SOME CLASSES OF MONOMIAL GROUPS

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STATEMENT

The work contained in this thesis is my own except where otherwise stated.

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ABSTRACT

This thesis is concerned with *nM*-groups (and *sM*-groups), finite groups whose complex irreducible characters are all induced from linear characters of normal (subnormal) subgroups. By a classical theorem of Taketá, all such groups are solvable. Our aim is to find group theoretic properties of these groups (that is, properties which are not defined in terms of characters). Isaacs and Passman proved that all metabelian groups are *nM*-groups. We show that all abelian by nilpotent groups are *sM*-groups. The class of all *nM*-groups (or *sM*-groups) is closed under taking factor groups, direct products, or normal Hall-subgroups. Normal subgroups and subdirect products of *nM*-groups need not be *nM*-groups, but all subgroups of *nM*-groups are *sM*-groups. The corresponding question concerning *sM*-groups are still open.

We prove that if K/L is a complemented chief factor of an *sM*-group G, then all elements of K/L have subnormal centralizers in G. The p-length of an *sM*-group is at most 1, for each prime p. All subgroups of G are *sM*-groups if and only if all chief factors of G (not only the complemented ones) satisfy the subnormal centralizer condition mentioned above, and every non-nilpotent section of G has a non-central minimal normal subgroup.

A (finite solvable) group G is an *nM*-group if and only if all its factor groups H satisfy the following condition: if A is an abelian normal subgroup of maximal order in H, if g is an element of H outside A, and C a subgroup of A such that A/C is cyclic, g normalizes C and acts trivially on A/C, then C must contain some non-trivial normal subgroup of H. If G is an *nM*-group, then each element of G acts on each chief factor of G either trivially or fixed point free; all subgroups of G are *sM*-groups; the Frattini factor group of G is a subdirect

product of Frobenius groups whose kernels are abelian and whose complements have cyclic derived groups; the Fitting factor group G/F of G is metabelian, supersolvable, and the odd order Sylow subgroups of G/F are abelian. These conclusions say nothing when G is a p-group; all we can do is to present examples which show, for each prime p, that there exist non-metabelian p-groups which are nM-groups, but not all p-groups are nM-groups.

An A-group is a (finite solvable) group whose Sylow subgroups are all abelian. We determine precisely which A-groups are nM-groups or sM-groups. In particular, an A-group is an nM-group if and only if it is a subdirect product of Frobenius groups. The class of these A-groups which are nM-groups (sM-groups) is closed under taking subgroups, factor groups, direct products; if the Frattini factor group of an A-group is in this class, so is the group.

We construct an A-group of derived length 5 which is an sM-group. It should be possible to build sM-groups of arbitrary nilpotent length by the same method.

CONTENTS

CHAPTER 1

INTRODUCTION

All groups considered in this thesis are finite groups.

A *monomial group* is one all of whose irreducible complex characters are *monomial characters* (see Definition 2.14).

The study of *monomial groups* is stimulated by the study of Artin's L-function arising from number theory. Artin defined the L-function in 1923 (see Heilbronn [11]). Roughly speaking, the L-function $L(s, \chi, K/k)$ is a function of a complex variable s, and depends on a complex character χ of the Galois group of a Galois extension K/k of an algebraic number field k of finite degree. In general, L is meromorphic in the whole complex plane. Artin conjectured that the L-function corresponding to any non-trivial character is always an entire function. One positive answer known concerns the case when χ is a *monomial character*.

Already in 1930, Taketa [20] proved that *every monomial group is solvable*. Interest in monomial groups seems to have been sustained ever since. For instance, a few years ago, van der Waall [23], [24], [25] listed all monomial groups of order up to 200. Price [17] and van der Waall [26] characterized all minimal non-M-groups. For other relatively recent results, see Dornhoff [5], Seitz [19], Schacher and Seitz [18], Chapter 6 of Isaacs [13], Winter and Murphy [22].

Despite all these efforts, there is as yet no satisfactory group theoretic characterization of monomial groups. We are still unable to answer some simple question, for instance, *whether odd order normal subgroups of monomial groups are monomial groups*. For details, see Chapter 2.

The complexity of monomial groups can also be seen from a theorem of

Dade, namely, *every solvable group can be embedded into a monomial group.* For instance, Dade's Theorem shows that there is no constant upper bound for the derived length, p-length, or nilpotent length of a monomial group.

The purpose of this thesis is to study two special classes of monomial groups, namely, the class of *normal monomial groups* (nM-groups; see Definition 3.2) and the class of *subnormal monomial groups* (sM-groups; see Definition 3.1). These are also defined in character theoretic terms, and the problem is to understand these classes in group theoretic terms (that is, without reference to characters).

The thesis is divided into eight chapters. Chapter 1 is the introduction. Chapter 2 collects some prerequisites from character theory and surveys results of monomial groups. Proofs are omitted if convenient references are available.

In Chapter 3 we report on what we know about questions on normal and subnormal monomial groups which arise by analogy with the results on monomial groups, though only the more elementary results are actually proved in this chapter. We show that the classes of nM-groups or sM-groups are closed under taking factor groups, direct products, or normal Hall subgroups. All metabelian groups are nM-groups, and all abelian-by-nilpotent groups are sM-groups. We also present some examples.

In Chapter 4 we study sM-groups in detail. In particular, we find that their p-length is at most 1, for each prime p. The highlight of the chapter is the result that all subgroups of a group G are sM-groups if and only if G satisfies the following two conditions:

(1) each non-nilpotent section of G has a non-central minimal normal subgroup;

(2) if K/L is a chief factor of G and $kL \in K/L$, then the centralizer of kL in G is subnormal.

The class of such groups is closed under subgroups, factor groups, and direct products.

Chapter 5 is devoted to *nM*-groups. We obtain a structural characterization of these groups, though it is too complicated to state here. It implies that the Frattini factor group of such a group is a subdirect product of (cyclic groups and) Frobenius groups whose kernels are abelian and whose complements have cyclic derived groups. In particular, the Fitting factor group is metabelian, supersolvable, and its odd order Sylow subgroups are cyclic. All subgroups of *nM*-groups are *sM*-groups. However, none of these conclusions says anything for p-groups, and in that case our structural characterization is much harder to relate to familiar concepts. For instance, we cannot decide whether the derived lengths of *nM*-groups are bounded. We give examples to show, for each prime p, that not all p-groups are *nM*-groups but there exist non-metabelian p-groups which are *nM*-groups.

In Chapter 6 we determine precisely which A-groups (solvable groups with all Sylow subgroups abelian) are *nM*-groups or *sM*-groups. The classes of these groups are closed under subgroups, factor groups, and direct products. Moreover, if the Frattini factor group of an A-group is an *nM*-group or an *sM*-group, so is the group itself.

Chapter 7 takes up the question of the nilpotent length of *sM*-groups. We show that under a suitable additional condition, the second derived group of an *sM*-group is nilpotent. However, we also construct an A-group of nilpotent length 5 which is an *sM*-group. We believe that it should be possible to construct *sM*-groups of arbitrary nilpotent length by the same method.

The thesis concludes with a Postscript on possible future directions.

CHAPTER 2

PRELIMINARIES

In the first half of this chapter, we collect some results from character theory that are needed in this thesis. Basic definitions and classical theorems (such as Frobenius reciprocity, orthogonality relations of irreducible characters, Mackey Subgroup Theorem) will be taken for granted.

The general theorems we state are results relatively recently obtained (for example, Isaacs' "Going Down Theorem"), results we want to have in the form most suitable for our purpose (for example, Clifford's Theorem, Tensor Product Theorem), and a special case of a result which has a much simpler proof than the general one given in the literature.

We also need some theorems on characters of specific groups (Frobenius groups, extra-special p-groups), and detailed information on the structure of the regular module of a cyclic p-group over a field of characteristic p.

In the second half of the chapter, we survey well-known results on M-groups. Although many of them can be found in standard texts like Huppert's [12], they are so important for us that this summary seems necessary. There is only one item here for which we offer a proof, as none exists in print: namely, that a Frobenius group is an M-group if and only if the derived group of its complement is cyclic.

We repeat that all groups G considered are assumed to be finite. The notation we use is the same as that in Isaacs [13]. Unless otherwise stated, all characters are over the complex field.

If N is a normal subgroup of G, there is a natural permutation action by G on the set $\mathrm{Irr}(N)$ of irreducible characters of N, defined as follows. For $\varphi \in \mathrm{Irr}(N)$ and $g \in G$, define the function φ^g from N to the complex field C by $\varphi^g(n) = \varphi(gng^{-1})$. Though not completely obvious, it is a fact that $\varphi^g \in \mathrm{Irr}(N)$ (see Isaacs [13], Lemma 6.1). We call φ^g a G-conjugate of φ. The next theorem tells how an irreducible character of G behaves, when restricted to N.

THEOREM 2.1 (Clifford's Theorem). *Let* χ *be an irreducible character of* G *and* $N \trianglelefteq G$. *Then*

(1) $\chi_N = k \sum_{i=1}^{t} \varphi_i$, *where* $\varphi_1, \varphi_2, \ldots, \varphi_t$ *is a complete* G*-orbit of irreducible characters of* N;

(2) $I_G(\varphi_1) = \{g \in G \mid \varphi_1^g = \varphi_1\}$ *defines a subgroup of* G, *and* $|G:I_G(\varphi_1)| = t$.

Proof. See Isaacs [13], Theorem 6.2.

Remark. Clifford's Theorem is also valid in terms of direct sum of irreducible representations over arbitrary fields. We state it in the present form because most of the time, we use it in the context of complex characters.

THEOREM 2.2. *Let* $N \trianglelefteq G$, $\theta \in \mathrm{Irr}(N)$, *and* $T = I_G(\theta)$. *Let*

$$A = \{\psi \in \mathrm{Irr}(T) \mid [\psi_N, \theta] \neq 0\}, \quad B = \{\chi \in \mathrm{Irr}(G) \mid [\chi_N, \theta] \neq 0\}.$$

Then

(1) *the map* $\psi \mapsto \psi^G$ *is a bijection of* A *onto* B: *in particular,* ψ^G *is irreducible;*

(2) *if* $\psi^G = \chi$ *with* $\psi \in A$, *then* ψ *is the unique irreducible constituent of* χ_T *which lies in* A;

(3) *if* $\psi^G = \chi$ *with* $\psi \in A$, *then* $[\psi_N, \theta] = [\chi_N, \theta]$.

Proof. See Isaacs [13], Theorem 6.11.

Let $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$. We say that θ is invariant in G if $I_G(\theta) = G$.

THEOREM 2.3 (Going Down Theorem). *Let K/L be an abelian chief factor of G. Suppose $\theta \in \mathrm{Irr}(K)$, and θ is invariant in G. Then one of the following holds:*

(1) $\theta_L \in \mathrm{Irr}(L)$;

(2) $\theta_L = e\varphi$ *for some* φ *in* $\mathrm{Irr}(L)$, *and* $e^2 = |K:L|$;

(3) $\theta_L = \sum_{i=1}^{t} \varphi_i$, *where the* φ_i *are distinct elements of* $\mathrm{Irr}(L)$, *and* $t = |K:L|$.

Proof. See Isaacs [13], Theorem 6.18.

Remark. It follows from Clifford's Theorem that $\varphi_1, \varphi_2, \ldots, \varphi_t$ is a complete K-orbit (G-orbit) of irreducible characters of L .

Theorems 2.2 and 2.3 combine to give a powerful tool in character theory. We apply it as follows. Consider a chief series of a solvable group G , say $G = G_0 > G_1 > \ldots > G_n = \{1\}$. Let χ be a non-linear irreducible character of G . Now we know that the restriction of χ to G_n reduces, and we can consider the smallest index i such that the restriction of χ to G_i reduces. Let $L = G_i$, $K = G_{i-1}$. We have χ_K is irreducible, and obviously χ_K is invariant in G . Then the "Going Down Theorem" tells us how χ_L decomposes into a sum of irreducible characters of L . In particular, if $|K:L|$ is not a perfect square, the only possiblity is that $\chi_L = \sum_{i=1}^{|K:L|} \varphi_i$. Theorem 2.2 then tells us that there is an irreducible character ψ of $I_G(\varphi_1)$ such that $\psi^G = \chi$, and by part (2) of Clifford's Theorem, $I_G(\varphi_1)$ is a proper subgroup of index $|K:L|$ in G. This often makes it possible to use induction on $|G|$.

We can say something more about Clifford's Theorem. Part (1) of Clifford's Theorem asserts that $\chi_N = k \sum_{i=1}^{t} \varphi_i$. Obviously each φ_i has the same degree as φ_1, so we have $\chi(1) = kt\varphi_1(1)$. It is a deep theorem the proof of which involves projective representations that $\chi(1)/\varphi_1(1) = kt$ divides $|G:N|$ (see Isaacs [13], Theorem 11.29). However, what we actually need in this thesis is a special case of this general theorem, namely we can assume that G is solvable. The proof is then relatively easy.

THEOREM 2.4. *Let G be a solvable group. Let $N \trianglelefteq G$. Suppose $\chi \in \mathrm{Irr}(G)$, $\theta \in \mathrm{Irr}(N)$, and $[\chi_N, \theta] \neq 0$. Then $\chi(1)/\theta(1)$ divides $|G:N|$.*

Proof. We proceed by induction on $|G:N|$. There is nothing to prove if $|G:N| = 1$. If N is a maximal normal subgroup of G, then the theorem follows immediately from the "Going Down Theorem". If N is not maximal, let K be a maximal normal subgroup of G containing N. Then there is an irreducible character φ of K such that $[\chi_K, \varphi] \neq 0 \neq [\varphi_N, \theta]$. Now $|G:K|$ and $|K:N|$ are both less than $|G:N|$. By the inductive hypothesis, $\chi(1)/\varphi(1)$ divides $|G:K|$ and $\varphi(1)/\theta(1)$ divides $|K:N|$. Thus $\chi(1)/\theta(1)$ divides $|G:K||K:N| = |G:N|$. The proof is complete.

THEOREM 2.5. *Let $G = H \times K$. Then*

(1) *every irreducible character of G has the unique form $\alpha\beta$ (that is, for all $h \in H$ and $k \in K$, $\chi(h \cdot k) = \alpha(h)\beta(k)$) with $\alpha \in \mathrm{Irr}(H)$ and $\beta \in \mathrm{Irr}(K)$;*

(2) *if $\alpha \in \mathrm{Irr}(H)$ and $\beta \in \mathrm{Irr}(K)$, then $\chi = \alpha\beta \in \mathrm{Irr}(G)$;*

(3) *let $A \leq H$, $B \leq K$; let $\gamma \in \mathrm{Irr}(A)$ and $\delta \in \mathrm{Irr}(B)$:*

then $(\gamma\delta)^G = \gamma^H\delta^K$.

Proof. For the proof of (1) and (2), see Dornhoff [6], Theorem 10.3. Part (3) follows directly from the formula for calculating induced characters.

THEOREM 2.6 (Tensor Product Theorem). *Let* $N \leq G$. *Let* $\gamma \in \text{Irr}(N)$ *and* $\chi \in \text{Irr}(G)$. *Then* $(\chi_N\gamma)^G = \chi\gamma^G$.

Proof. This follows directly from the formula for calculating induced characters.

Remark. Let G be any group. Let χ and ζ be two characters of G . Then $\chi\zeta$ may be regarded either as a character of G or as a character of $G \times G$. These two views are not unrelated, but they are clearly different. When the situation arises, the context always makes it clear which one is intended.

THEOREM 2.7. *Let* $H \leq G$ *and* χ *in* $\text{Irr}(H)$. *If* χ^G *is irreducible, then* $C_G(H) \leq H$.

Proof. If $C_G(H) \nleq H$, then $HC_G(H)$ properly contains H , and $\chi^{HC_G(H)}$ is irreducible. Without loss of generality, we may assume $G = HC_G(H)$. Thus χ is invariant in G . By Frobenius reciprocity,

$1 = [\chi^G, \chi^G] = \left[\chi, (\chi^G)_H\right] = [\chi, |G:H|\chi] = |G:H|$, a contradiction.

In this thesis, we are dealing mainly with complex characters. In fact, the two classes of groups we shall study are defined in terms of some properties of their complex characters. However, on one occasion in Chapter 7, we encounter a situation where a p'-group H acts on an elementary abelian p-group N , thus affording a modular representation of H . In order to relate this to complex characters, we need to extend the ground field to a splitting field of H , and then exploit the relation between

Brauer characters of H and the complex characters of H . The relevant results are the following.

Let F be any field of characteristic p , and E a Galois extension over F of finite degree. Let W be an irreducible FG-module where G is any group. Let V be an irreducible EG-submodule of $W \otimes_F E$, affording the character χ (with values in E). Denote by $F(\chi)$ the subfield of E obtained by adjoining all values of χ to F .

THEOREM 2.8. *The EG-module $W \otimes_F E$ is completely reducible, with all irreducible direct summands Galois conjugate to V . The number of different Galois conjugates of V is exactly the degree of the extension $F(\chi)/F$.*

Proof. See Curtis and Reiner [2], Theorem 70.15.

THEOREM 2.9. *Let $\mathrm{IBr}(G)$ denote the set of irreducible Brauer characters of G . If $p \nmid |G|$, then $\mathrm{IBr}(G) = \mathrm{Irr}(G)$.*

Proof. See Isaacs [13], Theorem 15.13.

Next, we turn to results on particular groups and characters.

THEOREM 2.10. *Let $\chi \in \mathrm{Irr}(G)$. Then $\chi(1)^2 = |G:Z(G)|$ if and only if χ vanishes outside $Z(G)$.*

Proof. See Isaacs [13], Corollary 2.30.

THEOREM 2.11. *Let G be an extra-special p-group of order p^{2n+1} . Then any non-linear irreducible character of G is faithful and has degree p^n . It is induced from some linear character of any maximal abelian normal subgroup of G .*

Proof. See Huppert [12], Chapter V, Satz 16.14.

Frobenius groups play an important role in this thesis. It will be convenient for us to think of a Frobenius group as a split extension K split H . Here K is called the Frobenius kernel and H the Frobenius complement. There are many different definitions of Frobenius group, for example, see Passman [16], Dornhoff [6], Isaacs [13], Huppert [12]. Near the end of this chapter, we shall quote results on the structure of Frobenius groups; here we state a theorem concerning their characters.

THEOREM 2.12. *If* χ *is any irreducible character of a Frobenius group* K split H *, then either* $\operatorname{Ker} \chi \geq K$ *or* χ *is induced from some irreducible character of* K *.*

Proof. See Dornhoff [6], Theorem 13.8.

Next, we prove a fact about the regular module of a cyclic p-group over a field of characteristic p .

THEOREM 2.13. *Let* G *be a cyclic group of order* p^n *, and* F *a field of characteristic* p *. Then the regular* FG*-module (also denoted by* FG *) is uniserial, that is, the submodules of* FG *form a chain. In fact, each submodule of* FG *has the form* $FG(g-1)^i$ *where* g *is a (fixed) generator of* G *and* $0 \leq i \leq p^n$ *.*

Proof. Let $F[x]$ denote the polynomial algebra with indeterminate x . The map $x \longmapsto g$ defines an algebra homomorphism from $F[x]$ onto FG with kernel containing $x^{p^n} - 1$. It is enough to show that the ideals of $F[x]$ containing $x^{p^n} - 1$ form a chain. Since $\operatorname{char} F = p$, $x^{p^n} - 1 = (x-1)^{p^n}$. Each ideal of $F[x]$ containing $x^{p^n} - 1$ is generated by a divisor of $(x-1)^{p^n}$, as $F[x]$ is a principal ideal domain. Now $F[x]$ is certainly a unique factorization domain, so the divisors of $(x-1)^{p^n}$ are (apart from unit factors) exactly the $(x-1)^i$, $0 \leq i \leq p^n$. Thus the claim is proved. In particular, the submodules of FG are exactly the

$FG(g-1)^i$ where $0 \le i \le p^n$.

Remark. It follows that the fixed point space of g in FG is $FG(g-1)^{p^n-1}$, while in $FG/FG(g-1)^{p^n-1}$ it is $FG(g-1)^{p^n-2}/FG(g-1)^{p^n-1}$. We shall need this fact later on.

Finally, we turn to M-groups.

DEFINITION 2.14. Let G be a group. An irreducible character χ of G is called a *monomial character* (M-character) if it is induced from a linear character of some subgroup of G . The group G is called a *monomial group* (M-group) if every irreducible character of G is monomial. We denote the class of all M-groups by M .

As was mentioned in the Introduction, the first celebrated result on M-groups was due to Taketa. This is the following.

THEOREM 2.15 (Taketa). *All M-groups are solvable groups.*

Proof. See Huppert [12], Chapter V, Satz 18.6.

The converse of Taketa's Theorem is false. SL(2, 3) is a solvable group, but not an M-group (see Huppert [12], Chapter V, Satz 18.7). We shall prove in Theorem 3.5 that all metabelian groups are M-groups. It follows that SL(2, 3) is a solvable non-M-group of smallest order, since all groups of order at most 23 are metabelian. (Van der Waall [24] has shown that SL(2, 3) is in fact unique with respect to being a non-M-group of smallest order.)

It is quite obvious that factor groups of M-groups are M-groups. Direct products of M-groups are M-groups. This is a consequence of Theorem 2.5; see Huppert [12], Chapter V, Satz 18.8. However, subgroups of an M-group need not be M-groups. This fact could be easily seen from a theorem of Dade.

THEOREM 2.16 (Dade). *Every solvable group can be embedded into an M-group.*

Proof. See Huppert [12], Chapter V, Satz 18.11.

Dade's Theorem shows that, for instance, SL(2, 3) which is not an M-group, can be embedded into an M-group. In fact, this can be seen more directly than by Dade's general construction: if we take the central product of two copies of Q_8, and split extend it by an element of order 3 which acts nontrivially on each of these central factors, we obtain an M-group. It contains a subgroup isomorphic to SL(2, 3).

Dornhoff [5] has shown that *normal Hall subgroups of an M-group are M-groups.* Unfortunately, the condition of being a Hall subgroup cannot be removed from the hypothesis of Dornhoff's Theorem. Dade [3] constructed an M-group which contains a normal subgroup that is not an M-group. But Dade's example depends on the prime 2 in a fundamental way so that no odd analogue seems possible. It is still an open question *whether normal subgroups of M-groups of odd order are M-groups.*

Another direction of research on M-groups is to investigate subgroup-closed classes of M-groups. Huppert proved that *all extensions of A-groups by supersolvable groups are M-groups* (see Huppert [12], Chapter V, Satz 18.4). This is a class of M-groups that is subgroup closed. Price [17] investigated the structure of minimal non-M-groups, that is, groups which are not M-groups but whose proper homomorphic images and proper subgroups are all M-groups. Van der Waall [26] recently completed the characterization of the structure of minimal non-M-groups. The largest subgroup-closed class of M-groups consists of all the (finite solvable) groups which have no section isomorphic to a minimal non-M-group. In view of the results of Price and van der Waall, this class may now be understood without reference to characters.

We conclude this chapter by establishing the following.

THEOREM 2.17. *A Frobenius group is an M-group if and only if the* *derived group of its complement is cyclic.*

An essentially equivalent result, namely that a Frobenius group is an *M*-group if and only if its complement is supersolvable, was proved by Seitz; however, he gave no proof in his paper [19] but referred to the unpublished part of his PhD thesis which we have not seen. The proof we give here is our own.

The key fact seems to be the following. Let G be a finite solvable group, $O(G)$ the largest normal subgroup of odd order in G, and suppose the Sylow 2-subgroups of G are cyclic or generalized quaternion. If $G/O(G)$ is not a 2-group, then it is either SL(2, 3) or the binary octahedral group (of order 48), and neither of these is an *M*-group. However, we have no convenient reference for this, so we give a short proof of the variant we really need.

LEMMA 2.18. *Let* G *be an* *M-group with cyclic or generalized quaternion Sylow* 2*-subgroups. Then* $G/O(G)$ *is a* 2*-group.*

Proof. We can assume without loss of generality that $O(G) = 1$. By Taketa's Theorem, G is solvable; hence if F denotes its Fitting subgroup, then $C_G(F) \leq F$ (see Huppert [12], Chapter III, Satz 4.2). So G/F is isomorphic to a subgroup of the outer automorphism group $\mathrm{Out}(F)$. It is well-known that normal subgroups of cyclic or generalized quaternion groups are cyclic or generalized quaternion, and that the outer automorphism group $\mathrm{Out}(F)$ of such a 2-group F is a 2-group unless $F \simeq Q_8$ in which case $\mathrm{Out}(F) \simeq S_3$, the symmetric group of degree 3 (see Passman [16], Propositions 9.9, 9.10). Thus we are done unless $F \simeq Q_8$ and $|G/F|$ is 3 or 6. We show that in this case a contradiction follows. For $|G/F'| = |G| - |G/F'| = \sum \chi(1)^2$ where χ runs through those irreducible characters of G whose kernel does not contain F'. As F' is the unique minimal normal subgroup of F, these χ must be faithful on F, so $\chi(1) > 1$ for each of them. Now $|G:F'|$ is 12 or 24, and simple trial and error shows neither of these numbers can be written as a sum of perfect squares without 1^2 or 2^2 among the summands. Hence $\chi(1) = 2$ for some χ in $\mathrm{Irr}(G)$ with $F' \nleq \mathrm{Ker}\,\chi$. As G is an M-group, this must be induced from a linear character of some subgroup A of index 2, and then $A' \leq \mathrm{Ker}\,\chi$. Now A is a normal subgroup containing a Sylow 3-subgroup which acts nontrivially on F/F', so even $F \leq A' \leq \mathrm{Ker}\,\chi$, contradicting $F' \nleq \mathrm{Ker}\,\chi$. This completes the proof.

LEMMA 2.19. *A Frobenius complement G is an M-group if and only if G' is cyclic.*

Proof. We need the fact that all Sylow subgroups of a Frobenius complement G are either cyclic or generalized quaternion (see Passman [16], Proposition 18.1). Thus if G is an M-group, $G/O(G)$ is a 2-group by the previous lemma. As all Sylow subgroups of $O(G)$ are cyclic, this implies that G is supersolvable. Then G' is nilpotent (see Hall [10], Theorem 10.5.4). On the other hand, if Q is a Sylow 2-subgroup of G, then $G = O(G)Q$ and $G' \leq O(G)Q'$. Since Q' is cyclic, all Sylow subgroups of the nilpotent group G' are cyclic, so G' is cyclic.

The converse part of the lemma follows from the fact that all supersolvable groups are M-groups.

Proof of Theorem 2.17. In view of the last lemma, all that remains to be proved is that if K split H is a Frobenius group with H' cyclic, then K split H is an M-group. Let $\chi \in \mathrm{Irr}(K \text{ split } H)$. By Theorem 2.12, either $\mathrm{Ker}\,\chi \geq K$ or χ is induced from K. In the first case, χ is an M-character because H is an M-group. In the second case, we appeal to the celebrated theorem of Thompson that all Frobenius kernels are nilpotent (see Passman [16], Theorem 17.4). Thus K is an M-group, and again χ is an M-character.

CHAPTER 3

ELEMENTARY RESULTS AND EXAMPLES

This chapter is a report on what we know about the questions on normal and subnormal M-groups which arise by analogy with M-group results surveyed in the previous chapter. To maintain continuity of the report, we defer all the proofs to the second half of the chapter. Because of Taketa's Theorem, no generality is lost if we restrict our attention to solvable groups. We consider only complex characters in this chapter.

DEFINITION 3.1. A character of a group G is called a *subnormal monomial character* (sM-character) if it is induced from a linear character of some subnormal subgroup of G. A group G is called a *subnormal monomial group* (sM-group) if all its irreducible characters are sM-characters. We denote the class of all sM-groups by $\mathfrak{s}\mathrm{M}$.

DEFINITION 3.2. A character of a group G is called a *normal monomial character* (nM-character) if it is induced from a linear character of some normal subgroup of G. A group G is called a *normal monomial group* (nM-group) if all its irreducible characters are nM-characters. We denote the class of all nM-groups by $n\mathrm{M}$.

The definitions yield the following inclusions:

$$n\mathrm{M} \subset \mathfrak{s}\mathrm{M} \subset \mathrm{M}$$

Each of the above inclusions is proper. The symmetric group S_4 of degree 4 is an M-group but not an sM-group, since it has an irreducible character of degree 3, but none of its Sylow 2-subgroups is subnormal. We shall construct an sM-group which is not an nM-group later in this chapter; see Example 3.11. First, let us examine some closure properties of $\mathfrak{s}\mathrm{M}$ and $n\mathrm{M}$.

THEOREM 3.3. *Both* $\mathfrak{s}\mathrm{M}$ *and* $n\mathrm{M}$ *are homomorphic image closed and* *direct product closed.*

THEOREM 3.4. *Both sM and nM are normal Hall subgroup closed.*

We shall see in Chapter 5 that nM is neither closed under normal subgroups (let alone under subgroups), nor closed under subdirect products. However, the corresponding questions for sM are still open. The difficulty in attempting to test whether normal subgroups of an sM-group are sM-groups could be seen in Example 3.12. The group we construct in this example has an irreducible sM-character such that none of the irreducible constituents of its restriction to a particular normal subgroup is an M-character (let alone an sM-character). This shows that we must somehow find a way to make full use of the assumption that *all* irreducible characters of G are sM-characters.

We shall obtain results in Chapter 4 which show that the analogues of Dade's Theorem are false. We shall obtain in Chapter 6 conclusive structural characterizations of the sM-groups and nM-groups which are A-groups. As for supersolvable groups, we can only say that they need not be sM-groups; see Example 3.10.

The analogues we can get for Huppert's Theorem are the following.

THEOREM 3.5. *All metabelian groups are nM-groups.*

THEOREM 3.6. *All extensions of abelian groups by nilpotent groups are sM-groups.*

These two theorems provide subgroup closed subclasses of nM and sM, respectively. But we shall see in Example 3.11 that an nM-group, all of whose subgroups are nM-groups, need not be metanilpotent (let alone metabelian). In Theorem 4.14 we give a structural characterization of the groups whose subgroups are all sM-groups. In Corollary 5.10 we show that all subgroups of nM-groups are sM-groups.

THEOREM 3.7. *If G is a Frobenius group such that the derived group* *of its complement is cyclic, then all subgroups of* G *are* sM*-groups.*

Of course, we know that if G in sM is a Frobenius group, then its Frobenius complement has cyclic derived group (see Theorem 2.17). This and Theorem 3.7 show that in the case of Frobenius groups, sM-group and M-group are the same. We do not have any necessary and sufficient condition for a Frobenius group to be an nM-group; all we have to offer is the following.

THEOREM 3.8. *Let* G *be a Frobenius group such that the kernel of* G *is abelian and the derived group of the complement is cyclic. Then all subgroups of* G *are* nM*-groups.*

We now turn to proofs and examples.

Proof of Theorem 3.3. We shall prove the case of sM. The proof for the case of nM is exactly the same. (Just replace each occurrence of "subnormal" by "normal".)

Let $G \in sM$. Let G/N be any homomorphic image of G. Let $\chi \in \text{Irr}(G/N)$. Regard χ as an irreducible character of G with kernel containing N. As such, χ is induced from a linear character λ of some subnormal subgroup S of G. $N \leq \text{Ker}\,\chi = \text{core}_G(\text{Ker}\,\lambda)$ implies that $N \leq \text{Ker}\,\lambda \leq S$. We can regard λ as a linear character of S/N which is subnormal in G/N. Obviously then $\lambda^{G/N} = \chi$, so χ is an sM-character.

Let $G, H \in sM$. If $\chi \in \text{Irr}(G \times H)$, then by Theorem 2.5, $\chi = \eta\zeta$ where $\eta \in \text{Irr}(G)$ and $\zeta \in \text{Irr}(H)$. By assumption, η and ζ are induced from linear characters λ, γ of subnormal subgroups A, B of G, H, respectively. Now $A \times B$ is subnormal in $G \times H$, $\lambda\gamma$ is a linear character of $A \times B$, and (by part (3) of Theorem 2.5) $(\lambda\gamma)^{G\times H} = \lambda^G\gamma^H = \eta\zeta = \chi$. The proof is complete.

Remark. In the above proof, we in fact proved that if χ is an sM or an nM-character of G with kernel containing a normal subgroup N, then χ is also an sM or an nM-character, respectively, as a character of G/N. The converse of this is also obviously true, namely, if χ is an sM or an nM-character of G/N, then χ regarded as a character of G, is also an sM or an nM-character, respectively. On many occasions later on, we need to show that a certain character χ of G is an sM or an nM-character. If the hypotheses are inherited by homomorphic images of G, then the above discussion allows us to assume that χ is faithful.

Proof of Theorem 3.4. We shall prove the case of sM. The proof of the case nM is clearly the same.

Let $G \in sM$ and N a normal Hall subgroup of G. Let θ be any non-linear irreducible character of N. Let χ be an irreducible constituent of θ^G. By Frobenius reciprocity $[\chi_N, \theta] \neq 0$ and by Theorem 2.4, $\chi(1)/\theta(1)$ divides $|G:N|$.

By assumption, χ is induced from a linear character λ of some subnormal subgroup S of G. Therefore

$$\chi(1)/\theta(1) = |G:S|/\theta(1) = |G:NS|\,|NS:S|/\theta(1)$$

divides $|G:N|$. But $|NS:S| = |N : N \cap S|$ is relatively prime to $|G:N|$, so that $|NS:S|$ is a factor of $\theta(1)$, in particular $|N : N \cap S| \leq \theta(1)$. Now $\lambda^{NS} \subseteq \chi_{NS}$ so that $(\lambda^{NS})_N \subseteq \chi_N$. Thus $(\lambda^{NS})_N$ is a sum of some G conjugates of θ. By Mackey's Subgroup Theorem, $(\lambda^{NS})_N = (\lambda_{N\cap S})^N$ and so $(\lambda^{NS})_N(1) = |N : N \cap S| \leq \theta(1)$. Thus $(\lambda_{N\cap S})^N$ is a conjugate of θ. Since $N \cap S$ is subnormal in N, this conjugate of θ is an sM-character. *A fortiori*, θ is an sM-character. The proof is complete.

Proof of Theorem 3.5. See Isaacs and Passman [14], Proposition 1.3.

Proof of Theorem 3.6. Let G be a group, A an abelian normal subgroup of G and G/A is nilpotent. We use induction on $|G|$ to conclude that all proper subgroups of G are sM-groups. Let χ be any non-linear irreducible character of G . Let $G > G_1 > \ldots > G_n = A$ be a chief series of G through A . We know that χ_A reduces, so we can consider the smallest index i such that the restriction of χ to G_i reduces. Let $L = G_i$, $K = G_{i-1}$. As G/A is nilpotent, $|K/L|$ is a prime p . By the Going Down Theorem, $\chi_L = \sum_{i=1}^{p} \varphi_i$ so that $I_G(\varphi_1)$ is a proper subgroup of index p in G . By Theorem 2.2, there is a ψ in $\mathrm{Irr}(I_G(\varphi_1))$ such that $\psi^G = \chi$. Now $I_G(\varphi_1) \geq A$, so it is subnormal in G . The result then follows, as $I_G(\varphi_1)$ is an *sM*-group.

Proof of Theorem 3.7. Let $G = K$ split H be any Frobenius group with H' cyclic. Let $\chi \in \mathrm{Irr}(G)$. By Theorem 2.12, either $\mathrm{Ker}\,\chi \geq K$ so that $\chi \in \mathrm{Irr}(H)$, or χ is induced from an irreducible character of K . In the first case, χ is an *sM*-character since H is metabelian, hence an *nM*-group by Theorem 3.5. In the second case, χ is an *sM*-character because the Frobenius kernel is nilpotent and hence an *sM*-group by Theorem 3.6. Therefore G is an *sM*-group.

Any subgroup N of G is either a subgroup of K so that N is nilpotent, or a subgroup of H so that N is metabelian, or itself a Frobenius group whose complement has cyclic derived group. In any case, N is an *sM*-group. The proof is complete.

Proof of Theorem 3.8. Let $G = K$ split H be a Frobenius group with K abelian and H' cyclic. Let $\chi \in \mathrm{Irr}(G)$; then either $\mathrm{Ker}\,\chi \geq K$ or χ is induced from a linear character of K . In either case, χ is an *nM*-character. The same argument as in the proof of Theorem 3.7 completes the proof.

To construct a supersolvable group which is not an *sM*-group, we need the following lemma.

LEMMA 3.9. *Let* $\chi \in \mathrm{Irr}(G)$ *be a nonlinear sM-character. Then* $\chi_{G'}$ *reduces.*

Proof. If not, let $\chi_{G'} = \zeta \in \mathrm{Irr}(G')$. By assumption, χ is induced from a linear character λ of some proper subnormal subgroup S of G. By Mackey's Subgroup Theorem, $\zeta = \chi_{G'} = (\lambda^G)_{G'} = \sum_x \left(\lambda^x_{S^x \cap G'}\right)^{G'}$ where x runs through a set of double coset representatives of S and G' in G, and this forces $SG' = G$. Take a maximal normal subgroup N of G that contains S. Then G/N is cyclic, so $G' \leq N$, contrary to $SG' = G$.

EXAMPLE 3.10. Let p be any odd prime. Let E be an extra-special p-group of exponent p, order p^3. Let $G = E$ split $\langle g \rangle$ where g has order 2 and it inverts some pair of generators of E. Then g acts trivially on the centre of E. By Theorems 2.10 and 2.11, any non-linear irreducible character ζ of E vanishes outside $Z(E)$, and so ζ is invariant in G. Let $\chi \in \mathrm{Irr}(G)$ be such that $[\chi_N, \zeta] \neq 0$. By the Going Down Theorem, $\chi_E = \zeta$. This shows that χ is not an sM-character by the previous lemma, since $E = G'$. Nevertheless, G is supersolvable as $G > E > \langle x, Z(E) \rangle > Z(E) > \{1\}$, where x is any generator of E, is a chief series of G with cyclic chief factors.

EXAMPLE 3.11. Let K be a relatively free group on 3 generators, with exponent 43 and class 2; that is,

$$K = \langle u, v, w \mid \text{exponent } 43, \text{ class } 2 \rangle .$$

K' is an elementary abelian group of order 43^3 and is generated by the commutators $[v, w]$, $[w, u]$ and $[u, v]$. Let

$$H = \langle x, y \mid x^7 = y^9 = 1, x^y = x^4 \rangle .$$

We define an action of H on K as follows. Choose an integer k whose multiplicative order modulo 43 is 21 , and define

$$u^x = u^{k^3}\ ,\quad v^x = v^{k^6}\ ,\quad w^x = w^{k^{12}}\ ,\quad u^y = v\ ,\quad v^y = w\ ,\quad w^y = u^{k^7}\ .$$

Routine calculation shows that this is a well-defined action. We claim that $G = K$ split H is a Frobenius group. To do so, it is enough to prove that any non-trivial element of H acts fixed point free on K/K' and on K' . Observe that with respect to the basis uK', vK', wK' of K/K' , x and y act on K/K' as the linear transformations defined by the matrices

$$x = \begin{pmatrix} k^3 & 0 & 0 \\ 0 & k^6 & 0 \\ 0 & 0 & k^{12} \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k^7 & 0 & 0 \end{pmatrix}$$

so that y^3 acts on K/K' as the linear transformation defined by

$$y^3 = \begin{pmatrix} k^7 & 0 & 0 \\ 0 & k^7 & 0 \\ 0 & 0 & k^7 \end{pmatrix} .$$

Since uK', vK', wK' are eigenvectors of x and y^3 and neither x nor y^3 has a trivial eigenvalue, they act fixed point free on K/K' . Now each nontrivial element of H has a power equal to x or y^3 , so it must also act fixed point free on K/K' .

We apply the same argument to show that any non-trivial element of H acts fixed point free on K' . To this end, we take $[v, w]$, $[w, u]$, $[u, v]$ as basis of K' ; routine calculation shows that x, y, y^3 act on K' as the linear transformations defined by the matrices

$$x = \begin{pmatrix} k^{18} & 0 & 0 \\ 0 & k^{15} & 0 \\ 0 & 0 & k^9 \end{pmatrix} ,\quad y = \begin{pmatrix} 0 & k^7 & 0 \\ 0 & 0 & k^7 \\ 1 & 0 & 0 \end{pmatrix} ,\quad y^3 = \begin{pmatrix} k^{14} & 0 & 0 \\ 0 & k^{14} & 0 \\ 0 & 0 & k^{14} \end{pmatrix} .$$

Obviously, neither x nor y^3 has a trivial eigenvalue. We conclude that the split extension K split H formed with this action is a Frobenius group.

We obtain several facts about $G = K$ split H .

(1) G is an sM-group since it is a Frobenius group with H' cyclic.

(2) G is not metanilpotent. This shows that a group whose subgroups are all sM-groups need not be metanilpotent, let alone abelian-by-nilpotent.

(3) G is not an nM-group. The reason is as follows. Let χ be an irreducible character of G such that $\mathrm{Ker}\ \chi \not\geq K'$. Suppose $\chi = \lambda^G$ where λ is a linear character of a normal subgroup N of G . Then $N' \leq \mathrm{Ker}\ \chi$, so $N' \not\geq K'$. However, K' is the unique minimal normal subgroup of G , so we must have $N' = 1$. Since K' is the only abelian normal subgroup of G , then $N \leq K'$. Now λ^K is irreducible, contrary to Theorem 2.7, as N is central and proper in K .

(4) G/K' is a Frobenius group with abelian kernel K/K' and complement H with cyclic derived group. By Theorem 3.8, all subgroups of G/K' are nM-groups, but G/K' is not metanilpotent.

EXAMPLE 3.12. Let p and q be prime numbers such that $p \mid (q^2-1)$ but $p \nmid (q-1)$. $\mathrm{GF}(q^2)$ is a splitting field for C_p , but $\mathrm{GF}(q)$ is not. Thus C_p has a 2-dimensional irreducible module over $\mathrm{GF}(q)$. By assumption, $p \neq 2$.

Let Q be an extra-special q-group of exponent q and order q^3 when q is odd, and let Q be the quaternion group when $q = 2$. Let P be an extra-special p-group of exponent p and order p^3 . Let C be an abelian normal subgroup of P so that P/C is isomorphic to C_p . Since $Q/Z(Q) \simeq C_q \times C_q$, the above discussion shows that we can let P/C act irreducibly on $Q/Z(Q)$; or equivalently, P acts irreducibly on $Q/Z(Q)$

with kernel C . When q is odd, Q is a relatively free group, so this action can be extended to Q . Otherwise we appeal to the fact that the quaternion group of order 8 has an automorphism of order 3 . Thus P acts on Q with kernel C and obviously P acts trivially on $Z(Q)$. Let $G = Q$ split P .

Let φ be a faithful irreducible character of Q , so $\varphi(1) = q$; and let γ be an irreducible character of C whose kernel avoids P' , and is therefore not normal in P . Now $\varphi\gamma$ is an irreducible character of $Q \times C$ which is not invariant in G , for its kernel is $\ker \gamma$ which is not normal in G . As $Q \times C$ is a maximal subgroup of G , it follows that $I_G(\varphi\gamma) = Q \times C$. Hence $(\varphi\gamma)^G$ is irreducible. Since $Q \times C$ is normal and nilpotent, $(\varphi\gamma)^G$ is an sM-character.

Let D be a subgroup of order p in P , not contained in C . Then D acts non-trivially on Q in such a way that QD has no subgroup of index q . By Mackey's Subgroup Theorem,

$$\left((\varphi\gamma)^G\right)_{QDP'} = \left((\varphi\gamma)_{QC\cap QDP'}\right)^{QDP'} = \left((\varphi\gamma)_{Q\times P'}\right)^{QDP'} = \left(\varphi\gamma_{P'}\right)^{QD\times P'} = \varphi^{QD}\gamma_{P'}$$

by Theorem 2.5. As D acts trivially on Q' we know that φ is invariant in QD , hence by Mackey's Subgroup Theorem, $\left(\varphi^{QD}\right)_Q = p\varphi$, and so by the Going Down Theorem, each irreducible constituent ψ of φ^{QD} is such that $\psi_Q = \varphi$. In particular, each constituent $\psi\gamma_{P'}$ of $\left((\varphi\gamma)^G\right)_{QDP'}$ has degree q . However, QDP' has no subgroup of index q , so none of these constituents can be M-characters.

CHAPTER 4

SUBNORMAL M-GROUPS

The first result of this chapter is that if G is an sM-group, K/L is a complemented chief factor of G, and $kL \in K/L$, then the centralizer $C_G(kL)$ of kL is subnormal in G. We find that sM-groups have p-length at most 1, for each prime p. This contrasts with the fact that there exist M-groups of arbitrarily large p-length (for by Dade's Theorem every solvable group is embeddable into some M-group). However, supersolvable groups have all the above mentioned properties, yet we have seen in Example 3.10 that not all supersolvable groups are sM-groups. We are still far away from having a complete structural characterization of sM-groups. Finally in Theorem 4.14, we give a structural characterization of groups all of whose subgroups are sM-groups.

We shall need some structural results first. All groups considered are finite solvable groups.

LEMMA 4.1. *If a subnormal subgroup* R *of* H *contains a Sylow* p*-subgroup* P *of* H*, then* R *contains* $O^{p'}(H)$*, the normal closure of* P *in* H.

Proof. We use induction on $|H|$. If $R = H$, there is of course nothing to prove; so we suppose N is a proper normal subgroup of H containing R. Then P is a Sylow p-subgroup of N, and its normal closure $O^{p'}(N)$ is characteristic in N, hence normal in H. This obviously shows that $O^{p'}(H) = O^{p'}(N)$. By the inductive hypothesis, $O^{p'}(N) \leq R$. The proof is complete.

Let G be a group. The socle of G, denoted by $\sigma(G)$, is the subgroup of G generated by all minimal normal subgroups of G. The following theorem gives some well-known properties of $\sigma(G)$.

THEOREM 4.2. (1) *$\sigma(G)$ is a characteristic subgroup of G .*

(2) *$\sigma(G)$ is abelian.*

(3) *$\sigma(G)$ is the direct product of some of the minimal normal subgroups of G .*

(4) $\sigma(G \times H) = \sigma(G) \times \sigma(H)$.

(5) *For each normal subgroup N of G contained in $\sigma(G)$, there is a normal subgroup M in G such that $\sigma(G) = N \times M$.*

(6) *If F and Φ denote the Fitting and Frattini subgroups of G , respectively, then F/Φ is the socle of G/Φ .*

LEMMA 4.3. *If $C_G(\sigma(G)) = \sigma(G)$ and N is a maximal normal subgroup of G , then*

(1) $\sigma(N) = N \cap \sigma(G)$,

(2) $C_N(\sigma(N)) = \sigma(N)$, *and*

(3) $\sigma(G) = (N \cap \sigma(G)) \times L$ *where L is central in G .*

Proof. Consider first the case $\sigma(G) \leq N$. Then (3) is trivial with $L = \{1\}$. As $\sigma(N)$ is an abelian normal subgroup of G (it is normal, as it is characteristic in N), so it will contain or avoid, and hence centralize, each minimal normal subgroup of G . Thus $\sigma(N)$ centralizes $\sigma(G)$, and by assumption, we have $\sigma(N) \leq \sigma(G)$. By part (5) of Theorem 4.2, $\sigma(G) = \sigma(N) \times M$ for some normal subgroup M of G . However, we have now M is normal in N but avoids $\sigma(N)$, which forces $M = \{1\}$, so $\sigma(N) = \sigma(G)$. Thus (1) and (2) follow immediately.

Next we suppose $\sigma(G) \nleq N$, and let L be a minimal normal subgroup of G which is not contained in N . Since N is a maximal normal subgroup, we have $G = N \times L$ and L is central in G . By part (4) of Theorem 4.2, $\sigma(G) = \sigma(N) \times \sigma(L) = \sigma(N) \times L$, so that (1) and (3) hold. As L is central, $C_G(\sigma(N)) = C_G(\sigma(G)) = \sigma(G)$, and (2) follows. The proof is complete.

LEMMA 4.4. *If* $C_G(\sigma(G)) = \sigma(G)$ *, and* R *is a subnormal subgroup of* G *, then* R *is a direct factor of* $R\sigma(G)$.

Proof. We use induction on $|G|$. If $R = G$, there is nothing to prove; so we may assume R is contained in some maximal normal subgroup N of G . By part (2) of Lemma 4.3, the inductive hypothesis applies to N , and we conclude that $R\sigma(N) = R \times K$ for some K . By (1) and (2) of Lemma 4.3, $\sigma(G) = (N \cap \sigma(G)) \times L = \sigma(N) \times L$, with L central in G . Thus $R\sigma(G) = R(\sigma(N) \times L) = R\sigma(N) \times L$ (note that $R\sigma(N) \cap L \le N \cap L = \{1\}$), and hence $R\sigma(G) = (R \times K) \times L = R \times (K \times L)$. The proof is complete.

LEMMA 4.5. *If* $C_G(\sigma(G)) = \sigma(G)$ *and* ρ *is a character of a subnormal subgroup* R *of* G *such that* ρ^G *is irreducible, then* $R \ge \sigma(G)$.

Proof. By Lemma 4.4, we see that $R\sigma(G) = R \times K$. Since ρ^G is irreducible, Theorem 2.7 asserts that $C_G(R) \le R$ which certainly shows that $K = \{1\}$ and $R \ge \sigma(G)$.

Next, we come to a key lemma of this chapter.

LEMMA 4.6. *Let* G *be an sM-group with a self centralizing, complemented, minimal normal subgroup* K .

(1) *If* $\mu \in \mathrm{Irr}(K)$ *and* $T = I_G(\mu)$ *, then* T *is subnormal in* G .

(2) *If* K *is a* p*-group, then* G/K *is a* p'*-group.*

(3) *If* $k \in K$ *, then* $C_G(k)$ *is subnormal in* G .

Proof. (1) By assumption, $G = K$ split H , where H is a maximal subgroup of G , so that T which contains K , is a semi-direct product of K and $(T \cap H)$. By Theorem 51.15 in Curtis and Reiner [2], T has an irreducible character τ such that $\tau_K = \mu$. By Theorem 2.2, τ^G is irreducible, so that $\tau^G = \rho^G$ for some linear character ρ of a subnormal subgroup R of G . Observe that $K = C_G(K)$ implies that K is the unique

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The author is grateful to Professor Isaacs for pointing out that the arguments can be considerably streamlined if Lemmas 4.3, 4.4, 4.5 are replaced by the following, equally straightforward, Lemma: if R is subnormal in G and $C_G(R) \le R$, then $R \ge \sigma(G)$.

minimal normal subgroup of G, and so $K = \sigma(G)$, and $C_G(\sigma(G)) = \sigma(G)$. We may then apply Lemma 4.5 to conclude that $K \leq R$. By Mackey's Subgroup Theorem, $(\tau^G)K = (\rho^G)K$ is a sum of G-conjugates of μ, and also a sum of G-conjugates of ρ_K, so that $\mu = (\rho_K)^g$ for some g in G. Now $R^g \leq T$, and $|G:R^g| = |G:R| = \rho^G(1) = \tau^G(1) = |G:T|$, so $R^g = T$. This proves (1).

(2) Consider $\mathrm{Irr}(K)$ as an H-module. Note that $\mathrm{Irr}(K)$ is (isomorphic to K as group and) dual to K as H-module: hence it is faithful and irreducible. Let P be a Sylow p-sugbroup of H, and consider its permutation action on the set $\mathrm{Irr}(K)\setminus\{1_K\}$ whose cardinality is relatively prime to p: it must fix at least one nontrivial element in $\mathrm{Irr}(K)$, say μ. We have then $P \leq T \cap H$ where $T = I_G(\mu)$. By part (1), $T \cap H$ is subnormal in H, so we can apply Lemma 4.1 to conclude that $\mathcal{O}^{p'}(H) \leq T \cap H$. Now $\mathcal{O}^{p'}(H)$ is normal in H, so that the set of fixed points of $\mathcal{O}^{p'}(H)$ in $\mathrm{Irr}(K)$ is an H-submodule. Since μ lies in it, and as $\mathrm{Irr}(K)$ is an irreducible H-module, $\mathcal{O}^{p'}(H)$ acts trivially on $\mathrm{Irr}(K)$. But we know that $\mathrm{Irr}(K)$ is a faithful H-module, so that $\mathcal{O}^{p'}(H) = \{1\}$. This proves (2).

(3) Let $k \in K$ and $C = H \cap C_G(k)$ so that $C_G(k) = KC$. Let $M = C_K(C)$. By (2), $(|H|, |K|) = 1$, so that Maschke's Theorem yields that C acts completely reducibly on K. In particular, $K = M \times L$, where L is the sum of non-trivial irreducible C-submodules of K. We claim that $L = [K, C]$. If W is an irreducible submodule of L, then obviously $[W, C] = W$, so $W \leq [K, C]$, and *a fortiori* $L \leq [K, C]$. Conversely $[K, C] = [M \times L, C] = [L, C] \leq L$ and the claim is proved. We write now $K = M \times [K, C]$.

Let $B = \{\mu \in \mathrm{Irr}(K) \mid \mathrm{Ker}\,\mu \geq [K, C]\}$. Obviously, $\bigcap_{\mu \in B} \mathrm{Ker}\,\mu = [K, C]$. Let $S = \bigcap_{\mu \in B} I_G(\mu)$. By (1), each $I_G(\mu)$ is subnormal in G , and as the intersection of subnormal subgroups is a subnormal subgroup, we have S subnormal in G . By the choice of B , we know that $C \leq S \cap H$; hence $[K, C] \leq [K, S \cap H]$ and $C_K(S \cap H) \leq C_K(C) = M$. By the same argument as we used before, we apply (2) and Maschke's Theorem to conclude that $K = C_K(S \cap H) \times [K, S \cap H]$. Since the μ are linear, we have $[K, I_G(\mu)] \leq \mathrm{Ker}\,\mu$, and so

$$[K, S \cap H] \leq \bigcap_{\mu \in B} [K, I_G(\mu)] \leq \bigcap_{\mu \in B} \mathrm{Ker}\,\mu = [K, C] .$$

We have already established the converse inclusion, so $[K, S \cap H] = [K, C]$. Since $C_K(S \cap H)$ is contained in M and each of them complements $[K, C]$ in K , we have $C_K(S \cap H) = M$. Therefore $S \cap H \leq C$; the converse inclusion having been noted before, this implies $S \cap H = C$. Thus $S = K(S \cap H) = KC = C_G(k)$ is subnormal in G . The proof is complete.

THEOREM 4.7. *Let G be an sM-group, K/L a complemented chief factor of G and $k \in K$. Then $C_G(kL)$ is subnormal in G .*

Proof. We argue by contradiction. Suppose G is a counterexample of least possible order, and $C_G(kL)$ is not subnormal in G . Let H be a complement of K/L , that is, $K \cap H = L$ and $KH = G$. Let $M/L = C_{H/L}(K/L)$. Then M/L is normal in H/L and centralized by K/L , so it is normal in G/L . Moreover, KM/M is a chief factor of G/M complemented by H/M , and $C_{G/M}(kM) = C_G(kM)/M = C_G(kL)/M$, so $C_{G/M}(kM)$ is not subnormal in G/M . Thus G/M is also a counterexample. By the minimality of G , $M = \{1\}$. In particular, $L = \{1\}$, K is a minimal normal subgroup of G complemented by H, and $C_G(K) = KC_H(K) = KM = K$. Thus we have reached a contradiction with Lemma 4.6 (3).

DEFINITION 4.8. The p-length of a group G is defined as follows. Consider the series $\{1\} = P_0 \leq N_0 < P_1 < N_1 < \ldots < P_n \leq N_n = G$, where N_k/P_k is the greatest normal p'-group of G/P_k, and P_{k+1}/N_k is the greatest normal p-subgroup of G/N_k. Then n is the p-length of G. Denote the p-length of G by $l_p(G)$.

We need the following result on p-length.

LEMMA 4.9. *Let G be any group, such that any proper homomorphic image of G has p-length at most k, and $l_p(G) > k$. Then*

(a) $\Phi(G) = \{1\}$,

(b) the Fitting subgroup F of G is the unique minimal normal subgroup of G, and it is its own centralizer.

Proof. See Huppert [12], Chapter V, Hilfsatz 6.9.

THEOREM 4.10. *If G is an sM-group and p is any prime, then the p-length of G is at most 1.*

Proof. We need only to consider those primes which divide $|G|$. We argue by contradiction. Let G be a minimal counterexample with p-length greater than one for some prime p. Then all proper homomorphic images of G have p-length at most 1. By Lemma 4.9, $G = F$ split H, where F is the Fitting subgroup of G, and it is minimal normal and self-centralizing in G. If F is a p'-group, then G/F has p-length 1, implies that G has p-length 1. So F is a p-group. By Lemma 4.6 (2), G has p-length 1, a contradiction.

Finally, we consider groups whose subgroups are all sM-groups.

LEMMA 4.11. *If every subgroup of G is an sM-group, K/L is any chief factor of G, and $k \in K$, then $C_G(kL)$ is subnormal in G.*

Proof. We argue by contradiction. Let G be a minimal counterexample with $C_G(kL)$ not subnormal, then clearly $L = \{1\}$.

If M is a normal subgroup of G avoiding K , then KM/M is a chief factor of G/M and $C_{G/M}(kM) = C_G(kM)/M = C_G(k)/M$ is not subnormal in G/M . On the other hand, all subgroups of G/M are *sM*-groups, so G/M is a counterexample. As G was chosen minimal, $M = \{1\}$. Thus K is the unique minimal normal subgroup of G . Let p be the prime divisor of $|K|$, then G has no nontrivial p'-subgroup, and the Fitting subgroup F of G is a p-group, and Theorem 4.10 implies that G/F is a p'-group. Thus F is complemented by any Hall p'-subgroup H . If $KH = G$, we have a contradiction to Theorem 4.7, so assume $KH < G$. By assumption, all subgroups of KH are *sM*-groups. As K is central in F , it is a chief factor of KH . Thus $C_{KH}(k)$ is subnormal in KH by Theorem 4.7; it is immediate that $C_G(k)$ is subnormal in G , and this final contradiction completes the proof.

This lemma would be of no interest if its conclusion was not inherited by subgroups, but we can show that all is well in this respect. It will be convenient to have a name for groups for which the conclusion holds. We shall say that *a (finite solvable) group G is a chiefly sub-Frobenius group if $C_G(kL)$ is subnormal in G whenever kL is an element of a chief factor K/L of G .* Note that this is equivalent to saying that the centralizer of kL in $G/C_G(K/L)$ is subnormal in this factor group.

THEOREM 4.12. *All subgroups, factor groups, and direct products of chiefly sub-Frobenius groups are chiefly sub-Frobenius groups.*

Proof. The claim for factor groups is trivial, as each chief factor of G/N may be viewed as a chief factor of G . To see the claim for a direct product $G \times H$, we use that the Jordan-Hölder Theorem allows us to look only at the chief factors of a particular chief series. If K/L is a chief factor in a chief series of $G \times H$ through G, say, then either $K \leq G$ so K/L is a chief factor of G and $(G \times H)/C_{G\times H}(K/L)$ acts on K/L as $G/C_G(K/L)$ does, or $L \geq G$ in which case $(G \times H)/C_{G\times H}(K/L)$ acts on K/L just as $H/C_H(K\cap H/L\cap H)$ acts on the chief factor $(K \cap H)/(L \cap H)$ of H.

It is a little harder to deal with the case of a subgroup G_1 in a chiefly sub-Frobenius group G. Choose a chief series in G, intersect its members with G_1, and refine the resulting series of normal subgroups of G_1 to a chief series. By the Jordan-Hölder Theorem, it is sufficient to examine the factor K_1/L_1 of this chief series of G_1. Suppose K_1/L_1 arise by refining $(G_1 \cap K)/(G_1 \cap L)$ where K/L is a chief factor of G, so $G_1 \cap K \geq K_1 > L_1 \geq G_1 \cap L$. Then K_1/L_1 is G_1-isomorphic to the section K_1L/L_1L of K/L. Let p be the prime divisor of $|K/L|$. Now $G/C_G(K/L)$ acts faithfully and irreducibly on K/L, and the argument used for deducing part (2) from part (1) in the proof of Lemma 4.6 applies, with the conclusion that $G/C_G(K/L)$ is a p'-group. In particular, $G_1/C_{G_1}(K/L)$ is a p'-group, so by Maschke's Theorem K/L is completely reducible as G_1-module. Thus $K_1L/L = (L_1L/L) \times (M/L)$ for some M normalized by G_1, and K_1/L_1 is G_1-isomorphic to M/L. If now $k_1 \in K_1$ and mL is the element of M/L corresponding to k_1L_1 under some G_1-isomorphism $K_1/L_1 \simeq M/L$, we have $C_{G_1}(k_1L_1) = C_{G_1}(mL) = G_1 \cap C_G(mL)$. As $C_G(mL)$ is subnormal in G by assumption, we conclude that $C_{G_1}(k_1L_1)$ is subnormal in G_1. This completes the proof.

Remark. The step we repeated here from the proof of Lemma 4.6 may also be used to show that the p-length of a chiefly sub-Frobenius group cannot be greater than 1, for any prime p.

It remains an open question whether every sM-group is a chiefly sub-Frobenius group.

LEMMA 4.13. *If every subgroup of G is an sM-group, then every non-nilpotent section of G has a non-central minimal normal subgroup.*

Proof. Let G be a minimal counter example. G is non-nilpotent and $\sigma(G) \leq Z(G)$. If M is any minimal normal subgroup of G, choose N maximal among normal subgroups of G which avoid M. Now NM/N is the unique minimal normal subgroup of G/N. As $M \leq Z(G)$, MN/N is central in G/N, so either G/N is nilpotent or $N = \{1\}$ by minimality of G. If M is not the unique minimal normal subgroup of G, then $N > \{1\}$, and G/N is nilpotent. Since the interesection of all such N's is trivial, G is a subidrect product of such G/N's, and hence G is nilpotent. This contradiction shows that $\sigma(G)$ is the unique minimal normal subgroup of G; so $|\sigma(G)| = p$, a prime number. Hence $O_{p'}(G) = \{1\}$ and the Fitting subgroup F of G is a p-group. Since G has p-length 1 by Theorem 4.10, G/F is a p'-group. By Maschke's Theorem, G/F acts on $\sigma(F)$ completely reducibly, so that $\sigma(F) \leq \sigma(G)$. On the other hand, $\sigma(G) \leq F$ so $\sigma(G) \leq \sigma(F)$ and equality holds. Let $g \in G$ with prime order $q \neq p$; then $F\langle g\rangle$ is not nilpotent, for g cannot centralize the Fitting subgroup F. If $F\langle g\rangle < G$, then by the minimality of G, there is a minimal normal subgroup M of $F\langle g\rangle$ such that M is non-central in $F\langle g\rangle$; but then $M \cap \sigma(F) = \{1\}$ as $\sigma(F) = \sigma(G)$ is central, so $M \cap F = \{1\}$ and $F\langle g\rangle = F \times M$ follows, contrary to the non-nilpotence of $F\langle g\rangle$. We conclude that $G = F\langle g\rangle$. We know that $F = [F, g]C_F(g)$ (see Gorenstein [9], Chapter 5, Theorem 3.5), whence $[F, g] = [F, g, g] \trianglelefteq F$. If

$[F, g] < F$, then $[F, g]\langle g\rangle < G$. Let N be maximal among normal subgroups of $[F, g]\langle g\rangle$ which avoid $\sigma(G)$; then $N\sigma(G)/N$ is the unique minimal normal subgroup of $[F, g]\langle g\rangle/N$; it is central there, and has order p . By the minimality of G , this group is nilpotent and hence in fact a p-group. Thus $g \in N$ and $N \geq [F, g]$; hence $[F, g] \cap \sigma(F) = \{1\}$, a contradiction. We conclude that $[F, g] = F$. Let K/L be a chief factor of G with $F \geq K > L \geq \sigma(G)$ and choose M maximal among normal subgroups of G such that $M \cap K = L$. Then K/L is G-isomorphic to KM/M , and is the unique minimal normal subgroup of G/M . G/M is not nilpotent, otherwise $F = [F, g] \leq M$ contrary to $K \cap M = L$. Therefore K/L is non-central in G , and so it is a faithful irreducible $\langle g\rangle$-module. Thus $C_F(g) = \sigma(G)$. Now let F/K be a chief factor of G and choose N maximal among normal subgroups of $K\langle g\rangle$ which avoid $\sigma(G)$; the minimality of G yields, via the argument we have used repeatedly, that $K\langle g\rangle/N$ is a p-group, and $g \in N$. Thus $N \geq [K, g]$, and $[K, g] \cap \sigma(G) = \{1\}$. Now $K = [K, g]C_K(g) = [K, g] \times \sigma(G)$, and so $K\langle g\rangle = N \times \sigma(G)$. Let $\tau \in \mathrm{Irr}(K\langle g\rangle)$ with $\mathrm{Ker}\,\tau = N$. Then $T = I_G(\tau_K) \geq K\langle g\rangle$ which is maximal in G . If $T = K\langle g\rangle$, then $I_F(\tau_K) = T \cap F = K$ so that $(\tau_K)^F$ is irreducible. In particular, τ^G is irreducible as $(\tau^G)_F = (\tau_K)^F$. Now G is a subnormal M-group, so $\tau^G = \lambda^G$ where $\lambda \in \mathrm{Irr}(L)$, $\lambda(1) = 1$, and L is subnormal in G . So $|G:L| = |G:K\langle g\rangle|$ is a power of p , and $q \,\big|\, |L|$. Thus $L \geq O^{q'}(G)$ by Lemma 4.1, and so $L \geq O^{q'}(G) \geq [F, g] = F$, a contradiction. If $T = G$, then $N \cap K = \mathrm{Ker}\,\tau_K \lhd G$, so $N \cap K \cap \sigma(G) = \{1\}$ implies that $[K, g] \leq N \cap K = \{1\}$; hence $K = C_K(g) = \sigma(G) = \sigma(F)$. Thus $|\sigma(F)| = p$ and $F/\sigma(F)$ a chief factor of G implies that F is an extra special p-group. Let φ be a faithful irreducible character of F, so φ is G-invariant. There exists an irreducible character χ of G such that $\chi_F = \varphi$. Now $\chi = \lambda^G$ where $\lambda \in \mathrm{Irr}(L)$, $\lambda(1) = 1$ and L is subnormal in G. Thus L is of p-power index, so $q \,\big|\, |L|$, and by the argument we have used before, $L \geq O^{q'}(G) \geq [F, g] = F$, a final contradiction.

THEOREM 4.14. *All subgroups of G are sM-groups if and only if G is a chiefly sub-Frobenius group and each non-nilpotent section of G has a non-central minimal normal subgroup.*

Proof. The "only if" claim follows from Lemma 4.11 and Lemma 4.13. To prove the "if" claim, we argue by induction on $|G|$. We need only show that G itself is an *sM*-group. Let χ be a non-linear irreducible character of G. By the minimality of $|G|$, we can assume that χ is faithful. If G is nilpotent we are done, as nilpotent groups are *sM*-groups by Theorem 3.6. Otherwise, there exists a non-central minimal normal subgroup M in G. Then $C = C_G(M)$ is a proper normal subgroup of G. If M is a p-group, then G/C is a p'-group by the remark after Theorem 4.12. Let φ be an irreducible constituent of χ_C, and put $T = I_G(\varphi)$. As T/C is a p'-group and $M \cap \mathrm{Ker}\,\varphi \trianglelefteq T$, by Maschke's Theorem we have $M = (M \cap \mathrm{Ker}\,\varphi) \times Y$ with Y normal in T; as $[M, T] \leq M \cap \ker \varphi$, in fact $Y \leq Z(T)$. If $Y = \{1\}$, then χ is not faithful. Thus there is a non-trivial element y in Y and $C_G(y) \geq T$. Now $C_G(y) < G$, else $1 \neq y \in M \cap Z(G)$, so $M \leq Z(G)$ gives a contradiction. As $T \leq C_G(y)$, by Theorem 2.2, χ is induced from some irreducible character γ of $C_G(y)$, and $C_G(y)$ is subnormal in G by chiefly sub-Frobenius property of G. By the inductive hypothesis $C_G(y)$ is an *sM*-group, so it follows that χ is an *sM*-character as required.

Remark (added in proof). It is not too hard to see that the class of chiefly sub-Frobenius groups whose nonnilpotent sections all have non-central minimal normal subgroups, is direct product closed; it is obviously subgroup closed and factor group closed.

CHAPTER 5

NORMAL M-GROUPS

The main result of this chapter is a structural characterization of nM-groups (Theorem 5.4). Unfortunately, it is rather complicated to state and difficult to relate to any of the familiar structural properties. We derive from it that each nM-group G is "chiefly Frobenius" in the sense that if K/L is a non-central chief factor of G, then K/L split $G/C_G(K/L)$ is a Frobenius group (whose kernel is abelian and) whose complement has cyclic derived group; so the Fitting factor group G/F is metabelian, supersolvable, and its odd Sylow subgroups are abelian. We also prove that all subgroups of nM-groups are sM-groups. However, for p-groups these consequences of the characterisation are trivial, all we can do is to present some examples. These show, for each prime p, that not all p-groups are nM-groups but there exist non-metabelian p-groups which are nM-groups. We find nM-groups whose normal subgroups are not all nM-groups, and exhibit subdirect products of nM-groups which are not nM-groups.

LEMMA 5.1. *Let A be an abelian normal subgroup of maximal order in a group G.*

(1) *If χ is a faithful irreducible nM-character of G, then χ is induced from A; in particular,* $\chi(1) = |G:A|$.

(2) *If χ is an irreducible M-character of G with* $\chi(1) \geq |G:A|$ *then χ is induced from A; in particular it is an nM-character.*

Proof. (1) By assumption, $\chi = \beta^G$ for some linear character β of some normal subgroup B of G. As $B' \leq \mathrm{core}_G(\mathrm{Ker}\ \beta) = \mathrm{Ker}\ \chi = \{1\}$, we know that B is abelian, so $\chi(1) = |G:B| \geq |G:A|$. Let α be an irreducible constituent of χ_A . By Frobenius reciprocity, χ is a constituent of α^G , so that $\chi(1) \leq \alpha^G(1) = |G:A|$. Thus $\chi(1) = |G:A| = \alpha^G(1)$ and $\chi = \alpha^G$.

(2) Now $\chi = \gamma^G$ for some linear character γ of some subgroup H . As γ^G is irreducible, so is γ^{AH} . By Clifford's Theorem $(\gamma^{AH})_A = k \sum_{i=1}^{t} \alpha_i$ with $t = |AH:I_{AH}(\alpha)|$ where $\alpha = \alpha_1$ say. By Mackey's Subgroup Theorem, $(\gamma^{AH})_A = (\gamma_{H\cap A})^A$; so by Frobenius reciprocity,

$$k = \left[\alpha, (\gamma^{AH})_A\right] = \left[\alpha, (\gamma_{H\cap A})^A\right] = [\alpha_{H\cap A}, \gamma_{H\cap A}] \leq 1$$

as α and γ are linear. Thus

$$\begin{aligned} |AH:I_{AH}(\alpha)| &= \gamma^{AH}(1) = |AH:H| = |G:H|/|G:AH| \\ &= \chi(1)/|G:AH| \geq |G:A|/|G:AH| = |AH:A| , \end{aligned}$$

so $I_{AH}(\alpha) = A$. Now $\alpha^{AH} = \gamma^{AH}$ by Theorem 2.2, and so $\chi = \gamma^G = \alpha^G$. The proof is complete.

Let F denote the class of all (finite solvable) groups whose faithful irreducible characters are all nM-characters. (For groups which have no faithful irreducible characters, this condition is vacuous; these groups are regarded as members of F .) Obviously, *a group is an* nM*-group if and only if all its factor groups lie in* F . The key lemma is the following structural characterization of F .

LEMMA 5.2. *Let* A *be an abelian normal subgroup of maximal order in a group* G *. This group* G *lies in* F *if and only if, whenever* $g \in G\backslash A$ *and* C *is a subgroup of* A *such that* $C \geq [A, g]$ *and* A/C *is cyclic, we have* $\mathrm{core}_G(C) > \{1\}$.

Proof. The condition is clearly equivalent to the following: if A/C

is cyclic, $\text{core}_G(C) \not> \{1\}$, and $[A, g] \leq C$, then $g \in A$. The subgroups C with A/C cyclic are precisely the $\text{Ker}\ \alpha$ with α ranging through the irreducible characters of A . We know that $\text{core}_G(\text{Ker}\ \alpha) = \text{Ker}\ \alpha^G$, and that $[A, g] \leq \text{Ker}\ \alpha$ if and only if $g \in I_G(\alpha)$. Thus our task is to prove that $G \in \mathsf{F}$ if and only if $I_G(\alpha) = A$ whenever $\alpha \in \text{Irr}(A)$ and α^G is faithful.

Let χ be a faithful irreducible character of G , and α an irreducible constituent of χ_A . By Frobenius reciprocity, χ is a constituent of α^G , and so α^G is faithful. If $I_G(\alpha) = A$, Theorem 2.2 tells us that α^G is irreducible, so $\alpha^G = \chi$ and χ is an nM-character.

Conversely, suppose that $G \in \mathsf{F}$, and let $\alpha \in \text{Irr}(A)$ with α^G faithful. Consider an irreducible constituent χ of α^G . By Frobenius reciprocity and Clifford's Theorem, χ_A is a multiple of the sum of the G-conjugates of α , so $\text{Ker}\ \chi_A = \text{core}_G(\text{Ker}\ \alpha) = \text{Ker}\ \alpha^G = \{1\}$. Thus χ is faithful and hence an nM-character (as A is of maximal order). By Lemma 5.1 (1), $\chi(1) = |G:A|$ so $\chi = \alpha^G$. Put $T = I_G(\alpha)$; as α^G is irreducible, so is α^T . By Mackey's Subgroup Theorem, $(\alpha^T)_A = |T:A|\alpha$, so by Frobenius reciprocity, $1 = [\alpha^T, \alpha^T] = \left[\alpha, (\alpha^T)_A\right] = |T:A|$. Thus $I_G(\alpha) = A$ and the proof is complete.

For p-groups, this criterion takes a simpler form, for a p-group G which has a faithful irreducible character must have cyclic centre; equivalently if such a group G is non-trivial, then the socle of G , $\sigma(G)$, has order p .

LEMMA 5.3. *Let A be an abelian normal subgroup of maximal order in a p-group G . Then $G \in \mathsf{F}$ if and only if either $|\sigma(G)| \neq p$ or*

$$\bigcap_{g \in G \setminus A} [A, g] \geq \sigma(G) .$$

Proof. The "if" claim is immediate from the previous lemma. For the proof of the "only if" part, we argue by contradiction. Suppose $G \in F$, $|\sigma(G)| = p$, $g \in G\setminus A$, and $[A, g] \ngeq \sigma(G)$. Then $[A, g] \cap \sigma(G) = \{1\}$. Let C be a subgroup of A maximal with respect to containing $[A, g]$ and avoiding $\sigma(G)$. If $\langle aC \rangle$ is a subgroup of order p in A/C, then $\langle C, a \rangle > C$, so by the maximality of C, we have $\langle C, a \rangle \cap \sigma(G) \neq \{1\}$; thus $\langle C, a \rangle \geq \sigma(G)$ and in fact $\langle C, a \rangle = C\sigma(G)$. Thus the only subgroup of order p in A/C is $C\sigma(G)/C$. As A/C is an abelian p-group with only one subgroup of order p, it must be cyclic. By the previous lemma, $\mathrm{core}_G C > \{1\}$, so that $\mathrm{core}_G C \geq \sigma(G)$, contrary to the assumption that $C \cap \sigma(G) = \{1\}$. The proof is complete.

Remark. In each of the last two lemmas, A was an arbitrary abelian normal subgroup of maximal order. It follows that if one such subgroup satisfies the relevant condition, so does every other. Thus it makes no difference whether we require that at least one abelian normal subgroup of maximal order satisfies the condition, or that all such subgroups satisfy it. We have established the following criterion.

THEOREM 5.4. *A group (p-group) is an* nM*-group if and only if in every factor group* G *of our group, some abelian normal subgroup* A *of maximal order satisfies the conditions of Lemma* 5.2 *(Lemma* 5.3*).*

Unfortunately, these structural conditions are not easy to use or to relate to other structural properties. We proceed to discuss what we can obtain in this direction.

THEOREM 5.5. *If* G *is an* nM*-group,* $g \in G$*, and* K/L *is a chief factor of* G*, then* g *acts either trivially or fixed point free on* K/L*. Equivalently,* K/L split $G/C_G(K/L)$ *is a Frobenius group whenever* K/L *is a non-central chief factor of* G*.*

Remark. Note that by Theorem 2.17, each $G/C_G(K/L)$ has cyclic derived group. We shall refer to groups which satisfy the conclusions of this theorem and of this remark, as *chiefly Frobenius groups* . As Example 3.11 shows, the converse of Theorem 5.5 is false.

Proof. By passing to a factor group of G if necessary, we may assume that K is the unique minimal normal subgroup of G (and $L = \{1\}$); that is, $K = \sigma(G)$. Let p be the prime divisor of $|K|$ and F the Fitting subgroup of G , then G has no non-trivial normal p'-subgroup, so by Theorem 4.10, F is a Sylow p-subgroup of G . A Hall p'-subgroup H of G then complement F . Clearly, $Z(F)$ contains the unique minimal normal subgroup K , so if $g \in G$ and $g = fh$ with $f \in F$, $h \in H$, we have $C_K(g) = C_K(h)$. We need only pursue the case where $h \neq 1$.

Choose an abelian normal subgroup A of maximal order in G . As in the proof of Lemma 5.3, we see that, by Lemma 5.2, $[A, h] \geq \sigma(G) = K$. As A is an abelian p-group and h is a p'-element, we have $A = C_A(h) \times [A, h]$ (see Gorenstein [9], Chapter 5, Theorem 2.3). Thus h acts fixed point free on $[A, h]$ and also on K , and $C_K(g) = C_K(h) = \{1\}$. The proof is complete.

COROLLARY 5.6. *If G is a chiefly Frobenius group, so in particular if G is an nM-group, with Fitting subgroup F and Frattini subgroup Φ , then*

(1) *G/Φ is a subdirect product of groups of prime order and of Frobenius groups whose kernels are abelian and whose complements have cyclic derived groups;*

(2) *G/F is a subdirect product of Frobenius complements with cyclic derived groups; in particular, G/F is metabelian, supersolvable, and its odd order Sylow subgroups are all abelian.*

Proof. (1) We may assume $\Phi = \{1\}$, so that the intersection $\cap M$ of the maximal subgroups M of G is $\{1\}$. Thus also $\cap \operatorname{core}_G M = \{1\}$, that is, G is the subdirect product of the $G/\operatorname{core}_G M$. By Galois' Theorem (see Huppert [12], Chapter III, Satz 3.2), each $G/\operatorname{core}_G M$ is of the form K/L split $G/C_G(K/L)$.

(2) By Satz 4.3 in Chapter III of Huppert [12], G/F is a subdirect product of the $G/C_G(K/L)$.

We note the following analogue of Theorem 4.12.

THEOREM 5.7. *All subgroups, factor groups, and direct products of chiefly Frobenius groups are chiefly Frobenius groups.*

Proof. Observe that G is a chiefly Frobenius group if and only if the following hold for each chief factor K/L of G: the derived group of $G/C_G(K/L)$ is cyclic, and each nontrivial element of K/L has trivial centralizer in $G/C_G(K/L)$. The proof of Theorem 4.12 may now be repeated, *mutatis mutandis*, until the last substantive sentence, and then we proceed as follows. Since $C_G(mL) = C_G(K/L)$ by assumption, we conclude that $C_{G_1}(k_1L)$ centralises K_1/L_1, so $C_{G_1}(K_1/L_1) = C_{G_1}(k_1L_1)$. It remains to note that the derived group of $G_1/C_{G_1}(K_1/L_1)$ is cyclic, because this factor group is isomorphic to the subgroup $G_1C_G(K/L)/C_G(K/L)$ of $G/C_G(K/L)$. This completes the proof.

Next, we obtain a further consequence of Theorem 5.4.

LEMMA 5.8. *If H is a section of an nM-group and $\sigma(H)$ is central in H and has prime order p, then H is a p-group.*

Proof. We argue by contradiction. Suppose $\sigma(H)$ is central of order p, but H is not a p-group, and let G be an nM-group of least possible order with respect to G having a section S/R isomorphic to H. Put $\sigma(S/R) = Z/R$. The key step is to observe that

$$R(Z \cap N) = Z \textit{ for every nontrivial normal subgroup } N \textit{ of } G .$$

As $|Z/R| = p$, the only alternative is that $R(Z \cap N) = R$. In that case, $R(S \cap N) = R$ also, for $R(S \cap N)/R$ is normal in S/R but

$$R(S \cap N)/R \cap \sigma(S/R) = R(S \cap N)/R \cap Z/R = R(Z \cap N)/R = R/R .$$

Therefore G/N has a section isomorphic to H, namely

$$SN/RN = SRN/RN \simeq S/(S \cap RN) = S/(S \cap N)R = S/R \simeq H ,$$

contrary to the minimal choice of $|G|$. This proves our key step, which we shall use repeatedly.

Let A be an abelian normal subgroup of maximal order in G, and M be any minimal normal subgroup of G in A. Now from the key step above, $R(Z \cap M) = Z$, so that $Z/R = R(Z \cap M)/R \simeq (Z \cap M)/(R \cap M)$ and therefore M is a p-group. As this holds for every choice of M, we conclude that A is also a p-group. Let s be a p'-element of S; then $s \notin A$.

Next we prove that $[A, s](R \cap A) \ngeq Z \cap A$. Suppose this is false. Then $Z \cap A = (Z \cap [A, s])(R \cap A)$ and so

$$Z/R = R(Z \cap A)/R \simeq (Z \cap A)/(R \cap A) = (Z \cap [A, s])(R \cap A)/(R \cap A)$$
$$= (Z \cap [A, s])/(R \cap [A, s])$$

shows that s does have non-trivial fixed points in $B = [A, s]/(R \cap [A, s])$; that is, $C_B(s) > \{1\}$. As s is a p'-element acting on the abelian p-groups A and B, by Theorem 2.3 of Chapter 5 in Gorenstein [9], we know that $A = [A, s] \times C_A(s)$ and $B = [B, s] \times C_B(s)$. The first of these yields $[A, s] = [A, s, s]$; hence by the definition of B we have $B = [B, s]$, so in the direct decomposition of B we must have $C_B(s) = \{1\}$. This contradiction completes the proof of the present step.

Now we choose C in A maximal with respect to containing $[A, s](R \cap A)$ but not containing $Z \cap A$. If $A \geq D > C$, then $D \geq Z \cap A$

and $D/C \geq (Z \cap A)C/C$. Thus $(Z \cap A)C/C$ is the unique minimal subgroup of A/C and this implies that A/C is cyclic. By Lemma 5.2, C contains a minimal normal subgroup M of G . On the other hand, $R \cap A \leq R \cap C$ by the choice of C ; the converse inclusion holds because $A \geq C$; so $R \cap A = R \cap C$. Also $R \cap A = R \cap C \leq Z \cap C < Z \cap A$. As $(Z \cap A)/(R \cap A) = (Z \cap A)R/R = Z/R$ and $|Z/R| = p$, we conclude that $R \cap A = Z \cap C \geq Z \cap M$, so $Z = R(Z \cap M) = R$, a final contradiction.

THEOREM 5.9. *If G is an nM-group, then each non-nilpotent section of G has a non-central minimal normal subgroup.*

Proof. Let K be a non-nilpotent section of G , and suppose all minimal normal subgroups M_i of K are central. Let N_i be maximal among the normal subgroups of K which avoid M_i . Then K is the subdirect product of the K/N_i ; so at least one of these, say K/N_1 , is non-nilpotent. Put $H = K/N_1$. Now $\sigma(H)$ is M_1N_1/N_1 , so it is central and of prime order. By Lemma 5.8, H should be nilpotent, but we chose it so that it is not. This contradiction proves the theorem.

COROLLARY 5.10. *If G is an nM-group, then all subgroups of G are sM-groups.*

Proof. It is obvious that each chiefly Frobenius group is a chiefly sub-Frobenius group. Hence Theorem 5.5 and the remark which follows it, together with Theorem 5.9 and Theorem 4.14 give our claim.

At this stage, one might feel that we are very close to a convenient structural characterization of *nM*-groups. Indeed, we shall show in the next chapter that an *A*-group is an *nM*-group if and only if it is chiefly Frobenius.

However, for p-groups, Theorems 5.5 and Theorem 5.9 do not say anything at all. It is rather frustrating that we are unable to exploit the relative simplicity of Lemma 5.3 (as compared to Lemma 5.2) to find a more familiar and managable criterion for a p-group to be an nM-group. For instance, we cannot decide whether p-groups which are nM-groups can have arbitrarily high derived length. (If their derived lengths were bounded independent of the choice of p, say by n, then the derived lengths of arbitrary nM-groups would be bounded by $n + 2$. For a minimal counter-example G to this would have a unique minimal normal subgroup, so by Theorem 4.10, its Fitting subgroup would be a Sylow subgroup and hence, by Theorem 3.4, F would be also an nM-group.) All we can offer is some examples which show for each prime p, that not all p-groups are nM-groups but there exist non-metabelian p-groups which are nM-groups. A further example (which is not a p-group) shows that subdirect products of nM-groups need not be nM-groups.

EXAMPLE 5.11. Let p be a prime, $p \geq 5$, and T an extra-special group of order p^5 and exponent p, generated by a, b, c, d, such that $[a, b] = [c, d]$ is central and $[a, c] = [b, c] = [a, d] = [b, d] = 1$. Consider the map σ defined by $a^\sigma = a$, $b^\sigma = bc$, $c^\sigma = acd$, $d^\sigma = ad$. Routine calculation with the defining relations of T shows that σ extends to an automorphism of T, which we also denote by σ. Note that σ acts trivially on T'.

It is easy to prove by induction on k that

$$a^{\sigma^k} = a ,$$

$$d^{\sigma^k} = a^k d ,$$

$$c^{\sigma^k} = ca^{\binom{k+1}{2}} d^k ,$$

$$b^{\sigma^k} = ba^{\binom{k+1}{3}} c^k d^{\binom{k}{2}} [d, c]^{\binom{k}{3}} .$$

We shall give only the hardest step, the inductive step in the last item, assuming the others have already been dealt with. Thus

$$\begin{aligned} b^{\sigma^{k+1}} &= b^{\sigma^k} c^{\sigma^k} \\ &= ba^{\binom{k+1}{3}} c^k d^{\binom{k}{2}} [d, c]^{\binom{k}{3}} ca^{\binom{k+1}{2}} d^k \\ &= ba^{\binom{k+1}{3}+\binom{k+1}{2}} c^{k+1} d^{\binom{k}{2}+k} [d^{\binom{k}{2}}, c][d, c]^{\binom{k}{3}} \\ &= ba^{\binom{k+2}{3}} c^{k+1} d^{\binom{k+1}{2}} [d, c]^{\binom{k+1}{3}} \end{aligned}$$

since $\binom{k+1}{3} + \binom{k+1}{2} = \binom{k+2}{3}$, $\binom{k}{2} + k = \binom{k+1}{2}$, $[d^{\binom{k}{2}}, c] = [d, c]^{\binom{k}{2}}$, and $\binom{k}{2} + \binom{k}{3} = \binom{k+1}{3}$.

It can be seen from these formulas that σ^p is the identity on T. Thus we may form a group S as the split extension of T by a group of order p generated by an element s which induces σ on T. Note that $[[b, s], [c, s]] = [c, ad] = [c, d] \neq 1$, so that S is not metabelian. Thus if A is an abelian normal subgroup of maximal order in S, we must have $|S:A| > p^2$ (or else S/A would be abelian). On the other hand, consider a faithful irreducible character τ of T. Since s acts trivially on T', it follows from Theorems 2.10 and 2.11 that τ is invariant in S. If χ is an irreducible constituent of τ^S, then (by Frobenius reciprocity and the Going Down Theorem) $\chi_T = \tau$, so $\chi(1) = p^2$ and $T \cap \text{Ker } \chi = \{1\}$. However, since σ is an outer automorphism of T, we have $C_S(T) \leq T$, thus $\text{Ker } \chi = \{1\}$ follows. Thus χ is a faithful irreducible character of degree p^2 which cannot be induced from A (as $|S:A| > p^2$). By Lemma 5.1 (1), χ is not an *nM*-character, so S is not an *nM*-group.

Remark. The above group S was taken from Blackburn's list of groups of maximal class (see Blackburn [1]), though his description was unsuitable for characters.

EXAMPLE 5.12. It is immediate to check that the group S of Example 5.11 has an automorphism ρ of order p such that $t^{\rho} = t$ for all t in T while $s^{\rho} = s[a, b]$. Let R be the split extension of S by a group of order p generated by an element r which induces ρ on S . We claim that R *is an* nM-*group*; *it is obviously non metabelian, and its normal subgroup* S *is not an* nM-*group*.

It is clear that T is normal in R and R/T' is metabelian, so we only prove that if ψ is an irreducible character of R whose kernel does not contain T' , then ψ is an nM-character of R . If $\psi(1) \leq p$, this is automatic, for ψ is an M-character and all subgroups of index p are normal. The subgroup generated by a, d, $[a, b]$, and r is an abelian normal subgroup of index p^3 in R , so if B is an abelian normal subgroup of maximal order in R , then $|R:B| \leq p^3$ (alternatively, by Satz 7.3 (b) in Chapter III of Huppert [12], if $|\beta| = p^{\beta}$, then $\beta(\beta+1) \geq 14$; hence $\beta \geq 4$ and $|R:B| \leq p^3$). Thus Lemma 5.1 (2) shows that ψ is an nM-character unless $\psi(1) \leq p^2$; so we need only pursue the case $\psi(1) = p^2$. By assumption, ψ is non-trivial on T' , so ψ_T is a faithful irreducible character of T by Theorem 2.11. Hence ψ vanishes on $T \backslash T'$ by Theorem 2.10. It cannot vanish on $S \backslash T$, for $|S:T'| = p^5$ and $T' = Z(S)$ (see Theorem 2.10), thus $\psi(ts^m) \neq 0$ for some t in T and some m such that $0 < m < p$. Now $(ts^m)^r = ts^m[a, b]^m$, and hence $1 \neq [a, b]^m$ in $Z(R)$, so if X is a representation of R which affords

ψ , we have $X([a, b]^m) = \lambda I$ where λ is a complex number and I an identity matrix. As T' avoids Ker ψ , we know that $\lambda \neq 1$. Thus $X((ts^m)^r) = \lambda X(ts^m)$ and hence $\psi((ts^m)^r) = \lambda\psi(ts^m) \neq \psi(ts^m)$, a contradiction. Thus there is no ψ of degree p^2 which would not contain T' in its kernel, and our proof is complete.

At this stage, we have seen that if $p \geq 5$ then not all p-groups are *nM*-groups, but some of them are non-metabelian *nM*-groups. We defer the consideration of these statements for the remaining primes, to follow up what is perhaps a more interesting result here, that normal subgroups of *nM*-groups need not be *nM*-groups. In the next example we use R to show that subdirect products of *nM*-groups need not be *nM*-groups.

EXAMPLE 5.13. Let R be as in Example 5.12, and q a prime such that $q \equiv 1 \pmod{p}$, for instance, we may take $p = 5$ and $q = 11$. Then it is possible to form a split extension G of a cyclic group Q of order q by R , so that $C_R(Q) = S$. *This* G *is a subdirect product of* $G/Q = R$ *and* G/S *, both of which are nM-groups* (the latter because it is metabelian), but we claim that G *is not an nM-group.* To show this, we start with the faithful irreducible character χ of S which is not an *nM*-character (note that any non-faithful irreducible character of S is a character of $S/\text{Ker }\chi$ which is metabelian). Recall from Example 5.11 that χ_T is faithful irreducible and $\chi(1) = p^2$. If ψ is an irreducible constituent of χ^R then, by Frobenius reciprocity, χ is a constituent of ψ_S , so that $T' \nleq \text{Ker }\psi$. Thus by the last sentence of the argument concerning Example 5.12, $\psi(1) > p^2$. It follows that $\chi^R = \psi$.

Let λ be a faithful irreducible character of Q . By Theorem 2.5, $\lambda\chi$ is an irreducible character of QS . Now Ker $\lambda\chi$ avoids both Q and S and so must be trivial as $|Q|$ and $|S|$ are coprime. Thus $\lambda\chi$ is a faithful irreducible character of QS of degree p^2. By Mackey's Subgroup Theorem, $((\lambda\chi)^G)_R = ((\lambda\chi)_S)^R = \chi^R = \psi \in \mathrm{Irr}(R)$, so $(\lambda\chi)^G$ is a faithful irreducible character of degree p^3. Let A be an abelian normal subgroup of maximal order in G. By Lemma 5.1 (1), if $(\lambda\chi)^G$ is an nM-character, then $(\lambda\chi)^G = \alpha^G$ for some linear character α of A; in particular, $|G:A| = p^3$. Thus $Q \le A$, and hence $A \le C_G(Q) = QC_R(Q) = QS$. Thus $AS = QS$ and the r^i form a set of representatives of the A, S or Q, S double cosets in G. We can without loss of generality assume that $\alpha^{QS} = \lambda\chi$. By Mackey's Subgroup Theorem,

$$\sum_{i=1}^{p-1} \left[(\alpha^{r^i})_{A\cap S}\right]^S = (\alpha^G)_S = ((\lambda\chi)^G)_S = \sum_{i=1}^{p-1} ((\lambda\chi)^{r^i})_S = \sum_{i=1}^{p-1} \chi^{r^i} .$$

Thus $\chi = \left[(\alpha^{r^i})_{A\cap S}\right]^S$ for some i, contrary to the fact that χ is not an nM-character. This proves that $(\lambda\chi)^G$ is not an nM-character, and so G is not an nM-group.

We now turn to the problem of p-groups with $p < 5$.

EXAMPLE 5.14. *The wreath product of a group of order* 2 *and a quaternion group* Q *of order* 8 *is a non-metabelian* 2*-group which is an* nM*-group.* This is a split extension of an elementary abelian group A of order 2^8 by Q, where Q permutes regularly some basis of A. Now $G' = [A, Q]Q'$ and $G'' = [[A, Q], Q']$. If a is an element of the permuted basis of A, and h is an element of order 4 in Q, then $[[a, h], h^2] = [a^{-1}a^h, h^2] = a^{-h}aa^{-h^2}a^{h^3} \ne \{1\}$ (being the product of four distinct elements of the basis), so $G'' \ne \{1\}$. If $A < T \le G$ then $T = A(T \cap Q)$ with $T \cap Q > \{1\}$ and so $Q' \le T$. Thus if $\chi \in \mathrm{Irr}(G)$ and

$T = I_G(\alpha)$ where α is an irreducible constituent of χ_A, either $T = A$ in which case $\chi = \alpha^G$ by Theorem 2.2, or $T \geq Q'$ so that $G'' \leq [A, Q'] \leq [A, T] \leq \text{Ker}\,\alpha$ and hence $G'' \leq \text{Ker}\,\alpha$. In the later case, χ is really a character of G/G'' which is an nM-group by Theorem 3.5. Therefore in either case, χ is an nM-character, and so G is an nM-group.

It is not quite so easy to find a non-metabelian 3-group which is an nM-group. The only construction we have works without extra effort for any prime p in place of 3, so we describe it in this generality. It relies on the following.

LEMMA 5.15. (1) *If N is a normal subgroup of an arbitrary finite group G, and U is the regular G-module over an arbitrary field, one may view the set $C_U(N)$ of fixed points of N in U as G/N-module; as such, $C_U(N)$ is the regular G/N-module.*

(2) *Suppose in addition that U has prime characteristic p, and that N is a cyclic p-group in the centre of G. Put $V = U/C_U(N)$, then, as G/N-module, $C_V(N)$ is also regular.*

Proof. (1) By definition, U has a basis $\{u_g \mid g \in G\}$ such that $u_g h = u_{gh}$ for all g, h in G. Put $u_x = \sum_{g \in x} u_g$ for each element x in G/N. It is straightforward to see that $\{u_x \mid x \in G/N\}$ is a basis of $C_U(N)$, permuted regularly by G/N.

(2) A generator h of N permutes the given basis of U in cycles of length $|N|$, so as N-module, U is a direct sum of regular modules U_i $(i = 1, 2, \ldots, |G:N|)$. Thus $C_U(N) = \bigoplus_i C_{U_i}(N)$, and so $V = \bigoplus_i V_i$ where $V_i = U_i/C_{U_i}(N)$, and $C_V(N) = \bigoplus C_{V_i}(N)$. By Theorem 2.13, we must have

$C_{U_i}(N) = U_i(h-1)^{n-1}$ and $C_{V_i}(N) = V_i(h-1)^{n-2} = U_i(h-1)^{n-2}/U_i(h-1)^{n-1}$ where $n = |N|$, and $\dim C_{U_i}(N) = \dim C_{V_i}(N) = 1$. Clearly, $u \to u(h-1)$ maps $U_i(h-1)^{n-2}$ linearly onto $U_i(h-1)^{n-1}$ with kernel $U_i(h-1)^{n-1}$, and so yields an isomorphism of $C_V(N)$ onto $C_U(N)$. Since h is central in G, this is a G-module isomorphism. Now the claim follows by (1).

EXAMPLE 5.16. Let P be an extra-special p-group of order p^3, and R a maximal subgroup of P. Apply part (2) of the previous lemma to the regular P-module U over a field of order p, with $N = P'$, then $C_V(P')$ is a regular P/P'-module, which we may call W, and clearly $C_W(R/P') = C_V(R)$. From part (1) of our lemma applied to W and to the normal subgroup R/P' of P/P', we see that $C_V(R)$ is a regular P/R-module. As P/R is of order p, $C_V(R)$ is uniserial P-module of dimension p. Let $A/C_U(P')$ be the unique 2-dimensional submodule of $C_V(R)$. We claim that the split extension G of A by P is a non-metabelian nM-group of order p^{p^2+5}.

To see this, let us examine the module A a little further. Because P acts transitively on a basis of U, the fixed point space $C_U(P)$ is 1-dimensional, hence $C_A(P)$ is 1-dimensional. As P is a p-group and A has characteristic p, every irreducible P-submodule of A is trivial, so $C_A(P)$ is the unique minimal P-submodule of A. As $A/C_A(P')$ is a 2-dimensional uniserial module on which only the cyclic quotient P/R acts, we see that $[A/C_A(P'), P]$ is 1-dimensional. Let $1 \neq h \in P'$, as h is central in P, $a \mapsto a(h-1) = [a, h]$ is a P-module endomorphism of A with kernel $C_A(P')$. Thus the image $[[A, P], h]$ of $[A, P]$ is a

1-dimensional submodule. Hence $[[A, P], P']$ is the unique irreducible $C_A(P)$. Also, if h is any nontrivial element of P, we have $[A, h] \geq C_A(P)$. This is clear from our discussion when $h \in P'$, otherwise we appeal to the fact that $C_A(P')$ is a regular module for the abelian group P/P', so even $[C_A(P'), h]$ is a non-zero submodule and therefore must contain the unique irreducible. We are now ready to consider $G = A$ split P.

Since P acts faithfully on A (for P' does), we must have $\sigma(G) \leq C_A(P)$, so $\sigma(G) = C_A(P) = [[A, P], P']$. As $G' = [A, P]P'$ and $P'' = \{1\}$, we have $G'' = \sigma(G)$. Thus G is not metabelian, and A is an abelian normal subgroup of maximal order in G (for any normal subgroup of index less than p^3 must have abelian factor group and so cannot be abelian). If $g \in G \backslash A$, then $g = ah$ with $a \in A$, $1 \neq h \in P$, so $[A, g] = [A, h] \geq C_A(P) = \sigma(G)$. Thus, by Theorem 5.3, the faithful irreducible characters of G are all *nM*-characters. Non-faithful characters of G must contain the unique minimal normal subgroup G'' in their kernels, so they are *nM*-characters by Theorem 3.5. This completes the proof.

It remains to show that not all 2-groups or 3-groups are *nM*-groups. This can be easily seen from an example which works equally well for all primes p.

EXAMPLE 5.17. *The Sylow p-subgroup of* $\mathrm{GL}(n, p)$ *are nM-groups if and only if* $n \leq 4$, *that is, if and only if they are metabelian.*

To see this, we take a Sylow p-subgroup as the subgroup G consisting of upper unitriangular matrices. Let E_{ij} denote the $n \times n$ matrix with i, j entry 1 and all other entries 0, and I the $n \times n$ identity matrix. Now $I + E_{ij} \in G$ whenever $i < j$, and a straightforward calculation shows that if $i < j$ and $k < l$, then $[I+E_{ij}, I+E_{kl}] = I + \delta_{jk}E_{il}$ where δ_{jk} is the Kronecker delta. For each integer k with $1 \le k < n$, let G_k denote the subgroup of G generated by $\{I+E_{ij} \mid 1 \le i \le k < j \le n\}$. From the commutator relation above, we see that each G_k is abelian, and of course $|G_k| = p^{k(n-k)}$, as $(I+E_{ij})^p = 1$. It was shown by Goozeff in [8] that the G_k are precisely the maximal abelian normal subgroups of G. Since $G_1 \cap G_{n-1}$ is generated by $I + E_{1n}$, we see that $I + E_{1n}$ generates $Z(G)$. Put $A = G_{[n/2]}$; this is then a normal abelian subgroup of maximal order in G. It is easy to see from the commutator relation above that if $n > 5$, then G is not metabelian, while $[A, I+E_{n-2,n-1}]$ is the subgroup generated by $\{I+E_{i,n-1} \mid i \le [n/2]\}$ and hence avoids $Z(G)$. Thus, by Lemma 5.3, in this case, G is not an nM-group. On the other hand,

$$|G:A| = p^{(n^2-n)/2-[n/2](n-[n/2])}$$

so if $n \le 4$, then $|G:A| \le p^2$, and hence G/A is abelian, G is metabelian, and an nM-group by Theorem 3.5.

The example also illustrates that *not all maximal abelian normal subgroups of a p-group need be abelian normal subgroups of maximal order*, a fact which contributes to the difficulty of applying Lemma 5.2 and Lemma 5.3. For instance, choose G as above with $p > 2$ and $n = 4$; then G/G_1 is extra-special of order p^3. Let U be the regular G/G_1-module over the field of order p, regarded as G-module, and form the split extension H of U by G. Let $A = UG_1$, this is an abelian normal subgroup of H with H/A an extraspecial p-group of order p^3, so any normal subgroup of H properly containing A must contain AG'. Now U as a $G'G_1/G_1$-module is a direct sum of regulars which are uniserial of dimension p, whence it is easy to see that AG' has class precisely p. If B were an abelian normal subgroup of H not contained in A, then AB would contain AG' yet it would have class at most 2, for $(AB)' = [A, B] \leq A \cap B \leq Z(AB)$. Thus A is the unique maximal abelian normal subgroup of H. However, A/U is not contained in the unique abelian normal subgroup of maximal order in H/U namely in UG_2/U. This shows that Lemmas 5.2 and 5.3 are not suited for applications which involve induction on group order.

The trouble we have had to go to for examples of p-groups which are non-metabelian suggests that it would be very hard to find p-groups which are nM-groups and have derived length greater than 3, if indeed any such groups exist. As a further indication of how hard this task would be, we mention that such a group would need to have order at least p^{15}. For if there is an nM-group G with $G''' \neq \{1\}$ and $|G| \leq p^{14}$, this G has a factor group H with cyclic centre and $H''' \neq \{1\}$. This H has a faithful irreducible nM-character χ, which must be induced from a linear character of some abelian normal subgroup A, by Lemma 5.1 (1). Then H/A is an nM-group with $(H/A)'' \neq \{1\}$. By Corollary 2.30 of Isaacs [13], ~~Itô's Theorem (see Dornhoff [6], Theorem 22.3)~~ $\chi(1)^2 \leq |H:Z(H)| \leq p^{13}$ so $|H/A| \leq \chi(1) \leq p^6$. Repeating this argument with H/A in place of G, we find a non-metabelian nM-group K with an abelian normal subgroup of index dividing p^2, a contradiction.

CHAPTER 6

A-GROUPS

In the investigation of *nM*-groups we saw that the case of *p*-groups presented a largely intractable problem. To demonstrate that the difficulties in the way of a better understanding of *nM*-groups are essentially nilpotent in nature, we consider here *A*-groups, that is, solvable groups whose Sylow subgroups are all abelian. A further reason for looking at these is that all *A*-groups are *M*-groups by a special case of Huppert's Theorem (first established by Itô in [15]).

We show that an *A*-group is an *nM*-group if and only if it is a chiefly Frobenius group; equivalently, if and only if it is a subdirect product of Frobenius groups. It follows that *A*-groups which are *nM*-groups have derived length at most 3 , and that an *A*-group is an *nM*-group if (and only if) its Frattini factor group is an *nM*-group. The class of the *A*-groups which are *nM*-groups is closed under taking subgroups, factor groups, and direct products.

We also show that an *A*-group is an *sM*-group if and only if it is a chiefly sub-Frobenius group. The question of the derived lengths of such groups is left for the next chapter. An *A*-group is an *sM*-group if (and only if) its Frattini factor group is an *sM*-group. The class of the *A*-groups which are *sM*-groups is closed under taking subgroups, factor groups and direct products.

We shall need some elementary lemmas on *A*-groups. Let $\sigma(G)$ denote the socle of G .

LEMMA 6.1. *If G is an A-group, then $C_G(\sigma(G))$ is the Fitting subgroup F of G , and $\sigma(G) = \sigma(F)$.*

Proof. Let F_p be a Sylow p-subgroup of F . As the Sylow

p-subgroups of G are abelian, F_p is an abelian p-group and $G/C_G(F_p)$ is a p'-group. Thus $C_G(\sigma(F_p))$ acts trivially on F_p (see Chapter 5, Theorem 2.5 in Gorenstein [9]), so that $C_G(\sigma(F_p)) = C_G(F_p)$. Also, by Maschke's Theorem, $\sigma(F_p) \leq \sigma(G)$. These hold for all p , so $C_G(\sigma(F)) = C_G(F)$ and $\sigma(F) \leq \sigma(G)$. Since F is the Fitting subgroup, it must contain both $C_G(F)$ and $\sigma(G)$ (see Huppert [12], Chapter III, Satz 4.2 (b)). Thus $\sigma(G) = \sigma(F)$ and $C_G(\sigma(G)) = F$ as claimed.

LEMMA 6.2. *Let G be an A-group with $\sigma(G)$ minimal normal in G . Then (for some prime p) the Fitting subgroup F is a Sylow p-subgroup, complemented by a Hall p'-subgroup H . Also G has a normal subgroup P such that $\sigma(G)$ is G-isomorphic to F/P which is complemented by HP , and G/P has trivial Frattini subgroup.*

Proof. Since the Sylow subgroups of F are normal in G and $\sigma(G)$ is the unique minimal normal subgroup, F is a p-group. A Sylow p-subgroup of G must contain F and, being abelian, it centralizes F . However, by Lemma 6.1, F is its own centralizer, so F is a Sylow p-subgroup. It is then complemented by a Hall p'-subgroup H . Let p^{n+1} denote the exponent of F , and consider the map $\pi : f \mapsto f^{p^n}$. This is clearly a G-endomorphism of F , with $\{1\} < F\pi \leq \sigma(F)$. Now $F\pi$ is normal in G , hence $\sigma(G) \leq F\pi$. By Lemma 6.1, we have $F\pi = \sigma(G)$. Put $P = \operatorname{Ker} \pi$. Then $P \lhd G$, F/P is G-isomorphic to $\sigma(G)$, and so $C_{G/P}(F/P) = F/P$. Thus $\sigma(G/P) = F/P$ and HP/P is a maximal subgroup complementing F/P . If we let $\Phi(G/P)$ denote the Frattini subgroup of G/P , then $\Phi(G/P) \cap F/P \leq HP/P \cap F/P = \{1\}$, and so $\Phi(G/P) = \{1\}$. (It is easy to see that P is the Frattini subgroup of G , but we do not need that.)

LEMMA 6.3. *If K/L is any chief factor of an A-group G, then K/L is G-isomorphic to some complemented chief factor M/N of G; in particular, to one with N containing the Frattini subgroup Φ of G.*

Proof. Let S be maximal among the normal subgroups of G with $K \cap S = L$. Then K/L is G-isomorphic to KS/S and KS/S is the unique minimal normal subgroup of G/S. Thus by Lemma 6.2, KS/S is G-isomorphic to M/N where M is the Fitting subgroup of G/S and the Frattini subgroup of G/N is trivial, and M/N is complemented. Now $N\Phi/N$ is contained in the Frattini subgroup of G/N (see Huppert [12], Chapter III, Hilfsatz 3.4), so $\Phi \leq N$ as required.

COROLLARY 6.4. *Let G be an A-group and Φ the Frattini subgroup of G.*

(1) *If G/Φ is a chiefly Frobenius group, so is G.*

(2) *If G/Φ is an sM-group, then G is a chiefly sub-Frobenius group.*

Proof. (1) Let K/L be any chief factor of G. By Lemma 6.3, K/L is G-isomorphic to M/N where $N \geq \Phi$. Thus each element g of G acts on K/L as $g\Phi$ acts on M/N. By assumption, $g\Phi$ acts trivially or fixed point free on M/N, hence g acts trivially or fixed point free on K/L. Also $G/C_G(K/L) \simeq (G/\Phi)/C_{G/\Phi}(M/N)$, so that $G/C_G(K/L)$ has cyclic derived group as required.

(2) Let K/L be any chief factor of G. We use again Lemma 6.3 to conclude that K/L is G-isomorphic to a complemented chief factor M/N where $N \geq \Phi$. If $k \in K$ and kL corresponds to mN under the G-isomorphism $K/L \simeq M/N$, then $C_G(kL)/\Phi = C_{G/\Phi}(mN)$ and so $C_G(kL)$ is subnormal by Theorem 4.7.

LEMMA 6.5. *If an A-group H is a Frobenius complement, then its derived group is cyclic.*

Proof. Now all Sylow subgroups of H are cyclic, so that H is supersolvable, H' is nilpotent with cyclic Sylow subgroups, and thus H' is cyclic.

THEOREM 6.6. *If G is an A-group, then the following are equivalent:*

(1) G is an nM-group;

(2) G is a chiefly Frobenius group;

(3) G is a subdirect product of Frobenius groups.

Proof. By Theorem 5.5, (1) implies (2). From Lemma 6.5, Theorem 3.8, Theorem 5.5, and Theorem 5.7, we see that (3) implies (2).

We prove by induction on $|G|$ that (2) implies (1). By Theorem 5.7, we can assume that all proper factor groups of G are nM-groups; so by Theorem 5.4 we need only show that G satisfies the conditions of Lemma 5.2. The unique abelian normal subgroup of maximal order in an A-group is the Fitting subgroup F. By Lemma 6.1, if $g \in G \setminus F$, then g fails to centralize some minimal normal subgroup N of G. As G is a chiefly Frobenius group, g acts fixed point free on N, so that $[F, g] \geq [N, g] = N$. Thus for any C such that $F \geq C \geq [F, g]$, $\mathrm{core}_G C > \{1\}$.

To see that (2) implies (3), we argue by contradiction. By Theorem 5.7, a minimal counterexample G must be subdirectly irreducible, that is, $\sigma(G)$ is minimal normal in G. By the first statement of Lemma 6.2, $G = F$ split H. By Lemma 6.1 and the assumption that G is a chiefly Frobenius group, each non-trivial element h of H acts fixed point free on $\sigma(G) = \sigma(F)$. Thus $C_F(h) \cap \sigma(F) = \{1\}$ and so $C_F(h) = 1$. This proves that G is a Frobenius group, a contradiction.

COROLLARY 6.7. *If an A-group G is an nM-group, then $G''' = \{1\}$.*

Proof. G is a subdirect product of Frobenius A-groups K split H with K abelian and H' cyclic.

COROLLARY 6.8. *An A-group G is an nM-group if and only if its Frattini factor group G/Φ is an nM-group.*

Proof. The "only if" part follows from Theorem 3.3. The "if" part follows immediately from part (1) of Corollary 6.4 and the equivalence of (1) and (2) in Theorem 6.6.

COROLLARY 6.9. *All subgroups, factor groups and (finite) direct products of A-groups which are nM-groups, are also nM-groups.*

Proof. This follows from Theorem 5.7 and the equivalence (1) and (2) in Theorem 6.6.

Now we turn to *A*-groups which are *sM*-groups. The key result is the following.

THEOREM 6.10. *An A-group G is an sM-group if and only if it is a chiefly sub-Frobenius group.*

Proof. The "only if" part follows from Theorem 3.3 and part (2) of Corollary 6.4. The "if" part follows from Lemma 6.1 and Theorem 4.14.

COROLLARY 6.11. *An A-group G is an sM-group if and only if its Frattini factor group G/Φ is an sM-group.*

Proof. This follows from Theorem 3.3, part (2) of Corollary 6.4, and Theorem 6.10.

COROLLARY 6.12. *All subgroups, factor groups, and (finite) direct products of sM-groups* which are A-groups *are sM-groups.*

Proof. This follows from Theorem 4.12 and Theorem 6.10.

CHAPTER 7

THE NILPOTENT LENGTH OF SUBNORMAL M-GROUPS

This chapter contains two results concerning the nilpotent length (that is, the Fitting height) of sM-groups. The first gives an additional condition under which the second derived group of an sM-group must be nilpotent. While the condition is rather artificial, we include the result because its proof involves ideas which seem interesting even if we cannot use them to better effect. The second is a sketch of the construction of an sM-group of nilpotent length 5 ; we believe it should be possible to build sM-groups of arbitrary large nilpotent length by the same method.

THEOREM 7.1. *If G is an sM-group such that every chief factor K/L of G with $K \leq G''$ is cyclic or of prime rank, then G'' is nilpotent.*

Proof. We argue by contradiction. Let G be a counterexample of minimal order. Then all proper factor groups of G are nilpotent by metabelian but G is not; so G must have a unique minimal normal subgroup M . Now of course $M \leq G''$ so $|M| = p^r$ where r is 1 or a prime. Let F be the Fitting subgroup and Φ the Frattini subgroup of G , then $G'' \nleq F$. As F/Φ is the Fitting subgroup of G/Φ (see Huppert [12], Chapter III, Satz 4.2), $\Phi \neq \{1\}$ would imply $(G/\Phi)'' \leq F/\Phi$, contrary to $G'' \nleq F$. Thus $\Phi = \{1\}$. Let H be a maximal subgroup not containing M , so $G = M$ split H . As $C_H(M)$ is normal in G but avoids M , it must be trivial, so by Theorem 4.10, H is a p'-group.

Let e be the exponent of H and E a finite field of characteristic p containing a primitive eth root of 1 , so E is a splitting field for H (see Isaacs [13], Theorem 9.15). By Theorem 2.8, if we form $E \otimes M$ (over the field of order p) we obtain a direct sum of, say s , Galois conjugates of some irreducible EH-module V . As M is a faithful

H-module, so is V. Since r is 1 or a prime, and $r = s \dim V$, either $s = 1$ or $\dim V = 1$. If $\dim V = 1$, then H must be abelian; this is clearly impossible since G'' is non-nilpotent. Thus $\dim V > 1 = s$, so M is absolutely irreducible. Let u be the Brauer character of H afforded by M. By Theorem 2.9, u is an ordinary (irreducible) character, so by Theorem 3.9, $u_{H'}$ reduces. By Clifford's Theorem, the degrees of the irreducible constituents of $u_{H'}$ are proper divisors of the prime degree of u, so these constituents are all linear and hence H'' is in the kernel of u. As u is faithful, this means that $H'' = \{1\}$, so $G'' \leq M$, contrary to the assumption that G'' is nonabelian. This completes the proof.

EXAMPLE 7.2. Now we turn to *the construction of an sM-group of nilpotent length* 5. It will be an A-group, so by Theorem 6.10 all we need to prove about it is that it is a chiefly sub-Frobenius group of derived length 5.

We start with $\langle a \mid a^{16} = 1\rangle$ and $\langle b \mid b^{27} = 1\rangle$. Let a act on b invertingly, and form the corresponding split extension

$$G = \langle a, b \mid a^{16} = b^{27} = 1, b^a = b^{-1}\rangle .$$

Note that $G' = \langle b\rangle$, and $\langle a^2b\rangle$ is a cyclic subgroup of index 2; in particular, G is certainly a chiefly sub-Frobenius group. We shall need repeatedly the following fact. If $g \in G$, $i \geq 1$, and $a^{2^i} \notin \langle g\rangle$, $b^{3^{i-1}} \notin \langle g\rangle$, then $g \in \langle a^{2^{i+1}} b^{3^i}\rangle$.

Let G act on $\langle c, d \mid c^{49} = d^{49} = 1, cd = dc\rangle$ so that $c^a = d$, $d^a = c^{-1}$, $c^b = c^{18}$, $d^b = d^{30}$. As $18^3 \equiv 18 \times 30 \equiv 1 \pmod{49}$, it is straightforward to check that this definition is legitimate. One may also view $\langle c, d\rangle$ as the G-module induced from the $\langle a^2b\rangle$-module $\langle c\rangle$ such that $c^{a^2b} = c^{-18}$. As a^2 and b act fixed point free but a^4b^3 acts trivially on $\langle c, d\rangle$, the comment above with $i = 1$ makes it easy to check that $H = \langle c, d\rangle$ split G is a chiefly sub-Frobenius group. Also, $H' = \langle b, c, d\rangle$ and $H'' = \langle c, d\rangle$. The Fitting subgroup $F(H)$ is $\langle a^4, b^3, c, d\rangle$, and $F(H)/\langle a^8, b^9, c^7, d\rangle$ is cyclic of order 42 .

Consider a 1-dimensional faithful $F(H)/\langle a^8, b^9, c^7, d\rangle$-module U over the field of order 43 , as an $F(H)$-module, and form the induced H-module U^H . By Mackey's Subgroup Theorem, $(U^H)_{F(H)} = \bigoplus_{i=1}^{12} U_i$ where the U_i are the conjugates of U under a set of representatives of the cosets of $F(H)$ in H . Let the numbering be arranged so that $U_1 = U$ and $U_1, \ldots, U_6$ are the conjugates under $\langle a^2, b, c, d\rangle$; then $\langle a^2, b, c, d\rangle$ normalizes and a interchanges $\bigoplus_{i=1}^{6} U_i$ and $\bigoplus_{i=7}^{12} U_i$. As $\langle c\rangle$ and $\langle d\rangle$ are also normalized by $\langle a^2, b, c, d\rangle$ and interchanged by a , we readily see that c acts fixed point freely on the first sum and trivially on the second, while d acts trivially on the first and fixed point freely on the second. Of course, $\langle a^8, b^9, c^7, d^7\rangle$ acts trivially while a^4 and b^3 act fixed point freely on both. In particular, if $h \in \langle c, d\rangle$ then

$$[U^H, h] = \begin{cases} 1 & \text{if } h \in \langle c^7, d^7\rangle , \\ \bigoplus_{i=1}^{6} U_i & \text{if } h \in \langle c, d^7\rangle \setminus \langle c^7, d^7\rangle , \\ \bigoplus_{i=7}^{12} U_i & \text{if } h \in \langle c^7, d\rangle \setminus \langle c^7, d^7\rangle , \\ U^H & \text{otherwise.} \end{cases}$$

Now form $K = U^H$ split H . Then $K' = U^H H'$ and $K'' = U^H H'' = U^H\langle c, d\rangle$, so $K''' = U^H$. To verify that K is a chiefly sub-Frobenius group, we need only establish that if $0 \neq u \in U^H$, then $C_H(u)$ is subnormal in H . Since $C_H(u)$ is generated by its elements x of prime power order, and since joins of subnormal subgroups are subnormal (see Passman [16], Chapter 1, Theorem 6.6), this will be proved if we show that the subnormal closure of each such x centralizes u . When x is a 7-element, it lies in the normal Sylow 7-subgroup $\langle c, d\rangle$ of H , so $\langle x\rangle$ itself is subnormal and there is no work to do. When x is a 2-element or a 3-element, then some conjugate x^y lies in the Hall $\{2, 3\}$-subgroup G of H . As this x^y fixes the nonzero u^y , we have $a^4 \notin \langle x^y\rangle$ and $b^3 \notin \langle x^y\rangle$, so $x^y \in \langle a^8 b^9\rangle$. Thus if x^y is a 2-element, it is central in H , while if it is a 3-element, it is central in the normal subgroup $\langle b, c, d\rangle$ of H ; so $\langle x^y\rangle$ is subnormal. Consequently, so is $\langle x\rangle$, and we are done.

We shall need later on that if $h \in \langle c, d\rangle$ then the subnormal closure S of h in K is contained in $[U^H, h]\langle h\rangle$. To see this, observe that $U^H \leq \sigma(K)$, and recall that by Lemma 4.4, S is a direct factor of $S\sigma(K)$: therefore S is a direct factor of SU^H , and hence $[U^H, h] \leq [SU^H, S] \leq S$. ~~Conversely~~ Indeed, $[U^H, h]\langle h\rangle$ is normal in $U^H\langle h\rangle$ which is subnormal in K , so $S \leq [U^h, h]\langle h\rangle$.

Note that the Fitting subgroup $F(K)$ is $U^H\langle a^8, b^9, c^7, d^7\rangle$, of index 3528 in K . Choose a prime p such that $p \equiv 1 \bmod (2 \times 3 \times 7 \times 43)$ (this is psosible by Dirichlet's Theorem), and a faithful 1-dimensional module V for the cyclic group $F(K)/\left(\bigoplus_{i=2}^{11} U_i\right)\langle d^7\rangle$ of order $2 \times 3 \times 7 \times 43$, over the field of order p . Regard V as an $F(K)$-module, and consider the induced module V^K. By Mackey's Subgroup Theorem, $(V^K)_{F(K)}$ is the direct sum of 3528 conjugates V_i of V. Number these so that $V_1 = V$ and the 1764 conjugates under $U^H\langle a^2, b, c, d\rangle$ are listed first: let W_1 be the sum of these, and W_2 be the sum of the other V_i. Then $U^H\langle a^2, b, c, d\rangle$ normalizes and a interchanges W_1 and W_2, and also $\langle c^7\rangle$ and $\langle d^7\rangle$. It follows that c^7 acts fixed point freely on W_1 and trivially on W_2, while d^7 acts trivially on W_1 and fixed point freely on W_2. Similarly $\bigoplus_{i=7}^{12} U_i$ acts trivially on W_1, and $\bigoplus_{i=1}^{6} U_i$ acts trivially on W_2.

The proof of the fact that V^K split K is a chiefly sub-Frobenius group follows the previous pattern. The only step that is different is to show that if $0 \neq v \in V^K$ and $h \in \langle c, d\rangle \cap C_K(v)$, then the subnormal closure of h in K centralizes v. If $h \in F(K)$, then $\langle h\rangle$ is subnormal and there is nothing to prove. If $h \notin F(K)$, that is, $h \notin \langle c^7, d^7\rangle$, then h^7 is nontrivial and fixes the nonzero v, hence $h^7 \in \langle c^7\rangle$ or $\langle d^7\rangle$ (otherwise it would act fixed point free on both W_1, W_2). Say $\langle h^7\rangle = \langle c^7\rangle$; then $v \in W_2$. Also, $h \in \langle c, d^7\rangle \setminus \langle c^7, d^7\rangle$, so $S = [U^H, h]\langle h\rangle = \left(\bigoplus_{i=1}^{6} U_i\right)\langle h\rangle$, and as $\bigoplus_{i=1}^{6} U_i$ acts trivially on W_2, S does centralize v.

It remains to note that $K''' = U^H$ acts nontrivially on V^K and therefore K''' cannot lie in the Fitting subgroup of V^K split K; hence

$(V^K K)'''$ is non-abelian, and therfore the derived length of $V^K K$ is 5 .

CHAPTER 8

POSTSCRIPT

The principal aim in the study of M-groups, and certainly in the present thesis, has been to relate classes of finite solvable groups defined in character theoretic terms to classes defined in structural terms. The last two decades have seen an explosive development in the theory of (structurally defined) classes of finite solvable groups. This theory subdivides into chapters according to what closure properties the classes in question are required to have. One of the difficulties which prevents the adequate structural understanding of the class M of M-groups is that it has too few closure properties. In these circumstances, one might ask for classes with further closure properties which "approximate" M from above or from below. The determination of the unique largest subgroup closed class in M by Price and van der Waall may be seen as an example of such a result. (It should now be possible to decide whether that class is subdirect product closed.) On the other hand, no upper approximation seems to exist for M, beyond Taketa's Theorem, and Dade's Embedding Theorem shows that no better subgroup closed upper approximation is possible. In a sense, the key to the results in the present thesis has been the study of primitive nM or sM-groups (primitive in the sense of having a core free maximal subgroup). Perhaps a study of the primitive M-groups may pay similar dividends. Indeed, it is clear that not every primitive solvable group is an M-group (consider $C_3 \times C_3$ split SL(2, 3) formed with respect to the natural action), so the Schunck class generated by M is not the class of all finite solvable groups; this might be the way to find a better upper approximation for M than Taketa's. Our Lemma 4.6 provides only partial (but, as we have seen, still quite useful) information about the

Schunck class generated by $\mathfrak{s}M$, while Corollary 5.6 (1) may be viewed as a complete determination of the Schunck class generated by $\mathfrak{n}M$. Certain subgroup closed formations have also played a critical role in this thesis (Theorem 4.12 and 4.14; the last remark in Chapter 4; Theorem 5.5, the remark after it, and Theorem 5.7). We are inclined to conjecture that the Frattini factor group of a chiefly Frobenius group is always an $\mathfrak{n}M$-group. It seems likely that profitable directions of work could be identified by exploiting further concepts and ideas from the theory of classes of finite solvable groups.

Finally, we draw attention to the close relationship between Examples 5.12 and 5.13. While our failed attempts to decide whether $\mathfrak{s}M$ is normal subgroup closed or subdirect product closed have left us uncertain about which way the answers might go, we feel there is an intimate connection between the two questions, and conjecture that if one has an affirmative answer, so does the other.

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