ON 3-GROUPS OF SECOND MAXIMAL CLASS

by

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STATEMENT

The work in this thesis is my own,
except where otherwise stated.

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ABSTRACT

This thesis describes an algorithm and an implementation of this algorithm for generating finite p-groups. It is essentially an extension algorithm which given a group G calculates new groups $H_1, \ldots, H_m$ all of which have G as a certain kind of quotient. The new groups $H_1, \ldots, H_m$ can be used as input for the algorithm and more new groups are calculated. The p-groups generated in this way can be used to form a tree. These trees are frequently infinite. Parts of the trees are drawn as tree diagrams. These tree diagrams are a succinct way of representing some results about p-groups.

Detailed tree diagrams of 2-groups and 3-groups of maximal class and 2-~~class~~ groups of second maximal class are drawn using previously known results. The implementation of the algorithm has been used to calculate all 2-generator 3-groups of second maximal nilpotency class up to order 3^8 and many up to order 3^{10}. This is approximately 2,500 groups. The calculations were used to draw a tree diagram for some 2-generator 3-groups of second maximal class. Seventeen infinite branches of this tree are exhibited. These are of several different types. For one type, one infinite branch has been studied in detail and a complete description of the tree associated with this infinite branch is given.

The presentations of the p-groups calculated by the algorithm are thought of as being standard. The final chapter describes a process for recognizing such a presentation or a presentation which gives a group isomorphic to a group with a standard presentation.

CHAPTER 1

BACKGROUND AND INTRODUCTION

Over the years it has become apparent that major difficulties in the study of finite p-groups are the great diversity and large numbers of these groups. For this reason it is of interest to study certain classes of them. One aspect of this thesis is the study of 3-groups of second maximal class. Before any details are given about these groups background material is discussed. This leads to a brief outline of the contents of the thesis.

Let G be a group. The *lower central series*

$$G \geq \gamma_2(G) \geq \gamma_3(G) \geq \ldots \geq \gamma_{i-1}(G) \geq \gamma_i(G) \geq \ldots$$

of G is defined inductively by

$$\gamma_2(G) = [G, G] \ , \quad \gamma_i(G) = [\gamma_{i-1}(G), G] \quad \text{for} \quad i \in \{3, 4, \ldots\}^1 .$$

If there exists an integer c such that $\gamma_c(G) = E$, then G is *nilpotent* and if c is the least such integer then G is said to have *nilpotency class* $c - 1$.

It is well-known that a group of prime-power order, p^n , $n \geq 2$, can have nilpotency class no larger than $n - 1$. Groups of order p^n having nilpotency class exactly $n - 1$ are called groups of *maximal nilpotency class* (Huppert [1967], III, §14)[2]. These groups have been studied by various people. In 1904, 2-groups of maximal nilpotency class were already known to de Séguier [1904]. Burnside [1911, §98] studied groups with maximal size conjugacy classes. He showed these to have the properties of maximal nilpotency class groups.

The major task of classifying groups of maximal nilpotency class was commenced by Wiman [1946, 1952]. He introduced the name "maximal (nilpotency) class". For each group G , of maximal nilpotency class, he introduced a characteristic subgroup, $\gamma_1(G)$, of index p , defined as the largest subgroup of G such that $[\gamma_1(G), \gamma_2(G)] \leq \gamma_4(G)$. The subgroup

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[1] Notation used in this thesis is based on that used in Gorenstein's *Finite Groups*. A notation index is also given.

[2] References are given by the author and the date of publication which appears in square brackets.

$\gamma_1(G)$ is useful in determining the structure of G. Blackburn [1958] continued this study. He succeeded in classifying all 3-groups of maximal nilpotency class and all groups of order p^6 and nilpotency class 5. For his calculations he introduced the following concept. If G is a group of maximal nilpotency class and $[\gamma_i(G), \gamma_j(G)] \leq \gamma_{i+j+l}(G)$, $i, j \in \{1, 2, \ldots\}$ then G has *degree of commutativity* l. It is not assumed that l is the greatest such number.

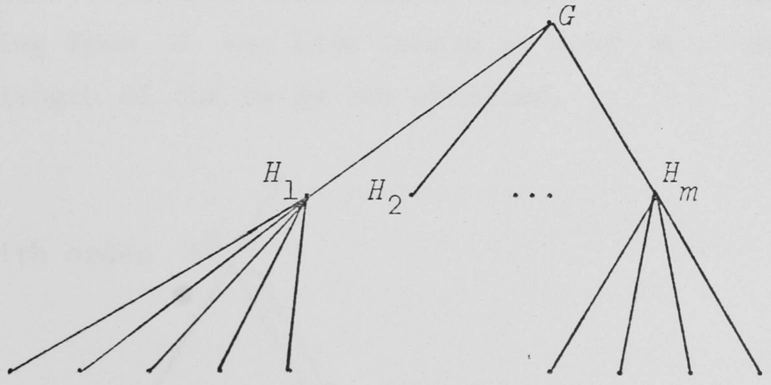
Leedham-Green and McKay [1976] constructed for all primes $p \geq 5$, and all integers $l \geq 0$ and $n \geq 4$ with $n \leq 2l + 4$, groups of order p^n, nilpotency class $n - 1$ and degree of commutativity l. Previously, this had been done independently by Miech [1974], using very different techniques. He also studied metabelian p-groups of maximal nilpotency class (Miech [1970]). Shepherd [1970] proved that if $n \geq 4$ and a group has order p^n, nilpotency class $n - 1$ and degree of commutativity l, then $2l \geq n-3p+6$; he later improved this to $2l \geq n-2p+4$. This work was also done independently by Leedham-Green and McKay. The first mentioned result of Leedham-Green and McKay shows that Shepherd's bound is almost best possible. From Shepherd's result it follows that if G is a p-group of maximal nilpotency class then G is soluble of length at most $[\log_2(3p-3)]$. This same bound was also calculated by Leedham-Green and McKay. The existence of such a bound had been shown previously by Alperin [1962].

Maung, a student of Leedham-Green, is reported to have calculated all 5-groups of maximal nilpotency class up to order 5^{16}. This was done with the aid of a computer. Work by Leedham-Green and McKay on the classification of p-groups of maximal nilpotency class is in progress.

Leedham-Green and McKay and their co-workers at Queen Mary College have generalized the study of groups of maximal class to that of groups of large class, that is groups of order p^n and class $n - r$, for fixed r and varying n (r a positive integer). A considerable amount of work is being done on these groups, in particular the suggestion that such groups have solubility length bounded by p and r alone is being investigated. This has been proved for groups of maximal nilpotency class, as seen above. When r is 2 the groups, of order p^n and class $n - 2$, are called groups of second maximal nilpotency class. For p equal to 2 these groups have been enumerated by James [1975] who called them groups of almost maximal class. Implicit in his work is that these groups have solubility length not exceeding 3 .

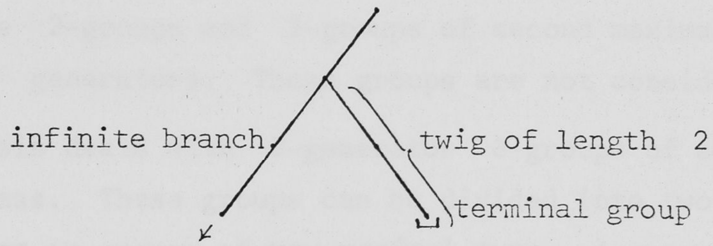
This brings us to 3-groups of second maximal nilpotency class. I first became interested in these groups through Leedham-Green. He visited Canberra during September 1976 and the question of bounding the solubility length of large class groups, and in particular 3-groups of second maximal nilpotency class, was then in his mind. To assist in the study of these groups all 2-generator 3-groups of second maximal nilpotency class were calculated up to order 3^8 and many up to order 3^{10} . This amounted to approximately two thousand five hundred groups and the calculations were done with the aid of a computer (Ascione, Havas, Leedham-Green [1977]). These calculations showed that various cases which might arise do not in fact occur. Using this Leedham-Green [in preparation b] was able to prove that 3-groups of second maximal nilpotency class have solubility length bounded by 4 .

The method used to calculate the 3-groups of second maximal nilpotency class is described in Chapter 3. In theory it can be used to calculate any finite p-group. It is essentially an extension algorithm which given a group G calculates new groups $H_1, \ldots, H_m$ all of which have G as a particular kind of quotient. The group G is said to *give rise* to the groups $H_1, \ldots, H_m$. Such a group, G , is said to be *capable* while a group which does not give rise to any groups is said to be *terminal*. The groups $H_1, \ldots, H_m$, called descendants of G are used as input for the algorithm and more new groups are calculated and so on. This information can be conveniently displayed in a tree diagram as follows:



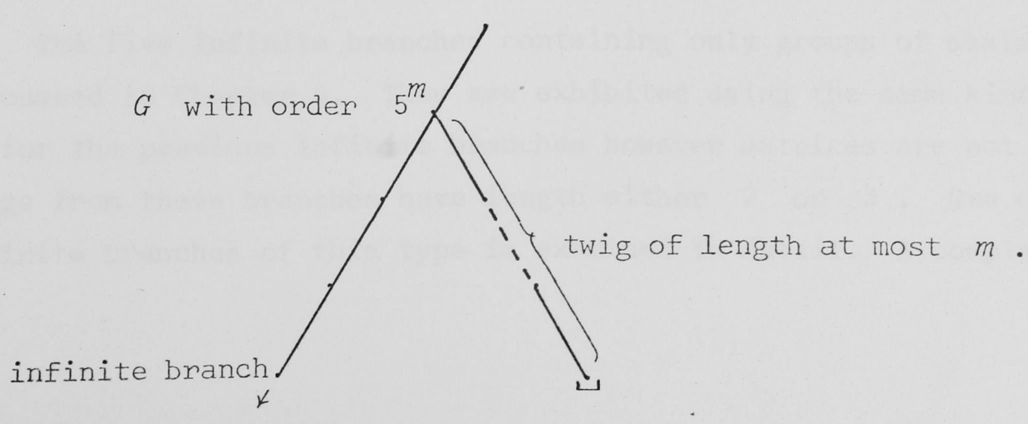
Tree diagrams are described in more detail in Chapter 4 and many examples are given. It is felt that tree diagrams are important as they give a clear, concise picture of the groups being described and allow the information to be assimilated more easily. The word branch will be used to denote part of a tree.

The idea of using tree diagrams to describe results about p-groups is of fairly recent origin. This idea suggests questions about the nature of the tree diagram. It can be shown that some trees have infinite branches. Many branches end with terminal groups and these branches are necessarily finite. Finite branches coming from an infinite branch are called *twigs*. The length of a twig is the number of groups on it excluding the group on



the infinite branch. The term *infinite branch* is used to mean just the string of capable groups and not the twigs coming from the infinite branch. The first questions asked about tree diagrams concern the number of infinite branches and the length of the twigs coming from them. For groups of maximal nilpotency class previous results can be re-interpreted to answer these questions. In this case an upper bound on the length of the twigs can be calculated using the degree of commutativity of the groups.

The tree of 2-groups of maximal nilpotency class has one infinite branch with twigs of length 1. Blackburn's results show that the trees of 3-groups and 5-groups of maximal nilpotency class each have one infinite branch. He has also calculated the degree of commutativity of these groups. This shows that the infinite branch in the tree of 3-groups of maximal nilpotency class has twigs of length at most 1. If G is a 5-group of maximal nilpotency class and of order 5^m on the infinite branch then a twig coming from G can have length at most m. In both cases the bounds for the length of the twigs are obtained.



Shepherd's results show in general that, for each p , $p \geq 5$, the tree of p-groups of maximal nilpotency class has exactly one infinite branch. The existence of at least one infinite branch had been known for quite some time (Baumslag and Blackburn [1960]). Shepherd's results also show that if G is a group of maximal nilpotency class and order p^m on the infinite branch then a twig coming from G can have length at most $m + 3p - 11$.

James' work on 2-groups of second maximal nilpotency class shows that the tree of 2-generator 2-groups of second maximal nilpotency class has 4 infinite branches with twigs of length 1 and 2 .

There are 2-groups and 3-groups of second maximal nilpotency class which have 3 generators. These groups are not considered here.

This thesis deals with 2-generator 3-groups of second maximal nilpotency class. These groups can be divided into two types called groups of *maximal type* or groups of *non-maximal type*. A group of maximal type can be constructed as a pullback whereas a group of non-maximal type can not be. This classification gives two types of infinite branches in the tree of 2-generator 3-groups of second maximal nilpotency class.

A complete tree diagram of these groups, up to order 3^9 , is shown in Chapter 4. This tree diagram consists of many parts most of which give details of the groups up to order 3^{10} . Fourteen infinite branches are exhibited in this tree. Five of these infinite branches contain only groups of maximal type. The remaining nine infinite branches contain only groups of non-maximal type.

These nine infinite branches are discussed in Chapter 5 and are exhibited using infinite groups of matrices. It is shown that there is an infinite set of finite 3-quotients of these groups of matrices and all the finite 3-quotients are different 2-generator 3-groups of second maximal nilpotency class. The twigs from these infinite branches appear quite complicated.

The five infinite branches containing only groups of maximal type are discussed in Chapter 6. They are exhibited using the same kind of argument as for the previous infinite branches however matrices are not used. The twigs from these branches have length either 2 or 3 . One of the infinite branches of this type is examined in detail. A complete description of this infinite branch and its twigs is given. It is also shown how these groups are constructed as pullbacks.

It is shown that the fourteen infinite branches exhibited are all different. A result of Leedham-Green shows that there are no more.

The final chapter describes a process for determining whether two finite p-groups are isomorphic or not. This process is based on the method used for generating p-groups. Where this is machine implemented it provides a practical method for determining isomorphism. Calculating p-groups using the method for generating them and drawing a tree diagram, in some sense gives a classification of the groups. The process described allows other groups of the same type, not calculated in the same way, to be recognized in the tree diagram.

CHAPTER 2

PRELIMINARY RESULTS

This chapter contains results which are used later but best dealt with here.

In Chapter 1 nilpotency class is defined. For this thesis class is defined using a different series. The algorithm for generating p-groups uses the lower-exponent-p-central series. This series reflects both the commutator and pth power structure of the groups and so is preferred to the lower-central series. It is also more convenient for computer programming. The lower-exponent-p-central series underlies all the work in this thesis.

Let G be a group.

DEFINITION 2.1. The *lower-exponent-p-central series* is a descending series of subgroups

$$G = P_0(G) \geq P_1(G) \geq P_2(G) \geq \ldots \geq P_{c-1}(G) \geq P_c(G) \geq \ldots$$

where $P_c(G) = [P_{c-1}(G), G](P_{c-1}(G))^p$ for $c \geq 1$.

DEFINITION 2.2. The *upper-exponent-p-central series* is an ascending series of subgroups

$$Q_0(G) = E \leq Q_1(G) \leq Q_2(G) \leq \ldots \leq Q_i(G) \leq Q_{i+1}(G) \leq \ldots$$

where $Q_{i+1}(G)$ is defined so that $Q_{i+1}(G)/Q_i(G)$ is the exponent-p-centre of $G/Q_i(G)$. (The exponent-p-centre consists of central elements with order p.)

LEMMA 2.3. *If* $P_c(G) = P_{c+1}(G)$ *then* $P_{c+1}(G) = P_{c+2}(G)$.

Proof. This is clear since, if $P_c(G) = P_{c+1}(G)$ then

$$P_{c+1}(G) = [P_c(G), G](P_c(G))^p = [P_{c+1}(G), G](P_{c+1}(G))^p = P_{c+2}(G) .$$

LEMMA 2.4. *If* G *is finitely generated then* $G/P_1(G)$ *is elementary abelian and if the number of generators is* d *then the order of* $G/P_1(G)$

is at most p^d .

Proof. Suppose G is generated by $\{g_1, \ldots, g_d\}$ then $P_1(G)$ is generated by $\left\{[g_j, g_i], g_i^p, P_2(G); i, j \in \{1, \ldots, d\}, i < j\right\}$. Now $G/P_1(G)$ is generated by $\{g_1P_1(G), \ldots, g_dP_1(G)\}$. It is clear that $(g_iP_1(G))^p = P_1(G)$ and $[g_jP_1(G), g_iP_1(G)] = P_1(G)$ for $i, j \in \{1, \ldots, d\}$, $i < j$. If $g_i^m = e$ for some $i \in \{1, \ldots, d\}$ and m is not a multiple of p then $(g_iP_1(G))^p = (g_iP_1(G))^m = P_1(G)$ and hence $g_iP_1(G) = P_1(G)$. Thus $G/P_1(G)$ is elementary abelian of order at most p^d .

LEMMA 2.5. *If* θ *is a homomorphism of* G *, then* $P_c(G)\theta = P_c(G\theta)$.

Proof. The proof is by induction on c . If θ is a homomorphism of G then

$$P_1(G)\theta = \left([G, G]G^p\right)\theta = [G\theta, G\theta](G\theta)^p = P_1(G\theta) ,$$

from a property of homomorphisms. Similarly if $P_c(G)\theta = P_c(G\theta)$ then $P_{c+1}(G)\theta = P_{c+1}(G\theta)$.

COROLLARY 2.6. *Each term* $P_c(G)$ *is fully invariant in* G .

COROLLARY 2.7. *If* N *is a normal subgroup in* G *then* $P_c(G/N) = \left(P_c(G).N\right)/N$.

Proof. The homomorphism θ is $\theta : G \to G/N$.

LEMMA 2.8. *If* G *is finitely generated then* $G/P_c(G)$ *is a finite p-group.*

Proof. Suppose G is generated by $\{g_1, \ldots, g_d\}$ then, by Lemma 2.4, $G/P_1(G)$ is elementary abelian of order at most p^d . For $i \geq 1$,

$$P_1\left(P_i(G)\right) = [P_i(G), P_i(G)]\left(P_i(G)\right)^p \leq P_{i+1}(G) = [P_i(G), G]\left(P_i(G)\right)^p .$$

Thus $P_i(G)/P_{i+1}(G)$ is a quotient of $P_i(G)/P_1\left(P_i(G)\right)$ which is elementary abelian of order some power of p and so $G/P_c(G)$ is a finite p-group.

LEMMA 2.9. *If G is a finite p-group, then $P_c(G) = E$ and $Q_c(G) = G$ for some positive integer c.*

Proof. Suppose G is a finite p-group then G has non-trivial centre and so $Q_1(G) \neq E$. If $G/Q_1(G)$ is trivial then $G = Q_1(G)$. Thus G is an elementary abelian group and hence $P_1(G) = E$. In this case c is 1.

If $G/Q_1(G)$ is non-trivial it is a finite p-group and so $Q_2(G) \neq Q_1(G)$. Thus, there is an integer c such that $Q_c(G) = G$. Suppose that c is the least such integer. It is now shown by induction that $P_i(G) \leq Q_{c-i}(G)$ for $i \in \{0, \ldots, c\}$. When i is 0, $P_0(G) = G = Q_c(G)$. Suppose $P_i(G) \leq Q_{c-i}(G)$ for some i. By definition $Q_{c-i}(G)/Q_{c-i-1}(G)$ is the exponent-p-centre of $G/Q_{c-i-1}(G)$ and by the inductive hypothesis $P_i(G)/Q_{c-i-1}(G)$ is contained in the exponent-p-centre of $G/Q_{c-i-1}(G)$. This means that for each g_i in $P_i(G)$, $[g_i, g] \in Q_{c-i-1}(G)$ for all g in G and $g_i^p \in Q_{c-i-1}(G)$. This is equivalent to $P_{i+1}(G) \leq Q_{c-i-1}(G)$. Thus $P_c(G) = E$. If $P_{c-1}(G) = E$ then $P_{c-2}(G)$ contains only central elements of order p. Thus $P_{c-2}(G) \leq Q_1(G)$ but this is not true. Hence $P_{c-1}(G) \neq E$.

DEFINITION 2.10. A group G has *exponent-p-central class* c if $P_c(G) = E$ but $P_{c-1}(G) \neq E$.

In the rest of this thesis, class will denote exponent-p-central class and nilpotency class will be specified as such.

LEMMA 2.11. *The group $G/P_c(G)$ has class at most c and if $G/P_c(G)$ has class c then $G/P_{c-1}(G)$ has class $c - 1$.*

Proof. Using Corollary 2.7,

$$P_c\big(G/P_c(G)\big) = \big(P_c(G).P_c(G)\big)/P_c(G) = E .$$

Thus $G/P_c(G)$ has class at most c. Now suppose $G/P_c(G)$ has class exactly c then, using the definition of class, $P_{c-1}\big(G/P_c(G)\big) \neq E$. It follows, using Corollary 2.7, that $P_c(G) < P_{c-1}(G)$. Now $P_{c-1}(G) \leq P_{c-2}(G)$ however $P_{c-1}(G) \neq P_{c-2}(G)$ since if they were equal, Lemma 2.3 gives $P_{c-1}(G) = P_c(G)$ and this is not true. Again using Corollary 2.7 it is shown that $P_{c-1}\big(G/P_{c-1}(G)\big) = E$ but $P_{c-2}\big(G/P_{c-1}(G)\big) \neq E$ and hence $G/P_{c-1}(G)$ has class $c - 1$.

LEMMA 2.12. *If* N *is a normal subgroup of* G *and* G/N *has class* c *, then* $P_c(G) \leq N$ *and* $P_{c-1}(G) \nleq N$.

Proof. Since G/N has class c it follows from the definition of class that $P_c(G/N) = E$ but $P_{c-1}(G/N) \neq E$. Using Corollary 2.7, $P_c(G/N) = \big(P_c(G).N\big)/N$. However, this is the identity hence $P_c(G) \leq N$. In a similar way it can be shown that $P_{c-1}(G) \nleq N$. □

A subgroup H of a group G is *omissible* if every subset of G which with H generates G, generates G by itself. Recall that $\Phi(G)$ the Frattini subgroup of G is defined as the intersection of G with all its maximal subgroups. Recall also that $\Phi(G)$ consists precisely of those elements x of G that can be omitted from every system of generators for G in which they occur. Thus the Frattini subgroup $\Phi(G)$ is omissible (M. Hall [1959, p. 157]).

LEMMA 2.13. *If* G *is a finite* p*-group then* $P_1(G) = \Phi(G)$ *and hence* $P_1(G)$ *is omissible in* G.

Proof. Since G is a finite p-group, G is nilpotent. Every maximal subgroup, A, of a nilpotent group is a normal subgroup of prime index. Thus $|G/A| = p$ and so G/A is the cyclic group of order p and hence abelian. Thus $[G, G] \leq A$. The group G/A does not contain any non-trivial pth powers and so $G^p \leq A$. Thus $[G, G]G^p \leq A$, that is $P_1(G) \leq A$. Hence $P_1(G)$ is contained in every maximal subgroup of G and so $P_1(G) \leq \Phi(G)$. However $G/P_1(G)$ is elementary abelian and so $P_1(G)$ is the intersection of some maximal subgroups and so $\Phi(G) \leq P_1(G)$. Thus $\Phi(G) = P_1(G)$. □

If G is a finite p-group and the minimum number of generators for G is d then $G/P_1(G)$ is elementary abelian of order p^d. In this thesis the term *d-generator groups* will be used to denote such groups. It is clear that all d-generator finite p-groups have the elementary abelian group of order p^d as a quotient. This is important for the next chapter.

In the context of class, as opposed to nilpotency class, it is possible for a group to have order p^n and class n. This only occurs however for the cyclic group of order p^n. If a group has order p^n and class c it is said to have *co-class* $n - c$.

It is clear that the class of a group is greater than or equal to a group's nilpotency class. Thus a group of maximal nilpotency class must have co-class 1. The converse is not true. The groups $C_p \times C_{p^n}$ have co-class 1 but also nilpotency class 1. Similarly groups of second maximal nilpotency class have co-class 2 but there exist groups with co-class 2 and very low nilpotency class.

The concept of co-class can be extended to infinite groups. This is done by giving an alternative definition.

DEFINITION 2.14. If G is a group let $r_i(G)$ be the dimension of $P_{i-1}(G)/P_i(G)$ for $i \in \{1, \ldots\}$, as a linear space.

DEFINITION 2.15. If G is a group then the co-class of G is

$$m(G) = \sum_{i=1}^{c} \left(r_i(G)-1\right)$$

where c is the class of G or ∞ if the class does not exist.

It is clear that if G is a finite group $m(G)$ equals the co-class of G as previously defined.

The following result, due to Blackburn, shows that for groups of maximal and second maximal nilpotency class the non-zero terms in the co-class sum must occur in the first p terms.

In Blackburn's [1958] paper he defines $ECF(m, n, p)$ as the set of all groups G of order p^n and class $m - 1$ in which $|\gamma_{i-1}(G)/\gamma_i(G)| = p$,

$i \in \{3, 4, \ldots, m\}$ and $G/\gamma_2(G)$ is elementary abelian. He then proves the following theorem (3.9).

Let G be a p-group of (nilpotency) class at least $p + 1$, and suppose that $G/\gamma_{p+1}(G) \in ECF(p+1, n, p)$. Then $G \in ECF(m, n', p)$ for some m, n'.

Applying this theorem to a p-group, G, of order p^n and having maximal nilpotency class shows that $|\gamma_{i-1}(G)/\gamma_i(G)| = p$ for $i \in \{3, \ldots, n\}$. This means that $P_i(G) = [P_{i-1}(G), G]$ for $i \in \{2, \ldots, n-1\}$. Thus $r_i(G) = 1$ for $i \in \{2, \ldots, n-1\}$ and $m(G) = 1$, since $G/P_1(G) \cong C_p \times C_p$ and has rank 2.

If G is a two-generator p-group with second maximal nilpotency class then $G/P_1(G)$ is isomorphic to $C_p \times C_p$ and so the first term in the co-class sum is non-trivial. If the next $p - 1$ terms in the co-class sum are trivial then Blackburn's theorem forces the group to have maximal nilpotency class which is not the case. Thus the other non-trivial term must be in the first p terms and so $P_{i+1}(G) = [P_i(G), G]$ for i greater than or equal to p. Thus when p is 3 and G has second maximal nilpotency class c either

$$P_0(G)/P_1(G) \cong P_1(G)/P_2(G) \cong C_3 \times C_3$$

and

$$P_2(G)/P_3(G) \cong \ldots \cong P_{c-1}(G)/P_c(G) \cong C_3$$

or

$$P_0(G)/P_1(G) \cong P_2(G)/P_3(G) \cong C_3 \times C_3$$

and

$$P_1(G)/P_2(G) \cong P_3(G)/P_4(G) \cong \ldots \cong P_{c-1}(G)/P_c(G) \cong C_3 .$$

Finally, it is worth noting that there is an infinite group associated with each infinite branch of a tree. Suppose that $G_1, G_2, \ldots, G_n, \ldots$ are groups on an infinite branch and that for every n a homomorphic mapping φ_n of G_{n+1} onto G_n is given. If $g_{n+1} \in G_{n+1}$, then $g_{n+1}\varphi_n \in G_n$ and for every $g_n \in G_n$ there is a g_{n+1} such that

$g_n = g_{n+1}\varphi_n$.

Define a *thread* to be a sequence of elements

$$\gamma = g_1, g_2, \ldots, g_n, \ldots$$

for which $g_n \in G_n$ and $g_{n+1}\varphi_n = g_n$ for $n \in \{1, 2, \ldots\}$. Then

$$\gamma^{-1} = g_1^{-1}, g_2^{-1}, \ldots, g_n^{-1}, \ldots$$

is also a thread and if

$$\gamma' = g_1', g_2', \ldots, g_n', \ldots$$

is another thread then

$$\gamma\gamma' = g_1g_1', g_2g_2', \ldots, g_ng_n', \ldots$$

is also a thread. Under this multiplication the threads are easily seen to form a group (Kurosh [1955, p. 227]). The group is the *inverse limit* or *projective limit* of the sequence of groups $G_1, G_2, \ldots, G_n, \ldots$ and the homomorphisms φ_n , $n \in \{1, 2, \ldots\}$. These limit groups are not used in this thesis. The infinite groups discussed in Chapters 5 and 6 are not limit groups of this type. However, there is a mapping from these infinite groups on to the limit group as defined above.

CHAPTER 3

AN ALGORITHM FOR GENERATING FINITE p-GROUPS

In a lecture presented to a mini-conference on group theory M.F. Newman [1977] described "a procedure which given a prime p and a positive integer d generates a list of descriptions of all d-generator finite p-groups". An important feature of this procedure is that it lists each group only once. This chapter is broken into four sections and discusses the procedure in more detail than given by Newman. The first section gives a general description of the procedure. This goes over the same ground covered by Newman however the two presentations are rather different. The second section describes the type of group presentations used by the procedure and, in some cases, how to calculate them. At the completion of the second section it is possible to give an example. This comprises the third section. The last section describes the computer implementation of the procedure. The implementation was written in order to calculate 3-groups of second maximal nilpotency class. Certain features of these groups were used to simplify the task. Thus, while the discussion is kept as general as possible, certain restrictions are introduced. The example of Section 3 is used to illustrate parts of the implementation.

3.1. A Theoretical Description of the Generating Algorithm

In this chapter P is a d-generator, finite p-group with class c, (p a prime, c and d are positive integers). The group P is presented in a way that makes c and d easy to calculate. This type of presentation is described in Section 2.

DEFINITION 3.1. A group Q is a *descendent* of P if Q has d generators, $Q/P_c(Q)$ is isomorphic to P and $P_c(Q)$ is non-trivial. A group Q is an *immediate descendant* of P if it is a descendant of P and has class $c + 1$.

Since P has d generators, $P/P_1(P)$ is an elementary abelian group of order p^d and hence P is a descendant of this elementary abelian group.

Also since,

$$\big(P/P_j(P)\big)/P_i\big(P/P_j(P)\big) \cong \big(P/P_j(P)\big)/\big(\big(P_i(P)P_j(P)\big)/P_j(P)\big) \cong P/P_i(P) \ ,$$

using Corollary 2.7 and that $P_j(P) < P_i(P)$, it follows that $P/P_j(P)$ is a descendant of $P/P_i(P)$ for $i < j \le c$. (It is clear that $P/P_j(P)$ and $P/P_i(P)$ both have d generators and that $P_i\big(P/P_j(P)\big) \ne E$.) Since P has class exactly c it follows from Lemma 2.11 that $P/P_k(P)$ has class exactly k for $k \in \{1, \ldots, c\}$. Thus, $P/P_{i+1}(P)$ is an immediate descendant of $P/P_i(P)$ for $i < c$. Hence, it is possible to calculate P using an iterative process of calculating immediate descendants starting with the elementary abelian group of order p^d. Since any group, P, can be calculated in this way a complete list of such groups can be obtained by calculating a complete set of immediate descendants at each step. In order that the list is complete but contains exactly one copy of each group the set of immediate descendants calculated for a given group must be complete and irredundant, that is the set must include all immediate descendants, no two of which are isomorphic. The process of calculating such a set of immediate descendants is loosely referred to as the *generating algorithm*. Iterating the generating algorithm gives the procedure described by Newman. The theorems which follow show how to calculate the required set of immediate descendants.

While the procedure generates, in theory, a list of all d-generator finite p-groups such a list can never be obtained as it is infinite. The number of groups of a given order grows rapidly with increasing order and so an efficient generating algorithm is needed. No formal tests to check this have been done but the machine implementation has allowed calculations to be performed which would not otherwise have been possible. The calculation of all 2-generator 3-groups of second maximal class up to order 3^8 is an example of this.

THEOREM 3.2. *For every d-generator finite p-group, P, with class c, there exists a d-generator finite p-group, P^*, of class at most $c+1$ such that every immediate descendant of P is isomorphic to a quotient group of P^*.*

Proof. Let F be a free group of rank d, freely generated by

$\{a_1, \ldots, a_d\}$. Since P is a d-generator group there exists a homomorphism θ from F onto P. Let R be the kernel of this map so that F/R is isomorphic to P. Define R^* to be $[R, F]R^p$ and P^* to be F/R^*.

Now P^* has d generators since P has d generators, F has rank d and $R^* \leq R$. Using Corollary 2.7 initially and then the definition of $P_{c+1}(F)$,

$$P_{c+1}(F/R^*) = \left(P_{c+1}(F).R^*\right)/R^* = \left\{[P_c(F), F]\left(P_c(F)\right)^p R^*\right\}/R^* .$$

Since F/R has class c, it follows immediately from Lemma 2.12 that $P_c(F) \leq R$ and hence

$$P_{c+1}(F/R^*) \leq \left([R, F]R^p.R^*\right)/R^* = E .$$

Thus P^* has class at most $c + 1$.

Now suppose Q is an immediate descendant of P and so, in particular $Q/P_c(Q)$ is isomorphic to P. Let φ be a homomorphism from Q onto P and so $\ker \varphi = P_c(Q)$. Define $q_1, \ldots, q_d$ to be elements of Q such that $q_i\varphi = a_i\theta$ for $i \in \{1, \ldots, d\}$. Since the $a_i\theta$ are generators for P it follows that $Q = \langle q_1, \ldots, q_d, P_c(Q)\rangle$. However $P_c(Q) \leq P_1(Q)$ and hence is omissible; thus $Q = \langle q_1, \ldots, q_d\rangle$. Let ψ be the homomorphism from F onto Q defined by $a_i\psi = q_i$ for $i \in \{1, \ldots, d\}$. Let M be the kernel of ψ and then F/M is isomorphic to Q. It is enough to show that R^* is a subgroup of M since then Q is isomorphic to $(F/R^*)/(M/R^*)$ and therefore isomorphic to $P^*/(M/R^*)$.

By definition $a_i\psi = q_i$ and $q_i\varphi = a_i\theta$ and hence $a_i\psi\varphi = a_i\theta$ for $i \in \{1, \ldots, d\}$. Since $\ker \theta = R$ it follows that $R\psi\varphi = E$ and hence $R\psi \leq \ker \varphi = P_c(Q)$. Now

$$R^*\psi = \left([R, F]R^p\right)\psi \leq [P_c(Q), Q]\left(P_c(Q)\right)^p = E .$$

Thus $R^* \leq \ker \psi$, that is R^* is a subgroup of M as required.

3.3. *The group* P^* *is determined up to isomorphism by* P.

Proof. The group P is a given p-group with d generators and class c. Let F be a free group of rank d, as before, and R_1 and R_2 normal subgroups such that $F/R_1 \cong P$ and $F/R_2 \cong P$. Set $F/R_1 = P_1$ and $F/R_2 = P_2$. Define R_1^* as $[R_1, F]R_1^p$ and R_2^* as $[R_2, F]R_2^p$ and P_1^* as F/R_1^* and P_2^* as F/R_2^*. The groups P_1^* and P_2^* are now shown to be isomorphic.

There is a homomorphism, h, from P_1^* onto P_1 such that $w(a_1, \ldots, a_d)R_1^* h = w(a_1, \ldots, a_d)R_1$. There is an isomorphism, j, between P_1 and P_2. Define $b_i(a_1, \ldots, a_d)$ such that $b_i(a_1, \ldots, a_d)R_1 j = a_i R_2$ for $i \in \{1, \ldots, d\}$. Clearly $P_1 = \langle b_1 R_1, \ldots, b_d R_1 \rangle$ thus $P_1^* = \langle b_1 R_1^*, \ldots, b_d R_1^*, R_1/R_1^* \rangle$. However $R_1/R_1^* \leq \theta_1(P_1^*)$ and thus is omissible.

Hence $P_1^* = \langle b_1 R_1^*, \ldots, b_d R_1^* \rangle$. Now define φ_1 to be hj such that

$b_i R_1^* \varphi_1 = b_i R_1^* hj = b_i R_1 j = a_i R_2$ for $i \in \{1, \ldots, d\}$ and so φ_1 is a homomorphism from P_1^* onto P_2 and $\ker \varphi_1 = R_1/R_1^*$. Define ψ_2 be the homomorphism from F onto P_1^* such that $a_i\psi_2 = b_i R_1^*$, $i \in \{1, \ldots, d\}$. Let M_2 be the kernel of ψ_2 and so F/M_2 is isomorphic to P_1^*. By the definition of ψ_2 and φ_1 it follows that $a_i\psi_2\varphi_1 = a_i R_2$ for $i \in \{1, \ldots, d\}$. Thus $R_2\psi_2\varphi_1 = E$ and hence $R_2\psi_2 \leq \ker \varphi_1$ and hence

$$R_2^*\psi_2 = \left([R_2, F]R_2^p\right)\psi_2 \leq [\ker \varphi_1, P_1^*]\left(\ker \varphi_1\right)^p .$$

However $\ker \varphi_1 = R_1/R_1^*$ which consists only of central elements of order p. Thus $R_2^*\psi_2 = E$ and hence $R_2^* \leq \ker \psi_2 = M_2$. This means that $P_2^*/\left(M_2/R_2^*\right)$ is isomorphic to P_1^*. In a similar way it can be shown that $P_1^*/\left(M_1/R_1^*\right)$ is isomorphic to P_2^* and hence P_1^* and P_2^* are isomorphic.

DEFINITION 3.4. The group P^* defined in Theorem 3.2 is called the *p-covering group* of P.

The first step in the generating algorithm is, given P, to calculate P^*. This is done using the p-covering algorithm which is described in Section 2. The following discussion deals with calculating the appropriate quotients of P^* in order to get all the immediate descendants of P. This is part of the next step of the generating algorithm.

DEFINITION 3.5. The *p-multiplicator* of P ($\cong F/R$) is R/R^*, where $R^* = [R, F]R^p$ as in the proof of Theorem 3.2.

DEFINITION 3.6. The *nucleus* of P is $P_c(P^*)$.

DEFINITION 3.7. A group P is *capable* if the nucleus of P is non-trivial and *terminal* if the nucleus of P is trivial.

It is shown later that this definition means that P is capable if it has immediate descendants and terminal if it does not. Terminal groups are of little interest.

DEFINITION 3.8. Proper subgroups of the p-multiplicator of P which supplement the nucleus of P are called *allowable subgroups*.

Thus M/R^* is an allowable subgroup if $M/R^* \,.\, P_c(F/R^*) = R/R^*$.

THEOREM 3.9. *The subgroup M/R^* is an allowable subgroup if and only if M/R^* is the kernel of a map from P^* onto an immediate descendant of P .*

Proof. This proof uses all the notation of Theorem 3.2. Let M/R^* be the kernel of the map from P^* onto Q , where Q is an immediate descendant of P and Q is isomorphic to F/M as in Theorem 3.2.

It is first shown that M/R^* is a proper subgroup of the p-multiplicator of P . To do this it is only necessary to show that M is a proper subgroup of R as it has already been shown in Theorem 3.2 that R^* is a subgroup of M . Let $w(a_1, \ldots, a_d)$ be a word in M and calculate $w(a_1, \ldots, a_d)\theta$.

$$\begin{aligned} w(a_1, \ldots, a_d)\theta &= w(a_1\theta, \ldots, a_d\theta) \\ &= w(q_1, \ldots, q_d)P_c(Q) \\ &= w(a_1, \ldots, a_d)\psi P_c(Q) . \end{aligned}$$

But M is $\ker \psi$, thus the image of $w(a_1, \ldots, a_d)$ under θ is the identity coset. That is, $w(a_1, \ldots, a_d)$ is in $\ker \theta$ which is R and so M is a subgroup of R . Now $(F/R^*)/(M/R^*)$ is isomorphic to F/M which has class $c+1$ but $(F/R^*)/(R/R^*)$ is isomorphic to F/R which has class c and so M is a proper subgroup of R .

It is now shown that M/R^* supplements the nucleus of P . Since F/R has class c , $P_c(F)$ is a subgroup of R (Lemma 2.12). However, M is a subgroup of R and hence $M.P_c(F)$ is a subgroup of R . It was proved in Theorem 3.2 that $R\psi$ is a subgroup of $P_c(Q)$. However F/R and hence $F\psi/R\psi$ have class c , thus $P_c(F\psi)$ is a subgroup of $R\psi$ (Lemma 2.8) and so $R\psi$ is equal to $P_c(Q)$. It is clear that $R\psi$ is also equal to R/M and that $P_c(Q)$ is equal to $(P_c(F).M)/M$. Thus R/M is equal to $(P_c(F).M)/M$ and hence R is equal to $M.P_c(F)$. Now R^* is contained in M and R but not $P_c(F)$ and can be factored out to show $(M/R^*).(P_c(F)R^*)/R^*$ is equal to R/R^* . However, using Corollary 2.6, this becomes $(M/R^*).P_c(F/R^*)$ is equal to R/R^* as required.

Now suppose M/R^* is an allowable subgroup. It must be shown that M/R^* is the kernel of a map from P^* onto an immediate descendant of P . This is equivalent to showing that $P^*/(M/R^*)$ is an immediate descendant of P . The group $P^*/(M/R^*)$ is isomorphic to F/M . Since M/R^* is an allowable subgroup $(P_c(F).M)/R^*$ is equal to R/R^* and hence $(P_c(F).M)/M$ is equal to R/M . Corollary 2.6 shows that $(P_c(F).M)/M$ is equal to $P_c(F/M)$ and hence $P_c(F/M)$ equals R/M . It is now clear that F/M has d generators (since F/M is a quotient of F/R), that $(F/M)/P_c(F/M)$ is isomorphic to P and that $P_c(F/M)$ is non-trivial. This shows that F/M is a descendant of P . It remains to show that F/M has class $c + 1$. For this it is only necessary to show that $P_{c+1}(F/M)$ is trivial. Again, using Corollary 2.6 and the definition of $P_{c+1}(F)$,

$$P_{c+1}(F/M) = (P_{c+1}(F).M)/M = \left([P_c(F), F](P_c(F))^p.M\right) / M .$$

However, F/R , which is isomorphic to P , has class c and hence $P_c(F)$ is a subgroup of R . Thus $P_{c+1}(F)$ is a subgroup of $(R^*M)/M$ which is trivial and so F/M is an immediate descendant of P . □

3.10. *A group* P *is capable if and only if* P^* *has class exactly* $c + 1$.

Proof. It has already been shown that P^* has class at most $c + 1$ and hence P^* has class exactly $c + 1$ if and only if $P_c(P^*)$ is non-trivial. However, this is the condition for P to be capable. □

If P^* does not have class exactly $c + 1$ then $P_c(P^*)$ is trivial. This is the condition for P to be terminal and it is clear that in this case P can not have any immediate descendants.

If P^* does have class exactly $c + 1$ then $P_c(P^*)$ is non-trivial. Choose a proper, non-trivial subgroup, M/R^*, of R/R^* such that $(M/R^*).P_c(F/R^*)$ is equal to R/R^* and then by Theorem 3.9 the group $P^*/(M/R^*)$ is an immediate descendant of P. Thus, a capable group is one which has immediate descendants.

The first step in the generating algorithm, as previously said, is, given P, to calculate the p-covering group P^*. This is done by the p-covering algorithm. The next step is to determine whether P is capable or not. This is done by calculating $P_c(P^*)$, the nucleus of P. If the nucleus is trivial there is nothing further to be done as P is terminal. If P is capable the next step is to calculate the allowable subgroups. This is straightforward given P^*. Theorems 3.2 and 3.9 show that a complete set of immediate descendants is obtained by factoring allowable subgroups from P^*. In general this set is not irredundant.

DEFINITION 3.11. Two allowable subgroups M/R^* and N/R^* are *equivalent* if and only if F/M is isomorphic to F/N, where F is as before.

This is clearly an equivalence relation and so divides the allowable subgroups into equivalence classes. A complete and irredundant set of immediate descendants is obtained by factoring from P^* one representative of each equivalence class. Thus, having calculated the allowable subgroups, it remains to arrange them into equivalence classes. This is done using Aut P, the automorphism group of P. Thus, input for the generating algorithm consists of the group P and also Aut P. It is now shown how to associate with each automorphism in Aut P a permutation of allowable subgroups. It is then shown that calculating the orbits of these permutations is equivalent to calculating the equivalence classes of the allowable subgroups.

Let β be an automorphism of F/R ($\cong P$). Recall that F is freely generated by $\{a_1, \ldots, a_d\}$. The action of β is given by $a_iR\beta = u_iR$, $i \in \{1, \ldots, d\}$. This defines u_i, $i \in \{1, \ldots, d\}$ which is a word in $a_1, \ldots, a_d$. Define β^* as follows. Let $w(a_1, \ldots, a_d) \in F$; then

$$w(a_1, \ldots, a_d)R^*\beta^* = w(u_1, \ldots, u_d)R^* .$$

To prove that β^* is an automorphism of F/R^* ($\cong P^*$) it is first necessary to show that β^* is well-defined. It is equivalent to show that if $w(a_1, \ldots, a_d) \in R^*$ then $w(u_1, \ldots, u_d) \in R^*$. Suppose $w(a_1, \ldots, a_d) \in R$, $w(a_1, \ldots, a_d)R\beta = w(u_1, \ldots, u_d)R$. However $R\beta = R$ thus $w(u_1, \ldots, u_d) \in R$. Thus $(R/R^*)\beta^* \leq R/R^*$. Now since $R^* = [R, F]R^p$ it follows that if $w(a_1, \ldots, a_d) \in R^*$ then so is $w(u_1, \ldots, u_d)$ and β^* is well-defined.

It is clear that β^* is a homomorphism and so it remains to show that β^* is onto. Since β is an automorphism of F/R and $a_iR\beta = u_iR$, $i \in \{1, \ldots, d\}$ it follows that F/R is generated by $\{u_1R, \ldots, u_dR\}$, that is $F/R = \langle u_1R, \ldots, u_dR\rangle$. Thus $F = \langle u_1, \ldots, u_d, R\rangle$ and hence $F/R^* = \langle u_1R^*, \ldots, u_dR^*, R/R^*\rangle$. However $R/R^* \leq P_1(F/R^*)$ and hence, by Lemma 2.13, is omissible. Thus

$$F/R^* = \langle u_1R^*, \ldots, u_dR^*\rangle = \langle a_1R^*\beta^*, \ldots, a_dR^*\beta^*\rangle$$

and hence β^* is an automorphism of F/R^*.

Note that β^* is not uniquely determined by β, however $\beta^*|_{R/R^*}$ is. Suppose $a_iR\beta = u_iR = u_ir_iR = v_iR$ for non trivial $r_i \in R$. Then there are two mappings β_1^* and β_2^* where $w(a_1, \ldots, a_d)R^*\beta_1^* = w(u_1, \ldots, u_d)R^*$ and $w(a_1, \ldots, a_d)R^*\beta_2^* = w(v_1, \ldots, v_d)R^*$. It has already been shown that $(R/R^*)\beta^* \leq R/R^*$ and since β^* is an automorphism $(R/R^*)\beta^* = R/R^*$. Now restricting β_1^* and β_2^* to R/R^* shows that $w(u_1, \ldots, u_d)$ and $w(v_1, \ldots, v_d)$ are in R. However, it is shown in the next section of this chapter that words in R are a product of pth powers and commutators. Now, it is clear that $[v_j, v_i]R^* = [u_j, u_i]R^*$ and $v_i^pR^* = u_i^pR^*$ hence $w(u_1, \ldots, u_d)R^* = w(v_1, \ldots, v_d)R^*$. Thus $\beta^*|_{R/R^*}$ is uniquely determined by β.

LEMMA 3.12. *The automorphism* β^* *induces a permutation on the set of allowable subgroups.*

Proof. The nucleus of F/R, $P_c(F/R^*)$, is fully invariant in F/R^* (Corollary 2.5), and hence is characteristic. Let M/R^* be an allowable subgroup, thus $M/R^* \,.\, P_c(F/R^*) = R/R^*$. Calculate $(M/R^*)\beta^* \,.\, P_c(F/R^*)$:

$$(M/R^*)\beta^* \,.\, P_c(F/R^*) = \big((M/R^*).P_c(F/R^*)\big)\beta^* = R/R^* .$$

Thus $(M/R^*)\beta^*$ is also an allowable subgroup. □

Denote the permutation induced, by β^* , on the set of allowable subgroups by β' . Let G denote the group of permutations which contains all permutations β' corresponding to automorphisms, β , of F/R . The claim is that the orbits of the allowable subgroups under G correspond to the equivalence classes. Before proving this claim it is first shown that there is a homomorphism from $\mathrm{Aut}(F/R)$ to the group of permutations induced on the allowable subgroups. Define the map $'$ where $' : \beta \to \beta'$ where β is an automorphism of F/R and β' is a permutation induced on the allowable subgroups as described above.

LEMMA 3.13. *The map* $'$ *is a homomorphism.*

Proof. It is first shown that $'$ is well-defined. Suppose that

$$a_iR\beta = u_iR = u_ir_iR = v_iR ,$$

then there are two choices β_1^* and β_2^* such that

$$w(a_1, \ldots, a_d)R^*\beta_1^* = w(u_1, \ldots, u_d)$$

or

$$w(a_1, \ldots, a_d)R^*\beta_2^* = w(v_1, \ldots, v_d) .$$

As shown previously, when restricted to R/R^* , these mappings are identical and hence the induced permutations are also.

Let β_1 and β_2 be automorphisms of F/R where $a_iR\beta_1 = u_iR$ and $a_iR\beta_2 = v_iR$ for $i \in \{1, \ldots, d\}$, where u_i and v_i are words in $a_1, \ldots, a_d$. Define x_i to be $u_i(v_1, \ldots, v_d)$ for $i \in \{1, \ldots, d\}$. Let $w(a_1, \ldots, a_d)$ be a word in R , then

$$\begin{aligned} w(a_1, \ldots, a_d)R^*\beta_1^*|_{R/R^*}\beta_2^*|_{R/R^*} &= w(u_1, \ldots, u_d)R^*\beta_2^*|_{R/R^*} \\ &= w(x_1, \ldots, x_d)R^* . \end{aligned}$$

Also $a_iR\beta_1\beta_2$ equals $u_iR\beta_2$ equals x_iR for $i \in \{1, \ldots, d\}$ and so

$$w(a_1, \ldots, a_d)R^*(\beta_1\beta_2)^*|_{R/R^*} = w(x_1, \ldots, x_d)R^* .$$

Thus $\beta_1'\beta_2'$ equals $(\beta_1\beta_2)'$ and so $'$ is a homomorphism.

THEOREM 3.14. *Suppose F/R, which is isomorphic to P, is a p-group as described above. The orbits of the allowable subgroups under G correspond exactly to the equivalence classes of allowable subgroups.*

Proof. Suppose M/R^* and N/R^* are allowable subgroups in the same equivalence class. Then F/M and F/N are isomorphic. Let θ_1 be an isomorphism from F/M to F/N and let f_i be defined by

$$a_iM\theta_1 = f_iN \quad \text{for} \quad i \in \{1, \ldots, d\} .$$

Now $P_c(F/M)$ equals R/M and $P_c(F/N)$ equals R/N and

$$P_c(F/M)\theta_1 \leq P_c(F/N) \quad \text{while} \quad P_c(F/N)\theta_1^{-1} \leq P_c(F/M) .$$

Thus $(R/M)\theta_1$ equals R/N and θ_1 induces an automorphism, θ, on F/R. Now $'$ is applied to θ to get θ' and it is shown that $(M/R^*)\theta'$ equals N/R^*. Let $w(a_1, \ldots, a_d)$ be an element of M; then

$$w(a_1, \ldots, a_d)R^*\theta^*|_{R/R^*} = w(f_1, \ldots, f_d)R^*$$

where $\theta^*|_{R/R^*}$ is the restriction of θ^* to R/R^*. Now

$$\begin{aligned} w(f_1, \ldots, f_d)N &= w(f_1N, \ldots, f_dN) \\ &= w(a_1M\theta_1, \ldots, a_dM\theta_1) \\ &= w(a_1, \ldots, a_d)M\theta_1 \\ &= eM\theta_1 \\ &= N \end{aligned}$$

where eM is the identity coset. Thus $w(f_1, \ldots, f_d)$ is an element of N and so $(M/R^*)\theta^*|_{R/R^*}$ is a subgroup of N/R^*. However, $(M/R^*)\theta^*|_{R/R^*}$ and N/R^* have the same index in F/R^* and so they are equal. Thus $(M/R^*)\theta'$ and N/R^* are equal. Since θ is in Aut F/R it follows that M/R^* and N/R^* are in the same orbit.

Now suppose M/R^* and N/R^* are two allowable subgroups which are in the same orbit. Thus, there is a γ_1, such that $(M/R^*)\gamma_1$ equals N/R^*.

This permutation γ_1 corresponds to γ , an automorphism of F/R . Define a map γ_2 from F/M to F/N by

$$w(a_1, \ldots, a_d)M\gamma_2 = (w(a_1, \ldots, a_d)R^*)\gamma^*N$$

where $w(a_1, \ldots, a_d)$ is a word in F and γ^* is an automorphism of F/R^* corresponding to γ in the way previously defined. It is now shown that γ_2 is an isomorphism.

Suppose $w(a_1, \ldots, a_d)$ is an element of M then $(w(a_1, \ldots, a_d)R^*)\gamma^*$ is an element of N/R^* since this is the original supposition about γ_1 . Thus $(w(a_1, \ldots, a_d)R^*)\gamma^*N$ equals N and so γ_2 is well-defined. The mapping γ_2 is clearly a homomorphism and so it remains to show that γ_2 is onto. Since γ^* is an automorphism of F/R^* it has an inverse γ^{*-1} . Let $w(a_1, \ldots, a_d)R^*\gamma^{*-1}$ equal $v(a_1, \ldots, a_d)R^*$. Then

$$\begin{aligned} v(a_1, \ldots, a_d)M\gamma_2 &= (v(a_1, \ldots, a_d)R^*)\gamma^*N \\ &= w(a_1, \ldots, a_d)R^*N \\ &= w(a_1, \ldots, a_d)N . \end{aligned}$$

Thus γ_2 is onto and the theorem is proved. □

Once the orbits are calculated an orbit representative from each orbit of allowable subgroups is chosen. Let O be a set of orbit representatives; then the set $\{P^*/(M/R^*) ; M/R^* \in O\}$ is a complete and irredundant set of immediate descendants of P .

It has already been said that input for the generating algorithm consists of P and Aut P . After calculating the immediate descendants of P their automorphism groups could be calculated independently and then the generating algorithm could be applied again. It is more convenient to calculate the automorphism groups of the immediate descendants as part of the generating algorithm. The structure of these automorphism groups allows this to be done.

THEOREM 3.15. *If* F/M *is an immediate descendant of* P *then*

$$\text{Aut } F/M = \hat{S}.K$$

where K *is the group of automorphisms of* F/M *which are trivial on* F/R *and* $\hat{S}$ *is defined as follows. Let* S *be the subgroup of* Aut F/R *containing those* β*'s for which* β^* *fixes* M/R^* *. Then* $\hat{S}$ *consists of those automorphisms,* β^* *, restricted to* F/M *.*

Proof. Suppose γ_1 is an automorphism of F/M . Now γ_1 fixes $P_c(F/M)$ and hence γ_1 can be restricted to F/R . Denote this restricted automorphism by γ .

If γ is the identity then γ_1 is in K .

If γ is not the identity then calculate γ^* . It is now shown that γ^* fixes F/R^* . First let

$$a_iM\gamma_1 = u_iM \quad \text{for} \quad i \in \{1, \ldots, d\}$$

so that the u_i's , words in $a_1, \ldots, a_d$, are defined by this. Then

$$w(a_1, \ldots, a_d)R^*\gamma^* = w(u_1, \ldots, u_d)R^* .$$

Now let $w(a_1, \ldots, a_d)$ be in M :

$$\begin{aligned} w(u_1, \ldots, u_d)M &= w(u_1M, \ldots, u_dM) \\ &= w(a_1M\gamma_1, \ldots, a_dM\gamma_1) \\ &= w(a_1, \ldots, a_d)M\gamma_1 \\ &= eM\gamma_1 \\ &= M . \end{aligned}$$

Thus γ^* fixes M/R^* and hence can be restricted to F/M . Denote this restricted automorphism by $\hat{\gamma}$. It is clear that $\hat{\gamma}$ is in $\hat{S}$. Now

$$a_iM\hat{\gamma} = u_ir_iM \quad \text{for} \quad i \in \{1, \ldots, d\}$$

where r_i is a word in R .

Define γ_2 to be such that

$$u_iM\gamma_2 = u_ir_iM \quad \text{for} \quad i \in \{1, \ldots, d\}$$

then clearly γ_2 is in K and $\hat{\gamma}$ equals $\gamma_1\gamma_2$. Thus γ_1 equals $\hat{\gamma}\gamma_2^{-1}$

and $\hat{\gamma}$ is in $\hat{S}$ and γ_2^{-1} is in K . □

If β is an automorphism of F/R such that β^* acts trivially on R/R^* then β' is the identity permutation and plays no part in determining the orbits. The inner automorphisms are of this type.

3.16. *In the calculation of orbits it is not necessary to use inner automorphisms of* F/R .

Proof. Suppose β is an inner automorphism of F/R , that is

$$a_i R\beta = a^{-1}a_i aR \quad \text{for} \quad i \in \{1, \ldots, d\}$$

where a is a word in $a_1, \ldots, a_d$. Suppose $w(a_1, \ldots, a_d)$ is a word in R ; then

$$\begin{aligned} w(a_1, \ldots, a_d)R^*\beta^* &= w\left(a^{-1}a_1a, \ldots, a^{-1}a_da\right)R^* \\ &= a^{-1}w(a_1, \ldots, a_d)aR^* \\ &= w(a_1, \ldots, a_d)R^* . \end{aligned}$$

The last step follows since words in R are central modulo R^* . Thus β' is the identity permutation and is not necessary for calculating orbits.

To summarize this section the steps of the generating algorithm are listed.

1. Given P , the p-covering group P^* is calculated.

2. The nucleus of P , $P_c(P^*)$ is calculated to determine whether P is capable or terminal. If P is terminal there is nothing else to do. If P is capable the following steps are performed.

3. A list of allowable subgroups is made.

4. For each automorphism β in Aut P the permutation β' is calculated.

5. The orbits of allowable subgroups are calculated and an orbit representative is chosen.

6. A description of each immediate descendant is calculated.

7. A description of the automorphism group of each immediate descendant is calculated.

3.2. A Practical Description of the Generating Algorithm

The previous section describes the theory of the generating algorithm without giving any of the practicalities involved in actually carrying it out. This section describes the type of descriptions for P and Aut P which are used by the generating algorithm. Enough details are given to allow the generating algorithm to be carried out by hand.

The input for the generating algorithm is a description of P and a description of Aut P. The type of presentation used for P is first discussed. This is followed by a description of the method used for calculating P^* and $P_c(P^*)$. Then Aut P and how to calculate the automorphism group of an immediate descendant are discussed. The type of presentation used for these groups is also given.

POWER-COMMUTATOR PRESENTATIONS

The input description for P is a power-commutator presentation.

DEFINITION 3.17. A *power-commutator presentation* of a group has the following form:

$$\Big\langle a_1, \ldots, a_n ; \; a_i^p = \prod_{k=i+1}^{n} a_k^{\alpha(i,k)}, \; 0 \le \alpha(i, k) < p,$$

$$[a_j, a_i] = \prod_{k=j+1}^{n} a_k^{\alpha(i,j,k)}, \; 0 \le \alpha(i, j, k) < p, \; i < j \Big\rangle .$$

(Any powers or commutators not shown are assumed to be trivial.)

Presentations of this type were first mentioned by Sylow [1872] and he in fact proved that every group of order p^n has a power-commutator presentation on n generators. The converse is not true, a power-commutator presentation on n generators may not present a group of order p^n; the order may be less.

It is not difficult to prove that in groups with a power-commutator presentation every element can be written in the form $\prod_{k=1}^{n} a_k^{\xi(k)}$ with $0 \le \xi(k) < p$. Words of this form are called *normal* words. Thus every word in $a_1, \ldots, a_n$ is equivalent to a normal word. The method for calculating a normal word equivalent to a given word is as follows. First notice that inverses do not need to be considered as they can always be eliminated using the power relations. If a word is not normal it has a subword, a_i^p, $i \in \{1, \ldots, d\}$ or $a_j a_i$, $i, j \in \{1, \ldots, d\}$, $i < j$. These subwords can be replaced by the words $\prod_{k=i+1}^{n} a_k^{\alpha(i,k)}$, $0 \leq \alpha(i, k) < p$, and $a_i a_j \prod_{k=j+1}^{n} a_k^{\alpha(i,j,k)}$, $0 \leq \alpha(i, j, k) < p$, respectively. By repeatedly replacing the first type of subwords by the second type a non-normal word is made normal. It can be shown that this process terminates. Such a process is called *collection*. For more details about collection processes see M.F. Newman [1976] or Havas and Nicolson [1976].

DEFINITION 3.18. A power-commutator presentation on n generators which presents a group of order p^n is called *consistent*.

The following theorem shows how to recognize a consistent presentation. This was first proved by Wamsley [1974].

THEOREM 3.19. Consistency Theorem. *If, in a power-commutator presentation, the following words,*

$$a_k a_j a_i, \quad 1 \leq i < j < k \leq n,$$

$$\left.\begin{matrix} a_j^p a_i \\ a_j a_i^p \end{matrix}\right\}, \quad 1 \leq i < j \leq n,$$

$$a_i^{p+1}, \quad 1 \leq i \leq n,$$

when collected in two essentially different ways give the same normal word then the presentation is consistent.

The essentially different ways of collecting are as follows. Brackets indicate the subwords to be replaced first.

$$(a_k a_j) a_i \quad \text{and} \quad a_k (a_j a_i),$$

$$\left(a_j^p\right) a_i \quad \text{and} \quad a_j^{p-1} (a_j a_i),$$

$$a_j \left(a_i^p\right) \quad \text{and} \quad (a_j a_i) a_i^{p-1},$$

$$a_i \left(a_i^p\right) \quad \text{and} \quad \left(a_i^p\right) a_i .$$

Collecting these words is referred to as performing the consistency checks. The consistency theorem is not proved here.

If a presentation is found not to be consistent then the element which caused the two normal words to be different is eliminated. An example serves to illustrate this. Consider the group presentation

$$\Big\langle a_1, a_2, \ldots, a_9;\ a_1^3 = e,\ a_2^3 = a_4^2,\ a_3^3 = a_6^2,\ a_4^3 = e,\ a_5^3 = a_7,\ a_6^3 = a_8,$$
$$a_7^3 = a_8^3 = a_9^3 = e,$$
$$[a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_5,\ [a_4, a_1] = a_6,\ [a_4, a_2] = a_9,$$
$$\text{all other simple commutators are trivial}\Big\rangle .$$

Performing the consistency checks (in reverse order) shows that this presentation is not consistent since

$$a_2\left(a_2^3\right) = a_2a_4^2 \quad \text{but} \quad \left(a_2^3\right)a_2 = a_2a_4^2a_9^2 .$$

Thus a_9 must be eliminated if the presentation is to be consistent. Eliminating a_9 gives a new presentation as follows.

$$\Big\langle a_1, \ldots, a_8;\ a_1^3 = e,\ a_2^3 = a_4^2,\ a_3^3 = a_6^2,\ a_4^3 = e,\ a_5^3 = a_7,\ a_6^3 = a_8,$$
$$a_7^3 = a_8^3 = e,$$
$$[a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_5,\ [a_4, a_1] = a_6,$$
$$\text{all other simple commutators are trivial}\Big\rangle .$$

It appears that the consistency checks must be performed for this presentation. However, those checks that have already been performed do not have to be redone. In the case above a_1^4 and a_2^4 were checked. In the new presentation both these words give the same normal words when collected in two essentially different ways. Thus performing all the consistency checks once and eliminating those generators which cause the two normal words to be different will give a consistent presentation. The presentation above is still not consistent since

$$a_4\left(a_1^3\right) = a_4 \quad \text{but} \quad \left(a_4a_1\right)a_1^2 = a_4a_8$$

and

$$\left(a_3^3\right)a_2 = a_2a_6^2 \quad \text{but} \quad a_3^2\left(a_3a_2\right) = a_2a_6^2a_7 \ .$$

All the other consistency checks have the normal words equal or show that a_7 and a_8 must be eliminated.

The presentation below is a consistent power-commutator presentation:

$$\Big\langle a_1, \ldots, a_6;\ a_1^3 = e,\ a_2^3 = a_4^2,\ a_3^3 = a_6^2,\ a_4^3 = a_5^3 = a_6^3 = e,$$
$$[a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_5,\ [a_4, a_1] = a_6\Big\rangle \ .$$

This is a presentation for the group O which is used as an example in the next section. This group is also mentioned in Chapter 4.

Consistent power-commutator presentations of d-generator p-groups of order p^n and class c are now discussed. Thus, in addition to assuming that P is a d-generator p-group of class c it is also assumed in the rest of this chapter that P has order p^n. Suppose P is such a group with a consistent power-commutator presentation. Since P is generated by $a_1, \ldots, a_d$ the elements $a_{d+1}, \ldots, a_n$ have definitions in terms of $a_1, \ldots, a_d$. These definitions appear among the relations of the presentation and are called *defining relations*. Defining relations, for p-groups, can ~~only~~ always be of two special types. They are as follows:

$$a_k = \begin{cases} a_i^p & ,\ i < k\ , \text{ and } a_i \text{ is a } p\text{th power of some element or } i \le d\ , \\ [a_j, a_i] & ,\ i < j < k\ ,\ i \le d\ . \end{cases}$$

To every generator, a_i, of P a weight, $\mathrm{wt}(a_i)$, is assigned as follows:

$$\mathrm{wt}(a_i) = 1 \quad \text{for} \quad i \in \{1, \ldots, d\}\ ,$$
$$\mathrm{wt}[a_j, a_i] = \mathrm{wt}(a_j) + \mathrm{wt}(a_i)\ ,$$
$$\mathrm{wt}\left(a_i^p\right) = \mathrm{wt}(a_i) + 1\ .$$

If u is any word in P then u is equivalent to a normal word say $\prod_{k=1}^{n} a_k^{\xi(k)}$. Suppose $\xi(l)$ is the first non-trivial exponent then

$\mathrm{wt}(u) = \mathrm{wt}(a_l)$. The identity element is defined to have infinite weight. It is clear that $P_i(P)$ contains elements of weight $i + 1$ and higher. This follows by induction since $P_0(P) = P$ contains elements of weight 1 and higher.

The definitions of $a_{d+1}, \ldots, a_n$ ensure that there is at least one element of each possible weight. Thus, let all the generators of weight i be $a_{d(i-1)+1}, \ldots, a_{d(i)}$ for $i \in \{2, \ldots, c\}$. It follows that $d(1)$ is d and $d(c)$ is n . It is clear that all generators of weight i are in $P_{i-1}(P)$.

Using the definitions it follows that $\mathrm{wt}(a_n) = c$ and hence elements of P can not have weight greater than c . Thus in the presentation for P ,

$$[a_j, a_i] = e \quad \text{if} \quad \mathrm{wt}(a_j) + \mathrm{wt}(a_i) \geq c + 1$$

and

$$a_i^p = e \quad \text{if} \quad \mathrm{wt}(a_i) = c \ .$$

THE p-COVERING ALGORITHM

Given a consistent power-commutator presentation for P the p-covering algorithm calculates a consistent power-commutator presentation for P^* , the p-covering group. This section describes the p-covering algorithm.

Suppose P has the following consistent presentation:

$$\Big\langle a_1, \ldots, a_n;\ a_i^p = \prod_{k=i+1}^{n} a_k^{\alpha(i,k)},\ 0 \leq \alpha(i, k) < p,$$

$$[a_j, a_i] = \prod_{k=j+1}^{n} a_k^{\alpha(i,j,k)},\ 0 \leq \alpha(i, j, k) < p,\ i < j \Big\rangle \ .$$

Among the relations there are $n - d$ defining relations and hence $d + \binom{n}{2}$ non-defining relations. Tietze transformations can be applied so that P is presented on $a_1, \ldots, a_d$. The presentation is

$$\Big\langle a_1, \ldots, a_d;\ u_i = v_i,\ i \in \Big\{1, \ldots, d+\binom{n}{2}\Big\}\Big\rangle$$

where u_i and v_i are words in $a_1, \ldots, a_d$ and each relation corresponds to a non-defining relation in the original presentation for P. Since P is F/R and F is a free group of rank d it follows that

$$R = \left\langle u_i v_i^{-1}, \ i \in \left\{1, \ldots, d+\binom{n}{2}\right\}\right\rangle .$$

Now R^* equals $[R, F]R^p$ and hence

$$R^* = \left\langle \left[u_i v_i^{-1}, a_j\right], \left(u_i v_i^{-1}\right)^p, \ i \in \left\{1, \ldots, d+\binom{n}{2}\right\}, \ j \in \{1, \ldots, d\}\right\rangle .$$

Thus the p-covering group P^*, which is F/R^* has the following presentation:

$$\left\langle a_1, \ldots, a_d; \ \left[u_i v_i^{-1}, a_j\right] = e, \ \left(u_i v_i^{-1}\right)^p = e, \ i \in \left\{1, \ldots, d+\binom{n}{2}\right\}\right\rangle .$$

Now Tietze transformations are applied to write this presentation in terms of $a_1, \ldots, a_n$ where the original definitions are used for $a_{d+1}, \ldots, a_n$. Also the relations $u_i v_i^{-1} = b_i$ are added. Thus the presentation is

$$\left\langle a_1, \ldots, a_n, b_1, \ldots, b_{d+\binom{n}{2}}; \ \left.\begin{array}{r} a_l^p = \prod_{k=l+1}^{n} a_k^{\alpha(l,k)} b_i, \\ 0 \le \alpha(l, k) < p, \\ [a_j, a_l] = \prod_{k=j+1}^{n} a_k^{\alpha(l,j,k)} b_i, \\ 0 \le \alpha(l, j, k) < p, \end{array}\right| \begin{array}{l} \text{where each } b_i \\ \text{appears depends on the} \\ \text{original non-defining} \\ \text{relation.} \end{array}\right.$$

$$\text{original definitions for } a_{d+1}, \ldots, a_n .$$

$$\text{the } b_i\text{'s are central of order } p \Big\rangle .$$

In general this power-commutator presentation is not consistent and so consistency checks need to be performed and the presentation made consistent. A consistent power-commutator presentation for P^* is as follows:

$$\left\langle a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+q}; \ a_i^p = \prod_{k=i+1}^{n+q} a_k^{\alpha(i,k)}, \ 0 \le \alpha(i, k) < p ,\right.$$

$$[a_j, a_i] = \prod_{k=j+1}^{n+q} a_k^{\alpha(i,j,k)}, \ 0 \le \alpha(i, j, k) < p$$

$$a_{n+1}, \ldots, a_{n+q} \text{ are central of order } p \Big\rangle .$$

The elements $a_{n+1}, \ldots, a_{n+q}$ correspond to the elements among $b_1, \ldots, b_{d+\binom{n}{2}}$ not eliminated during consistency checking. The p-multiplicator of P is generated by $a_{n+1}, \ldots, a_{n+q}$ and has rank q. The definitions for $a_{d+1}, \ldots, a_n$ remain unchanged. The definitions for $a_{n+1}, \ldots, a_{n+q}$ are of two types. Either

$$a_{n+l} = \left(\prod_{k=i+1}^{n} a_k^{\alpha(i,k)}\right)^{-1} a_i^p \quad \text{or} \quad a_{n+l} = \left(\prod_{k=j+1}^{n} a_k^{\alpha(i,j,k)}\right)^{-1} [a_j, a_i] .$$

The element a_{n+l} can not have weight 1 and so

$$a_{n+l} = \prod_{k=d+1}^{n} a_k^{\xi(k)} a_i^p \quad \text{or} \quad a_{n+l} = \prod_{k=d+1}^{n} a_k^{\xi(k)} [a_j, a_i] , \quad 0 \le \xi(k) < p .$$

Recalling the definitions of $a_{d+1}, \ldots, a_n$ it follows that a_{n+l} is a product of pth powers and commutators. (This is required earlier; see p. 21.)

Weights of the generators $a_1, \ldots, a_n$ are defined as before. The weights of the generators $a_{n+1}, \ldots, a_{n+q}$ are defined as follows. The generator a_{n+l} has a definition of the form $u = va_{n+l}$ and $\mathrm{wt}(a_{n+l}) = \mathrm{wt}(u)$ for $l \in \{1, \ldots, q\}$. The weights of other elements and the identity are defined as before. Also, as before, it can be shown that $P_i(P^*)$ contains elements of weight $i+1$ and higher.

The above shows that in its most basic form the p-covering algorithm has the following two steps.

(1) Add new generators, which are central and of order p, to each non-defining relation.

(2) Perform consistency checks on the resulting presentation eliminating generators until it is consistent.

This is the method employed for hand calculations although special properties of the particular group usually reduce the work in the consistency checks.

The machine implementation of the p-covering algorithm uses the two basic steps however the details are different. The new generators are added in class by class and as much information as possible is calculated about them before they are added in. This reduces the time spent on consistency checks. The machine implementation of the p-covering algorithm is a part of the nilpotent quotient algorithm (NQA) (Wamsley [1974], Newman [1976]).

A METHOD FOR CALCULATING THE NUCLEUS OF P

The following theorem gives a method for reading off $P_c(P^*)$ once P^* has been calculated.

THEOREM 3.20. *The nucleus of* P *is generated by* $[a_j, a_i]$ *and* a_j^p *for* $i \in \{1, \ldots, d\}$ *and* $j \in \{d(c-1)+1, \ldots, n\}$, $i < j$.

Proof. By definition $P_c(P^*) = [P_{c-1}(P^*), P^*](P_{c-1}(P^*))^p$. It is known that $P_c(P^*)$ contains elements of weight $c+1$ and hence

$$\left\langle [a_j, a_i], a_j^p ;\ i \in \{1, \ldots, d\}, j \in \{d(c-1)+1, \ldots, n\} \right\rangle \le P_c(P^*) .$$

Now $P_{c-1}(P^*)$ contains elements of weight c and higher. Let u be an element of $P_{c-1}(P^*)$; then $u = \prod_{k=d(c-1)+1}^{n} a_k^{\alpha(k)} a$, where $\mathrm{wt}(a)$ is $c+1$ or a is the identity.

Now $P_c(P^*)$ is generated by

$$\left\langle [u, g], u^p ;\ u \in P_c(P^*), g \in P^* \right\rangle .$$

However $u^p = \prod_{k=d(c-1)+1}^{n} \left(a_k^{\alpha(k)}\right)^p a^p$ (provided c is greater than or equal to 3), but $a^p = e$ and hence a_j^p , $j \in \{d(c-1)+1, \ldots, n\}$ is enough to generate these elements. Now consider the elements $[u, g]$. Since $\mathrm{wt}(u)$ is c it is only necessary to consider elements g with weight 1 . For g with weight greater than or equal to 2 , $[u, g]$ is trivial.

Using weight considerations it is enough to consider g as a word in $a_1, \ldots, a_d$. Again using weight considerations it follows that

$$[u, w_1(a_1, \ldots, a_d)] = w_2([u, a_1], \ldots, [u, a_d])$$

and

$$\left[\prod_{k=d(c-1)+1}^{n} a_k^{\alpha(k)}, a_i\right] = \prod_{k=d(c-1)+1}^{n} [a_k, a_i]^{\alpha(k)} \quad \text{for } i \in \{1, \ldots, d\} .$$

Thus

$$P_c(P^*) \leq \left\langle [a_j, a_i], a_j^p; \; j \in \{d(c-1)+1, \ldots, n\}, \; i \in \{1, \ldots, d\} \right\rangle$$

and the theorem is proved. □

Thus, once P^* is calculated, $P_c(P^*)$ can be read off and it can be determined whether the group is capable or terminal. If P is capable it is straightforward to calculate the allowable subgroups.

It is now shown that the immediate descendants of P have the same type of presentation as P. To do this it is only necessary to show that the elements of weight $c + 1$ have definitions of the required type. Theorem 3.20 shows that the definitions for elements of weight $c + 1$ are either as $[a_j, a_i]$ or a_j^p where $j \in \{d(c-1)+1, \ldots, n\}$ and $i \in \{1, \ldots, d\}$. This is the required form. When factoring out the allowable subgroups the other elements are written in terms of those with weight $c + 1$. Thus the immediate descendants of P have the same type of presentation as P.

AUTOMORPHISM GROUPS

Attention is now turned to $\operatorname{Aut} P$, how to calculate $\operatorname{Aut} F/M$ and how to describe these automorphism groups.

Suppose $\{\beta_1, \ldots, \beta_t\}$ is a generating set for $\operatorname{Aut} P$. The first step is to calculate the permutations of the allowable subgroups. To do this it is first necessary to calculate β^* for each automorphism β in the generating set. Suppose

$$a_iR\beta = u_i \quad \text{for } i \in \{1, \ldots, d\}$$

where u_i is a word in $a_1, \ldots, a_d$. Then β^* has the following action:

$a_iR^*\beta^* = u_iR^*$ for $i \in \{1, \ldots, d\}$,

$a_iR^*\beta^* = w(a_1, \ldots, a_{i-1})R^*\beta^* = w(u_1, \ldots, u_{i-1})R^* = u_iR^*$

for $i \in \{d+1, \ldots, n+q\}$.

The definitions of $a_{d+1}, \ldots, a_{n+q}$ give the words of elements with lower subscripts. Since $(R/R^*)\beta^*$ equals R/R^* it follows that

$$u_i = a_{n+1}^{\mu_{i-n,1}} a_{n+2}^{\mu_{i-n,2}} \ldots a_{n+q}^{\mu_{i-n,q}} \quad \text{for} \quad i \in \{n+1, \ldots, n+q\} ,$$

and $0 \le \mu_{i-n,j} < p$.

Anticipating Section 4, a $q \times q$ matrix M_{β^*} with entries $\mu_{i-n,j}$ is used to represent the automorphism β^* in the machine implementation of the generating algorithm. In hand calculations there is little value in using this matrix. Once $\{\beta_1^*, \ldots, \beta_t^*\}$ is calculated it is straightforward, though rather tedious, to calculate $\{\beta_1', \ldots, \beta_t'\}$. The orbits can now be calculated and this again is straightforward.

Suppose M/R^* is an orbit representative. Theorem 3.15 shows that Aut F/M is $\hat{S}.K$. Let $\{\gamma_1', \ldots, \gamma_r'\}$ be a generating set for the stabilizer of M/R^* and so

$$\hat{S} = \langle \hat{\gamma}_1, \ldots, \hat{\gamma}_r \rangle .$$

The subgroup K is generated by elements of the form

$$\gamma_{ij} : \begin{array}{ll} a_i \longmapsto a_i a_{n+j} , & i \in \{1, \ldots, d\} , \quad j \in \{1, \ldots, u\} , \\ a_k \longmapsto a_k , & k \in \{1, \ldots, d\} \backslash \{i\} , \end{array}$$

and $a_{n+1}, \ldots, a_{n+u}$ are elements of $F/M \setminus F/R$. Thus

$$\text{Aut } F/M = \langle \hat{\gamma}_1, \ldots, \hat{\gamma}_r, \gamma_{ij} \text{ (as above)} \rangle ,$$

and hence to calculate it the stabilizer of M/R^* is determined and the automorphisms in K are added. (In hand calculations there is no particular method for calculating the stabilizer of M/R^* .)

DEFINITION 3.21. An *adequate set* A of automorphisms for P is a set of automorphisms of Aut P which contains enough automorphisms to allow the generating algorithm to be successfully iterated. More explicitly A has the following properties.

(1) The orbits calculated by the set $\{\beta' ; \beta \in A\}$ correspond to the equivalence classes of allowable subgroups.

(2) Suppose F/M is an immediate descendant of P . Let S_A be the stabilizer of M/R^* using the set $\{\beta' ; \beta \in A\}$ and let T_P be a set of automorphisms of F/M which depends only on P . Then $\hat{S}_A \cup T_P$ is an adequate set for F/M .

Clearly the set $\{\beta_1, \ldots, \beta_t\}$, which is a generating set of Aut P , is an adequate set for P . In this case T_P is the empty set.

Suppose $\beta_{s+1}, \ldots, \beta_t$ are inner automorphisms and $\{\beta_1, \ldots, \beta_s\}$ is a set of coset representatives for Out P , where Out P = (Aut P)/(Inn P) .

3.22. *The set* $\{\beta_1, \ldots, \beta_s\}$ *is an adequate set for* P .

Proof. It is shown in 3.16 that inner automorphisms are not necessary for calculating orbits. Suppose β is an inner automorphism of F/R such that

$$a_iR\beta = a^{-1}a_iaR \quad \text{for} \quad i \in \{1, \ldots, d\}$$

as in 3.16. Since β' is the identity permutation it is certainly in the stabilizer of each orbit representative and hence $\hat{\beta}$ is in the generating set of the automorphism group of each immediate descendant. Suppose one immediate descendant is F/M . Theorem 3.15 shows that

$$a_iM\hat{\beta} = a^{-1}a_iar_iM \quad \text{for} \quad i \in \{1, \ldots, d\}$$

where r_i is a word in R . The following automorphisms γ and δ are automorphisms of F/M such that

$$a_iM\gamma = a_ir_iM \quad \text{and} \quad a_iM\delta = a^{-1}a_iaM \quad \text{for} \quad i \in \{1, \ldots, d\} .$$

The automorphism γ is in K and is a product of elements in K in the generating set for Aut F/M . Clearly $\hat{\beta}$ equals $\gamma\delta$ and so $\hat{\beta}$ can be replaced in the generating set by δ . However, δ is an inner automorphism and so can be omitted. In this case also T_P is the empty set. Thus $\{\beta_1, \ldots, \beta_s\}$ is an adequate set for P . □

A DESCRIPTION FOR AUT P

Before discussing the method of describing Aut P a restriction is made. Only groups P for which Aut P is soluble are considered. All 3-groups of second maximal class have soluble automorphism groups. The restriction simplifies the calculation of orbits in the machine implementation. It is possible to calculate orbits without having a soluble automorphism group. The group system "Cayley" (Cannon [1976]) has such a routine however in the interests of efficiency it is better to use the specialized one for soluble automorphism groups where this is possible.

3.23. *If* Aut F/R *is soluble then all descendants of* F/R *have a soluble automorphism group.*

Proof. In Theorem 3.15 it is shown that Aut F/M equals $\hat{S}.K$ where F/M is an immediate descendant of F/R. It follows that (Aut $F/M)/K$ is isomorphic to S, where S is as defined in Theorem 3.15. Thus if Aut F/R is soluble, so is Aut F/M. □

Since Aut F/R is soluble it has a composition series which can be made to go through Inn F/R, the group of inner automorphisms of F/R. An automorphism from each term in the series can be chosen to form a generating set for Aut F/R, say $\{\beta_1, \ldots, \beta_t\}$. This generating set is used to write a presentation of Aut F/R. The presentation has the form

$$\Big\langle \beta_1, \ldots, \beta_t;\ \beta_i^{\rho(i)} = \prod_{k=i+1}^{t} \beta_k^{\alpha(i,k)},\ 0 \le \alpha(i, k) < \rho(k),\ \rho(i) \text{ a prime,}$$

$$\beta_j\beta_i = \beta_i\beta_j \prod_{k=i+1}^{t} \beta_k^{\alpha(i,j,k)},\ 0 \le \alpha(i, j, k) < \rho(k) \Big\rangle .$$

This type of presentation is called a *power-commutation* presentation and is a generalization of a power-commutator presentation (Newman [1976]). It can be shown that a finite soluble group always has a power-commutation presentation.

Again suppose $\beta_{s+1}, \ldots, \beta_t$ are inner automorphisms. Factoring these from the presentation above leaves a presentation on $\beta_1, \ldots, \beta_s$ with the same form as the one above. It is this type of presentation which is used to describe an adequate set of automorphisms for P.

3.3. An Example

Suppose P is a group with the following consistent power-commutator presentation:

$$\Big\langle a_1, a_2, a_3, a_4, a_5, a_6;\ a_1^3 = e,\ a_2^3 = a_4^2,\ a_3^3 = a_6^2,\ a_4^3 = a_5^3 = a_6^3 = e,$$

$$[a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_5,\ [a_4, a_1] = a_6 \Big\rangle .$$

This is a presentation for the 3-group O which has order 3^6 and class 4. It is shown in the diagrams appearing in Chapter 4.

The 3-covering group of P has the following consistent power-commutator presentation:

$$\Big\langle a_1, a_2, \ldots, a_{10};\ a_1^3 = a_9,\ a_2^3 = a_4^2 a_{10},\ a_3^3 = a_6^2 a_7 a_8,\ a_4^3 = a_7^2,\ a_5^3 = a_8^2,$$
$$[a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_5,\ [a_4, a_1] = a_6,$$
$$[a_4, a_3] = a_8^2,\ [a_5, a_1] = a_8^2,\ [a_5, a_2] = a_7^2,\ [a_6, a_1] = a_7,\ [a_6, a_2] = a_8\Big\rangle .$$

The 3-multiplicator of P is

$$\Big\langle a_7, a_8, a_9, a_{10};\ a_7^3 = a_8^3 = a_9^3 = a_{10}^3 = e, \text{ abelian}\Big\rangle$$

and the rank of the 3-multiplicator is 4.

The nucleus of P which is $P_4(P^*)$ is $\langle a_7, a_8\rangle$ and hence P is capable. The allowable subgroups are now calculated. All the maximal subgroups of the 3-multiplicator are given by

$$\langle a_8, a_9, a_{10}\rangle,\ \Big\langle a_7 a_8^{\alpha}, a_9, a_{10}\Big\rangle,\ \Big\langle a_7 a_9^{\alpha}, a_8 a_9^{\beta}, a_{10}\Big\rangle,\ \Big\langle a_7 a_{10}^{\alpha}, a_8 a_{10}^{\beta}, a_9 a_{10}^{\gamma}\Big\rangle$$

where $\alpha, \beta, \gamma \in \{0, 1, 2\}$. There are forty maximal subgroups; however those that contain the nucleus must be eliminated. They are

$$\langle a_7, a_8, a_{10}\rangle,\ \langle a_7, a_8, a_9\rangle,\ \langle a_7, a_8, a_9 a_{10}\rangle,\ \Big\langle a_7, a_8, a_9 a_{10}^2\Big\rangle .$$

Any immediate descendants calculated by factoring out these allowable subgroups have order 3^7. To obtain immediate descendants of order 3^8 the allowable subgroups are

$$\Big\langle a_7^{\alpha} a_8^{\beta} a_9,\ a_7^{\gamma} a_8^{\delta} a_{10}\Big\rangle \quad \text{where } \alpha, \beta, \gamma, \delta \in \{0, 1, 2\} .$$

It is not possible to have immediate descendants of P with order 3^9. In this example only immediate descendants of order 3^7 are calculated and so the required allowable subgroups are:

1. $\langle a_8, a_9, a_{10}\rangle$
2. $\langle a_7a_8^2, a_9, a_{10}\rangle$
3. $\langle a_7a_8, a_9, a_{10}\rangle$
4. $\langle a_7a_9^2, a_8, a_{10}\rangle$
5. $\langle a_7a_9^2, a_8a_9^2, a_{10}\rangle$
6. $\langle a_7a_9^2, a_8a_9, a_{10}\rangle$
7. $\langle a_7a_9, a_8, a_{10}\rangle$
8. $\langle a_7a_9, a_8a_9, a_{10}\rangle$
9. $\langle a_7a_9, a_8a_9^2, a_{10}\rangle$
10. $\langle a_7a_{10}^2, a_8, a_9\rangle$
11. $\langle a_7a_{10}^2, a_8a_{10}^2, a_9\rangle$
12. $\langle a_7a_{10}^2, a_8a_{10}, a_9\rangle$
13. $\langle a_7a_{10}^2, a_8, a_9a_{10}^2\rangle$
14. $\langle a_7a_{10}^2, a_8a_{10}^2, a_9a_{10}^2\rangle$
15. $\langle a_7a_{10}^2, a_8a_{10}, a_9a_{10}^2\rangle$
16. $\langle a_7a_{10}^2, a_8, a_9a_{10}\rangle$
17. $\langle a_7a_{10}^2, a_8a_{10}^2, a_9a_{10}\rangle$
18. $\langle a_7a_{10}^2, a_8a_{10}, a_9a_{10}\rangle$
19. $\langle a_7a_{10}, a_8, a_9\rangle$
20. $\langle a_7a_{10}, a_8a_{10}, a_9\rangle$
21. $\langle a_7a_{10}, a_8a_{10}^2, a_9\rangle$
22. $\langle a_7a_{10}, a_8, a_9a_{10}\rangle$
23. $\langle a_7a_{10}, a_8a_{10}, a_9a_{10}\rangle$
24. $\langle a_7a_{10}, a_8a_{10}^2, a_9a_{10}\rangle$
25. $\langle a_7a_{10}, a_8, a_9a_{10}^2\rangle$
26. $\langle a_7a_{10}, a_8a_{10}, a_9a_{10}^2\rangle$
27. $\langle a_7a_{10}, a_8a_{10}^2, a_9a_{10}^2\rangle$
28. $\langle a_7, a_9, a_{10}\rangle$
29. $\langle a_7, a_8a_9^2, a_{10}\rangle$
30. $\langle a_7, a_8a_9, a_{10}\rangle$
31. $\langle a_7, a_8a_{10}^2, a_9\rangle$
32. $\langle a_7, a_8a_{10}^2, a_9a_{10}^2\rangle$
33. $\langle a_7, a_8a_{10}^2, a_9a_{10}\rangle$
34. $\langle a_7, a_8a_{10}, a_9\rangle$
35. $\langle a_7, a_8a_{10}, a_9a_{10}\rangle$
36. $\langle a_7, a_8a_{10}, a_9a_{10}^2\rangle$.

(All allowable subgroups are elementary abelian and so only the generators are shown.)

An adequate set A of automorphisms for P is:

$$\left\{\begin{array}{llll}\beta_1 : a_1 \mapsto a_1a_4, & \beta_2 : a_1 \mapsto a_1 \quad, & \beta_3 : a_1 \mapsto a_1a_5, & \beta_4 : a_1 \mapsto a_1 \\ \quad\;\; a_2 \mapsto a_2 & \quad\;\; a_2 \mapsto a_2a_4 & \quad\;\; a_2 \mapsto a_2 & \quad\;\; a_2 \mapsto a_2^2\end{array}\right\}.$$

These are extended to automorphisms of P^* and their action on the p-multiplicator is

$$\beta_1^* : \begin{array}{l} a_7 \mapsto a_7 \\ a_8 \mapsto a_8 \\ a_9 \mapsto a_9 \\ a_{10} \mapsto a_8a_{10} \end{array} \qquad \beta_2^* : \begin{array}{l} a_7 \mapsto a_7 \\ a_8 \mapsto a_8 \\ a_9 \mapsto a_9 \\ a_{10} \mapsto a_{10} \end{array} \qquad \beta_3^* : \begin{array}{l} a_7 \mapsto a_7 \\ a_8 \mapsto a_8 \\ a_9 \mapsto a_8^2a_9 \\ a_{10} \mapsto a_{10} \end{array} \qquad \beta_4^* : \begin{array}{l} a_7 \mapsto a_7^2 \\ a_8 \mapsto a_8 \\ a_9 \mapsto a_9 \\ a_{10} \mapsto a_{10}^2 \end{array} .$$

The corresponding permutations of the allowable subgroups are

β_1' : (2 20 11)(3 12 21)(5 23 14)(6 15 24)(8 26 17)(9 18 27)(28 34 31)(29 35 32)(30 36 33);

β_2' : is the identity permutation;

β_3' : (2 5 8)(3 9 6)(11 14 17)(12 18 15)(20 23 26)(21 27 24)(28 29 30)(31 32 33)(34 35 36) ;

β_4' : (2 3)(4 7)(5 9)(6 8)(11 12)(13 16)(14 18)(15 17)(20 21)(22 25)(23 27)(24 26)(31 34)(32 35)(33 36) .

Under these permutations the allowable subgroups form the following equivalence classes:

1;
2, 3, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27;
4, 7;
10;
13, 16;
19;
22, 25;
28, 29, 30, 31, 32, 33, 34, 35, 36.

Therefore P has eight immediate descendants of order 3^7. They are $P^*/1$, $P^*/2$, $P^*/4$, $P^*/10$, $P^*/13$, $P^*/19$, $P^*/22$, $P^*/28$, where the numbers represent the allowable subgroups.

The stabilizer of allowable subgroup 1 is generated by β_1', β_2', β_3', β_4' . An adequate set of automorphisms for $P^*/1$ is

$$\left\{\begin{matrix} a_1 \mapsto a_1 & a_1 \mapsto a_1a_4 & a_1 \mapsto a_1 & a_1 \mapsto a_1a_5 & a_1 \mapsto a_1 \\ a_2 \mapsto a_2a_6 & a_2 \mapsto a_2 & a_2 \mapsto a_2a_4 & a_2 \mapsto a_2 & a_2 \mapsto a_2^2 \end{matrix}\right\} .$$

The last four are in $\hat{S}_A$ and the first is in T_P . For all immediate descendants of P ,

$$T_P = \left\{\begin{matrix} a_1 \mapsto a_1 \\ a_2 \mapsto a_2a_6 \end{matrix}\right\} .$$

The stabilizer of allowable subgroup 2 is generated by β_2' and so an adequate set of automorphisms for $P^*/2$ is

$$\left\{\begin{matrix} a_1 \mapsto a_1 & a_1 \mapsto a_1 \\ a_2 \mapsto a_2a_6 , & a_2 \mapsto a_2a_4 \end{matrix}\right\} .$$

The stabilizer of allowable subgroup 28 is generated by β_2', β_4' and so an adequate set of automorphisms for $P^*/28$ is

$$\left\{\begin{matrix} a_1 \mapsto a_1 & a_1 \mapsto a_1 & a_1 \mapsto a_1 \\ a_2 \mapsto a_2a_6 , & a_2 \mapsto a_2a_4 , & a_2 \mapsto a_2^2 \end{matrix}\right\} .$$

The rest are calculated similarly.

3.4. Machine Implementation of the Generating Algorithm

This section deals with the machine implementation of the generating algorithm, or more correctly the parts of the generating algorithm which were implemented to calculate 3-groups of second maximal class. The p-covering algorithm is part of the NQA and already existed. This is discussed in Section 2. The routines discussed here are MATPER, ORBIT and STABILIZER. These calculate the permuations, orbits and stabilizers respectively. Calculating enough automorphisms for an adequate set is also discussed. The ideas for this implementation were communicated by C.R. Leedham-Green and I believe they are due to him and his colleagues at Queen Mary College. Most of the code for the machine implementation of the generating algorithm was written by George Havas, the remainder being written by W.A. Alford and J.B. Ascione.

The machine implementation can only deal with a group P which is a

2-generator group and has a soluble automorphism group. If P has order p^n then the only immediate descendants the machine implementation calculates are those with order p^{n+1}.

A METHOD FOR REPRESENTING ALLOWABLE SUBGROUPS

At the end of Section 2 it is shown how the automorphisms β^* are represented by a $q \times q$ matrix M_{β^*}, where q is the rank of the p-multiplicator. The method for representing the allowable subgroups is more complicated. Since only immediate descendants of order p^n are calculated all allowable subgroups are maximal subgroups of the p-multiplicator.

The maximal subgroups of the p-multiplicator, $\langle a_{n+1}, \ldots, a_{n+q} \rangle$ can be represented in the following way:

$$\langle a_{n+2}, \ldots, a_{n+q} \rangle, \left\langle a_{n+1}^{\alpha_1} a_{n+2}, a_{n+3}, \ldots, a_{n+q} \right\rangle, \ldots, \tag{*}$$

$$\left\langle a_{n+1}^{\alpha_1} a_{n+i}, a_{n+2}^{\alpha_2} a_{n+i}, \ldots, a_{n+i-1}^{\alpha_{i-1}} a_{n+i}, a_{n+i+1}, \ldots, a_{n+q} \right\rangle, \ldots,$$

$$\left\langle a_{n+1}^{\alpha_1} a_{n+q}, \ldots, a_{n+q-1}^{\alpha_{q-1}} a_{n+q} \right\rangle, \text{ where } 0 \le \alpha_j < p, \quad j \in \{1, \ldots, q\}.$$

The total number of these subgroups is $(p^q-1)/(p-1)$ however, in general, not all of them are allowable as they may not all supplement $P_c(P^*)$. Recall that elements of the p-multiplicator are labelled so that the elements of $P_c(P^*)$ have the lowest subscripts. Suppose $P_c(P^*)$ is generated by $a_{n+1}, \ldots, a_{n+l}$ $(0 < l \le q)$, then those subgroups with $\alpha_1 = \ldots = \alpha_l = 0$ are not allowable but the remainder are. Thus once the ranks of the p-multiplicator and the nucleus are calculated the allowable subgroups are determined.

The allowable subgroups can be represented as either matrices or vectors. A vector representation is more efficient however it is first necessary to describe a matrix representation. Each maximal subgroup is generated by $q - 1$ elements. Each of these elements can be written as a vector with the entries being the exponents so that the element

$a_{n+1}^{\xi_1} a_{n+2}^{\xi_2} \cdots a_{n+q}^{\xi_q}$ is represented by $(\xi_1, \xi_2, \ldots, \xi_q)$. A $(q-1) \times q$ matrix representing a subgroup has the vectors corresponding to the generating elements as rows. Clearly, by taking different generating elements for the subgroup a different matrix is obtained. However, by echelonizing the matrix in the usual way a standard matrix is obtained. The standard matrices correspond to the subgroups as presented in (*). Thus the matrix corresponding to the subgroup

$$\left\langle a_{n+1} a_{n+i}^{\alpha_1},\ a_{n+2} a_{n+i}^{\alpha_2},\ \ldots,\ a_{n+i-1} a_{n+i}^{\alpha_{i-1}},\ a_{n+i+1},\ \ldots,\ a_{n+q} \right\rangle$$

is

$$\begin{array}{c} \qquad\qquad i\text{th col} \\ \qquad\qquad\downarrow \\ \begin{bmatrix} 1 & 0 & 0 & \cdots & & \alpha_1 & 0 & & \cdots & 0 \\ 0 & 1 & 0 & \cdots & & \alpha_2 & 0 & & \cdots & 0 \\ & & & \ddots & & & & & & \\ 0 & & & 0 & 1 & \alpha_{i-1} & 0 & & \cdots & 0 \\ 0 & & \cdots & & & 0 & 1 & 0 & \cdots & 0 \\ 0 & & & \cdots & & & 0 & 1 & 0 \cdots & 0 \\ & & & & & & & & \ddots & \\ 0 & & & & \cdots & & & & 0 & 1 \end{bmatrix} \end{array}$$

and this is a standard matrix.

Let S denote the standard matrix representing a subgroup. A vector v which represents the subgroup is one for which $Sv^t = 0 \pmod p$. A *normal vector* is one whose first (that is, left-most) non-zero entry is 1.

3.24. *There is exactly one normal vector* v *such that* $Sv^t = 0$ $\pmod p$.

Proof. This is clear since the rank of S is $q - 1$. □

Normalizing a vector is achieved by adding, modulo p, a vector to itself and then adding the vector to the result until the first non-zero entry is 1. This should be achieved in at most $p - 2$ steps.

A label can be given to each allowable subgroup using the normal vector representing that subgroup.

DEFINITION 3.25. If V is a normal vector with the first k entries zero and q entries altogether, then

$$\text{label}(V) = 1 + V(k+2) + pV(k+3) + p^2V(k+4) + \ldots$$
$$+ p^{q-k-2}V(q) + \left((p^k-1)/(p-1)\right)p^{q-k} .$$

EXAMPLE. Suppose p is 3 then if $V = (1, 0, 0, 0)$ label$(V) = 1$, if $V = (1, 2, 2, 2)$, label$(V) = 27$, if $V = (0, 1, 0, 0)$, label$(V) = 28$ and if $V = (0, 1, 2, 2)$, label$(V) = 36$. As in the example in Section 3 the labels are identified with the allowable subgroups.

Before describing how to calculate the permutations the following observation is made.

3.26. *If P is a 2-generator group of second maximal nilpotency class with order p^n, $n \geq 3$, then $P_c(P^*) = \langle [a_n, a_1], [a_n, a_2] \rangle$.*

Proof. Theorem 3.20 shows that in this case

$$P_c(P^*) = \left\langle [a_n, a_1], [a_n, a_2], a_n^p \right\rangle$$

since a_n is the only generator of weight c. Since P has second maximal nilpotency class a_n is defined as a commutator - this is shown in Chapter 2. Thus

$$\begin{aligned} a_n^p &= [a_{n-1}, a_i]^p \qquad i \in \{1, 2\} \\ &= \left[a_{n-1}^p, a_i\right] \quad \text{(provided } p \neq 2\text{)} \\ &= \left[a_n^{\alpha(n-1,n)}, a_i\right] \\ &= [a_n, a_i]^{\alpha(n-1,n)} . \end{aligned}$$

Thus a_n^p is written in terms of $[a_n, a_1]$ or $[a_n, a_2]$ and so

$$P_c(P^*) = \langle [a_n, a_1], [a_n, a_2] \rangle . \qquad \square$$

Since the machine implementation deals only with 2-generator groups and is only concerned with calculating groups of second maximal nilpotency class the following discussion assumes that the rank of the nucleus is at most 2. When the rank of the nucleus is 1 there are p^{q-1} allowable subgroups and when it is 2 there are $p^{q-2}(p+1)$ allowable subgroups.

THE MATPER ROUTINE

MATPER stands for changing matrices into permutations. This routine starts with matrices $M_{\beta_1^*}, \ldots, M_{\beta_s^*}$, where $\{\beta_1, \ldots, \beta_s\}$ is an adequate set of automorphisms for P, and calculates $(\beta_1')^{-1}, \ldots, (\beta_s')^{-1}$. The permutations are written in terms of the labels identified with the allowable subgroups. The routine also needs the ranks of the p-multiplicator and nucleus.

Suppose M/R^* is an allowable subgroup and β^* is an automorphism of F/R^*. Suppose also that $(M/R^*)\beta^*$ equals N/R^* another allowable subgroup. If S is a matrix corresponding to M/R^* then SM_{β^*} is a matrix corresponding to N/R^*. This is clear if the matrix multiplication is performed. Thus the smallest set of $(q-1) \times q$ matrices which contains S and is closed with respect to right multiplication by $M_{\beta_1^*}, \ldots, M_{\beta_s^*}$ contains exactly the matrices which correspond to the allowable subgroups in the orbit containing M/R^*.

Suppose v is the normal vector which represents M/R^* then by definition

$$Sv^t = 0 .$$

Since the matrix M_{β^*} represents an automorphism, it is invertible and hence

$$SM_{\beta^*}M_{\beta^*}^{-1}v^t = 0 .$$

Thus a vector corresponding to N/R^* is $M_{\beta^*}^{-1}v^t$. Since the inverse of an automorphism is an automorphism the matrices $M_{\beta_1^*}, \ldots, M_{\beta_s^*}$ give the same orbits as the matrices $M_{\beta_1^*}^{-1}, \ldots, M_{\beta_s^*}^{-1}$. Thus the smallest set of vectors which contains v and is closed with respect to left multiplication by $M_{\beta_1^*}, \ldots, M_{\beta_s^*}$ contains exactly the vectors which correspond to the allowable subgroups in the orbit containing M/R^*. Notice that $M_{\beta^*}v^t$ is a vector corresponding to $(M/R^*)(\beta^*)^{-1}$.

The labelling of the vectors has been arranged so that the results of multiplying the matrices and vectors is achieved by adding columns of the matrices.

Suppose the rank of the nucleus is 2 . There are $p^{q-2}(p+1)$ allowable subgroups. These are represented by the following normal vectors arranged in order according to their labels:

$(1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, p-1, 0, \ldots, 0),$
$(1, 0, 1, 0, \ldots, 0), (1, 1, 1, 0, \ldots, 0), \ldots, (1, p-1, 1, 0, \ldots, 0),$
$(1, 0, 2, 0, \ldots, 0), (1, 1, 2, 0, \ldots, 0), \ldots, (1, p-1, p-1, 0, \ldots, 0),$
$(1, 0, 0, 1, 0, \ldots, 0), \ldots, (1, p-1, p-1, \ldots, p-1), (0, 1, 0, \ldots, 0),$
$(0, 1, 1, 0, \ldots, 0), \ldots, (0, 1, p-1, 0, \ldots, 0), (0, 1, 0, 1, 0, \ldots, 0),$
$(0, 1, 1, 1, 0, \ldots, 0), \ldots, (0, 1, p-1, 1, 0, \ldots, 0),$
$\ldots, (0, 1, p-1, p-1, \ldots, p-1)$.

If $\mathrm{v} = (v_1, \ldots, v_q)$ is a vector then it is well-known that

$$M_{\beta^*}(v_1, \ldots, v_q)^t = (M_{\beta^*})_1 v_1 + (M_{\beta^*})_2 v_2 + \ldots + (M_{\beta^*})_q v_q$$

where $(M_{\beta^*})_i$ is the ith column of the matrix M_{β^*} .

The appropriate sequence of adding columns of the matrix to give the permutation action corresponding to M_{β^*} is as follows.

The image of the first vector, $(1, 0, \ldots, 0)$ is $(M_{\beta^*})_1$.

The image of the second vector $(1, 1, 0, \ldots, 0)$ is $(M_{\beta^*})_1 + (M_{\beta^*})_2$.

The image of the third vector $(1, 2, 0, \ldots, 0)$ is $((M_{\beta^*})_1 + (M_{\beta^*})_2) + (M_{\beta^*})_2$; to obtain this the second column is added to the vector which was already calculated as the image of the second vector. After each image vector is obtained the label is calculated. The flow chart (Figure 3.27) describes the process in detail.

To sum up, the process described above takes the matrices $M_{\beta_1^*}, \ldots, M_{\beta_s^*}$, the ranks of the p-multiplicator and nucleus and calculates the action of the permutations $(\beta_1')^{-1}, \ldots, (\beta_s')^{-1}$. This action is

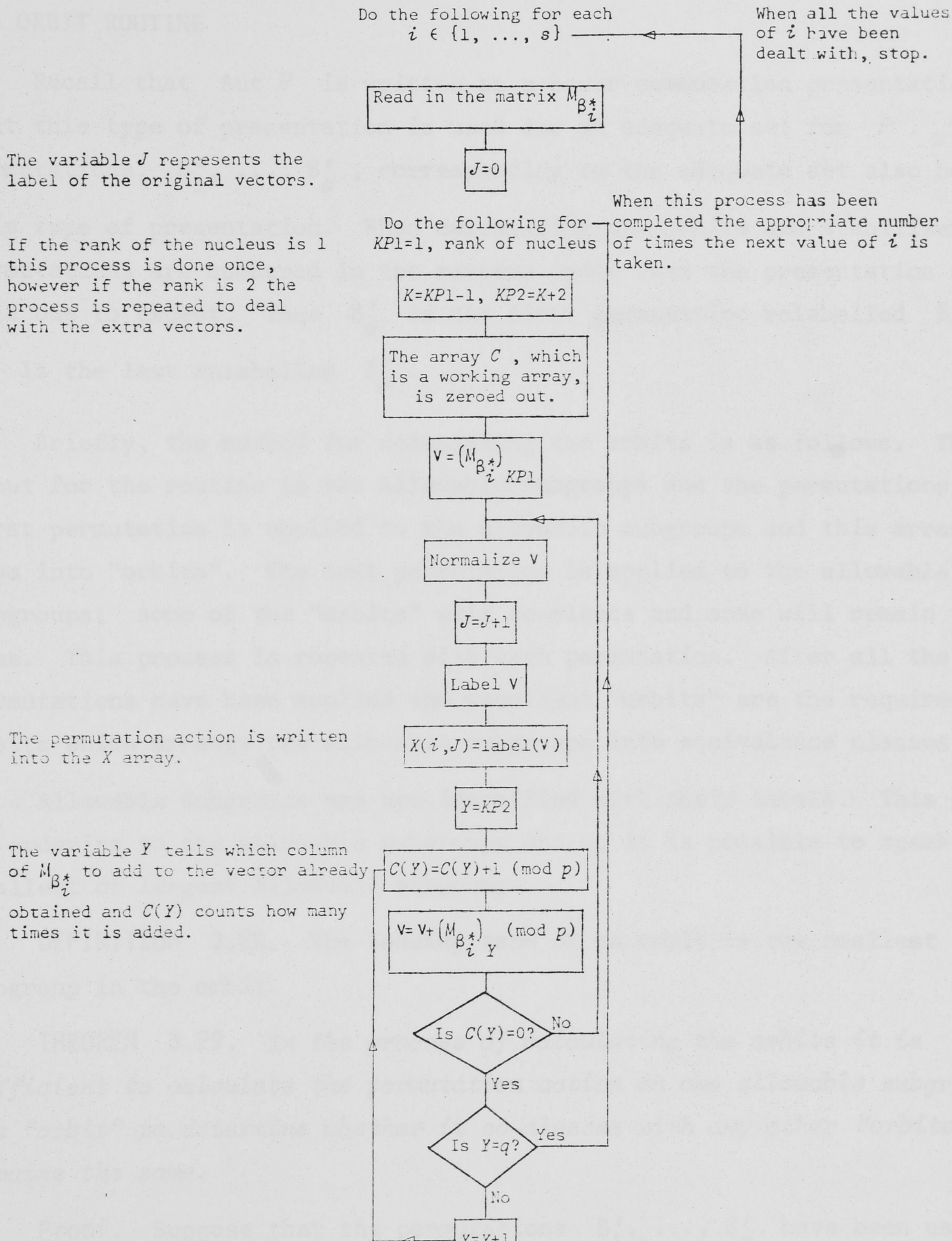


FIGURE 3.27. Flowchart for the MATPER routine

described by an array X where $X(i, j)$ gives the image, as a label, of the allowable subgroup with label j, under the ith permutation.

THE ORBIT ROUTINE

Recall that Aut P is written as a power-commutation presentation and that this type of presentation is used for an adequate set for P. The permutations, $\beta_1', \ldots, \beta_s'$, corresponding to the adequate set also have this type of presentation. When calculating the orbits and stabilizers the permutations are numbered in the reverse order than the presentation would lead one to expect. Thus β_s' is the first permutation relabelled β_1' and β_1' is the last relabelled β_s'.

Briefly, the method for calculating the orbits is as follows. The input for the routine is the allowable subgroups and the permutations. The first permutation is applied to the allowable subgroups and this arranges them into "orbits". The next permutation is applied to the allowable subgroups; some of the "orbits" will co-alesce and some will remain the same. This process is repeated with each permutation. After all the permutations have been applied the resultant "orbits" are the required orbits which arrange the allowable subgroups into equivalence classes.

Allowable subgroups are now identified with their labels. This gives an ordering to the allowable subgroups and so it is possible to speak of the smallest or largest allowable subgroup.

DEFINITION 3.28. The *leading term* of an orbit is the smallest subgroup in the orbit.

THEOREM 3.29. *In the process of calculating the orbits it is sufficient to calculate the permutation action on one allowable subgroup in the "orbit" to determine whether it co-alesces with any other "orbits" or remains the same.*

Proof. Suppose that the permutations $\beta_1', \ldots, \beta_r'$ have been used to calculate orbits. The next permutation is β_{r+1}'. A typical orbit with leading term j is taken and $j\beta_{r+1}'$ is calculated. There are two possibilities. Either $j\beta_{r+1}'$ is in the orbit with leading term j or it is not.

Suppose $j\beta'_{r+1}$ is not in the same orbit as j . Since the set of adequate permutations has a power-commutation presentation there is a prime ρ such that $(\beta'_{r+1})^{\rho} = \prod_{k=1}^{r} (\beta'_{k})^{\alpha(r+1,k)}$ (remembering that the numbering of permutations is reversed and hence $(\beta'_{r+1})^{\rho}$ is in the orbit with leading term j). Now suppose $j(\beta'_{r+1})^{l}$ is in the orbit with leading term j and l is less than ρ . Then

$$j(\beta'_{r+1})^{l} = j \prod_{k=1}^{r} (\beta'_{k})^{\alpha_k}$$

and hence j is fixed by

$$(\beta'_{r+1})^{l} \prod_{k=1}^{r} (\beta'_{r+1-k})^{\zeta(k)} \; . \tag{1}$$

Similarly, it can be shown that j is also fixed by

$$(\beta'_{r+1})^{\rho} \prod_{k=1}^{r} (\beta'_{r+1-k})^{\eta(k)} \; . \tag{2}$$

Since ρ is a prime and l is less than ρ , l and $\overset{\rho}{\cancel{q}}$ are co-prime. Thus by taking the appropriate combination of (1) and (2) it follows that j is fixed by $\beta'_{r+1} \prod_{k=1}^{r} (\beta'_{r+1-k})^{\nu(k)}$. This means that $j\beta'_{r+1}$ is in the same orbit as j and so contradicts the original assumption. Thus ρ is the smallest number for which $j(\beta'_{r+1})^{\rho}$ is in the orbit with leading term j .

Now suppose $j(\beta'_{r+1})^{\lambda}$ is in the same orbit as $j(\beta'_{r+1})^{\mu}$. It can be assumed, without loss of generality, that μ is less than or equal to λ and both are less than ρ . Using a similar argument as before it can be shown that j is fixed by $(\beta'_{r+1})^{\lambda-\mu} \prod_{k=1}^{r} (\beta'_{r+1-k})^{\omega(k)}$. This means that $j(\beta'_{r+1})^{\lambda-\mu}$ is in the same orbit as j and hence λ equals μ . Thus $j, j\beta'_{r+1}, j(\beta'_{r+1})^{2}, \ldots, j(\beta'_{r+1})^{\rho-1}$ all lie in different orbits.

Now suppose that i is a different allowable subgroup from j and that it is in the same orbit as j. Using the same argument as above it can be shown that $i, i\beta'_{r+1}, i(\beta'_{r+1})^2, \ldots, i(\beta'_{r+1})^{\rho-1}$ are all in different orbits. Suppose that $j(\beta'_{r+1})^{\lambda}$ and $i(\beta'_{r+1})^{\lambda}$ are in different orbits. Then

$$j(\beta'_{r+1})^{\lambda}(\beta'_{r+1})^{t}\prod_{k=1}^{r}(\beta'_k)^{\alpha(k)} = i(\beta'_{r+1})^{\lambda}$$

and hence

$$j(\beta'_{r+1})^{t}\prod_{k=1}^{r}(\beta'_k)^{\xi(k)} = i\ .$$

Thus i is in the same orbit as $j(\beta'_{r+1})^{t}$, that is t equals 0 and hence $j(\beta'_{r+1})^{\lambda}$ is in the same orbit as $i(\beta'_{r+1})^{\lambda}$.

Suppose $j\beta'_{r+1}$ is in the same orbit as j. Let i be another allowable subgroup, different from j, and in the same orbit as j. Suppose $i\beta'_{r+1}$ is not in the same orbit as j. Then

$$j\beta'_{r+1}(\beta'_{r+1})^{t}\prod_{k=1}^{r}(\beta'_k)^{\alpha(k)} = i\beta'_{r+1}$$

and hence

$$j(\beta'_{r+1})^{t}\prod_{k=1}^{r}(\beta'_k)^{\xi(k)} = i\ .$$

Thus i is in the same orbit as $j(\beta'_{r+1})^{t}$; that is t equals 0 and hence $i\beta'_{r+1}$ is in the orbit with leading term j. □

The method used to calculate orbits in the machine implementation of the generating algorithm is now described and an example is worked. Three arrays are used in the orbit calculation. They are A, B and C where $A(j)$ is the leading subgroup of the orbit containing j; $B(j)$ is the subgroup following j, if it exists, otherwise $B(j)$ equals 0; $C(j)$ is the number of times j has been the last term in the orbit with leading term $A(j)$. The C array is calculated in the orbit routine but is used in the stabilizer routine.

Initially the arrays are set as follows:

$$A(j) = j \ , \quad B(j) = 0 \ , \quad C(j) = 1 \ .$$

The first step in the orbit routine is to apply the first permutation, β_1', to the allowable subgroups. Suppose j is the lowest available subgroup which has not yet been processed. The entry in the B array $B(j)$ is examined. Since this is the first step $B(j)$ equals 0. The image of j, $j\beta_1'$ is calculated. Again, since this is the first step, $A(j\beta_1')$ equals $j\beta_1'$. If, however, $A(j\beta_1')$ equals j then β_1' fixes j and j remains in an orbit by itself. In the arrays, $B(j)$ is put equal to 0 and $C(j)$ is put equal to 1. If $A(j\beta_1')$ is greater than j then $B(j)$ is put equal to $A(j\beta_1')$, $A(j\beta_1')$ is put equal to j and $C(j\beta_1')$ is put equal to 0. Notice that $A(j\beta_1')$ can not be less than j since the way j is chosen makes this impossible. Let $j\beta_1'$ equal k. Again $B(k)$ equals 0 and so $k\beta_1'$ is calculated. If $A(k\beta_1')$ equals j it means the end of the orbit has been reached and $B(k)$ is put equal to 0, $C(k)$ is put equal to 1. If $A(k\beta_1')$ is greater than j then the orbit is extended and $B(k)$ is put equal to $A(k\beta_1')$, $A(k\beta_1')$ is put equal to j, and $C(k\beta_1')$ is put equal to 0.

This process is repeated in the obvious way and so after applying the first permutation the orbits correspond to the cycles of the permutation.

Recall that in the example given in Section 3 the permutations are

β_1' : (2 20 11)(3 12 21)(5 23 14)(6 15 24)(8 26 17)(9 18 27)(28 34 31)
(29 35 32)(30 36 33) ;

β_3' : (2 5 8)(3 9 6)(11 14 17)(12 18 15)(20 23 26)(21 27 24)(28 29 30)
(31 32 33)(34 35 36) ;

β_4' : (2 3)(4 7)(5 9)(6 8)(11 12)(13 16)(14 18)(15 17)(20 21)(22 25)
(23 27)(24 26)(31 34)(32 35)(33 36) .

(Note that the machine implementation actually uses the inverses of these permutations.)

After β_1' is applied to the allowable subgroups the arrays A, B and C are as follows:

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$A(j)$	1	2	3	4	5	6	7	8	9	10	2	3	13	5	6	16	8	9	19	2	3	22	5	6
$B(j)$	0	20	12	0	23	15	0	26	18	0	0	21	0	0	24	0	0	27	0	11	0	0	14	0
$C(j)$	2	1	1	2	1	1	2	1	1	2	1	0	2	1	0	2	1	0	2	0	1	2	0	1

j	25	26	27	28	29	30	31	32	33	34	35	36
$A(j)$	25	8	9	28	29	30	28	29	30	28	29	30
$B(j)$	0	17	0	34	35	36	0	0	0	31	32	33
$C(j)$	2	0	1	1	1	1	1	1	1	0	0	0

Now suppose $\beta_1', \ldots, \beta_r'$ have been applied to the allowable subgroups and "orbits" calculated. The permutation β_{r+1}' is now to be applied. Suppose j is the lowest available subgroup which has not been processed. The orbit containing j is traced through until a subgroup, say k, is found for which $B(k)$ equals 0. This is the last subgroup in the orbit. (Note that k may actually be j.) Now $k\beta_{r+1}'$ and $A(k\beta_{r+1}')$ are found. If $A(k\beta_{r+1}')$ equals j then β_{r+1}' does not extend the orbit containing j and so $B(k)$ remains 0 and $C(k)$ is increased by one.

If $A(k\beta_{r+1}')$ is greater than j then two orbits will co-alesce. In the arrays $B(k)$ is put equal to $A(k\beta_{r+1}')$ and the orbit with leading term $A(k\beta_{r+1}')$ is traced through. For each subgroup, i, examined in the tracing process $A(i)$ is put equal to j and $C(i)$ is put equal to 0. The tracing procedure ends when a subgroup is found for which $B(i)$ equals 0. Then $i\beta_{r+1}'$ and $A(i\beta_{r+1}')$ are calculated. If $A(i\beta_{r+1}')$ equals j then the orbit will not grow any more and so $C(i)$ is put equal to 1. If $A(i\beta_{r+1}')$ is greater than j then another orbit will co-alesce with the orbit containing j. Then $B(i)$ is put equal to $A(i\beta_{r+1}')$ and the tracing procedure resumes. The process continues in the obvious way. The orbits can be read off from the A array. Theorem 3.29 guarantees that this process does indeed calculate the orbits. After the orbits are calculated the number of orbits and the leading term of each orbit is determined by finding those j for which $A(j) = j$. This is used to calculate the stabilizer.

A flow chart describing the orbit routine is given in Figure 3.30.

In the example β_2' is the identity and so when it is applied only the C array changes. The arrays after β_2' and β_3' have been applied are

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$A(j)$	1	2	3	4	2	3	7	2	3	10	2	3	13	2	3	16	2	3	19	2	3	22	2	3
$B(j)$	0	20	12	0	23	15	0	26	18	0	5	21	0	8	24	0	0	27	0	11	9	0	14	0
$C(j)$	4	1	1	4	0	0	4	0	0	4	2	0	4	0	0	4	1	0	4	0	2	4	0	1

j	25	26	27	28	29	30	31	32	33	34	35	36
$A(j)$	25	2	3	28	28	28	28	28	28	28	28	28
$B(j)$	0	17	6	34	35	36	29	30	0	31	32	33
$C(j)$	4	0	0	1	0	0	2	0	1	0	0	0

.

The arrays after β_4' has been applied are

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$A(j)$	1	2	2	4	2	2	4	2	2	10	2	2	13	2	2	13	2	2	19	2	2	22	2	2
$B(j)$	0	20	12	7	23	15	0	26	18	0	5	21	16	8	24	0	3	27	0	11	9	25	14	0
$C(j)$	5	1	0	4	0	0	1	0	0	5	2	0	4	0	0	1	1	0	5	0	0	4	0	1

j	25	26	27	28	29	30	31	32	33	34	35	36
$A(j)$	22	2	2	28	28	28	28	28	28	28	28	28
$B(j)$	0	17	6	34	35	36	29	30	0	31	32	33
$C(j)$	1	0	0	1	0	0	2	0	2	0	0	0

THE STABILIZER ROUTINE

The next step is to calculate the stabilizing elements generated by $\beta_1', \ldots, \beta_s'$, of the orbit representative of each orbit. The leading subgroup of the orbit is chosen as the orbit representative however there is no reason why a different subgroup could not be chosen.

The STABILIZER routine works on the following principle. Consider the orbit with leading term j. Suppose that for each permutation applied this orbit has increased in length, that is, co-alesced with another orbit. It is clear that no subgroup in this orbit, and in particular the leading

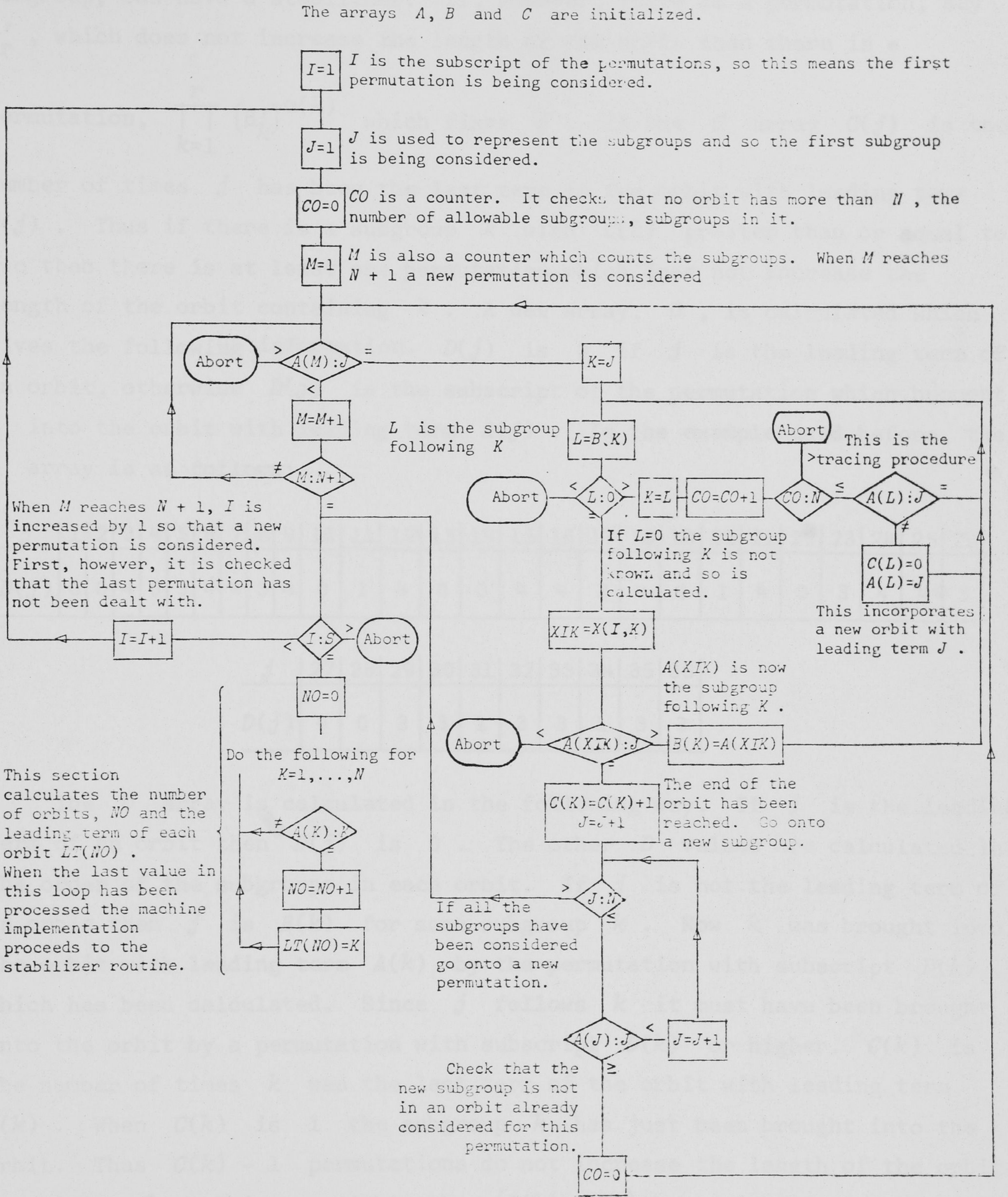


FIGURE 3.30. Flow Chart for the ORBIT Routine

subgroup, can have a stabilizer. If, however, there is a permutation, say β'_r , which does not increase the length of the orbit then there is a permutation, $\prod_{k=1}^{r} (\beta'_k)^{\alpha(k)}$ which fixes j . In the C array $C(j)$ is the number of times j has been the last term in the orbit with leading term $A(j)$. Thus if there is a subgroup k with $C(k)$ greater than or equal to two then there is at least one permutation which does not increase the length of the orbit containing k . A new array, D , is calculated which gives the following information. $D(j)$ is 0 if j is the leading term of an orbit, otherwise $D(j)$ is the subscript of the permutation which brought j into the orbit with leading term $A(j)$. In the example used before, the D array is as follows:

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
$D(j)$	0	0	4	0	3	4	4	3	4	0	1	4	0	3	4	4	3	4	0	1	4	0	3	4	4	3

j	27	28	29	30	31	32	33	34	35	36
$D(j)$	4	0	3	3	1	3	3	1	3	3

.

The D array is calculated in the following way. If j is the leading term of an orbit then $D(j)$ is 0 . The other D values are calculated in the order of the subgroups in each orbit. If j is not the leading term of an orbit then j is $B(k)$ for some subgroup k . Now k was brought into the orbit with leading term $A(k)$ by the permutation with subscript $D(k)$, which has been calculated. Since j follows k it must have been brought into the orbit by a permutation with subscript $D(k)$ or higher. $C(k)$ is the number of times k was the last term of the orbit with leading term $A(k)$. When $C(k)$ is 1 the subgroup k has just been brought into the orbit. Thus $C(k) - 1$ permutations do not increase the length of the orbit containing k . This means that the $\left(D(k)+C(k)\right)$th permutation will bring j into the orbit containing k and hence $D(j)$ is $D(k) + C(k)$.

Once a subgroup k is found for which $C(k)$ is greater than or equal to 2 it follows that the $\left(D(k)+1\right)$st permutation up to the $\left(D(k)+C(k)-1\right)$th permutation do not increase the length of the orbit containing k . These permutations are used to calculate the stabilizing elements of $A(k)$. The first permutation, that is the $\left(D(k)+1\right)$st is applied to $A(k)$. If this fixes $A(k)$ the first stabilizing element is

$\beta'_{D(k)+1}$

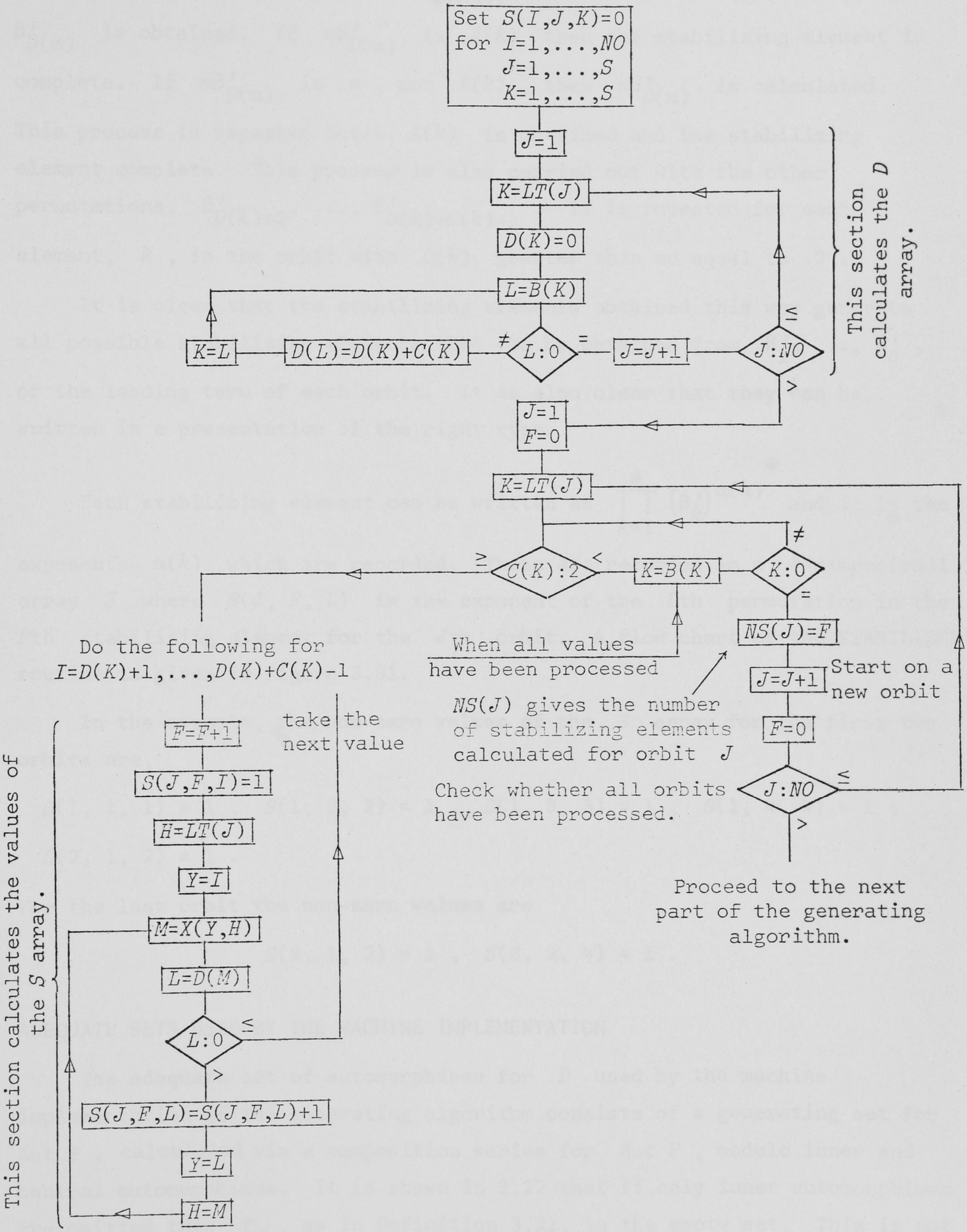


FIGURE 3.31. Flow Chart for the STABILIZER Routine

If $A(k)$ is not fixed, say $A(k)\beta'_{D(k)+1}$ is m then the image of m under $\beta'_{D(m)}$ is obtained. If $m\beta'_{D(m)}$ is $A(k)$ then the stabilizing element is complete. If $m\beta'_{D(m)}$ is n, not $A(k)$, then $n\beta'_{D(n)}$ is calculated. This process is repeated until $A(k)$ is obtained and the stabilizing element complete. This process is also carried out with the other permutations, $\beta'_{D(k)+2}, \ldots, \beta'_{D(k)+C(k)-1}$. It is repeated for each element, k, in the orbit with $C(k)$ greater than or equal to 2.

It is clear that the stabilizing elements obtained this way generate all possible stabilizing elements that can be obtained from $\beta'_1, \ldots, \beta'_s$, of the leading term of each orbit. It is also clear that they can be written in a presentation of the right type.

Each stabilizing element can be written as $\prod_{k=1}^{s} (\beta'_k)^{\alpha(k)}$ and it is the exponents $\alpha(k)$ which are recorded. These are recorded in a 3-dimensional array S where $S(J, F, L)$ is the exponent of the Lth permutation in the Fth stabilizing element for the Jth orbit. A flow chart of the STABILIZER routine is given in Figure 3.31.

In the example, the non-zero values of the S array for the first two orbits are,

$S(1, 1, 1) = 1$, $S(1, 2, 2) = 1$, $S(1, 3, 3) = 1$, $S(1, 4, 4) = 1$;

$S(2, 1, 2) = 1$.

For the last orbit the non-zero values are

$$S(8, 1, 2) = 1 , \quad S(8, 2, 4) = 1 .$$

ADEQUATE SETS USED BY THE MACHINE IMPLEMENTATION

The adequate set of automorphisms for P used by the machine implementation of the generating algorithm consists of a generating set for Aut P, calculated via a composition series for Aut P, modulo inner and central automorphisms. It is shown in 3.22 that if only inner automorphisms are omitted then T_P, as in Definition 3.21, is the empty set. This is not so if central automorphisms are also omitted. The following argument, showing how to calculate T_P, is only true for n greater than or equal to

8 . Prior to that the adequate sets of automorphisms are adjusted by hand.

In P there are three possibilities for the commutators $[a_{n-1}, a_1]$ and $[a_{n-1}, a_2]$, either

$$\begin{matrix} [a_{n-1}, a_1] = a_n & & [a_{n-1}, a_1] = a_n & & [a_{n-1}, a_1] = e \,, \\ & \text{or} & & \text{or} & \\ [a_{n-1}, a_2] = e & & [a_{n-1}, a_2] = a_n & & [a_{n-1}, a_2] = a_n \,. \end{matrix}$$

In the first two cases

$$T_P = \left\{ \begin{matrix} a_1 \mapsto a_1 \\ \\ a_2 \mapsto a_2 a_n \end{matrix} \right\}$$

and in the third case

$$T_P = \left\{ \begin{matrix} a_1 \mapsto a_1 a_n \\ \\ a_2 \mapsto a_2 \end{matrix} \right\} .$$

For immediate descendants of P the automorphisms in K are all either inner or central and so not included in the adequate set for P . The automorphisms in T_P correspond to the automorphisms of K from the previous step. The other automorphisms of K are not included as they are either inner automorphisms or can be obtained as the product of an inner automorphism and the automorphism in T_P .

The possibility of pruning the adequate set of automorphisms for P still further has not been investigated. However in some cases (see Chapter 6) it appears that the adequate set could be reduced considerably.

CHAPTER 4

TREES AND TREE DIAGRAMS

This chapter is split into two sections. The first section describes a tree of groups associated with a p-group P, which represents the descendants of P. Known results about 2-groups and 3-groups of maximal class and 2-groups and some 3-groups of second maximal class are presented in terms of these trees. This is to fill in background information and hopefully to convince the reader of the merit of this approach.

The second section details the computer results ~~of~~ for 3-groups of second maximal nilpotency class calculated by the generating algorithm. Again the results are presented in the form of trees.

4.1. Tree Diagrams calculated from previously known results

A *tree* is defined in the usual way to be a set of *nodes* and *links* in which there are no cycles. A *directed tree* is a tree in which the links have a direction.

Each p-group P is thought of as having an associated directed tree. This tree represents the descendants of P. In this tree the nodes represent groups with P being represented by the *root node*. A directed link from a node Q to another node R indicates that R is an immediate descendant of Q. The groups represented by the nodes are those calculated by the generating algorithm. The consistent power-commutator presentations of such groups depend on orbit representatives however the tree of groups is independent of these.

All the trees described in this chapter are associated with 2-generator groups. These trees are all subtrees of the tree associated with $C_p \times C_p$. The following *tree diagram* in Figure 4.1 shows a very small part of the tree associated with $C_p \times C_p$.

The conventions used in this and all other tree diagrams are as follows:

(1) The root node (here $C_p \times C_p$) appears at the top of the diagram.

(2) Groups of the same order, and only that order, appear on the same

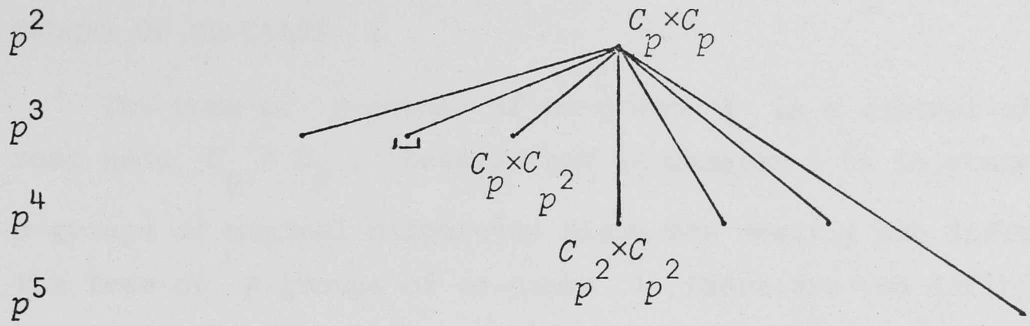


FIGURE 4.1. This represents all the immediate descendants (up to isomorphism) of $C_p \times C_p$.

level. The order of the groups at each level is shown on the left hand side of the diagram.

(3) A node marked ⊔ indicates that the group represented by that node is terminal.

(4) The arrangement of groups in each set of immediate descendants is unimportant. For convenience and to facilitate checking it is usual that it is the same as the computer output - where this is available.

(5) Easily identifiable groups (in this case the abelian ones) are indicated in the diagram.

Further conventions will be introduced when necessary.

It can be shown that there exist groups with an infinite number of capable descendants. The trees associated with these groups are therefore infinite. The group $C_p \times C_p$ is such a group since the groups $C_{p^r} \times C_{p^r}$, $r \in \{2, 3, \ldots\}$ are all descendants of $C_p \times C_p$ and are all capable. This chain of groups is called an *infinite branch* of the tree with root node $C_p \times C_p$. Also, each group $C_{p^r} \times C_{p^r}$ has an infinite number of capable descendants, namely $C_{p^r} \times C_{p^s}$, $s \in \{r+1, r+2, \ldots\}$. Each of these chains is also called an infinite branch and thus the tree with root node $C_p \times C_p$ has an infinite number of infinite branches.

Restricting the nodes of the tree with root node $C_p \times C_p$, to those which represent groups of co-class 1 or 2 considerably reduces the number of infinite branches.

GROUPS OF CO-CLASS 1

The tree of p-groups of co-class 1 is a subtree of the tree with root node $C_p \times C_p$. Recall that in Chapter 1 it is stated that the tree of p-groups of maximal nilpotency class has exactly one infinite branch. In the tree of p-groups of co-class 1 there are two infinite branches. One branch is the branch for groups of maximal nilpotency class and the other contains abelian groups. The trees for p equal to 2 and 3 are discussed here. When p is larger the tree diagrams become much more complicated.

The beginning of the tree diagram for groups of co-class 1 and p equal to 2 is shown in Figure 4.2.

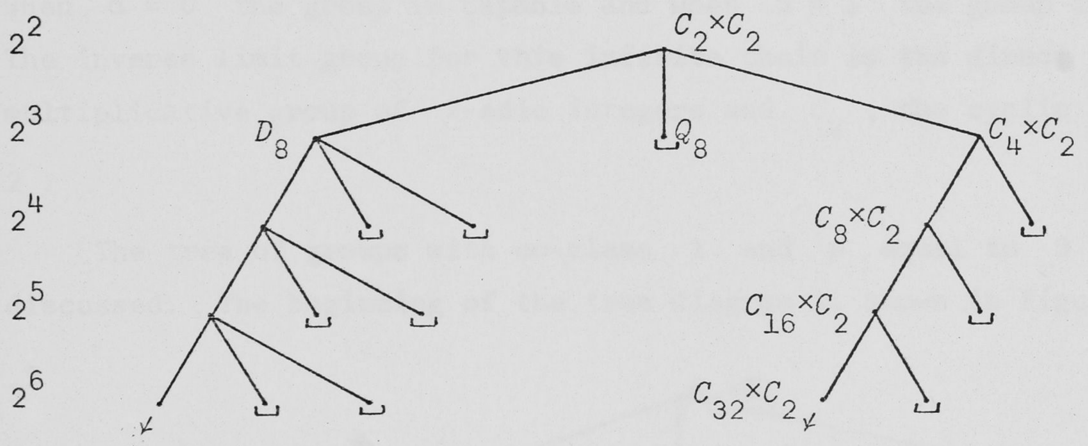


FIGURE 4.2 showing 2-groups of co-class 1 .

The labels D_8 and Q_8 denote the dihedral group of order 8 and the quaternion group of order 8 respectively. The infinite branch (indicated by an arrow) beginning with D_8 contains groups of maximal nilpotency class. At each level there is one capable group and two terminal groups. The presentations for these groups at the 2^n level are as follows:

$$\left\langle a_1, \ldots, a_n;\ a_1^2 = a_n^\alpha,\ a_2^2 = a_n^\beta,\ a_3^2 = a_4a_5,\ a_4^2 = a_5a_6,\ \ldots,\ a_{n-2}^2 = a_{n-1}a_n, \right.$$
$$a_{n-1}^2 = a_n,\ a_n^2 = e,\ [a_2, a_1] = a_3,\ [a_3, a_1] = [a_3, a_2] = a_4,$$
$$\left. [a_4, a_1] = [a_4, a_2] = a_5,\ \ldots,\ [a_{n-1}, a_1] = [a_{n-1}, a_2] = a_n \right\rangle .$$

The parameters α and β have the obvious range, that is either 0 or 1 . When $\alpha = \beta = 0$ the presentation gives the capable group. The presentations with $\alpha = 1$, $\beta = 0$, and $\alpha = \beta = 1$ give the two terminal groups. When $\alpha = 0$, $\beta = 1$, the group is isomorphic to that with $\alpha = 1$, $\beta = 0$.

The infinite branch beginning with $C_4 \times C_2$ contains groups of co-class 1 but not maximal nilpotency class. At the 2^n level there are two groups - a capable group, $C_{2^{n-1}} \times C_2$, and a terminal group. The presentations are as follows:

$$\left\langle a_1, \ldots, a_n;\ a_1^2 = e,\ a_2^2 = a_3,\ a_3^2 = a_4,\ a_4^2 = a_5,\ \ldots,\ a_{n-1}^2 = a_n,\ a_n^2 = e,\ [a_2, a_1] = a_n^{\alpha} \right\rangle .$$

When $\alpha = 0$ the group is capable and when $\alpha = 1$ the group is terminal. The inverse limit group for this infinite chain is the direct product of the multiplicative group of 2-adic integers and C_2 , the cyclic group of order 2 .

The tree of groups with co-class 1 and p equal to 3 is now discussed. The beginning of the tree diagram is shown in Figure 4.3.

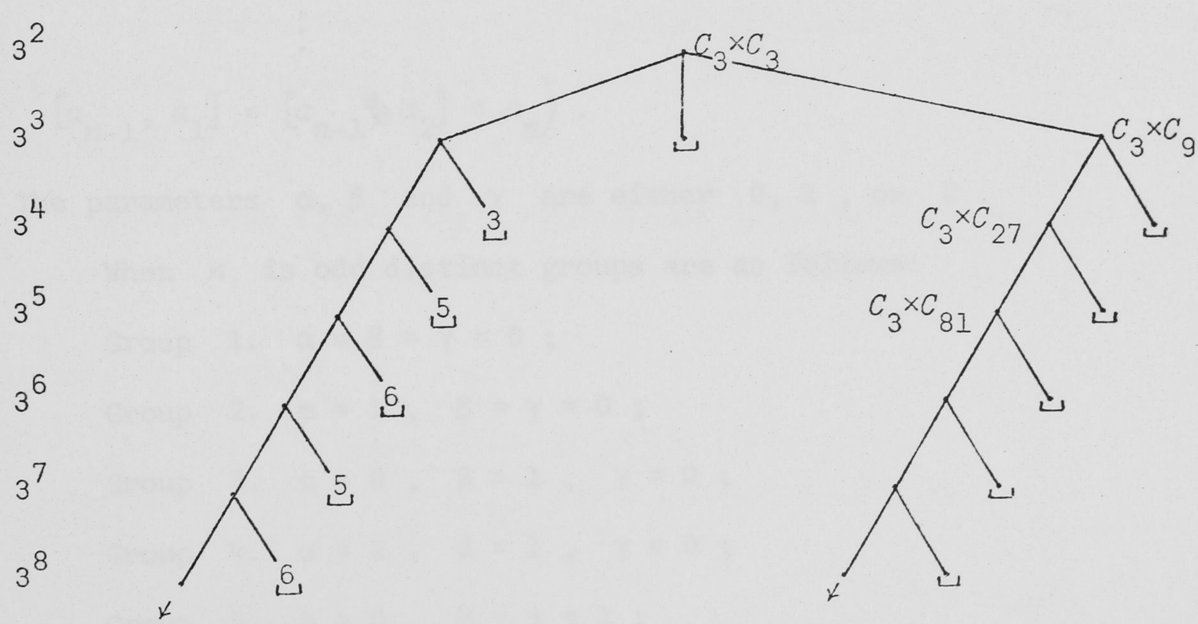


FIGURE 4.3 showing 3-groups of co-class 1 .

The symbol 5 indicates that there are five terminal groups - the nodes representing these groups have been suppressed.

The 3-groups of maximal nilpotency class were completely determined by Blackburn [1958]. In his paper he has the following two theorems (4.2 and 4.3): "For $n \geq 5$ and $p > 2$ the number of types of metabelian groups of order p^n and (nilpotency) class $n - 1$ in which $[\gamma_1(G), \gamma_2(G)] = \gamma_{n-1}(G)$ is 3 for $p = 3$ and $1 + (2n-6, p-1) + (n-2, p-1)$ for $p > 3$."

"For $n \geq 4$ the number of types of p-groups of maximal (nilpotency) class of order p^n which possess an abelian maximal subgroup is $2 + (n-2, p-1)$." This second theorem had previously been proved by Wiman [1946].

Thus, for $p = 3$ there are six groups when n is greater than or equal to 5 and odd and seven groups when n is greater than or equal to 6 and even. Presentations for these groups were calculated by the generating algorithm and are as follows:

$$\Big\langle a_1, \ldots, a_n;\ a_1^3 = a_n^\beta,\ a_2^3 = a_n^\gamma,\ a_3^3 = a_5^2a_6,\ a_4^3 = a_6^2a_7,\ \ldots,\ a_{n-3}^3 = a_{n-1}^2a_n,$$

$$a_{n-2}^3 = a_n^2,\ a_{n-1}^3 = a_n^3 = e,\ [a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_4a_n^\alpha,$$

$$[a_4, a_1] = [a_4, a_2] = a_5,$$

$$[a_5, a_1] = [a_5, a_2] = a_6,$$

$$\vdots$$

$$[a_{n-1}, a_1] = [a_{n-1}, a_2] = a_n\Big\rangle .$$

The parameters α, β and γ are either 0, 1 , or 2 .

When n is odd distinct groups are as follows:

Group 1. $\alpha = \beta = \gamma = 0$;

Group 2. $\alpha = 1$, $\beta = \gamma = 0$;

Group 3. $\alpha = 0$, $\beta = 1$, $\gamma = 0$;

Group 4. $\alpha = 2$, $\beta = 1$, $\gamma = 0$;

Group 5. $\alpha = 0$, $\beta = \gamma = 1$;

Group 6. $\alpha = \beta = \gamma = 1$.

Group 1 is capable and those groups with $\alpha = 0$ have an abelian maximal subgroup, namely that generated by $\left\{a_1^2a_2, a_3, a_4, \ldots, a_n\right\}$.

When n is even distinct groups are as follows:

Group 1. $\alpha = \beta = \gamma = 0$;

Group 2. $\alpha = 1$, $\beta = \gamma = 0$;

Group 3. $\alpha = 0$, $\beta = 1$, $\gamma = 0$;

Group 4. $\alpha = 2$, $\beta = 1$, $\gamma = 0$;

Group 5. $\alpha = \beta = 0$, $\gamma = 1$;

Group 6. $\alpha = 0$, $\beta = \gamma = 1$;

Group 7. $\alpha = \beta = \gamma = 1$.

Again Group 1 is capable and those with $\alpha = 0$ have an abelian maximal subgroup.

The other infinite branch in Figure 4.3 starts with $C_3 \times C_9$. There are two groups at each level - one capable and one terminal. At the 3^n level the presentations are as follows:

$$\left\langle a_1, \ldots, a_n ; a_1^3 = e, a_2^3 = a_3, a_3^3 = a_4, a_4^3 = a_5, \ldots, a_n^3 = e, [a_2, a_1] = a_n^{\alpha} \right\rangle .$$

When $\alpha = 0$ the group is capable and when $\alpha = 1$ or 2 the groups are isomorphic and terminal.

The inverse limit group of this infinite branch is the direct product of the multiplicative group of 3-adic integers and C_3 , the cyclic group of order 3 .

Notice the similarity in both these infinite branches with the infinite branches in the 2-group case.

GROUPS OF CO-CLASS 2

The tree of 2-generator p-groups of co-class 2 is a subtree of the tree with root node $C_p \times C_p$. This subtree is described for p equal to 2 .

Figure 4.4 shows the beginning of the tree diagram. The branches are labelled for easy identification. There are four infinite branches containing groups with second maximal nilpotency class. These are labelled $J1$, $J2$, $J3$, $J4$. There are three infinite branches containing groups with co-class 2 but not having second maximal nilpotency class. These are labelled A, B and C . These three branches are their twigs are now described.

The first infinite branch labelled A begins with $C_4 \times C_4$. There are

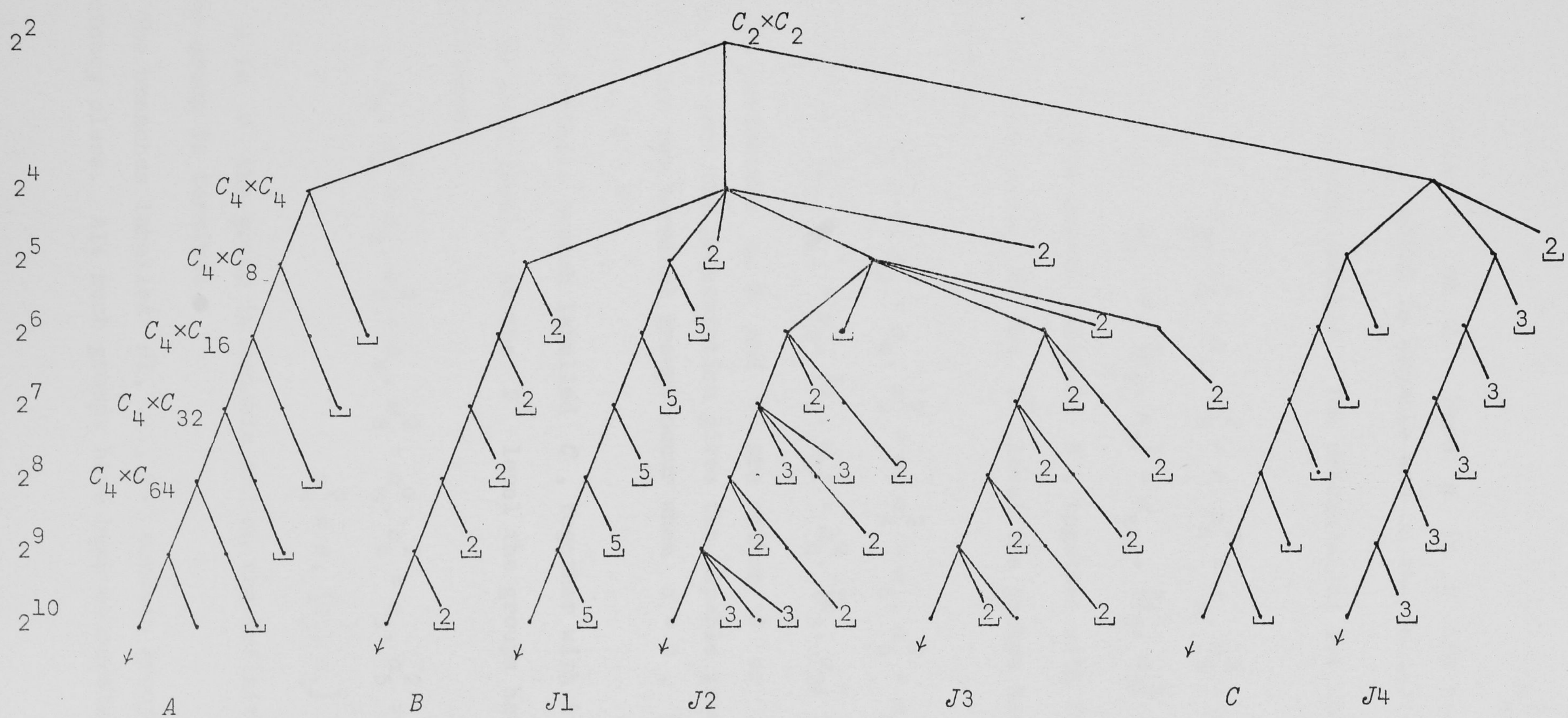


FIGURE 4.4 showing 2-groups of co-class 2 .

three groups at each level for n greater than or equal to 6 . At the 2^n level the groups have the following presentations:

$$\left\langle a_1, \ldots, a_n;\ a_1^2 = a_3,\ a_2^2 = a_4,\ a_3^2 = e,\ a_4^2 = a_5,\ a_5^2 = a_6,\ \ldots,\ a_{n-1}^2 = a_n,\ a_n^2 = e,\ [a_2, a_1] = a_n^{\alpha} \right\rangle .$$

Here α is either 0 or 1 . When α is 0 the presentation gives the group $C_4 \times C_{2^{n-2}}$ which is capable and on the infinite branch. When α is 1 the group is also capable. The presentation for the terminal group is as follows:

$$\left\langle a_1, \ldots, a_n;\ a_1^2 = a_3,\ a_2^2 = a_4,\ a_3^2 = e,\ a_4^2 = a_5,\ a_5^2 = a_6,\ \ldots,\ a_{n-1}^2 = a_n,\ a_n^2 = e,\ [a_2, a_1] = a_{n-1},\ [a_3, a_2] = a_n,\ [a_4, a_1] = a_n \right\rangle .$$

The infinite branch labelled B , together with its twigs has three groups at each level. At the 2^n level the groups have the following presentations:

$$\left\langle a_1, \ldots, a_n;\ a_1^2 = e,\ a_2^2 = a_4,\ a_3^2 = e,\ a_4^2 = a_5,\ a_5^2 = a_6,\ \ldots,\ a_{n-1}^2 = a_n,\ a_n^2 = e,\ [a_2, a_1] = a_3,\ [a_3, a_1] = a_n^{\alpha},\ [a_3, a_2] = a_n^{\beta},\ [a_4, a_1] = a_n^{\gamma} \right\rangle .$$

The parameters α, β and γ are either 0 or 1 . When α, β and γ are all zero the presentation gives the capable group on the infinite branch. The two terminal groups occur when $\alpha = 0$, $\beta = \gamma = 1$ and $\alpha = \beta = 1$, $\gamma = 0$.

The infinite branch labelled C , together with its twigs has two groups at each level. At the 2^n level the groups have the following presentations:

$$\left\langle a_1, \ldots, a_n;\ a_1^2 = a_3,\ a_2^2 = a_4,\ a_3^2 = a_n^{\alpha},\ a_4^2 = a_5,\ a_5^2 = a_6,\ \ldots,\ a_{n-1}^2 = a_n,\ a_n^2 = e,\ [a_2, a_1] = a_3,\ [a_3, a_2] = a_n^{\alpha} \right\rangle .$$

When α is 0 the group is capable and on the infinite branch. When α is 1 the group is terminal.

The branches labelled $J1, \ldots, J4$ contain groups of second maximal nilpotency class. All such groups have been enumerated by James [1975] and on the basis of his calculations these groups have been arranged into a tree. This is shown in the tree diagram in Figure 4.4. The infinite branch labelled $J1$ together with its twigs has six groups at the 2^n level, for n greater than or equal to 6 and three groups at the 2^5 level. The presentations for these groups are as follows and use the notation in James' paper:

$$\Big\langle s, s_1, s_2, \ldots, s_{n-2};\ s^2 = s_{n-2}^{\alpha},\ s_1^4 = s_3 s_{n-2}^{\beta},\ s_2^2 = s_3^{-1},\ s_3^2 = s_4^{-1},\ \ldots,$$
$$s_{n-3}^2 = s_{n-2}^{-1},\ s_{n-2}^2 = e,\ [s_1, s] = s_2,\ [s_1, s_2] = s_{n-2}^{\gamma},$$
$$[s_1, s_i] = e \text{ for } i > 2,\ [s_i, s] = s_{i+1} \text{ for } i \in \{2, \ldots, n-2\}\Big\rangle .$$

For $n \geq 6$ the distinct groups are given by:

(1) $\alpha = \beta = \gamma = 0$;

(2) $\alpha = 1$, $\beta = \gamma = 0$;

(3) $\alpha = 0$, $\beta = 1$, $\gamma = 0$;

(4) $\alpha = \beta = 0$, $\gamma = 1$;

(5) $\alpha = 0$, $\beta = \gamma = 1$;

(6) $\alpha = \beta = \gamma = 1$.

The first presentation with $\alpha = \beta = \gamma = 0$ gives the capable group. The remaining groups are terminal.

When n is 5 , γ may be taken as 0 and so the three groups are given by the first three presentations.

The infinite branch labelled $J2$ together with its twigs has two groups when n is 6 , four groups when n is 7 , ten groups when n is greater than 7 and even and six groups when n is greater than 7 and odd. The presentations are as follows:

$$\Big\langle s,\ s_1,\ s_2,\ s_3,\ \ldots,\ s_{n-2};\ s^4 = s_{n-2}^{\alpha},\ s_1^2 = s_{n-2}^{\delta}$$

(where $\delta = 0$ if n is odd),

$$s_2^2 = s_{n-3}^{\beta} s_{n-2}^{\gamma},\ s_3^2 = s_5 s_6,\ s_i^2 = s_{i+2} s_{i+3},\ i \geq 4,\ [s_1, s] = s_2,\ [s_2, s] = s_3,$$

$$[s_3, s] = s_4,\ \ldots,\ [s_{n-3}, s] = s_{n-2},\ [s_2, s_1] = s_{n-3}^{\beta} s_{n-2}^{\gamma},$$

$$[s_{4j-1}, s_1] = s_{4j+1}^{-1} s_{4j+2}^{-1},\ [s_{4j}, s_1] = s_{4j+1} s_{4j+2},\ [s_{4j+1}, s_1] = e,$$

$$[s_{4j+2}, s_1] = s_{4j+3} \text{ for } j \geq 1,$$

$$[s_2, s_3] = s_5 s_6 s_{n-2}^{\beta},\ [s_2, s_i] = s_i^2 \text{ for } i \geq 4,\ \gamma_3 \text{ is abelian}\Big\rangle .$$

When n is 6, all elements with subscripts greater than 4 are trivial. In this case $\alpha = \beta = \gamma = 0$ and there are two groups depending on whether δ is 0 or 1. The capable group has $\delta = 0$ while the terminal group has $\delta = 1$. When n is 7, $\beta = \delta = 0$, α may be 0 or 1 and γ may be 0 or 1 thus giving four groups. The group with $\alpha = \beta = \gamma = \delta = 0$ is capable and has eight immediate descendants. The group with $\alpha = \beta = \delta = 0$ and $\gamma = 1$ is capable and two immediate descendants. The remaining groups are terminal.

When n is greater than 7 and even there are ten groups. Distinct groups are given as follows:

(1) $\alpha = 0,\ \beta = 1,\ \gamma = 0,\ \delta = 0$, (6) $\alpha = 1,\ \beta = 1,\ \gamma = 0,\ \delta = 0$,
(2) $\alpha = 0,\ \beta = 0,\ \gamma = 1,\ \delta = 1$, (7) $\alpha = 1,\ \beta = 0,\ \gamma = 1,\ \delta = 1$,
(3) $\alpha = 0,\ \beta = 0,\ \gamma = 1,\ \delta = 0$, (8) $\alpha = 1,\ \beta = 0,\ \gamma = 1,\ \delta = 0$,
(4) $\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \delta = 1$, (9) $\alpha = 1,\ \beta = 0,\ \gamma = 0,\ \delta = 1$,
(5) $\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \delta = 0$, (10) $\alpha = 1,\ \beta = 0,\ \gamma = 0,\ \delta = 0$.

Group 5 is capable and has four immediate descendants. Group 3 is capable and has two immediate descendatns. The remaining groups are terminal.

When n is greater than 7 and odd there are six groups of order 2^n. Distinct groups are given as follows:

(1) $\alpha = 0,\ \beta = 0,\ \gamma = 1,\ \delta = 0$, (4) $\alpha = 1,\ \beta = 0,\ \gamma = 0,\ \delta = 0$,
(2) $\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \delta = 0$, (5) $\alpha = 0,\ \beta = 1,\ \gamma = 0,\ \delta = 0$,
(3) $\alpha = 1,\ \beta = 0,\ \gamma = 1,\ \delta = 0$, (6) $\alpha = 1,\ \beta = 1,\ \gamma = 0,\ \delta = 0$.

Group 2 is capable and has eight immediate descendants. Group 1 is capable and has two immediate descendants. The remaining groups are terminal.

The infinite branch labelled $J3$ together with its twigs has three groups when n is 5 , four groups when n is 6 and six groups for n greater than or equal to 7 . The presentations are as follows:

$$\Big\langle s, s_1, s_2, \ldots, s_{n-2};\ s^4 = s_{n-2}^{\alpha},\ s_1^2 = s_{n-3}^{\beta} s_{n-2}^{\gamma},\ s_2^2 = s_4 s_5 s_{n-2}^{\delta},$$

$$s_i^2 = s_{i+2} s_{i+3},\ i \in \{3, \ldots, n-4\},$$

$$s_{n-3}^2 = e,\ s_{n-2}^2 = e,\ [s_1, s] = s_2,\ [s_2, s] = s_3,\ \ldots,\ [s_{n-3}, s] = s_{n-2},$$

$$[s_1, s_2] = s_4 s_5 s_{n-2}^{\beta+\delta},\ [s_1, s_i] = s_{i+2} s_{i+3},\ i \in \{3, \ldots, n-4\}\Big\rangle .$$

When n is 5 the distinct groups are given as follows:

(1) $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$,

(2) $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 0$,

(3) $\alpha = 1$, $\beta = 0$, $\gamma = 1$, $\delta = 0$.

Group 1 is capable and has four immediate descendants. The other groups are terminal.

When n is 6 the distinct groups are given as follows:

(1) $\alpha = 0, \beta = 0, \gamma = 0, \delta = 0$, (3) $\alpha = 1, \beta = 0, \gamma = 0, \delta = 0$,

(2) $\alpha = 0, \beta = 0, \gamma = 1, \delta = 0$, (4) $\alpha = 1, \beta = 0, \gamma = 1, \delta = 0$.

Group 1 is capable having four immediate descendants and group 2 is capable having two immediate descendants. The other groups are terminal.

When n is greater than or equal to 7 the distinct groups are given as follows:

(1) $\alpha = 0, \beta = 0, \gamma = 0, \delta = 0$, (4) $\alpha = 1, \beta = 0, \gamma = 1, \delta = 0$,

(2) $\alpha = 0, \beta = 0, \gamma = 1, \delta = 0$, (5) $\alpha = 0, \beta = 1, \gamma = 0, \delta = 0$,

(3) $\alpha = 1, \beta = 0, \gamma = 0, \delta = 0$, (6) $\alpha = 1, \beta = 1, \gamma = 0, \delta = 0$.

Group 1 is capable having four immediate descendants; group 2 is capable having two immediate descendants. The other groups are terminal.

The infinite branch labelled $J4$ together with its twigs has three groups when n is 5 and four groups when n is greater than or equal to 6 . The presentations are as follows:

$$\left\langle s,\ s_1,\ \ldots,\ s_{n-2};\ s^4 = s_{n-2}^{\alpha},\ s_1^2 = s_2^{-1}s_{n-3}^{\beta}s_{n-2}^{\gamma},\ s_2^2 = s_3^{-1}s_{n-2}^{\delta},\ s_3^2 = s_4^{-1},\right.$$
$$s_4^2 = s_5^{-1},\ \ldots,\ s_{n-3}^2 = s_{n-2}^{-1},\ s_{n-2}^2 = e,$$
$$\left.[s_1,\ s] = s_2,\ [s_2,\ s] = s_3,\ \ldots,\ [s_{n-3},\ s] = s_{n-2},\ \left[s_1,\ s^2\right] = s_{n-2}^{\delta}\right\rangle .$$

The distinct groups are

(1) $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$,

(2) $\alpha = 0$, $\beta = 0$, $\gamma = 1$, $\delta = 0$,

(3) $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 1$,

(4) $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 0$.

Group 1 is capable and the others are terminal. When n is 5 , Group 4 is isomorphic to Group 2.

There are three infinite branches in the tree of 2-generator 3-groups of co-class 2 but not second maximal nilpotency class. These are discussed here but 3-groups of second maximal nilpotency class are discussed in the next section.

Figure 4.5 shows a tree diagram of part of the tree of 2-generator 3-groups of co-class 2 but not second maximal nilpotency class. This tree is a subtree of the tree of 3-groups of co-class 2 .

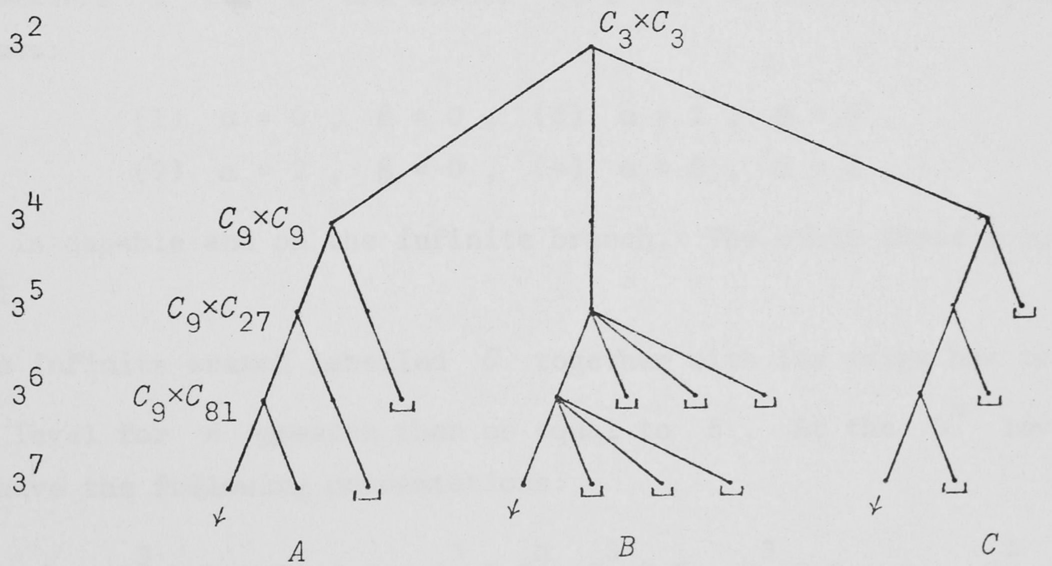


FIGURE 4.5 showing 3-groups of co-class 2 but not second maximal nilpotency class.

The infinite branches are labelled A, B and C and correspond to the branches so labelled in the 2-group case.

The infinite branch labelled A together with its twigs has three groups at each level for n greater than or equal to 6 . At the 3^n level the groups have the following presentations:

$$\langle a_1, \ldots, a_n;\ a_1^3 = a_3,\ a_2^3 = a_4,\ a_3^3 = e,\ a_4^3 = a_5,\ a_5^3 = a_6,\ \ldots,\ a_{n-1}^3 = a_n,\ a_n^3 = e,\ [a_2, a_1] = a_n^{\alpha} \rangle .$$

When α is 0 the group is capable and on the infinite branch. When α is either 1 or 2 the groups are isomorphic and capable. The terminal group has the following presentation:

$$\langle a_1, \ldots, a_n;\ a_1^3 = a_3,\ a_2^3 = a_4,\ a_3^3 = e,\ a_4^3 = a_5,\ a_5^3 = a_6,\ \ldots,\ a_{n-1}^3 = a_n,\ a_n^3 = e,\ [a_2, a_1] = a_{n-1},\ [a_3, a_2] = a_n^2,\ [a_4, a_1] = a_n \rangle .$$

The infinite branch labelled B together with its twigs has four groups at each level for n greater than or equal to 6 . At the 3^n level the groups have the following presentations:

$$\langle a_1, \ldots, a_n;\ a_1^3 = e,\ a_2^3 = a_4,\ a_3^3 = e,\ a_4^3 = a_5,\ a_5^3 = a_6,\ \ldots,\ a_{n-1}^3 = a_n,\ a_n^3 = e,\ [a_2, a_1] = a_3,\ [a_3, a_1] = a_n^{\alpha},\ [a_3, a_2] = a_n^{\beta} \rangle .$$

The parameters α and β are either 0, 1 or 2 and distinct groups are as follows:

(1) $\alpha = 0$, $\beta = 0$, (3) $\alpha = 1$, $\beta = 0$,

(2) $\alpha = 2$, $\beta = 0$, (4) $\alpha = 0$, $\beta = 1$.

Group 1 is capable and on the infinite branch. The other three groups are terminal.

The infinite branch labelled C together with its twigs has two groups at each level for n greater than or equal to 5 . At the 3^n level the groups have the following presentations:

$$\langle a_1, \ldots, a_n;\ a_1^3 = a_3,\ a_2^3 = a_4,\ a_3^3 = a_n^{\alpha},\ a_4^3 = a_5,\ a_5^3 = a_6,\ \ldots,\ a_{n-1}^3 = a_n,\ a_n^3 = e,\ [a_2, a_1] = a_3,\ [a_3, a_2] = a_n^{\beta},\ [a_4, a_1] = a_n^{\alpha} \rangle .$$

When α and β are 0 the group is capable and on the infinite branch. The terminal group is given when α is 1 and β is 2 .

4.2. Tree Diagrams Calculated Using the Generating Algorithm

Infinite branches in the tree of 2-generator 3-groups of second maximal nilpotency class are the subject of Chapters 5 and 6. Here, the extent to which the tree diagram has been calculated, using the generating algorithm, is illustrated. Figure 4.6 shows the tree diagram of all 2-generator 3-groups of either co-class 1 or 2 up to order 3^5 and some groups of order 3^6. Some groups in the diagram have immediate descendants of co-class 3. These are not shown. The five infinite branches in the diagram have already been discussed. Now 3-groups of second maximal nilpotency class are considered. Recall that in Chapter 2 it is shown that all 2-generator 3-groups, P, of second maximal nilpotency class, c, have either

$$P_0(P)/P_1(P) \cong P_1(P)/P_2(P) \cong C_3 \times C_3$$

and

$$P_2(P)/P_3(P) \cong \ldots \cong P_{c-1}(P)/P_c(P) \cong C_3$$

or

$$P_0(P)/P_1(P) \cong P_2(P)/P_3(P) \cong C_3 \times C_3$$

and

$$P_1(P)/P_2(P) \cong P_3(P)/P_4(P) \cong \ldots \cong P_{c-1}(P)/P_c(P) \cong C_3 .$$

Groups of the first type are called *CF-groups* and groups of the second type are called *non CF-groups*. This term was introduced by Blackburn. Figure 4.6 shows the classification of 3-groups of second maximal nilpotency class into *CF*-groups and non *CF*-groups. Non *CF*-groups of order 3^6 and *CF*-groups of order 3^5 are labelled by letters of the alphabet. This is shown in Figure 4.6. Even though some letters are used for both types of groups no confusion should arise as a different style of type is employed. Some of the groups labelled by letters are terminal. Of those that are capable many do not have descendants of order 3^9. These groups and their descendants are shown in Figure 4.7.

For the non *CF*-groups this leaves B, H, I, Q, U having descendants of order 3^{10}. In fact all these groups give rise to infinite branches. For the *CF*-groups A, E, G give rise to infinite branches. Figures 4.8 to

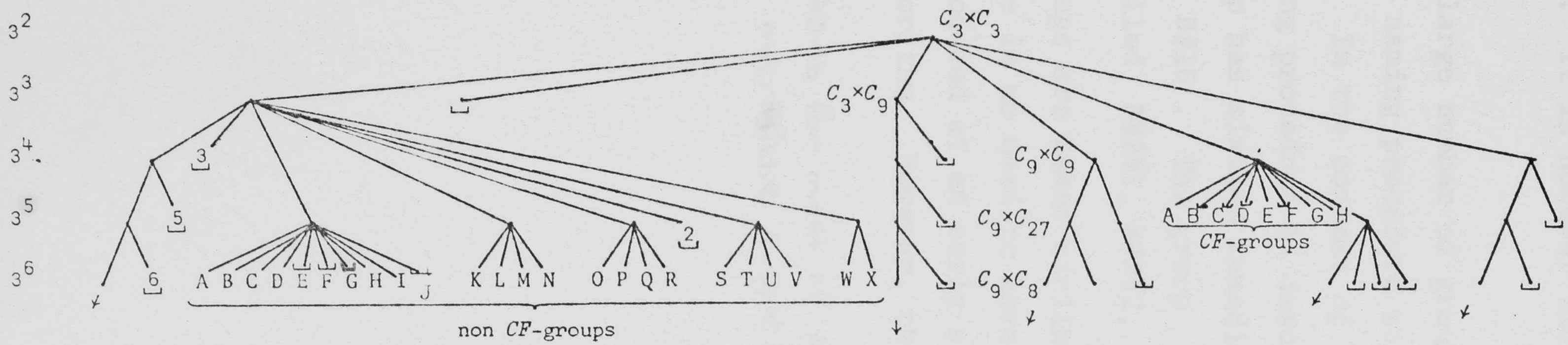


FIGURE 4.6 showing 2-generator 3-groups of co-class 1 and 2 .

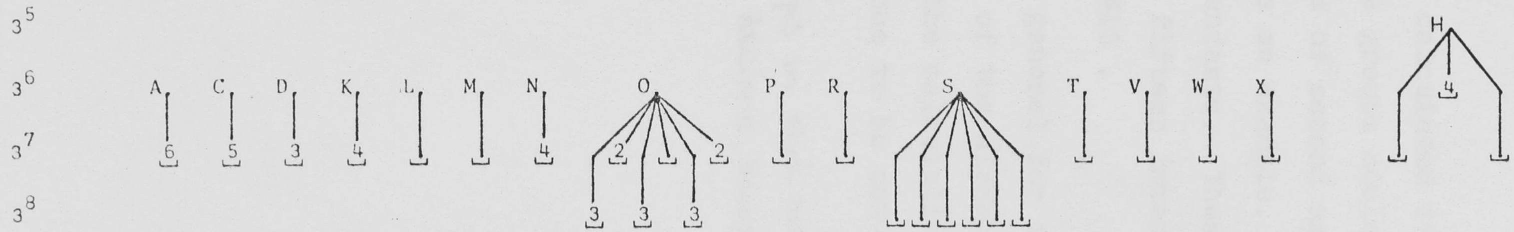


FIGURE 4.7 showing 2-generator 3-groups of co-class 2 which do not have descendants of order 3^9 .

4.12 show the extent of the tree diagrams calculated for the trees corresponding to these groups. Figure 4.11 shows the tree diagram of the tree associated with group I . This diagram is slightly different from that appearing in Ascione, Havas, Leedham-Green [1977]; however Figure 4.11 gives the correct version. This tree diagram has been calculated further than indicated. It is however too cumbersome to draw. This is also true of Figure 4.12.

With the large number of groups being calculated it was found necessary to introduce a naming procedure so that the groups could be easily distinguished. In the context of 3-groups of second maximal nilpotency class the naming procedure is described via an example. Consider the group B ; this group has sixteen immediate descendants. These are labelled B#1, B#2, ..., B#16 . The group B#2 has fifteen immediate descendants and these are labelled B#2#1, B#2#2, ..., B#2#15 .

Other groups are named similarly. In general for this naming procedure there is no need to have letters of the alphabet. These were *ad hoc* names introduced at an early stage of the calculations with the generating algorithm. However, they continue to be convenient and so are retained.

Presentations for most of the groups in this section can be found in the microfiche supplement to Ascione, Havas and Leedham-Green [1977].

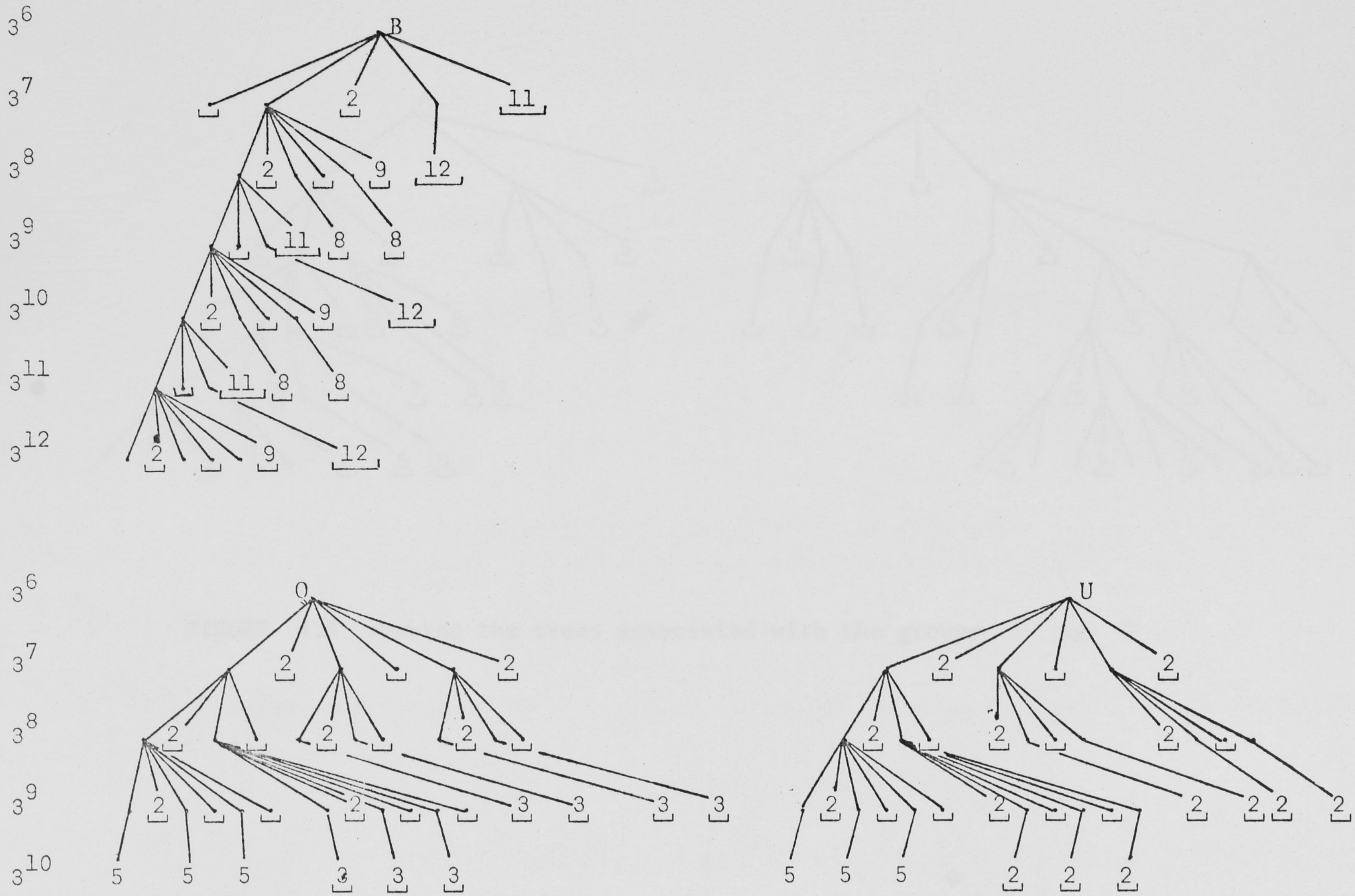


FIGURE 4.8 showing the trees associated with the groups B, Q, U .

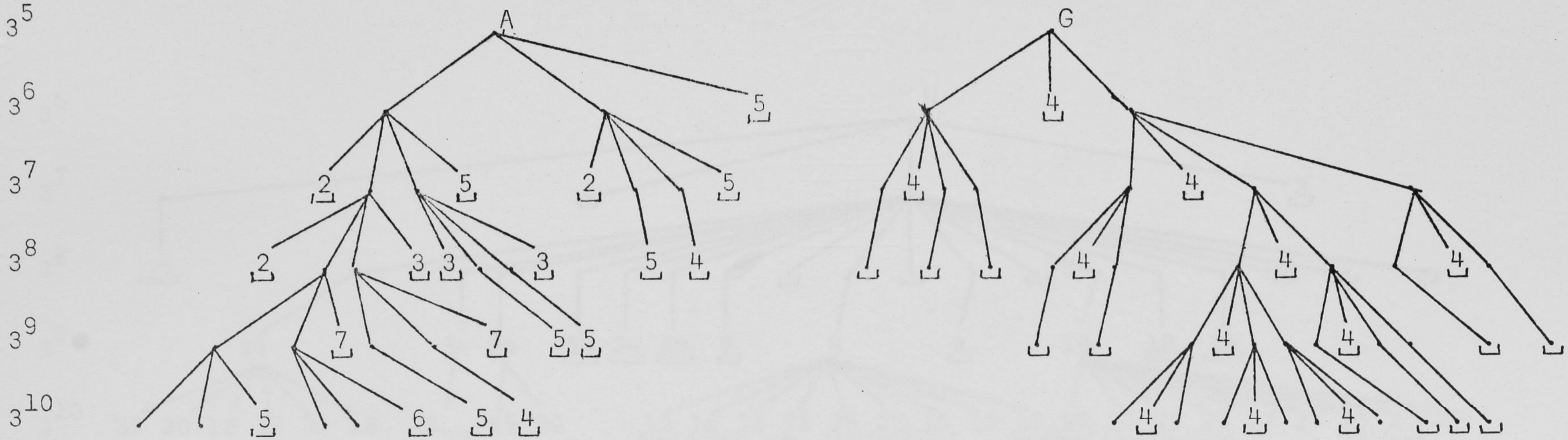


FIGURE 4.9 showing the trees associated with the groups A and G .

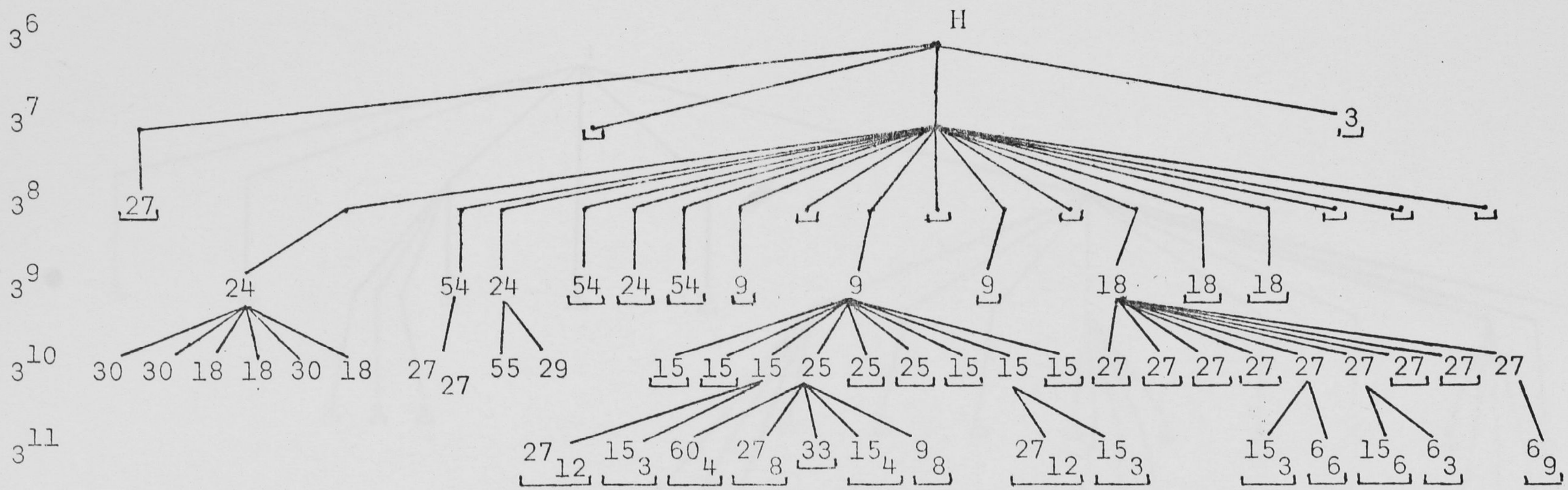


FIGURE 4.10 showing the tree associated with group H .

Note: b_m indicates that there are m sets of immediate descendants each of b groups.

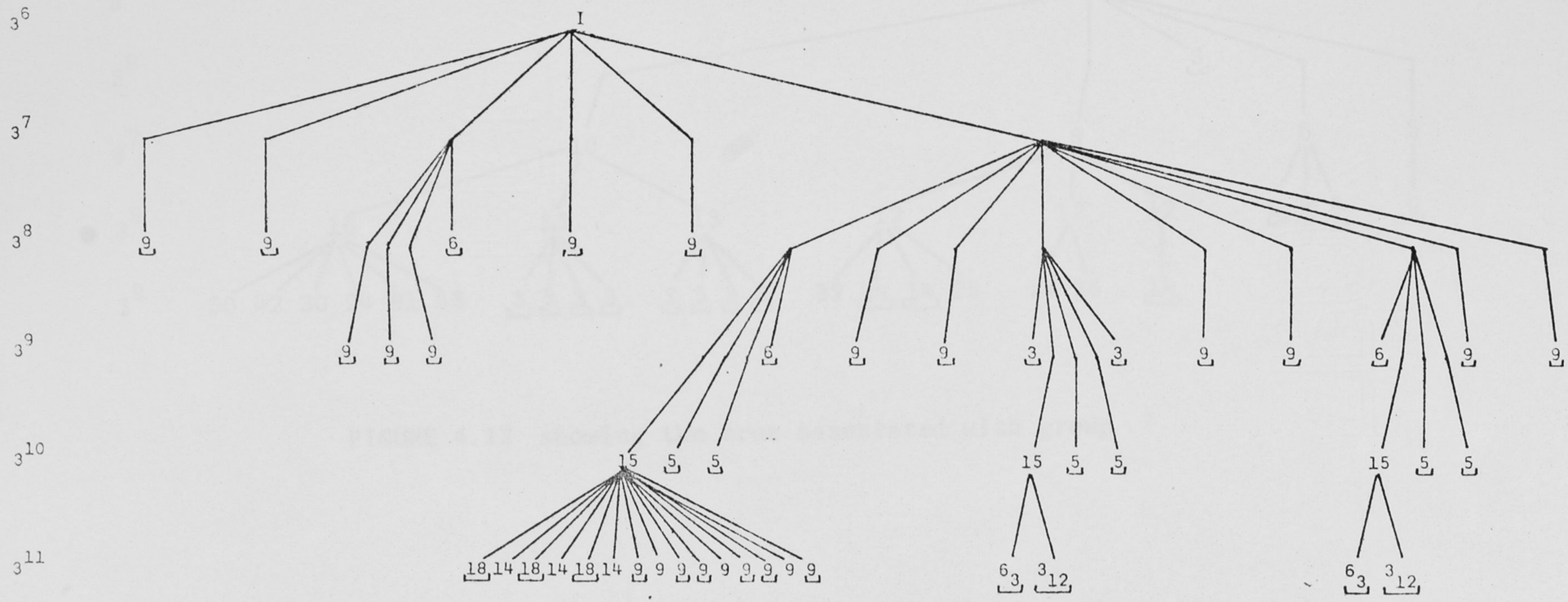


FIGURE 4.11 showing the tree associated with group I .

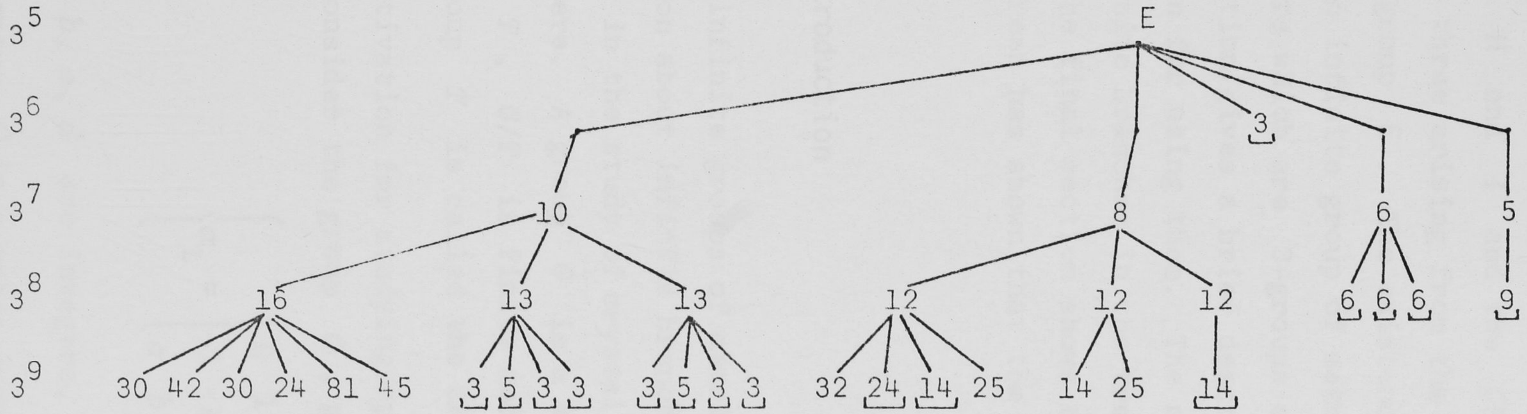


FIGURE 4.12 showing the tree associated with group E .

CHAPTER 5

SOME INFINITE BRANCHES

This chapter deals with the infinite branches arising from the non *CF*-groups H and I and the *CF*-group E . There are nine such infinite branches, three arising from the group H , one from the group I and five from the group E . The existence of these infinite branches is shown by finding an infinite group of matrices which has an infinite number of finite 3-quotients which are 3-groups of second maximal nilpotency class. The first section gives a brief description of the infinite groups and a motivation for using them. The next three sections calculate groups which give infinite branches in the tree of 3-groups of second maximal nilpotency class. The final section shows that all the groups calculated are distinct. Leedham-Green has shown that the groups calculated are all the possible ones.

5.1. Introduction

The infinite groups of matrices studied in this chapter to give information about infinite branches are space groups. Space groups arise naturally in the study of crystallography, however these considerations are ignored here. A group G is a *space group* if G has a free abelian normal subgroup T , G/T is finite and T is equal to its centralizer in G . The subgroup T is called the *translation subgroup*.

A motivation for studying space groups comes from 3-groups of maximal class. Consider the group G , generated by

$$\left\{a_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ a & b & 1 \end{pmatrix}, a_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & d & 1 \end{pmatrix}\right\},$$

where a, b, c, d are integers, $c^2 + d^2 \neq 0$.

The group G is a space group however this is not proved here. Elements in G are defined as follows: $a_{i+1} = [a_i, a_1]$ for $i \in \{2, \ldots\}$. Let T be the subgroup of G generated by $\{a_2, a_3\}$. It can be shown that

T contains all the matrices of G which have I_2 in the top left-hand block. Consider the following series of subgroups in T :

$$T > [T, G] > [T, G, G] > \ldots > [T, mG] > \ldots$$

where $[T, mG] = [T, \underbrace{G, G, \ldots, G}_{m \text{ times}}]$.

It can be shown that

$$|[T, mG]/[T, (m+1)G]| = 3 \quad \text{for } m \in \{1, \ldots\} .$$

The group $G/[T, mG]$ is the 3-group of maximal nilpotency class of order 3^{m+1} on the infinite branch, that is, it is the group which has infinitely many descendants.

The above group G consists of the cyclic group of order 3 acting on a free abelian group of rank 2 . For 3-groups of second maximal nilpotency class this suggests looking at a free abelian group of rank 6 acted on by either $C_3 \text{ wr } C_3$, or a subgroup of $C_3 \text{ wr } C_3$ which has order 27 and exponent 9 , or C_9 . These groups were studied initially to give a set of examples of infinite branches in the tree of 2-generator 3-groups of second maximal nilpotency class. The above mentioned groups are space groups and can be generated by the following pairs of matrices:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ d_1 & d_2 & e_1 & e_2 & f_1 & f_2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ a_1 & a_2 & b_1 & b_2 & c_1 & c_2 & 1 \end{pmatrix}$$

(the top left-hand 6×6 blocks generate $C_3 \text{ wr } C_3$);

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ d_1 & d_2 & e_1 & e_2 & f_1 & f_2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ a_1 & a_2 & b_1 & b_2 & c_1 & c_2 & 1 \end{pmatrix}$$

(the top left-hand 6×6 blocks generate the group of order 27);

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ d_1 & d_2 & e_1 & e_2 & f_1 & f_2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ d_1 & d_2 & e_1 & e_2 & f_1 & f_2 & 1 \end{pmatrix}$$

(the top left-hand 6×6 blocks generate C_9).

In all the above matrices, $a_1, a_2, b_1, \ldots, f_2$ are integers and not all zero. Extending the domain of coefficients enables the size of the matrices to be reduced. This is done by adjoining a primitive cube root of unity, ω , and working over $\mathbb{Q}(\omega)$ instead of $\mathbb{Q}$. The matrices are reduced to 4×4 matrices and these were found more convenient to work with. In the reduced form they are as follows:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ d & e & f & 1 \end{pmatrix}, \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix}$$

(the top left-hand 3×3 blocks generate $C_3 \text{ wr } C_3$);

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega & 0 & 0 & 0 \\ d & e & f & 1 \end{pmatrix}, \begin{pmatrix} \omega^2 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix}$$

(the top left-hand 3×3 blocks generate the group of order 27);

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega & 0 & 0 & 0 \\ a & b & c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ d & e & f & 1 \end{pmatrix}$$

(the top left-hand 3×3 blocks generate C_9).

In all the above matrices $a, b, \ldots, f$ are in $\mathbb{Z}(\omega)$.

Each pair of matrices is discussed separately in the following sections of this chapter.

5.2. One Infinite Branch

Consider the group $G(a, b, c, d, e, f)$ of 4×4 matrices over $\mathbb{Q}(\omega)$ generated by

$$\left\{ a_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ d & e & f & 1 \end{pmatrix}, \; a_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega & 0 & 0 & 0 \\ a & b & c & 1 \end{pmatrix} \right\}$$

where $a, \ldots, f$ and ω are as before and $d^2 + e^2 + f^2 \neq 0$. There is a set of finite 3-quotients of G which constitutes an infinite branch in the tree of 3-groups of second maximal nilpotency class. Exactly one infinite branch arises in this way. These two claims are now proved.

The group G is a space group but this is not proved here. Let T be the subgroup of G which consists of all those matrices in G with I_3 in the top left-hand block. Thus T is an abelian, normal subgroup of G. It can be shown that T is the translation subgroup of G. Elements of T have only the first three entries in the bottom row which are unspecified. These entries are written as a row vector to represent the element in T. Thus

$$\left(\begin{array}{ccc:c} & & & 0 \\ & I_3 & & 0 \\ & & & 0 \\ \hdashline g & h & i & 1 \end{array} \right)$$

is written as (g, h, i). Multiplication of two elements in T is written as addition of the vectors.

Implicit in the following calculations is that G/T is isomorphic to C_9. A method for naming elements is also shown.

$$a_1^3 = (3d, 3e, 3f), \quad a_2^3 = a_4 = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega & 0 \\ a+\omega(b+c) & a+b+\omega c & a+b+c & 1 \end{pmatrix},$$

$$[a_2, a_1] = a_3 = (-f\omega+d, -d+e, -e+f),$$

$$a_3^3 = (-3f\omega+3d, -3d+3e, -3e+3f),$$

$$[a_3, a_1] = e ,$$

$$[a_3, a_2] = (-e\omega+2f\omega-d,\ -f\omega+2d-e,\ -d+2e-f) = a_5 ,$$

$$a_4^3 = e ,$$

$$[a_4, a_1] = \big(d(1-\omega),\ e(1-\omega),\ f(1-\omega)\big) ,$$

$$[a_4, a_2] = e ,$$

$$[a_4, a_3] = \big(d(1-\omega)+f(-2\omega-1),\ d(\omega-1)+e(1-\omega),\ e(\omega-1)+f(1-\omega)\big) , \text{ and}$$

$$a_{i+1} = [a_i, a_2] \quad \text{for} \quad i \in \{5, 6, \ldots\} .$$

$$[(g, h, i), a_1] = e ,$$

$$[(g, h, i), a_2] = (i\omega-g,\ g-h,\ h-i) ,$$

$$[(g, h, i), a_3] = e ,$$

$$[(g, h, i), a_4] = \big(g(\omega-1),\ h(\omega-1),\ i(\omega-1)\big) ,$$

$$(g, h, i)^3 = (3g, 3h, 3i)$$

where (g, h, i) is any element in T . Thus

$$a_6 = \big(d(1-\omega)+3e\omega-3f\omega,\ -3d+e(1-\omega)+3f\omega,\ 3d-3e+f(1-\omega)\big) ,$$

$$a_7 = (4d\omega-d-6e\omega+5f\omega+f,\ 4d-d\omega+4e\omega-e-6f\omega,\ -6d+4e-e\omega+4f\omega-f) ,$$

$$a_8 = (-10d\omega+d+11e\omega+e-10f\omega-5f,\ 5d\omega-5d-10e\omega+e+11f\omega+f,\ 10d-d\omega+5e\omega-5e-10f\omega+f) ,$$

$$a_9 = (21d\omega-21e\omega-6e+21e\omega+15e,\ -15d\omega+6d+21e\omega-21f\omega-6f,\ 6d\omega-15d-15e\omega+6e+21f\omega) .$$

It is now shown that the subgroup T is generated by $a_1, a_3, a_5, a_6, a_7, a_8$. This is done by first considering the following elements of T :

$$t_1 = (d, e, f) ,$$

$$t_2 = (f\omega, d, e) ,$$

$$t_3 = (e\omega, f\omega, d) ,$$

$$t_4 = (d\omega, e\omega, f\omega) ,$$

$$t_5 = (-f\omega - f,\ d\omega,\ e\omega)\ ,$$

$$t_6 = (-e\omega - e,\ -f\omega - f,\ d\omega)\ ,$$

$$t_7 = (-d\omega - d,\ -e\omega - e,\ -f\omega - f) = -t_1 - t_4\ ,$$

$$t_8 = (f,\ -d\omega - d,\ -e\omega - e) = -t_2 - t_5\ ,$$

$$t_9 = (e,\ f,\ -d\omega - d) = -t_3 - t_6\ ,$$

$$t_n = \begin{cases} t_{m(\mathrm{mod}9)} & \text{if } m \text{ is not a multiple of } 9\ , \\ t_9 & \text{if } m \text{ is a multiple of } 9\ . \end{cases}$$

Now $a_1 = t_1$, $a_3 = t_1 - t_2$ and for n greater than or equal to 5 ,

$$a_n = (-1)^n \sum_{k=0}^{n-3} (-1)^k \binom{n-3}{k} t_{k+1}\ .$$

This is easily proved by induction using that

$$[t_i,\ a_2] = t_{i+1} - t_i \quad \text{for } i \in \{1, \ldots\}$$

and

$$[t_i t_j,\ a_2] = [t_i,\ a_2][t_j,\ a_2] \quad \text{for } i,\ j \in \{1, \ldots\}\ .$$

Notice that t_{i+3} is just a multiple of ω of t_i and hence $[t_i,\ a_4] = t_{i+3} - t_i$.

This is enough to show that T is generated by $\{t_1, t_2, t_3, t_4, t_5, t_6\}$. However, an equivalent generating set is $\{a_1, a_3, a_5, a_6, a_7, a_8\}$. In general $[T, mG]$ is generated by $\{a_{m+3}, \ldots, a_{m+8}\}$ and $[T, (m+1)G]$ is generated by $\{a_{m+4}, \ldots, a_{m+9}\}$; the element a_{m+3} is not in $[T, (m+1)G]$. It is now shown by induction that a_{m+3}^3 is in $[T, (m+1)G]$. Calculations show that

$$a_1^3 = 3t_1 = -2(t_2 - t_1) - 2(t_3 - t_2) - 2(t_4 - t_3) - (t_5 - t_4) - (t_6 - t_5) - (-t_1 - t_4 - t_6)\ .$$

Each of the terms in brackets is in $[T, G]$ and hence a_1^3 is in $[T, G]$.

Now $a_3^3 = [a_2, a_1^3] = [a_1^3, a_2]^{-1}$ and since a_1^3 is in $[T, G]$ it follows that a_3^3 is in $[T, G, G]$. Similarly $a_5^3 = [a_3^3, a_2]$ and hence a_5^3 is in $[T, 3G]$. Suppose a_{m+2}^3 is in $[T, mG]$. Now $a_{m+3}^3 = [a_{m+2}^3, a_2]$ and hence a_{m+3}^3 is in $[T, (m+1)G]$. Thus in T is the following series of subgroups:

$$T > [T, G] > [T, G, G] > \dots > [T, mG] > \dots$$

such that $|[T, mG]/[T, (m+1)G]| = 3$.

Now $G/[T, G]$ has the following consistent power-commutator presentation:

$$\langle a_1, a_2, a_3, a_4, a_5;\ a_1^3 = e,\ a_2^3 = a_4,\ a_3^3 = e,\ a_4^3 = e,\ a_5^3 = e,$$
$$[a_2, a_1] = a_3,\ [a_3, a_2] = a_5 \rangle .$$

This represents a 3-group of second maximal nilpotency class. In fact the presentation gives *CF*-group E .

It is clear that $G/[T, (m+1)G]$ is an immediate descendant of $G/[T, mG]$ and hence the following set is an infinite set of finite 3-quotients of G which constitutes an infinite branch in the tree of 3-groups of second maximal nilpotency class:

$$\{G/[T, G], G/[T, G, G], \dots, G/[T, mG], \dots\} .$$

For $m \in \{1, \dots\}$ the group $G/[T, mG]$ is independent of the values of $a, \dots, f$ and hence only one possible infinite branch can arise from this case.

5.3. Six Infinite Branches

Consider the group $G(a, b, c, d, e, f)$ of 4×4 matrices over $\mathbb{Q}(\omega)$ generated by

$$\left\{ a_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ d & e & f & 1 \end{pmatrix},\ a_2 = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix} \right\}$$

where $a, \ldots, f$ and ω are as before and $(d+e+f)^2 + b^2 + c^2 \neq 0$. There is a set of finite 3-quotients of G which constitutes an infinite branch in the tree of 3-groups of second maximal nilpotency class. There are six non-isomorphic groups G, namely,

$$G(0, 0, 1, 0, 0, 0),$$
$$G(0, 0, 0, 1, 0, 0),$$
$$G(0, 0, 1, 1, 0, 0),$$
$$G(0, 1, 1, 0, 0, 0),$$
$$G(0, 2, 1, 0, 0, 0),$$
$$G(0, 1, 1, 1-\omega, 0, 0).$$

It is first shown that there is a set of finite 3-quotients of G which constitutes an infinite branch in the tree of 3-groups of second maximal nilpotency class. This is done by the same basic method used in the previous section. The group G is a space group but this is not proved here. Let T be the subgroup of G consisting of all those matrices in G with I_3 in the top left-hand block. Thus T is an abelian, normal subgroup and it can be shown that T is the translation subgroup of G.

Calculations show that $a_1, a_2, [a_2, a_1] = a_3$, $[a_2, a_1, a_1] = a_4$ are not in T; however $[a_2, a_1, a_2], [a_2, a_1, a_1, a_1], [a_2, a_1, a_1, a_2], a_1^3, a_2^3, [a_2, a_1]^3$ and $[a_2, a_1, a_1]^3$ are all in T. Thus G/T has order at least 81, class 3 and exponent 3; thus G/T is isomorphic to $C_3 \text{ wr } C_3$. Again, elements of T are written as vectors containing the first three entries in the bottom row.

It is now convenient to regard T as a faithful $\mathbb{Z}(G/T)$-module under conjugation with coset representatives. The coset representatives a_1 and a_2 have the following action:

$$(g, h, i)^{a_1} = (i, g, h),$$

$$(g, h, i)^{a_2} = (g\omega, h, i),$$

for (g, h, i) in T.

Define δ to be the sum $d + e + f$. As a module

$$T = \langle (\delta, \delta, \delta), \big(c(\omega-1), b(1-\omega), 0\big) \rangle ,$$

$$[T, G] = \langle \big(\delta(\omega-1), 0, 0\big), \big(c(1-\omega), (b+c)(\omega-1), b(1-\omega)\big) \rangle ,$$

$$[T, G, G] = \langle \big(\delta(1-\omega), \delta(\omega-1), 0\big), \big((b-c)(1-\omega), (b-c)(1-\omega), (b-c)(1-\omega)\big), (3b, 0, 0), (3c, 0, 0) \rangle ,$$

$$[T, 3G] = \langle \big(\delta(\omega-1), \delta(\omega-1), \delta(\omega-1)\big), (-3b, 3b, 0), (-3c, 3c, 0), \big(3(b-c), 0, 0\big) \rangle ,$$

$$[T, 4G] = \langle (3\delta, 0, 0), (3b, 3b, 3b), (3c, 3c, 3c), \big(-3(b-c), 3(b-c), 0\big) \rangle ,$$

$$[T, 5G] = \langle (-3\delta, 3\delta, 0), \big(3b(\omega-1), 0, 0\big), \big(3c(\omega-1), 0, 0\big), \big(3(b-c), 3(b-c), 3(b-c)\big) \rangle ,$$

$$[T, 6G] = \langle (3\delta, 3\delta, 3\delta), \big(-3b(\omega-1), 3b(\omega-1), 0\big), \big(-3c(\omega-1), 3c(\omega-1), 0\big), \big(3(b-c)(\omega-1), 0, 0\big) \rangle .$$

It is easy to see that $\big(3c(\omega-1), 3b(1-\omega), 0\big)$ is in $[T, 6G]$ and hence $3T \leq [T, 6G]$. It is now shown that $[T, 6G] \leq 3T$ and this is used to show that $\big|[T, mG]/[T, (m+1)G]\big| = 3$.

First, however, $P_3(G)$ is calculated. It is clear that elements in $P_2(G)$ have the form $a_4^{\lambda}t$ where $\lambda \in \{0, 1, 2\}$ and $t \in T$.

$$\left(a_4^{\lambda}.t\right)^3 = t.t^{a_4^{\lambda}}.t^{a_4^{2\lambda}}$$

$$= t^3 \text{ or } e \text{ depending on } \lambda .$$

Calculating $[t, a_1]$, $[t, a_2]$, $[a_4, a_1]$, $[a_4, a_2]$ it follows that (as a module)

$$P_3(G) = \langle \big(\delta(\omega-1), 0, 0\big), \big(c(1-\omega), (b+c)(\omega-1), b(1-\omega)\big) \rangle .$$

Thus $P_3(G) = [T, G]$.

Now T is a subgroup of $\mathbb{Z}(\omega) \times \mathbb{Z}(\omega) \times \mathbb{Z}(\omega)$ and so T has at most six generators. Thus T/T^3, that is $T/3T$, has order at most 3^6. Since G/T has order 3^4 it follows that $G/3T$ has order at most 3^{10}. Now $G/P_3(G)$ has order 3^5 and class 3 and hence $G/3T$ has class at most 8. From this it follows that $[G, 8G] \leq 3T$. However since $[T, G] = P_3(G) = [G, 3G]$ it follows that $[T, 6G] = [G, 8G]$. Thus $[T, 6G] \leq 3T$.

It follows that $[T, 6G] = 3T$ and hence $T/[T, G]$ has order at most $3^{\cancel{6}}$. This means that $|[T, mG]/[T, (m+1)G]| = 3$ for $m \in \{1, \ldots, 5\}$. If this is not the case then $[T, mG] = [T, (m+1)G]$ for some $m \in \{1, \ldots, 5\}$. All subsequent terms $[T, (m+2)G]$, $[T, (m+3)G]$, ... are then equal and this is not possible.

Each term of the vectors in $[T, 6G]$ is essentially a multiple of $(\omega-1)$ of the corresponding terms of the vectors in $[T, 3G]$. This is a repeating pattern and hence $|[T, mG]/[T, (m+1)G]| = 3$ for $m \in \{1, \ldots\}$.

It has already been said that $G/[T, G]$ is a 3-group of second maximal nilpotency class. Thus, the following infinite set of finite

$$\{G/[T, G], G/[T, G, G], G/[T, 3G], \ldots, G/[T, mG], \ldots\}$$

3-quotients of G constitutes an infinite branch in the tree of 3-groups of second maximal nilpotency class.

It is now shown that there at most six non-isomorphic groups, G , of the type described at the beginning of the section. To simplify this a few lemmas are first proved.

The above calculations show that the elements of T do not depend on the value of a or the individual values of d, e, f only the sum $d + e + f$ which is denoted by δ . This leads to the following lemma.

LEMMA 5.1. *The group* $G(a, b, c, d, e, f)$ *as defined above is isomorphic to the group* $G(0, b, c, \delta, 0, 0)$.

Proof. Denote $G(a, b, c, d, e, f)$ by Group 1 and $G(0, b, c, \delta, 0, 0)$ by Group 2 and let T_1 and T_2 be the translation subgroups of these groups respectively. Denote the matrices generating Group 1 by a_1 and a_2 and define a_3 to be $[a_2, a_1]$, a_4 to be $[a_3, a_1]$ (as above) and the matrices generating Group 2 by b_1 and b_2 and define b_3 to be $[b_2, b_1]$, b_4 to be $[b_3, b_1]$. Now every element in Group 1 can be written uniquely as $a_1^{\alpha}a_2^{\beta}a_3^{\gamma}a_4^{\delta}t$ where α, β, γ, δ are either 0, 1 or 2 and t is in T_1 . Similarly every element in Group 2 can be written uniquely as $b_1^{\alpha}b_2^{\beta}b_3^{\gamma}b_4^{\delta}t$ where α, β, γ, δ are either 0, 1 or 2 and t is in T_2 . It will be established that T_1 and T_2 are identical.

Let θ be a map from Group 1 to Group 2 where

$$\theta : a_1^{\alpha} a_2^{\beta} a_3^{\gamma} a_4^{\delta} t \mapsto b_1^{\alpha} b_2^{\beta} b_3^{\gamma} b_4^{\delta} t \, .$$

It is clear that θ is one-to-one and onto and so to prove the lemma it remains to show that θ is a homomorphism.

Put $g_1 = a_1^{\alpha_1} a_2^{\beta_1} a_3^{\gamma_1} a_4^{\delta_1} t_1$ and $g_2 = a_1^{\alpha_2} a_2^{\beta_2} a_3^{\gamma_2} a_4^{\delta_2} t_2$; then

$$g_1 g_2 = a_1^{\alpha_1+\alpha_2} a_2^{\beta_1+\beta_2} \left[a_2^{\beta_1}, a_1^{\alpha_2}\right] a_3^{\gamma_1+\gamma_2} \left[a_3^{\gamma_1}, a_1^{\alpha_2}\right] a_4^{\delta_1+\delta_2} t_1 t_2 t_{12}$$

where

$$t_{12} = t_{11}\left[t_{11}, a_4^{\delta_2}\right]\left[t_1, a_4^{\delta_2}\right] ,$$

$$t_{11} = t_{10}\left[t_{10}, a_4^{\delta_1}\right] t_9 ,$$

$$t_{10} = t_7\left[t_7, \left[a_3^{\gamma_1}, a_1^{\alpha_2}\right]\right] t_8 ,$$

$$t_9 = \left[a_4^{\gamma_1}, a_3^{\gamma_2}\right]\left[t_1, a_3^{\gamma_2}\right] t_6 \left[t_6, a_3^{\gamma_2}\right] ,$$

$$t_8 = \left[a_3^{\gamma_1}, a_1^{\alpha_2}, a_3^{\gamma_2}\right] t_5 \left[t_5, a_3^{\gamma_2}\right] ,$$

$$t_7 = t_3\left[t_3, a_3^{\gamma_2}\right]\left[t_3, a_3^{\gamma_1}\right]\left[t_3, a_3^{\gamma_1}, a_3^{\gamma_2}\right] t_4 \left[t_4, a_3^{\gamma_2}\right] ,$$

$$t_6 = \left[a_4^{\delta_1}, a_2^{\beta_2}\right]\left[a_4^{\delta_1}, a_1^{\alpha_2}\right]\left[a_4^{\delta_1}, a_1^{\alpha_2}, a_2^{\beta_2}\right]\left[t_1, a_2^{\beta_2}\right]\left[t_1, a_1^{\alpha_2}\right]\left[t_1, a_1^{\alpha_2}, a_2^{\beta_2}\right] ,$$

$$t_5 = \left[a_3^{\gamma_1}, a_1^{\alpha_2}, a_2^{\beta_2}\right] ,$$

$$t_4 = \left[a_3^{\gamma_1}, a_2^{\beta_2}\right] ,$$

$$t_3 = \left[a_2^{\beta_1}, a_1^{\alpha_2}, a_2^{\beta_2}\right] .$$

Now, $\theta(g_1) = b_1^{\alpha_1} b_2^{\beta_1} b_3^{\gamma_1} b_4^{\delta_1} t_1$ and $\theta(g_2) = b_1^{\alpha_2} b_2^{\beta_2} b_3^{\gamma_2} b_4^{\delta_2} t_2$ and

$$\theta(g_1)\theta(g_2) = b_1^{\alpha_1+\alpha_2} b_2^{\beta_1+\beta_2} \left[b_2^{-\beta_1}, b_1^{\alpha_2}\right] b_3^{\gamma_1+\gamma_2} \left[b_3^{-\gamma_1}, b_1^{\alpha_2}\right] b_4^{\delta_1+\delta_2} t_1 t_2 t'_{12} ,$$

where t'_{12} is the same as t_{12} except that a_i is replaced by b_i, $i \in \{1, \ldots, 4\}$, in the expressions for $t_3, \ldots, t_{12}$. The expression for g_1g_2 is not yet in an appropriate form to allow $\theta(g_1g_2)$ to be calculated. Before this is done it is first shown that t_{12} equals t'_{12}. Let (g, h, i) be an element of T_1 and also T_2, then $(g, h, i)^{a_1}$ is (i, g, h) and $(g, h, i)^{b_1}$ is (i, g, h). Thus $[t, a_1]$ equals $[t, b_1]$ and hence $\left[t, a_1^2\right]$ equals $\left[t, b_1^2\right]$ where t is an element of T_1 or T_2. Also $(g, h, i)^{a_2}$ is $(g\omega, h, i)$ and $(g, h, i)^{b_2}$ is $(g\omega, h, i)$. Thus $[t, a_2]$ equals $[t, b_2]$ and $\left[t, a_2^2\right]$ equals $\left[t, b_2^2\right]$. Since a_3 and a_4 are defined in terms of a_1 and a_2 and b_3 and b_4 are defined in terms of b_1 and b_2 it follows that $[t, a_3]$ equals $[t, b_3]$, $\left[t, a_3^2\right]$ equals $\left[t, b_3^2\right]$, $[t, a_4]$ equals $[t, b_4]$ and $\left[t, a_4^2\right]$ equals $\left[t, b_4^2\right]$ for any element t of T_1 or T_2. There are eight more types of commutators to check; namely:

$$\left[a_2^{-\beta_1}, a_1^{\alpha_1}, a_2^{\beta_2}\right] ,$$

$$\left[a_3^{-\gamma_1}, a_2^{\beta_2}\right] ,$$

$$\left[a_3^{-\gamma_1}, a_1^{\alpha_2}, a_2^{\beta_2}\right] ,$$

$$\left[a_4^{-\delta_2}, a_2^{\beta_2}\right] ,$$

$$\left[a_4^{-\delta_1}, a_1^{\alpha_2}\right] ,$$

$$\left[a_4^{\delta_1}, a_1^{\alpha_2}, a_2^{\beta_2}\right] ,$$

$$\left[a_3^{\gamma_1}, a_1^{\alpha_2}, a_3^{\gamma_2}\right] ,$$

$$\left[a_4^{\delta_1}, a_3^{\gamma_2}\right] .$$

Calculations with the matrices show that all these commutators are equal to the corresponding commutators in Group 2 and so t_{12} equals t'_{12} .

Recall that

$$g_1g_2 = a_1^{\alpha_1+\alpha_2} a_2^{\beta_1+\beta_2}\left[a_2^{\beta_1}, a_1^{\alpha_2}\right]a_3^{\gamma_1+\gamma_2}\left[a_3^{\gamma_1}, a_1^{\alpha_2}\right]a_4^{\delta_1+\delta_2}t_1t_2t_{12} .$$

Taking all possible values for $\left[a_2^{\beta_1}, a_1^{\alpha_2}\right]$ and $\left[a_3^{\gamma_1}, a_1^{\alpha_2}\right]$ there are twelve possibilities for g_1g_2 . However, the above is enough to show that whatever the possibility, $\theta(g_1g_2) = \theta(g_1)\theta(g_2)$. Thus θ is an isomorphism and the lemma is proved. $\square$

All further work in this section deals with the group $G(0, b, c, \delta, 0, 0)$ which is now denoted $G(\delta, b, c)$. (The order of δ, b, c may seem unnatural however this order is used throughout this chapter.) The generators of $G(\delta, b, c)$ are

$$a_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \delta & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & b & c & 1 \end{pmatrix} ,$$

where δ, b, c are in $\mathbb{Z}(\omega)$ and $\delta^2 + b^2 + c^2 \neq 0$.

LEMMA 5.2. *The group* $G(\delta, b, c)$ *is isomorphic to the group* $G(\lambda\delta, \lambda b, \lambda c)$ *where* λ *is in* $\mathbb{Q}(\omega)$.

Proof. The isomorphism is achieved by conjugating a_1 and a_2 by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix} .$$

LEMMA 5.3. *The group* $G(0, b, c)$ *is isomorphic to the group* $G(0, c, b)$.

Proof. Conjugating a_1 and a_2 by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gives

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & c & b & 1 \end{pmatrix} .$$

However, squaring the first matrix gives

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and squaring this gives the first matrix. Thus a group isomorphic to $G(0, b, c)$ is generated by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & c & b & 1 \end{pmatrix} ,$$

but these matrices generate $G(0, c, b)$. □

Before showing that for all choices of δ, b and c there are at most six distinct groups $G(\delta, b, c)$ a few remarks about $\mathbb{Z}(\omega)$ are made. The integers of $\mathbb{Q}(\omega)$ make up $\mathbb{Z}(\omega)$. It can be shown that these integers have the form $r + s\omega$, where r and s are integers. It can also be shown that $1 - \omega$ is a prime in $\mathbb{Z}(\omega)$ and hence any element in $\mathbb{Z}(\omega)$ can be written as $(1-\omega)^{\rho} a_1$, where $\rho \in \{0, 1, \dots\}$ and a_1 contains no powers of $1 - \omega$. Thus

$$a_1 = \begin{Bmatrix} \pm 1 \\ \pm\omega \\ \pm(1+\omega) \end{Bmatrix} + (1-\omega)\chi$$

where χ is in $\mathbb{Z}(\omega)$. For the purposes of the following classification it is enough to take

$$a_1 = \begin{Bmatrix} 1 \\ 1+\omega \end{Bmatrix} + (1-\omega)\chi \quad \text{where} \quad \chi \quad \text{is in} \quad \mathbb{Z}(\omega) .$$

Finally, it can be shown that $\mathbb{Z}(\omega)$ is a Euclidean integral domain. This is used extensively in the following classification.

In defining the matrices a_1 and a_2 which generate $G(\delta, b, c)$ the case $\delta = b = c = 0$ is excluded since then a_1 and a_2 generate C_3 wr C_3 . To consider all possible choices for δ, b and c the following cases are considered.

(1) One of δ, b and c is non-zero.

(2) One of δ, b and c is zero.

(3) None of δ, b and c is zero.

CASE 1 - ONE OF δ, b AND c IS NON-ZERO

In this case Lemma 5.2 is used and λ is chosen appropriately so that the only groups to consider are, $G(1, 0, 0)$, $G(0, 1, 0)$ and $G(0, 0, 1)$. However, Lemma 5.3 shows that $G(0, 1, 0)$ and $G(0, 0, 1)$ are isomorphic. This case yields two groups, $G(1, 0, 0)$ and $G(0, 0, 1)$.

CASE 2 - ONE OF δ, b AND c IS ZERO

This case must itself be split into three different cases, namely:

(i) $b = 0$;

(ii) $c = 0$;

(iii) $\delta = 0$.

Suppose b is zero. Write δ as $(1-\omega)^{\rho}\delta_1$ where δ_1 does not contain any powers of $1 - \omega$ and write c as $(1-\omega)^{\tau}c_1$ where c_1 does not contain any powers of $1 - \omega$. Suppose $\rho < \tau$; applying Lemma 5.2 and choosing λ appropriately the group to consider is,

$$G\left(\delta_1, 0, (1-\omega)^{\tau-\rho}c_1\right)$$

where δ_1 and c_1 have no common factors. Since δ_1 and c_1 contain no powers of $1 - \omega$ then δ_1 and $c_1(1-\delta)^{\tau-\rho}$ have no common factors and so $\alpha\delta_1 + \beta c_1(1-\omega)^{\tau-\rho}$ is 1 for some α and β in $\mathbb{Z}(\omega)$. The group $G\left(\delta_1, 0, (1-\omega)^{\tau-\rho}c_1\right)$ contains the elements $\left(\delta_1(1-\omega), 0, 0\right)$ and $\left(c_1(1-\omega)^{\tau-\rho+1}, 0, 0\right)$ and hence $\left((1-\omega)\left(\alpha\delta_1+\beta c_1(1-\omega)^{\tau-\rho}\right), 0, 0\right)$. Thus the group contains $(1-\omega, 0, 0)$. Since δ_1 has no powers of $1 - \omega$ it can be written as either $1 + (1-\omega)\chi$ or $1 + \omega + (1-\omega)\chi$ where χ is in $\mathbb{Z}(\omega)$.
Notice that since the group contains $(1-\omega, 0, 0)$ it also contains $\left((1-\omega)\chi, 0, 0\right)$, $\left((1-\omega)^{\tau-\rho}c_1, 0, 0\right)$ and the inverses of these elements.
The group $G\left(\delta_1, 0, (1-\omega)^{\tau-\rho}c_1\right)$ is generated by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & (1-\omega)^{\tau-\rho}c_1 & 1 \end{pmatrix}.$$

However, the group also contains the following elements

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \delta_1 & 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc:c} & I_3 & & 0 \\ & & & 0 \\ & & & 0 \\ \hdashline -(1-\omega)\chi & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \left\{\begin{matrix} 1 \\ \text{or } 1+\omega \end{matrix}\right\} & 0 & 0 & 1 \end{pmatrix}$$

depending on the original δ_1

and

$$\begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & (1-\omega)^{\tau-\rho}c_1 & 1 \end{pmatrix} \left(\begin{array}{ccc:c} & I_3 & & 0 \\ & & & 0 \\ & & & 0 \\ \hdashline 0 & 0 & (1-\omega)^{\tau-\rho}c_1 & 1 \end{array}\right) = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus depending on δ_1, $G(1, 0, 0)$ or $G(1+\omega, 0, 0)$ are subgroups of

$G\left(\delta_1, 0, (1-\omega)^{\tau-\rho}c_1\right)$. However, both $G(1, 0, 0)$ and $G(1+\omega, 0, 0)$ contain the element $(1-\omega, 0, 0)$ and hence both contain $G\left(\delta_1, 0, (1-\omega)^{\tau-\rho}c_1\right)$ as a subgroup. Thus $G\left(\delta_1, 0, (1-\omega)^{\tau-\rho}c_1\right)$ is isomorphic to $G(1, 0, 0)$. If $\tau < \rho$ the group to consider is $G\left((1-\omega)^{\rho-\tau}\delta_1, 0, c_1\right)$ and an argument similar to the one above shows that this group is isomorphic to $G(0, 0, 1)$. When $\tau = \rho$ the group to consider is $G\left(\delta_1, 0, c_1\right)$. Again a similar argument is employed and depending on δ_1 and c_1 this group is isomorphic to one of

$$G(1, 0, 1) ,$$
$$G(1, 0, 1+\omega) ,$$
$$G(1+\omega, 0, 1) .$$

However $G(1, 0, 1+\omega)$ is $G(1, 0, 2)$ since both groups contain $(1-\omega, 0, 0)$. Similarly $G(1+\omega, 0, 1)$ is $G(2, 0, 1)$. By Lemma 5.2, $G(2, 0, 1)$ is isomorphic to $G(-2, 0, -1)$ and this is $G(1, 0, 2)$ since both these groups contain $(3, 0, 0)$.

To show that $G(1, 0, 1)$ and $G(1, 0, 2)$ are isomorphic $\Phi(\omega)$ is considered. In $\Phi(\omega)$ the map $\omega \mapsto \omega^2$ is clearly an isomorphism. Now $G(1, 0, 1)$ is generated by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and hence $G(1, 0, 1)$ is isomorphic to the group generated by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} \omega^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} .$$

However, another pair of generators for this group is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix} .$$

Thus $G(1, 0, 1)$ is isomorphic to $G(1, 0, 2)$.

Suppose c is zero. Write δ as $(1-\omega)^{\rho}\delta_1$ where δ_1 has no powers of $1 - \omega$ and write b as $(1-\omega)^{\sigma}b_1$ where b_1 has no powers of $1 - \omega$. The argument for this case proceeds exactly as for the case when b is zero. Thus, when ρ is less than σ the group is isomorphic to $G(1, 0, 0)$. When σ is less than ρ the group is isomorphic to $G(0, 1, 0)$. By Lemma 5.3 this group is isomorphic to $G(0, 0, 1)$. When σ equals ρ , depending on δ_1 and b_1 the group is isomorphic to one of

$$G(1, 1, 0)\ ,$$
$$G(1, 1+\omega, 0)\ ,$$
$$G(1+\omega, 1, 0)\ .$$

The group $G(1, 1, 0)$ contains $(1, 1, 1)$ and hence $(-1, -1, -1)$. The generator a_2 can be multiplied by this to show that $G(1, 1, 0)$ is $G(1, 0, -1)$; that is $G(1, 0, 2)$. However this group is isomorphic to $G(1, 0, 1)$. Now $G(1, 1+\omega, 0)$ is $G(1, 2, 0)$ and an argument as above shows that this is $G(1, 0, 1)$. Finally $G(1+\omega, 1, 0)$ is $G(2, 1, 0)$ which is isomorphic to $G(1, 2, 0)$.

Suppose δ is zero. Write b as $(1-\omega)^{\sigma}b_1$ where b_1 contains no powers of $1 - \omega$ and write c as $(1-\omega)^{\tau}c_1$ where c_1 contains no powers of $1 - \omega$. Using Lemma 5.3 it can always be arranged that $\tau \le \sigma$. When $\tau < \sigma$ by applying Lemma 5.2 the group to consider is

$$G\left(0,\ (1-\omega)^{\sigma-\tau}b_1,\ c_1\right)\ .$$

Provided λ in Lemma 5.2 is chosen appropriately, b_1 and c_1 have no common factors. Using an argument similar to that above it can be shown that the group contains $\left((1-\omega)^2, 0, 0\right)$. If $\sigma - \tau$ is greater than or equal to 2 then the group is isomorphic to $G(0, 0, 1)$. If $\sigma - \tau$ equals one then the group is $\left(0, (1-)b_1, c_1\right)$. Possibilities for c_1 are

$$\begin{Bmatrix} 1 \\ 1+\omega \end{Bmatrix} + (1-\omega)\begin{Bmatrix} 0 \\ 1 \\ 1+\omega \end{Bmatrix} + (1-\omega)^2\chi$$

where χ is in $\mathbb{Z}(\omega)$. Using the same type of argument as before, depending on b_1 and c_1 , the group is isomorphic to one of the following groups:

$$\begin{array}{ll} G(0,\ 1-\omega,\ 1)\ , & G(0,\ \omega+2,\ 1)\ , \\ G(0,\ 1-\omega,\ -\omega+2)\ , & G(0,\ \omega+2,\ -\omega+2)\ , \\ G(0,\ 1-\omega,\ \omega+3)\ , & G(0,\ \omega+2,\ \omega+3)\ , \\ G(0,\ 1-\omega,\ 1+\omega)\ , & G(0,\ \omega+2,\ 1+\omega)\ , \\ G(0,\ 1-\omega,\ 2)\ , & G(0,\ \omega+2,\ 2)\ , \\ G(0,\ 1-\omega,\ 2\omega+3)\ , & G(0,\ \omega+2,\ 2\omega+3)\ . \end{array}$$

By examining the different elements in these groups it can be shown that they are all isomorphic to $G(0,\ 0,\ 1)$.

If σ equals τ the group to consider is $G\left(0,\ b_1,\ c_1\right)$. Using the same argument, it appears that there are thirty-six possible groups depending on b_1 and c_1 . However, Lemma 5.3 reduces this to twenty-one groups. They are as follows:

$$\begin{array}{ll} G(0,\ 1,\ 1)\ , & G(0,\ 2,\ 1)\ , \\ G(0,\ 1,\ -\omega+2)\ , & G(0,\ 1,\ 1+\omega)\ , \\ G(0,\ 1,\ \omega+3)\ , & G(0,\ 1,\ 2\omega+3)\ , \\ G(0,\ -\omega+2,\ -\omega+2)\ , & G(0,\ -\omega+2,\ 1+\omega)\ , \\ G(0,\ -\omega+2,\ \omega+3)\ , & G(0,\ -\omega+2,\ 2)\ , \\ G(0,\ \omega+3,\ \omega+3)\ , & G(0,\ -\omega+2,\ 2\omega+3)\ , \\ G(0,\ 1+\omega,\ 1+\omega)\ , & G(0,\ \omega+3,\ 1+\omega)\ , \\ G(0,\ 1+\omega,\ 2)\ , & G(0,\ \omega+3,\ 2)\ , \\ G(0,\ 1+\omega,\ 2\omega+3)\ , & G(0,\ \omega+3,\ 2\omega+3)\ , \\ G(0,\ 2,\ 2)\ , & \\ G(0,\ 2,\ 2\omega+3)\ , & \\ G(0,\ 2\omega+3,\ 2\omega+3)\ . & \end{array}$$

The groups in the left-hand column are all isomorphic to $G(0,\ 1,\ 1)$ and the groups in the right-hand column are all isomorphic to $G(0,\ 2,\ 1)$. Thus case 2, when one of δ, b and c is zero yields three groups; $G(1,\ 0,\ 1)$ when b is zero and $G(0,\ 1,\ 1)$ and $G(0,\ 2,\ 1)$ when δ is zero.

CASE 3: NONE OF δ, b AND c IS ZERO

Write δ as $(1-\omega)^{\sigma}\delta_1$, b as $(1-\omega)^{\rho}b_1$ and c as $(1-\omega)^{\tau}c_1$ where

δ_1, b_1 and c_1 do not contain any powers of $1 - \omega$. The arguments employed in this case are exactly the same as those used in the previous case. There are seven cases to consider depending on the relative sizes of σ, ρ and τ. In each of these seven cases there are three parts depending on which two of δ_1, b_1 and c_1 have no common factors. The results are summarized below.

σ is the smallest

When δ_1 and b_1, and δ_1 and c_1 have no common factors all possible groups are isomorphic to $G(1, 0, 0)$.

ρ is the smallest

When δ_1 and b_1, and b_1 and c_1 have no common factors all possible groups are isomorphic to $G(0, 0, 1)$.

τ is the smallest

When δ_1 and c_1, and b_1 and c_1 have no common factors all possible groups are isomorphic to $G(0, 0, 1)$.

$\sigma = \rho < \tau$

All possible groups are isomorphic to $G(1, 0, 1)$.

$\sigma = \tau < \rho$

All possible groups are isomorphic to $G(1, 0, 1)$.

$\rho = \tau < \sigma$

All possible groups are isomorphic to either $G(0, 1, 1)$, $G(0, 2, 1)$ or $G(1-\omega, 1, 1)$.

$\sigma = \rho = \tau$

All possible groups are isomorphic to either $G(1, 0, 0)$ or $G(1, 0, 1)$.

The first three cases are not quite complete. The missing parts are now dealt with. Suppose σ is less than ρ and τ and b_1 and c_1 have no common factors. Applying Lemma 5.2 the group to consider is $G\left(\delta_1, (1-\omega)^{\rho-\sigma}b_1, (1-\omega)^{\tau-\sigma}c_1\right)$. Suppose τ is less than ρ, then the group contains $\left((1-\omega)^{\tau-\sigma+2}, 0, 0\right)$. If ρ is greater than or equal to

$\tau + 2$ then the group is isomorphic to $G\left(\delta_1, 0, (1-\omega)^{\tau-\sigma}c_1\right)$. Lemma 5.2 can be applied so that δ_1 and c_1 have no common factors. This case has already been done.

If ρ equals $\tau + 1$ depending on b_1 and c_1 the group is isomorphic to one of the following:

$$G\left(\delta_1, (1-\omega)^{\tau-\sigma+1}, (1-\omega)^{\tau-\sigma}\right), \qquad G\left(\delta_1, (1-\omega)^{\tau-\sigma+1}(1+\omega), (1-\omega)^{\tau-\sigma}\right),$$

$$G\left(\delta_1, (1-\omega)^{\tau-\sigma+1}, (1-\omega)^{\tau-\sigma}(-\omega+2)\right), \quad G\left(\delta_1, (1-\omega)^{\tau-\sigma+1}(1+\omega), (1-\omega)^{\tau-\sigma}(-\omega+2)\right),$$

$$G\left(\delta_1, (1-\omega)^{\tau-\sigma+1}, (1-\omega)^{\tau-\sigma}(\omega+3)\right), \quad G\left(\delta_1, (1-\omega)^{\tau-\sigma+1}(1+\omega), (1-\omega)^{\tau-\sigma}(\omega+3)\right),$$

$$G\left(\delta_1, (1-\omega)^{\tau-\sigma+1}, (1-\omega)^{\tau-\sigma}(1+\omega)\right), \quad G\left(\delta_1, (1-\omega)^{\tau-\sigma+1}(1+\omega), (1-\omega)^{\tau-\sigma}(1+\omega)\right),$$

$$G\left(\delta_1, (1-\omega)^{\tau-\sigma+1}, (1-\omega)^{\tau-\sigma}.2\right), \qquad G\left(\delta_1, (1-\omega)^{\tau-\sigma+1}(1+\omega), (1-\omega)^{\tau-\sigma}.2\right),$$

$$G\left(\delta_1, (1-\omega)^{\tau-\sigma+1}(2\omega+3)\right), \qquad G\left(\delta_1, (1-\omega)^{\tau-\sigma+1}(1+\omega), (1-\omega)^{\tau-\sigma}(2\omega+3)\right).$$

Groups in the left-hand column now fall into the category of δ_1 and b_1 having no common factors and these groups have already been dealt with. The first group in the right-hand column has δ_1 and c_1 with no common factors. Now

$$-\omega + 2 = (1+\omega)(-3\omega-1),$$

$$\omega + 3 = (1+\omega)(-2\omega+1),$$

$$2 = (1+\omega)(-2\omega),$$

$$2\omega + 3 = (1+\omega)(-\omega+2).$$

Applying Lemma 5.2, with λ equal to $(1+\omega)^{-1}$, to the last five groups in the right-hand column shows that these groups fall into a case that has already been dealt with.

If ρ is less than τ then the argument is similar to that used when τ is less than ρ. If ρ equals τ there are thirty-six different groups to consider. All of these can be shown to fall into cases already considered by either using the method above or modifying the generators by elements in the group.

The other two missing cases in the summary are dealt with in the same way. Thus case 3 yields one group - $G(1-\omega, 1, 1)$.

This section has shown that there are at most six non-isomorphic groups in which the top left-hand 3×3 blocks generate $C_3 \text{ wr } C_3$. In 5.5 it is shown that these six groups are all different.

5.4. Two Infinite Branches

Consider the group $G(a, b, c, d, e, f)$ of 4×4 matrices over $\mathbb{Q}(\omega)$ generated by

$$\left\{ a_1 = \begin{bmatrix} \omega^2 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{bmatrix},\ a_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega & 0 & 0 & 0 \\ d & e & f & 1 \end{bmatrix} \right\}$$

where $a, \ldots, f$ and ω are as before and $a, \ldots, f$ are not all zero. There is a set of finite 3-quotients of G which constitutes an infinite branch in the tree of 3-groups of second maximal nilpotency class. There are two non-isomorphic groups, namely,

$$G(0, 0, 0, \omega, 0, 0) ,$$
$$G(1, -\omega, 0, 0, 1, 1) .$$

Consider the group generated by the following pair of matrices:

$$b_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \begin{matrix} a(\omega+2)+b(\omega+2) \\ +c(\omega+2)+3d(-\omega-1)-3e \end{matrix} & \begin{matrix} a(\omega-1)+b(\omega+2) \\ +c(\omega-1) \end{matrix} & \begin{matrix} a(\omega-1)+b(\omega+2) \\ +c(\omega+2)-3e+3f \end{matrix} & 1 \end{bmatrix},$$

$$b_2 = \begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \begin{matrix} a(1-\omega)+b(1-\omega) \\ +c(1-\omega)+3e\omega \end{matrix} & \begin{matrix} a(1-\omega)-b(\omega+2) \\ +c(1-\omega)+3e \end{matrix} & \begin{matrix} a(1-\omega)-b(\omega+2) \\ -c(\omega+2)+3e \end{matrix} & 1 \end{bmatrix}.$$

A subgroup of index 3 in this group is generated by $[b_2, b_1]$ and b_1b_2 . However,

$$[b_1, b_2] = \begin{pmatrix} \omega^2 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3a & 3b & 3c & 1 \end{pmatrix} \quad \text{and} \quad b_1b_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega & 0 & 0 & 0 \\ 3d & 3e & 3f & 1 \end{pmatrix} .$$

Conjugating these matrices by

$$\left(\begin{array}{ccc:c} & & & 0 \\ & I_3 & & 0 \\ & & & 0 \\ \hdashline 0 & 0 & 0 & 3 \end{array}\right)$$

shows that the subgroup generated by $[b_2, b_1]$ and b_1b_2 is isomorphic to $G(a, b, c, d, e, f)$, the first group mentioned in this section.

The group generated by b_1 and b_2 is a group of the type studied in Section 5.3. Recall that a group of this type is denoted $G(\delta, b, c)$. A group of this type has two subgroups of index 3 which modulo the translation subgroup have exponent 9 . It can be shown that these two subgroups are isomorphic. A proof similar to that given in Lemma 5.1 may be used however this is not given here. Recall also that there are at most six non-isomorphic groups $G(\delta, b, c)$. Thus, there are at most six non-isomorphic groups $G(a, b, c, d, e, f)$ and these can be determined by studying the appropriate subgroup (shown above) of each of the six groups $G(\delta, b, c)$. The groups obtained are as follows:

$G(\delta, b, c)$	Subgroup of index 3
$G(1, 0, 0)$	$G(0, 0, 0, \omega, 0, 0)$
$G(1, 0, 1)$	$G(1, 0, -1, \omega, 0, 1)$
$G(0, 1, 1)$	$G(1, -\omega, 0, 0, 1, 1)$
$G(0, 0, 1)$	$G(1, 0, -1, 0, 0, 1)$
$G(0, 2, 1)$	$G(1, -2\omega, 1, 0, 2, 1)$
$G(1-\omega, 1, 1)$	$G(1, -\omega, 0, 2\omega+1, 1, 1)$

In fact, the first two groups (in the right-hand column) are isomorphic and the last four groups are isomorphic and different from the first two. For each of the two non-isomorphic groups, $G(0, 0, 0, \omega, 0, 0)$ and $G(1, -\omega, 0, 0, 1, 1)$ the following infinite set

$$\{G/[T, G], G/[T, G, G], \ldots, G/[T, mG], \ldots\}$$

constitutes an infinite branch in the tree of 3-groups of second maximal nilpotency class. Again, T is the subgroup of G consisting of all those matrices of G with I_3 in the top left-hand block.

As before G is a space group but this is not proved here. The importance of G being a space group is that it allowed a space group program to show that there are only two non-isomorphic groups $G(a, b, c, d, e, f)$.

The space group program is based on the ideas of Zassenhaus and the implementation, by K.-J. Köhler, is based on the work of Brown (Brown *et al* [1978]). I am indebted to K.-J. Köhler for running the program for me. The space group program calculated all the distinct space groups of the types considered in this chapter. The results confirmed those of Sections 5.2 and 5.3 and showed that there are two distinct groups $G(a, b, c, d, e, f)$.

5.5. Conclusion

In this chapter nine space groups have been discussed. The space group program shows that they are all distinct. It is now shown that the infinite branch obtained from each space group is different. (This also shows that the space groups themselves are distinct.) Suppose G is a space group. The following table shows for each G the first 3-quotient $G/[T, mG]$, that is the 3-quotient with the lowest order, which occurs on only one infinite branch.

Space Group	Lowest order 3-quotient occurring on only one infinite branch
Group from Section 5.2	E#1#5#1#15
$G(0, 0, 1)$	H#~~2~~3#2
$G(1, 0, 0)$	E#1#5#1#1
$G(1, 0, 1)$	E#2#5#1#16
$G(0, 1, 1)$	H#3#3
$G(0, 2, 1)$	H#~~1~~3#1
$G(1-\omega, 1, 1)$	I
$G(0, 0, 0, \omega, 0, 0)$	E#1#5#2
$G(1, -\omega, 0, 0, 1, 1)$	E#2#5#1#31

(Recall from Chapter 4 the method for naming groups.)

Finally, C.R. Leedham-Green has shown that the infinite branches described in this chapter are all the possible infinite branches which arise from the groups H, I and E. This result stems from Leedham-Green's proof that 3-groups of second maximal nilpotency class have solubility length bounded by 4 and is still in preparation. It also depends on some results of S. McKay.

CHAPTER 6

SOME MORE INFINITE BRANCHES

This chapter deals with the infinite branches arising from the non *CF*-groups B, Q and U and the *CF*-groups A and G . Groups on these infinite branches can be constructed as pullbacks. These groups are said to be of *maximal type*. Groups on the infinite branches discussed in Chapter 5 can not be constructed as pullbacks. These groups are said to be of *non-maximal type*. This is the distinguishing feature between the infinite branches of Chapter 5 and the infinite branches to be discussed here.

The first section shows that there is an infinite branch arising from each of the groups B, Q, U, A and G . The second section examines in detail the infinite branch and its twigs arising from B . The third and final section looks at the ways in which 2-generator 3-groups of second maximal class can be constructed as pullbacks. The descendants of B are used in an example to illustrate this.

6.1. Five Infinite Branches

This section shows that there is an infinite branch arising from each of the groups B, Q, U, A and G . A detailed proof showing the existence of an infinite branch arising from U is given; the other proofs are similar and dealt with briefly.

The idea of the proof is as follows. The group U_∞ with presentation

$$\Big\langle a_1, a_2, a_3, a_4, a_5;\ a_1^3 = a_5^2,\ a_2^3 = a_3^{-3}a_4^{-1},\ a_5^3 = e,\ [a_2, a_1] = a_3,$$
$$[a_3, a_1] = a_4,\ [a_3, a_2] = a_5,\ [a_4, a_1] = a_3^{-3}a_4^{-3},\ [a_4, a_2] = [a_4, a_3] = e,$$
$$[a_5, a_1] = [a_5, a_2] = [a_5, a_3] = [a_5, a_4] = e\Big\rangle,$$

is shown to be an infinite group which has an infinite number of finite quotients each of which is a 3-group of second maximal nilpotency class and a descendant of U .

The motivation for considering the group U_∞ is as follows. Consider the group U_n with consistent power-commutator presentation

$$\Big\langle a_1, \ldots, a_n;\ a_1^3 = a_5^2,\ a_2^3 = a_4^2a_6,\ a_3^3 = a_6^2a_7,\ a_4^3 = a_7^2a_8,\ a_5^3 = e,\ a_6^3 = a_8^2a_9,$$
$$a_7^3 = a_9^2a_{10},\ \ldots,\ a_{n-3}^3 = a_{n-1}^2a_n,\ a_{n-2}^3 = a_n^2,\ [a_2, a_1] = a_3,\ [a_3, a_1] = a_4,$$
$$[a_3, a_2] = a_5,\ [a_4, a_1] = a_6,\ [a_6, a_1] = a_7,\ [a_7, a_1] = a_8,\ \ldots,$$
$$[a_{n-1}, a_1] = a_n\Big\rangle .$$

This group is a descendant of U . It is shown that U_n is on an infinite branch arising from U . This is done by exhibiting an infinite group which has U_n as a factor for all n greater than or equal to 6 . The following presentation clearly gives such a group providing it is infinite:

$$\Big\langle a_1, \ldots;\ a_1^3 = a_5^2,\ a_2^3 = a_4^2a_6,\ a_3^3 = a_6^2a_7,\ a_4^3 = a_7^2a_8,\ a_5^3 = e,\ a_k^3 = a_{k+2}^2a_{k+3}$$
$$\text{for } k \geq 6,$$
$$[a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_5,\ [a_4, a_1] = a_6,\ [a_k, a_1] = a_{k+1}$$
$$\text{for } k \geq 6,$$
$$\text{all other simple commutators are trivial}\Big\rangle .$$

However, for $k \geq 9$, $a_k = a_{k-3}^3a_{k-1}^{-2}$ and $a_8 = a_4^3a_7^{-2}$, $a_7 = a_3^3a_6^{-2}$, $a_6 = a_2^3a_4^{-2}$, $a_3^3a_4^3a_6 = e$. Using Tietze transformations the presentation for U_∞ above is arrived at.

Consider the group U_∞ with presentation as above. Simple calculations show that the terms in the lower-exponent-p-central series for U_∞ are as follows:

$$P_1(U_\infty) = \langle a_3, a_4, a_5\rangle ,$$

$$P_2(U_\infty) = \langle a_3^3, a_4, a_5\rangle ,$$

$$P_3(U_\infty) = \langle a_3^3, a_4^3\rangle ,$$

$$\left.\begin{array}{l} P_{2m}(U_\infty) = \left\langle a_3^{3^m}, a_4^{3^{(m-1)}}\right\rangle \\[2ex] P_{2m+1}(U_\infty) = \left\langle a_3^{3^m}, a_4^{3^m}\right\rangle \end{array}\right\}\ m \in \{2, 3, \ldots\} .$$

The group U_∞ can not be trivial since it has non-trivial quotients.

Clearly $P_c(U_\infty) > P_{c+1}(U_\infty)$ and hence U_∞ is infinite. By applying the Tietze transformations in reverse it can be shown that U_n is a factor of U_∞ for all n greater than or equal to 6 .

The existence of infinite branches arising from the other groups follows using a similar argument as above. The groups, with presentations listed below, are used:

$$B_\infty : \Big\langle a_1, a_2, a_3, a_4, a_5;\ a_1^3 = e,\ a_2^3 = a_3^{-3}a_4^{-1}a_5,\ a_5^3 = e,\ [a_2, a_1] = a_3,$$
$$[a_3, a_1] = a_4,\ [a_3, a_2] = a_5,\ [a_4, a_1] = a_3^{-3}a_4^{-3},\ [a_4, a_2] = [a_4, a_3] = e,$$
$$[a_5, a_1] = [a_5, a_2] = [a_5, a_3] = [a_5, a_4] = e\Big\rangle ,$$

$$Q_\infty : \Big\langle a_1, a_2, a_3, a_4, a_5;\ a_1^3 = e,\ a_2^3 = a_3^{-3}a_4^{-1},\ a_5^3 = e,\ [a_2, a_1] = a_3,$$
$$[a_3, a_1] = a_4,\ [a_3, a_2] = a_5,\ [a_4, a_1] = a_3^{-3}a_4^{-3},\ [a_4, a_2] = [a_4, a_3] = e,$$
$$[a_5, a_1] = [a_5, a_2] = [a_5, a_3] = [a_5, a_4] = e\Big\rangle ,$$

$$A_\infty : \Big\langle a_1, a_2, a_3, a_4;\ a_1^9 = a_3^{-9}a_4^{-3},\ a_2^3 = e,\ [a_1, a_2] = a_3,\ [a_3, a_2] = a_4,$$
$$[a_4, a_2] = a_3^{-3}a_4^{-3},\ [a_3, a_1] = e,\ [a_4, a_1] = [a_4, a_3] = e\Big\rangle ,$$

$$G_\infty : \Big\langle a_1, a_2, a_3, a_4;\ a_1^3 = a_3^3a_4,\ a_2^9 = e,\ [a_2, a_1] = a_3,\ [a_3, a_2] = a_4,$$
$$[a_4, a_2] = a_3^{-3}a_4^{-3},\ [a_3, a_1] = e,\ [a_4, a_1] = [a_4, a_3] = e\Big\rangle .$$

6.2. The Infinite Branch Arising from B

This section deals with the descendants of the group B . These groups have been calculated, using the machine implementation of the generating algorithm, up to order 3^{10} . The following calculations enable the tree of descendants of B to be described. The method employed to do this is a hand calculation of the generating algorithm for a general case.

Let B_n be the group with the following consistent power-commutator presentation:

$$\Big\langle a_1, \ldots, a_n;\ a_1^3 = e,\ a_2^3 = e,\ a_3^3 = a_6^2a_7,\ a_4^3 = a_7^2a_8,\ a_5^3 = a_7^2a_8,\ a_6^3 = a_8^2a_9,$$

$$a_7^3 = a_9^2a_{10}, \ldots, a_{n-3}^3 = a_{n-1}^2a_n,\ a_{n-2}^3 = a_n^2,\ a_{n-1}^3 = a_n^3 = e,$$

$$[a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_5,$$

$$[a_4, a_1] = [a_4, a_2] = [a_5, a_1] = [a_5, a_2] = a_6,$$

$$[a_6, a_1] = [a_6, a_2] = a_7,$$

$$[a_7, a_1] = [a_7, a_2] = a_8,$$

$$\vdots$$

$$[a_{n-1}, a_1] = [a_{n-1}, a_2] = a_n\Big\rangle .$$

It is assumed that n is greater than or equal to 7. The immediate descendants of B_n are now calculated. The 3-covering group B_n^* is now required. The following is a presentation for B_n^*, however it is not consistent:

$$\Big\langle a_1, \ldots, a_n, a_{1,3}, \ldots, a_{n,n-1};\ a_1^3 = a_{1,3}, a_2^3 = a_{2,3}, a_3^3 = a_6^2a_7a_{3,3},$$

$$a_4^3 = a_7^2a_8a_{3,4}, a_5^3 = a_7^2a_8a_{3,5},$$

$$a_6^3 = a_8^2a_9a_{3,6}, a_7^3 = a_9^2a_{10}a_{3,7}, \ldots, a_{n-3}^3 = a_{n-1}^2a_na_{3,n-3},\ a_{n-2}^3 = a_n^2a_{3,n-2},$$

$$a_{n-1}^3 = a_{3,n-1},\ a_n^3 = a_{3,n},$$

$$[a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_5,$$

$$[a_4, a_1] = a_6,\ [a_4, a_2] = a_6a_{4,2}, [a_4, a_3] = a_{4,3},$$

$$[a_5, a_1] = a_6a_{5,1}, [a_5, a_2] = a_6a_{5,2}, [a_5, a_3] = a_{5,3}, [a_5, a_4] = a_{5,4},$$

$$[a_j, a_1] = a_{j+1},\ [a_j, a_2] = a_{j+1}a_{j,2}, [a_j, a_i] = a_{j,i},$$

$$6 \le j \le n-1,\ 3 \le i < j,$$

$$\vdots$$

$$[a_n, a_1] = a_{n,1}, [a_n, a_2] = a_{n,2}, \ldots, [a_n, a_i] = a_{n,i}, 3 \le i \le n-1\Big\rangle .$$

Now the consistency checks are performed. All these checks have been performed; however only those which eliminate generators are shown here. The order in which the checks are performed is important to reduce work.

For $i \in \{7, \ldots, n-1\}$, collect $a_n a_{i-1} a_1$ in two different ways:

$$a_n(a_{i-1}a_1) = a_1 a_{i-1} a_i a_n a_{n,1} a_{n,i-1} a_{n,i} ,$$

$$(a_n a_{i-1})a_1 = a_1 a_{i-1} a_i a_n a_{n,1} a_{n,i-1} .$$

Thus $a_{n,i} = e$ for $i \in \{7, \ldots, n-1\}$. Similarly, by performing consistency checks on $a_n a_5 a_1$, $a_n a_3 a_2$, $a_n a_3 a_1$, $a_n a_2 a_1$ the generators $a_{n,6}$, $a_{n,5}$, $a_{n,4}$, $a_{n,3}$, respectively are found to be trivial.

Now consider $a_{n-i}^3 a_j$ for $j \in \{3, \ldots, n-i-1\}$:

$$(a_{n-i})^3 a_j = a_j a_{n-i+2}^2 a_{n-i+3} a_{n-i+2,j}^2 a_{n-i+3,j} a_{3,n-i} ,$$

$$a_{n-i}^2(a_{n-i}a_j) = a_j a_{n-i+2}^2 a_{n-i+3} a_{3,n-i} .$$

Thus $a_{n-i+2,j}^2 a_{n-i+3,j} = e$. This covers a large number of cases since $j \in \{3, \ldots, n-i+1\}$ and for each j , $i \in \{3, \ldots, \min[n-j+1, n-5]\}$. When $j = 3$, $i \in \{3, \ldots, n-5\}$ and more specifically when $j = 3$ and $i = 3$, $a_{n-1,3}^2 a_{n,3} = e$. However it has already been shown that $a_{n,3} = e$ thus $a_{n-1,3} = e$. When $j = 3$ and $i = 4$ it is shown that $a_{n-2,3} = e$. In this way the following are all shown to be trivial:

for $j \in \{3, \ldots, 6\}$, $a_{n-1,j}, \ldots, a_{7,j}$ are trivial;

for $j \in \{7, \ldots, n-2\}$, $a_{n-1,j}, \ldots, a_{j+1,j}$ are trivial.

Now consider $(a_{n-i})^3 a_1$ for $i \in \{4, \ldots, n-6\}$:

$$(a_{n-i})^3 a_1 = a_1 a_{n-i+2}^2 a_{n-i+3}^3 a_{n-i+4} a_{3,n-i} ,$$

$$a_{n-i}^2(a_{n-i}a_1) = a_1 a_{n-i+2}^2 a_{n-i+3}^3 a_{n-i+4} a_{3,n-i} a_{3,n-i+1} .$$

Thus $a_{3,n-i+1} = e$ for $i \in \{4, \ldots, n-6\}$. Collecting the following specific cases gives the following results:

$$a_{n-1}^3 a_1 \Rightarrow \quad a_{3,n} = e \ ,$$

$$a_{n-2}^3 a_1 \Rightarrow a_{3,n-1} = a_{n,1}^2 \ ,$$

$$a_{n-3}^3 a_1 \Rightarrow a_{3,n-2} = a_{n,1} \ ,$$

$$a_4^3 a_1 \Rightarrow \quad a_{3,6} = e \ ,$$

$$a_3^3 a_1 \Rightarrow \quad a_{3,4} = e \ ,$$

$$a_3^3 a_2 \Rightarrow \quad a_{3,5} = a_{6,2}^2 a_{7,2} \ .$$

Now consider $a_{n-i}^3 a_2$ for $i \in \{4, \ldots, n-6\}$:

$$\left(a_{n-i}\right)^3 a_2 = a_2 a_{n-i+2}^2 a_{n-i+3}^3 a_{n-i+4} a_{n-i+2,2}^2 a_{n-i+3,2} \ ,$$

$$a_{n-i}^2 \left(a_{n-i} a_2\right) = a_2 a_{n-i+2}^2 a_{n-i+3}^3 a_{n-i+4} \ .$$

Thus $a_{n-i+2,2}^2 a_{n-i+3,2} = e$ for $i \in \{4, \ldots, n-6\}$. Before making use of this it is necessary to consider the two special cases below:

$$a_{n-2}^3 a_2 \Rightarrow \quad a_{n,1} = a_{n,2} \ ,$$

$$a_{n-3}^3 a_2 \Rightarrow a_{n-1,2} = e \ .$$

Combining this with $a_{n-i+2,2}^2 a_{n-i+3,2} = e$ for $i \in \{4, \ldots, n-6\}$ shows that $a_{j,2} = e$ for $j \in \{8, \ldots, n-1\}$.

Finally consider the following cases:

$$a_4^3 a_2 \Rightarrow \qquad a_{7,2} = e \,,$$

$$a_2 a_1^3 \Rightarrow \qquad a_{3,3} a_{4,3} = e \,,$$

$$a_2^3 a_1 \Rightarrow a_{5,2} a_{5,3}^2 a_{6,3}^2 a_{6,5} a_{3,3} a_{6,2}^2 = e \,,$$

$$a_3 a_2 a_1 \Rightarrow \qquad a_{5,1} = a_{4,2} a_{6,2} \,,$$

$$a_4 a_2 a_1 \Rightarrow \qquad a_{4,3} = a_{6,2}^2 \,,$$

$$a_5 a_2 a_1 \Rightarrow \qquad a_{5,3} a_{6,2} a_{6,3}^2 = e \,,$$

$$a_6 a_2 a_1 \Rightarrow \qquad a_{6,3} = e \,,$$

$$a_4 a_3 a_1 \Rightarrow \qquad a_{6,4} = e \,,$$

$$a_5 a_3 a_1 \Rightarrow \qquad a_{5,4} = e \,,$$

$$a_6 a_3 a_2 \Rightarrow \qquad a_{6,5} = e \,.$$

All other consistency checks have the two normal words equal.

The remaining generators are relabelled as follows: $a_{n,1} = a_{n+1}$, $a_{6,2} = a_{n+2}$, $a_{4,2} = a_{n+3}$, $a_{1,3} = a_{n+4}$, $a_{2,3} = a_{n+5}$ and the following is a consistent power commutator presentation for B_n^* :

$$\Big\langle a_1, \ldots, a_{n+5};\ a_1^3 = a_{n+4},\ a_2^3 = a_{n+5},\ a_3^3 = a_6^2 a_7 a_{n+2},\ a_4^3 = a_7^2 a_8,$$

$$a_5^3 = a_7^2 a_8 a_{n+2}^2,\ a_6^3 = a_8^2 a_9,$$

$$a_7^3 = a_9^2 a_{10},\ \ldots,\ a_{n-2}^3 = a_n^2 a_{n+1},\ a_{n-1}^3 = a_{n+1}^2,$$

$$[a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_5,$$

$$[a_4, a_1] = a_6,\ [a_4, a_2] = a_6 a_{n+3},\ [a_4, a_3] = a_{n+2}^2,$$

$$[a_5, a_1] = a_6 a_{n+2} a_{n+3},\ [a_5, a_2] = a_6 a_{n+2}^2,\ [a_5, a_3] = a_{n+2}^2,$$

$$[a_6, a_1] = a_7,\ [a_6, a_2] = a_7 a_{n+2},$$

$$\vdots$$

$$[a_i, a_1] = [a_i, a_2] = a_{i+1},\ 7 \le i \le n\Big\rangle .$$

The rank of the 3-multiplicator of B_n is 5. The following is an adequate set of automorphisms for B_n:

$$\left\{ \beta_1 : \begin{array}{l} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 a_{n-1} \end{array}, \; \beta_2 : \begin{array}{l} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 a_{n-2} \end{array}, \; \ldots, \; \beta_{n-7} : \begin{array}{l} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 a_7 \end{array}, \right.$$

$$\beta_{n-6} : \begin{array}{l} a_1 \mapsto a_1 a_6 \\ a_2 \mapsto a_2 \end{array}, \; \beta_{n-5} : \begin{array}{l} a_1 \mapsto a_1 a_4 \\ a_2 \mapsto a_2 \end{array}, \; \beta_{n-4} : \begin{array}{l} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 a_4 \end{array}, \; \beta_{n-3} : \begin{array}{l} a_1 \mapsto a_1 a_5 \\ a_2 \mapsto a_2 \end{array},$$

$$\left. \beta_{n-2} : \begin{array}{l} a_1 \mapsto a_1^2 \\ a_2 \mapsto a_2^2 \end{array}, \; \beta_{n-1} : \begin{array}{l} a_1 \mapsto a_2^2 \\ a_2 \mapsto a_1^2 \end{array} \right\}.$$

The matrix M corresponding to each automorphism is calculated. For $\beta_1, \ldots, \beta_{n-5}$ the matrix $M = I_5$, the identity matrix. The other matrices are as follows:

$$M_{\beta^*_{n-4}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad M_{\beta^*_{n-3}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{\beta^*_{n-2}} = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad \text{where } x = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even,} \end{cases}$$

$$M_{\beta^*_{n-1}} = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \quad \text{where } x = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

When n is odd the allowable subgroups are divided into fifteen equivalence classes with the following representative subgroups:

1. $\langle a_{n+2}, a_{n+3}, a_{n+4}, a_{n+5} \rangle$,

2. $\left\langle a_{n+1} a_{n+2}^2, a_{n+3}, a_{n+4}, a_{n+5} \right\rangle$,

3. $\langle a_{n+1}a_{n+2}, a_{n+3}, a_{n+4}, a_{n+5}\rangle$,

4. $\left\langle a_{n+1}a_{n+3}^2, a_{n+2}, a_{n+4}, a_{n+5}\right\rangle$,

5. $\left\langle a_{n+1}a_{n+3}^2, a_{n+2}a_{n+3}, a_{n+4}, a_{n+5}\right\rangle$,

6. $\langle a_{n+1}a_{n+3}, a_{n+2}, a_{n+4}, a_{n+5}\rangle$,

7. $\left\langle a_{n+1}a_{n+4}^2, a_{n+2}, a_{n+3}, a_{n+5}\right\rangle$,

8. $\left\langle a_{n+1}a_{n+4}^2, a_{n+2}, a_{n+3}a_{n+4}^2, a_{n+5}\right\rangle$,

9. $\left\langle a_{n+1}a_{n+4}^2, a_{n+2}, a_{n+3}a_{n+4}, a_{n+5}\right\rangle$,

10. $\left\langle a_{n+1}a_{n+5}^2, a_{n+2}, a_{n+3}, a_{n+4}a_{n+5}^2\right\rangle$,

11. $\left\langle a_{n+1}a_{n+5}^2, a_{n+2}, a_{n+3}a_{n+5}^2, a_{n+4}a_{n+5}^2\right\rangle$,

12. $\left\langle a_{n+1}a_{n+5}^2, a_{n+2}, a_{n+3}a_{n+5}, a_{n+4}a_{n+5}^2\right\rangle$,

13. $\left\langle a_{n+1}a_{n+5}^2, a_{n+2}, a_{n+3}, a_{n+4}a_{n+5}\right\rangle$,

14. $\left\langle a_{n+1}a_{n+5}^2, a_{n+2}, a_{n+3}a_{n+5}^2, a_{n+4}a_{n+5}\right\rangle$,

15. $\left\langle a_{n+1}a_{n+5}^2, a_{n+2}, a_{n+3}a_{n+5}, a_{n+4}a_{n+5}\right\rangle$.

When n is even the allowable subgroups are divided into fourteen equivalence classes with the following representative subgroups:

1. $\langle a_{n+2}, a_{n+3}, a_{n+4}, a_{n+5}\rangle$,

2. $\left\langle a_{n+1}a_{n+2}^2, a_{n+3}, a_{n+4}, a_{n+5}\right\rangle$,

3. $\left\langle a_{n+1}a_{n+3}^2, a_{n+2}, a_{n+4}, a_{n+5}\right\rangle$,

4. $\left\langle a_{n+1}a_{n+3}^2, a_{n+2}a_{n+3}, a_{n+4}, a_{n+5}\right\rangle$,

5. $\left\langle a_{n+1}a_{n+4}^2, a_{n+2}, a_{n+3}, a_{n+5}\right\rangle$,

6. $\left\langle a_{n+1}a_{n+4}^2, a_{n+2}, a_{n+3}a_{n+4}^2, a_{n+5}\right\rangle$,

7. $\langle a_{n+1}a_{n+4}, a_{n+2}, a_{n+3}, a_{n+5}\rangle$,

8. $\langle a_{n+1}a_{n+4}, a_{n+2}, a_{n+3}a_{n+4}, a_{n+5}\rangle$,

9. $\left\langle a_{n+1}a_{n+5}^2, a_{n+2}, a_{n+3}, a_{n+4}a_{n+5}^2\right\rangle$,

10. $\left\langle a_{n+1}a_{n+5}^2, a_{n+2}, a_{n+3}a_{n+5}^2, a_{n+4}a_{n+5}^2\right\rangle$,

11. $\left\langle a_{n+1}a_{n+5}^2, a_{n+2}, a_{n+3}, a_{n+4}a_{n+5}\right\rangle$,

12. $\left\langle a_{n+1}a_{n+5}^2, a_{n+2}, a_{n+3}a_{n+5}^2, a_{n+4}a_{n+5}\right\rangle$,

13. $\langle a_{n+1}a_{n+5}, a_{n+2}, a_{n+3}, a_{n+4}a_{n+5}\rangle$,

14. $\langle a_{n+1}a_{n+5}, a_{n+2}, a_{n+3}a_{n+5}, a_{n+4}a_{n+5}\rangle$.

The representative subgroups are chosen using the selection procedure of the generating algorithm.

The above shows that when n is odd B_n has fifteen immediate descendants and when n is even B_n has fourteen immediate descendants. The machine implementation of the generating algorithm calculates adequate sets of automorphisms for all immediate descendants. Here, adequate sets of automorphisms are only calculated for those immediate descendants which are capable. To determine which immediate descendants are capable the 3-covering group of each is calculated. Instead of calculating this separately for each of the twenty-nine immediate descendants all the 3-covering groups can be calculated at once. This is done by calculating a presentation for the 3-covering group of the 3-covering group of B_n , that is B_n^{**} . The calculation of B_n^{**} is similar to the calculation of B_n^* . A power-commutator presentation for B_n^{**} is as follows:

$$\langle a_1, a_2, \ldots, a_{n+5}, a_{1,3}, a_{2,3}, a_{3,3}, a_{3,5}, a_{n+1,1}, a_{4,2}, a_{4,3}, a_{5,1}, a_{5,2}, a_{5,3}, a_{6,2};$$

$$a_1^3 = a_{n+4}a_{1,3}, a_2^3 = a_{n+5}a_{2,3}, a_3^3 = a_6^2 a_7 a_{n+2} a_{3,3}, a_4^3 = a_7^2 a_8,\ a_5^3 = a_7^2 a_8 a_{3,5},$$

$$a_6^3 = a_8^2 a_9,\ a_7^3 = a_9^2 a_{10},\ \ldots,\ a_{n-2}^3 = a_n^2 a_{n+1},\ a_{n-1}^3 = a_{n+1}^2 a_{n+1,1},$$

$$a_n^3 = a_{n+1,1}^2, a_{n+1}^3 = e,$$

$$[a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_5,$$

$$[a_4, a_1] = a_6,\ [a_4, a_2] = a_6 a_{n+3} a_{4,2}, [a_4, a_3] = a_{n+2}^2 a_{4,3},$$

$$[a_5, a_1] = a_6 a_{n+2} a_{n+3} a_{5,1}, [a_5, a_2] = a_6 a_{n+2}^2 a_{5,2}, [a_5, a_3] = a_{n+2}^2 a_{5,3},$$

$$[a_6, a_1] = a_7,\ [a_6, a_2] = a_7 a_{n+2} a_{6,2},$$

$$[a_7, a_1] = [a_7, a_2] = a_8,$$

$$[a_8, a_1] = [a_8, a_2] = a_9,$$

$$\vdots$$

$$[a_n, a_1] = [a_n, a_2] = a_{n+1},$$

$$[a_{n+1}, a_1] = [a_{n+1}, a_2] = a_{n+1,1}\rangle .$$

To obtain a presentation for the 3-covering group of an immediate descendant of B_n the orbit representative for that immediate descendant is factored from B_n^{**}. The following additional relations must also be used:

$$a_{n+2}a_{3,5} = a_{6,2}^2[a_{n+2}, a_2] ,$$

$$a_{3,3}a_{4,3}[a_{n+4}, a_2] = e ,$$

$$a_{4,3}a_{6,2} = [a_{n+3}, a_1] ,$$

$$[a_{n+5}, a_1] = a_{n+2}a_{3,3}a_{3,5}a_{5,3}^2 a_{5,2} ,$$

$$a_{5,3}a_{6,2}[a_{n+2}, a_2][a_{n+3}, a_2] = e ,$$

$$a_{5,1} = a_{4,2}a_{4,3}a_{5,3} ,$$

$$[a_{n+2}, a_1] = e .$$

In these relations $a_{n+2}, \ldots, a_{n+5}$ are either the identity or rewritten in terms of a_{n+1} as determined by the orbit representative.

Recall that a group P, of order p^n and class c, is capable if and only if $P_c(P^*)$ is non-trivial. Recall also Theorem 3.20 which shows that $P_c(P^*)$ is generated by $[a_n, a_1]$, $[a_n, a_2]$ and a_n^3. This theorem is used to determine which immediate descendants are terminal. Since the immediate descendants have order 3^{n+1} the elements to check in their 3-covering groups are $[a_{n+1}, a_1]$, $[a_{n+1}, a_2]$ and a_{n+1}^3. The presentation for B_n^{**} shows that in all these groups a_{n+1}^3 is trivial and $[a_{n+1}, a_1]$ equals $[a_{n+1}, a_2]$. Thus it is only necessary to check the element $[a_{n+1}, a_1]$. The results are summarized below.

FOR n ODD

Group 2 - a consistency check on $a_6a_2a_1$ shows that $[a_{n+1}, a_1] = e$.

Groups 3 and 5 - $a_{n+2} = a_{n+1}^2$, but $[a_{n+2}, a_1] = e$ and hence $[a_{n+1}, a_1]^2 = e$, that is

$$[a_{n+1}, a_1] = e .$$

Groups 7, 8, ..., 15 - either $a_{n+4} = a_{n+1}$ or $a_{n+4} = a_{n+1}^2$ and in both cases performing consistency checks on a_1^4 gives $[a_{n+1}, a_1] = e$.

FOR n EVEN

Group 2 - same as group 2 for n odd.

Group 4 - same as group 5 for n odd.

Groups 5, 6, ..., 14 - either $a_{n+4} = a_{n+1}$ or $a_{n+4} = a_{n+1}^2$ and in both cases performing consistency checks on a_1^4 gives $[a_{n+1}, a_1] = e$.

The groups not shown here are in fact capable. This is shown by exhibiting their immediate descendants.

When n is either odd or even $B_n\#1$ is isomorphic to B_{n+1}. This is clear since they have identical presentations and hence $B_n\#1$ is capable.

If β is an automorphism in the set of adequate automorphisms for B_n then β^* fixes $\langle a_{n+2}, a_{n+3}, a_{n+4}, a_{n+5}\rangle$. An examination of the matrices M shows this to be true. Thus, there is no need to calculate the immediate descendants of $B_n\#1$ as the calculations above have done this already.

Consider $B_n\#4$ when n is odd and $B_n\#3$ when n is even. Both these groups are capable and their 3-covering groups have the following consistent power-commutator presentation:

$$\Big\langle a_1, \ldots, a_{n+6};\ a_1^3 = a_{n+5},\ a_2^3 = a_{n+6},\ a_3^3 = a_6^2 a_7 a_{n+2}^2 a_{n+3},\ a_4^3 = a_7^2 a_8,$$
$$a_5^3 = a_7^2 a_8 a_{n+3}^2,\ a_6^3 = a_8^2 a_9,\ a_7^3 = a_9^2 a_{10},\ \ldots,\ a_{n-1}^3 = a_{n+1}^2 a_{n+2},\ a_n^3 = a_{n+2}^2,$$
$$[a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_5,$$
$$[a_4, a_1] = a_6,\ [a_4, a_2] = a_6 a_{n+1} a_{n+4},\ [a_4, a_3] = a_{n+2} a_{n+3}^2,$$
$$[a_5, a_1] = a_6 a_{n+1} a_{n+3} a_{n+4},\ [a_5, a_2] = a_6 a_{n+3}^2,\ [a_5, a_3] = a_{n+2}^2 a_{n+3}^2,$$
$$[a_6, a_1] = a_7,\ [a_6, a_2] = a_7 a_{n+3},$$
$$[a_7, a_1] = a_8,\ [a_7, a_2] = a_8,$$
$$\vdots$$
$$[a_{n+1}, a_1] = [a_{n+1}, a_2] = a_{n+2}\Big\rangle .$$

For n odd a stabilizer calculation shows that the following set is an adequate set of automorphisms for $B_n\#4$:

$$\left\{ \beta_1 : \begin{array}{l} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 a_n \end{array},\ \beta_2 : \begin{array}{l} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 a_{n-1} \end{array},\ \ldots,\ \beta_{n-6} : \begin{array}{l} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 a_7 \end{array},\right.$$
$$\beta_{n-5} : \begin{array}{l} a_1 \mapsto a_1 a_6 \\ a_2 \mapsto a_2 \end{array},\ \beta_{n-4} : \begin{array}{l} a_1 \mapsto a_1 a_4 \\ a_2 \mapsto a_2 \end{array},\ \beta_{n-3} : \begin{array}{l} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 a_4 \end{array},$$
$$\left. \beta_{n-2} : \begin{array}{l} a_1 \mapsto a_1 a_5 \\ a_2 \mapsto a_2 \end{array},\ \beta_{n-1} : \begin{array}{l} a_1 \mapsto a_1^2 \\ a_2 \mapsto a_2^2 \end{array},\ \beta_n : \begin{array}{l} a_1 \mapsto a_2^2 \\ a_2 \mapsto a_1^2 \end{array} \right\} .$$

For $\beta_1, \ldots, \beta_{n-4}$, the associated matrix M is I_5. Also

$$M_{\beta^*_{n-3}} = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 1&1&0&0&1 \end{pmatrix}, \quad M_{\beta^*_{n-2}} = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 1&2&0&1&0\\ 0&0&0&0&1 \end{pmatrix},$$

$$M_{\beta^*_{n-1}} = \begin{pmatrix} 2&0&0&0&0\\ 0&2&0&0&0\\ 2&2&1&0&0\\ 0&0&0&2&0\\ 0&0&0&0&2 \end{pmatrix}, \quad M_{\beta^*_{n}} = \begin{pmatrix} 1&0&0&0&0\\ 0&2&0&0&0\\ 1&0&2&0&0\\ 0&0&0&0&2\\ 0&0&0&2&0 \end{pmatrix}.$$

The allowable subgroups fall into eight equivalence classes with the following representative subgroups:

1. $\langle a_{n+3}, a_{n+4}, a_{n+5}, a_{n+6} \rangle$,

2. $\left\langle a_{n+2}a^2_{n+3}, a_{n+4}, a_{n+5}, a_{n+6} \right\rangle$,

3. $\langle a_{n+2}a_{n+3}, a_{n+4}, a_{n+5}, a_{n+6} \rangle$,

4. $\langle a_{n+2}a_{n+4}, a_{n+3}, a_{n+5}, a_{n+6} \rangle$,

5. $\left\langle a_{n+2}a^2_{n+5}, a_{n+3}a^2_{n+5}, a_{n+4}, a_{n+6} \right\rangle$,

6. $\left\langle a_{n+2}a^2_{n+5}, a_{n+3}a^2_{n+5}, a_{n+4}a^2_{n+5}, a_{n+6} \right\rangle$,

7. $\langle a_{n+2}a_{n+5}, a_{n+3}a_{n+5}, a_{n+4}, a_{n+6} \rangle$,

8. $\left\langle a_{n+2}a_{n+6}, a_{n+3}a_{n+6}, a_{n+4}a_{n+6}, a_{n+5}a^2_{n+6} \right\rangle$.

These groups B_n#4#1, ..., B_n#4#8 are all terminal. The proof is omitted but similar to that used for previously showing that groups are terminal.

For n even a stabilizer calculation shows that the following set is an adequate set of automorphisms for B_n#3 :

$$\left\{ \beta_1 : \begin{matrix} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 a_n \end{matrix},\ \beta_2 : \begin{matrix} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 a_{n-1} \end{matrix},\ \ldots,\ \beta_{n-6} : \begin{matrix} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 a_7 \end{matrix},\right.$$

$$\beta_{n-5} : \begin{matrix} a_1 \mapsto a_1 a_6 \\ a_2 \mapsto a_2 \end{matrix},\ \beta_{n-4} : \begin{matrix} a_1 \mapsto a_1 a_4 \\ a_2 \mapsto a_2 \end{matrix},\ \beta_{n-3} : \begin{matrix} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 a_4 \end{matrix},\ \beta_{n-2} : \begin{matrix} a_1 \mapsto a_1 a_5 \\ a_2 \mapsto a_2 \end{matrix},$$

$$\left. \beta_{n-1} : \begin{matrix} a_1 \mapsto a_2 \\ a_2 \mapsto a_1 \end{matrix} \right\} .$$

For $\beta_1, \ldots, \beta_{n-4}$, M equals I_5 :

$$M_{\beta^*_{n-3}} = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 1&1&0&0&1 \end{pmatrix} , \quad M_{\beta^*_{n-2}} = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 1&2&0&1&0 \\ 0&0&0&0&1 \end{pmatrix} , \quad M_{\beta^*_{n-1}} = \begin{pmatrix} 2&0&0&0&0 \\ 0&1&0&0&0 \\ 0&1&2&0&0 \\ 0&0&0&0&1 \\ 0&0&0&1&0 \end{pmatrix} .$$

The allowable subgroups fall into twelve equivalence classes with the following representative subgroups:

1. $\langle a_{n+3}, a_{n+4}, a_{n+5}, a_{n+6} \rangle$,
2. $\langle a_{n+2} a_{n+3}^2, a_{n+4}, a_{n+5}, a_{n+6} \rangle$,
3. $\langle a_{n+2} a_{n+3}, a_{n+4}, a_{n+5}, a_{n+6} \rangle$,
4. $\langle a_{n+2} a_{n+4}^2, a_{n+3}, a_{n+5}, a_{n+6} \rangle$,
5. $\langle a_{n+2} a_{n+4}^2, a_{n+3} a_{n+4}, a_{n+5}, a_{n+6} \rangle$,
6. $\langle a_{n+2} a_{n+4}, a_{n+3}, a_{n+5}, a_{n+6} \rangle$,
7. $\langle a_{n+2} a_{n+5}^2, a_{n+3} a_{n+5}^2, a_{n+4}, a_{n+6} \rangle$,
8. $\langle a_{n+2} a_{n+5}^2, a_{n+3} a_{n+5}^2, a_{n+4} a_{n+5}^2, a_{n+6} \rangle$,
9. $\langle a_{n+2} a_{n+5}^2, a_{n+3} a_{n+5}^2, a_{n+4} a_{n+5}, a_{n+6} \rangle$,
10. $\langle a_{n+2} a_{n+5}, a_{n+3} a_{n+5}, a_{n+4}, a_{n+6} \rangle$,
11. $\langle a_{n+2} a_{n+5}, a_{n+3} a_{n+5}, a_{n+4} a_{n+5}, a_{n+6} \rangle$,

12. $\langle a_{n+2}a_{n+5},\ a_{n+3}a_{n+5},\ a_{n+4}a_{n+5}^2,\ a_{n+6}\rangle$.

These groups $B_n\#3\#1, \ldots, B_n\#3\#12$ are all terminal.

When n is odd, $B_n\#6$ is capable and $(B_n\#6)^*$ has the following consistent power-commutator presentation:

$$\begin{aligned}
\langle a_1, \ldots, a_{n+6};\ & a_1^3 = a_{n+5},\ a_2^3 = a_{n+6},\ a_3^3 = a_6^2a_7a_{n+2}a_{n+3},\ a_4^3 = a_7^2a_8, \\
& a_5^3 = a_7^2a_8a_{n+3}^2,\ a_6^3 = a_8^2a_9,\ a_7^3 = a_9^2a_{10}, \\
& a_8^3 = a_{10}^2a_{11},\ \ldots,\ a_{n-1}^3 = a_{n+1}^2a_{n+2},\ a_n^3 = a_{n+2}^2, \\
& [a_2, a_1] = a_3,\ [a_3, a_1] = a_4,\ [a_3, a_2] = a_5, \\
& [a_4, a_1] = a_6,\ [a_4, a_2] = a_6a_{n+1}^2a_{n+4},\ [a_4, a_3] = a_{n+2}^2a_{n+3}^2, \\
& [a_5, a_1] = a_6a_{n+1}^2a_{n+3}a_{n+4},\ [a_5, a_2] = a_6a_{n+3}^2,\ [a_5, a_3] = a_{n+2}a_{n+3}^2, \\
& [a_6, a_1] = a_7,\ [a_6, a_2] = a_7a_{n+3}, \\
& [a_7, a_1] = a_8,\ [a_7, a_2] = a_8, \\
& \vdots \\
& [a_{n+1}, a_1] = a_{n+2},\ [a_{n+1}, a_2] = a_{n+2}\rangle .
\end{aligned}$$

A stabilizer calculation shows that the following set is an adequate set of automorphisms for $B_n\#6$:

$$\left\{\beta_1 : \begin{matrix} a_1 \mapsto a_1 \\ a_2 \mapsto a_2a_n \end{matrix},\ \beta_2 : \begin{matrix} a_1 \mapsto a_1 \\ a_2 \mapsto a_2a_{n-1} \end{matrix},\ \ldots,\ \beta_{n-6} : \begin{matrix} a_1 \mapsto a_1 \\ a_2 \mapsto a_2a_7 \end{matrix},\right.$$

$$\beta_{n-5} : \begin{matrix} a_1 \mapsto a_1a_6 \\ a_2 \mapsto a_2 \end{matrix},\ \beta_{n-4} : \begin{matrix} a_1 \mapsto a_1a_4 \\ a_2 \mapsto a_2 \end{matrix},\ \beta_{n-3} : \begin{matrix} a_1 \mapsto a_1 \\ a_2 \mapsto a_2a_4 \end{matrix},$$

$$\left.\beta_{n-2} : \begin{matrix} a_1 \mapsto a_1a_5 \\ a_2 \mapsto a_2 \end{matrix},\ \beta_{n-1} : \begin{matrix} a_1 \mapsto a_1^2 \\ a_2 \mapsto a_2^2 \end{matrix},\ \beta_n : \begin{matrix} a_1 \mapsto a_2^2 \\ a_2 \mapsto a_1^2 \end{matrix}\right\} .$$

For $\beta_1, \ldots, \beta_{n-4}$, M equals I_5 :

$$M_{\beta^*_{n-3}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad M_{\beta^*_{n-2}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

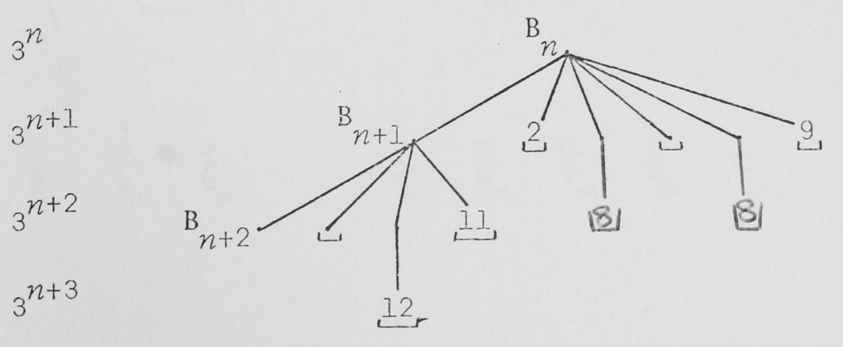
$$M_{\beta^*_{n-1}} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad M_{\beta^*_{n}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

The allowable subgroups fall into eight equivalence classes with the following representative subgroups:

1. $\langle a_{n+3}, a_{n+4}, a_{n+5}, a_{n+6} \rangle$,
2. $\langle a_{n+2}a_{n+3}^2, a_{n+4}, a_{n+5}, a_{n+6} \rangle$,
3. $\langle a_{n+2}a_{n+3}, a_{n+4}, a_{n+5}, a_{n+6} \rangle$,
4. $\langle a_{n+2}a_{n+4}^2, a_{n+3}, a_{n+5}, a_{n+6} \rangle$,
5. $\langle a_{n+2}a_{n+5}^2, a_{n+3}a_{n+5}, a_{n+4}, a_{n+6} \rangle$,
6. $\langle a_{n+2}a_{n+5}^2, a_{n+3}a_{n+5}, a_{n+4}a_{n+5}, a_{n+6} \rangle$,
7. $\langle a_{n+2}a_{n+5}, a_{n+3}a_{n+5}^2, a_{n+4}, a_{n+6} \rangle$,
8. $\langle a_{n+2}a_{n+5}, a_{n+3}a_{n+5}^2, a_{n+4}a_{n+5}^2, a_{n+6} \rangle$.

The groups $B_n\#6\#1, \ldots, B_n\#6\#8$ are all terminal.

Suppose n is odd then the following diagram sums up the above calculations:



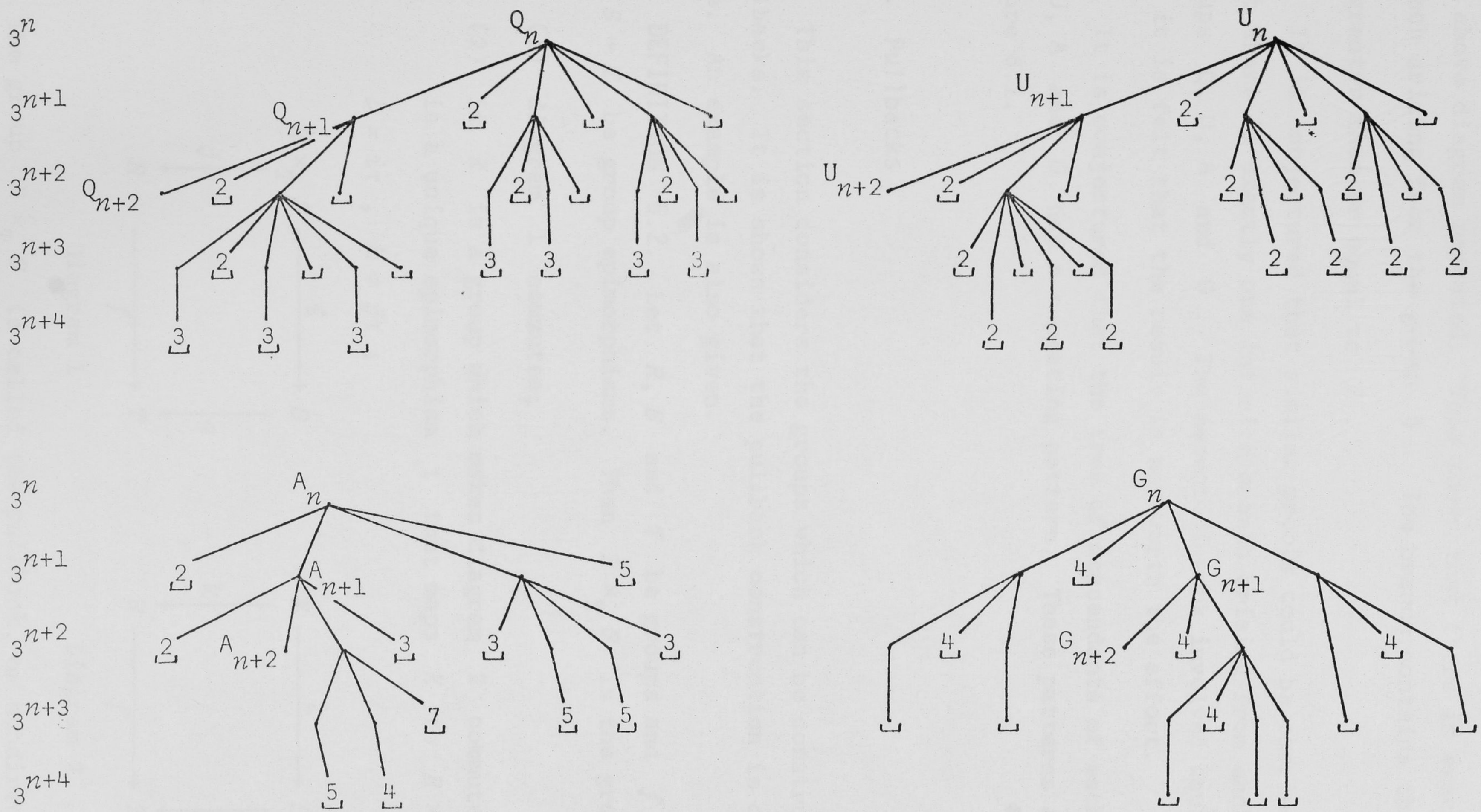


FIGURE 6.1 showing the conjectured patterns for the trees of descendants of Q, U, A and G . (Here n is even.)

The machine calculations show that for n equal to 7, 8 and 9 the calculations above are correct. Thus, an induction argument shows that for n greater than or equal to 7 the tree of descendants of B consists of the above diagram repeated. This shows that there is exactly one infinite branch arising from the group B . The branch contains the groups B_n for n greater than or equal to 7 .

It is conjectured that similar proofs could be carried through to show that there is exactly one infinite branch arising from each of the other groups Q, U, A and G . The amount of work involved would be considerable and it is felt that the result is not worth the effort.

It is conjectured that the tree of descendants of each of the groups Q, U, A and G has a repeating pattern. These patterns are shown in Figure 6.1.

6.3. Pullbacks

This section considers the groups which can be constructed as pullbacks. It is shown that the pullback construction is of a very special type. An example is also given.

DEFINITION 6.2. Let R, S and T be groups and $f : R \to T$, $g : S \to T$ be group epimorphisms. Then $R \times_T S$ is the group such that:

(1) diagram 1 commutes;

(2) if K is a group which makes diagram 2 commute then there is a unique epimorphism τ that maps K to $R \times_T S$ and $l = i\tau$, $k = j\tau$;

$$
\begin{tikzcd}
R \times_T S \arrow[r, "i"] \arrow[d, "j"'] & S \arrow[d, "g"] \\
R \arrow[r, "f"'] & T
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
K \arrow[r, "l"] \arrow[d, "k"'] & S \arrow[d, "g"] \\
R \arrow[r, "f"'] & T
\end{tikzcd} .
$$

Diagram 1 Diagram 2

The group $R \times_T S$ is called the *pullback*, or subdirect product, of R and S over T and

$$R \times_T S = \langle (r, s);\ r \in R,\ s \in S \text{ and } f(r) = g(s) \rangle .$$

LEMMA 6.3. *Suppose G is a group which contains two non-trivial normal subgroups L and M which intersect trivially, then G is isomorphic to $G/L \times_{G/LM} G/M$.*

Proof. ~~This proof assumes that G is a p-group which is always the case here.~~ Let h be a map between G and $G/L \times_{G/LM} G/M$ such that $h(g) = (gL, gM)$ for g in G. Now (gL, gM) is in $G/L \times_{G/LM} G/M$ since G/LM equals G/ML. It is clear that h is a homomorphism. Suppose that $h(g_1) = h(g_2)$ for g_1, g_2 in G. Then $(g_1L, g_1M) = (g_2L, g_2M)$ and so $g_2 = g_1 l = g_1 m$ for some l in L and some m in M. Thus $l = m$, but L and M intersect trivially and hence $l = m = e$. Thus $g_1 = g_2$ and the mapping h is one-to-one. Consider (g_1L, g_2M), an element of $G/L \times_{G/LM} G/M$. Since this group is a pullback $g_1LM = g_2LM$ and hence $g_2 = g_1 lm$ for some l in L and m in M. As cosets $g_1L = g_1 lL$ and $g_2M = g_1 lmM = g_1 lM$. Thus $h(g_1 l) = (g_1L, g_2M)$ and hence h is onto. Thus h is an isomorphism. $\square$

If G is a group which satisfies Lemma 6.3 then it is said that G can be *constructed as a pullback*.

THEOREM 6.4. *Suppose P is a 2-generator 3-group of order 3^n and second maximal class which can be constructed as a pullback. If P has second maximal nilpotency class then P can be constructed as a pullback of two groups with maximal nilpotency class, one of order 3^{n-1}, the other of order less than or equal to 3^4. If P does not have second maximal nilpotency class then the ways in which P can be constructed as a pullback are shown.*

Proof. Consider the upper and lower exponent-3-central series of P. Since P has second maximal class $Q_1(P)$ is a subgroup of $C_3 \times C_3 \times C_3$. Suppose $Q_1(P)$ equals $C_3 \times C_3 \times C_3$, then $Q_{i+1}(P)/Q_i(P)$ is isomorphic to C_3 for $i \in \{2, \ldots, n-3\}$ and so $P/Q_1(P)$ has co-class 0 and hence is cyclic. Thus P is generated by $\{x, Q_1(P)\}$ and so is abelian. Now if P is abelian and $Q_1(P)$ is $C_3 \times C_3 \times C_3$ then P is a 3-generator group. Thus $Q_1(P)$ is a subgroup of $C_3 \times C_3$. Since P contains at least two non-trivial normal subgroups which intersect trivially it follows that $\Omega_1(P)$ contains $C_3 \times C_3$. Thus $\Omega_1(P)$ equals $C_3 \times C_3$.

Choose L and M to be non-trivial normal subgroups which intersect trivially and choose them to be maximal with respect to this intersecting property. Since P contains the normal subgroup $P_{n-3}(P)$ it is possible to choose M such that $P_{n-3}(P)$ is a subgroup of M. Let L_1 and M_1 be subgroups of L and M respectively such that $\Omega_1(P)$ equals $L_1 \times M_1$. Thus M_1 never equals M since $P_{n-3}(P)$ is contained in $P_{n-4}(P)$. Thus, there are two possibilities:

(1) $L_1 = L$, $M_1 \neq M$;

(2) $L_1 \neq L$, $M_1 \neq M$.

Case 1. $L_1 = L$, $M_1 \neq M$. It has already been shown that $\Omega_1(P)$ equals $C_3 \times C_3$ and hence it has four subgroups. One of these is $P_{n-3}(P)$ and another is L.

Let l be an element of L; then l is not in $P_{n-3}(P)$. Suppose l is in $P_i(P)$ but not in $P_{i+1}(P)$ for i greater than or equal to 3.

Suppose P has second maximal nilpotency class; then $[P_i(P), P]$ equals $P_{i+1}(P)$ for i greater than or equal to three. Also $[L, P]$ equals E since $L \leq \Omega_1(P) \leq Z(P)$, the centre of P. Now $P_{i+1}(P) \leq P_i(P)$ and so $LP_{i+1}(P) \leq LP_i(P)$. However l is a generator of L and l is in $P_i(P)$ thus $LP_i(P)$ equals $P_i(P)$. Thus $LP_{i+1}(P) \leq P_i(P)$ but since l is not in $P_{i+1}(P)$ and has order 3, $LP_{i+1}(P) = P_i(P)$.

This means that

$$[P_i(P), P] = P_{i+1}(P) ;$$

but

$$[LP_{i+1}(P), P] = P_{i+2}(P) ,$$

which is a contradiction. Hence l is not in $P_i(P)$ for i greater than or equal to 3 and in particular l is not in $P_3(P)$.

If P is a non-CF-group then $P_2(P)/P_3(P)$ equals $C_3 \times C_3$ and so $P_2(P)$ has four subgroups containing $P_3(P)$. One of these subgroups must be M as it can not be any larger. One of the remaining three contains L . Thus P/M is a group of order 3^4 and maximal nilpotency class and P/L is a group of order 3^{n-1} and maximal nilpotency class.

If P is a CF-group then $[P_2(P), P]$ equals $P_3(P)$ and so by the argument above l is in L but not in $P_2(P)$. Also $P_1(P)/P_2(P)$ equals $C_3 \times C_3$ and so $P_1(P)$ has four subgroups which contain $P_2(P)$. One of these subgroups must be M and one of the remaining three contains L . Thus P/M is a group of order 3^3 and maximal nilpotency class and P/L is a group of order 3^{n-1} and maximal nilpotency class.

Case 2. $L_1 \neq L$, $M_1 \neq M$. Let L_2 and M_2 be subgroups of L and M respectively such that L_1 is in L_2 and M_1 is in M_2 . Then $Q_2(P)/Q_1(P)$ equals $C_3 \times C_3$ and $P/Q_2(P)$ is cyclic. An examination of 2-generator 3-groups of second maximal class but not second maximal nilpotency class shows that there are three types of groups in this case. For all three types the two generators of the group are a and b such that a has order 3^{n-2} and b has order 9 . The groups are as follows:

(1) $C_9 \times C_{3^{n-2}}$ where $L = \langle b \rangle$ and $M = \langle a \rangle$;

(2) $\langle a,b;\ a^{3^{n-2}} = e,\ b^9 = e,\ a^b = a^{1+3^{n-3}} \rangle$ where $L = \langle b^3 \rangle$ and $M = \langle a^3 \rangle$.

This group is an immediate descendant of $C_9 \times C_{3^{n-3}}$.

(3) $\langle a,\ b;\ a^{3^{n-2}} = e,\ b^9 = e,\ b^a = b^7 \rangle$ where $L = \langle b^3 \rangle$ and $M = \langle a^3 \rangle$.

This group is the capable group in branch C of Figure 4.5.

It must happen that L_2 equals L since if there were an L_3 and

M_3 containing L_2 and M_2 respectively then $\mathcal{Q}_3(P)/\mathcal{Q}_2(P)$ would equal $C_3 \times C_3$. This is not possible since P has second maximal class. □

Descendants of the group B are now examined to determine those that can be constructed as pullbacks. As before B_n denotes the group on the infinite branch of order 3^n. If n is odd then B_n#1, B_n#7, B_n#10, B_n#13 can be constructed as pullbacks. For the other groups $\mathcal{Q}_1$ is C_3 and so they can not be constructed as pullbacks.

If n is even then B_n#1, B_n#5, B_n#7, B_n#9, B_n#11, B_n#13 can be constructed as pullbacks. The other groups have $\mathcal{Q}_1$ equal to C_3. Notice that the capable immediate descendants of B_n can not be constructed as pullbacks; neither can any of their immediate descendants.

The above groups are constructed in the following way. In the notation of Definition 6.2 the group R is the capable maximal class group of order 3^4. Depending on whether n is odd or even the group S is one of the six or seven groups of maximal nilpotency class. These groups are numbered as in Chapter 4. The group S is the capable group of maximal nilpotency class of order 3^3. Different pullback groups can be obtained depending on the maps f and g. The results are summarized below.

For n odd: group S	resultant pullbacks	For n even: group S	resultant pullbacks
1	B_n#1	1	B_n#1
2	B_n#1, B_n#1, B_n#7	2	B_n#1, B_n#1, B_n#5, B_n#7
3	B_n#7, B_n#13	3	B_n#5, B_n#11
4	B_n#7, B_n#10	4	B_n#5, B_n#7, B_n#9
5	B_n#10	5	B_n#7, B_n#13
6	B_n#10, B_n#13, B_n#13	6	B_n#9
		7	B_n#9, B_n#11, B_n#11, B_n#13, B_n#13

The resultant pullback groups all have the following normal subgroup lattice shown in Figure 6.5.

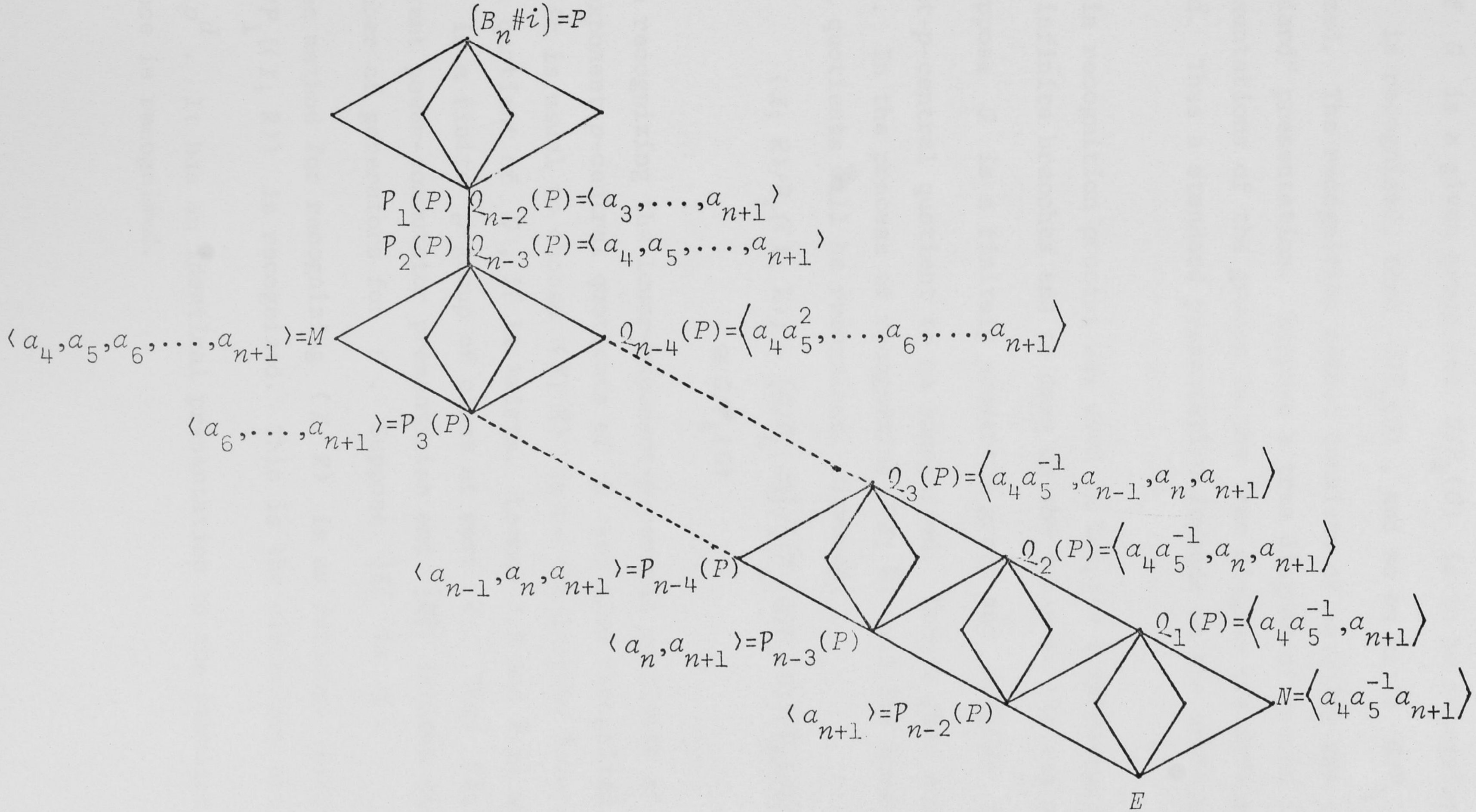


FIGURE 6.5 showing the normal subgroup lattice of pullback groups.

CHAPTER 7

A RECOGNITION PROCESS

This chapter describes a method for recognizing the lower-exponent-p-central quotients of a given group. The quotients are recognized in order. Thus, if G is a given group and $G/P_k(G)$ is to be recognized, first $G/P_1(G)$ is recognized, then $G/P_2(G)$, and so on until $G/P_k(G)$ is recognized. The recognition method consists of matching the quotients with a "standard" presentation. Suppose a tree diagram of groups is calculated. The presentations of the groups in the tree diagram are defined to be *standard*. Thus a standard presentation depends on the orbit representative chosen.

This recognition process was used in Chapter 5 to recognize the groups on the infinite branches and to draw up the table at the end of Chapter 5.

Suppose G is a finitely presented group and $G/P_k(G)$ is the lower-exponent-p-central quotient to be recognized. Denote this quotient $\langle X;\, R\rangle$. In the process of recognizing $\langle X;\, R\rangle$ all its lower-exponent-p-central quotients will be recognized. However,

$$\begin{aligned}\langle X;\, R\rangle/P_i(\langle X;\, R\rangle) &= \big(G/P_k(G)\big)/\big(\big(P_i(G)P_k(G)\big)/P_k(G)\big)\\ &\cong G/P_i(G)\end{aligned}$$

Thus in recognizing the lower-exponent-p-central quotients of $\langle X;\, R\rangle$ the lower-exponent-p-central quotients of G are also recognized. For this reason it is usual to choose $\langle X;\, R\rangle$ to be the largest lower-exponent-p-central quotient of G, if it exists. Lemmas 2.8 and 2.11 show that $\langle X;\, R\rangle$ is a finite p-group of class at most k. Thus $\langle X;\, R\rangle$ has a consistent power-commutator presentation and $|X|$ is less than or equal to the number of generators for G. Suppose $|X|$ is d.

The method for recognizing $\langle X;\, R\rangle$ is as follows. First $\langle X;\, R\rangle/P_1(\langle X;\, R\rangle)$ is recognized. This is the elementary abelian group of order p^d. It has an identical presentation to the standard presentation and hence is recognized.

Now suppose that $\langle X;\ R\rangle/P_1(\langle X;\ R\rangle),\ \ldots,\ \langle X;\ R\rangle/P_c(\langle X;\ R\rangle)$ have been recognized but $\langle X;\ R\rangle/P_{c+1}(\langle X;\ R\rangle)$ has not been recognized. In recognizing the first c quotients the presentation for $\langle X;\ R\rangle$ may have changed. Let X denote the group isomorphic to $\langle X;\ R\rangle$ whose presentation is such that $X/P_c(X)$ is standard. The method for recognizing $X/P_{c+1}(X)$ is as follows. Put $X/P_c(X) = P = \langle a_1,\ \ldots,\ a_n;\ \tilde{R}\rangle$. Let $P = F/R$ where F is a free group of rank d . Now P^* can be calculated and then M/R^* is calculated such that $P^*/(M/R^*) = X/P_{c+1}(X)$. If M/R^* happens to be the orbit representative then the presentation for $X/P_{c+1}(X)$ is standard.

Suppose however that M/R^* is not the orbit representative. Then the presentation for $X/P_{c+1}(X)$ is not standard. Suppose L/R^* is the orbit representative. Then there exists a β' such that $(M/R^*)\beta' = L/R^*$, where β' is a permutation of the set of allowable subgroups. An automorphism, β , of P which corresponds to β' can be calculated. Suppose $a_i\beta = u_i(a_1,\ \ldots,\ a_d)$ for $i \in \{1,\ \ldots,\ d\}$. Now M/R^* and L/R^* are generated by words in $a_{n+1},\ \ldots,\ a_{n+q}$ but these can be written in terms of $a_1,\ \ldots,\ a_d$. Suppose

$$M/R^* = \langle w_1(a_1,\ \ldots,\ a_d),\ \ldots,\ w_j(a_1,\ \ldots,\ a_d)\rangle$$

then

$$L/R^* = \langle w_1(u_1,\ \ldots,\ u_d),\ \ldots,\ w_j(u_1,\ \ldots,\ u_d)\rangle .$$

Suppose the non-standard presentation for $X/P_{c+1}(X)$ is $\langle a_1,\ \ldots,\ a_{n+1};\ R_a\rangle$. Consider the following presentation,

$$\langle b_1,\ \ldots,\ b_d,\ b_{d+1},\ \ldots,\ b_{d+n},\ b_{d+n+1};\ R_b,\ b_{d+i} = u_i(b_1,\ \ldots,\ b_d),\ i \in \{1,\ \ldots,\ d\},\ P_{c+1} = \emptyset\rangle , \quad (*)$$

where R_b represents the same relations as R_a only in $b_{d+1},\ \ldots,\ b_{d+n+1}$, and $P_{c+1} = \emptyset$ indicates that the group has class $c+1$. It is first shown that the group with presentation (*) is isomorphic to $X/P_{c+1}(X)$. Then it is shown that (*) is a standard presentation for $X/P_{c+1}(X)$.

Since β is an automorphism of F/R it follows that $F = \langle u_1, \ldots, u_d, R\rangle$. It can also be shown that $F = \langle u_1, \ldots, u_d, P_{c+1}(F)\rangle$. Thus

$$a_i = v_i(u_1, \ldots, u_d)x_i \text{ , where } x_i \in P_{c+1}(F) \text{ ,}$$
$$= v_i(u_1(a_1, \ldots, a_d), \ldots, u_d(a_1, \ldots, a_d))x_i \text{ for } i \in \{1, \ldots, d\} \text{ .}$$

However, $a_1, \ldots, a_d$ are generators of a free group and $b_1, \ldots, b_d$ are generators for another free group of rank d. Hence

$$b_i = v_i(u_1(b_1, \ldots, b_d), \ldots, u_d(b_1, \ldots, b_d))x_i \text{ for } i \in \{1, \ldots, d\} \text{ ,}$$

where x_i is also written in terms of $b_1, \ldots, b_d$. These relations for b_i are used to perform Tietze transformations on the presentation labelled (*). Since $P_{c+1} = \emptyset$ the x_i disappear. Since $u_i(b_1, \ldots, b_d) = b_{d+i}$ for $i \in \{1, \ldots, d\}$, the b_i are written in terms of $b_{d+1}, \ldots, b_{2d}$. Thus the presentation becomes $\langle b_{d+1}, \ldots, b_{d+n+1}; R_b, P_{c+1} = \emptyset\rangle$ which is clearly isomorphic to $X/P_{c+1}(X)$.

The subgroup corresponding to M/R^* in the group with presentation (*) is $\langle w_1(b_{d+1}, \ldots, b_{2d}), \ldots, w_j(b_{d+1}, \ldots, b_{2d})\rangle$. However, rewriting this in terms of $b_1, \ldots, b_d$ the subgroup becomes,

$$\langle w_1(u_1(b_1, \ldots, b_d), \ldots, u_d(b_1, \ldots, b_d)), \ldots, w_j(u_1(b_1, \ldots, b_d), \ldots, u_d(b_1, \ldots, b_d))\rangle \text{ .}$$

Thus when written in terms of $b_1, \ldots, b_d$ the subgroup corresponds to L/R^*. Thus, when (*) is written in terms of $b_1, \ldots, b_d$ it becomes a standard presentation for $X/P_{c+1}(X)$.

EXAMPLE. Consider the following group $B1$, given by a consistent power-commutator presentation,

$$\left\langle a_1, a_2, a_3, a_4, a_5;\ a_1^3 = a_4a_5,\ a_3^3 = a_5^2,\ [a_2, a_1] = a_3,\ [a_3, a_1] = a_5^2,\ [a_3, a_2] = a_4,\ [a_4, a_2] = a_5\right\rangle \text{ .}$$

Now $B1/P_1(B1) = C_3 \times C_3$ and this is standard. Also

$$B1/P_2(B1) = \langle a_1, a_2, a_3;\ [a_2, a_1] = a_3 \rangle$$

and this is standard. Now

$$B1/P_3(B1) = \left\langle a_1, a_2, a_3, a_4;\ a_1^3 = a_4,\ [a_2, a_1] = a_3,\ [a_3, a_2] = a_4 \right\rangle$$

and this is not standard - at least not on my particular tree diagram. Put $B1/P_2(B1) = P$ and then

$$P^* = \left\langle a_1, a_2, a_3, a_4, a_5, a_6, a_7;\ a_1^3 = a_6,\ a_2^3 = a_7,\ [a_2, a_1] = a_3, \right.$$
$$\left. [a_3, a_1] = a_4,\ [a_3, a_2] = a_5 \right\rangle$$

and $M/R^* = \left\langle a_4, a_5a_6^2, a_7 \right\rangle$. The orbit representative L/R^* is $\langle a_4a_7, a_5, a_6 \rangle$ and β is such that $a_1\beta = a_1(a_1a_2)^2 = u_1$, $a_2\beta = a_1a_2 = u_2$. Consider the presentation

$$B = \left\langle b_1, b_2, b_3, b_4, b_5, b_6, b_7;\ b_3 = b_1(b_1b_2)^2,\ b_4 = b_1b_2, \right.$$
$$b_3^3 = b_6b_7,\ b_5^3 = b_7^2,\ [b_4, b_3] = b_5,\ [b_5, b_3] = b_7^2,$$
$$\left. [b_5, b_4] = b_6,\ [b_6, b_4] = b_7 \right\rangle .$$

Now this presentation is rewritten as a power-commutator presentation in terms of b_1, b_2 and b_4 and b_5 are eliminated. This could be done by hand but in practice it is done by applying a machine implementation of the NQA.

The presentation obtained is

$$\left\langle b_1, \ldots, b_5;\ b_1^3 = b_5^2,\ b_2^3 = b_4^2b_5,\ b_3^3 = b_5^2, \right.$$
$$\left. [b_2, b_1] = b_3,\ [b_3, b_1] = b_4,\ [b_3, b_2] = b_5,\ [b_4, b_1] = b_5 \right\rangle .$$

With this presentation for $B1$ the lower-exponent-p-central quotients $B1/P_1(B1)$, $B1/P_2(B1)$, $B1/P_3(B1)$ and $B1/P_4(B1) = B1$ are all standard and hence the group $B1$ is recognized.

This example arose from a comparison of lists of groups compiled by Bender and James. It is in this type of exercise that the recognition process can be most useful.

To use the recognition process it is not necessary to have standard presentations already defined. The process can be used to determine whether two finite p-groups are isomorphic or not. Suppose G and H are finite p-groups with class c. The quotients $G/P_1(G), \ldots, G/P_c(G)$ are defined to be standard and the quotients $H/P_1(H), \ldots, H/P_c(H)$ are matched with them.

INDEX OF NOTATION

Aut G is the automorphism group of G

β is an automorphism of F/R

β^* is an automorphism of F/R^*

β' is a permutation of the allowable subgroups

$\langle a_1, \ldots, a_n \rangle$ indicates the group generated by $a_1, \ldots, a_n$, relations may or may not be given; also indicates the normal subgroup generated by $a_1, \ldots, a_n$.

$\langle a_1, \ldots, a_n, B \rangle$ indicates the group generated by $a_1, \ldots, a_n$ and all the elements of B

$[x, y] = x^{-1}y^{-1}xy$

$[G, H] = \langle [g, h];\ g \in G, h \in H \rangle$

$G^p = \langle g^p;\ g \in G \rangle$

e is the identity element

E is the trivial group

F is a free group of rank d

$\gamma_i(G)$ is the $(i-1)$th term of the lower-central series of G

I_n is the $n \times n$ identity matrix

Inn(G) is the group of inner automorphisms of G

$[x]$ is the integer part of x

M_{β^*} is the matrix corresponding to β^*

Out(G) = Aut(G)/Inn(G)

P is a d-generator, finite p-group of order p^n and class c

P^* is the p-covering group of P

$P_i(G)$ is the ith term in the lower-exponent-p-central series of G

$R \times_T S$ is the pullback of R and S over T

$\mathbb{Q}$ represents the rational numbers

$Q_i(G)$ is the ith term of the upper-exponent-p-central series of G

ω is a primitive cube root of unity

$w(a_1, \ldots, a_d)$ is a word in $a_1, \ldots, a_d$; u, v, x are also used for words

$\mathbb{Z}$ represents the integers

$Z(G)$ is the centre of G

INDEX OF DEFINITIONS

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