{A}})\underline{\underline{V}}) .$$

In [5, 1.4] Bryant and Kovács have characterized the groups in $S((\underline{\underline{U}}\wedge\underline{\underline{A}})\underline{\underline{V}})$ so the above theorem gives a complete description of the groups in $S(\underline{\underline{U}}\underline{\underline{V}})$. By Cossey [8] a join irreducible Cross variety in $\underline{\underline{U}}$ is generated by a single critical group.

Derivation of 6.3. Let $\{\underline{\underline{U}}_\lambda \mid \lambda \in \Lambda\}$ be the set of nonabelian join irreducible Cross subvarieties of $\underline{\underline{U}}$. Now $\underline{\underline{U}} = \bigvee_{\lambda\in\Lambda} \underline{\underline{U}}_\lambda \vee (\underline{\underline{U}}\wedge\underline{\underline{A}})$ so by [20, 21.23],

$$\underline{\underline{U}}\underline{\underline{V}} = \bigvee_{\lambda} (\underline{\underline{U}}_\lambda\underline{\underline{V}}) \vee ((\underline{\underline{U}}\wedge\underline{\underline{A}})\underline{\underline{V}})$$

and by 4.1,

$$S(\underline{\underline{U}}\underline{\underline{V}}) \subseteq \bigcup_{\lambda} S(\underline{\underline{U}}_\lambda\underline{\underline{V}}) \cup S((\underline{\underline{U}}\wedge\underline{\underline{A}})\underline{\underline{V}}) .$$

By 6.1, $S(\underline{\underline{U}}_\lambda\underline{\underline{V}}) \subseteq S(\underline{\underline{U}}\underline{\underline{V}})$ and by [5, 1.2], $S((\underline{\underline{U}}\wedge\underline{\underline{A}})\underline{\underline{V}}) \subseteq S(\underline{\underline{U}}\underline{\underline{V}})$. By 6.2, $S(\underline{\underline{U}}_\lambda\underline{\underline{V}}) = F(\underline{\underline{U}}_\lambda\underline{\underline{V}})$ so

$$S(\underline{\underline{U}}\underline{\underline{V}}) = \bigcup_{\lambda} F(\underline{\underline{U}}_\lambda\underline{\underline{V}}) \cup S((\underline{\underline{U}}\wedge\underline{\underline{A}})\underline{\underline{V}}) . \qquad //$$

Theorem 6.1 is a consequence of the following theorem.

6.4 THEOREM. *Suppose* p *is a prime and* α *is a positive integer. Suppose* $\underline{\underline{U}}$ *is a nontrivial locally finite variety such that for some variety* $\underline{\underline{W}}$ *of* p'*-exponent*

$$\underline{\underline{W}} \subseteq \underline{\underline{U}} \subseteq \underline{\underline{A}}_{p^\alpha}\underline{\underline{W}}$$

and $\underline{\underline{U}}$ *is generated by critical groups not in* $\underline{\underline{W}}$ *. If* $\underline{\underline{V}}$ *is a locally finite variety then*

$S(\underline{\underline{U}}\underline{\underline{V}}) = \text{Qs}\{G \mid G \in M(\underline{\underline{U}}\underline{\underline{V}})$ *and there is an* $N \triangleleft G$, $N \in \underline{\underline{U}}\backslash\underline{\underline{W}}$ *and* $G/N \in \underline{\underline{V}}\}$.

Also $S(\underline{\underline{UV}})$ *generates* $\underline{\underline{UV}}$.

The following example shows that, with the assumptions of 6.4, $S(\underline{\underline{UV}})$ need not equal $\mathrm{QS}C(\underline{\underline{UV}})$.

6.5 EXAMPLE. Let $q, \underline{\underline{U}}_1, \underline{\underline{U}}_2$ and $\underline{\underline{U}}_3$ be as in 4.2. Let p be a prime different from q . Suppose $G \in M(\underline{\underline{U}}_1 \cup \underline{\underline{U}}_2)$ and let $A(G)$ be an irreducible Z_pG-module on which G acts faithfully; one exists by 3.7. Let

$$\underline{\underline{U}} = \mathrm{var}\{GA(G) \mid G \in M(\underline{\underline{U}}_1 \cup \underline{\underline{U}}_2)\} .$$

If $\underline{\underline{W}} = \underline{\underline{U}}_1 \vee \underline{\underline{U}}_2$ then $\underline{\underline{W}}$ is a variety of p'-exponent such that

$$\underline{\underline{W}} \subseteq \underline{\underline{U}} \subseteq \underline{\underline{A}}_p\underline{\underline{W}} .$$

Since G acts faithfully and irreducibly on $A(G)$, $GA(G)$ is monolithic and by [17], critical. Hence $\underline{\underline{U}}$ satisfies the conditions of 6.4.

Suppose $\underline{\underline{V}}$ is a locally finite variety of exponent coprime to pq . Let H be a group in $\underline{\underline{U}}_3$ which is not in $\underline{\underline{U}}_1 \wedge \underline{\underline{U}}_2$. We show $H \notin S(\underline{\underline{UV}})$. Suppose by way of contradiction that $H \in S(\underline{\underline{UV}})$. By 4.3,

$$S(\underline{\underline{UV}}) \subseteq \mathrm{QS}\{GA(G) \text{ wr } F_n(\underline{\underline{V}}) \mid G \in M(\underline{\underline{U}}_1 \cup \underline{\underline{U}}_2) \text{ and } n = 1, 2, \ldots\} .$$

Then for some $G \in M(\underline{\underline{U}}_1 \cup \underline{\underline{U}}_2)$ and some positive integer n , $H \in \mathrm{QS}(GA(G) \text{ wr } F_n(\underline{\underline{V}}))$. Since H is a q-group it must be a section of a Sylow q-subgroup S of $GA(G)$ wr $F_n(\underline{\underline{V}})$.

Clearly $S \cong G^r$ where $r = |F_n(\underline{\underline{V}})|$. Thus $S \in \underline{\underline{U}}_1 \cup \underline{\underline{U}}_2$ so $H \in \underline{\underline{U}}_3 \cap (\underline{\underline{U}}_1 \cup \underline{\underline{U}}_2)$. Now $\underline{\underline{U}}_3 \cap (\underline{\underline{U}}_1 \cup \underline{\underline{U}}_2) = (\underline{\underline{U}}_3 \cap \underline{\underline{U}}_1) \cup (\underline{\underline{U}}_3 \cap \underline{\underline{U}}_2) = \underline{\underline{U}}_1 \wedge \underline{\underline{U}}_2$ since the pairwise meets of $\underline{\underline{U}}_1, \underline{\underline{U}}_2$ and $\underline{\underline{U}}_3$ are equal. Thus $H \in \underline{\underline{U}}_1 \wedge \underline{\underline{U}}_2$ contradicting the choice of H . It follows, in particular, that any critical group in $\underline{\underline{U}}_3$ but not in $\underline{\underline{U}}_1 \wedge \underline{\underline{U}}_2$ is not in $S(\underline{\underline{UV}})$. However, by 6.4, $S(\underline{\underline{UV}})$ generates $\underline{\underline{UV}}$. //

In the rest of this section it will be shown that to prove 6.1 it suffices to prove 6.4. Thus for the rest of this section assume 6.4 is true.

Suppose $\underline{\underline{U}}$ is a variety of A-groups of exponent $p(1)^{\alpha(1)} \ldots p(r)^{\alpha(r)}$ where $p(1), \ldots, p(r)$ are distinct primes and $\alpha(1), \ldots, \alpha(r)$ are positive integers. Let $\underline{\underline{U}}_i$ be the variety generated by the monolithic groups in $\underline{\underline{U}}$ whose monoliths are $p(i)$-groups for $i = 1, \ldots, r$, and let $\underline{\underline{U}}_0$ be generated by the monolithic groups in $\underline{\underline{U}}$ with nonabelian monoliths. (We adhere to the convention that even the empty class of groups generates $\underline{\underline{E}}$.) Clearly

$$\underline{\underline{U}} = \bigvee_{i=0}^{r} \underline{\underline{U}}_i .$$

For $i > 0$, let $\underline{\underline{W}}_i$ be generated by the groups in $\underline{\underline{U}}_i$ of $p(i)$-prime exponent. Suppose $\underline{\underline{V}}$ is a locally finite variety.

6.6 LEMMA. *If* $i > 0$ *then* $S(\underline{\underline{U}}_i\underline{\underline{V}}) \subseteq S(\underline{\underline{UV}})$.

Proof. It is enough to prove the lemma for $i = 1$. If every section closed class of groups generating $\underline{\underline{UV}}$ contains a subclass generating $\underline{\underline{U}}_1\underline{\underline{V}}$ then $S(\underline{\underline{U}}_1\underline{\underline{V}}) \subseteq S(\underline{\underline{UV}})$.

Let $p = p(1)$ and $\alpha = \alpha(1)$. By 4.3, $\underline{\underline{U}}_1\underline{\underline{V}}$ is generated by groups $L = G \text{ wr } H$ where $G \in M(\underline{\underline{U}})$, σG is a p-group and $H \in F(\underline{\underline{V}})$. Since σG is a p-group and $\sigma L = (\sigma G)^H$, σL is also a p-group.

Suppose G is a section closed class of groups generating $\underline{\underline{UV}}$ and let

$$L \cong M/N , \quad M \leq G_1 \times \ldots \times G_t , \quad G_i \in G \text{ for } i = 1, \ldots, t$$

be a minimal representation of L on G. We show $G_i \in \underline{\underline{U}}_1\underline{\underline{V}}$ for all i.

Let $\underline{\underline{Y}} = \underline{\underline{W}}_1\underline{\underline{V}}$. As $L \in \underline{\underline{U}}_1\underline{\underline{V}}$, $\underline{\underline{Y}}(L) \in \underline{\underline{A}}_{p^\alpha}$. By 2.9 there is an $N_i \leq \sigma L$ such that N_i is similar in L to σG_i in G_i. Then

$$L/C_L(N_i) \cong G_i/\sigma^* G_i .$$

Since $\sigma L \leq \underline{\underline{Y}}(L) \in \underline{\underline{A}}_{p^\alpha}$, $C_L(N_i) \geq \underline{\underline{Y}}(L)$. Thus $G_i/\sigma^* G_i \in \underline{\underline{Y}}$ and so $\sigma^* G_i \geq \underline{\underline{Y}}(G_i)$ for each i. Fix i and let $K = \underline{\underline{Y}}(G_i)$. If $K = 1$ then $G_i \in \underline{\underline{Y}} \subseteq \underline{\underline{U}}_1\underline{\underline{V}}$ and we are done. Suppose $K > 1$.

Since $K \leq \underline{\underline{V}}(G_i) \in \underline{\underline{U}}$, K is an A-group. By [15, VI 14.3 (b)], $K' \cap Z(K) = 1$. As K centralizes σG_i, $\sigma G_i \leq Z(K)$ so $Z(K) \neq 1$. As G_i is monolithic, $K' = 1$ and K is abelian. It follows that each Sylow subgroup of K is normal in G_i and so K must be of prime power order. Now $K \geq \sigma G_i$ which is isomorphic to N_i so K is a p-group and thus $K \in \underline{\underline{A}}_{p^\alpha}$. Hence

$G_i \in \underline{\underline{A}}_{p^\alpha}\underline{\underline{Y}} \wedge \underline{\underline{UV}} = \left(\underline{\underline{A}}_{p^\alpha}\underline{\underline{W}}_1 \wedge \underline{\underline{U}}\right)\underline{\underline{V}}$. If we show $\underline{\underline{A}}_{p^\alpha}\underline{\underline{W}}_1 \wedge \underline{\underline{U}} = \underline{\underline{U}}_1$ we are done.

Clearly $\underline{\underline{U}}_1 \subseteq \underline{\underline{A}}_{p^\alpha}\underline{\underline{W}}_1 \wedge \underline{\underline{U}}$. A monolithic group in $\underline{\underline{A}}_{p^\alpha}\underline{\underline{W}}_1 \wedge \underline{\underline{U}}$ but not in $\underline{\underline{W}}_1$ must have a p-group for its monolith and be in $\underline{\underline{U}}_1$ by definition of $\underline{\underline{U}}_1$. A monolithic group in $\underline{\underline{W}}_1$ is in $\underline{\underline{U}}_1$ since $\underline{\underline{W}}_1 \subseteq \underline{\underline{U}}_1$. This completes the proof. //

6.7 LEMMA. *Let* $\underline{\underline{U}}$ *be a variety of* A*-groups and* $\underline{\underline{V}}$ *be a locally finite variety and* $G \in M(\underline{\underline{UV}})\backslash\underline{\underline{V}}$ *with* σG *a* p*-group. Let* $\underline{\underline{U}}^*$ *be generated by the groups in* $M(\underline{\underline{U}})$ *with monolith a* p*-group.*

Then $G \in \underline{\underline{U}}^*\underline{\underline{V}}$.

Proof. Let $\sigma = \sigma G$, $\sigma^* = \sigma^* G$ and $V = \underline{\underline{V}}(G)$. Since $G \notin \underline{\underline{V}}$, $V \geq \sigma$. Clearly $\sigma^* \geq \sigma$ so $\sigma^* \cap V \geq \sigma$. As σ^* centralizes σ , $\sigma \leq Z(\sigma^* \cap V)$. As V is an A-group so is $\sigma^* \cap V$ and by [15, VI 14.3 (b)],

$$Z(\sigma^* \cap V) \cap (\sigma^* \cap V)' = 1 .$$

Also $(\sigma^* \cap V)'$ char $(\sigma^* \cap V) \triangleleft G$. As G is monolithic, $(\sigma^* \cap V)' = 1$. Thus $\sigma^* \cap V$ is abelian and its Sylow subgroups are normal in G . Therefore $\sigma^* \cap V$ is a p-group.

Since V is an A-group, a Sylow p-subgroup S of V containing σ is abelian so

$$\sigma \leq S \leq \sigma^* \cap V .$$

Because $\sigma^* \cap V$ is a p-group, $S = \sigma^* \cap V$. Thus $S \triangleleft G$.

Suppose N is a normal subgroup of V avoiding S . Then

$$[N, \sigma] \leq [N, S] \leq N \cap S = 1 ,$$

so $N \leq \sigma^* \cap V = S$ and $N = 1$. Thus $S \geq \sigma V$, the socle of V . It follows that V is a subdirect product of monolithic groups each with monolith a p-group. Hence $V \in \underline{\underline{U}}^*$ and $G \in \underline{\underline{U}}^*\underline{\underline{V}}$. //

Derivation of 6.1. Suppose $G \in M(\underline{\underline{UV}})$. If σG is not abelian then by 4.4, $G \in S(\underline{\underline{UV}})$. If $\underline{\underline{V}}(G) = 1$ then by 4.5, $G \in S(\underline{\underline{UV}})$. Suppose σG is abelian and $\underline{\underline{V}}(G) \neq 1$. Then for some i , σG is a $p(i)$-group. By 6.7, $G \in \underline{\underline{U}}_i\underline{\underline{V}} \backslash \underline{\underline{W}}_i\underline{\underline{V}}$. Thus G has a normal subgroup N such that $N \in \underline{\underline{U}}_i \backslash \underline{\underline{W}}_i$ and $G/N \in \underline{\underline{V}}$. By 6.4, $G \in S(\underline{\underline{U}}_i\underline{\underline{V}})$ and by 6.6, $G \in S(\underline{\underline{UV}})$. //

7. $R_\alpha G$-Modules

Let p be a prime held fixed for the rest of this chapter and $\underline{\underline{U}}$ be a locally finite variety. In the next section we shall take $\underline{\underline{U}}$ as in 6.4 but that restriction is unnecessary here. Let G be a group and V be a normal p'-subgroup of G. Let

$$C(V) = \{A \mid A \text{ is an } R_\alpha V\text{-module and } VA \in \underline{\underline{U}}\} .$$

As G and V are fixed in this section we may write C for $C(V)$.

7.1 DEFINITION. If $A \leq B$ and $B \in C$ then $A \in C$ and $B/A \in C$. If $A_1, A_2 \in C$, then $A_1 \oplus A_2 \in C$. Thus if $A_1, A_2 \leq B$ then $A_1, A_2 \in C$ implies $A_1 + A_2 \in C$ while $B/A_1, B/A_2 \in C$ implies

$$B/A_1 \cap A_2 \in C .$$

Thus we may define the *C-radical of* A, C-rad A, of an $R_\alpha V$-module A to be the largest submodule of A in C. The *C-residual of* A, C-res A, is the smallest submodule of A such that the factor module is in C. Notice that by 3.12,

$$A/(\text{C-res } A) \cong \text{C-rad } A .$$

7.2 REMARK. For $R_\alpha V$-modules A, B since $A \oplus B \twoheadrightarrow A$ induces C-rad$(A \oplus B) \twoheadrightarrow$ C-rad A we find

$$\text{C-rad}(A \oplus B) = (\text{C-rad } A) \oplus (\text{C-rad } B) .$$

Let C = C-res$(A \oplus B)$. Then

$$C \subseteq [(\text{C-res } A) \oplus B] \cap [A \oplus (\text{C-res } B)] = (\text{C-res } A) \oplus (\text{C-res } B) .$$

Also $A/A \cap C \cong (A+C)/C \in C$ so $A \cap C \supseteq$ C-res A. It follows that

$$\text{C-res}(A \oplus B) = (\text{C-res } A) \oplus (\text{C-res } B) .$$

If A is an $R_\alpha G$-module, $B \leq A_V$ and $g \in G$, then by 3.8, $B \in C$ implies $Bg \in C$, and in fact C-rad A_V admits G. More generally,

$(\mathcal{C}\text{-rad } Bg)g^{-1} \in \mathcal{C}$ and so, since $(\mathcal{C}\text{-rad } Bg)g^{-1} \leq B$,

$$(\mathcal{C}\text{-rad } Bg)g^{-1} \subseteq \mathcal{C}\text{-rad } B \in \mathcal{C} .$$

Thus

$$\mathcal{C}\text{-rad } Bg = (\mathcal{C}\text{-rad } B)g .$$

Similarly

$$\left(\frac{Bg}{\mathcal{C}\text{-res } Bg}\right)g^{-1} = \frac{B}{(\mathcal{C}\text{-res } Bg)g^{-1}} \in \mathcal{C}$$

and

$$\frac{Bg}{(\mathcal{C}\text{-res } B)g} = \left(\frac{B}{\mathcal{C}\text{-res } B}\right)g \in \mathcal{C}$$

so

$$\mathcal{C}\text{-res } Bg = (\mathcal{C}\text{-res } B)g .$$

7.3 DEFINITION. Let $\mathcal{B}(G)$ be the class of $R_\alpha G$-modules A such that the restriction $A_V \in \mathcal{C}$. In this section G is fixed and we write $\mathcal{B}$ for $\mathcal{B}(G)$. It is easy to see that $\mathcal{B}$ has the closure properties of $\mathcal{C}$ described in 7.1 and $\mathcal{B}$-rad A and $\mathcal{B}$-res A may be defined in the obvious ways. Because $(\mathcal{B}\text{-rad } A)_V \subseteq \mathcal{C}\text{-rad } A_V$ and because $\mathcal{C}$-rad A_V admits the action of G we have

$$(\mathcal{B}\text{-rad } A)_V = \mathcal{C}\text{-rad } A_V .$$

Similarly, $\mathcal{C}$-res A_V admits G and so

$$(\mathcal{B}\text{-res } A)_V = \mathcal{C}\text{-res } A_V .$$

Furthermore arguments for $\mathcal{C}$ can be adapted to show

$$\mathcal{B}\text{-rad}(A \oplus B) = (\mathcal{B}\text{-rad } A) \oplus (\mathcal{B}\text{-rad } B)$$

and

$$\mathcal{B}\text{-res}(A \oplus B) = (\mathcal{B}\text{-res } A) \oplus (\mathcal{B}\text{-res } B)$$

for any $R_\alpha G$-modules A and B .

7.4 THEOREM. *If an* $R_\alpha G$*-module* A *is a direct sum of* *homocyclic* $R_\alpha G$-*modules then*

$$B\text{-rad } A \cong A/(B\text{-res } A) .$$

Proof. Suppose first that A is indecomposable and let

$$A_V = \bigoplus_i A_i$$

where A_i is the C_i-component of A_V for some irreducible $R_\alpha V$-module C_i. For $g \in G$ the composition factors of $A_1 g$ are isomorphic to $C_1 \otimes g$. Thus if $C_1 \otimes g \cong C_i$ then, since $C_1 \otimes g$ cannot be isomorphic to C_j for $j \neq i$, the projection of A_V onto A_j determined by the direct sum must send each composition factor of $A_1 g$ to 0 and so the projection of A_V onto A_i must send each composition factor of $A_1 g$ isomorphically. Thus $A_1 g \subseteq A_i$. Since

$$A_i g^{-1} \supseteq (A_1 g) g^{-1} = A_1$$

a similar argument shows $A_i g^{-1} \subseteq A_1$ and consequently $A_i = A_1 g$. It follows that G permutes the A_i, and if G has more than one orbit then A is decomposable, which is a contradiction. Hence G permutes the A_i transitively.

If A is indecomposable and homocyclic of exponent p^β then it follows from the proof of 3.12 that A_i is a direct sum of isomorphic indecomposable $R_\alpha V$-modules. By 3.11 there is an $\alpha(i)$ such that $C\text{-rad } A_i = p^{\alpha(i)} A_i$ and by 7.1, $C\text{-res } A_i = p^{\beta-\alpha(i)} A_i$. Let $g \in G$ be such that $A_i g = A_j$. Then

$$p^{\alpha(i)} A_j = \left(p^{\alpha(i)} A_i\right) g = (C\text{-rad } A_i) g = C\text{-rad } A_j = p^{\alpha(j)} A_j .$$

Thus $\alpha(i) = \alpha(j)$ for all i, j. Let $\alpha = \alpha(i)$. Now by 7.2,

$$\mathcal{C}\text{-rad } A_V = p^{\alpha} A_V \quad \text{and} \quad \mathcal{C}\text{-res } A_V = p^{\beta-\alpha} A_V \ .$$

By 7.3, $(\mathcal{B}\text{-rad } A)_V = \mathcal{C}\text{-rad } A_V$ so $\mathcal{B}\text{-rad } A = p^{\alpha}A$ and similarly $\mathcal{B}\text{-res } A = p^{\beta-\alpha}A$. Thus the theorem is true if A is homocyclic and indecomposable.

Suppose now that A is a direct sum of homocyclic modules. We can write $A = \bigoplus_i A_i$ where the A_i are homocyclic and indecomposable. By 7.3 and the last paragraph we have

$$\mathcal{B}\text{-rad } A = \bigoplus_i (\mathcal{B}\text{-rad } A_i) \cong \bigoplus_i \{A_i/\mathcal{B}\text{-res } A_i\} \cong A/(\mathcal{B}\text{-res } A) \ ,$$

which completes the proof. //

Let $I(G) = \mathcal{B}\text{-rad } R_{\alpha}G$ and $J(G) = R_{\alpha}G/\{\mathcal{B}\text{-res } R_{\alpha}G\}$. As G is fixed in this section we write I and J for $I(G)$ and $J(G)$ respectively. By the last theorem, $I \cong J$ and by its proof any indecomposable direct summand of I is monolithic.

A module $A \in \mathcal{B}$ is said to be $\mathcal{B}$-*injective* if whenever $B \leq C$ and $C \in \mathcal{B}$ then every homomorphism $B \to A$ can be extended to a

C homomorphism $C \to A$. Suppose $B \leq$ ~~A~~, $C \in \mathcal{B}$ and $\theta : B \to I$ is a homomorphism. Then $I \leq R_{\alpha}G$ so $\theta : B \to I \leq R_{\alpha}G$ can be extended to $\pi : C \to R_{\alpha}G$. As $C \in \mathcal{B}$, $C\pi \in \mathcal{B}$ so $C\pi \leq I$. Thus I is $\mathcal{B}$-injective. By an argument similar to [10, 57.3] it can be shown that any direct summand of I is $\mathcal{B}$-injective.

A module $A \in \mathcal{B}$ is said to be $\mathcal{B}$-*projective* if whenever $\pi : B \twoheadrightarrow C$ is a homomorphism of B onto C, $B \in \mathcal{B}$ and there is a homomorphism $\theta : A \to C$ then there is a homomorphism $\mu : A \to B$ such that $\mu\pi = \theta$. Suppose $\pi : B \twoheadrightarrow C$, $B \in \mathcal{B}$ and $\theta : J \to C$. As J is a factor module of $R_{\alpha}G$, θ induces $\overline{\theta} : R_{\alpha}G \to C$. As $R_{\alpha}G$ is a projective $R_{\alpha}G$-module there is a homomorphism $\overline{\mu} : R_{\alpha}G \to B$

such that $\overline{\mu}\pi = \overline{\theta}$. Now $R_\alpha G/\ker \overline{\mu} \in \mathcal{B}$ so by the minimality of $\mathcal{B}$-res $R_\alpha G$, $\ker \overline{\mu} \geq \mathcal{B}$-res $R_\alpha G$. Thus there is a homomorphism $\mu : J \to B$ such that if $\chi : R_\alpha G \twoheadrightarrow J$ is the obvious map, then $\chi\mu = \overline{\mu}$. Now $\overline{\theta} = \chi\theta$ so $\chi\mu\pi = \chi\theta$ and, since χ is onto, $\mu\pi = \theta$. Thus J is $\mathcal{B}$-projective. By an argument similar to that of [10, 56.5] it can be shown that any direct summand of J is $\mathcal{B}$-projective.

7.5 COROLLARY. *A direct summand of* I *is* $\mathcal{B}$*-projective and* $\mathcal{B}$*-injective.*

Proof. By 7.4, $I \cong J$ and a direct summand of I is isomorphic to a direct summand of J , which is $\mathcal{B}$-projective. //

The following lemma is similar to [5, 2.2].

7.6 LEMMA. *Suppose* H *is an extension of a module* $B \in \mathcal{B}$ *by* G *where the action of* G *on* B *by conjugation is the module action, and* $A \leq B$ *for some monolithic direct summand* A *of* I *. If* $N \leq H$ *is maximal such that* $N \triangleleft H$ *and* $N \cap \sigma A = 1$ *then*

$$H/N \cong GA/\ker A .$$

Proof. The factor group H/N is an extension of BN/N by H/BN , and BN/N is an $R_\alpha G$-module. As $N \cap \sigma A = 1$, $N \cap A = 1$ and $A \lesssim BN/N$. As $BN/N \in \mathcal{B}$ and A is $\mathcal{B}$-injective, A is isomorphic to a direct summand of BN/N . By the choice of N , $A \cong BN/N$.

Since $H/B \cong G$, H/BN is isomorphic to a factor group of G : let $K \triangleleft G$ be such that $H/BN \cong G/K$. Since A is an $R_\alpha G$-module and, by the last paragraph, an $R_\alpha(G/K)$-module via the isomorphism, we have $K \leq \ker A$. By the maximality of N , $K = \ker A$.

It follows from the proof of 7.4 that A may be considered as an injective $R_\beta G$-module for some β . It is not hard to see this implies A is an injective $R_\beta(G/\ker A)$-module. Since H/N is isomorphic to an extension of A by $G/\ker A$, we have by [5, 2.1],

$$H/N \cong GA/\ker A \text{ .} \qquad //$$

7.7 LEMMA. *If A is a monolithic module in $\mathcal{B}$ then there is a direct summand A_1 of I and an integer β such that A and A_1 can be considered as $R_\beta G$-modules and A_1 is isomorphic to the $R_\beta G$-injective hull of A.*

Proof. Let B be the $R_\alpha G$-injective hull of A. Since A is monolithic so is B and thus B is principal indecomposable. Let C be a complement for B in $R_\alpha G$. By 7.3,

$$I = \mathcal{B}\text{-rad } R_\alpha G = (\mathcal{B}\text{-rad } B) \oplus (\mathcal{B}\text{-rad } C) \text{ .}$$

Now $A \in \mathcal{B}$ and $A \leq B$ so $A \leq \mathcal{B}\text{-rad } B$.

Since B is monolithic so is $\mathcal{B}$-rad B. Thus $\mathcal{B}$-rad B is an indecomposable direct summand of I. By the proof of 7.4 there is an integer β such that $\mathcal{B}$-rad B, considered as $R_\beta G$-module, is injective. It follows that, considered as $R_\beta G$-modules, $A_1 = \mathcal{B}\text{-rad } B$ is the $R_\beta G$-injective hull of A. //

8. The Skeleton of a Product Variety

Let $\underline{\underline{U}}$, $\underline{\underline{V}}$ and $\underline{\underline{W}}$ be as in the statement of Theorem 6.4. Let $\underline{\underline{Y}} = \underline{\underline{WV}}$ and Y be the (infinite) free group of countably infinite rank of $\underline{\underline{Y}}$ generated by $y_1, y_2, \ldots$, and Y_n the free subgroup of Y generated by $y_1, \ldots, y_n$, and let $V_n = \underline{\underline{V}}(Y_n)$. Let $\underline{\underline{Z}} = \underline{\underline{UV}}$, Z its free group freely generated by $z_1, z_2, \ldots$ and Z_n the subgroup generated by $z_1, \ldots, z_n$. When we write $\mathcal{B}(H)$ or $I(H)$

with $H \in \underline{\underline{Y}}$ then H and $\underline{\underline{V}}(H)$ correspond respectively to G and V of the last section.

8.1 PROPOSITION. *The variety* $\underline{\underline{Z}}$ *is generated by*

$$H = \{Y_n A/\ker A \mid A \text{ is a principal indecomposable } R_\beta Y_n\text{-module such that } V_n A \in \underline{\underline{U}},\ 1 \le \beta \le \alpha,\ n = 1, 2, \ldots\}.$$

Proof. Observe that if $GA/\ker A \in H$ then $\underline{\underline{V}}(GA/\ker A) \le \underline{\underline{V}}(G/\ker A)A \in \underline{\underline{U}}$ so $GA/\ker A \in \underline{\underline{UV}}$ and $H \subseteq \underline{\underline{UV}}$.

We show that there is a class of groups in var H which generates $\underline{\underline{UV}}$. By the Schur Zassenhaus Theorem, 2.4, a critical group in $\underline{\underline{U}}$ but not in $\underline{\underline{W}}$ is the split extension of its $\underline{\underline{W}}$-verbal subgroup, H say, by a group in $\underline{\underline{W}}$, K say. As KH is monolithic, $C_K(H) = 1$. Because $H \in \underline{\underline{A}}_{p^\alpha}$ and $K \in \underline{\underline{W}}$, K is isomorphic to a p'-group of automorphisms of H. Furthermore K acts indecomposably on H so by [11, 5.2.2], H is homocyclic, say of exponent p^β, $\beta \le \alpha$.

By 4.3, $\underline{\underline{UV}}$ is generated by groups $KH \text{ wr } F_r(V) = M$, KH as above, $r = 1, 2, \ldots$. In order to prove 8.1, it therefore suffices to show that each such M is contained in var H. Let $A = H^{F_r(\underline{\underline{V}})}$ be the Sylow p-subgroup of the base group of M. Let $F = \text{gp}(K, F_r(\underline{\underline{V}}))$ so that F is a complement for A in M. As $F \in \underline{\underline{Y}}$, for some n there is a homomorphism

$$\pi : Y_n \twoheadrightarrow F$$

of Y_n onto F. Let $G = Y_n$ and $V = \underline{\underline{V}}(G)$. We regard A as an $R_\beta G$-module via π; that is, for $a \in A$ and $g \in G$ define $ag = a(g^\pi)$.

Because A and $\underline{\underline{V}}(F)$ are in the base group of M,

$\underline{\underline{V}}(F)A \in \underline{\underline{U}}$. Now the restriction $\pi|_V$ of π to V maps V onto $\underline{\underline{V}}(F)$. The groups $\ker \pi|_V$ and A are normal subgroups of VA such that $A \cap \ker \pi|_V = 1$. It follows that VA is a subdirect product of $\underline{\underline{V}}(F)A$ and V so $VA \in \underline{\underline{U}}$. Thus $A_V \in C(V)$. Now $(A_V)^G{}_V$ is a direct sum of conjugates of A_V which are all isomorphic by 3.8, so $(A_V)^G{}_V \in C(V)$ and therefore $(A_V)^G \in \mathcal{B}(G)$. This implies that $G(A_V)^G \in \underline{\underline{UV}}$.

By 2.3, $H = \sigma^*(KH)$ so if $K \neq 1$ then 2.6 may be invoked to give $C_F(A) = 1$; if $K = 1$ this is obvious. Thus $\ker \pi = C_G(A) = \ker A$ so $FA \cong GA/\ker A$. By 3.5, $A \lesssim (A_V)^G$ so by 3.6, FA is a section of $G(A_V)^G/\ker(A_V)^G$. By 3.9, $(A_V)^G \cong \bigoplus_{i=1}^{s} A_i$ where the A_i are principal indecomposable $R_\beta G$-modules. Since $\bigcap_i \ker A_i = \ker(A_V)^G$, $G(A_V)^G/\ker(A_V)^G$ is a subdirect product of the $GA_i/\ker A_i$. Since $V(A_V)^G \in \underline{\underline{U}}$, $VA_i \in \underline{\underline{U}}$ for each i . Thus $GA_i/\ker A_i \in H$ for each i so H generates $\underline{\underline{UV}}$. //

Let F_∞ be the (infinite) free group of countably infinite rank of $\underline{\underline{A}}_p \underline{\underline{Y}}_\alpha$ freely generated by $f_1, f_2, \ldots$ and let F_n be the subgroup of F generated by $f_1, f_2, \ldots, f_n$. Define $\delta : Y_{n+1} \twoheadrightarrow Y_n$ by $y_i^\delta = y_i$ for $i \leq n$ and $y_{n+1}^\delta = 1$.

8.2 LEMMA. *Let* $A = \underline{\underline{Y}}(F_{n+1})$ *and regard* A *as a* Y_{n+1}*-module via the homomorphism* $\zeta : F_{n+1} \twoheadrightarrow Y_{n+1}$ *which sends* $f_i \mapsto y_i$ *for* *all* $i = 1, \ldots, n+1$. *Then* A *contains a submodule* B *such that* $B_{Y_n} \cong R_\alpha Y_n$ *and* $\ker \delta$ *acts trivially on* B .

Proof. Notice $\ker \zeta = A$. Let T be a right transversal for $\mathrm{gp}(f_{n+1}, A)$ in the complete inverse image of $\ker \delta$ under ζ . Then each element x of $\ker \delta$ can be written uniquely as a product yt^ζ with y an element of $\mathrm{gp}(y_{n+1})$ and t an element of T . Let C be a multiplicatively written regular $R_\alpha Y_{n+1}$-module generated by c . Observe that the submodule of C_{Y_n} generated by $\prod\{c^x \mid x \in \ker \delta\}$ is a regular $R_\alpha Y_n$-module. Since $Y_{n+1}C \in \underline{\underline{A}}_p \underline{\underline{\alpha Y}}$, there is a homomorphism $\varphi : F_{n+1} \to Y_{n+1}C$ such that $f_i^\varphi = y_i$ for all $i \le n$ and $f_{n+1}^\varphi = cy_{n+1}$. Let e be the exponent of $\underline{\underline{Y}}$, $f = f_{n+1}^e$ and $h = \prod_{t \in T} f^t$. Notice $h \in A$ and $\ker \delta$ acts trivially on h . Let B be the $R_\alpha Y_{n+1}$-module generated by h . By the preceding remark $\ker \delta$ acts trivially on B , and B_{Y_n} is also generated by h . For each $g \in F_n$ we have

$$h^g \circ \varphi = \prod_t (f \circ \varphi)^{(t\circ\varphi)(g\circ\varphi)} = \prod_t \left(\prod_y c^y\right)^{(t\circ\zeta)(g\circ\zeta)} = \left(\prod_x c^x\right)^{g\circ\zeta}$$

where y ranges through $\mathrm{gp}(y_{n+1})$ and x through $\ker \delta$. This shows that $h \mapsto \prod_x c^x$ extends to a homomorphism of B_{Y_n} onto the regular submodule of C_{Y_n} generated by $\prod_x c^x$; hence B_{Y_n} is a regular submodule of A_{Y_n} . //

8.3 COROLLARY. *If* $I(Y_n)$ *is regarded as a* Y_{n+1}*-module via* δ *and* $\underline{\underline{Y}}(Z_{n+1})$ *is regarded as a* Y_{n+1}*-module via the homomorphism* $Z_{n+1} \twoheadrightarrow Y_{n+1}$ *such that* $z_i \mapsto y_i$ *for all* $i = 1, \ldots, n+1$ *then*

$$I(Y_n) \lesssim \underline{\underline{Y}}(Z_{n+1})$$

Proof. Let A and B be as in 8.2. Then $I(Y_n) \lesssim B \leq A$. Also the split extension $V_{n+1}I(Y_n)$ is a subdirect product of V_{n+1} and $V_nI(Y_n)$ and is therefore in $\underline{\underline{U}}$. Thus $I(Y_n) \lesssim B(Y_{n+1})\text{-rad } A$. By [20, 21.13], A is a free group in $\underline{\underline{A}}_{p^\alpha}$ so we can apply 7.4 to get

$$I(Y_n) \lesssim B(Y_{n+1})\text{-rad } A \cong A/(B(Y_{n+1})\text{-res } A) .$$

Since $A = \underline{\underline{Y}}(F_{n+1})$ and $B(Y_{n+1})\text{-res } A = \underline{\underline{Z}}(F_{n+1})$,

$$I(Y_n) \lesssim B(Y_{n+1})\text{-rad } A \cong \underline{\underline{Y}}(F_{n+1})/\underline{\underline{Z}}(F_{n+1}) .$$

The homomorphism $F_{n+1} \twoheadrightarrow Z_{n+1}$ such that $f_i \mapsto z_i$ for all $i = 1, \ldots, n+1$ has kernel $\underline{\underline{Z}}(F_{n+1})$ and induces a module isomorphism

$$I(Y_n) \lesssim \underline{\underline{Y}}(F_{n+1})/\underline{\underline{Z}}(F_{n+1}) \cong \underline{\underline{Y}}(Z_{n+1}) .$$

This completes the proof. //

The next lemma gives one description of $S(\underline{\underline{Z}})$.

8.4 LEMMA. *The skeleton* $S(\underline{\underline{Z}}) = QsH$ *with* H *as in* 8.1.

Proof. Suppose G is a section closed class of groups generating $\underline{\underline{Z}}$. We show $H \subseteq G$. If $Y_nB/\ker B \in H$ then by 7.7, $B \lesssim A$ for some monolithic direct summand A of $I(Y_n)$. By 8.3, $A \lesssim \underline{\underline{Y}}(Z_{n+1})$. Identify A with a subgroup of $\underline{\underline{Y}}(Z_{n+1})$ and consider the subgroup $H = \text{gp}(\underline{\underline{Y}}(Z_{n+1}), Z_n)$ of Z_{n+1}. Since Z_{n+1} is a subdirect product of groups in G, so is H. Since σA is a minimal normal subgroup of H there is a homomorphism θ of H onto a group in G such that $\sigma A \cap \ker\theta = 1$. It follows that $A \cap \ker\theta = 1$. Let $N \geq \ker\theta$ and be maximal in H such that $N \triangleleft H$ and $A \cap N = 1$. Then $H/N \in G$. Since H is an extension of $\underline{Y}(Z_{n+1})$ by Y_n and $\underline{Y}(Z_{n+1}) \in B(Y_n)$, 7.6 gives

$$H/N \cong Y_n A/\ker A .$$

As $B \lesssim A$, $Y_n B/\ker B$ is a section of $Y_n A/\ker A$ by 3.6. Thus $Y_n B/\ker B \in G$ and $H \subseteq G$. Now by 8.1,

$$S(\underline{Z}) \subseteq \mathrm{QS}H \subseteq \cap\{G \mid G \text{ SCQ}\underline{Z}\} = S(\underline{Z}) . \qquad //$$

8.5 LEMMA. *Let* H *be as in* 8.1 *and*

$K = \{G \mid G \in M(\underline{Z})$ *and there is an* $N \triangleleft G$ *such that* $N \in \underline{U}\backslash\underline{W}$ *and* $G/N \in \underline{V}\}$.

Then $\mathrm{QS}H = \mathrm{QS}K$.

Proof. Let $G \in K$ and $N \triangleleft G$ such that $N \in \underline{U}\backslash\underline{W}$ and $G/N \in \underline{V}$. We show $G \in \mathrm{QS}H$. Let $A = \underline{W}(N)$ so that A is the unique Sylow p-subgroup of N. The subgroup $\mathrm{gp}(A, \underline{V}(G))$ of N is in $\underline{U}$ since N is. Clearly $A \in S_p(\mathrm{gp}(A, \underline{V}(G)))$ and has a complement isomorphic to $\underline{V}(G/A)$. Let $K = G/A$. The split extension $\underline{V}(K)A$ is in $\underline{U}$. If we regard A as an $R_\alpha K$-module then $A \in B(K)$ (taking K for G and $\underline{V}(K)$ for V of the last section). By 7.7 there is a $\beta \in \{1, \ldots, \alpha\}$ such that if A is regarded as an $R_\beta K$-module then the $R_\beta K$-injective hull B of A is isomorphic to a direct summand of $I(K)$. Form an extension G^* of B by K using the same factor set as in the extension G of A by K. Then $G \leq G^*$. As B is an injective $R_\beta K$-module, G^* splits over B, $G^* = KB$, by [5, 2.1]. Let $M = C_K(B)$. Clearly $M \triangleleft G^*$. Since $\sigma G \leq A \leq B$, $M \cap B = 1$ implies $M \cap \sigma G = 1$ and $M \cap G = 1$. Thus $G \lesssim G^*/M$.

By the choice of B the split extension $\underline{\underline{V}}(K)B$ is in $\underline{\underline{U}}$. In a natural way B is a K/M-module and it follows that $\underline{\underline{V}}(K/M)B \in \underline{\underline{U}}$. As $K \in \underline{\underline{Y}}$, $K/M \in \underline{\underline{Y}}$ so there is an n for which there is a homomorphism θ of Y_n onto K/M, and we regard B as a Y_n-module via θ. Now $V_n \le Y_n$ and the split extension V_nB is a subdirect product of $\underline{\underline{V}}(K/M)B$ and V_n, so $V_nB \in \underline{\underline{U}}$. Taking Y_n and V_n for the G and V of the last section, 7.7 implies there is $\gamma \in \{\beta, \beta+1, \ldots, \alpha\}$ such that if B is regarded as an $R_\gamma Y_n$-module then the $R_\gamma Y_n$-injective hull C of B is a direct summand of $I(Y_n)$. Now $KB/M \cong Y_nB/\ker B$ so by 3.6, KB/M is a section of $Y_nC/\ker C \in \mathfrak{H}$. Since $G \lesssim G^*/M = KB/M$, $G \in QS\mathfrak{H}$ and so $\mathfrak{K} \subseteq QS\mathfrak{H}$.

On the other hand suppose $H = Y_nA/\ker A \in \mathfrak{H}$. Let $N = \mathrm{gp}(V_n, \ker A)A/\ker A$. Then $N \lhd H$, $N \in \underline{\underline{U}}\backslash\underline{\underline{W}}$ and $H/N \in \underline{\underline{V}}$ so $\mathfrak{H} \subseteq QS\mathfrak{K}$. //

Theorem 6.4 is a consequence of Proposition 8.1 and Lemmas 8.4 and 8.5.

APPENDIX

The main theorem in the following paper deals with the lattice of varieties of groups rather than with section closed classes of groups. It is included because it provides another application of the main technical theorem of this chapter, Theorem 6.4. *However to make the appendix self-contained the specific case of the theorem needed is proved here. It is interesting to contrast the ease of proof of this special case with the complexity of the proof of Theorem* 6.4. *The reference numbers in the appendix refer to the references at the end of it rather than to those at the end of the thesis.*

A PRODUCT VARIETY OF GROUPS WITH DISTRIBUTIVE LATTICE

L.F. Harris

Abstract. By a variety of A-groups is meant a locally finite variety of groups whose nilpotent groups are abelian. It is shown that if $\underline{\underline{U}}$ is a variety of A-groups and $\underline{\underline{V}}$ is a locally finite variety whose lattice of subvarieties is distributive and the exponents of $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are coprime, then the lattice of subvarieties of the product variety $\underline{\underline{UV}}$ is distributive.

1. Introduction.

The *lattice of a variety* $\underline{\underline{V}}$ *of groups* is the lattice of subvarieties of $\underline{\underline{V}}$ partially ordered by inclusion. It is modular because the lattice of the variety of all groups is dual to the lattice of fully invariant subgroups of the free group of countably infinite rank. For any positive integer m let $\underline{\underline{A}}_m$, $\underline{\underline{B}}_m$ and $\underline{\underline{N}}_m$ denote respectively the variety of all abelian groups of exponent dividing m, the variety of all groups of exponent dividing m, and the variety of all groups which are nilpotent of class at most m. A *variety of A-groups* is defined to be a locally finite variety whose nilpotent groups are abelian. G. Higman [7, 54.24] gave the first example of a variety with a nondistributive lattice. R.A. Bryce [3, 6.2.5] showed that for a prime p the product variety $\underline{\underline{A}}_{p^2}\underline{\underline{A}}_{p^2}$ has a nondistributive lattice but that a variety of metabelian groups of bounded exponent in which, for each p, the p-groups have class at most p has distributive lattice. He also showed that if m is nearly prime to n (*i.e.* if a prime p divides m then p^2 does not divide n) then the lattice of $\underline{\underline{A}}_m\underline{\underline{A}}_n$ is distributive.

M. S. Brooks [2] showed that the lattice of $\underline{\underline{A}}_3\underline{\underline{A}}_9$ is not distributive. The main result here generalizes one of John Cossey [4] who showed that the lattice of varieties of A-groups is distributive. The *exponent* of a locally finite variety is defined to be the order of the free group on one generator of the variety.

THEOREM 1. *Suppose* $\underline{\underline{U}}$ *is a variety of* A*-groups and* $\underline{\underline{V}}$ *is a locally finite variety with distributive lattice and the exponents of* $\underline{\underline{U}}$ *and* $\underline{\underline{V}}$ *are coprime. Then the lattice of* $\underline{\underline{UV}}$ *is distributive.*

Notation and terminology not here defined are as in Hanna Neumann [7]. In view of Theorem 1 it is worth noting that L.G. Kovács has an unpublished example which shows that although the lattice of the meet $\underline{\underline{B}}_8 \wedge \underline{\underline{N}}_3$ is distributive, that of $(\underline{\underline{B}}_8 \wedge \underline{\underline{N}}_3)\underline{\underline{A}}_3$ is not.

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2. A Theorem on Skeletons.

By a *section* of a group is meant a factor group of a subgroup of it. If G is a class of groups then sG and QG denote the classes of all groups isomorphic to, respectively, subgroups and factor groups of groups in G. A class G of groups is said to be *section closed* if $QG \subseteq G$ and $sG \subseteq G$. It is well known and easy to see that if G is a class of groups then QsG is section closed. The *skeleton* $S(\underline{\underline{V}})$ *of a variety* $\underline{\underline{V}}$

is defined (in Bryant and Kovács [2]) to be the intersection of the section closed classes of groups generating $\underline{\underline{V}}$. A *monolithic group* is defined to be a finite group with a unique minimal normal subgroup, called the *monolith*. To prove Theorem 1 we need the following result.

THEOREM 2. *Suppose* p *is a prime and* $\underline{\underline{Y}}$ *is a locally finite variety containing a variety* $\underline{\underline{X}}$ *of* p'*-exponent such that for some positive integer* α , $\underline{\underline{Y}}$ *is contained in* $\underline{\underline{A}}_{p^\alpha}\underline{\underline{X}}$,

$$\underline{\underline{X}} \subseteq \underline{\underline{Y}} \subseteq \underline{\underline{A}}_{p^\alpha}\underline{\underline{X}} ,$$

and $\underline{\underline{Y}}$ *is generated by monolithic groups not in* $\underline{\underline{X}}$. *Then*

$$S(\underline{\underline{Y}}) = \mathrm{Qs}\{G \mid G \in \underline{\underline{Y}},\ G \notin \underline{\underline{X}} \text{ and } G \text{ is monolithic}\} ,$$

and $S(\underline{\underline{Y}})$ *generates* $\underline{\underline{Y}}$.

Proof. Let G be a monolithic group in $\underline{\underline{Y}}$ but not in $\underline{\underline{X}}$, let σG be the monolith of G , σ^*G be the centralizer of σG in G , $Z(G)$ be the center of G , $X = \underline{\underline{X}}(G)$ be the $\underline{\underline{X}}$-verbal subgroup of G , and G' be the derived group of G . We write $H \triangleleft G$ if H is a normal subgroup of G .

Notice X is the Sylow p-subgroup of σ^*G ; we show they are equal. If σ^*G is not abelian then

$$\sigma G \le (\sigma^*G)' \cap Z(\sigma^*G) \cap X = 1$$

by [6, IV 2.2], which is a contradiction. Thus σ^*G is abelian and, since G is monolithic, σ^*G is of prime power order. Because

$$\sigma G \le X \le \sigma^*G ,$$

we have $X = \sigma^*G$.

Let H be a section closed class of groups generating $\underline{\underline{Y}}$. To prove the theorem it suffices to show $G \in H$. We shall use some properties of the minimal representation defined in [7, p. 163 *ff*].

Let

$$G \cong H/K \ , \quad H \leq H_1 \times \ldots \times H_r \ , \quad H_i \in \mathsf{H} \quad \text{for} \quad i = 1, \ldots, r$$

be a minimal representation of G on H. Then each H_i is monolithic and $\sigma H_i \cong \sigma G$ so σH_i is a p-group. By the last paragraph $\sigma^* H_i = \underline{\underline{X}}(H_i)$. By the Schur Zassenhaus Theorem there is a complement, K_i say, for $\sigma^* H_i$ in H_i. Since H_i is monolithic, $\sigma^* H_i$ is an indecomposable K_i-group so by [5, 5.2.2], $\sigma^* H_i$ is a homocyclic p-group. For some j the exponent of $\sigma^* H_j$ is greater than or equal to the exponent of $\sigma^* G$. Let n be the exponent of $\sigma^* G$. It follows as in Lemma 3 of Cossey [4] that

$$G \cong H_j / (\sigma^* H_j)^n \ ,$$

and $G \in \mathsf{H}$, proving the theorem.

3. Proof of Theorem 1. Let $\underline{\underline{U}}_1, \underline{\underline{U}}_2 \leq \underline{\underline{UV}}$. We first show

$$(\underline{\underline{V}} \wedge \underline{\underline{U}}_1) \vee (\underline{\underline{V}} \wedge \underline{\underline{U}}_2) = \underline{\underline{V}} \wedge (\underline{\underline{U}}_1 \vee \underline{\underline{U}}_2) \ . \qquad (*)$$

Since $(\underline{\underline{V}} \wedge \underline{\underline{U}}_1) \subseteq \underline{\underline{V}} \wedge (\underline{\underline{U}}_1 \vee \underline{\underline{U}}_2)$ it suffices to prove that if F is a finite free group of $\underline{\underline{V}} \wedge (\underline{\underline{U}}_1 \vee \underline{\underline{U}}_2)$ then $F \in (\underline{\underline{V}} \wedge \underline{\underline{U}}_1) \vee (\underline{\underline{V}} \wedge \underline{\underline{U}}_2)$. Let $\underline{\underline{U}}_1 \cup \underline{\underline{U}}_2$ denote the set theoretic union of $\underline{\underline{U}}_1$ and $\underline{\underline{U}}_2$. Let

$$F \cong H/K \ , \quad H \leq H_1 \times \ldots \times H_r \ , \quad H_i \in \underline{\underline{U}}_1 \cup \underline{\underline{U}}_2 \quad \text{for} \quad i = 1, \ldots, r$$

be a minimal representation of F on $\underline{\underline{U}}_1 \cup \underline{\underline{U}}_2$. Because $F \in \underline{\underline{V}}$, σH_i has exponent dividing that of $\underline{\underline{V}}$. Since the exponents of $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are relatively prime it follows that $H_i \in \underline{\underline{V}}$ for all i. As $H_i \in \underline{\underline{U}}_1 \cup \underline{\underline{U}}_2$ we have $H_i \in (\underline{\underline{V}} \cap \underline{\underline{U}}_1) \cup (\underline{\underline{V}} \cap \underline{\underline{U}}_2)$. It follows that $F \in (\underline{\underline{V}} \wedge \underline{\underline{U}}_1) \vee (\underline{\underline{V}} \wedge \underline{\underline{U}}_2)$, proving (*).

We need a lemma.

LEMMA. *If* G *is a monolithic group in* $\underline{\underline{U}}_1 \vee \underline{\underline{U}}_2$ *but not in* $\underline{\underline{V}}$ *then* $G \in \underline{\underline{U}}_1 \cup \underline{\underline{U}}_2$.

Proof. If σG is not abelian then by taking a minimal representation of G on $\underline{\underline{U}}_1 \cup \underline{\underline{U}}_2$ and arguing as in [7, 53.31] the result follows. Thus we may assume σG is an abelian p-group for some prime p . Let

$$G \cong H/K , \quad H \leq H_1 \times \ldots \times H_r , \quad H_i \in \underline{\underline{U}}_1 \cup \underline{\underline{U}}_2 \quad \text{for } i = 1, \ldots, r$$

be a minimal representation of G on $\underline{\underline{U}}_1 \cup \underline{\underline{U}}_2$. Let $V_i = (\sigma^* H_i) \cap \underline{\underline{V}}(H_i)$ and observe that the Sylow p-subgroups of the H_i are in V_i and $\sigma H_i \leq Z(V_i)$. As V_i is an A-group,

$$Z(V_i) \cap V_i' = 1 .$$

Since H_i is monolithic, $V_i' = 1$. Thus V_i is abelian and must be a p-group.

Let $\underline{\underline{Y}}$ be the variety generated by $H_1, \ldots, H_r$ and $\underline{\underline{X}}$ be the variety generated by $H_1/V_1, \ldots, H_r/V_r$. Then by Theorem 2,

$$S(\underline{\underline{Y}}) = Qs\{H \mid H \in \underline{\underline{Y}},\ H \notin \underline{\underline{X}} \text{ and } H \text{ is monolithic}\} .$$

It follows that

$$G \in S(\underline{\underline{Y}}) \subseteq Qs\{H_1, \ldots, H_r\} \subseteq \underline{\underline{U}}_1 \cup \underline{\underline{U}}_2$$

proving the lemma.

To prove Theorem 1 it suffices to show that if $\underline{\underline{W}} \subseteq \underline{\underline{UV}}$ then

$$\underline{\underline{W}} \wedge (\underline{\underline{U}}_1 \vee \underline{\underline{U}}_2) = (\underline{\underline{W}} \wedge \underline{\underline{U}}_1) \vee (\underline{\underline{W}} \wedge \underline{\underline{U}}_2) .$$

Since $\underline{\underline{W}} \wedge (\underline{\underline{U}}_1 \vee \underline{\underline{U}}_2) \supseteq \underline{\underline{W}} \wedge \underline{\underline{U}}_1$ it suffices to show that if G is a monolithic group in $\underline{\underline{W}} \wedge (\underline{\underline{U}}_1 \vee \underline{\underline{U}}_2)$ then G is in $(\underline{\underline{W}} \wedge \underline{\underline{U}}_1) \vee (\underline{\underline{W}} \wedge \underline{\underline{U}}_2)$.

Suppose first that $G \notin \underline{\underline{V}}$. Then by the lemma $G \in \underline{\underline{U}}_1 \cup \underline{\underline{U}}_2$.

As $G \in \mathfrak{W}$,

$$G \in \mathfrak{W} \cap \left(\mathfrak{U}_1 \cup \mathfrak{U}_2\right) = \left(\mathfrak{W} \cap \mathfrak{U}_1\right) \cup \left(\mathfrak{W} \cap \mathfrak{U}_2\right) \subseteq \left(\mathfrak{W} \wedge \mathfrak{U}_1\right) \vee \left(\mathfrak{W} \wedge \mathfrak{U}_2\right) .$$

Suppose $G \in \mathfrak{V}$. Using the fact that $\mathfrak{V}$ has distributive lattice and applying (*) twice, we have

$$\begin{aligned}
\mathfrak{V} \wedge \mathfrak{W} \wedge \left(\mathfrak{U}_1 \vee \mathfrak{U}_2\right) &= \mathfrak{V} \wedge \mathfrak{W} \wedge \left[\mathfrak{V} \wedge \left(\mathfrak{U}_1 \vee \mathfrak{U}_2\right)\right] \\
&= (\mathfrak{V} \wedge \mathfrak{W}) \wedge \left[\left(\mathfrak{V} \wedge \mathfrak{U}_1\right) \vee \left(\mathfrak{V} \wedge \mathfrak{U}_2\right)\right] \\
&= \left[(\mathfrak{V} \wedge \mathfrak{W}) \wedge \left(\mathfrak{V} \wedge \mathfrak{U}_1\right)\right] \vee \left[(\mathfrak{V} \wedge \mathfrak{W}) \wedge \left(\mathfrak{V} \wedge \mathfrak{U}_2\right)\right] \\
&= \left(\mathfrak{V} \wedge \mathfrak{W} \wedge \mathfrak{U}_1\right) \vee \left(\mathfrak{V} \wedge \mathfrak{W} \wedge \mathfrak{U}_2\right) \\
&= \mathfrak{V} \wedge \left[\left(\mathfrak{W} \wedge \mathfrak{U}_1\right) \vee \left(\mathfrak{W} \wedge \mathfrak{U}_2\right)\right] .
\end{aligned}$$

This completes the proof of the theorem.

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CHAPTER FOUR

HYPOCRITICAL AND SINCERE GROUPS

It is equivalent to the definition given in Section 1 to say that a group is hypocritical if whenever it is in a locally finite variety generated by a section closed class of groups then it is in the class. Clearly a hypocritical group is critical. It follows immediately from the definition that a locally finite variety generated by any class of hypocritical groups is generated by its spine. One reason for our interest in varieties generated by their spines, and hence in hypocritical groups, is the following. If a variety $\underline{\underline{V}}$ generated by its spine is contained in a locally finite join, $\bigvee_{\lambda} \underline{\underline{V}}_{\lambda}$, of a possibly infinite number of varieties, then a consideration of the finite free groups of the $\underline{\underline{V}}_{\lambda}$ shows $\underline{\underline{V}} = \bigvee_{\lambda} (\underline{\underline{V}} \wedge \underline{\underline{V}}_{\lambda})$. In particular if $\underline{\underline{V}}$ and all its subvarieties are generated by their spines then

$$\underline{\underline{U}} \wedge \left(\bigvee_{\lambda} \underline{\underline{V}}_{\lambda} \right) = \bigvee_{\lambda} (\underline{\underline{U}} \wedge \underline{\underline{V}}_{\lambda})$$

whenever $\underline{\underline{U}} \subseteq \underline{\underline{V}}$ and $\bigvee_{\lambda} \underline{\underline{V}}_{\lambda}$ is locally finite. (By Cossey [9] any variety of A-groups is generated by its spine relative to the class of varieties of A-groups, so the lattice of varieties of A-groups has this infinite distributivity.)

A finite group which is not hypocritical is said to be *sincere*. A variety generated by a single sincere critical group is not generated by its spine. Furthermore if the skeleton $S(\underline{\underline{V}})$ of a locally finite variety $\underline{\underline{V}}$ contains a sincere group H which is not in $\text{Qs}\{S(\underline{\underline{V}}) \backslash H\}$, equivalently $S(\underline{\underline{V}}) \backslash H$ is section closed, then, taking $K_1, \ldots, K_s$ such that $H \notin \mathrm{Qs}K_i$ for $i = 1, \ldots, s$ and $H \in \mathrm{Qs}(K_1 \times \ldots \times K_s)$, we have

$$T(\underline{\underline{V}}) \subseteq \mathrm{Qs}(\{K_1, \ldots, K_s\} \cup (S(\underline{\underline{V}}) \backslash H)) \cap S(\underline{\underline{V}}) \subset S(\underline{\underline{V}})$$

so $\underline{\underline{V}}$ is not generated by its spine.

In this chapter a class of critical groups is considered and it is shown that some groups in it are hypocritical and some are sincere. The class consists of those critical groups which, for some prime p , are an extension of a nontrivial abelian p-group by a p'-group. Thus any subclass defined by a fixed prime p which generates a locally finite variety in fact generates a variety which satisfies the conditions of $\underline{\underline{U}}$ in Theorem 6.4.

In Section 9 the main theorems are stated and it is shown that certain groups are hypocritical. In Section 10 a method is developed for showing groups are sincere and is applied to some groups. In Section 11 the method is further applied to illustrate the difficulties which arise in showing that a group is sincere.

9. Some Hypocritical Groups

Let p be a prime and G^* an irreducible linear p'-group of degree k over the field of p elements. Let α be a positive integer and let S be a k-generator homocyclic group of exponent p^α . Then $S/\Phi S$ becomes an irreducible $Z_p G^*$-module in an obvious way. By 2.7, an action of G^* on S can be defined such that the action induced on $S/\Phi S$ is the original action and, by 2.8, the split extension $G(p^\alpha, G^*)$ of S by G^* is unique up to isomorphism. The groups $G(p^\alpha, G^*)$ will be the central concern of this chapter.

Obviously $G(p^{\alpha}, G^*)$ is monolithic and by [17, 1.65], it is critical. However as the following theorem shows, $G(p^{\alpha}, G^*)$ is often not hypocritical.

9.1 THEOREM. *If* $\alpha = 1$ *or* G^* *has degree* 1 *then* $G(p^{\alpha}, G^*)$ *is hypocritical. If* G^* *has degree at least* 2 *then there is an* α *such that* $G(p^{\alpha}, G^*)$ *is sincere. If* $G(p^{\alpha}, G^*)$ *is sincere then so is* $G(p^{\alpha+1}, G^*)$.

The obvious problem is to find the smallest α such that $G(p^{\alpha}, G^*)$ is sincere. This problem is not solved here but a number of partial results are given which illustrate its difficulty. In particular, consider the case when G^* is cyclic of order n, in which case $G(p^{\alpha}, n)$ is used to denote $G(p^{\alpha}, G^*)$; it is well defined by [21]. Let $\alpha(p, n)$ be the smallest integer such that $G(p^{\alpha}, n)$ is sincere for all $\alpha \geq \alpha(p, n)$.

9.2 THEOREM. *If* p *does not divide* n *and* n *does not divide* $p - 1$ *then* $2 \leq \alpha(p, n) \leq 3$. *Let* k *be the smallest positive integer such that* n *divides* $p^k - 1$. *If either*

(a) *there is a nonconstant sequence* $a(1), \ldots, a(r)$ *of integers with* $r \leq p$, $0 \leq a(i) \leq k-1$ *for all* i *and* $p^{a(1)} + \ldots + p^{a(r)} \equiv 1$ *(modulo* n*), or*

(b) n *is prime and some prime divisor of* k *is less than* $p - 1$, *or*

(c) $p = 3$ *and there exist integers* $a(1), a(2), a(3)$ *and* $a(4)$ *such that* $0 \leq a(i) \leq k-1$ *for all* i, $a(1) < a(2) < a(3)$ *and* $3^{a(1)} + 3^{a(2)} + 3^{a(3)} + 3^{a(4)} \equiv 1$ *(modulo* n*)*, *then* $\alpha(p, n) = 2$.

Since a variety generated by a single sincere critical group is not generated by its spine, and since, by Cossey [8], $G(p^{\alpha}, n)$ generates $\underline{\underline{A}}_{p^{\alpha}}\underline{\underline{A}}_{n}$, we have a corollary.

9.3 COROLLARY. *If* n *is not divisible by a prime* p *and does not divide* $p - 1$ *and if* $\alpha \geq 3$ *then* $\underline{\underline{A}}_{p^{\alpha}}\underline{\underline{A}}_{n}$ *is not generated by its spine.* //

In the rest of this section it will be shown that certain groups are hypocritical. The first lemma, due to Bryant and Kovács, implies that if p is a prime and $\underline{\underline{V}}$ a locally finite variety of p'-exponent then $\underline{\underline{A}}_{p}\underline{\underline{V}}$ is generated by its spine. This contrasts interestingly with the last corollary.

9.4 LEMMA (R.M. Bryant and L.G. Kovács, unpublished). *If* G *is a monolithic group and the monolith* σG *is a* p*-group for some prime* p *while the factor group* $G/\sigma G$ *is a* p'*-group then* G *is hypocritical.*

Proof. Suppose H is a section closed class of groups such that var H is locally finite and contains G . Since there is a minimal representation of G on H , there is an H in H such that σH in H is similar to σG in G . Take such an H of the smallest possible order. Let K be a minimal supplement for $\sigma^{*}H$ in H .

Since σH in $\mathrm{gp}(\sigma H, K)$ is similar to σG in G , $H = \mathrm{gp}(\sigma H, K)$. It follows that $\sigma^{*}H = \mathrm{gp}(\sigma H, (\sigma^{*}H) \cap K)$. By the choice of K , $(\sigma^{*}H) \cap K \leq \Phi K$ so $(\sigma^{*}H) \cap K$ is nilpotent and therefore $\sigma^{*}H$ is nilpotent. It follows that $\sigma^{*}H$ is a p-group and, by similarity, $H/\sigma^{*}H$ is a p'-group. By the Schur Zassenhaus

Theorem there is a complement H_1 for σ^*H in H. Now $\mathrm{gp}(\sigma H, H_1)$ is isomorphic to G. //

9.5 LEMMA (R.M. Bryant). *If G is monolithic and σ^*G is cyclic then G is hypocritical.*

Proof. Since G is monolithic and σ^*G is abelian it must be a p-group for some prime p. It follows that σ^*G is a Sylow p-subgroup of G. Suppose $\mathfrak{H}$ is a section closed class of groups such that var $\mathfrak{H}$ is locally finite and contains G. Let

$$G \cong H/K, \quad H \leq H_1 \times \dots \times H_t, \quad H_i \in \mathfrak{H} \text{ for } i = 1, \dots, t$$

be a minimal representation of G on $\mathfrak{H}$. By 2.10, a Sylow p-subgroup T of H is normal in H and $K \leq \Phi T$. Since the Sylow p-subgroup σ^*G of G is cyclic so is T. Let π_i be the projection of H onto H_i defined by the subdirect product and T_i be the image of T under π_i. For some j the exponent of T_j is equal to the exponent of T and is therefore greater than or equal to the exponent of σ^*G.

Since $T \triangleleft H$ there is a complement H^* for T in H by the Schur Zassenhaus Theorem. Let H_j^* be the image of H^* under π_j. Now

$$|H_j^*| \leq |H^*| = |G/\sigma^*G|$$

and H_j^* is a complement for T_j in H_j. Since $T_j \leq \sigma^*H_j$ and $T_j \triangleleft H_j$,

$$|H_j^*| = |H_j/T_j| \geq |H_j/\sigma^*H_j| = |G/\sigma^*G|$$

by 2.9. It follows that $T_j = \sigma^*H_j$ and if e is the exponent of σ^*G then $G \cong H_j/T_j^{\,e}$. //

9.6 LEMMA. *The group $G(4, 3)$ is hypocritical.*

Proof. Let $G = G(4, 3)$ and $\mathcal{H}$ be a section closed class of groups such that var $\mathcal{H}$ is locally finite and contains G. Let

$$G \cong K/L \ , \quad K \leq K_1 \times \ldots \times K_t \ , \quad K_i \in \mathcal{H} \text{ for } i = 1, \ldots, t$$

be a minimal representation of G on $\mathcal{H}$.

Let G be generated by $\bar{a}, \bar{b}$ such that $|\bar{a}| = 4$ and $|\bar{b}| = 3$. Identify G with K/L via the above isomorphism. By 2.10, L is in the Frattini subgroup of the (normal) Sylow 2-subgroup $S(K)$ of K. Thus K is generated by elements a and b such that $aL = \bar{a}$, $bL = \bar{b}$, $|a| = 2^n$ for some n, $|b| = 3$, and $a^{2b}L \neq a^2L$. Therefore $a^{2b} \neq a^2$. Let $\pi(i)$ be the projection of K onto K_i defined by the subdirect product and let $a(i) = a^{\pi(i)}$ and $b(i) = b^{\pi(i)}$. Then $K_i = \mathrm{gp}(a(i), b(i))$ and for some j, $a(j)^{2b(j)} \neq a(j)^2$. Furthermore the Sylow 2-subgroup of G is generated by $\bar{a}$ and $\bar{a}^{\bar{b}}$ so $S(K)$ is generated by a, a^b and the Sylow 2-subgroup of K_j is generated by $a(j), a(j)^{b(j)}$. Let H be a minimal section of K_j of the form

$$H = \mathrm{gp}(f, h), \ |f| = 2^n \text{ for some } n, \ |h| = 3$$

and $S = \mathrm{gp}(f, f^h)$, $f^2 \neq f^{2h}$ for $S \in S_2(H)$. Then $H \in \mathcal{H}$ and it suffices to show $H \cong G$.

To simplify notation let $g = f^h$. If $g^{-2}f^2 \notin Z(S)$ then it is easy to see that $H/Z(S)$ has the same form as H, so by the choice of H, $g^{-2}f^2 \in Z(S)$. Now

$$1 = [g^{-2}f^2, g] = [f^2, g]$$

so $f^2 \in Z(S)$ and $g^2 \in Z(S)$. Thus $\mathrm{gp}(f^2, g^2) \triangleleft S$ and $S/\mathrm{gp}(f^2, g^2)$

is a dihedral group. By [15, I.56) p. 94] the only dihedral group whose automorphism group is not a 2-group is the direct product of two cyclic groups of order 2. Thus $S^2 \leq Z(S)$ and S has class at most 2. Therefore $|S'| \leq 2$. Now S/S' is a 2-generator 2-group with an automorphism of order 3 and must be homocyclic. Since $f^2 \neq g^2$ and $|S'| \leq 2$, $|S/S'| > 4$ and $G \cong H/S'$. //

10. Some Sincere Groups

In this section necessary and sufficient conditions are given for a group to be sincere. To apply these some information is needed about a modification of the associated Lie ring of a group. It is then shown that for large enough α, $G(p^\alpha, G^*)$ is sincere, and some other applications are given. The *Fitting subgroup* $F(H)$ of a group H is the join of the normal nilpotent subgroups of H.

10.1 THEOREM. *The group* $G = G(p^\alpha, G^*)$ *is sincere if and only if there is a monolithic group* H *such that* σH *in* H *is similar to* σG *in* G, $\sigma^* H = F(H)$, $\sigma H \leq F(H)'$, $F(H)/\Phi H$ *is similar in* $H/\Phi H$ *to* σG *in* G *and* p^α *does not divide the exponent of* $F(H)/F(H)'$.

Proof. Suppose that the conditions hold and take H minimal to satisfy them. Then in any chief series of H at most α chief factors are similar to σG. Let F be a relatively free p-group on the minimal number of generators of $F(H)$ of exponent the larger of p^α and the exponent of $F(H)$ and of class the class of $F(H)$. Then there is a homomorphism π of F onto $F(H)$. Let $R = \ker \pi$ so $F/R \cong F(H)$ and let F/R be a G^*-group via this isomorphism.

Now $R \leq \Phi F$ so by 2.7 we may make F into a G^*-group such that the action induced on F/R is the original action.

Let S in F be such that $S \geq R$ and $S/R \cong \sigma H$ as G^*-groups. For any positive integer γ let

$$A(\gamma) = \underline{\underline{A}}_{p^\gamma}(F) .$$

Then as G^*-groups

$$S/R \cong \sigma H \cong F(H)/\Phi H \cong F/\Phi F \cong A(\alpha-1)/A(\alpha) .$$

Since p^α does not divide the exponent of $F(H)/F(H)'$, $RA(\alpha) \geq A(\alpha-1)$. Clearly $A(\alpha) \not\leq R$ so $RA(\alpha)/R$ contains the monolith S/R of the split extension G^*F/R and thus $RA(\alpha) \geq S$. By the modular law

$$R\big(A(\alpha)\cap S\big) = RA(\alpha) \cap S = S$$

and

$$\big(A(\alpha-1)\cap R\big)A(\alpha) = A(\alpha-1) \cap RA(\alpha) = A(\alpha-1) .$$

It follows that

$$S/R = R\big(A(\alpha)\cap S\big)/R \cong A(\alpha)\cap S/A(\alpha)\cap R$$

and

$$A(\alpha-1)/A(\alpha) = \big(A(\alpha-1)\cap R\big)A(\alpha)/A(\alpha) \cong A(\alpha-1)\cap R/A(\alpha)\cap R .$$

Let $\bar{F} = F/A(\alpha)\cap R$, $A_1 = A(\alpha)\cap S/A(\alpha)\cap R$ and $A_2 = A(\alpha-1)\cap R/A(\alpha)\cap R$. Notice $\bar{F}$ is a G^*-group and in any chief series of the split extension $G^*\bar{F}$, $\alpha + 1$ chief factors are similar to σG. Since S/R and $A(\alpha-1)/A(\alpha)$ are central G^*-invariant factors of F, so are A_1 and A_2. Consequently A_1 and A_2 are in the center of $\bar{F}$ and are G^*-invariant. By the last two paragraphs

$$A_1 \cong F/\Phi F \cong A_2$$

as G^*-groups. Let μ be a G^*-isomorphism from A_1 to A_2. Let

$$N_1 = \mathrm{gp}\left(a(a^\mu) \mid a \in A_1\right)$$

and let $N_2 = A_2$. Then N_i , for $i = 1, 2$, is a G^*-invariant central subgroup of $\bar{F}$. Let H_i be the split extension of $F_i = \bar{F}/N_i$ by G^* for $i = 1, 2$. Then in any chief series of H_i , α chief factors are similar to σG . Thus in any chief series of $H_i/\sigma H_i$ and $\mathrm{gp}(\Phi F_i, G^*)$ there are only $\alpha - 1$ chief factors similar to σG . Since ΦF_i is the unique maximal G^*-invariant subgroup of F_i , it follows that $G \notin \mathrm{QS} H_i$. ∴

We show $G \in \mathrm{QS}(H_1 \times H_2)$. Because $N_1 \cap N_2 = 1$, $G^*\bar{F}$ is a section of $H_1 \times H_2$. As $\bar{F} = F/A(\alpha)\cap \not{Z}$, it has a homomorphic R

image $F/A(\alpha)$ and $G \cong G^*F/A(\alpha)$.

For the converse let G be sincere. Then there is a class $\mathcal{H}$ of groups generating a locally finite variety containing G such that $G \notin \mathrm{QS}\mathcal{H}$. Choose n minimal such that G is a section of a direct product of n groups from $\mathcal{H}$. Let $\mathcal{G}$ be the section closure of the class of direct products of fewer than n groups from $\mathcal{H}$, so that for some $H_1, H_2 \in \mathcal{G}$,

$$G \in \mathrm{QS}(H_1 \times H_2) \tag{1}$$

but $G \notin \mathcal{G}$. Now choose H_1 and H_2 minimal in the sense that neither can be replaced by a proper section without violating (1), and choose $H \in \mathrm{S}(H_1 \times H_2)$ minimal subject to $G \in \mathrm{Q}H$, say $G \cong H/K$. Observe that for some i , say $i = 1$, p^α divides the exponent of H_i . By [17], H_i is monolithic and σH_i is similar in H_i to σG

in G . Now by an argument similar to that used in the proof of Lemma 2.10 ~~2.7~~ it can be shown that for $T \in S_p(H)$, we have $T \lhd H$ and $K \leq \Phi T$. Hence a complement H^* for T in H is isomorphic to G^* . Writing T_1 and $H_1{}^*$ for the projections of T and H^* respectively determined by the subdirect product, similarity implies $H_1{}^* \cong G^*$. It follows that $T_1 = \sigma^* H_1 = F(H_1)$. Because p^α divides the exponent of H_1 , and hence of $F(H_1)$, and $G \notin QsH_1$, $\sigma H_1 \leq F(H_1)'$. The projection of H onto H_1 sends $\Phi H = \underline{\underline{A}}_p(F(H))$ onto $\underline{\underline{A}}_p(F(H_1)) = \Phi H_1$ so $F(H)/\Phi H$ and $F(H_1)/\Phi H_1$ are G^*-isomorphic. Thus $F(H_1)/\Phi H_1$ is similar in $H_1/\Phi H_1$ to σG in G . //

The theorem has an immediate corollary.

10.2 COROLLARY. *If* $G(p^\alpha, G^*)$ *is sincere then so is* $G(p^{\alpha+1}, G^*)$. //

In order to apply Theorem 10.1 we use a modified form of the associated Lie ring of a group (*cf.* Higman [13]). We shall use basic facts from the first half of Chapter 5 of Magnus, Karrass, Solitar [19] without further reference. Let p be a prime and n a positive integer not divisible by p such that the smallest positive integer k for which n divides $p^k - 1$ is greater than 1 . As is well known (*cf.* [21]), a cyclic group of order n has a faithful irreducible representation of dimension k over Z_p . Let F be an absolutely free group on k generators and let

$$L_i = F_{(i)}/F_{(i+1)}F_{(i)}{}^p .$$

The group L_i is abelian and, hereafter, written additively. If $a_i \in L_i$ and $a_j \in L_j$ then $[a_i, a_j]$ is defined to be

$[b_i, b_j]F_{(i+j+1)}F_{(i+j)}^{p}$ where b_i and b_j are elements of F in the cosets a_i and a_j respectively; it is well defined by an argument similar to that of [11, 5.6.1]. The sum $\bigoplus_{i=1}^{\infty} L_i$, with the Lie multiplication $[a, b]$ extended by linearity, is called the *associated Lie* Z_p*-algebra* and denoted by L . Because F is free and by [19, Theorem 5.12], L is a free Lie Z_p-algebra. By an obvious modification of [19, Corollary 5.12], L_r has a basis, as Z_p-space, of basic Lie elements of degree r (defined in [19, Theorem 5.8]).

Let $GL(k, p) = \text{Aut } L_1$ so that $GL(k, p)$ is isomorphic to the general linear group of nonsingular $k \times k$ matrices over Z_p . Let the p'-part of the exponent of $GL(k, p)$ be m and let Λ be the field obtained by adjoining a primitive mth root of unity to Z_p . By [10, 70.24], Λ is a splitting field for every subgroup of $GL(k, p)$. For each $i = 1, 2, \ldots$, let

$$L_i^* = L_i \otimes_{Z_p} \Lambda .$$

Under the natural embedding of L_i in L_i^* , L_i spans the Λ-space L_i^* , so the definition of $[a_i, a_j]$ can be extended by linearity to $[a_i^*, a_j^*]$ for $a_i^* \in L_i^*$ and $a_j^* \in L_j^*$. Under the bracket operation, $L^* = \bigoplus_{i=1}^{\infty} L_i^*$ becomes a Lie Λ-algebra which is free because L is free. By a modification of [19, Corollary 5.12], any Λ-basis of L_1^* leads to a Λ-basis of L_r^* , consisting of the basic Lie elements of degree r .

Let $\operatorname{End} F$ and $\operatorname{End} L_i$ be the monoids formed by the endomorphisms of F and L_i respectively. Let $\pi_i : \operatorname{End} F \to \operatorname{End} L_i$ be the map induced by the restriction of endomorphisms of F to $F_{(i)}$. Since F is free, π_1 is onto. If two endomorphisms have the same image under π_1, it is easy to see they also do under π_i for $i \geq 1$. Hence there is a monoid homomorphism $\mu_i : \operatorname{End} L_1 \to \operatorname{End} L_i$ such that $\pi_1\mu_i = \pi_i$. Under μ_i, $GL(k, p)$ is sent to a subgroup of $\operatorname{Aut} L_i$, and so L_i becomes a $Z_p GL(k, p)$-module. For $a_1, \ldots, a_r \in L_1$, $\theta \in GL(k, p)$ and f_j in the coset $a_j\theta$ of F, the image of a left-normed element of L_i is given explicitly by

$$[a_1, \ldots, a_r]\theta = [f_1, \ldots, f_r]F_{(r+1)}F_{(r)}{}^p .$$

Extending this definition by linearity, $L_r{}^*$ becomes a $\Lambda GL(k, p)$-module, and if $b_i \in L_i{}^*$ and $b_j \in L_j{}^*$ then

$$[b_i, b_j]\theta = [b_i\theta, b_j\theta] .$$

That is, $GL(k, p)$ may operate on L^* by Lie algebra automorphisms.

The following is an unpublished theorem of L.G. Kovács which will be useful in applying Theorem 10.1. Its proof involves the Witt formula and some ideas from [6].

10.3 THEOREM (L.G. Kovács). *There exists an $r > 1$ (which may depend on k and p) such that L_r has a submodule isomorphic to L_1.* //

10.4 THEOREM. *Suppose $1 < r < p^{\alpha-1}$, G^* is an irreducible p'-subgroup of $GL(k, p)$, and $(L_1)_{G^*} \lesssim (L_r)_{G^*}$. Then $G(p^\alpha, G^*)$ is sincere.*

Proof. Let F be as above, $A = F_{(r)}$, $B = \underline{\underline{B}}_{p^{\alpha-1}}(F)$ and $C = F_{(r+1)}F_{(r)}^{\ p}$. The first step of the proof of [19, Theorem 5.13B] can easily be adapted to show $A \cap B \leq C$. Since $C \leq A$, the modular law yields $A \cap BC = C$. Let $D \leq A$ be such that D/C and L_1 are G^*-isomorphic. Then $C \leq D \leq A$ so $D \cap BC = C$. Put $F/BC = \bar{F}$; then $\bar{F}$ is a finite relatively free p-group with Frattini factor group naturally isomorphic to L_1. By 2.7, it is now possible to turn $\bar{F}$ into a G^*-group such that the action on $F/\Phi F$ is the same as that obtained from the action on L_1 via the natural isomorphism. Moreover as

$$L_r = A/C = A/A \cap BC \cong AB/BC = \bar{F}_{(r)},$$

we also have that L_r is G^*-isomorphic to $\bar{F}_{(r)}$. In particular

$$F/\Phi F \cong D/C \cong DB/BC \leq \bar{F}_{(r)}$$

as G^*-modules. Let M be a normal G^*-subgroup of $\bar{F}$ maximal with respect to $M \cap (DB/BC) = 1$. The split extension H of $\bar{F}/M$ by G^* satisfies the conditions of Theorem 10.1 so $G(p^\alpha, G^*)$ is sincere. //

If G^* has degree at least 2 then by Theorem 10.3 there is an $r > 1$ such that $(L_1)_{G^*} \lesssim (L_r)_{G^*}$ so if $p^{\alpha-1} > r$ it follows by the last theorem that $G(p^\alpha, G^*)$ is sincere.

10.5 COROLLARY. *If* G^* *has degree at least* 2 *then there is an* α *such that* $G(p^\alpha, G^*)$ *is sincere.* //

The first sentence of Theorem 9.1 follows from Lemmas 9.4 and 9.5, the second is just Corollary 10.5, and the final sentence follows from Corollary 10.2. Thus the proof of Theorem 9.1 is complete.

As mentioned earlier the obvious problem is to find the smallest α such that $G(p^{\alpha}, G^*)$ is sincere. In fact it would be nice to know if there is a bound on such α which is independent of G^*. There is such a bound, 3, if G^* is cyclic, and the main lemma which is needed in the proof of that can also be applied to show that for many cyclic G^* the bound is in fact 2.

By the choice of n and k there is an irreducible cyclic subgroup T of order n in $\mathrm{GL}(k, p)$. Let $T = \mathrm{gp}(\theta)$. The representations of T over Λ are absolutely irreducible and, since T is abelian, they are all one dimensional by [12, 16.6.7]. By [11, 5.6.3], for some primitive nth root λ of unity in Λ, the characteristic roots of θ on $L_1{}^*$ are λ^{p^i} for $i = 0, 1, \ldots, k-1$. It follows that there is a basis $u_0, u_1, \ldots, u_{k-1}$ for $L_1{}^*$ such that

$$u_i\theta = \lambda^{p^i} u_i \quad \text{for all } i .$$

10.6 THEOREM. *If there exists a nonconstant sequence,* $a(1), \ldots, a(r)$ *of integers with* $0 \le a(i) \le k-1$ *for all* i *and*

$$p^{a(1)} + \ldots + p^{a(r)} \equiv 1 \ (\text{modulo } n)$$

then $(L_1)_T \lesssim (L_r)_T$. *In view of Theorem* 10.4, *if* $r < p^{\alpha-1}$ *then* $G(p^{\alpha}, n)$ *is sincere.*

Proof. Let there be such a sequence and, by renaming if necessary, let

$$a(1) > a(2) \le \ldots \le a(r) .$$

Then $c = [u_{a(1)}, u_{a(2)}, \ldots, u_{a(r)}]$ is a basic Lie element in L^*, so $c \ne 0$. By the choice of the u_i,

$$c\theta = [u_{a(1)}\theta, u_{a(2)}\theta, \ldots, u_{a(r)}\theta]$$
$$= \lambda^{l}[u_{a(1)}, u_{a(2)}, \ldots, u_{a(r)}]$$

where

$$l = p^{a(1)} + p^{a(2)} + \ldots + p^{a(r)} \equiv 1 \text{ (modulo } n) .$$

Since λ has order n, λ is an eigenvalue of θ on $(L_r{}^*)_T$.

Since θ has the common eigenvalue λ on $(L_1{}^*)_T$ and $(L_r{}^*)_T$, they have a common composition factor and, by [10, 29.6], so do $(L_1)_T$ and $(L_r)_T$. However $(L_1)_T$ is irreducible and $(L_r)_T$ is completely reducible so $(L_1)_T \lesssim (L_r)_T$. //

If $(L_1)_T \lesssim (L_r)_T$ then it is equally easy to see that the converse of the first statement of Theorem 10.6 holds, but as this is not needed it is not proved here.

Because

$$(p-1)p^{k-1} + p(p^{k-2}) = p^k \equiv 1 \text{ (modulo } n)$$

and $2p-1 < p^2$, Theorem 10.6 has a corollary.

10.7 COROLLARY. *If* p *does not divide* n *and* n *does not divide* $p - 1$ *then* $G(p^3, n)$ *is sincere.* //

As the following theorem shows, we can do slightly better than Theorem 10.6 would suggest.

10.8 THEOREM. *If there exists a nonconstant sequence of at most* p *integers* $a(1), \ldots, a(r)$ *with* $0 \le a(i) \le k-1$ *for all* i, *and*

$$p^{a(1)} + \ldots + p^{a(r)} \equiv 1 \text{ (modulo } n)$$

then $G(p^2, n)$ *is sincere.*

Proof. For $r < p$ the result is part of Theorem 10.6. Suppose then that $r = p$. By [18, 4.06],

(1) $$\underline{\underline{A}}_p\underline{\underline{A}}_p(F)\cdot F_{(p+1)} \cap F_{(p)} = F_{(p)}{}^pF_{(p+1)}\left(F''\cap F_{(p)}\right)$$

and, since $[y, x^p] = [y, px]$ is a law of $\underline{\underline{A}}_p\underline{\underline{A}}_p$ by [18, 4.02],

(2) $$\underline{\underline{A}}_p\underline{\underline{A}}_p \wedge \underline{\underline{N}}_p \subseteq [\underline{\underline{B}}_p, \underline{\underline{E}}] \text{ .}$$

Let $A = F_{(p)}$, $B = \underline{\underline{A}}_p\underline{\underline{A}}_p(F)\cdot F_{(p+1)}$ and $C = F_{(p+1)}F_{(p)}{}^p$ and observe $B \geq C$. By (1), $A \cap B = C(A\cap F'')$. By renaming if necessary, let $a(1) > a(2) \leq \ldots \leq a(r)$ and $c = [u_{a(1)}, u_{a(2)}, \ldots, u_{a(r)}]$. Then c is a left-normed basic Lie element in $L_p{}^*$ but, by [20, 36.33], not in $L_p{}^* \cap [L_2{}^*, L_2{}^*]$. ⊕L*_n / ⊕L*_n n≥2
Thus θ has an eigenvalue λ on $L_p{}^*/\left(L_p{}^*\cap[L_2{}^*, L_2{}^*]\right)$. Observe ⊕L*_n / ⊕L*_n n≥2

$$L_p/\left(L_p\cap[L_2, L_2]\right) = A/A\cap B \cong AB/B \text{ .}$$ ⊕L_n / ⊕L_n n≥2

Let $\bar{F} = F/B$ and turn $\bar{F}$ into a T-group, as in the proof of Theorem 10.4. Then $L_1 \lesssim AB/B$ so take $A_1 \leq AB/B$ such that L_1 and A_1 are isomorphic as T-groups. As $Z(\bar{F}) \leq \Phi\bar{F}$, $Z(\bar{F})$ is elementary abelian. Let D be a normal T-subgroup of $\bar{F}$ containing a T-complement for A_1 in $Z(\bar{F})$ and maximal such that $D \cap A_1 = 1$. Since, by (2), $\underline{\underline{B}}_p(\bar{F}) \leq Z(\bar{F}) \leq DA_1$, we have

$$\underline{\underline{B}}_p(\bar{F}/D) \leq A_1D/D \leq (\bar{F}/D)' \text{ .}$$

It follows that the split extension H of $\bar{F}/D$ by T satisfies the conditions of Theorem 10.1, so $G(p^2, n)$ is sincere. //

For the next item, we restrict attention further to the case where n is a prime.

10.9 COROLLARY. *If n is a prime and some prime divisor of k is less than $p - 1$ then $G(p^2, n)$ is sincere.*

Proof. Suppose r is a prime divisor of k which is less than

$p - 1$. The rank of L_r is given by the Witt formula, 2.11, as $\frac{1}{r}(k^r - k)$. Notice k does not divide the rank of L_r. Since the only irreducible modules for the cyclic group $T = \mathrm{gp}(\theta)$ of order n over a field of p elements are the trivial module and the rank k modules, there is a trivial T-module in L_r. It follows that there is a basic Lie element u of weight r such that $u\theta = u$. Because $[u, u_{k-1}]$ is basic, it is nonzero. Now

$$[u, u_{k-1}]\theta = \left[u, \lambda^{p^{k-1}} u_{k-1}\right] = \lambda^{p^{k-1}} [u, u_{k-1}] ,$$

so $\lambda^{p^{k-1}}$ is a common eigenvalue of θ on L_1 and L_{r+1} and hence, as in the proof of Lemma 10.6, we have that $(L_1)_T \lesssim (L_{r+1})_T$. Since $1 < r+1 < p$, Theorem 10.4 implies $G(p^2, n)$ is sincere. //

Lemma 9.4, Corollaries 10.7, 10.8 and 10.9 prove the first statement and parts *(a)* and *(b)* of Theorem 9.2. In the next section part *(c)* is proved.

11. On 3-Groups and Automorphisms

~~Observe that for $p > 3$, Theorem 9.2 *(c)* follows from Theorem 10.6.~~ The part of Theorem 9.2 which remains to be proved is restated here for convenience.

11.1 THEOREM. *Let n be an integer greater than and not divisible by 3; let k be the smallest positive integer such that n divides $3^k - 1$. If there exist integers $a(1)$, $a(2)$, $a(3)$ and $a(4)$ such that*

$$0 \le a(i) \le k-1 \text{ for all } i,\ a(1) < a(2) < a(3) ,$$

and

$$3^{a(1)} + 3^{a(2)} + 3^{a(3)} + 3^{a(4)} \equiv 1 \ (modulo\ n)$$

then $G(9, n)$ *is sincere.*

Part of the interest of this theorem lies in the fact that the prime, 3 , is less than the sequence length, 4 , and that the groups in question remain sincere. That this is not always the case is demonstrated by the hypocritical group $G(4, 3)$, for which $p = 2$ and there is a nonconstant sequence 0, 0, 1 of length $3 = p + 1$ such that $2^0 + 2^0 + 2^1 \equiv 1$ (modulo 3) .

To prove Theorem 11.1, we construct groups H as described in Theorem 10.1, and so work in the variety $[\underline{\underline{B}}_3, \underline{\underline{E}}]$. We also work in $\underline{\underline{B}}_3\underline{\underline{A}}_3$ so that the derived group has exponent 3 .

A group G in a locally finite variety $\underline{\underline{V}}$ is said to be *hypocritical relative to* $\underline{\underline{V}}$ if it is in every section closed class of groups which generates a subvariety of $\underline{\underline{V}}$ containing it. One could then restate Cossey's result [9] as: an A-group is hypocritical relative to any variety of A-groups containing it. One additional step to the proof of Theorem 11.1 shows that the group $G(9, q)$ is hypocritical relative to $[\underline{\underline{B}}_3, \underline{\underline{E}}] \wedge \underline{\underline{B}}_3\underline{\underline{A}}_3$ if and only if none of the sufficient conditions of its sincerity given in Theorems 10.8 and 11.1 can be satisfied.

In fact a computer check has confirmed that $G(9, n)$ is sincere for all primes $n < 1093$ by checking that the conditions of Theorems 10.8 11.1 are satisfied for all such n but are not satisfied for $n = 1093$. Thus the group $G(9, 1093)$ is hypocritical relative to $[\underline{\underline{B}}_3, \underline{\underline{E}}] \wedge \underline{\underline{B}}_3\underline{\underline{A}}_3$ but it is not known if $G(9, 1093)$ is hypocritical in general.

Notice that the conditions of the theorem imply $k \geq 3$. Let F

be the absolutely free group of rank k and let $T = \mathrm{gp}(\theta) \le \mathrm{GL}(k, p)$ with T of order n. Let $\underline{\underline{V}} = [\underline{\underline{B}}_3, \underline{\underline{E}}] \wedge \underline{\underline{B}}_3\underline{\underline{A}}_3$, $V = \underline{\underline{V}}(F)$ and $\bar{F} = F/V$. Since $\underline{\underline{B}}_3 \le \underline{\underline{N}}_3$ by [15, III 6.6], $[\underline{\underline{B}}_3, \underline{\underline{E}}] \le \underline{\underline{N}}_4$ and so $\bar{F} \in \underline{\underline{N}}_4$. On account of a result of Magnus (36.32 in [20]), the second derived group $\bar{F}''$ of $\bar{F}$ is generated by the basic non-left-normed commutators of weight 4. The next lemma implies $\bar{F}''$ is freely generated by them. We write $[a, b; c, d]$ for $[[a, b], [c, d]]$.

11.2 LEMMA. *The order* $|\bar{F}''| = 3^{\frac{1}{2}a(a-1)}$ *where* $a = \frac{1}{2}k(k-1)$.

Proof. We first show $\bar{F}'' \ne 1$ by constructing a 3-generator group G in $[\underline{\underline{B}}_3, \underline{\underline{E}}] \wedge \underline{\underline{B}}_3\underline{\underline{A}}_3$ such that $G'' \ne 1$. Let $\underline{\underline{T}}_3 = \underline{\underline{N}}_2 \wedge \underline{\underline{B}}_3$ and let S be the free group of $\underline{\underline{T}}_3$ freely generated by a, b, c, d. Define an automorphism f of S by

$$a^f = ac, \quad b^f = bd, \quad c^f = c \text{ and } d^f = d.$$

It is easy to check that f has order 3. Let G be the split extension of S by $\mathrm{gp}(f)$. Then $G \in \underline{\underline{T}}_3\underline{\underline{A}}_3 \le \underline{\underline{B}}_3\underline{\underline{A}}_3$ and $G = \mathrm{gp}(a, b, f)$. Since

$$[a, f; b, f] = [c, d] \ne 1,$$

$G'' \ne 1$. It remains to show that $G \in [\underline{\underline{B}}_3, \underline{\underline{E}}]$. As $S \in \underline{\underline{B}}_3$, an element of order greater than 3 in G must be of the form $f^{\pm 1}t$ with $t \in S$. Let $h = (f^{\pm 1}t)^3$. An easy calculation shows G/S' has exponent 3 so $h \in S' = Z(S)$. Since h is also centralized by $f^{\pm 1}t$, $h \in Z(G)$. Thus $G \in [\underline{\underline{B}}_3, \underline{\underline{E}}]$ and $\bar{F}'' \ne 1$.

Take first the case $k = 3$, and observe that $L_4 \cap [L_2, L_2]$ is a 3-dimensional Z_3-space. End F induces the action of $\mathrm{GL}(3, 3)$

on $L_4 \cap [L_2, L_2]$. It is easy to check that the subgroup $SL(3, 3)$ of $GL(3, 3)$ acts trivially on every reducible $Z_3GL(3, 3)$-module, but not on $L_4 \cap [L_2, L_2]$, so this module must be irreducible. Hence no verbal subgroup can lie properly between $F_{(4)}{}^3F_{(5)}$ and $F''F_{(4)}{}^3F_{(5)}$, so we conclude $\bar{F}''$ is 3-dimensional.

In the general case a standard argument (like [20, 33.45]) involving deletions shows that if there is a nontrivial relation modulo V among the non-left-normed basic commutators of weight 4 in F then there is one such that the commutators which occur in it nontrivially all involve the same free generators. By the last paragraph there are no nontrivial relations modulo V among commutators involving only three generators. Let a, b, c, d be among the distinct free generators of F. It is now sufficient to show that in any relation of the type

$$[a, b; c, d]^{\alpha}[a, c; b, d]^{\beta}[a, d; b, c]^{\gamma} \in V$$

we must have $\alpha \equiv \beta \equiv \gamma \equiv 0 \pmod 3$. Using an endomorphism of F sending $a \mapsto b$ and fixing the other generators, we see $\beta \equiv \gamma$; using one sending $a \mapsto c$ and fixing the other generators, we see $\alpha \equiv -\gamma$; and using one sending $a \mapsto d$ and fixing the other generators, we see $\alpha \equiv \beta$. Thus

$$-\gamma \equiv \alpha \equiv \beta \equiv \gamma \pmod 3$$

so

$$\alpha \equiv \beta \equiv \gamma \equiv 0 \pmod 3,$$

and consequently there are no nontrivial relations among the basic non-left-normed commutators in $\bar{F}''$. //

The proof of Theorem 11.1 now comes without difficulty. Let

$A = F''F_{(5)}F_{(4)}{}^3$ and $C = F_{(5)}F_{(4)}{}^3$. By the last lemma VA/VC has the same order as A/C so $A \cap VC = C$.

Let $K = A/C$ so that $K = L_4 \cap [L_2, L_2]$. Then K is freely generated by the non-left-normed basic Lie elements of weight 4, so the same is true of $K \otimes \Lambda \cong L_4{}^* \cap [L_2{}^*, L_2{}^*]$ relative to any Λ-basis of $L_1{}^*$. If there exist integers satisfying the conditions of Theorem 11.1, then there is a non-left-normed basic commutator c of weight 4 obtained by a suitable ordering and bracketing of $u_{a(1)}$, $u_{a(2)}$, $u_{a(3)}$, and $u_{a(4)}$. Hence $c\theta = \lambda c$ so θ has the common eigenvalue λ on $L_1{}^*$ and $K \otimes \Lambda$. Thus $(L_1)_T \lesssim (K)_T$.

As in the proof of Theorem 10.4, make $\tilde{F} = F/VC$ into a T-group such that L_1 and $\tilde{F}/\Phi\tilde{F}$ are T-isomorphic, and L_1 is isomorphic to a submodule of $\tilde{F}''$. Then an adaption of the proof of Theorem 10.8 completes the proof of Theorem 11.1.

Finally it is shown that the conditions of Theorems 10.8 and 11.1 determine hypocrisy relative to $[\underline{\underline{B}}_3, \underline{\underline{E}}] \wedge \underline{\underline{B}}_3\underline{\underline{A}}_3$. In view of the fact that L_1 is a submodule of L_r or K if and only if the relevant congruence is satisfied, it suffices to show $\bar{F}'' = \bar{F}_{(4)}$. By some elementary commutator calculations it is verified that $\bar{F}/\bar{F}'' \in \underline{\underline{A}}_3\underline{\underline{A}}_3$, so it is in $\underline{\underline{A}}_3\underline{\underline{A}}_3 \wedge [\underline{\underline{B}}_3, \underline{\underline{E}}]$ which is a proper subvariety of $\underline{\underline{A}}_3\underline{\underline{A}}_3 \wedge \underline{\underline{N}}_4$ and, by [18], must be contained in $\underline{\underline{A}}_3\underline{\underline{A}}_3 \wedge \underline{\underline{N}}_3$. Thus $\bar{F}'' \geq \bar{F}_{(4)}$ and, since the other inclusion is easy to see, we are done.

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