Asymptotic Enumeration of Bipartite Graphs, Tournaments, Digraphs and Eulerian Digraphs with Multiple Edges

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Statement

I hereby state that this thesis contains only my original research except where explicit reference has been made to the work of others.

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Abstract

Combinatorial enumeration has been a very active research field in combinatorics. Asymptotic methods were originally used in analysis and were applied to combinatorics as combinatorialists became more and more interested in the asymptotic behaviour of various combinatorial objects. Asymptotics in combinatorics have undergone major development during the past fifteen years.

In this thesis, we are concerned with the asymptotic enumeration of various labelled graphs.

Firstly, we determine the asymptotic number of labelled bipartite graphs with a given degree sequence for the case where the maximum degree is $o(|E(G)|^{1/3})$. In particular, if $k = o(n^{1/2})$, the number of regular bipartite graphs of degree k with each part having n vertices is asymptotically

$$\frac{(nk)!}{(k!)^{2n}} \exp\Big(-\frac{(k-1)^2}{2} - \frac{k^3}{6n} + O\big(\frac{k^2}{n}\big)\Big).$$

The previous best result required $k = o(n^{1/3})$. This problem is essentially that of the asymptotic enumeration of 0-1 matrices with prescribed row and column sums which has been drawing great attention since it was put forth in 1963.

Secondly, we consider the asymptotic number of tournaments with a given excess sequence. A *tournament* is a digraph in which, for each pair of distinct vertices v and w, either (v, w) or (w, v) is an edge, but not both. A tournament is *regular* if the in-degree is equal to the out-degree at each vertex. Let $v_1, v_2, \ldots, v_n$

be the vertices of a labelled tournament and let d_j^-, d_j^+ be the in-degree and out-degree of v_j for $1 \le j \le n$. Define $\delta_j = d_j^+ - d_j^-$ and call $\delta_1, \delta_2, \ldots, \delta_n$ the *excess sequence* of the tournament. We also define $d_1^+, d_2^+, \ldots, d_n^+$ to be the *score sequence*. Let $NT(n; \delta_1, \ldots, \delta_n)$ be the number of labelled tournaments with n vertices and excess sequence $\delta_1, \ldots, \delta_n$ and let $RT(n) = NT(n; 0, \ldots, 0)$ be the number of labelled regular tournaments with n vertices. The first attack that we are aware of on the asymptotics of regular tournaments was due to Spencer. In particular, Spencer evaluated $RT(n)$ to within a factor of $(1+o(1))^n$. Also, Spencer obtained

$$NT(n; \delta_1, \ldots, \delta_n) = RT(n) \exp\Big(\big(-\frac{1}{2} + o(1)\big) \frac{\sum_{j=1}^n \delta_j^2}{n}\Big).$$

Recently, McKay improved the result greatly and obtained that as $n \to \infty$, for any $\epsilon > 0$,

$$RT(n) = \Big(\frac{2^{n+1}}{\pi n}\Big)^{(n-1)/2} n^{1/2} e^{-1/2} \big(1 + O(n^{-1/2+\epsilon})\big) \qquad (n \quad odd).$$

We identify $NT(n; \delta_1, \ldots, \delta_n)$ as a coefficient in a n-variable power series, and estimate it by applying the saddle-point method to the integral provided by Cauchy's Theorem. Since the parameter which is tending to infinity is the number of dimensions, the application of the saddle-point method has an analytic flavour different from that of most fixed-dimensional problems. In this thesis, we find a much more accurate formula for the asymptotic number of labelled tournaments with a given excess sequence.

Thirdly, we use similar method to determine the asymptotic number of labeled digraphs with a given excess sequence. As far as we know, this is the first significant research on this problem.

Finally, we again employ the saddle-point method to obtain an asymptotic formula for the number of labelled eulerian digraphs with n vertices in which the multiplicity of each edge is bounded by a fixed integer.

Contents

Chapter 1

Introduction

Combinatorial enumeration has been being a very active research field in combinatorics. Many typical problems in this area are described by Goulden and Jackson [14]. Asymptotic methods were originally used in analysis. There is a long history in mathematics of analytic problems being solved by asymptotic methods. Detailed information on asymptotic methods can be found in de Bruijn [9].

The first survey article about asymptotics in combinatorics that we are aware of is Bender [3], in which quite a few applications of asymptotic methods to combinatorial problems were described. Some other asymptotic methods for combinatorics were also discussed in Canfield [10], and Meir and Moon [28, 29]. Asymptotics in combinatorics have undergone major development during the past fifteen years.

Harary and Palmer [15] included many early results on problems relating to enumeration of graphs, in which one chapter was about the asymptotic enumeration of graphs, but only some preliminary results were presented there.

We are concerned with the asymptotic enumeration of various labelled graphs by degree sequences. Much work has been done in this area. The first significant results on the number of labelled graphs by degree sequence were obtained by Read [32, 33]. The enumeration of labelled cubic graphs and labelled connected cubic graphs was done by Read [32]. Asymptotic formulae for these numbers can also be found in [32]. An asymptotic formula for the number of labelled graphs with arbitrary but bounded degrees was established by Bender and Canfield [4] and Wormald [38] independently. Bollobás [5] obtained an asymptotic formula for the number of regular graphs and general labelled graphs in which the degrees can increase with the number of vertices. This was improved by McKay [22], who obtained an asymptotic formula for the number of labelled graphs, provided each degree is $o(E^{1/4})$. Also, Wormald [38] found the asymptotic formulae for the numbers of the r-regular graphs for any fixed r and the labelled regular graphs with given fixed degree and girth. Asymptotic formulae for the numbers of labelled r-regular graphs which are k-connected or cyclically k-connected, for any fixed k and r can also be found in [38]. Wormald [38–40], Bollobás [6], Bollobás and McKay [7], McKay [16, 19, 23], Robinson and Wormald [34, 35] and Fenner and Frieze [13] obtained some properties of random locally restricted graphs. McKay [18] and McKay and Wormald [26] studied random locally restricted graphs by a method based on switching edges. By switching arguments, McKay and Wormald [24] improved McKay's result [22], establishing an asymptotic formula for the number of labelled graphs in which each degree is $o(E^{1/3})$. In particular, an asymptotic formula for the number of regular graphs of degree of k and order n was obtained if $k = o(n^{1/2})$. The previous best result obtained by McKay [22] required $k = o(n^{1/3})$. Most recently, McKay and Wormald [27] showed how to generate k-regular graphs on n vertices uniformly at random in expected time $O(nk^3)$, provided $k = O(n^{1/3})$, by using a modification of the switching argument.

A simpler proof of the formula given by McKay [22] for the asymptotic number of k-regular graphs for $k = o(n^{1/3})$ can also be found in [27]. By the saddle point method, McKay and Wormald [25] obtained an asymptotic formula for general labeled graphs in which each degree is approximately a constant times n, which included the degree sequences of almost all graphs. Asymptotic formulae for the numbers of k-regular graphs with high degree and the total number of regular graphs were also obtained. Also by the saddle point method, McKay [17] proved the asymptotic formulae for the numbers of regular tournaments, eulerian digraphs and eulerian oriented graphs. Some numerical results of the accurate numbers of labelled regular tournaments, eulerian digraphs, eulerian oriented graphs, regular graphs and regular bipartite graphs were obtained by McKay [20].

In this thesis, we obtain an asymptotic formula for the number of bipartite graphs by degree sequence by using the switching arguments. By the saddle point method, we obtain asymptotic formulae for tournaments, digraphs and eulerian digraphs with multiple edges.

For the terminology for graphs we follow Bondy and Murty [8] and for that in probability theory we follow Feller [12].

Chapter 2 discusses the asymptotic number of bipartite graphs by degree sequence. This is essentially the enumeration of $n \times m$ 0–1 matrices with prescribed row and column sums. As Ryser [36] pointed out, the number of such 0–1 matrices must be a very complicated function of the row and column sums. After twenty years, no explicit formula for the value has been obtained despite much effort.

Therefore, researchers turned to the asymptotic version of the problem. The first result of interest to us was that of Read [32], who obtained an asymptotic formula to a factor of $1+o(1)$ for the case where the row and column sums are all 3. The same result with fixed constant row and column sums was established by Everett and Stein [11] and extended to arbitrary but bounded row and column sums by Békéssy, Békéssy and Komlós [1], Bender [2] and Wormald [38]. The first attempt to allow the row and column sums to increase with $n+m$ was done by O'Neil [31], provided that the row and column sums are independently equal and have a certain upper bound. This was improved by Mineev and Pavlov [30] and Bollobás and McKay [7], who obtained an asymptotic formula to a factor of $1+O(n^{-3/4})$. Most recently, McKay [21] improved the previous result greatly. In particular, he got an asymptotic formula for the number of $n \times n$ 0–1 matrices with row and column sums k, provided $k = o(n^{1/3})$. Also, McKay [20] obtained some numerical results for the number of labelled regular bipartite graphs. In Chapter 2, we obtain an asymptotic formula for the number of labelled bipartite graphs with a given degree sequence for the case where the maximum degree is $o(|E(G)|^{1/3})$. In particular, if $k = o(n^{1/2})$, the number of regular bipartite graphs of degree k with each part having n vertices is asymptotically

$$\frac{(nk)!}{(k!)^{2n}} \exp\Big(-\frac{(k-1)^2}{2} - \frac{k^3}{6n} + O\big(\frac{k^2}{n}\big)\Big).$$

In Chapter 4, we obtain an asymptotic formula for the number of labelled tournaments. A *tournament* is a digraph in which, for each pair of distinct vertices v and w, either (v,w) or (w,v) is an edge, but not both. A tournament is *regular* if the in-degree is equal to the out-degree at each vertex. Let $v_1, v_2, \ldots, v_n$ be the vertices of a labelled tournament and let d_j^-, d_j^+ be the in-degree and out-degree of v_j for $1 \le j \le n$. Define $\delta_j = d_j^+ - d_j^-$ and call $\delta_1, \delta_2, \ldots, \delta_n$ the *excess sequence* of the tournament. We also define $d_1^+, d_2^+, \ldots, d_n^+$ to be the *score sequence.* Let $NT(n; \delta_1, \ldots, \delta_n)$ be the number of labelled tournaments with n vertices and excess sequence $\delta_1, \ldots, \delta_n$. As in McKay [17], let $RT(n) = NT(n; 0, \ldots, 0)$ be the number of labelled regular tournaments with n vertices.

The first attack that we are aware of on the asymptotics of the regular tournaments was due to Joel Spencer [37]. In particular, Spencer evaluated $RT(n)$ to within a factor of $(1 + o(1))^n$. Also, Spencer obtained

$$NT(n, \delta_1, \ldots, \delta_n) = RT(n) \exp\Big(\big(-\tfrac{1}{2} + o(1)\big) \frac{\sum_{j=1}^n d_j^2}{n}\Big).$$

Recently, B. D. McKay [17] obtained the following much more accurate estimate of $RT(n)$, as $n \to \infty$, for any $\epsilon > 0$,

$$RT(n) = \Big(\frac{2^{n+1}}{\pi n}\Big)^{(n-1)/2} n^{1/2} e^{-1/2} \big(1 + O(n^{-1/2+\epsilon})\big) \qquad (n \quad odd).$$

It is easy to see that $RT(n) = 0$ if n even. Exact values of $RT(n)$ for $n \le 21$ were also obtained by McKay [20].

We are concerned with the asymptotic value of $NT(n; \delta_1, \ldots, \delta_n)$. We identify the required quantity as a coefficient in a n-variable power series, and estimate it by applying the saddle-point method to the integral provided by Cauchy's Theorem. With the help of the asymptotic estimation of an integral obtained in Chapter 3, we obtain that for the case where $\delta = \max\{|\delta_1|, \ldots, |\delta_n|\} = o(n^{3/4})$, as $n \to \infty$,

$$NT(n; \delta_1, \ldots, \delta_n) = n^{1/2}\Big(\frac{2^{n+1}}{n\pi}\Big)^{(n-1)/2} \exp\Big(-\frac{1}{2} - \frac{1}{2n}\sum_{j=1}^{n}\delta_j^2 + \frac{1}{n^2}\sum_{j=1}^{n}\delta_j^2 - \frac{1}{12n^3}\sum_{j=1}^{n}\delta_j^4 - \frac{1}{4n^4}\Big(\sum_{j=1}^{n}\delta_j^2\Big)^2 - \frac{1}{30n^5}\sum_{j=1}^{n}\delta_j^6 - \frac{1}{6n^6}\Big(\sum_{j=1}^{n}\delta_j^3\Big)^2 - \frac{1}{2n^7}\Big(\sum_{j=1}^{n}\delta_j^2\Big)^3 + O\Big(\frac{\delta^4}{n^3} + n^{-1/4+\epsilon}\Big)\Big),$$

for any $\epsilon > 0$.

In Chapter 5, we estimate the asymptotic number of labelled digraphs. Let $NDG(n; \delta_1, \ldots, \delta_n)$ be the number of labelled simple loop-free digraphs with n vertices and excess sequence $\delta_1, \ldots, \delta_n$. Using a similar method to that of Chapter 4, we obtain that for the case where $\delta = \max\{|\delta_1|, \ldots, |\delta_n|\} = o(n^{3/4})$, as $n \to \infty$,

$$NDG(n; \delta_1, \ldots, \delta_n) = n^{1/2}\Big(\frac{4^n}{\pi n}\Big)^{(n-1)/2} \exp\Big(-\frac{1}{4} - \frac{1}{n}\sum_{j=1}^{n}\delta_j^2 + \frac{1}{n^2}\sum_{j=1}^{n}\delta_j^2 - \frac{1}{6n^3}\sum_{j=1}^{n}\delta_j^4 - \frac{1}{2n^4}\Big(\sum_{j=1}^{n}\delta_j^2\Big)^2 - \frac{1}{15n^5}\sum_{j=1}^{n}\delta_j^6 - \frac{1}{3n^6}\Big(\sum_{j=1}^{n}\delta_j^3\Big)^2 - \frac{1}{n^7}\Big(\sum_{j=1}^{n}\delta_j^2\Big)^3 + O\Big(\frac{\delta^4}{n^3} + n^{-1/4+\epsilon}\Big)\Big),$$

for any $\epsilon > 0$. As far as we know, this is the first significant research on this problem.

In Chapter 6, we evaluate the asymptotic number of eulerian digraphs with multiple edges. By an *eulerian* digraph we mean a digraph in which the in-degree is equal to the out-degree at each vertex. Let $EDME(n, t)$ be the number of labelled loop-free eulerian digraphs with n vertices in which the multiplicity of each edge is at most t. Allowing loops would multiply $EDME(n, t)$ by exactly $(t + 1)^n$, since loops do not affect the eulerian property. For the case where $t = 1$, McKay [17] obtained that as $n \to \infty$, for any $\epsilon > 0$,

$$EDME(n,1) = \Big(\frac{4^n}{\pi n}\Big)^{(n-1)/2} n^{1/2} e^{-1/4}\big(1 + O(n^{-1/2+\epsilon})\big).$$

Numerical results for the number of eulerian digraphs with up to 16 vertices can be found in [20].

As before, we will identify $EDME(n,t)$ as a coefficient in a n-variable power series, and estimate it by applying the saddle-point method to the integral provided by Cauchy's Theorem. The choice of contour is trivial but substantial work is required to demonstrate that the parts of contour where the integrand is small contribute negligibly to the result. We will show that as $n \to \infty$,

$$EDME(n,t) = \Big(\frac{3(t+1)^{2n}}{t(t+2)\pi n}\Big)^{(n-1)/2} n^{1/2} \exp\Big(-\frac{3(t^2+2t+2)}{20t(t+2)} + O(n^{-1/2+\epsilon})\Big)$$

for any $\epsilon > 0$.

Chapter 2

Asymptotic Enumeration of Bipartite Graphs by Degree Sequence

2.1. Introduction.

Let B be a bipartite graph with parts X and Y, where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$. We also assume that x_i has degree s_i for $1 \le i \le n$ and y_j has degree t_j for $1 \le j \le m$. For any integer $t \ge 0$, define $[x]_t = x(x-1)\cdots(x-t+1)$. Define $s_{\max} = \max_{i=1}^{n} s_i$, $t_{\max} = \max_{j=1}^{m} t_j$ and $k_{\max} = \max\{s_{\max}, t_{\max}\}$. For any $r \ge 0$, further define $S_r = \sum_{i=1}^{n} [s_i]_r$ and $T_r = \sum_{j=1}^{m} [t_j]_r$. For simplicity, write $S = S_1$ and $T = T_1$. It is obvious that $S = T$.

Let $\mathcal{B}(\mathbf{s}, \mathbf{t})$ be the set of all labelled simple bipartite graphs with parts X, Y and degree sequence $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ and $\mathbf{t} = (t_1, t_2, \ldots, t_m)$ respectively, where x_i has degree s_i and y_j has degree t_j for all i, j. Define $G(\mathbf{s}, \mathbf{t}) = |\mathcal{B}(\mathbf{s}, \mathbf{t})|$. The labellings we consider are those related by independent permutations of X and Y. We are concerned with the asymptotic value of $G(\mathbf{s}, \mathbf{t})$ as $n, m \to \infty$. We will determine the asymptotic value of $G(\mathbf{s}, \mathbf{t})$ when $s_{\max} = o(S^{1/3})$ and $t_{\max} = o(T^{1/3})$.

Let $\mathcal{U}_{n,m}(\mathbf{s}, \mathbf{t})$ denote the set of all the $n \times m$ 0-1 matrices whose i-th row sum is s_i for $1 \le i \le n$ and j-th column sum is t_j for $1 \le j \le m$. $|\mathcal{U}_{n,m}(\mathbf{s}, \mathbf{t})|$ denotes the cardinality of the set. As Ryser [36] pointed out the value of $|\mathcal{U}_{n,m}(\mathbf{s}, \mathbf{t})|$ must be a very complicated function of $s_1, \ldots, s_n, t_1, \ldots, t_m$. After twenty years, no explicit formula for the value has been obtained despite much effort. Therefore, researchers turned to the asymptotic enumeration of the problem. It is obvious that $|\mathcal{U}_{n,m}(\mathbf{s},\mathbf{t})| = G(\mathbf{s},\mathbf{t})$. Define $\alpha = S_2T_2/(2S^2)$ and define $Q(\mathbf{s},\mathbf{t})$ by

$$|\mathcal{U}_{n,m}(\mathbf{s},\mathbf{t})| = \frac{S!}{s_1!\,s_2!\cdots s_n!\,t_1!\,t_2!\cdots t_m!}Q(\mathbf{s},\mathbf{t}).$$

The first result of interest to us was that of Read [32], who proved that $Q(\mathbf{s},\mathbf{t}) = e^{-\alpha+o(1)}$ if $s_i = t_j = 3$ for all i and j. The same result with fixed constant s_i and constant t_j for $1 \le i \le n$ and $1 \le j \le m$ was established by Everett and Stein [11] and extended to arbitrary but bounded row and column sums by Békéssy, Békéssy and Komlós [1], Bender [2] and Wormald [38]. The first attempt to allow $k_{\max}$ to increase with $n+m$ was by O'Neil [31], who proved that $Q(\mathbf{s},\mathbf{t}) = e^{-\alpha}(1 + O(n^{-1+\delta}))$, provided that the row and column sums are independently equal and $k_{\max} \le (\log n)^{1/4-\epsilon}$. This was improved by Mineev and Pavlov [30], who obtained the following:

(1) If $n = m$ and $s_i = t_j = k$ for all i and j, then $Q(\mathbf{s},\mathbf{t}) = e^{-\alpha} + O(n^{-1+\gamma^2/2})$, provided $1 < k \le \gamma \log^{1/2} n$ and $0 < \gamma < 1$ (γ fixed).

(2) If $s_i = k$ for all i and $t_j = l$ for all j where $l > 1$, then $Q(\mathbf{s},\mathbf{t}) = e^{-\alpha} + O(\log^3 n/n^{1-\gamma})$, provided $1 < k \le (l-1)^{-1}\gamma \log n$ and $0 < \gamma < 1$ (γ fixed).

(3) $Q(\mathbf{s},\mathbf{t}) = e^{-\alpha} + O(n^{\gamma^4/4-1/2}\log^2 n)$ if $k_{\max} \le \gamma \log^{1/4} n$ and $0 < \gamma < (2/3)^{1/4}$ (γ fixed).

Bollobás and McKay [7] proved that $Q(\mathbf{s},\mathbf{t}) = e^{-\alpha}(1 + O(n^{-3/4}))$ if $s_i = t_j = k$ for all i, j and $k = O(\log^{1/3} n)$. Most recently, McKay [21] showed that

$$Q(\mathbf{s},\mathbf{t}) = \exp\Big(-\frac{S_2T_2}{2S^2} + O\big(\frac{\hat{\Delta}^2}{S}\big)\Big),$$

where $\hat{\Delta} = 3 + 4k_{\max}(k_{\max} - 1)$, provided that $k_{\max} \geq 1$ and $\hat{\Delta} \leq \epsilon S$, where $\epsilon < \frac{2}{3}$.

In this Chapter, we use almost the same method as in [21] to obtain an asymptotic formula for $G(\mathbf{s}, \mathbf{t})$ for the case $k_{\max} = o(S^{1/3})$.

2.2. The model.

Consider a set of $S+T$ *points* arranged in *cells* $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m$ of size $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_m$ respectively. Take a partition P (called a *pairing*) of the $S+T$ points into $(S+T)/2 = S = T$ parts (called *pairs*) of size 2 each with the form (x, y) where $x \in x_i$ and $y \in y_j$ for some i and some j. The *degree* of cell x_i is s_i and that of cell y_j is t_j.

Two pairs are *parallel* if they involve the same cells. The *multiplicity* of a pair is the number of pairs (include itself) parallel to it. A *single* pair is a pair of multiplicity one. A *double* pair is a set of two parallel pairs of multiplicity two, while a *triple* pair is a set of three parallel pairs of multiplicity three. If p is a point, then $v(p)$ is the cell containing that point.

The *bipartite multigraph* $B(P)$ *associated with* P has parts X and Y. The edges of $B(P)$ are in correspondence with the pairs of P; the pair (x, y) corresponds to an edge (x_i, y_j) if $x \in x_i$ and $y \in y_j$.

For d, $t \geq 0$, define $\mathcal{C}_{d,t} = \mathcal{C}_{d,t}(\mathbf{s}, \mathbf{t})$ to be the set of all pairings with degrees $\mathbf{s}$ and $\mathbf{t}$, and exactly d double pairs and t triple pairs, but no pairs of multiplicity greater than three.

We will make use of the following two operations on pairs.

I d-switching:

Take a double pair $\{\{p_1, p_1'\}, \{p_2, p_2'\}\}$ and two single pairs $\{p_3, p_3'\}$ and $\{p_4, p_4'\}$, where $\{v(p_1), \ldots, v(p_4)\} \in X$ and $\{v(p_1'), \ldots, v(p_4')\} \in Y$, such that six distinct cells are involved. Replace these four pairs by $\{p_1, p_3'\}$, $\{p_2, p_4'\}$, $\{p_1', p_3\}$ and $\{p_2', p_4\}$. Other than the original double pair, all of the pairs created or destroyed must be single. (See Figure 1.)

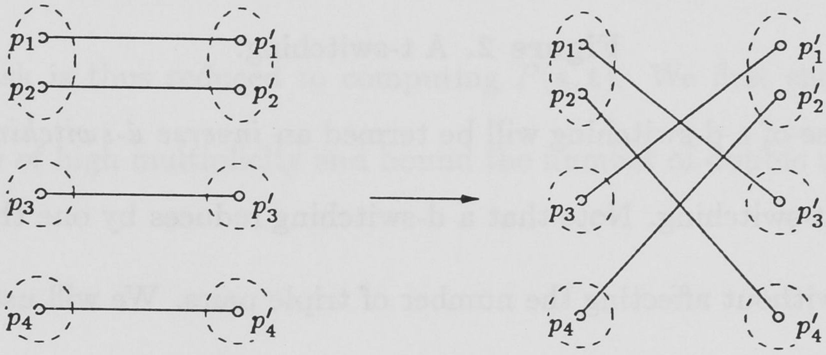

Figure 1. A d-switching.

II t-switching:

Take a triple pair $\{\{p_1, p_1'\}, \{p_2, p_2'\}, \{p_3, p_3'\}\}$ and three single pairs $\{p_4, p_4'\}$, $\{p_5, p_5'\}$ and $\{p_6, p_6'\}$, where $\{v(p_1), \ldots, v(p_6)\} \in X$ and $\{v(p_1'), \ldots, v(p_6')\} \in Y$, such that eight distinct cells are involved. Replace these six pairs by $\{p_1, p_4'\}$, $\{p_2, p_5'\}$, $\{p_3, p_6'\}$, $\{p_1', p_4\}$, $\{p_2', p_5\}$, and $\{p_3', p_6\}$. Other than the original triple pair, all of the pairs created or destroyed must be single. (See Figure 2.)

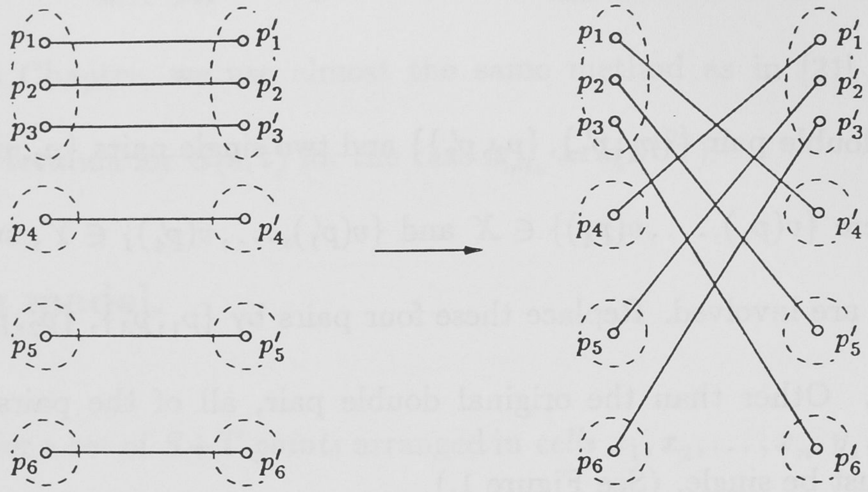

Figure 2. A t-switching.

The inverse of a d-switching will be termed an *inverse d-switching*, and similarly with the t-switching. Note that a d-switching reduces by one the number of double pairs, without affecting the number of triple pairs. We will use this fact to estimate the relative cardinalities of $\mathcal{C}_{d,t}$ and $\mathcal{C}_{d-1,t}$. The number of triple pairs is similarly affected by t-switching.

2.3. Preliminary results.

Let P be a random pairing with degrees $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_m$, where $1 \le k_{\max} = o(S^{1/3})$. We will begin with some elementary bounds on the probability that P has certain substructures.

Lemma 2.3.1. *The probability of r given pairs occuring in P is*

$$\frac{1}{[S]_r} \le (S-r)^{-r}.$$

Proof. The total number of pairings is $S!$ and the number of pairings containing the r given pairs is $(S-r)!$. ∎

Define $P(\mathbf{s},\mathbf{t})$ to be the probability that P contains no pairs of multiplicity greater than one. Since each separately labelled bipartite graph of degree $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_m$ corresponds to exactly $s_1!\,s_2!\cdots s_n!\,t_1!\,t_2!\cdots t_m!$ pairings, we have

$$G(\mathbf{s},\mathbf{t}) = \frac{S!}{s_1!\,s_2!\cdots s_n!\,t_1!\,t_2!\cdots t_m!} P(\mathbf{s},\mathbf{t}). \tag{1}$$

Our task is thus reduced to computing $P(\mathbf{s},\mathbf{t})$. We first show that we can ignore pairs of high multiplicity and bound the number of double and triple pairs. Define

$$N_2 = \max\big(\lceil \log(ST)\rceil, \lceil 28S_2T_2/S^2\rceil\big)$$

and

$$N_3 = \max\big(\lceil \log(ST)\rceil, \lceil 28S_3T_3/S^3\rceil\big).$$

In the following lemma, and for the remainder of the paper, the notations "$O(\,)$" and "$o(\,)$" refer to the passage of S to infinity within the constraint that $k_{\max} = o(S^{1/3})$. The implied constants will be uniform over all free variables unless otherwise stated.

Lemma 2.3.2.

$$\frac{1}{P(\mathbf{s},\mathbf{t})} = \big(1 + O(\frac{k_{\max}^3}{S})\big) \sum_{d=0}^{N_2} \sum_{t=0}^{N_3} \frac{|\mathcal{C}_{d,t}|}{|\mathcal{C}_{0,0}|}.$$

Proof. (1) Denote P_1 to be the probability that P contains a pair of multiplicity greater than three. By Lemma 2.3.1, we have

$$P_1 \leq 24(S-4)^{-4} \sum_{i=1}^{n} \sum_{j=1}^{m} \binom{s_i}{4}\binom{t_j}{4}$$

$$= 24(S-4)^{-4} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{s_i(s_i-1)(s_i-2)(s_i-3)}{4!} \frac{t_j(t_j-1)(t_j-2)(t_j-3)}{4!}$$

$$\leq 24(S-4)^{-4} s_{\max}^3 t_{\max}^3 \sum_{i=1}^{n} \frac{s_i}{4!} \sum_{j=1}^{m} \frac{t_j}{4!}$$

$$= O\left(\frac{s_{\max}^3 t_{\max}^3}{ST}\right).$$

(2) Denote P_2 to be the probability that there are more than N_2 double pairs in P. Let $d = N_2 + 1$.

(a) Let R_d be the number of possible sets of d doubles in P. We have

$$R_d \leq \binom{S_2T_2/2}{d}$$

$$\leq \frac{S_2^d T_2^d}{2^d d!}.$$

By Stirling's formula,

$$n! \geq \left(\frac{n}{e}\right)^n,$$

we have

$$R_d \leq \left(\frac{eS_2T_2}{2d}\right)^d.$$

(b) Let $P(d)$ be the probability that each of a set of d doubles occurs. Since $\log(ST) = 2\log S = o(S^{2/3})$, $S_2^2/S^2 \leq k_{\max}^2 S^2/S^2 = o(S^{2/3})$ and similarly $S_2T_2/S^2 = o(S^{2/3})$, we have $d = o(S^{2/3})$. So by Lemma 2.3.1, we have

$$P(d) \leq (S-2d)^{-2d}$$

$$= S^{-2d}(1-2d/S)^{-2d}$$

$$= O(1)S^{-2d}\left(\exp((2d)/S)\right)^{2d}.$$

(c) Define $P(a)$ to be the probability of all the d doubles occur. Then, we have

$$\begin{aligned}
P(a) &= O\Big(\Big(\frac{eS_2T_2}{2dS^2}\Big)^d\Big(\exp((2d)/S)\Big)^{2d}\Big) \\
&= O\Big(\Big(\frac{eS_2T_2}{2dS^2}\Big)^d e^{2d}\Big) \qquad (since \quad 2d \le S) \\
&= O\Big(\Big(\frac{e^3S_2T_2}{2dS^2}\Big)^d\Big) \\
&= O\Big(\big(e^3/56\big)^d\Big) \qquad (since \quad d \ge \frac{28S_2T_2}{S^2}) \\
&= O\Big(\frac{1}{e^d}\Big) \qquad (since \quad e^4 < 56) \\
&= O(\frac{1}{ST}) \qquad (since \quad d \ge \log(ST)).
\end{aligned}$$

Therefore, $P_2 = O(1/(ST))$.

(3) Similarly, we have that the probability that there are more than N_3 triple pairs in P is $P_3 = O(1/(ST))$.

(4) Let A be the set of all the pairings and B be the set of all the pairings other than those in any $\mathcal{C}_{d,t}$ for $0 \le d \le N_2$ and $0 \le t \le N_3$. Then we have

$$P(\mathbf{s},\mathbf{t}) = \frac{|\mathcal{C}_{0,0}|}{|A|}.$$

Hence,

$$\begin{aligned}
\frac{1}{P(\mathbf{s},\mathbf{t})} &= \frac{|A|-|B|}{|\mathcal{C}_{0,0}|}\Big(\frac{|A|}{|A|-|B|}\Big) \\
&= \sum_{d=0}^{N_2}\sum_{t=0}^{N_3}\frac{|\mathcal{C}_{d,t}|}{|\mathcal{C}_{0,0}|}\Big(1+\frac{|B|\,/\,|A|}{1-|B|\,/\,|A|}\Big). \qquad (2)
\end{aligned}$$

Since

$$\begin{aligned}\frac{|B|}{|A|} &= \frac{|B|}{\sum_{d=0}^{N_2}\sum_{t=0}^{N_3}|\mathcal{C}_{d,t}|+|B|}\\ &\le P_1+P_2+P_3\\ &\le O\Big(\frac{s_{\max}^3 t_{\max}^3}{ST}\Big)+O\Big(\frac{1}{ST}\Big)+O\Big(\frac{1}{ST}\Big)\\ &= O\Big(\frac{s_{\max}^3 t_{\max}^3}{ST}\Big),\end{aligned}$$

the lemma follows by equation (2). ∎

We will estimate $|\mathcal{C}_{d,t}| \,/\, |\mathcal{C}_{0,0}|$ via estimates on the terms of the expansion

$$\frac{|\mathcal{C}_{d,t}|}{|\mathcal{C}_{0,0}|} = \frac{|\mathcal{C}_{d,t}|}{|\mathcal{C}_{d,t-1}|}\cdots\frac{|\mathcal{C}_{d,1}|}{|\mathcal{C}_{d,0}|}\frac{|\mathcal{C}_{d,0}|}{|\mathcal{C}_{d-1,0}|}\cdots\frac{|\mathcal{C}_{1,0}|}{|\mathcal{C}_{0,0}|}.$$

Each of these terms can be estimated by means of one of the switchings.

If K is a bipartite multigraph, let $e(K)$ denote its number of edges (counting multiplicities). If xx' is an edge of K, then $\mu_K(xx')$ denotes its multiplicity, i.e., the number of edges parallel to xx' including itself. If K and K' are bipartite multigraphs with the same vertex set, then $K+K'$ is the bipartite multigraph with the same vertex set such that $\mu_{K+K'}(xx') = \mu_K(xx') + \mu_{K'}(xx')$ for all $\{x, x'\}$. Similarly, $2K$ means $K+K$ and $K+xx'$ is K with the multiplicity of xx' increased by one.

Let L be a simple bipartite graph with parts X and Y, and let H be a bipartite multigraph on the same vertex set with the restriction that if any edge xx' has $\mu_H(xx') \ge 1$, then xx' is an edge of L. Let $l_{\max}$ denote the maximum degree of L. Define $\mathcal{C}(L,H) = \mathcal{C}(L,H;\mathbf{s},\mathbf{t})$ to be the set of all pairings P with degrees $\mathbf{s}$, $\mathbf{t}$ such that the following are true for all $\{x, x'\}$:

(a) If xx' is an edge of L, then $\mu_{B(P)}(xx') = \mu_H(xx')$.

(b) If xx' is not an edge of L, then $\mu_{B(P)}(xx') \leq 1$.

In other words, $B(P)$ must be simple outside L and match H inside L.

From now on, let $k_i = s_i$ for $1 \leq i \leq n$ and $k_{n+j} = t_j$ for $1 \leq j \leq m$.

Lemma 2.3.3. *Suppose that L is as defined above, and that H and $H + J$ satisfy the requirements given above for H. Let h_i be the degree of x_i and h_{n+j} be the degree of y_j of H for $1 \leq i \leq n$ and $1 \leq j \leq m$. $j_1, j_2, \ldots, j_{n+m}$ are similarly defined for J. Then, if $k_{\max}(k_{\max} + l_{\max})e(J) = o(S)$, $e(H) = o(S)$, and $\mathcal{C}(L, H) \neq \emptyset$, we have*

$$\frac{|\mathcal{C}(L, H + J)|}{|\mathcal{C}(L, H)|} = \frac{\prod_{i=1}^{n+m}[k_i - h_i]_{j_i}}{[S - e(H)]_{e(J)} \prod_{\{x,x'\}}[\mu_{H+J}(xx')]_{\mu_J(xx')}} \times \left(1 + O\left(\frac{k_{\max}(k_{\max} + l_{\max})e(J)}{S}\right)\right).$$

Proof. This is a special case of the combination of Theorems 3.4 and 3.8 of [21].

∎

We will use Lemma 2.3.3 to analyse the structure of $\mathcal{C}_{d,0}$. For a pairing $P \in \mathcal{C}_{d,0}$, let $D(P)$ be the simple bipartite graph with parts X and Y and just those edges which correspond in position to the d double pairs of P.

Lemma 2.3.4. *Let $D = D(P)$ for some $P \in \mathcal{C}_{d,0}$, where $0 \leq d \leq N_2$. Let A be a simple bipartite graph with parts X and Y which is edge-disjoint from D. Let d_i be the degree of x_i and d_{n+j} be the degree of y_j of D for $1 \leq i \leq n$ and $1 \leq j \leq m$. $a_1, a_2, \ldots, a_{n+m}$ are similarly defined for A. Suppose that $e(A)k_{\max}^2 = o(S)$. Then the probability that $A \subseteq B(P)$ when P is chosen at random from those $P \in \mathcal{C}_{d,0}$*

such that $D(P) = D$ *is*

$$\frac{\prod_{i=1}^{n+m}[k_i - 2d_i]_{a_i}}{[S]_{e(A)}} \exp\Bigl(O\bigl(\frac{e(A)(k_{\max}^2 + d)}{S}\bigr)\Bigr).$$

Proof. The lemma is trivially true if $a_i > k_i - 2d_i$ for any i, so suppose that this is not the case. Define the bipartite graph L which has the edges of D and A. Then, for any $J \subseteq A$, Lemma 2.3.3 tells us that

$$\frac{|\mathcal{C}(L, 2D + J)|}{|\mathcal{C}(L, 2D)|} = f(J),$$

where

$$f(J) = \frac{\prod_{i=1}^{n+m}[k_i - 2d_i]_{j_i}}{[S - 2d]_{e(J)}} \exp\Bigl(O\bigl(\frac{k_{\max}^2 e(J)}{S}\bigr)\Bigr),$$

and $j_1, j_2, \ldots, j_{n+m}$ are the degrees of J. Now, the required probability can be written as

$$\frac{f(A)}{\sum_{J \subseteq A} f(J)},$$

and since the denominator is $1 + O\bigl(e(A)k_{\max}^2/S\bigr)$, the lemma follows. ∎

In the following, we will find it convenient to write k_v in place of s_i if $v = x_i$ $(1 \leq i \leq n)$ or t_j if $v = y_j$ $(1 \leq j \leq m)$.

Lemma 2.3.5. *Suppose that* $0 \leq d \leq N_2$ *and* $S_2, T_2 \geq S$. *Choose* $v \in \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$ *and* $r \geq 0$. *Then, if* P *is chosen at random from* $\mathcal{C}_{d,0}$, *cell* v *is incident with exactly* r *double pairs with probability* $Q_v(r)/\sum_{i=0}^{\lfloor k_v/2 \rfloor} Q_v(i)$, *where*

$$Q_v(i) = \begin{cases} \dfrac{[d]_i[k_v]_{2i}}{i!\,S_2^i} \exp\Bigl(O\bigl(\dfrac{ik_{\max}^2}{S} + \dfrac{i^2k_{\max}^2 + idk_{\max}}{S_2} + \dfrac{i^2k_{\max}^2 + idk_{\max}}{T_2}\bigr)\Bigr), & \text{if } v \in X; \\ \dfrac{[d]_i[k_v]_{2i}}{i!\,T_2^i} \exp\Bigl(O\bigl(\dfrac{ik_{\max}^2}{S} + \dfrac{i^2k_{\max}^2 + idk_{\max}}{S_2} + \dfrac{i^2k_{\max}^2 + idk_{\max}}{T_2}\bigr)\Bigr), & \text{if } v \in Y. \end{cases}$$

Proof. Firstly, we assume that $v \in X$. Suppose that $D = D(P)$ for some $P \in \mathcal{C}_{d,0}$, and let w be a neighbour of v in D. Let $x \in X$ and $y \in Y$ and $x \neq v$, such that xy is not an edge of D, and let L be the bipartite graph with the edges of D and xy. Let $R = D - vw$, $0 \leq \alpha \leq 2$ and $0 \leq \beta \leq 2$, and, for vertex u, let r_u denote the degree of u in R. Then

$$\frac{|\mathcal{C}(L, 2R + \alpha vw + \beta xy)|}{|\mathcal{C}(L, 2R)|} = \frac{f_R(\alpha, \beta; v, w, x, y)}{\alpha!\,\beta!\,[S - 2d + 2]_{\alpha+\beta}}\Big(1 + O\big(\frac{k_{\max}^2}{S}\big)\Big),$$

where, by Lemma 2.3.3,

$$f_R(\alpha, \beta; v, w, x, y) = \begin{cases} [k_v - 2r_v]_\alpha [k_w - 2r_w]_\alpha [k_x - 2r_x]_\beta [k_y - 2r_y]_\beta, & \text{if } w \neq y, \\ [k_v - 2r_v]_\alpha [k_w - 2r_w]_{\alpha+\beta} [k_x - 2r_x]_\beta, & \text{if } w = y. \end{cases}$$

For any simple bipartite graph B, let $N[B]$ denote the number of pairings $P \in \mathcal{C}_{d,0}$ such that $D(P) = B$. Then $N[R + vw] = |\mathcal{C}(L, 2R + 2vw) \cup \mathcal{C}(L, 2R + 2vw + xy)|$, and similarly for $N[R + xy]$. Thus, when $N[R + vw] \neq 0$,

$$\begin{aligned} \frac{N[R + xy]}{N[R + vw]} &= \frac{f_R(0, 2; v, w, x, y)}{f_R(2, 0; v, w, x, y)}\Big(1 + O\big(\frac{k_{\max}^2}{S}\big)\Big) \\ &= \frac{[k_x - 2r_x]_2 [k_y - 2r_y]_2}{[k_v - 2r_v]_2 [k_w - 2r_w]_2}\Big(1 + O\big(\frac{k_{\max}^2}{S}\big)\Big). \end{aligned} \tag{3}$$

since the terms involving $f_R(1, 2; v, w, x, y)$ and $f_R(2, 1; v, w, x, y)$ are small enough to be incorporated into the error term.

Suppose $1 \leq i \leq \lfloor k_v/2 \rfloor$. Define $\mathcal{R}(i)$ to be the set of all simple bipartite graphs with parts X and Y with exactly $d - 1$ edges, of which exactly $i - 1$ are incident with v. For $R \in \mathcal{R}(i)$, let $\mathcal{X}(R)$ denote the set of all distinct pairs $\{x, y\}$ such that $x \in X$, $y \in Y$ where $x \neq v$ and xy is not an edge of R. Similarly, let $\mathcal{W}(R)$ denote the set of all $w \in Y$ such that vw is not an edge of R.

If n_i denotes the number of pairings $P \in \mathcal{C}_{d,0}$ such that exactly i double pairs are incident with v, then

$$n_{i-1} = \frac{1}{d-i+1} \sum_{R \in \mathcal{R}(i)} \sum_{xy \in \mathcal{X}(R)} N[R + xy]$$

and

$$n_i = \frac{1}{i} \sum_{R \in \mathcal{R}(i)} \sum_{w \in \mathcal{W}(R)} N[R + vw].$$

From (3) we find that, for any w and $R \in \mathcal{R}(i)$ for which the denominator is non-zero,

$$\begin{aligned}
&\frac{\sum_{xy \in \mathcal{X}(R)} N[R+xy]}{N[R+vw]} \\
&= \frac{\sum_{xy \in X \times Y} N[R+xy] - \sum_{xy \notin \mathcal{X}(R)} N[R+xy]}{N[R+vw]} \\
&= \frac{\sum_{xy \in X \times Y} N[R+xy] - \sum_{x=v \text{ or } xy \in R} N[R+xy]}{N[R+vw]} \\
&= \Big(\frac{\sum_{xy \in X \times Y} [k_x - 2r_x]_2 [k_y - 2r_y]_2}{[k_v - 2r_v]_2 [k_w - 2r_w]_2} \\
&\quad - \frac{\sum_{x=v \text{ or } xy \in R} [k_x - 2r_x]_2 [k_y - 2r_y]_2}{[k_v - 2r_v]_2 [k_w - 2r_w]_2} \Big) \Big(1 + O\big(\frac{k_{\max}^2}{S}\big)\Big) \\
&= \Big(\frac{\sum_{xy \in X \times Y} \big([k_x]_2 - O(r_x k_{\max})\big)\big([k_y]_2 - O(r_y k_{\max})\big)}{[k_v - 2r_v]_2 [k_w - 2r_w]_2} \\
&\quad - \frac{\sum_{x=v \text{ or } xy \in R} O\big([k_x]_2 [k_y]_2\big)}{[k_v - 2r_v]_2 [k_w - 2r_w]_2} \Big) \Big(1 + O\big(\frac{k_{\max}^2}{S}\big)\Big) \\
&= \Big(\frac{\sum_{xy \in X \times Y} [k_x]_2 [k_y]_2 + O\big(\sum_{xy \in X \times Y} (r_x k_{\max} [k_y]_2 + r_y k_{\max} [k_x]_2\big)}{[k_v - 2r_v]_2 [k_w - 2r_w]_2} \\
&\quad + \frac{O\big(\sum_{xy \in X \times Y} r_x r_y k_{\max}^2 + k_{\max}^2 T_2 + d k_{\max}^4\big)\big)}{[k_v - 2r_v]_2 [k_w - 2r_w]_2} \Big) \Big(1 + O\big(\frac{k_{\max}^2}{S}\big)\Big) \\
&= \frac{S_2 T_2 + O\big(d k_{\max} T_2 + d k_{\max} S_2 + d^2 k_{\max}^2 + k_{\max}^2 T_2 + d k_{\max}^4\big)}{[k_v - 2r_v]_2 [k_w - 2r_w]_2} \\
&\quad \times \Big(1 + O\big(\frac{k_{\max}^2}{S}\big)\Big)
\end{aligned}$$

$$= \frac{S_2T_2}{[k_v - 2r_v]_2[k_w - 2r_w]_2}$$
$$\times \Big(1 + O(\frac{k^2_{\max}}{S}) + O(\frac{dk_{\max}}{S_2} + \frac{dk_{\max}}{T_2} + \frac{d^2k^2_{\max}}{S_2T_2} + \frac{k^2_{\max}}{S_2} + \frac{dk^4_{\max}}{S_2T_2})\Big)$$
$$= \frac{S_2T_2}{[k_v - 2r_v]_2[k_w - 2r_w]_2}$$
$$\times \Big(1 + O(\frac{k^2_{\max}}{S} + \frac{dk_{\max} + k^2_{\max}}{S_2} + \frac{dk_{\max} + k^2_{\max}}{T_2})\Big).$$

We can sum over w in a similar way to obtain, for any $R \in \mathcal{R}(i)$ for which the denominator is non-zero,

$$\frac{\sum_{xy \in \mathcal{X}(R)} N[R + xy]}{\sum_{w \in \mathcal{W}(R)} N[R + vw]}$$
$$= \frac{S_2}{[k_v - 2(i-1)]_2}\Big(1 + O(\frac{k^2_{\max}}{S} + \frac{ik^2_{\max} + dk_{\max}}{S_2} + \frac{ik^2_{\max} + dk_{\max}}{T_2})\Big).$$

(Note that for S sufficiently large, both the numerator and denominator must be non-zero since $d = o(S^{2/3})$.) Since this is uniform over R, we conclude that

$$\frac{n_i}{n_{i-1}} = \frac{(d-i+1)[k_v - 2(i-1)]_2}{iS_2}\Big(1 + O(\frac{k^2_{\max}}{S} + \frac{ik^2_{\max} + dk_{\max}}{S_2} + \frac{ik^2_{\max} + dk_{\max}}{T_2})\Big).$$

Identifying n_r/n_0 as the quantity $Q_v(r)$, we now obtain the lemma on taking the product over i.

Similarly, we have the result for the case $v \in Y$. ∎

2.4. Analysis of the switchings.

We now analyse each of the switching types in turn, under the assumptions of Section 2.3.

Lemma 2.4.1. *Suppose* $0 \le d \le N_2$, $0 < t \le N_3$ *and* $S_3T_3 > 0$. *Then,*

$$\frac{|\mathcal{C}_{d,t}|}{|\mathcal{C}_{d,t-1}|} = \frac{S_3T_3}{6tS^3}\Big(1 + O\big(\frac{(k_{\max}^4 + k_{\max}^2(d+t))(S_3+T_3)}{S_3T_3}\big)\Big).$$

Proof. Choose an arbitrary $P \in \mathcal{C}_{d,t}$. Define N to be the number of t-switchings which can be applied to P. We can choose a triple pair and its labelling in $6t$ ways, and choose three distinct labelled simple pairs $\{p_4, p_4'\}$, $\{p_5, p_5'\}$ and $\{p_6, p_6'\}$ in $[S - 2d - 3t]_3$ ways. Unwanted coincidences like $v(p_1) = v(p_4)$ account for $O(tk_{\max}S^2)$ choices, while those like $v(p_4) = v(p_5)$ account for $O(tS(S_2+T_2))$. The forbidden cases where, for example, P already has a pair involving $v(p_1)$ and $v(p_4')$ account for $O(ts_{\max}t_{\max}S^2)$. Overall, we find that

$$N = 6tS^3\Big(1 + O\big(\frac{s_{\max}t_{\max} + d + t}{S}\big)\Big).$$

Now, choose an arbitrary $P' \in \mathcal{C}_{d,t-1}$, and let $N' = N'(P)$ be the number of inverse t-switchings which can be applied to it. We can choose two distinct 3-stars (one in X, the other in Y) in S_3T_3 ways. Of these choices, we must eliminate those not permitted. Unwanted coincidences, like $v(p_1) = v(p_4)$ and $v(p_1') = v(p_4')$ account for $O(s_{\max}^3T_3 + t_{\max}^3S_3)$. Cases where P' already has a pair involving $v(p_1)$ and $v(p_1')$ or $v(p_4)$ and $v(p_4')$, for example, account for $O(t_{\max}s_{\max}^3T_3 + s_{\max}t_{\max}^3S_3)$. Finally, cases where either of the 3-stars include a non-simple pair account for $O(s_{\max}^2(d+t)T_3 + t_{\max}^2(d+t)S_3)$. Overall, we find that

$$N' = S_3T_3\Big(1 + O\big(\frac{t_{\max}s_{\max}^3T_3 + s_{\max}t_{\max}^3S_3 + (d+t)(s_{\max}^2T_3 + t_{\max}^2S_3)}{S_3T_3}\big)\Big).$$

The error term for N' dominates that for N, so the lemma follows on considering the ratio N'/N. ∎

Lemma 2.4.2. *Suppose* $1 \le d \le N_2$. *Then,*

$$\frac{|\mathcal{C}_{d,0}|}{|\mathcal{C}_{d-1,0}|} = \frac{S_2T_2}{2dS^2}\Big(1 + O\big(\frac{(k^3_{\max} + dk_{\max})(S_2+T_2)}{S_2T_2}\big)\Big).$$

Proof. Choose an arbitrary $P \in \mathcal{C}_{d,0}$. Define N to be the number of d-switchings which can be applied to P. We can choose a double pair and its labelling in $2d$ ways, and choose two distinct labelled simple pairs $\{p_3, p_3'\}$ and $\{p_4, p_4'\}$ in $[S-2d]_2$ ways. Unwanted coincidence like $v(p_1) = v(p_3)$ account for $O(dk_{\max}S)$ choices, while those like $v(p_3) = v(p_4)$ account for $O(d(S_2+T_2))$. The forbidden cases where, for example, P already has a pair involving $v(p_1)$ and $v(p_3')$ account for $O(ds_{\max}t_{\max}S)$. Overall, we find that

$$N = 2dS^2\Big(1 + O\big(\frac{s_{\max}t_{\max} + d}{S}\big)\Big).$$

Now, choose an arbitrary $P' \in \mathcal{C}_{d-1,0}$, and let $N' = N'(P)$ be the number of inverse d-switchings which can be applied to it. We can choose two distinct 2-stars (one in X, the other in Y) in S_2T_2 ways. Of these choices, we must eliminate those not permitted. Unwanted coincidences, like $v(p_1) = v(p_3)$, $v(p_1') = v(p_3')$, $v(p_3) = v(p_4)$ and $v(p_3') = v(p_4')$ account for $O(s^2_{\max}T_2 + t^2_{\max}S_2)$. Cases where P' already has a pair involving $v(p_1)$ and $v(p_1')$ or $v(p_3)$ and $v(p_3')$, for example, account for $O(t_{\max}s^2_{\max}T_2 + s_{\max}t^2_{\max}S_2)$. Finally, cases where either of the 2-stars include non-simple pair account for $O(ds_{\max}T_2 + dt_{\max}S_2)$. Overall, we find that

$$N' = S_2T_2\Big(1 + O\big(\frac{t_{\max}s^2_{\max}T_2 + s_{\max}t^2_{\max}S_2 + d(s_{\max}T_2 + t_{\max}S_2)}{S_2T_2}\big)\Big).$$

The error term for N' dominates that for N, so the lemma follows on considering the ratio N'/N. ∎

Whilst we will use Lemma 2.4.2 in one special case, it is not sufficiently accurate for us in general. The reason is that the number of double pairs in a random pairing is in general much higher than the numbers of triple pairs. However, Lemma 2.4.3 is the best that can be done using uniform counts over arbitrary members of $\mathcal{C}_{d,0}$ and $\mathcal{C}_{d-1,0}$. In order to do better, we need to consider averages over $\mathcal{C}_{d,0}$ and $\mathcal{C}_{d-1,0}$.

Lemma 2.4.3. *Suppose* $1 \le d \le N_2$, $S_2 \ge S$ *and* $T_2 \ge S$. *Then*

$$\frac{|\mathcal{C}_{d,0}|}{|\mathcal{C}_{d-1,0}|} = \frac{S_2T_2}{2dS^2}\Big(1 + \frac{2S_2T_3}{S^2T_2} + \frac{2T_2S_3}{S^2S_2} + \frac{4d}{S} - \frac{S_3T_3}{SS_2T_2} - \frac{2S_2T_2}{S^3} - \frac{4dS_3}{S_2^2} - \frac{4dT_3}{T_2^2} + O\big(\frac{k_{\max}^2 + d}{S_2} + \frac{k_{\max}^2 + d}{T_2}\big)\Big).$$

Proof. Define N to be the average number of possible d-switchings, where the average is over all $P \in \mathcal{C}_{d,0}$. We can choose $\{p_1, p_1', p_2, p_2'\}$ in $2d$ ways and then $\{p_3, p_3', p_4, p_4'\}$ in at most $[S-2]_2$ ways. This gives us the initial overcount $N \le N^* = 2d[S-2]_2 = 2dS^2(1 + O(1/S))$. However, some of these choices are not legal. We can divide the set of illegal choices into three families:

X_1: These are choices involving too few vertices, for example if $v(p_1) = v(p_3)$ or $v(p_3') = v(p_4')$.

X_2: These are the choices for which the pairing already has a pair involving $v(p_1)$ and $v(p_3')$ or the three other similar cases. However, we exclude any choice which belongs to X_1.

X_3: These are choices for which either $\{p_3, p_3'\}$ or $\{p_4, p_4'\}$ has multiplicity two. However, we exclude any choice which belongs to X_1.

In each case, we will consider the probability that randomly choosing one the possibilities leading to N^* lies in the given family, where the probability is averaged over random P. We will also bound the intersection of X_2 and X_3.

Case X_1: The probability of landing in X_1 is easily seen to be at most $O(k_{\max}/S)$, by just counting the cases.

Case X_2: Let P_i $(i = 1, 2)$ be the probability that there is a pair $\{x, x'\}$ of multiplicity i such that $v(x) = v(p_1)$ and $v(x') = v(p_3')$. Note that our conditions on d, $k_{\max}$, S, S_2 imply that $dk_{\max}^2/S_2 = o(1)$. From Lemma 2.3.5, the expected number of pairs of adjacent double pairs with the adjacency in X is $O(d^2 S_4/S_2^2) = O(d^2 k_{\max}^2/S_2)$. Allowing $k_{\max}$ for the choices of p_3 and S for the choice of $\{p_4, p_4'\}$, we find that $P_2 = O\big(dk_{\max}^3/(SS_2)\big) = O(k_{\max}/S)$.

P_1 is more involved. For any choice of $D(P)$, p_1, p_1', p_2 and p_2', there are on average

$$T_2(k_v - 2r)\Big(1 + O\big(\frac{k_{\max}^2 + d}{S} + \frac{dk_{\max}}{T_2}\big)\Big)$$

choices for p_3, p_3', p_4 and p_4', where $v = v(p_1)$ and r is the number of double pairs incident with v. This follows from Lemma 2.3.4 on summing over all the possibilities. If K denotes the expected number of configurations included in the value of P_1, then by Lemma 2.3.5,

$$\begin{aligned} K &= 2T_2\Big(1 + O\big(\frac{k_{\max}^2 + d}{S} + \frac{dk_{\max}}{T_2}\big)\Big) \sum_{v\in X}\sum_{r\geq 1} r(k_v - 2r)\frac{Q_v(r)}{\sum_{i\geq 0} Q_v(i)} \\ &= \frac{2dT_2}{S_2}\Big(1 + O\big(\frac{k_{\max}^2 + d}{S} + \frac{dk_{\max}}{T_2}\big)\Big) \sum_{v\in X} [k_v]_3 \frac{\sum_{i\geq 0} S_v(i)}{\sum_{i\geq 0} Q_v(i)}, \end{aligned}$$

where

$$S_v(i) = \frac{[d-1]_i[k_v-3]_{2i}}{i!\,S_2^i}\exp\Big(O\big(\frac{ik_{\max}^2}{S}+\frac{i^2k_{\max}^2+idk_{\max}}{S_2}+\frac{i^2k_{\max}^2+idk_{\max}}{T_2}\big)\Big).$$

Since $\sum_{i\geq 0} Q_v(i) \geq 1$,

$$\frac{\sum_{i\geq 0} S_v(i)}{\sum_{i\geq 0} Q_v(i)} = 1 + O\big(\sum_{i\geq 0}(Q_v(i)-S_v(i))\big).$$

Putting $z = ik_{\max}^2/S + (i^2k_{\max}^2 + idk_{\max})/S_2 + (i^2k_{\max}^2 + idk_{\max})/T_2$, we have

$$Q_v(i) = \frac{[d]_i[k_v]_{2i}}{i!S_2^i}e^{z_1},$$

and

$$S_v(i) = \frac{[d-1]_i[k_v-3]_{2i}}{i!S_2^i}e^{z_2},$$

where $z_1 = O(z)$, $z_2 = O(z)$.

$$Q_v(i) - S_v(i) = C(i)X(i),$$

where

$$C(i) = \frac{[d-1]_{i-1}[k_v-3]_{2i-3}}{i!S_2^i},$$

$$X(i) = d[k_v]_3e^{z_1} - (d-i)[k_v-2i]_3e^{z_2}.$$

Since

$$X(i) = d[k_v]_3(e^{z_1}-e^{z_2}) + e^{z_2}\big(d[k_v]_3 - (d-i)[k_v-2i]_3\big),$$

$$e^x = 1 + O(xe^x),$$

and

$$\mid d[k_v]_3 - (d-i)[k_v-2i]_3 \mid = \mid d([k_v]_3 - [k_v-2i]_3) + i[k_v-2i]_3 \mid$$

$$= O(dik_v^2 + ik_v^3),$$

we have that

$$|X(i)| \leq aY(i),$$

for some $a > 0$, where $Y(i) = (dk_v^3 z + idk_v^2 + ik_v^3)e^{az}$.

Define $T(i) = C(i)Y(i)$. Since

$$\frac{C(i+1)}{C(i)} \le \frac{dk_v^2}{iS_2},$$

and

$$\begin{aligned}\frac{Y(i+1)}{Y(i)} &\le 4\exp\big(a(z(i+1) - z(i))\big) \\ &= 4\exp\big(a(\frac{k_{\max}^2}{S} + \frac{dk_{\max} + k_{\max}^2 + 2ik_{\max}^2}{S_2} + \frac{dk_{\max} + k_{\max}^2 + 2ik_{\max}^2}{T_2})\big) \\ &= 4\big(1 + o(1)\big),\end{aligned}$$

we have

$$\begin{aligned}\frac{T(i+1)}{T(i)} &\le \frac{4dk_v^2}{iS_2}(1 + o(1)) \\ &\le \frac{4dk_v^2}{S_2}.\end{aligned}$$

Since $d \le N_2$ and $k_v \le k_{\max}$, we have $dk_v^2/S_2 = o(1)$. Hence, we obtain

$$\begin{aligned}\sum_{i\ge0}\big(Q_v(i) - S_v(i)\big) &\le \sum_{i\ge0} T(i) \\ &\le \sum_{i\ge0} T(1)(\frac{4dk_v^2}{S_2})^i \\ &= \frac{T(1)}{1 - (4dk_v^2)/S_2}.\end{aligned}$$

Since

$$\begin{aligned}T(1) &= \frac{O(1)}{(k_v+2)S_2}\big(d[k_v]_3 - (d-1)[k_v - 2]_3\big) \\ &= O\Big(\frac{k_{\max}^2 + dk_{\max}}{S_2}\Big),\end{aligned}$$

we finally have

$$K = \frac{2dT_2S_3}{S_2}\Big(1 + O(\frac{k_{\max}^2 + d}{S} + \frac{k_{\max}^2 + dk_{\max}}{S_2} + \frac{dk_{\max}}{T_2})\Big).$$

Hence,

$$\begin{aligned}P_1 &= \frac{K}{N^*}\\ &= \frac{T_2S_3}{S^2S_2}\Big(1+O(\frac{k_{\max}^2+d}{S}+\frac{k_{\max}^2+dk_{\max}}{S_2}+\frac{dk_{\max}}{T_2}+\frac{1}{S})\Big)\\ &= \frac{T_2S_3}{S^2S_2}+O(\frac{k_{\max}}{S}).\end{aligned}$$

Define $D_i(i=1,2)$ to be the probability that there is a pair $\{x,x'\}$ of multiplicity i such that $v(x)=v(p_2)$ and $v(x')=v(p_4')$. E_i and $F_i(i=1,2)$ are definded similarly for $v(p_1'), v(p_3)$ and $v(p_2'), v(p_4)$.

Similarly to previous argument, we have $D_2 = O(k_{\max}/S)$ and

$$D_1 = \frac{T_2S_3}{S^2S_2}+O(\frac{k_{\max}}{S}).$$

Similarly, we have $E_2 = F_2 = O(k_{\max}/S)$ and

$$E_1 = F_1 = \frac{S_2T_3}{S^2T_2}+O(\frac{k_{\max}}{S}).$$

Any two of the eight events counted in X_2 (single or double pair in any of four positions) occur together with probability $O(k_{\max}/S)$, so altogether we find that X_2 occurs with probability

$$\frac{2S_2T_3}{S^2T_2}+\frac{2T_2S_3}{S^2S_2}+O(\frac{k_{\max}}{S}).$$

Case X_3: With the help of Lemma 2.3.4 and 2.3.5, a routine calculation gives the probability of this case as $4d/S+O(k_{\max}/S)$.

Events X_2 and X_3 occur together with probability $O(k_{\max}/S)$, by similar reasoning. Thus we have altogether that

$$N = 2dS^2\Big(1-\frac{2S_2T_3}{S^2T_2}-\frac{2T_2S_3}{S^2S_2}-\frac{4d}{S}+O(\frac{k_{\max}}{S})\Big).$$

Conversely, define N' to be the average number of possible inverse d-switchings, where the average is over all $P \in \mathcal{C}_{d-1,0}$. For each choice of $v = v(p_1)$, there are at most $[k_v]_2$ ways to choose p_1 and p_2. A similar bound holds for $v(p_1')$, and so we have an initial overcount $N' \le S_2T_2$. However, some of these choices are not legal and, as before, we divide these into a number of cases:

Y_1: These are choices involving too few vertices, for example $v(p_1) = v(p_3)$ or $v(p_1') = v(p_3')$.

Y_2: These are choices where there is already a pair involving $v(p_1)$ and $v(p_1')$, excluding anything in case Y_1.

Y_3: These are choices where there is already a pair involving $v(p_3)$ and $v(p_3')$, or $v(p_4)$ and $v(p_4')$. Again, we exclude anything in case Y_1.

Y_4: These are choices for which one or more of the pairs chosen have multiplicity two, except any choice in case Y_1.

These four cases can be analysed using the same method used for X_1–X_3. For cases Y_2 and Y_3, we can simply sum over all the possibilities using Lemma 2.3.4. For Case Y_4, we need Lemma 2.3.5. We will merely state the probability in each case.

Case Y_1: $\quad O\left(\frac{k_{\max}^2}{S_2} + \frac{k_{\max}^2}{T_2}\right).$

Case Y_2: $\quad \frac{S_3T_3}{SS_2T_2} + O\left(\frac{k_{\max}^2 + d}{S_2} + \frac{k_{\max}^2 + d}{T_2}\right).$

Case Y_3: $\quad \frac{2S_2T_2}{S^3} + O\left(\frac{k_{\max}^2}{S_2} + \frac{k_{\max}^2}{T_2}\right).$

Case Y_4: $\quad \frac{4dS_3}{S_2^2} + \frac{4dT_3}{T_2^2} + O\left(\frac{k_{\max}^2 + d}{S_2} + \frac{k_{\max}^2 + d}{T_2}\right).$

The conjunction of any two of these cases gives no new error terms, so overall we have

$$N' = S_2T_2\Big(1 - \frac{S_3T_3}{SS_2T_2} - \frac{2S_2T_2}{S^3} - \frac{4dS_3}{S_2^2} - \frac{4dT_3}{T_2^2} + O\big(\frac{k_{\max}^2 + d}{S_2} + \frac{k_{\max}^2 + d}{T_2}\big)\Big).$$

The lemma now follows on comparing N to N'. ∎

2.5. Consolidation.

With the aid of the lemmas in Section 2.4, we can now apply Lemma 2.3.2 to estimate $P(\mathbf{s}, \mathbf{t})$.

Lemma 2.5.1.

$$P(\mathbf{s}, \mathbf{t}) = \exp\Big(-\frac{S_2T_2}{2S^2} + \frac{S_3T_3}{3S^3} - \frac{S_2^2T_3}{2S^4} - \frac{T_2^2S_3}{2S^4} + \frac{S_2^2T_2^2}{2S^5} + O\big(\frac{k_{\max}^3}{S}\big)\Big).$$

Proof. Let $0 \le d \le N_2$ and $0 \le t \le N_3$. By Lemma 2.4.1,

$$\frac{|\mathcal{C}_{d,t}|}{|\mathcal{C}_{d,0}|} = \frac{S_3^tT_3^t}{6^tS^{3t}t!}\exp\Big(O\big(\frac{(k_{\max}^4 t + k_{\max}^2(dt + t^2))(S_3 + T_3)}{S_3T_3}\big)\Big).$$

Define

$$A = \frac{S_3T_3}{6S^3},$$

$$B = \frac{(k_{\max}^4 + k_{\max}^2 d)(S_3 + T_3)}{S_3T_3},$$

$$C = \frac{k_{\max}^2(S_3 + T_3)}{S_3T_3},$$

$$U_t = \frac{A^t}{t!}\exp\Big(O\big(tB + t^2C\big)\Big),$$

and

$$V_t = \frac{A^t}{t!}.$$

We have

$$\begin{aligned}\sum_{t=0}^{N_3}(V_t - U_t) &= \sum_{t=0}^{N_3}\Big(1-\exp\big(O(tB+t^2C)\big)\Big)\frac{A^t}{t!}\\ &= \sum_{t=0}^{N_3}\frac{A^t}{t!}(O(tB+t^2C))\\ &= O(B)\sum_{t=0}^{N_3}\frac{tA^t}{t!} + O(C)\sum_{t=0}^{N_3}\frac{t^2A^t}{t!}.\end{aligned}$$

Because

$$\begin{aligned}\sum_{t=0}^{N_3}\frac{tA^t}{t!} &= A\sum_{t=1}^{N_3}\frac{A^{t-1}}{(t-1)!}\\ &= O\big(A\exp(A)\big),\end{aligned}$$

and

$$\begin{aligned}\sum_{t=0}^{N_3}\frac{t^2A^t}{t!} &= \sum_{t=1}^{N_3}\frac{(t-1)A^t}{(t-1)!} + \sum_{t=1}^{N_3}\frac{A^t}{(t-1)!}\\ &= A^2\sum_{t=2}^{N_3}\frac{A^{t-2}}{(t-2)!} + A\sum_{t=1}^{N_3}\frac{A^{t-1}}{(t-1)!}\\ &= O\big(A^2\exp(A)\big) + O\big(A\exp(A)\big),\end{aligned}$$

we have

$$\sum_{t=0}^{N_3}(V_t - U_t) = O\big(AB + A^2C + AC\big)\exp(A).$$

Since

$$\begin{aligned}&O\big(AB + A^2C + AC\big)\\ &= O\Big(\frac{(k_{\max}^4 + k_{\max}^2 d)(S_3+T_3)}{S^3} + \frac{S_3T_3k_{\max}^2(S_3+T_3)}{S^6} + \frac{k_{\max}^2(S_3+T_3)}{S^3}\Big)\\ &= O\Big(\frac{k_{\max}^4(k_{\max}^2+d)}{S^2} + \frac{k_{\max}^8}{S^3} + \frac{k_{\max}^4}{S^2}\Big)\\ &= O\Big(\frac{k_{\max}^3 + k_{\max}d}{S}\Big),\end{aligned}$$

we have

$$\begin{aligned}\sum_{t=0}^{N_3}U_t &= \sum_{t=0}^{N_3}V_t + O\Big(\frac{k_{\max}^3 + k_{\max}d}{S}\Big)\exp(A)\\ &= \exp(A) - Q(A) + O\Big(\frac{k_{\max}^3 + k_{\max}d}{S}\Big)\exp(A),\end{aligned}$$

where $Q(A)$ is the remainder of Taylor's expansion for $\exp(A)$. As we know, we can have

$$Q(A) = \frac{\exp(\theta A)}{(N_3+1)!} A^{N_3+1},$$

where $0 < \theta < 1$.

By Stirling's formula, we have

$$(N_3+1)! \geq \Big(\frac{N_3+1}{e}\Big)^{N_3+1},$$

and since we have that

$$N_3 \geq \frac{e^2 S_3 T_3}{6S^3}$$

and

$$N_3 \geq \log(ST),$$

we obtain that

$$\begin{aligned}
\exp(-A)Q(A) &\leq \exp\big((\theta-1)A\big)\Big(\frac{eS_3T_3}{(N_3+1)6S^3}\Big)^{N_3+1} \\
&\leq \big(\frac{1}{e}\big)^{N_3+1} \\
&\leq \frac{1}{eST} \\
&= O(\frac{1}{S}).
\end{aligned}$$

Now we have

$$\begin{aligned}
\sum_{t=0}^{N_3} \frac{|\mathcal{C}_{d,t}|}{|\mathcal{C}_{d,0}|} &= \sum_{t=0}^{N_3} U_t \\
&= \exp(A) + O(\frac{1}{S})\exp(A) + O(\frac{k_{\max}^3 + k_{\max}d}{S})\exp(A) \\
&= \exp(A)\Big(1 + O(\frac{k_{\max}^3 + k_{\max}d}{S})\Big) \\
&= \exp(A)\exp\Big(O(\frac{k_{\max}^3 + k_{\max}d}{S})\Big). \qquad (4)
\end{aligned}$$

Now assuming $S_2 \geq S$, $T_2 \geq S$ and by Lemma 2.4.3, we have

$$\frac{|\mathcal{C}_{d,0}|}{|\mathcal{C}_{0,0}|} = \frac{S_2^d T_2^d}{2^d S^{2d} d!} \exp\Big(\frac{2S_2T_3d}{S^2T_2} + \frac{2T_2S_3d}{S^2S_2} + \frac{2d^2}{S} - \frac{S_3T_3d}{SS_2T_2} - \frac{2S_2T_2d}{S^3} - \frac{2d^2S_3}{S_2^2} - \frac{2d^2T_3}{T_2^2} + O\big(\frac{k_{\max}^2 d + d^2}{S_2} + \frac{k_{\max}^2 d + d^2}{T_2}\big)\Big).$$

Combining this with (4) and summing over d with the help of the approximation $d^2 \approx (dS_2T_2)/(2S^2)$, we obtain

$$\sum_{d=0}^{N_2}\sum_{t=0}^{N_3} \frac{|\mathcal{C}_{d,t}|}{|\mathcal{C}_{0,0}|} = \exp\Big(\frac{S_2T_2}{2S^2} - \frac{S_3T_3}{3S^3} + \frac{S_2^2T_3}{2S^4} + \frac{T_2^2S_3}{2S^4} - \frac{S_2^2T_2^2}{2S^5} + O\big(\frac{k_{\max}^3}{S}\big)\Big). \quad (5)$$

In the case where $0 < S_2 < S$ or $0 < T_2 < S$, Lemma 2.4.2 gives the same result to within the same error. In the trivial case $S_2 = 0$ or $T_2 = 0$ (which implies $S_3 = 0$ or $T_3 = 0$), Equation (5) again holds.

We now have the result we have been seeking by (1) and Lemma 2.5.1.

Theorem 2.5.2. *If $k_{\max} = o(S^{1/3})$, then*

$$G(\mathbf{s},\mathbf{t}) = \frac{S!}{s_1!\,s_2!\cdots s_n!\,t_1!\,t_2!\cdots t_m!} \times \exp\Big(-\frac{S_2T_2}{2S^2} + \frac{S_3T_3}{3S^3} - \frac{S_2^2T_3}{2S^4} - \frac{T_2^2S_3}{2S^4} + \frac{S_2^2T_2^2}{2S^5} + O\big(\frac{k_{\max}^3}{S}\big)\Big)$$

uniformly as $S \to \infty$. ∎

Corollary 2.5.3. *If $k = o(n^{1/2})$, the number of separately labelled regular bipartite graphs of degree k with each part having n vertices is asymptotically*

$$\frac{(nk)!}{(k!)^{2n}} \exp\Big(-\frac{(k-1)^2}{2} - \frac{k^3}{6n} + O\big(\frac{k^2}{n}\big)\Big),$$

as $n \to \infty$. ∎

Corollary 2.5.4. *Let $1 \leq p = p(n) = o(n^{1/2})$ and $1 \leq q = q(m) = o(m^{1/2})$ and suppose $n, m \to \infty$ with $pn = qm$. The number of separately labelled bipartite* *graphs with parts X and Y, where X has n vertices each of degree p and Y has m vertices each of degree q, is asymptotically*

$$\frac{(np)!}{(p!)^n (q!)^m} \exp\Big(-\frac{(p-1)(q-1)}{2} - \frac{pq^2}{6n} + O\big(\frac{p^2}{n} + \frac{q^2}{m}\big)\Big),$$

as $n, m \to \infty$. ∎

Chapter 3

An Integral

3.1. Introduction.

In this chapter, we estimate the asymptotic value of a n-dimensional integral by the averaging method [25]. The averaging method was first proposed by McKay and Wormald [25]. They obtained the asymptotic value of a n-demensional integral which was then applied to prove an asymptotic formula for the number of graphs of high degree. We modify their proof to obtain an integral theorem and we will make use of the integral theorem to estimate the asymptotic numbers of tournaments and digraphs in Chapters 4 and 5.

3.2. Averaging method.

In this section, we will use the averaging method [25] to evaluate a n-dimensional integral. We define $U_n(t) = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \mid |x_i| \le t, i = 1, 2, \ldots, n\}$ and we will need the following lemma, which is well known.

Lemma 3.2.1. *The surface area of a n-dimensional sphere of radius ρ is $2\pi^{n/2}\rho^{n-1}/\Gamma(n/2)$.* ∎

Theorem 3.2.2. *Let $0 < \epsilon < 1/6$. Let t and t' be constants with $t' > 0$, and let i be the complex unit. Suppose that*

(i) $B_k(n) = B'_k(n)i$ *with* $B'_k(n) = O(n^{-1/4})$ *uniformly for* $1 \le k \le n-1$,

(ii) $C_{jk}(n) = C'_{jk}(n)i$ *with* $C'_{jk}(n) = O(n^{-1/4})$ *uniformly for* $1 \le j, k \le n-1$,

(iii) $A(n)$ *is real-valued function with* $t' \le A(n) \le t$, *and*

(iv) $B'_k(n), C'_{jk}(n), D_{jk}(n)$, $E_k(n)$ *and* $F(n)$ *are all real-valued functions whose absolute values are bounded by* t *for* $1 \le j, k \le n-1$.

Suppose that $\delta > 0, 0 < \Delta < 1/4 - \epsilon/2$, *and that*

$$f(\boldsymbol{x}) = \exp\Big(-A(n)n\sum_{k=1}^{n-1} x_k^2 + n\sum_{k=1}^{n-1} B_k(n)x_k^3 + \sum_{j\ne k} C_{jk}(n)x_k^2 x_j$$

$$+\sum_{j\ne k} D_{jk}(n)x_k^3 x_j + n\sum_{k=1}^{n-1} E_k(n)x_k^4 + F(n)\Big(\sum_{k=1}^{n-1} x_k^2\Big)^2 + O(n^{-\delta})\Big)$$

is integrable for $\boldsymbol{x} \in U_{n-1}(n^{-1/2+\epsilon})$. *Then*

$$\int_{U_{n-1}(n^{-1/2+\epsilon})} f(\boldsymbol{x})\,d\boldsymbol{x}$$
$$= \Big(\frac{\pi}{A(n)n}\Big)^{(n-1)/2}$$
$$\times \exp\Big(\frac{3\sum_{k=1}^{n-1} E_k(n)}{4A(n)^2 n} + \frac{F(n)}{4A(n)^2} + O\big(n^{-1/2+8\epsilon} + n^{-\Delta} + n^{-\delta}\big)\Big).$$

Proof. Define $\mu_2 = \sum_{k=1}^{n-1} x_k^2$ and for $\rho \ge 0$, define $W_{n-1}(\rho) = U_{n-1}(n^{-1/2+\epsilon}) \cap \{\boldsymbol{x} \mid \mu_2 = \rho^2\}$. We approach the integral by considering integration first over $W_{n-1}(\rho)$ and then over ρ, although this is not the way we obtain the final estimate. Note first that $W_{n-1}(\rho) = \emptyset$ if $\rho > n^\epsilon$.

For $\boldsymbol{x} \in W_{n-1}(\rho)$ and $\rho \le n^\epsilon$, we have

$$\Big|n\sum_{k=1}^{n-1} B_k(n)x_k^3\Big| \le t\rho^2 n^{1/2+\epsilon},$$
$$\Big|\sum_{j\ne k} C_{jk}(n)x_k^2 x_j\Big| \le t\rho^2 n^{1/2+\epsilon},$$

$$\Big|\sum_{j\neq k} D_{jk}(n)x_k^3x_j\Big| \le t\rho^2 n^{2\epsilon},$$

$$\Big|n\sum_{k=1}^{n-1} E_k(n)x_k^4\Big| \le t\rho^2 n^{2\epsilon},$$

and

$$\Big|F(n)\Big(\sum_{k=1}^{n-1} x_k^2\Big)^2\Big| \le t\rho^2 n^{2\epsilon}.$$

We now divide the region of integration into three parts. Define

$$K_1 = U_{n-1}(n^{-1/2+\epsilon}) \cap \{\, \boldsymbol{x} \mid 0 \le \rho < (2A(n))^{-1/2}(1-n^{-\Delta})\,\},$$

$$K_2 = U_{n-1}(n^{-1/2+\epsilon}) \cap$$
$$\{\, \boldsymbol{x} \mid (2A(n))^{-1/2}(1-n^{-\Delta}) \le \rho \le (2A(n))^{-1/2}(1+n^{-\Delta})\,\}, \text{ and}$$

$$K_3 = U_{n-1}(n^{-1/2+\epsilon}) \cap \{\, \boldsymbol{x} \mid (2A(n))^{-1/2}(1+n^{-\Delta}) < \rho \le n^{\epsilon}\,\}.$$

When $\boldsymbol{x} \in W_{n-1}(\rho)$, $f(\boldsymbol{x}) = \exp\big(-A(n)n\rho^2 + O(\rho^2 n^{1/2+\epsilon} + n^{-\delta})\big)$. Also by Lemma 3.2.1, the area of $W_{n-1}(\rho)$ is at most $O(1)(2\pi e/(n-1))^{(n-1)/2}\rho^{n-2}$. Thus

$$\Big|\int_{K_1} f(\boldsymbol{x})\,d\boldsymbol{x}\Big|$$
$$\le O(1)\Big(\frac{2\pi e}{n-1}\Big)^{(n-1)/2}$$
$$\times \int_0^{(2A(n))^{-1/2}(1-n^{-\Delta})} \rho^{n-2}\exp\big(-A(n)n\rho^2 + O(\rho^2 n^{1/2+\epsilon} + n^{-\delta})\big)\,d\rho.$$

Apart from the $O(\ )$ term, the integrand is unimodal, with its maximum at $\rho^2 = (n-2)/(2A(n)n)$, so we can bound the integral by the length of its range times its maximum value, where the latter is achieved near $\rho = (2A(n))^{-1/2}(1-n^{-\Delta})$. Using $\log(1-x) < -x - \frac{1}{2}x^2$ for $x = n^{-\Delta}$ and n^{-1}, we find that

$$\Big|\int_{K_1} f(\boldsymbol{x})\,d\boldsymbol{x}\Big| \le \Big(\frac{\pi}{A(n)n}\Big)^{(n-1)/2}\exp\big(-n^{1-2\Delta} + O(n^{1/2+\epsilon})\big).$$

The same bound can be derived for the absolute value of the integral over K_3. The integral over $K_1 \cup K_3$ will turn out to be negligible compared to that over K_2, which we now consider.

The function $f(\boldsymbol{x})$ shows a lot of variation on $W_{n-1}(\rho)$, $\rho \approx (2A(n))^{-1/2}$, making direct estimation of the integral difficult. Instead, we take advantage of the fact that an integral over a region symmetrical about the origin is invariant under averaging of its integrand over sign changes of the arguments.

For $1 \le m \le n$, define

$$\begin{aligned}\psi_m(\boldsymbol{x}) = \exp\Bigl(&-A(n)n\sum_{k=1}^{n-1} x_k^2 + n\sum_{k=1}^{n-1} E_k(n)x_k^4 + F(n)\Bigl(\sum_{k=1}^{n-1} x_k^2\Bigr)^2 + n\sum_{k=m}^{n-1} B_k(n)x_k^3\\ &+\sum_{k=1}^{n-1}\sum_{j=m}^{n-1} C_{jk}(n)x_k^2x_j + \sum_{k=m}^{n-1}\sum_{j=m}^{n-1} D_{jk}(n)x_k^3x_j + \frac{n^2}{2}\sum_{k=1}^{m-1} B_k(n)^2x_k^6\Bigr)\end{aligned}$$

and

$$\bar{\psi}_m(\boldsymbol{x}) = \tfrac{1}{2}\bigl(\psi_m(\boldsymbol{x}) + \psi_m(x_1,\ldots,x_{m-1},-x_m,x_{m+1},\ldots,x_n)\bigr).$$

Further define $\eta = \frac{3}{2} - 6\epsilon$. Then we have

$$\int_{U_{n-1}(n^{-1/2+\epsilon})} \bar{\psi}_m(\boldsymbol{x})\,d\boldsymbol{x} = \int_{U_{n-1}(n^{-1/2+\epsilon})} \psi_m(\boldsymbol{x})\,d\boldsymbol{x}. \qquad (2.1)$$

For $\boldsymbol{x} \in U_{n-1}(n^{-1/2+\epsilon})$, since

$$\begin{aligned}\psi_m(\boldsymbol{x}) = \psi_{m+1}(\boldsymbol{x})\exp\Bigl(&B_m(n)nx_m^3 + \sum_{k=1}^{n-1} C_{mk}(n)x_k^2x_m - \tfrac{1}{2}B_m(n)^2n^2x_m^6\\ &+\sum_{k=m+1}^{n-1} D_{mk}(n)x_k^3x_m + \sum_{j=m+1}^{n-1} D_{jm}(n)x_m^3x_j + D_{mm}(n)x_m^4\Bigr),\end{aligned}$$

$$\begin{aligned}&\psi_m(x_1,\ldots,x_{m-1},-x_m,x_{m+1},\ldots,x_n)\\ &= \psi_{m+1}(\boldsymbol{x})\exp\Bigl(-B_m(n)nx_m^3 - \sum_{k=1}^{n-1} C_{mk}(n)x_k^2x_m - \tfrac{1}{2}B_m(n)^2n^2x_m^6\Bigr)\end{aligned}$$

$$- \sum_{k=m+1}^{n-1} D_{mk}(n)x_k^3 x_m - \sum_{j=m+1}^{n-1} D_{jm}(n)x_m^3 x_j + D_{mm}(n)x_m^4\Big),$$

and $\frac{1}{2}(e^x + e^{-x}) = \exp\big(\frac{1}{2}x^2 + O(x^4)\big)$ for small x, we have

$$\begin{aligned}
\bar{\psi}_m(\boldsymbol{x}) &= \tfrac{1}{2}\psi_{m+1}(\boldsymbol{x})\exp\big(-\tfrac{1}{2}B_m(n)^2 n^2 x_m^6 + D_{mm}(n)x_m^4\big) \\
&\quad \times \Big(\exp\big(B_m(n)nx_m^3 + \sum_{k=1}^{n-1} C_{mk}(n)x_k^2 x_m \\
&\qquad + \sum_{k=m+1}^{n-1} D_{mk}(n)x_k^3 x_m + \sum_{j=m+1}^{n-1} D_{jm}(n)x_m^3 x_j\big) \\
&\qquad + \exp\big(-B_m(n)nx_m^3 - \sum_{k=1}^{n-1} C_{mk}(n)x_k^2 x_m \\
&\qquad - \sum_{k=m+1}^{n-1} D_{mk}(n)x_k^3 x_m - \sum_{j=m+1}^{n-1} D_{jm}(n)x_m^3 x_j\big)\Big) \\
&= \psi_{m+1}(\boldsymbol{x})\exp\big(D_{mm}(n)x_m^4\big) \\
&\quad \times \exp\Big(B_m(n)nx_m^4 \sum_{k=1}^{n-1} C_{mk}(n)x_k^2 + B_m(n)nx_m^4 \sum_{k=m+1}^{n-1} D_{mk}(n)x_k^3 \\
&\qquad + B_m(n)nx_m^6 \sum_{j=m+1}^{n-1} D_{jm}(n)x_j + x_m^2 \sum_{k=1}^{n-1} C_{mk}(n)x_k^2 \sum_{k=m+1}^{n-1} D_{mk}(n)x_k^3 \\
&\qquad + x_m^4 \sum_{k=1}^{n-1} C_{mk}(n)x_k^2 \sum_{j=m+1}^{n-1} D_{jm}(n)x_j \\
&\qquad + x_m^4 \sum_{k=m+1}^{n-1} D_{mk}(n)x_k^3 \sum_{j=m+1}^{n-1} D_{jm}(n)x_j \\
&\qquad + \tfrac{1}{2}\big(\sum_{k=1}^{n-1} C_{mk}(n)x_k^2 x_m)\big)^2 + \tfrac{1}{2}\big(\sum_{k=m+1}^{n-1} D_{mk}(n)x_k^3 x_m)\big)^2 \\
&\qquad + \tfrac{1}{2}\big(\sum_{j=m+1}^{n-1} D_{jm}(n)x_m^3 x_j)\big)^2 \\
&\qquad + O(H^4)\Big) \\
&= \psi_{m+1}(\boldsymbol{x})\exp\big(O(n^{-\eta})\big)
\end{aligned} \tag{2.2}$$

uniformly over m, where

$$H = B_m(n)nx_m^3 + \sum_{k=1}^{n-1} C_{mk}(n)x_k^2x_m + \sum_{k=1}^{n-1} D_{mk}(n)x_k^3x_m + \sum_{j=m+1}^{n-1} D_{jm}(n)x_m^3x_j.$$

Also

$$f(\boldsymbol{x}) = \psi_1(\boldsymbol{x})\exp\big(O(n^{-\delta}+n^{-1/2+3\epsilon})\big), \tag{2.3}$$

$$\psi_n = \exp\Big(-A(n)n\sum_{k=1}^{n-1} x_k^2 + n\sum_{k=1}^{n-1} E_k(n)x_k^4 + F(n)\Big(\sum_{k=1}^{n-1} x_k^2\Big)^2 + \frac{1}{2}\sum_{k=1}^{n-1} B_k(n)^2n^2x_k^6\Big).$$

In K_2 we have $\mu_2 = (2A(n))^{-1}(1+O(n^{-\Delta}))$, so

$$\psi_n(\boldsymbol{x}) = \exp\Big(-A(n)n\sum_{k=1}^{n-1} x_k^2 + n\sum_{k=1}^{n-1} E_k(n)x_k^4 + \frac{1}{2}\sum_{k=1}^{n-1} B_k(n)^2n^2x_k^6 + \frac{F(n)}{4A(n)^2} + O(n^{-\Delta})\Big).$$

The integral of ψ_n over $U_{n-1}(n^{-1/2+\epsilon})$ differs from that over K_2 by at most

$$\Big(\frac{\pi}{A(n)n}\Big)^{(n-1)/2} \exp\big(-n^{1-2\Delta} + O(n^{1/2+\epsilon})\big),$$

as in the estimation of the integral of f over $K_1 \cup K_3$. Furthermore,

$$\int_{U_{n-1}(n^{-1/2+\epsilon})} \psi_n(\boldsymbol{x})\,d\boldsymbol{x} = \exp\Big(\frac{F(n)}{4A(n)^2} + O(n^{-\Delta})\Big) \times \prod_{k=1}^{n-1} \int_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} \exp\big(-A(n)nx^2 + E_k(n)nx^4 + \tfrac{1}{2}B_k(n)^2n^2x^6\big)\,dx$$

$$
\begin{aligned}
&= \exp\Bigl(\frac{F(n)}{4A(n)^2} + O(n^{-\Delta})\Bigr) \\
&\quad \times \prod_{k=1}^{n-1} \int_{-n^{-1/2+\epsilon}}^{n^{-1/2+\epsilon}} e^{-A(n)nx^2}\bigl(1 + E_k(n)nx^4 + \tfrac{1}{2}B_k(n)^2 n^2 x^6 + O(n^{-2+8\epsilon})\bigr)\, dx \\
&= \exp\Bigl(\frac{F(n)}{4A(n)^2} + O(n^{-\Delta})\Bigr)\Bigl(\frac{\pi}{A(n)n}\Bigr)^{(n-1)/2} \\
&\quad \times \prod_{k=1}^{n-1} \exp\Bigl(\frac{3E_k(n)n}{4A(n)^2n^2} + \frac{15B_k(n)^2}{16A(n)^3 n} + O(n^{-2+8\epsilon})\Bigr) \\
&= \Bigl(\frac{\pi}{A(n)n}\Bigr)^{(n-1)/2} \\
&\quad \times \exp\Bigl(\frac{3\sum_{k=1}^{n-1} E_k(n)}{4A(n)^2 n} + \frac{F(n)}{4A(n)^2} + O(n^{-1/2+8\epsilon} + n^{-\Delta})\Bigr), \qquad (2.4)
\end{aligned}
$$

since

$$
\int_{-\infty}^{\infty} x^{2k} e^{-A(n)nx^2}\, dx = \frac{(2k)!}{k!\,(4A(n)n)^k}\sqrt{\frac{\pi}{A(n)n}} \quad \text{for } k \geq 0.
$$

In the following, any expression Q^* denotes the expression Q with all occurrences of $B_k(n)$ and $C_{jk}(n)$ replaced by 0. Also, all integrals will be over $U_{n-1}(n^{-1/2+\epsilon})$.

Since $|\psi_1| = \psi_1^*$, (2.1) and (2.2) imply that

$$
\int |\psi_1| = \exp\bigl(O(n^{1-\eta})\bigr) \int \psi_n^*, \qquad (2.5)
$$

since all the integrands involved are real. We also have for $2 \leq m \leq n$

$$
\begin{aligned}
\int |\bar{\psi}_m| &\leq \int \tfrac{1}{2}\bigl(|\psi_m(\boldsymbol{x})| + |\psi_m(x_1, \ldots, x_{m-1}, -x_m, x_{m+1}, \ldots, x_n)|\bigr) \\
&= \int |\psi_m| \\
&= \exp\bigl(O(n^{-\eta})\bigr) \int |\bar{\psi}_{m-1}|, \quad \text{by (2.2)},
\end{aligned}
$$

which implies that

$$
\int |\bar{\psi}_m| \leq \exp\bigl(O(n^{1-\eta})\bigr) \int |\psi_1|
$$

for $m = 1, 2, \ldots, n$.

From (2.2) we now have, for $m = 1, 2, \ldots, n-1$,

$$\begin{aligned}\left|\int \bar{\psi}_m - \int \psi_{m+1}\right| &= O(n^{-\eta}) \int |\bar{\psi}_m| \\ &= O(n^{-\eta}) \int |\psi_1| \\ &= O(n^{-\eta}) \int \psi_n^*, \quad \text{by (2.5).}\end{aligned}$$

Similarly, by (2.3),

$$\begin{aligned}\left|\int f - \int \psi_1\right| &= O(n^{-\delta}) \int |\psi_1| \\ &= O(n^{-\delta}) \int \psi_n^*.\end{aligned}$$

Thus, by (2.1),

$$\begin{aligned}\left|\int f - \int \psi_n\right| &\leq \left|\int f - \int \psi_1\right| + \left|\int \bar{\psi}_1 - \int \psi_2\right| \\ &\quad + \left|\int \bar{\psi}_2 - \int \psi_3\right| + \cdots + \left|\int \bar{\psi}_{n-1} - \int \psi_n\right| \\ &= O(n^{1-\eta} + n^{-\delta}) \int \psi_n^*.\end{aligned}$$

That is,

$$\int f = \exp\big(O((n^{1-\eta} + n^{-\delta})Z')\big) \int \psi_n, \tag{2.6}$$

where $Z' = |\int \psi_n^* / \int \psi_n| = 1$.

The theorem follows from (2.4), (2.6), and the fact that the integral over $K_1 \cup K_3$ is negligible. ∎

Chapter 4

Asymptotic Enumeration of Tournaments with a Given Score Sequence

4.1. Introduction.

A *tournament* is a digraph in which, for each pair of distinct vertices v and w, either (v,w) or (w,v) is an edge, but not both. A tournament is *regular* if the in-degree is equal to the out-degree at each vertex. Let $v_1, v_2, \ldots, v_n$ be the vertices of a labelled tournament and let d_j^-, d_j^+ be the in-degree and out-degree of v_j for $1 \le j \le n$. Define $\delta_j = d_j^+ - d_j^-$ and call $\delta_1, \delta_2, \ldots, \delta_n$ the *excess sequence* of the tournament. We also define $d_1^+, d_2^+, \ldots, d_n^+$ to be the *score sequence*. Let $NT(n;\delta_1,\ldots,\delta_n)$ be the number of labelled tournaments with n vertices and excess sequence $\delta_1,\ldots,\delta_n$. As in [17], let $RT(n) = NT(n;0,\ldots,0)$ be the number of labelled regular tournaments with n vertices.

The first attack that we are aware of on the asymptotics of the regular tournaments was due to Joel Spencer [37]. In particular, Spencer evaluated $RT(n)$ to within a factor of $(1+o(1))^n$. Also, Spencer obtained

$$NT(n,\delta_1,\ldots,\delta_n) = RT(n)\exp\Big(\big(-\tfrac{1}{2}+o(1)\big)\frac{\sum_{j=1}^n d_j^2}{n}\Big).$$

Recently, B. D. McKay [17] obtained the following much more accurate estimate of $RT(n)$ as $n \to \infty$, for any $\epsilon > 0$,

$$RT(n) = \Big(\frac{2^{n+1}}{\pi n}\Big)^{(n-1)/2} n^{1/2} e^{-1/2}\big(1+O(n^{-1/2+\epsilon})\big) \qquad (n \quad odd).$$

It is easy to see that $RT(n) = 0$ if n even. Exact values of $RT(n)$ for $n \leq 21$ were also obtained by B. D. McKay [20].

We are concerned with the asymptotic value of $NT(n; \delta_1, \ldots, \delta_n)$. We identify the required quantity as a coefficient in a n-variable power series, and estimate it by applying the saddle-point method to the integral provided by Cauchy's Theorem.

4.2. Calculations.

In this section, we will identify $NT(n; \delta_1, \ldots, \delta_n)$ as a coefficient of the generating function of tournaments $\prod_{1 \leq j < k \leq n}(x_j^{-1}x_k + x_j x_k^{-1})$ and then apply Cauchy's Theorem to convert the quantity into a n-dimensional integral. By choosing suitable contours, we can make the linear items within the exponential of the integrand vanish. We then employ a few linear transformations in order to diagonalize the quadratic terms.

4.2.1. Saddle point method.

Since the generating function $\prod_{1 \leq j < k \leq n}(x_j^{-1}x_k + x_j x_k^{-1})$ enumerates tournaments by excess of out-degree over in-degree at each vertex, $NT(n; \delta_1, \ldots, \delta_n)$ is the coefficient of $x_1^{\delta_1} \cdots x_n^{\delta_n}$ in $\prod_{1 \leq j < k \leq n}(x_j^{-1}x_k + x_j x_k^{-1})$. By Cauchy's Theorem,

$$NT(n; \delta_1, \ldots, \delta_n) = \frac{1}{(2\pi i)^n} \oint \cdots \oint \frac{\prod_{1 \leq j < k \leq n}(x_j^{-1}x_k + x_j x_k^{-1})}{x_1^{\delta_1+1} \cdots x_n^{\delta_n+1}} dx_1 \cdots dx_n,$$

where each integration is around a simple closed contour encircling the origin once in the anticlockwise direction. We first choose the jth contour to be the circle of radius r_j and then substitute $x_j = r_j e^{i\theta_j}$ for $1 \le j \le n$. We obtain

$$NT(n;\delta_1,\ldots,\delta_n) = \frac{I}{(2\pi)^n \prod_{1\le j\le n} r_j^{\delta_j}},$$

where

$$I = \int_{U_n(\pi)} \frac{\prod_{1\le j<k\le n}\left(\frac{r_k}{r_j}\exp\big(i(\theta_k-\theta_j)\big) + \frac{r_j}{r_k}\exp\big(i(\theta_j-\theta_k)\big)\right)}{\exp\big(i\sum_{1\le j\le n}(\delta_j\theta_j)\big)}\, d\boldsymbol{\theta}. \quad (2.1)$$

Defining

$$g(\boldsymbol{\theta}) = \frac{\prod_{1\le j<k\le n}\left(\frac{r_k}{r_j}\exp\big(i(\theta_k-\theta_j)\big) + \frac{r_j}{r_k}\exp\big(i(\theta_j-\theta_k)\big)\right)}{\prod_{1\le j<k\le n}\left(\frac{r_k}{r_j}+\frac{r_j}{r_k}\right)\exp\big(i\sum_{1\le j\le n}(\delta_j\theta_j)\big)},$$

we have

$$I = I_1 \prod_{1\le j<k\le n} \left(\frac{r_k}{r_j}+\frac{r_j}{r_k}\right),$$

where

$$I_1 = \int_{U_n(\pi)} g(\boldsymbol{\theta})\, d\boldsymbol{\theta}. \quad (2.2)$$

For each j we have that $d_j^+ + d_j^- = n-1$ and $d_j^+ - d_j^- = \delta_j$, so we have $2d_j^+ = n-1+\delta_j$ for $1\le j\le n$. Thus if n is odd, all the δ_j will be even and if n is even all the δ_j will be odd. Therefore, we know that translation of any θ_j by π leaves the integrand unchanged.

We will begin the evaluation of I_1 with the part of the integrand which will turn out to give the major contribution. Let I_2 be the contribution to I_1 of those $\boldsymbol{\theta}$ such that either $|\theta_j-\theta_n| \le n^{-1/2+\epsilon/4}$ or $|\theta_j-\theta_n+\pi| \le n^{-1/2+\epsilon/4}$ for $1\le j\le n-1$, where θ_j values are taken mod 2π. We now prove that the contribution to I_2 with different values of θ_n are the same. By transformation $\theta_j \mapsto \phi_j + \phi_n$ for

$1 \le j \le n-1$ and $\theta_n \longmapsto \phi_n$, we have

$$\begin{aligned}\int_{|\theta_j-\theta_n|\le n^{-1/2+\epsilon/4}} g(\boldsymbol{\theta})\,d\boldsymbol{\theta} &= \int_{-\pi}^{\pi}\Bigl(\int_{|\theta_j-\theta_n|\le n^{-1/2+\epsilon/4}} g(\boldsymbol{\theta})\,d\boldsymbol{\theta}'\Bigr)d\theta_n \\ &= \int_{-\pi}^{\pi}\Bigl(\int_{|\phi_j|\le n^{-1/2+\epsilon/4}} g(\phi_1,\ldots,\phi_{n-1},0)\,d\boldsymbol{\phi}'\Bigr)d\phi_n \\ &= 2\pi\int_{|\phi_j|\le n^{-1/2+\epsilon/4}} g(\phi_1,\ldots,\phi_{n-1},0)\,d\boldsymbol{\phi}',\end{aligned}$$

where $\boldsymbol{\phi}'=(\phi_1,\ldots,\phi_{n-1})$ and $\boldsymbol{\theta}'=(\theta_1,\ldots,\theta_{n-1})$ since $\sum_{j=1}^{n}\delta_j=0$. Considering the other $2^{n-1}-1$ relevant regions similarly, we obtain

$$I_2 = 2^n\pi\int_{U_{n-1}(n^{-1/2+\epsilon/4})} g(\boldsymbol{\theta})\,d\boldsymbol{\theta}',$$

where $\boldsymbol{\theta}'=(\theta_1,\ldots,\theta_{n-1})$ with $\theta_n=0$.

Since

$$\exp(ix) = 1+ix-\tfrac{1}{2}x^2-\tfrac{1}{6}ix^3+\tfrac{1}{24}x^4+O(x^5),$$

putting

$$T_{jk}(\boldsymbol{\theta}) = \frac{\frac{r_k}{r_j}\exp\bigl(i(\theta_k-\theta_j)\bigr)+\frac{r_j}{r_k}\exp\bigl(i(\theta_j-\theta_k)\bigr)}{\frac{r_k}{r_j}+\frac{r_j}{r_k}},$$

we have

$$\begin{aligned}T_{jk}(\boldsymbol{\theta}) = 1 &+ \frac{r_k^2-r_j^2}{r_k^2+r_j^2}i(\theta_k-\theta_j) - \tfrac{1}{2}(\theta_k-\theta_j)^2 - \tfrac{1}{6}\frac{r_k^2-r_j^2}{r_k^2+r_j^2}i(\theta_k-\theta_j)^3 \\ &+ \tfrac{1}{24}(\theta_k-\theta_j)^4 + O(|\theta_k-\theta_j|^5).\end{aligned}$$

Because $\log(1+z)=z-\frac{1}{2}z^2+\frac{1}{3}z^3-\frac{1}{4}z^4+O(z^5)$ for complex z, we obtain

$$\begin{aligned}g(\boldsymbol{\theta}) &= \frac{\prod_{1\le j<k\le n}T_{jk}(\boldsymbol{\theta})}{\exp\bigl(i\sum_{1\le j\le n}(\delta_j\theta_j)\bigr)} \\ &= \exp\Bigl(\sum_{1\le j<k\le n}\log T_{jk}(\boldsymbol{\theta}) - i\sum_{1\le j\le n}(\delta_j\theta_j)\Bigr)\end{aligned}$$

$$= \exp\Big(\sum_{1\le j\le n} \big(\sum_{1\le k\le n} -\frac{r_k^2 - r_j^2}{r_k^2 + r_j^2} - \delta_j\big) i\theta_j$$

$$+ \sum_{1\le j<k\le n} \big(-\tfrac{1}{2} + \tfrac{1}{2}\big(\frac{r_k^2 - r_j^2}{r_k^2 + r_j^2}\big)^2\big)(\theta_k - \theta_j)^2$$

$$+ \sum_{1\le j<k\le n} \big(\tfrac{1}{3}\frac{r_k^2 - r_j^2}{r_k^2 + r_j^2} - \tfrac{1}{3}\big(\frac{r_k^2 - r_j^2}{r_k^2 + r_j^2}\big)^3\big) i(\theta_k - \theta_j)^3$$

$$+ \sum_{1\le j<k\le n} \big(-\tfrac{1}{12} + \tfrac{1}{3}\big(\frac{r_k^2 - r_j^2}{r_k^2 + r_j^2}\big)^2 - \tfrac{1}{4}\big(\frac{r_k^2 - r_j^2}{r_k^2 + r_j^2}\big)^4\big)(\theta_k - \theta_j)^4$$

$$+ O\big(\sum_{1\le j<k\le n} |\theta_k - \theta_j|^5 \big)\Big). \tag{2.3}$$

4.2.2. Choices of contours.

We choose suitable r_j for $1 \le j \le n$ so that the coefficient of the linear item of θ_j within the exponential in the integrand $g(\boldsymbol{\theta})$ will be zero for $1 \le j \le n$.

We need $r_1, \ldots, r_n$ such that

$$\sum_{k=1}^{n} \frac{r_j^2 - r_k^2}{r_k^2 + r_j^2} = \delta_j, \qquad (1 \le j \le n). \tag{2.4}$$

Substitute $r_j^2 = (1 + b_j)/(1 - b_j)$ for $1 \le j \le n$ to get

$$\sum_{k=1}^{n} \frac{b_j - b_k}{1 - b_j b_k} = \delta_j, \qquad (1 \le j \le n). \tag{2.5}$$

Let us consider the functions $f_1, \ldots, f_n$ defined by

$$f_j(\boldsymbol{b}) = \frac{\delta_j}{n} - \frac{1}{n}\sum_{k=1}^{n} \frac{b_j b_k (b_j - b_k)}{1 - b_j b_k}, \qquad (1 \le j \le n). \tag{2.6}$$

Further define $\boldsymbol{b}^{(0)} = (\delta_1/n, \ldots, \delta_n/n)$ and

$$b_j^{(i)} = f_j(\boldsymbol{b}^{(i-1)}), \qquad i = 1, 2, \ldots \tag{2.7}$$

We will prove that the sequence $\boldsymbol{b}^{(0)}, \boldsymbol{b}^{(1)}, \ldots$ converges to a vector $\boldsymbol{b}^{(\infty)}$.

Let $\|\cdot\|$ be the norm on $\mathbb{R}^n$, defined by $\|(x_1, \ldots, x_n)\| = \max_{1\le k\le n}|x_k|$. Define $E = \|(\delta_1, \ldots, \delta_n)\|/n$. Assume that $E = o(n^{-1/4+\epsilon})$ and $E \le 1/100$.

Lemma 4.2.1. $\|\boldsymbol{b}^{(i+1)} - \boldsymbol{b}^{(i)}\| \leq 3E^{3+2i}15^i$.

Proof. We prove the lemma by induction on i. For $i = 0$, we have

$$\|\boldsymbol{b}^{(1)} - \boldsymbol{b}^{(0)}\| = \frac{1}{n^4}\max_{1\leq j\leq n}\left|\sum_{k=1}^{n}\frac{\delta_j\delta_k(\delta_j - \delta_k)}{1-\delta_j\delta_k/n^2}\right|$$
$$\leq \frac{2E^3}{1-E^2} \leq 3E^3.$$

Now suppose the lemma is true for some $i \geq 0$. We prove the lemma is also true for $i+1$. We have

$$\begin{aligned}\|\boldsymbol{b}^{(i)}\| &\leq \|\boldsymbol{b}^{(0)}\| + \|\boldsymbol{b}^{(1)} - \boldsymbol{b}^{(0)}\| + \cdots + \|\boldsymbol{b}^{(i)} - \boldsymbol{b}^{(i-1)}\| \\ &\leq E + \sum_{k=0}^{\infty} 3E^{3+2k}15^k \\ &= E + \frac{3E^3}{1-15E^2}. \end{aligned} \tag{2.8}$$

Define $B = E + \frac{3E^3}{1-15E^2}$. Then, by the induction hypothesis $\|\boldsymbol{b}^{(i)} - \boldsymbol{b}^{(i-1)}\| \leq 3E^{2i+1}15^{i-1} \leq 3E^3$ and routine computation on $f_j(\boldsymbol{b}^{(i)}) - f_j(\boldsymbol{b}^{(i-1)})$, we obtain

$$\|\boldsymbol{b}^{(i+1)} - \boldsymbol{b}^{(i)}\| \leq \|\boldsymbol{b}^{(i)} - \boldsymbol{b}^{(i-1)}\|R,$$

where

$$R = \frac{2B(3B+B^2) + 6BE^3(1+B^2) + 6BE^3(2+B^2) + 18E^6(1+B^2)}{(1-B^2)(1-B^2-6BE^3-9E^6)}$$
$$\leq 2\big(2B(3B+B^2) + 6BE^3(1+B^2) + 6BE^3(2+B^2) + 18E^6(1+B^2)\big).$$

Denote the last quantity as H. Since $B \leq 2E$, we have

$$\frac{H}{E^2} \leq 2\big((6+8E) + +6BE(1+B^2) + 6BE(2+B^2) + 18E^4(1+B^2)\big) \leq 15.$$

Therefore

$$\|\boldsymbol{b}^{(i+1)} - \boldsymbol{b}^{(i)}\| \le 15E^2 \|\boldsymbol{b}^{(i)} - \boldsymbol{b}^{(i-1)}\|$$
$$\le 3E^{3+2i}15^i. \quad \blacksquare$$

Lemma 4.2.2. *$\boldsymbol{b}^{(0)}, \boldsymbol{b}^{(1)}, \ldots$ converges to a vector $\boldsymbol{b}^{(\infty)}$. Furthermore,*

$$\|\boldsymbol{b}^{(\infty)} - \boldsymbol{b}^{(i)}\| \le \frac{3E^{3+2i}15^i}{1 - 3E^{3+2i}15^i}.$$

for each $i \ge 0$.

Proof. For any $j > i$, by Lemma 4.2.1 we have

$$\|\boldsymbol{b}^{(j)} - \boldsymbol{b}^{(i)}\| \le \|\boldsymbol{b}^{(i+1)} - \boldsymbol{b}^{(i)}\| + \|\boldsymbol{b}^{(i+2)} - \boldsymbol{b}^{(i+1)}\| + \cdots + \|\boldsymbol{b}^{(j)} - \boldsymbol{b}^{(j-1)}\|$$
$$\le \frac{3E^{3+2i}15^i}{1 - 3E^{3+2i}15^i}.$$

Hence the series is Cauchy convergent and thus convergent. Hence $\boldsymbol{b}^{(\infty)}$ exists and the bound on $\|\boldsymbol{b}^{(\infty)} - \boldsymbol{b}^{(i)}\|$ follows on taking $j \to \infty$. $\blacksquare$

Lemma 4.2.3. *Let $\boldsymbol{b}^{(\infty)} = (b_1^\infty, \ldots, b_n^\infty)$, $r_j = \left((1+b_j^\infty)/(1-b_j^\infty)\right)^{1/2}$ for $1 \le j \le n$. Then $r_1, \ldots, r_n$ is a solution for equations (2.4).*

Proof. Since $f_1, \ldots, f_n$ are continuous, by (2.7) we have $b_j^\infty = f_j(\boldsymbol{b}^\infty)$. That is, for $1 \le j \le n$ we have

$$b_j^\infty = \frac{\delta_j}{n} - \frac{1}{n}\sum_{k=1}^{n} \frac{b_j^\infty b_k^\infty (b_j^\infty - b_k^\infty)}{1 - b_j^\infty b_k^\infty}$$
$$= \frac{\delta_j}{n} - \frac{1}{n}\sum_{k=1}^{n} \left(\frac{b_j^\infty - b_k^\infty}{1 - b_j^\infty b_k^\infty} + b_k^\infty - b_j^\infty\right),$$

hence

$$\delta_j = \sum_{k=1}^{n} \frac{b_j^\infty - b_k^\infty}{1 - b_j^\infty b_k^\infty} + \sum_{k=1}^{n} b_k^\infty,$$

and since $\sum_{j=1}^n \delta_j = 0$, we obtain

$$\begin{aligned}\sum_{j=1}^{n} b_j^\infty &= \sum_{j=1}^{n} \frac{\delta_j}{n} - \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{b_j^\infty b_k^\infty (b_j^\infty - b_k^\infty)}{1 - b_j^\infty b_k^\infty} \\ &= 0.\end{aligned}$$

So $b_1^\infty, \ldots, b_n^\infty$ satisfies (2.5) and thus $r_1, \ldots, r_n$ satisfies (2.4). ∎

For $1 \le j \le n$, define

$$\begin{aligned} w_j = \frac{\delta_j}{n} &+ \frac{\delta_j \sum_{k=1}^n \delta_k^2}{n^4} + \frac{-\delta_j^3 \sum_{k=1}^n \delta_k^2 + \delta_j^2 \sum_{k=1}^n \delta_k^3}{n^6} \\ &+ \frac{3\delta_j \big(\sum_{k=1}^n \delta_k^2\big)^2}{n^7} + \frac{-\delta_j^4 \sum_{k=1}^n \delta_k^3 + \delta_j^3 \sum_{k=1}^n \delta_k^4}{n^8} \\ &+ \frac{-6\delta_j^3 \big(\sum_{k=1}^n \delta_k^2\big)^2 + 6\delta_j^2 \sum_{k=1}^n \delta_k^2 \sum_{k=1}^n \delta_k^3}{n^9} \\ &+ \frac{2\delta_j \big(\sum_{k=1}^n \delta_k^3\big)^2 - 2\delta_j \sum_{k=1}^n \delta_k^2 \sum_{k=1}^n \delta_k^4}{n^9} \\ &+ \frac{12\delta_j \big(\sum_{k=1}^n \delta_k^2\big)^3}{n^{10}}. \end{aligned}$$

By computation, we have that $\|\boldsymbol{b}^{(4)} - \boldsymbol{w}\| = O(E^9)$. Therefore, by Lemma 4.2.2, we have

Lemma 4.2.4. $\|\boldsymbol{b}^{(\infty)} - \boldsymbol{w}\| = O(E^9)$. ∎

4.2.3. Transformations.

We diagonalize the quadratic form within the exponential in the integrand of (2.3) by transformations.

By Lemma 4.2.3, the integrand (2.3) has now been simplified as

$$g(\boldsymbol{\theta}) = \exp\Big(\sum_{1 \le j < k \le n} \big(-\tfrac{1}{2} + \tfrac{1}{2}\big(\frac{r_k^2 - r_j^2}{r_k^2 + r_j^2}\big)^2\big)(\theta_k - \theta_j)^2$$

$$+ \sum_{1\le j<k\le n} \left(\tfrac{1}{3}\frac{r_k^2 - r_j^2}{r_k^2 + r_j^2} - \tfrac{1}{3}\left(\frac{r_k^2 - r_j^2}{r_k^2 + r_j^2}\right)^3\right) i(\theta_k - \theta_j)^3$$
$$+ \sum_{1\le j<k\le n} \left(-\tfrac{1}{12} + \tfrac{1}{3}\left(\frac{r_k^2 - r_j^2}{r_k^2 + r_j^2}\right)^2 - \tfrac{1}{4}\left(\frac{r_k^2 - r_j^2}{r_k^2 + r_j^2}\right)^4\right)(\theta_k - \theta_j)^4$$
$$+ O\Big(\sum_{1\le j<k\le n} |\theta_k - \theta_j|^5\Big)\Big), \tag{2.9}$$

with $\theta_n = 0$. For convenience, we denote $\boldsymbol{b}^\infty = (b_1, \ldots, b_n)$, we obtain

$$g(\boldsymbol{\theta}) = \exp\Big(\sum_{1\le j<k\le n} \left(-\tfrac{1}{2} + \tfrac{1}{2}\left(\frac{b_k - b_j}{1 - b_j b_k}\right)^2\right)(\theta_k - \theta_j)^2$$
$$+ \sum_{1\le j<k\le n-1} \left(\tfrac{1}{3}\frac{b_k - b_j}{1 - b_j b_k} - \tfrac{1}{3}\left(\frac{b_k - b_j}{1 - b_j b_k}\right)^3\right) i(\theta_k - \theta_j)^3$$
$$+ \sum_{1\le j\le n-1} \left(-\tfrac{1}{3}\frac{b_n - b_j}{1 - b_j b_n} + \tfrac{1}{3}\left(\frac{b_n - b_j}{1 - b_j b_n}\right)^3\right) i\theta_j^3$$
$$+ \sum_{1\le j<k\le n-1} \left(-\tfrac{1}{12} + \tfrac{1}{3}\left(\frac{b_k - b_j}{1 - b_j b_k}\right)^2 - \tfrac{1}{4}\left(\frac{b_k - b_j}{1 - b_j b_k}\right)^4\right)(\theta_k - \theta_j)^4$$
$$+ \sum_{1\le j\le n-1} \left(-\tfrac{1}{12} + \tfrac{1}{3}\left(\frac{b_n - b_j}{1 - b_j b_n}\right)^2 - \tfrac{1}{4}\left(\frac{b_n - b_j}{1 - b_j b_n}\right)^4\right)\theta_j^4$$
$$+ O\Big(\sum_{1\le j<k\le n} |\theta_k - \theta_j|^5\Big)\Big)$$
$$= \exp\Big(\sum_{1\le j<k\le n} \left(-\tfrac{1}{2} + \tfrac{1}{2}\left(\frac{b_k - b_j}{1 - b_j b_k}\right)^2\right)(\theta_k - \theta_j)^2$$
$$+ \sum_{1\le j<k\le n-1} \left(\tfrac{1}{3}\frac{b_k - b_j}{1 - b_j b_k} - \tfrac{1}{3}\left(\frac{b_k - b_j}{1 - b_j b_k}\right)^3\right) i(\theta_k - \theta_j)^3$$
$$+ \sum_{1\le j<k\le n-1} \left(-\tfrac{1}{12} + \tfrac{1}{3}\left(\frac{b_k - b_j}{1 - b_j b_k}\right)^2 - \tfrac{1}{4}\left(\frac{b_k - b_j}{1 - b_j b_k}\right)^4\right)(\theta_k - \theta_j)^4$$
$$+ O\big(n^{-1/2+5\epsilon}\big)\Big), \tag{2.10}$$

when $\boldsymbol{\theta}' \in U_{n-1}(n^{-1/2+\epsilon})$ with $\theta_n = 0$.

Define $V = U_{n-1}(n^{-1/2+\epsilon})$ and let $T : \mathbb{R}^{n-1} \mapsto \mathbb{R}^{n-1}$ be the linear transformation defined by $T : \boldsymbol{\theta}' \mapsto \boldsymbol{y} = (y_1, y_2, \ldots, y_{n-1})$, where

$$y_j = \theta_j - \sum_{k=1}^{n-1} \theta_k/(n + n^{1/2})$$

for $1 \leq j \leq n-1$. Let $V_1 = T(V)$. By straightforward calculations we have

$$V_1 = \{\boldsymbol{y} \mid |y_j + \sum_{k=1}^{n-1} y_k/(n^{1/2}+1)| \leq n^{-1/2+\epsilon} \text{ for } 1 \leq j \leq n-1\},$$

and

$$\det(T) = n^{1/2}.$$

Putting $a_{kj} = (b_k - b_j)/(1 - b_j b_k)$ and $s = 1/(n^{1/2}+1)$, we obtain

$$\begin{aligned} g(\boldsymbol{\theta}) = \exp\Big(& \sum_{k=1}^{n-1}\Big(-\frac{n}{2} + \tfrac{1}{2}\sum_{j=1}^{n} a_{kj}^2 + a_{nk}^2 s + \tfrac{1}{2}s^2 \sum_{l=1}^{n-1} a_{nl}^2\Big) y_k^2 \\ & + \sum_{j \neq k}\Big(-\tfrac{1}{2}a_{kj}^2 + a_{nj}^2 s + \tfrac{1}{2}s^2 \sum_{l=1}^{n-1} a_{nl}^2\Big) y_j y_k \\ & + \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}\big(\tfrac{1}{3}a_{kj} - \tfrac{1}{3}a_{kj}^3\big)\Big) i y_k^3 \\ & + \sum_{j \neq k}(-a_{kj} + O(E^2)) i y_k^2 y_j \\ & + \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}\big(-\tfrac{1}{12} + \tfrac{1}{3}a_{kj}^2 - \tfrac{1}{4}a_{kj}^4\big) + \tfrac{1}{12}\Big) y_k^4 \\ & + \sum_{j \neq k} O(1) y_k^3 y_j \\ & + \sum_{j \neq k}(-\tfrac{1}{4} + O(E^2)) y_k^2 y_j^2 \\ & + O(n^{-1/2+5\epsilon})\Big), \end{aligned} \tag{2.11}$$

where $s = 1/(n^{1/2}+1)$, each $O(\ )$ term is uniform over the subscript set of the sum involved, S_3 is the set of 3-subsets of $\{1, 2, \ldots, n-1\}$ and S_4 is the set of 4-subsets of $\{1, 2, \ldots, n-1\}$.

Denote

$$u_k = -\frac{n}{2} + \tfrac{1}{2}\sum_{j=1}^{n} a_{kj}^2 + a_{nk}^2 s + \tfrac{1}{2}s^2 \sum_{l=1}^{n-1} a_{nl}^2,$$

$$v_{kj} = -\tfrac{1}{2}a_{kj}^2 + \tfrac{1}{2}(a_{nj}^2 + a_{nk}^2)s + \tfrac{1}{2}s^2 \sum_{l=1}^{n-1} a_{nl}^2, \qquad \text{for } k < j,$$

and $v_{kj} = v_{jk}$ for $k > j, v_{kk} = u_k$. Define A to be the diagonal matrix with entries $u_1, \ldots, u_{n-1}, V$ the $n-1 \times n-1$ matrix with entries v_{jk} and $B = V - A$, I the identity matrix, and define a linear transformation by

$$\begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} = (I + A^{-1}B)^{-1/2} \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \end{pmatrix}.$$

By the formula $\det(X) = \exp(\operatorname{tr}\log(X))$, we have

$$\begin{aligned}
&\det\big((I + A^{-1}B)^{-1/2}\big) \\
&\quad = \exp\big(-\tfrac{1}{2}\operatorname{tr}\log\big((I + A^{-1}B)\big)\big) \\
&\quad = \exp\big(-\tfrac{1}{2}\operatorname{tr}\big(A^{-1}B - \tfrac{1}{2}A^{-1}BA^{-1}B + \tfrac{1}{3}A^{-1}BA^{-1}BA^{-1}B - \cdots\big)\big).
\end{aligned}$$

By Lemma 4.2.2, we have $\|\boldsymbol{b}\| \le 2E$. Hence $|a_{kj}| \le 8E$ and this implies that $|v_{jl}/u_k| \le cE^2/n$ for $1 \le j, k, l \le n-1$ for some constant $c > 0$. Therefore, the absolute value of each entry of $(A^{-1}B)^k$ is bounded by $c^k E^{2k}/n$. Hence, we have

$$\begin{aligned}
&\det\big((I + A^{-1}B)^{-1/2}\big) \\
&\quad \le \exp\big(cE^2 + c^2E^4 + \cdots\big) \\
&\quad = 1 + O(E^2).
\end{aligned}$$

Because

$$(I+A^{-1}B)^{-1/2}=I+\sum_{k=1}^{\infty}\binom{-\frac{1}{2}}{k}(A^{-1}B)^k,$$

as a matrix, each entry of $\sum_{k=1}^{\infty}\binom{-1/2}{k}(A^{-1}B)^k$ is $n^{-1}O(E^2)$ and

$$(I+BA^{-1})^{-1/2}(A+B)(I+A^{-1}B)^{-1/2}=A,$$

we obtain

$$\begin{aligned}
g(\boldsymbol{\theta}) = \exp\Big(&\sum_{k=1}^{n-1} u_k z_k^2\\
&+\sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}\big(\tfrac{1}{3}a_{kj}-\tfrac{1}{3}a_{kj}^3\big)+O(E^2)\Big)iz_k^3\\
&+\sum_{1\le j\ne k\le n-1}\big(-a_{kj}+O(E^2)\big)iz_k^2 z_j\\
&+\sum_{\{j,k,l\}\in S_3} n^{-7/4+3\epsilon}O(1)iz_j z_k z_l\\
&+\sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}\big(-\tfrac{1}{12}+\tfrac{1}{3}a_{kj}^2-\tfrac{1}{4}a_{kj}^4\big)+\tfrac{1}{12}+O(E^2)\Big)z_k^4\\
&+\sum_{1\le j\ne k\le n-1} O(1)z_k^3 z_j\\
&+\sum_{1\le j\ne k\le n-1}\big(-\tfrac{1}{4}+O(E^2)\big)z_k^2 z_j^2\\
&+\sum_{\{j,k,l\}\in S_3} n^{-3/2+2\epsilon}O(1)z_j^2 z_k z_l\\
&+\sum_{\{j,k,l,m\}\in S_4} n^{-3+4\epsilon}O(1)z_j z_k z_l z_m\\
&+O(n^{-1/2+5\epsilon})\Big).
\end{aligned}$$

Further define a linear transformation by

$$z_k=\Big(\frac{n}{-2u_k}\Big)^{1/2}x_k,\qquad 1\le k\le n-1,$$

Clearly, the determinant of the transformation is

$$\big(\tfrac{n}{2}\big)^{(n-1)/2}\prod_{k=1}^{n-1}(-u_k)^{-1/2},$$

and we obtain

$$
\begin{aligned}
g(\boldsymbol{\theta}) = \exp\Big(& \sum_{k=1}^{n-1} -\frac{n}{2}x_k^2 \\
&+ \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}\big(\tfrac{1}{3}a_{kj} - \tfrac{1}{3}a_{kj}^3\big)\big(\frac{n}{-2u_k}\big)^{3/2} + O(E^2)\Big)ix_k^3 \\
&+ \sum_{1\le j\ne k\le n-1} (-a_{kj} + O(E^2))ix_k^2x_j \\
&+ \sum_{\{j,k,l\}\in S_3} n^{-7/4+3\epsilon}O(1)ix_jx_kx_l \\
&+ \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}\big(-\tfrac{1}{12} + \tfrac{1}{3}a_{kj}^2 - \tfrac{1}{4}a_{kj}^4\big)\big(\frac{n}{-2u_k}\big)^2 + \tfrac{1}{12} + O(E^2)\Big)x_k^4 \\
&+ \sum_{1\le j\ne k\le n-1} O(1)x_k^3x_j \\
&+ \sum_{1\le j\ne k\le n-1} (-\tfrac{1}{4} + O(E^2))x_k^2x_j^2 \\
&+ \sum_{\{j,k,l\}\in S_3} n^{-3/2+2\epsilon}O(1)x_j^2x_kx_l \\
&+ \sum_{\{j,k,l,m\}\in S_4} n^{-3+4\epsilon}O(1)x_jx_kx_lx_m \\
&+ O(n^{-1/2+5\epsilon})\Big).
\end{aligned}
\tag{2.12}
$$

Let $T' : \boldsymbol{\theta}' \mapsto \boldsymbol{x}$ be the transformation involved in this section and $V' = T'(V)$. We know that $V' \subseteq U_{n-1}(3n^{-1/2+\epsilon})$. The asymptotic value of the integral of $f(\boldsymbol{x})$ over $U_{n-1}(3n^{-1/2+\epsilon})$ will be the same with that over $U_{n-1}(n^{-1/2+\epsilon})$. Furthermore, similar argument to that of Theorem 3.2.2 shows that the asymptotic value of the integral of $f(\boldsymbol{x})$ over $U_{n-1}(3n^{-1/2+\epsilon})\backslash V'$ is negligible. Therefore, we still keep the region as $U_{n-1}(n^{-1/2+\epsilon})$. When $\boldsymbol{x} \in U_{n-1}(n^{-1/2+\epsilon})$, we have that

$$
\sum_{\{j,k,l\}\in S_3} n^{-7/4+3\epsilon}O(1)ix_jx_kx_l = O(n^{-1/4+6\epsilon}),
$$

$$\sum_{\{j,k,l\}\in S_3} n^{-3/2+2\epsilon}O(1)x_j^2 x_k x_l = O(n^{-1/2+6\epsilon}),$$

and

$$\sum_{\{j,k,l,m\}\in S_4} n^{-3+4\epsilon}O(1)x_j x_k x_l x_m = O(n^{-1+8\epsilon}).$$

Hence, $g(\boldsymbol{\theta})$ is now simplified as

$$\begin{aligned} g(\boldsymbol{\theta}) = \exp\Big(&\sum_{k=1}^{n-1} -\frac{n}{2}x_k^2 \\ &+\sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}(\tfrac{1}{3}a_{kj}-\tfrac{1}{3}a_{kj}^3)\big(\frac{n}{-2u_k}\big)^{3/2}+O(E^2)\Big)ix_k^3 \\ &+\sum_{1\le j\ne k\le n-1}(-a_{kj}+O(E^2))ix_k^2x_j \\ &+\sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}(-\tfrac{1}{12}+\tfrac{1}{3}a_{kj}^2-\tfrac{1}{4}a_{kj}^4)\big(\frac{n}{-2u_k}\big)^2+\tfrac{1}{12}+O(E^2)\Big)x_k^4 \\ &+\sum_{1\le j\ne k\le n-1}O(1)x_k^3x_j \\ &+\sum_{1\le j\ne k\le n-1}(-\tfrac{1}{4}+O(E^2))x_k^2x_j^2 \\ &+O(n^{-1/4+6\epsilon})\Big). \end{aligned} \tag{2.13}$$

From now on we denote $p(\boldsymbol{x})$ to be the right hand side of (2.13). Now define

$$\begin{aligned} b(\boldsymbol{x}) = \exp\Big(&\sum_{k=1}^{n-1} -\frac{n}{2}x_k^2 \\ &+\sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}(\tfrac{1}{3}a_{kj}-\tfrac{1}{3}a_{kj}^3)\big(\frac{n}{-2u_k}\big)^{3/2}+O(E^2)\Big)ix_k^3 \\ &+\sum_{1\le j\ne k\le n-1}(-a_{kj}+O(E^2))ix_k^2x_j \\ &+\sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}(-\tfrac{1}{12}+\tfrac{1}{3}a_{kj}^2-\tfrac{1}{4}a_{kj}^4)\big(\frac{n}{-2u_k}\big)^2+\tfrac{1}{12}+O(E^2)\Big)x_k^4 \\ &+\sum_{1\le j\ne k\le n-1}O(1)x_k^3x_j \end{aligned}$$

$$
\begin{aligned}
&- \tfrac{1}{4}\Big(\sum_{k=1}^{n-1} x_k^2\Big)^2 \\
&+ \tfrac{1}{4}\sum_{k=1}^{n-1} x_k^4 \\
&+ O(n^{-1/4+6\epsilon})\Big). \qquad (2.14)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\Big|\int_{U_{n-1}(n^{-1/2+\epsilon})} (p(\boldsymbol{x}) - b(\boldsymbol{x}))\, d\boldsymbol{x}\Big| \\
&\le \int_{U_{n-1}(n^{-1/2+\epsilon})} |b(\boldsymbol{x})| \times \Big|\exp\Big(\sum_{1\le j\ne k\le n-1} O(E^2)x_k^2 x_j^2\Big) - 1\Big|\, d\boldsymbol{x} \\
&\le \int_{U_{n-1}(n^{-1/2+\epsilon})} |b(\boldsymbol{x})| O\Big(\sum_{1\le j\ne k\le n-1} O(E^2)x_k^2 x_j^2\Big)\, d\boldsymbol{x} \\
&\le O\big(n^{-1/2+6\epsilon}\big)\int_{U_{n-1}(n^{-1/2+\epsilon})} |b(\boldsymbol{x})|\, d\boldsymbol{x},
\end{aligned}
$$

hence

$$
\begin{aligned}
&\int_{U_{n-1}(n^{-1/2+\epsilon})} p(\boldsymbol{x})\, d\boldsymbol{x} \\
&\quad = \int_{U_{n-1}(n^{-1/2+\epsilon})} b(\boldsymbol{x})\, d\boldsymbol{x} + O(n^{-1/2+6\epsilon})\int_{U_{n-1}(n^{-1/2+\epsilon})} |b(\boldsymbol{x})|\, d\boldsymbol{x}. \qquad (2.15)
\end{aligned}
$$

Therefore by applying Theorem 3.2.2 to $b(\boldsymbol{x})$ and $|b(\boldsymbol{x})|$, we obtain

$$
\begin{aligned}
\int_{U_{n-1}(n^{-1/2+\epsilon})} p(\boldsymbol{x})\, d\boldsymbol{x} &= \Big(\frac{2\pi}{n}\Big)^{(n-1)/2} \exp\Big(\frac{3}{n}\sum_{k=1}^{n-1} E_k(n) - \tfrac{1}{4} + O(n^{-1/4+6\epsilon})\Big), \\
&= \Big(\frac{2\pi}{n}\Big)^{(n-1)/2} \exp\big(-\tfrac{1}{2} + O(n^{-1/4+6\epsilon})\big), \qquad (2.17)
\end{aligned}
$$

where

$$
E_k(n) = \frac{1}{n}\Big(\sum_{j=1}^{n-1}\big(-\tfrac{1}{12} + \tfrac{1}{3}a_{kj}^2 - \tfrac{1}{4}a_{kj}^4\big)\Big(\frac{n}{-2u_k}\Big)^2 + \tfrac{1}{3} + O(E^2)\Big).
$$

4.3. Main results.

The following lemma can be proved easily by Taylor series.

Lemma 4.3.1. *For sufficiently small $\epsilon > 0$, real λ and x with $|\lambda - 1/2| \le \epsilon$ and $|x| \le 5\pi/16$,*

$$\left| 1 - \lambda + \lambda \cos(x) \right| \le \exp(-\tfrac{1}{2}\lambda x^2). \quad \blacksquare$$

By the arguments of the last section, we have

$$I_2 = 2^n \pi n^{1/2} (1 + O(E^2)) \left(\frac{n}{2}\right)^{\frac{n-1}{2}} \prod_{k=1}^{n-1} (-u_k)^{-1/2} \left(\frac{2\pi}{n}\right)^{(n-1)/2}$$
$$\times \exp\left(-\tfrac{1}{2} + O(n^{-1/4+\epsilon})\right),$$

where

$$u_k = -\frac{n}{2} + \tfrac{1}{2} \sum_{j=1}^{n} a_{kj}^2 + a_{nk}^2 s + \tfrac{1}{2} s^2 \sum_{l=1}^{n-1} a_{nl}^2,$$
$$s = 1/(n^{1/2} + 1).$$

Since

$$\frac{n}{-2u_k} = \left(1 - \frac{1}{n} \sum_{j=1}^{n} a_{kj}^2\right)^{-1} + O(n^{-1} E^2),$$

we have

$$\left(\frac{n}{2}\right)^{\frac{n-1}{2}} \prod_{k=1}^{n-1} (-u_k)^{-1/2} = \exp(O(E^2)) \prod_{k=1}^{n-1} \left(1 - \frac{1}{n} \sum_{j=1}^{n} a_{kj}^2\right)^{-1/2}.$$

Hence, we have proved that if $\max\{|\delta_1|, \ldots, |\delta_n|\} = o(n^{3/4+\epsilon})$,

$$I_2 = 2^n \pi n^{1/2} \prod_{k=1}^{n-1} \left(1 - \frac{1}{n} \sum_{j=1}^{n} a_{kj}^2\right)^{-1/2} \left(\frac{2\pi}{n}\right)^{(n-1)/2}$$
$$\times \exp\left(-\tfrac{1}{2} + O(n^{-1/4+6\epsilon})\right). \tag{3.1}$$

So our remaining work is to prove that the contribution to I_1 of the parts other than the region of the integration of I_2 is negligible.

By Lemma 4.3.1, we have that, for $1 \le j, k \le n$,

$$\begin{aligned}|T_{jk}(\boldsymbol{\theta})| &= \left(1 - \lambda_{jk} + \lambda_{jk}\cos(2(\theta_j - \theta_k))\right)^{1/2} \\ &\le \exp\left(-\lambda_{jk}(\theta_j - \theta_k)^2\right),\end{aligned}$$

where

$$\lambda_{jk} = \frac{2r_k^2 r_j^2}{\left(r_k^2 + r_j^2\right)^2} = \tfrac{1}{2} + o(n^{-1/4+\epsilon}).$$

For $0 \le j \le 31$, define the interval $A_j = [(j-1)\pi/16, j\pi/16]$. For any $\boldsymbol{\theta} \in U_n(\pi)$, at least one of the 16 intervals $A_0 \cup A_1, A_2 \cup A_3, \ldots, A_{30} \cup A_{31}$ contains $n/16$ or more of the θ_j. Let us suppose that this is true of $A_0 \cup A_1$ (thereby undercounting the possibilities by at most a factor of 16). Define $B = A_3 \cup \cdots \cup A_{14} \cup A_{19} \cup \cdots \cup A_{30}$. If $\theta_j \in B$ and $\theta_k \in A_0 \cup A_1$, then $|\cos(\theta_k - \theta_j)| \le \cos(\pi/16)$. From this it easily follows that the contribution to I_1 of all the cases where n^ϵ or more of the θ_j lie in B is at most $O(\exp(-c_1 n^{1+\epsilon}))I_2$ for some $c_1 > 0$. Thus, with an undercount of at most 16, we can suppose that at least $n - n^\epsilon$ of the θ_j lie in $A_{31} \cup A_0 \cup A_1 \cup A_2 \cup A_{15} \cup \cdots \cup A_{18}$. At the expense of another factor of 2^n, we can suppose that $|\theta_j| \le \pi/2$ for all j and that $|\theta_j| \le \pi/8$ for at least $n - n^\epsilon$ of the θ_j. Now define $I_3(r)$ to be the integral of $g(\boldsymbol{\theta})$ on the region with those $\boldsymbol{\theta}$ such that

(i) $3\pi/16 \le |\theta_j| \le \pi/2$ for r values of j,

(ii) $|\theta_j| \le \pi/8$ for at least $n - n^\epsilon$ values of j, and

(iii) $\pi/8 \le |\theta_j| \le 3\pi/16$ for any other values of j.

Clearly $I_3(r) = 0$ if $r > n^\epsilon$. If θ_j and θ_k are in classes (i) and (ii), respectively, then $|\cos(\theta_k - \theta_j)| \leq \cos(\pi/16)$, while if they are both in calsses (ii) and (iii), $|T_{jk}(\boldsymbol{\theta})| \leq \exp(-\lambda_{jk}(\theta_j - \theta_k)^2)$. Using $|T_{jk}(\boldsymbol{\theta})| \leq 1$ for the other cases, we find

$$|I_3(r)| \leq \pi^r \big(1 - 2\lambda + 2\lambda \cos^2(\pi/16)\big)^{r(n-n^\epsilon)/2} \times \sum_{\{k_1,k_2,\ldots,k_{n-r}\}\in S_{n-r}} |I_3'(n-r)|, \tag{3.2}$$

where

$$\lambda = \min\{\lambda_{jk}\},$$

$$I_3'(m) = \int_{U_m(3\pi/16)} \prod_{1\leq s<t\leq m} \exp\big(-\lambda_{k_s k_t}(\theta_{k_s} - \theta_{k_t})^2\big)\, d\theta_{k_1} \cdots d\theta_{k_m}$$

and S_m is the set of the m-subsets of $\{1, 2, \ldots, n\}$. Now define $f_s = \lambda_{k_s k_s} - \sum_{t=1}^{m} \lambda_{k_s k_t}$ for $1 \leq s \leq m$, F to be the $m \times m$ diagonal matrix with entries $f_1, \ldots, f_m$ and G to be the $m \times m$ matrix with $g_{tt} = 0, g_{st} = \lambda_{k_s k_t}$ for $s \neq t$, I_m to be the $m \times m$ identity matrix. Similar to the argument in Section 4.2.3, we know that the transformation T_1 defined as

$$\begin{pmatrix} \theta_{k_1} \\ \vdots \\ \theta_{k_m} \end{pmatrix} = (I_m + F^{-1}G)^{-1/2} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}.$$

will transform $\sum_{s<t} -\lambda_{k_s k_t}(\theta_{k_s} - \theta_{k_t})^2$ into $\sum_{s=1}^{m} f_s y_s^2$, and $\det\big((I_m + F^{-1}G)^{-1/2}\big) = O(n^{1/2})$. Therefore, we obtain

$$I_3'(m) \leq O(n^{1/2})\pi^{m/2}/\Big(\prod_{s=1}^{m} f_s\Big)^{1/2}.$$

Substituting back into (3.2) we find that

$$2^n \sum_{r=1}^{n^\epsilon} |I_3(r)| \leq |I_2| \exp(-c_2 n + o(n))$$

for some $c_2 > 0$, since we have $a_{kj}^2 = 1 - 2\lambda_{jk}$. We conclude that only substantial contribution must come from the case $r = 0$.

Next, define $I_4(h)$ to be the contribution to I_1 of those $\boldsymbol{\theta}$ such that

(i) $|\theta_n| \le 3\pi/16$,

(ii) $n^{-1/2+\epsilon/4} < |\theta_j - \theta_n| \le 3\pi/8$ for h values of j, and

(iii) $|\theta_j - \theta_n| \le n^{-1/2+\epsilon/4}$ for the remaining values of j.

Clearly $|I_4(h)| \le 3\pi/8 |I_4'(h)|$, where $|I_4'(h)|$ is the same integral over $\boldsymbol{\theta}'$ with $\theta_n = 0$. Now define $h_k = \lambda_{kk} - \sum_{j=1}^{n-1} \lambda_{jk}$ for $1 \le k \le n-1$, F_2 to be the $(n-1)\times(n-1)$ diagonal matrix with entries $h_1, \ldots, h_{n-1}$ and G_2 to be the $(n-1)\times(n-1)$ matrix with $g_{kk} = 0, g_{jk} = \lambda_{jk}$ for $j \ne k$, I_{n-1} to be the $(n-1)\times(n-1)$ identity matrix. Let T_2 be the transformation defined as

$$\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_{n-1} \end{pmatrix} = (I_{n-1} + F_2^{-1} G_2)^{-1/2} \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}.$$

Now apply the bound $|T_{jk}(\boldsymbol{\theta})| \le \exp(-\lambda_{jk}(\theta_j - \theta_k)^2)$ and apply the above transformation T_2 to transform the $\boldsymbol{\theta}'$ to $\boldsymbol{y}$ and the values of $\boldsymbol{\theta}'$ contributing to $I_4'(h)$ for $h \ge 1$ map to a subset of those $\boldsymbol{y}$ such that either $|\sum_{k=1}^{n-1}(1/2 + o(n^{-1/4+\epsilon}))y_k| > n^{1/2+\epsilon/4}/2 + o(n^{1/4+\epsilon/4})$ or $|y_k| > n^{-1/2+\epsilon/4}/2$ for some k. Since the contribution to

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-\sum_{k=1}^{n-1} h_k y_k^2)\, d\boldsymbol{y}$$

of those $\boldsymbol{y}$ is $O(n)\pi^{(n-1)/2}/\left(\prod_{k=1}^{n-1} h_k\right)^{1/2} \exp(-c_3 n^{\epsilon/2})$ for some $c_3 > 0$, we conclude that

$$2^n \sum_{h=1}^{n-1} |I_4(h)| \le O(n)\exp(-c_3 n^{\epsilon/2})|I_2|.$$

The remaining case, $h = 0$, is covered by I_2. Therefore we have completed our proof and we have that if $\max\{|\delta_1|,\ldots,|\delta_n|\} = o(n^{3/4+\epsilon})$,

$$\begin{aligned} &NT(n;\delta_1,\ldots,\delta_n) \\ &= n^{1/2}\Big(\frac{2}{n\pi}\Big)^{(n-1)/2} \prod_{k=1}^{n-1}\Big(1-\frac{1}{n}\sum_{j=1}^{n} a_{kj}^2\Big)^{-1/2} \prod_{1\le j<k\le n}\Big(\frac{r_k}{r_j}+\frac{r_j}{r_k}\Big) \prod_{1\le j\le n} r_j^{-\delta_j} \\ &\times \exp\big(-\tfrac{1}{2} + O(n^{-1/4+5\epsilon})\big). \end{aligned} \tag{3.3}$$

Now define

$$\begin{aligned} p_{kj} &= \frac{w_k - w_j}{1 - w_j w_k}, \\ s_j &= \Big(\frac{1+w_j}{1-w_j}\Big)^{1/2}. \end{aligned} \tag{3.4}$$

We have $a_{kj} = p_{kj} + O(E^9)$ and $s_j = r_j + O(E^9)$. Hence,

$$\prod_{k=1}^{n-1}\Big(1-\frac{1}{n}\sum_{j=1}^{n} a_{kj}^2\Big)^{-1/2} = \exp(O(nE^9)) \prod_{k=1}^{n-1}\Big(1-\frac{1}{n}\sum_{j=1}^{n} p_{kj}^2\Big)^{-1/2},$$

$$\prod_{1\le j<k\le n}\Big(\frac{r_k}{r_j}+\frac{r_j}{r_k}\Big) = \exp(O(n^2E^9)) \prod_{1\le j<k\le n}\Big(\frac{s_k}{s_j}+\frac{s_j}{s_k}\Big),$$

and

$$\prod_{1\le j\le n} r_j^{-\delta_j} = \exp\Big(O\Big(\sum_{j=1}^{n}\delta_j E^9\Big)\Big) \prod_{1\le j\le n} s_j^{-\delta_j} = \exp(O(n^2E^{10})) \prod_{1\le j\le n} s_j^{-\delta_j}.$$

Therefore, we have

$$\begin{aligned} &NT(n;\delta_1,\ldots,\delta_n) \\ &= n^{1/2}\Big(\frac{2}{n\pi}\Big)^{(n-1)/2} \prod_{k=1}^{n-1}\Big(1-\frac{1}{n}\sum_{j=1}^{n} p_{kj}^2\Big)^{-1/2} \prod_{1\le j<k\le n}\Big(\frac{s_k}{s_j}+\frac{s_j}{s_k}\Big) \prod_{1\le j\le n} s_j^{-\delta_j} \\ &\times \exp\big(-\tfrac{1}{2} + O(n^{-1/4+10\epsilon})\big), \end{aligned}$$

where s_j and p_{kj} are defined as in (3.4).

Let $\delta = \max\{|\delta_1|, \ldots, |\delta_n|\}$. For the case where $\delta = o(n^{3/4})$, by computation, we have

$$\prod_{1\le k\le n-1}\Big(1-\frac{1}{n}\sum_{j=1}^{n} p_{kj}^2\Big)^{-1/2} = \exp\Big(\frac{1}{n^2}\sum_{j=1}^{n}\delta_j^2 + O\big(\frac{\delta^4}{n^3}\big)\Big),$$

$$\prod_{1\le j<k\le n}\Big(\frac{s_k}{s_j}+\frac{s_j}{s_k}\Big) = 2^{n(n-1)/2}\exp\Big(\frac{1}{2n}\sum_{j=1}^{n}\delta_j^2 + \frac{1}{4n^3}\sum_{j=1}^{n}\delta_j^4 + \frac{3}{4n^4}\Big(\sum_{j=1}^{n}\delta_j^2\Big)^2 + \frac{1}{6n^5}\sum_{j=1}^{n}\delta_j^6 + \frac{5}{6n^6}\Big(\sum_{j=1}^{n}\delta_j^3\Big)^2 + \frac{5}{2n^7}\Big(\sum_{j=1}^{n}\delta_j^2\Big)^3 + O\big(\frac{\delta^8}{n^6}\big)\Big),$$

and

$$\prod_{1\le j\le n} s_j^{-\delta_j} = \exp\Big(-\frac{1}{n}\sum_{j=1}^{n}\delta_j^2 - \frac{1}{3n^3}\sum_{j=1}^{n}\delta_j^4 - \frac{1}{n^4}\Big(\sum_{j=1}^{n}\delta_j^2\Big)^2 - \frac{1}{5n^5}\sum_{j=1}^{n}\delta_j^6 - \frac{1}{n^6}\Big(\sum_{j=1}^{n}\delta_j^3\Big)^2 - \frac{3}{n^7}\Big(\sum_{j=1}^{n}\delta_j^2\Big)^3 + O\big(\frac{\delta^8}{n^6}\big)\Big).$$

Therefore, we have

Theorem 4.3.2. *Let $\epsilon > 0$ be sufficiently small and $\delta_1, \ldots, \delta_n$ be integers. If $\sum_{k=1}^{n}\delta_k = 0$ and $\delta = \max\{|\delta_1|, \ldots, |\delta_n|\} = o(n^{3/4})$, as $n \to \infty$ we have the number of tournaments with n vertices and excess sequence $\delta_1, \ldots, \delta_n$ is asymptotically*

$$NT(n;\delta_1,\ldots,\delta_n) = n^{1/2}\Big(\frac{2^{n+1}}{n\pi}\Big)^{(n-1)/2}\exp\Big(-\frac{1}{2} - \frac{1}{2n}\sum_{j=1}^{n}\delta_j^2 + \frac{1}{n^2}\sum_{j=1}^{n}\delta_j^2 - \frac{1}{12n^3}\sum_{j=1}^{n}\delta_j^4 - \frac{1}{4n^4}\Big(\sum_{j=1}^{n}\delta_j^2\Big)^2 - \frac{1}{30n^5}\sum_{j=1}^{n}\delta_j^6 - \frac{1}{6n^6}\Big(\sum_{j=1}^{n}\delta_j^3\Big)^2 - \frac{1}{2n^7}\Big(\sum_{j=1}^{n}\delta_j^2\Big)^3 + O\big(\frac{\delta^4}{n^3} + n^{-1/4+\epsilon}\big)\Big). \quad ∎$$

Chapter 5

Asymptotic Enumeration of Digraphs

5.1. Introduction.

Let $v_1, v_2, \ldots, v_n$ be the vertices of a labelled simple loop-free digraph and let d_j^-, d_j^+ be the in-degree and out-degree of v_j for $1 \leq j \leq n$. We define $\delta_j = d_j^+ - d_j^-$ and refer to $\delta_1, \ldots, \delta_n$ as the *excess sequence* of the digraph. Let $NDG(n; \delta_1, \ldots, \delta_n)$ be the number of labelled simple loop-free digraphs with n vertices and excess sequence $\delta_1, \ldots, \delta_n$.

We are concerned with the asymptotic value of $NDG(n; \delta_1, \ldots, \delta_n)$. We identify the required quantity as a coefficient in a n-variable power series, and estimate it by applying the saddle-point method to the integral provided by Cauchy's Theorem. Some techniques are necessary to choose the suitable contours and substantial work is required to simplify the integrand and to demonstrate that the parts of contours where the integrand is small contribute negligibly to the result.

5.2. Calculations.

In this section, we will identify $NDG(n; \delta_1, \ldots, \delta_n)$ as a coefficient of the generating function of digraphs $\prod_{1 \leq j < k \leq n}(1 + x_j^{-1} x_k)(1 + x_j x_k^{-1})$ and then apply Cauchy's Theorem to convert the quantity into a n-dimensional integral. By choosing the suitable contours, we can make the linear items within the exponential of the integrand vanish. We then employ a few linear transformations in order to diagonalize the quadratic terms.

5.2.1. Saddle point method.

Since the generating function $\prod_{1\le j<k\le n}(1+x_j^{-1}x_k)(1+x_jx_k^{-1})$ enumerates digraphs by excess of out-degree over in-degree at each vertex, $NDG(n;\delta_1,\dots,\delta_n)$ is the coefficient of $x_1^{\delta_1}\cdots x_n^{\delta_n}$ in $\prod_{1\le j<k\le n}(1+x_j^{-1}x_k)(1+x_jx_k^{-1})$. By Cauchy's Theorem,

$$NDG(n;\delta_1,\dots,\delta_n) = \frac{1}{(2\pi i)^n}\oint\cdots\oint \frac{\prod_{1\le j<k\le n}(1+x_j^{-1}x_k)(1+x_jx_k^{-1})}{x_1^{\delta_1+1}\cdots x_n^{\delta_n+1}}\,dx_1\cdots dx_n,$$

where each integration is around a simple closed contour encircling the origin once in the anticlockwise direction. We first choose the jth contour to be the circle of radius r_j and then substitute $x_j = r_je^{i\theta_j}$ for $1\le j\le n$. We obtain

$$NDG(n;\delta_1,\dots,\delta_n) = \frac{2^{n(n-1)}I}{(2\pi)^n\prod_{1\le j\le n} r_j^{\delta_j}},$$

where

$$I = \int_{U_n(\pi)} \frac{\prod_{1\le j<k\le n}\Big(\frac{1}{2}+\frac{r_k}{4r_j}\exp\big(i(\theta_k-\theta_j)\big)+\frac{r_j}{4r_k}\exp\big(i(\theta_j-\theta_k)\big)\Big)}{\exp\big(i\sum_{1\le j\le n}(\delta_j\theta_j)\big)}\,d\boldsymbol{\theta}. \quad (2.1)$$

Defining

$$t_{jk} = \tfrac{1}{2}+\tfrac{1}{4}\Big(\frac{r_k}{r_j}+\frac{r_j}{r_k}\Big)$$

and

$$g(\boldsymbol{\theta}) = \frac{\prod_{1\le j<k\le n}\Big(\frac{1}{2} + \frac{r_k}{4r_j}\exp\big(i(\theta_k-\theta_j)\big) + \frac{r_j}{4r_k}\exp\big(i(\theta_j-\theta_k)\big)\Big)}{\prod_{1\le j<k\le n} t_{jk}\exp\big(i\sum_{1\le j\le n}(\delta_j\theta_j)\big)},$$

we have

$$I = I_1 \prod_{1\le j<k\le n} t_{jk},$$

where

$$I_1 = \int_{U_n(\pi)} g(\boldsymbol{\theta})\, d\boldsymbol{\theta}. \tag{2.2}$$

We will begin the evaluation of I_1 with the part of the integrand which will turn out to give the major contribution. Let I_2 be the contribution to I_1 of those $\boldsymbol{\theta}$ such that $|\theta_j - \theta_n| \le n^{-1/2+\epsilon/4}$ for $1 \le j \le n-1$, where θ_j values are taken mod 2π. We now prove that the contribution to I_2 with different values of θ_n are the same. By transformation $\theta_j \mapsto \phi_j + \phi_n$ for $1 \le j \le n-1$ and $\theta_n \mapsto \phi_n$, we have

$$\begin{aligned}\int_{|\theta_j-\theta_n|\le n^{-1/2+\epsilon/4}} g(\boldsymbol{\theta})\, d\boldsymbol{\theta} &= \int_{-\pi}^{\pi}\Big(\int_{|\theta_j-\theta_n|\le n^{-1/2+\epsilon/4}} g(\boldsymbol{\theta})\, d\boldsymbol{\theta}'\Big) d\theta_n \\ &= \int_{-\pi}^{\pi}\Big(\int_{|\phi_j|\le n^{-1/2+\epsilon/4}} g(\phi_1,\ldots,\phi_{n-1},0)\, d\boldsymbol{\phi}'\Big) d\phi_n \\ &= 2\pi\int_{|\phi_j|\le n^{-1/2+\epsilon/4}} g(\phi_1,\ldots,\phi_{n-1},0)\, d\boldsymbol{\phi}',\end{aligned}$$

where $\boldsymbol{\phi}' = (\phi_1,\ldots,\phi_{n-1})$ and $\boldsymbol{\theta}' = (\theta_1,\ldots,\theta_{n-1})$ since $\sum_{j=1}^n \delta_j = 0$. We obtain

$$I_2 = 2\pi\int_{U_{n-1}(n^{-1/2+\epsilon/4})} g(\boldsymbol{\theta})\, d\boldsymbol{\theta}',$$

where $\boldsymbol{\theta}' = (\theta_1,\ldots,\theta_{n-1})$ with $\theta_n = 0$.

Since

$$\exp(ix) = 1 + ix - \tfrac{1}{2}x^2 - \tfrac{1}{6}ix^3 + \tfrac{1}{24}x^4 + O(x^5),$$

putting

$$T_{jk}(\boldsymbol{\theta}) = \tfrac{1}{2} + \frac{r_k}{4r_j}\exp\bigl(i(\theta_k - \theta_j)\bigr) + \frac{r_j}{4r_k}\exp\bigl(i(\theta_j - \theta_k)\bigr),$$

we have

$$\begin{aligned} T_{jk}(\boldsymbol{\theta}) &= \tfrac{1}{2} + \tfrac{1}{4}\Bigl(\frac{r_k}{r_j} + \frac{r_j}{r_k}\Bigr) + \tfrac{1}{4}\Bigl(\frac{r_k}{r_j} - \frac{r_j}{r_k}\Bigr)i(\theta_k - \theta_j) \\ &\quad - \tfrac{1}{8}\Bigl(\frac{r_k}{r_j} + \frac{r_j}{r_k}\Bigr)(\theta_k - \theta_j)^2 - \tfrac{1}{24}\Bigl(\frac{r_k}{r_j} - \frac{r_j}{r_k}\Bigr)i(\theta_k - \theta_j)^3 \\ &\quad + \tfrac{1}{96}\Bigl(\frac{r_k}{r_j} + \frac{r_j}{r_k}\Bigr)(\theta_k - \theta_j)^4 + O\bigl(|\theta_k - \theta_j|^5\bigr). \end{aligned}$$

Because $\log(a+z) = \log(a) + \frac{1}{a}z - \frac{1}{2a^2}z^2 + \frac{1}{3a^3}z^3 - \frac{1}{4a^4}z^4 + O(z^5)$ for constant a and complex z, we obtain

$$\begin{aligned} g(\boldsymbol{\theta}) &= \frac{\prod_{1\le j<k\le n} T_{jk}(\boldsymbol{\theta})}{\exp\bigl(i\sum_{1\le j\le n}(\delta_j\theta_j)\bigr)\prod_{1\le j<k\le n} t_{jk}} \\ &= \exp\Bigl(\sum_{1\le j<k\le n}\log T_{jk}(\boldsymbol{\theta}) - i\sum_{1\le j\le n}(\delta_j\theta_j) - \sum_{1\le j<k\le n}\log t_{jk}\Bigr) \\ &= \exp\Bigl(\sum_{1\le j\le n}\Bigl(\sum_{1\le k\le n}\frac{1}{4t_{jk}}\Bigl(\frac{r_j}{r_k} - \frac{r_k}{r_j}\Bigr) - \delta_j\Bigr)i\theta_j + \sum_{1\le j<k\le n} o_{jk}(\theta_k - \theta_j)^2 \\ &\quad + \sum_{1\le j<k\le n} p_{jk}i(\theta_k - \theta_j)^3 + \sum_{1\le j<k\le n} q_{jk}(\theta_k - \theta_j)^4 \\ &\quad + O\Bigl(\sum_{1\le j<k\le n}|\theta_k - \theta_j|^5\Bigr)\Bigr), \end{aligned} \tag{2.3}$$

where

$$o_{jk} = -\frac{1}{8t_{jk}}\Bigl(\frac{r_k}{r_j} + \frac{r_j}{r_k}\Bigr) + \frac{1}{32t_{jk}^2}\Bigl(\frac{r_k}{r_j} - \frac{r_j}{r_k}\Bigr)^2,$$

$$p_{jk} = -\frac{1}{24t_{jk}}\Bigl(\frac{r_k}{r_j} - \frac{r_j}{r_k}\Bigr) + \frac{1}{32t_{jk}^2}\Bigl(\frac{r_k^2}{r_j^2} - \frac{r_j^2}{r_k^2}\Bigr) - \frac{1}{192t_{jk}^3}\Bigl(\frac{r_k}{r_j} - \frac{r_j}{r_k}\Bigr)^3,$$

and

$$\begin{aligned} q_{jk} &= \frac{1}{96t_{jk}}\Bigl(\frac{r_k}{r_j} + \frac{r_j}{r_k}\Bigr) - \frac{1}{96t_{jk}^2}\Bigl(\frac{r_k}{r_j} - \frac{r_j}{r_k}\Bigr)^2 - \frac{1}{128t_{jk}^2}\Bigl(\frac{r_k}{r_j} + \frac{r_j}{r_k}\Bigr)^2 \\ &\quad + \frac{1}{128t_{jk}^3}\Bigl(\frac{r_k}{r_j} - \frac{r_j}{r_k}\Bigr)^2\Bigl(\frac{r_k}{r_j} + \frac{r_j}{r_k}\Bigr) - \frac{1}{1024t_{jk}^4}\Bigl(\frac{r_k}{r_j} - \frac{r_j}{r_k}\Bigr)^4. \end{aligned}$$

5.2.2. Choices of contours.

We choose suitable r_j for $1 \le j \le n$ so that the coefficient of the linear item of θ_j within the exponential in the integrand $g(\boldsymbol{\theta})$ will be zero for $1 \le j \le n$.

We need $r_1, \ldots, r_n$ such that

$$\sum_{k=1}^{n} \frac{1}{4t_{jk}}\left(\frac{r_j}{r_k} - \frac{r_k}{r_j}\right) = \delta_j, \qquad (1 \le j \le n),$$

that is

$$\sum_{k=1}^{n} \frac{r_j^2 - r_k^2}{(r_k + r_j)^2} = \delta_j, \qquad (1 \le j \le n). \tag{2.4}$$

Substitute $r_j = (1 + b_j)/(1 - b_j)$ for $1 \le j \le n$ to get

$$\sum_{k=1}^{n} \frac{b_j - b_k}{1 - b_j b_k} = \delta_j, \qquad (1 \le j \le n). \tag{2.5}$$

Let us consider the functions $f_1, \ldots, f_n$ defined by

$$f_j(\boldsymbol{b}) = \frac{\delta_j}{n} - \frac{1}{n}\sum_{k=1}^{n} \frac{b_j b_k (b_j - b_k)}{1 - b_j b_k}, \qquad (1 \le j \le n). \tag{2.6}$$

Further define $\boldsymbol{b}^{(0)} = (\delta_1/n, \ldots, \delta_n/n)$ and

$$b_j^{(i)} = f_j(\boldsymbol{b}^{(i-1)}), \qquad i = 1, 2, \ldots \tag{2.7}$$

Note that this is exactly the same with that in Section 4.2.2. We know that the n-dimensional vector sequence $\boldsymbol{b}^{(0)}, \boldsymbol{b}^{(1)}, \ldots$ converges to a vector $\boldsymbol{b}^{(\infty)}$ and analogous lemmas to Lemma 4.2.1–4.2.4 are all true.

Let $\|\cdot\|$ be the norm on $\mathbf{R}^n$ defined by $\|(x_1, \ldots, x_n)\| = \max_{1 \le k \le n} |x_k|$. Define $E = \|(\delta_1, \ldots, \delta_n)\|/n$. Assume that $E = o(n^{-1/4+\epsilon})$ and $E \le 1/100$.

5.2.3. Transformations.

We diagonalize the quadratic form within the exponential in the integrand by transformations.

By the results of the previous section, the integrand (2.3) has now been simplified as

$$\begin{aligned}
g(\boldsymbol{\theta}) &= \exp\Big(\sum_{1\le j<k\le n} o_{jk}(\theta_k-\theta_j)^2 + \sum_{1\le j<k\le n} p_{jk} i(\theta_k-\theta_j)^3 \\
&\quad + \sum_{1\le j<k\le n} q_{jk}(\theta_k-\theta_j)^4 + O\Big(\sum_{1\le j<k\le n} |\theta_k-\theta_j|^5\Big)\Big) \\
&= \exp\Big(\sum_{1\le j<k\le n} o_{jk}(\theta_k-\theta_j)^2 \\
&\quad + \sum_{1\le j<k\le n-1} p_{jk} i(\theta_k-\theta_j)^3 - \sum_{1\le j<n-1} p_{jn} i\theta_j^3 \\
&\quad + \sum_{1\le j<k\le n-1} q_{jk}(\theta_k-\theta_j)^4 + \sum_{1\le j<n-1} q_{jn}\theta_j^4 \\
&\quad + O\Big(\sum_{1\le j<k\le n} |\theta_k-\theta_j|^5\Big)\Big) \\
&= \exp\Big(\sum_{1\le j<k\le n} o_{jk}(\theta_k-\theta_j)^2 + \sum_{1\le j<k\le n-1} p_{jk} i(\theta_k-\theta_j)^3 \\
&\quad + \sum_{1\le j<k\le n-1} q_{jk}(\theta_k-\theta_j)^4 + O(n^{-1/2+5\epsilon})\Big) \\
&= \exp\Big(\sum_{1\le j<k\le n} (-\tfrac{1}{4}+\tfrac{1}{4}a_{kj}^2)(\theta_k-\theta_j)^2 \\
&\quad + \sum_{1\le j<k\le n-1} (\tfrac{1}{12}a_{kj}-\tfrac{1}{12}a_{kj}^3) i(\theta_k-\theta_j)^3 \\
&\quad + \sum_{1\le j<k\le n-1} (-\tfrac{1}{96}+\tfrac{1}{24}a_{kj}^2-\tfrac{1}{32}a_{kj}^4)(\theta_k-\theta_j)^4 \\
&\quad + O(n^{-1/2+5\epsilon})\Big),
\end{aligned} \tag{2.8}$$

when $\boldsymbol{x} \in U_{n-1}(n^{-1/2+\epsilon})$ with $\theta_n = 0$ and $a_{kj} = (b_k - b_j)/(1 - b_k b_j)$.

Define $V = U_{n-1}(n^{-1/2+\epsilon})$ and let $T : \mathbb{R}^{n-1} \mapsto \mathbb{R}^{n-1}$ be the linear transformation defined by $T : \boldsymbol{\theta}' \longmapsto \boldsymbol{y} = (y_1, y_2, \ldots, y_{n-1})$, where

$$y_j = \theta_j - \sum_{k=1}^{n-1} \theta_k/(n + n^{1/2})$$

for $1 \le j \le n-1$. Let $V_1 = T(V)$. By straightforward calculations we have

$$V_1 = \{\boldsymbol{y} \mid |y_j + \sum_{k=1}^{n-1} y_k/(n^{1/2}+1)| \le n^{-1/2+\epsilon} \text{ for } 1 \le j \le n-1\},$$

$$\det(T) = n^{1/2},$$

$$\begin{aligned} g(\boldsymbol{\theta}) = \exp\Big(&\sum_{k=1}^{n-1}\Big(-\frac{n}{4} + \sum_{j=1}^{n} a_{kj}^2 + \tfrac{1}{2}sa_{nk}^2 + \tfrac{1}{4}s^2 \sum_{l=1}^{n-1} a_{nl}^2\Big) y_k^2 \\ &+ \sum_{j \ne k}\Big(-\tfrac{1}{4}a_{kj}^2 + \tfrac{1}{2}sa_{nj}^2 + \tfrac{1}{4}s^2 \sum_{l=1}^{n-1} a_{nl}^2\Big) y_j y_k \\ &+ \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}(\tfrac{1}{12}a_{kj} - \tfrac{1}{12}a_{kj}^3)\Big) i y_k^3 \\ &+ \sum_{j \ne k}(-\tfrac{1}{4}a_{kj} + O(E^2)) i y_k^2 y_j \\ &+ \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}(-\tfrac{1}{96} + \tfrac{1}{24}a_{kj}^2 - \tfrac{1}{32}a_{kj}^4) + \tfrac{1}{96}\Big) y_k^4 \\ &+ \sum_{j \ne k} O(1) y_k^3 y_j \\ &+ \sum_{j \ne k}(-\tfrac{1}{32} + O(E^2)) y_k^2 y_j^2 \\ &+ O(n^{-1/2+5\epsilon})\Big), \end{aligned}$$

where $s = 1/(n^{1/2}+1)$, each $O(\ )$ term is uniform over the subscript set of the sum involved, S_3 is the set of 3-subsets of $\{1, 2, \ldots, n-1\}$ and S_4 is the set of 4-subsets of $\{1, 2, \ldots, n-1\}$.

Denote

$$u_k = -\frac{n}{4} + \sum_{j=1}^{n} a_{kj}^2 + \tfrac{1}{2}sa_{nk}^2 + \tfrac{1}{4}s^2 \sum_{l=1}^{n-1} a_{nl}^2,$$

$$v_{kj} = -\tfrac{1}{4}a_{kj}^2 + \tfrac{1}{4}s(a_{nj}^2 + a_{nk}^2) + \tfrac{1}{4}s^2 \sum_{l=1}^{n-1} a_{nl}^2 \quad \text{for } k < j,$$

and $v_{kj} = v_{jk}$ for $k > j, v_{kk} = u_k$. Define A to be the diagonal matrix with entries $u_1, \ldots, u_{n-1}, V$ the $n-1 \times n-1$ matrix with entries v_{jk} and $B = V - A$, I the identity matrix, and define a linear transformation by

$$\begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} = (I + A^{-1}B)^{-1/2} \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \end{pmatrix}.$$

By the formula $\det(X) = \exp\bigl(\operatorname{tr}\log(X)\bigr)$, we have

$$\begin{aligned} &\det\bigl((I + A^{-1}B)^{-1/2}\bigr) \\ &\quad = \exp\bigl(-\tfrac{1}{2}\operatorname{tr}\log\bigl((I + A^{-1}B)\bigr) \\ &\quad = \exp\bigl(-\tfrac{1}{2}\operatorname{tr}\bigl(A^{-1}B - \tfrac{1}{2}A^{-1}BA^{-1}B + \tfrac{1}{3}A^{-1}BA^{-1}BA^{-1}B - \cdots\bigr)\bigr). \end{aligned}$$

Since $\|\boldsymbol{b}\| \le 2E$, we have $|a_{kj}| \le 8E$ and this implies that $|v_{jl}/u_k| \le cE^2/n$ for $1 \le j, k, l \le n-1$ for some constant $c > 0$. Therefore, the absolute value of each entry of $(A^{-1}B)^k$ is bounded by $c^k E^{2k}/n$. Hence, we have

$$\begin{aligned} &\det\bigl((I + A^{-1}B)^{-1/2}\bigr) \\ &\quad \le \exp\bigl(cE^2 + c^2E^4 + \cdots\bigr) \\ &\quad = 1 + O(E^2). \end{aligned}$$

Because

$$(I + A^{-1}B)^{-1/2} = I + \sum_{k=1}^{\infty} \binom{-\frac{1}{2}}{k} (A^{-1}B)^k,$$

as a matrix, each entry of $\sum_{k=1}^{\infty} \binom{-1/2}{k}(A^{-1}B)^k$ is $n^{-1}O(E^2)$, and since

$$(I+BA^{-1})^{-1/2}(A+B)(I+A^{-1}B)^{-1/2}=A,$$

we obtain

$$\begin{aligned}
g(\boldsymbol{\theta}) = \exp\Big(&\sum_{k=1}^{n-1} u_k z_k^2 \\
&+\sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}(\tfrac{1}{12}a_{kj}-\tfrac{1}{12}a_{kj}^3)+O(E^2)\Big)iz_k^3 \\
&+\sum_{j\neq k}(-\tfrac{1}{4}a_{kj}+O(E^2))iz_k^2z_j \\
&+\sum_{\{j,k,l\}\in S_3} n^{-7/4+3\epsilon}O(1)iz_jz_kz_l \\
&+\sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}(-\tfrac{1}{96}+\tfrac{1}{24}a_{kj}^2-\tfrac{1}{32}a_{kj}^4)+\tfrac{1}{96}+O(E^2)\Big)z_k^4 \\
&+\sum_{j\neq k}O(1)z_k^3z_j \\
&+\sum_{j\neq k}(-\tfrac{1}{32}+O(E^2))z_k^2z_j^2 \\
&+\sum_{\{j,k,l\}\in S_3} n^{-3/2+2\epsilon}O(1)z_j^2z_kz_l \\
&+\sum_{\{j,k,l,m\}\in S_4} n^{-3+4\epsilon}O(1)z_jz_kz_lz_m \\
&+O(n^{-1/2+5\epsilon})\Big),
\end{aligned}$$

where $s=1/(n^{1/2}+1)$.

Further define a linear transformation by

$$z_k=\Big(\frac{n}{-4u_k}\Big)^{1/2}x_k, \qquad 1\le k\le n-1.$$

Clearly, the determinant of the transformation is

$$(\tfrac{n}{4})^{(n-1)/2}\prod_{k=1}^{n-1}(-u_k)^{-1/2},$$

and we obtain

$$\begin{aligned}
g(\boldsymbol{\theta}) = \exp\Big(&\sum_{k=1}^{n-1} -\frac{n}{4}x_k^2 \\
&+ \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}(\tfrac{1}{12}a_{kj} - \tfrac{1}{12}a_{kj}^3)(\frac{n}{-4u_k})^{3/2} + O(E^2)\Big)ix_k^3 \\
&+ \sum_{j\neq k}(-\tfrac{1}{4}a_{kj} + O(E^2))ix_k^2x_j \\
&+ \sum_{\{j,k,l\}\in S_3} n^{-7/4+3\epsilon}O(1)ix_jx_kx_l \\
&+ \sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}\Big)(-\tfrac{1}{96} + \tfrac{1}{24}a_{kj}^2 - \tfrac{1}{32}a_{kj}^4(\frac{n}{-4u_k})^2 + \tfrac{1}{96} + O(E^2)\Big)x_k^4 \\
&+ \sum_{j\neq k} O(1)x_k^3x_j \\
&+ \sum_{j\neq k}(-\tfrac{1}{32} + O(E^2))x_k^2x_j^2 \\
&+ \sum_{\{j,k,l\}\in S_3} n^{-3/2+2\epsilon}O(1)x_j^2x_kx_l \\
&+ \sum_{\{j,k,l,m\}\in S_4} n^{-3+4\epsilon}O(1)x_jx_kx_lz_m \\
&+ O(n^{-1/2+5\epsilon})\Big). \qquad (2.9)
\end{aligned}$$

Let $T' : \boldsymbol{\theta}' \mapsto \boldsymbol{x}$ be the transformation involved in this section and $V' = T'(V)$. We know that $V' \subseteq U_{n-1}(3n^{-1/2+\epsilon})$. The asymptotic value of the integral of $f(\boldsymbol{x})$ over $U_{n-1}(3n^{-1/2+\epsilon})$ will be the same with that over $U_{n-1}(n^{-1/2+\epsilon})$. Furthermore, similar argument to that of Theorem 3.2.2 shows that the asymptotic value of the integral of $f(\boldsymbol{x})$ over $U_{n-1}(3n^{-1/2+\epsilon})\backslash V'$ is negligible. Therefore, we still keep the region as $U_{n-1}(n^{-1/2+\epsilon})$. When $\boldsymbol{x} \in U_{n-1}(n^{-1/2+\epsilon})$, we have that

$$\sum_{\{j,k,l\}\in S_3} n^{-7/4+3\epsilon}O(1)ix_jx_kx_l = O(n^{-1/4+6\epsilon}),$$

$$\sum_{\{j,k,l\}\in S_3} n^{-3/2+2\epsilon}O(1)x_j^2x_kx_l = O(n^{-1/2+6\epsilon}),$$

and

$$\sum_{\{j,k,l,m\}\in S_4} n^{-3+4\epsilon}O(1)x_jx_kx_lx_m = O(n^{-1+8\epsilon}).$$

Hence, $g(\boldsymbol{\theta})$ is now simplified as

$$\begin{aligned} g(\boldsymbol{\theta}) = \exp\Big(&\sum_{k=1}^{n-1} -\frac{n}{4}x_k^2 \\ &+\sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}(\tfrac{1}{12}a_{kj}-\tfrac{1}{12}a_{kj}^3)(\frac{n}{-4u_k})^{3/2}+O(E^2)\Big)ix_k^3 \\ &+\sum_{j\neq k}(-\tfrac{1}{4}a_{kj}+O(E^2))ix_k^2x_j \\ &+\sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}(-\tfrac{1}{96}+\tfrac{1}{24}a_{kj}^2-\tfrac{1}{32}a_{kj}^4)(\frac{n}{-4u_k})^2+\tfrac{1}{96}+O(E^2)\Big)x_k^4 \\ &+\sum_{j\neq k}O(1)x_k^3x_j \\ &+\sum_{j\neq k}(-\tfrac{1}{32}+O(E^2))x_k^2x_j^2 \\ &+O(n^{-1/4+6\epsilon})\Big). \end{aligned} \tag{2.10}$$

From now on, we denote $p(\boldsymbol{x})$ to be the right hand side of (2.10). Now define

$$\begin{aligned} b(\boldsymbol{x}) = \exp\Big(&\sum_{k=1}^{n-1} -\frac{n}{4}x_k^2 \\ &+\sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}(\tfrac{1}{12}a_{kj}-\tfrac{1}{12}a_{kj}^3)(\frac{n}{-4u_k})^{3/2}+O(E^2)\Big)ix_k^3 \\ &+\sum_{j\neq k}(-\tfrac{1}{4}a_{kj}+O(E^2))ix_k^2x_j \\ &+\sum_{k=1}^{n-1}\Big(\sum_{j=1}^{n-1}(-\tfrac{1}{96}+\tfrac{1}{24}a_{kj}^2-\tfrac{1}{32}a_{kj}^4)(\frac{n}{-4u_k})^2+\tfrac{1}{96}+O(E^2)\Big)x_k^4 \\ &+\sum_{j\neq k}O(1)x_k^3x_j \end{aligned}$$

$$
\begin{aligned}
&- \tfrac{1}{32}\Big(\sum_{j\neq k} x_k^2\Big)^2 \\
&+ \tfrac{1}{32}\sum_{k=1}^{n-1} x_k^4 \\
&+ O(n^{-1/4+6\epsilon})\Big)\Big). \qquad (2.11)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\Big|\int_{U_{n-1}(n^{-1/2+\epsilon})} (p(\boldsymbol{x}) - b(\boldsymbol{x}))\, d\boldsymbol{x}\Big| \\
&\leq \int_{U_{n-1}(n^{-1/2+\epsilon})} |b(\boldsymbol{x})| \times \Big|\exp\big(\sum_{1\leq j\neq k\leq n-1} O(E^2)x_k^2 x_j^2\big) - 1\Big|\, d\boldsymbol{x} \\
&\leq \int_{U_{n-1}(n^{-1/2+\epsilon})} |b(\boldsymbol{x})| O\big(\sum_{1\leq j\neq k\leq n-1} O(E^2)x_k^2 x_j^2\big)\, d\boldsymbol{x} \\
&\leq O\big(n^{-1/2+6\epsilon}\big) \int_{U_{n-1}(n^{-1/2+\epsilon})} |b(\boldsymbol{x})|\, d\boldsymbol{x},
\end{aligned}
$$

hence

$$
\begin{aligned}
&\int_{U_{n-1}(n^{-1/2+\epsilon})} p(\boldsymbol{x})\, d\boldsymbol{x} \\
&\qquad \int_{U_{n-1}(n^{-1/2+\epsilon})} b(\boldsymbol{x})\, d\boldsymbol{x} + O(n^{-1/2+6\epsilon}) \int_{U_{n-1}(n^{-1/2+\epsilon})} |b(\boldsymbol{x})|\, d\boldsymbol{x}. \qquad (2.12)
\end{aligned}
$$

Therefore by applying Theorem 3.2.2 to $b(\boldsymbol{x})$ and $|b(\boldsymbol{x})|$, we obtain

$$
\begin{aligned}
&\int_{U_{n-1}(n^{-1/2+\epsilon})} p(\boldsymbol{x})\, d\boldsymbol{x} \\
&\quad = \Big(\frac{4\pi}{n}\Big)^{(n-1)/2} \exp\Big(\frac{12}{n}\sum_{k=1}^{n-1} E_k(n) - \frac{1}{8} + O(n^{-1/4+6\epsilon})\Big), \qquad (2.13)
\end{aligned}
$$

where

$$
E_k(n) = \frac{1}{n}\Big(\sum_{j=1}^{n-1}\big(-\tfrac{1}{96} + \tfrac{1}{24}a_{kj}^2 - \tfrac{1}{32}a_{kj}^4\big)\Big(\frac{n}{-4u_k}\Big)^2 + \tfrac{1}{24} + O(E^2)\Big).
$$

5.3. Digraphs.

The following Lemma can be proved easily by Taylor series.

Lemma 5.3.1. *For sufficiently small $\epsilon > 0$, real a, b and x with $|a - 1/2| \leq \epsilon, |b - 1/8| \leq \epsilon$ and $|x| \leq \pi/4$,*

$$| 1 - a - b + a\cos(x) + b\cos(2x) | \leq \exp(-(\tfrac{1}{2}a + 2b)x^2). \quad \blacksquare$$

By the arguments of the last section, we obtain

$$I_2 = 2\pi n^{1/2}\Big(\frac{4\pi}{n}\Big)^{(n-1)/2} (\frac{n}{4})^{\frac{n-1}{2}} \prod_{k=1}^{n-1} (-u_k)^{-1/2}$$
$$\times \exp\Big(\frac{12}{n}\sum_{k=1}^{n-1} E_k(n) - \frac{1}{8} + O(n^{-1/4+\epsilon})\Big),$$

where

$$u_k = -\frac{n}{4} + \sum_{j=1}^{n} a_{kj}^2 + \tfrac{1}{2}sa_{nk}^2 + \tfrac{1}{4}s^2\sum_{l=1}^{n-1} a_{nl}^2,$$
$$E_k(n) = \frac{1}{n}\Big(\sum_{j=1}^{n-1}(-\tfrac{1}{96} + \tfrac{1}{24}a_{kj}^2 - \tfrac{1}{32}a_{kj}^4)i(\frac{n}{-4u_k})^2 + \tfrac{1}{24} + O(E^2)\Big),$$
$$s = 1/(n^{1/2} + 1).$$

Since

$$\frac{n}{-4u_k} = (1 - \frac{1}{n}\sum_{j=1}^{n} a_{kj}^2)^{-1} + O(n^{-1}E^2),$$

we have

$$(\frac{n}{4})^{\frac{n-1}{2}} \prod_{k=1}^{n-1} (-u_k)^{-1/2} = \exp(O(E^2)) \prod_{k=1}^{n-1} (1 - \frac{1}{n}\sum_{j=1}^{n} a_{kj}^2)^{-1/2},$$

and since $E_k(n) = -1/96 + O(E^2)$, we obtain

$$I_2 = 2\pi n^{1/2}(\frac{4\pi}{n})^{(n-1)/2} \prod_{k=1}^{n-1} (1 - \frac{1}{n}\sum_{j=1}^{n} a_{kj}^2)^{-1/2}$$
$$\times \exp(-\frac{1}{4} + O(n^{-1/4+\epsilon})). \tag{3.1}$$

So our remaining work is to prove that the contribution to I_1 of the parts other than the region of the integration of I_2 is negligible.

First we have that, for $1 \le j, k \le n$,

$$
\begin{aligned}
|T_{jk}(\boldsymbol{\theta})| &= t_{jk}^{-1}\Big(\tfrac{1}{4} + \tfrac{1}{4}(\frac{r_k}{r_j} + \frac{r_j}{r_k})\cos(\theta_k - \theta_j) + \tfrac{1}{16}((\frac{r_k}{r_j})^2 + (\frac{r_j}{r_k})^2) \\
&\quad + \tfrac{1}{8}(\cos^2(\theta_k - \theta_j) - \sin^2(\theta_k - \theta_j))\Big)^{1/2} \\
&= t_{jk}^{-1}\Big(\tfrac{1}{4} + \tfrac{1}{4}(\frac{r_k}{r_j} + \frac{r_j}{r_k})\cos(\theta_k - \theta_j) + \tfrac{1}{16}((\frac{r_k}{r_j})^2 + (\frac{r_j}{r_k})^2) \\
&\quad + \tfrac{1}{8}\cos(2(\theta_k - \theta_j))\Big)^{1/2} \\
&\le t_{jk}^{-1}\Big(\tfrac{1}{4} + \tfrac{1}{4}(\frac{r_k}{r_j} + \frac{r_j}{r_k}) + \tfrac{1}{16}(\frac{r_k}{r_j} + \frac{r_j}{r_k})^2\Big)^{1/2} \\
&= 1.
\end{aligned}
$$

By Lemma 5.3.1, we have, for $1 \le j, k \le n$,

$$
|T_{jk}(\boldsymbol{\theta})| \le \exp(-\lambda_{jk}(\theta_j - \theta_k)^2),
$$

where

$$
\begin{aligned}
\lambda_{jk} &= \frac{r_j r_k}{(r_j + r_k)^2} \\
&= \frac{1}{4} + o(n^{-1/4+\epsilon}).
\end{aligned}
$$

We will prove that the integral of $g(\boldsymbol{\theta})$ on the parts other than the region of the integration of I_2 is negligible to the value $|I_2|$.

For $0 \le j \le 31$, define the interval $A_j = [(j-1)\pi/16, j\pi/16]$. For any $\boldsymbol{\theta} \in U_n(\pi)$, at least one of the 16 intervals $A_0 \cup A_1, A_2 \cup A_3, \ldots, A_{30} \cup A_{31}$ contains $n/16$ or more of the θ_j. Let us suppose that this is true of $A_0 \cup A_1$ (thereby undercounting the possibilities by at most a factor of 16). Define $B = A_3 \cup \cdots \cup A_{30}$. If $\theta_j \in B$

and $\theta_k \in A_0 \cup A_1$, then $\pi/16 \le |\theta_j - \theta_k| \le 17\pi/16$, hence, $|T_{jk}(\boldsymbol{\theta})| \le at_{jk}^{-1} \le a$ for some constant $0 < a < 1$. Define C to be the set consisting of such pairs (j,k) and I_3 to be the contribution to I_1 of all the cases where n^ϵ or more of the θ_j lie in B. Since, if $(j,k) \in C$, $|T_{jk}(\boldsymbol{\theta})| \le a$, and for any other pair (j,k), $|T_{jk}(\boldsymbol{\theta})| \le 1$, we have

$$I_3 \le (2\pi)^n a^{n^{(1+\epsilon)/16}}.$$

From this it easily follows that $I_3 = O(\exp(-c_1 n^{1+\epsilon}))I_2$ for some $c_1 > 0$. Thus with an undercount of at most 16, we can suppose that at least $n - n^\epsilon$ of the θ_j lie in $A_{31} \cup A_0 \cup A_1 \cup A_2$. Now define $I_3(r)$ to be the contribution to I_1 of those $\boldsymbol{\theta}$ such that

(i) $3\pi/16 \le |\theta_j| \le \pi$ for r values of j,

(ii) $\theta_j \in A_{31} \cup A_0 \cup A_1 \cup A_2$ for at least $n - n^\epsilon$ values of j, and

(iii) $\theta_j \in A_3 \cup A_{30}$ for any other values of j.

Clearly $I_3(r) = 0$ if $r > n^\epsilon$. If θ_j and θ_k are in classes (i) and (ii), respectively, then $\pi/16 \le |\theta_k - \theta_j| \le 9\pi/8$, hence, $|T_{jk}(\boldsymbol{\theta})| \le b$ for some constant $0 < b < 1$. While if they are both in calsses (ii) and (iii), $|T_{jk}(\boldsymbol{\theta})| \le \exp(-\lambda_{jk}(\theta_k - \theta_j)^2)$. Using $|T_{jk}(\boldsymbol{\theta})| \le 1$ for the other cases, we find

$$|I_3(r)| \le \pi^r b^{r(n-n^\epsilon)/2} \sum_{\{k_1,k_2,\ldots,k_{n-r}\} \in S_{n-r}} |I_3'(n-r)|, \tag{3.2}$$

where

$$I_3'(m) = \int_{U_m(\pi)} \prod_{1 \le s < t \le m} \exp(-\lambda_{k_s k_t}(\theta_{k_s} - \theta_{k_t})^2)\, d\theta_{k_1} \cdots d\theta_{k_m}$$

and S_m is the set of the m-subsets of $\{1, 2, \ldots, n\}$. Now define $f_s = \lambda_{k_s k_s} -$

$\sum_{t=1}^{m} \lambda_{k_s k_t}$ for $1 \le k \le m$, F to be the $m \times m$ diagonal matrix with entries $f_1, \ldots, f_m$ and G to be the $m \times m$ matrix with $g_{tt} = 0, g_{st} = \lambda_{k_s k_t}$ for $s \neq t$, I_m to be the $m \times m$ identity matrix. Similar to the argument in Section 5.2.3, we know that the transformation T_1 defined as

$$\begin{pmatrix} \theta_{k_1} \\ \vdots \\ \theta_{k_m} \end{pmatrix} = (I_m + F^{-1}G)^{-1/2} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

will transform $\sum_{s<t} -\lambda_{k_s k_t}(\theta_{k_s} - \theta_{k_t})^2$ into $\sum_{s=1}^{m} f_s y_s^2$, and $\det((I_m + F^{-1}G)^{-1/2}) = O(n^{1/2})$. Therefore, we obtain

$$I_3'(m) \le O(n^{1/2})\pi^{m/2}/(\prod_{s=1}^{m} f_s)^{1/2}.$$

Substituting back into (3.2) we find that

$$\sum_{r=1}^{n^{\epsilon}} |I_3(r)| \le |I_2| \exp(-c_2 n + o(n))$$

for some $c_2 > 0$ since $a_{kj}^2 = 1 - 4\lambda_{kj}$. We conclude that only substantial contribution must come from the case $r = 0$.

Next, define $I_4(h)$ to be the contribution to I_1 of those $\boldsymbol{\theta}$ such that

(i) $|\theta_n| \le 3\pi/16$,

(ii) $n^{-1/2+\epsilon/4} < |\theta_j - \theta_n| \le 3\pi/8$ for h values of j, and

(ii) $|\theta_j - \theta_n| \le n^{-1/2+\epsilon/4}$ for the remaining values of j.

Clearly $|I_4(h)| \le 3\pi/8|I_4'(h)|$, where $|I_4'(h)|$ is the same integral over $\boldsymbol{\theta}'$ with $\theta_n = 0$. Now define $r_k = \lambda_{kk} - \sum_{j=1}^{n-1} \lambda_{kj}$ for $1 \le k \le n-1$. F_2 to be the $(n-1) \times (n-1)$ diagonal matrix with entries $r_1, \ldots, r_{n-1}$ and G_2 to be the $(n-1) \times (n-1)$ matrix with $g_{kk} = 0, g_{jk} = \lambda_{jk}$ for $j \neq k$, I_{n-1} to be the $(n-1) \times (n-1)$ identity matrix.

Let T_2 be the transformation defined as

$$\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_{n-1} \end{pmatrix} = (I_{n-1} + F_2^{-1} G_2)^{-1/2} \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}.$$

Now apply the bound $|T_{jk}(\boldsymbol{\theta})| \leq \exp\big(-\lambda_{jk}(\theta_j - \theta_k)^2\big)$ and apply the above transformation T_2 to transform the $\boldsymbol{\theta}'$ to $\boldsymbol{y}$ and the values of $\boldsymbol{\theta}'$ contributing to $I_4'(h)$ for $h \geq 1$ map to a subset of those $\boldsymbol{y}$ such that either $|\sum_{k=1}^{n-1}(1/2 + o(n^{-1/4+\epsilon}))y_k| > n^{1/2+\epsilon/4}/2 + o(n^{1/4+\epsilon/4})$ or $|y_k| > n^{-1/2+\epsilon/4}/2$ for some k. Since the contribution to

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\Big(-\sum_{k=1}^{n-1} r_k y_k^2\Big)\, d\boldsymbol{y}$$

of those $\boldsymbol{y}$ is $O(n)\pi^{(n-1)/2}/(\prod_{k=1}^{n-1} r_k)^{1/2} \exp\big(-c_3 n^{\epsilon/2}\big)$ for some $c_3 > 0$, we conclude that

$$\sum_{h=1}^{n-1} |I_4(h)| \leq O(n) \exp\big(-c_3 n^{\epsilon/2}\big) |I_2|.$$

The remaining case, $h = 0$, is covered by I_2. Therefore we have proved

$$\begin{aligned} &NDG(n; \delta_1, \ldots, \delta_n) \\ &\quad = n^{1/2}\Big(\frac{4^n}{\pi n}\Big)^{(n-1)/2} \prod_{k=1}^{n-1}\Big(1 - \frac{1}{n}\sum_{j=1}^{n} a_{kj}^2\Big)^{-1/2} \prod_{1 \leq j \leq n} r_j^{-\delta_j} \prod_{1 \leq j < k \leq n} t_{jk} \\ &\quad \times \exp\Big(-\frac{1}{4} + O(n^{-1/4+\epsilon})\Big). \quad \blacksquare \end{aligned}$$

Now define

$$h_j = \frac{1 + w_j}{1 - w_j},$$

$$r_{jk} = \frac{1}{2} + \frac{1}{4}\Big(\frac{h_k}{h_j} + \frac{h_j}{h_k}\Big),$$
$$s_{jk} = 1 - \frac{4h_k h_j}{(h_k + h_j)^2}, \tag{3.3}$$

where w_j for $1 \le j \le n$ are defined as in Section 4.2.2. We have $a_{jk}^2 = s_{jk} + O(E^9)$, $t_{jk} = r_{jk} + O(E^9)$ and $r_j = h_j + O(E^9)$. Hence,

$$\prod_{k=1}^{n-1}\Big(1 - \frac{1}{n}\sum_{j=1}^{n} a_{kj}^2\Big)^{-1/2} = \exp(O(nE^9)) \prod_{k=1}^{n-1}\Big(1 - \frac{1}{n}\sum_{j=1}^{n} s_{kj}\Big)^{-1/2},$$
$$\prod_{1\le j<k\le n} t_{jk} = \exp(O(n^2E^9)) \prod_{1\le j<k\le n} r_{jk}$$

and

$$\prod_{1\le j\le n} r_j^{-\delta_j} = \exp\Big(O\Big(\sum_{j=1}^{n} \delta_j E^9\Big)\Big) \prod_{1\le j\le n} h_j^{-\delta_j} = \exp(O(n^2E^{10})) \prod_{1\le j\le n} h_j^{-\delta_j}.$$

Therefore, we have

$$\begin{aligned} &NDG(n;\delta_1,\ldots,\delta_n) \\ &\quad= n^{1/2}\Big(\frac{4^n}{\pi n}\Big)^{(n-1)/2} \prod_{k=1}^{n-1}\Big(1 - \frac{1}{n}\sum_{j=1}^{n} s_{kj}\Big)^{-1/2} \prod_{1\le j\le n} h_j^{-\delta_j} \prod_{1\le j<k\le n} r_{jk} \\ &\qquad \times \exp\Big(-\frac{1}{4} + O(n^{-1/4+10\epsilon})\Big). \end{aligned}$$

where h_j, r_{jk} and s_{kj} are defined as in (3.3).

Let $\delta = \max\{|\delta_1|,\ldots,|\delta_n|\}$. For the case where $\delta = o(n^{3/4})$, by computation, we have

$$\prod_{1\le k\le n-1}\Big(1 - \frac{1}{n}\sum_{j=1}^{n} s_{kj}\Big)^{-1/2} = \exp\Big(\frac{1}{n^2}\sum_{j=1}^{n}\delta_j^2 + O\Big(\frac{\delta^4}{n^3}\Big)\Big),$$
$$\begin{aligned} \prod_{1\le j<k\le n} r_{jk} = \exp\Big(&\frac{1}{n}\sum_{j=1}^{n}\delta_j^2 + \frac{1}{2n^3}\sum_{j=1}^{n}\delta_j^4 + \frac{3}{2n^4}\Big(\sum_{j=1}^{n}\delta_j^2\Big)^2 \\ &+ \frac{1}{3n^5}\sum_{j=1}^{n}\delta_j^6 + \frac{5}{3n^6}\Big(\sum_{j=1}^{n}\delta_j^3\Big)^2 + \frac{5}{n^7}\Big(\sum_{j=1}^{n}\delta_j^2\Big)^3 + O\Big(\frac{\delta^8}{n^6}\Big)\Big), \end{aligned}$$

and

$$\prod_{1\le j\le n} h_j^{-\delta_j} = \exp\Big(-\frac{2}{n}\sum_{j=1}^{n}\delta_j^2 - \frac{2}{3n^3}\sum_{j=1}^{n}\delta_j^4 - \frac{2}{n^4}(\sum_{j=1}^{n}\delta_j^2)^2$$
$$- \frac{2}{5n^5}\sum_{j=1}^{n}\delta_j^6 - \frac{2}{n^6}(\sum_{j=1}^{n}\delta_j^3)^2 - \frac{6}{n^7}(\sum_{j=1}^{n}\delta_j^2)^3 + O(\frac{\delta^8}{n^6})\Big).$$

Therefore, we have

Theorem 5.3.2. *Let $\epsilon > 0$ be sufficiently small and $\delta_1, \ldots, \delta_n$ be integers. If $\sum_{k=1}^{n}\delta_k = 0$ and $\delta = \max\{|\delta_1|, \ldots, |\delta_n|\} = o(n^{3/4})$, as $n \to \infty$ we have the number of digraphs with n vertices and excess sequence $\delta_1, \ldots, \delta_n$ is asymptotically*

$$NDG(n;\delta_1,\ldots,\delta_n) = n^{1/2}(\frac{4^n}{\pi n})^{(n-1)/2}\exp\Big(-\frac{1}{4} - \frac{1}{n}\sum_{j=1}^{n}\delta_j^2 + \frac{1}{n^2}\sum_{j=1}^{n}\delta_j^2$$
$$- \frac{1}{6n^3}\sum_{j=1}^{n}\delta_j^4 - \frac{1}{2n^4}(\sum_{j=1}^{n}\delta_j^2)^2 - \frac{1}{15n^5}\sum_{j=1}^{n}\delta_j^6$$
$$- \frac{1}{3n^6}(\sum_{j=1}^{n}\delta_j^3)^2 - \frac{1}{n^7}(\sum_{j=1}^{n}\delta_j^2)^3 + O(\frac{\delta^4}{n^3} + n^{-1/4+\epsilon})\Big). \quad \blacksquare$$

Chapter 6

Asymptotic Enumeration of Eulerian Digraphs with Multiple Edges

6.1. Introduction.

By an *eulerian* digraph we mean a digraph in which the in-degree is equal to the out-degree at each vertex. Let $EDME(n,t)$ be the number of labelled loop-free eulerian digraphs with n vertices in which the multiplicity of each edge is at most t. Allowing loops would multiply $EDME(n,t)$ by exactly $(t+1)^n$, since loops do not affect the eulerian property. For the case where $t=1$, McKay [17] obtained the asymptotic formula

$$EDME(n,1) = \left(\frac{4^n}{\pi n}\right)^{(n-1)/2} n^{1/2} e^{-1/4}\big(1 + O(n^{-1/2+\epsilon})\big).$$

for any $\epsilon > 0$. Accurate values of $EDME(n,1)$ for $n \leq 16$ were also obtained by McKay [20].

We will identify $EDME(n,t)$ as a coefficient in a n-variable power series, and estimate it by applying the saddle-point method to the integral provided by Cauchy's Theorem. Since the parameter which is tending to ∞ is the number of dimensions, the application of the saddle-point method has an analytic flavour different from that of most fixed-dimensional problems. In particular, the choice of contour is trivial but substantial work is required to demonstrate that the parts of contour where the absolute value of the integrand is small contribute negligibly to the result.

6.2. Main result.

For $s \geq 0$ and $n \geq 1$, define $U_n(s) = \{(x_1, x_2, \ldots, x_n) \mid |x_i| \leq s \text{ for } 1 \leq i \leq n\}$.

Lemma 6.2.1.

(i) For integer $t \geq 0$ and real x with $|x| < \pi/(t+1)$,

$$\mid (1 + \exp(ix) + \cdots + \exp(itx))/(t+1) \mid \leq \exp(-\frac{1}{24}t(t+2)x^2).$$

(ii) For integer $t \geq 0$ and any real x,

$$\big|1 + \exp(ix) + \cdots + \exp(itx)\big| \leq t - 1 + \big(2 + 2\cos(x)\big)^{1/2}.$$

Proof. The proof for (ii) is too elementary to include and the proof for (i) is as follows.

$$\begin{aligned}
\Big| \frac{1 - \exp(i(t+1)x)}{1 - \exp(ix)} \Big| &= \Big(\frac{1 - \cos((t+1)x)}{1 - \cos(x)}\Big)^{1/2} \\
&= \frac{\sin((t+1)|x|/2)}{\sin(|x|/2)} \\
&= \exp\Big(\log(\sin((t+1)|x|/2)) - \log\big(\sin(|x|/2)\big)\Big) \\
&\leq (t+1)\exp(-\frac{1}{24}t(t+2)x^2),
\end{aligned}$$

since for $0 < x < \pi$,

$$\log\big(\sin(x)\big) = \log(x) + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} B_{2k} x^{2k}}{k(2k)!},$$

where $\{B_n\}$ are the Bernoulli numbers, which satisfy $(-1)^k B_{2k} < 0$. ∎

The following Theorem 6.2.2 was obtained by McKay [17] which is useful for our estimation.

Theorem 6.2.2. *Let a, b and c be real numbers with $a > 0$. Let $0 < \epsilon < 1/8$, and let $n \geq 2$ be an integer. Define*

$$J = J(a,b,c,n) =$$

$$\int \exp\Big(-a \sum_{1\leq j<k\leq n} (\theta_j - \theta_k)^2 + b \sum_{1\leq j<k\leq n} (\theta_j - \theta_k)^4 + \frac{c}{n^2}\Big(\sum_{1\leq j<k\leq n} (\theta_j - \theta_k)^2\Big)^2\Big)\, d\theta',$$

where the integral is over $\boldsymbol{\theta}' \in U_{n-1}(n^{-1/2+\epsilon})$ with $\theta_n = 0$. Then, as $n \to \infty$,

$$J = n^{1/2}\Big(\frac{\pi}{an}\Big)^{(n-1)/2} \exp\Big(\frac{6b+c}{4a^2} + O(n^{-1/2+4\epsilon})\Big). \quad \blacksquare$$

Theorem 6.2.3. *For any $\epsilon > 0$, as $n \to \infty$,*

$$EDME(n,t) = \Big(\frac{3(t+1)^{2n}}{t(t+2)\pi n}\Big)^{(n-1)/2} n^{1/2} \exp\Big(-\frac{3(t^2+2t+2)}{20t(t+2)} + O(n^{-1/2+\epsilon})\Big).$$

Proof. Since $\prod_{1\leq j\neq k\leq n}(1 + x_j x_k^{-1} + x_j^2 x_k^{-2} + \cdots + x_j^t x_k^{-t})$ is the generating function for the digraphs in which the multiplicity of each edge is at most t, $EDME(n,t)$ is the constant term. By Cauchy's Theorem,

$$EDME(n,t)$$
$$= \frac{1}{(2\pi i)^n} \oint \cdots \oint \frac{\prod_{1\leq j\neq k\leq n}(1 + x_j x_k^{-1} + x_j^2 x_k^{-2} + \cdots + x_j^t x_k^{-t})}{x_1 x_2 \cdots x_n} dx_1 \cdots dx_n,$$

where each integration is around a simple closed contour encircling the origin once in the anticlockwise direction. We choose each contour to be the unit circle and substitute $x_j = e^{i\theta_j}$ for $1 \leq j \leq n$. We obtain

$$EDME(n,t) = \frac{1}{(2\pi)^n}$$
$$\times \int_{U_n(\pi)} \prod_{1\leq j\neq k\leq n} \Big(1 + \exp\big(i(\theta_j - \theta_k)\big) + \cdots + \exp\big(it(\theta_j - \theta_k)\big)\Big)\, d\theta.$$

Defining

$$T_{jk}(\boldsymbol{\theta}) = \frac{1 + \exp\bigl(i(\theta_j - \theta_k)\bigr) + \cdots + \exp\bigl(it(\theta_j - \theta_k)\bigr)}{t+1}$$

and

$$g(\boldsymbol{\theta}) = \prod_{1 \le j \ne k \le n} T_{jk}(\boldsymbol{\theta}),$$

we have

$$EDME(n,t) = \frac{(t+1)^{n^2-n} I}{(2\pi)^n},$$

where

$$I = \int_{U_n(\pi)} g(\boldsymbol{\theta})\, d\boldsymbol{\theta}.$$

We will begin the evaluation of I with the part of the integrand which will turn out to give the major contribution. Let I_1 be the contribution to I of those $\boldsymbol{\theta}$ such that $|\theta_j - \theta_n| \le n^{-1/2+\epsilon}$ for $1 \le j \le n-1$, where θ_j values are taken mod 2π. Since $g(\boldsymbol{\theta})$ is invariant under uniform translation of all the θ_j, we see that the contributions to I_1 from different values of θ_n are the same. Hence,

$$I_1 = 2\pi \int_{U_{n-1}(n^{-1/2+\epsilon})} g(\boldsymbol{\theta})\, d\boldsymbol{\theta}',$$

where $\boldsymbol{\theta}' = (\theta_1, \ldots, \theta_{n-1})$ with $\theta_n = 0$.

Since

$$\exp(ix) = 1 + ix - \tfrac{1}{2}x^2 - \tfrac{1}{6}ix^3 + \tfrac{1}{24}x^4 + O(x^5),$$

and

$$\log(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 + O(z^5),$$

for complex z, we obtain

$$\begin{aligned} g(\boldsymbol{\theta}) &= \prod_{1\le j\ne k\le n} T_{jk}(\boldsymbol{\theta}) \\ &= \exp\Big(\sum_{1\le j\ne k\le n} \log T_{jk}(\boldsymbol{\theta})\Big) \\ &= \exp\Big(-\frac{1}{24}t(t+2)\sum_{1\le j\ne k\le n}(\theta_j-\theta_k)^2 \\ &\quad -\frac{1}{2880}t(t+2)(t^2+2t+2)\sum_{1\le j\ne k\le n}(\theta_j-\theta_k)^4 \\ &\quad +O\big(\sum_{1\le j<k\le n}|\theta_j-\theta_k|^5\big)\Big). \end{aligned}$$

Applying Theorem 6.2.2, we have

$$I_1 = 2\pi n^{1/2}\Big(\frac{12\pi}{t(t+2)n}\Big)^{(n-1)/2}\exp\Big(-\frac{3(t^2+2t+2)}{20t(t+2)}+O(n^{-1/2+5\epsilon})\Big). \quad (2.1)$$

So our remaining work is to prove that the integral of $g(\boldsymbol{\theta})$ over the other parts of the region of integration is negligible compared to (2.1).

Let $\delta = \pi/6(t+1)$. For $j = 0,1,2,3,6t+4,6t+5$, define the interval $A_j = [(j-1)\delta, j\delta]$, and $B = [-\pi,-2\delta]\cup[2\delta,\pi]$. For any $\boldsymbol{\theta}\in U_n(\pi)$, let us suppose that $A_0\cup A_1$ contains $n/3(t+1)$ or more of the θ_j. (If not, we can make this true by suitable translation). If $\theta_j\in B$ and $\theta_k\in A_0\cup A_1$, then $\delta\le|\theta_j-\theta_k|\le\pi+\delta$. Define C to be the set consisting of such pairs (j,k) and I_2 to be the contribution to I of all the cases where n^ϵ or more of the θ_j lie in B. Since, if $(j,k)\in C$, $|T_{jk}(\boldsymbol{\theta})|\le (t-1+(2+2\cos(\delta))^{1/2})/(t+1)$, and for any other pair (j,k), $|T_{jk}(\boldsymbol{\theta})|\le 1$, we have that

$$|I_2|\le(2\pi)^n\big((t-1+(2+2\cos(\delta))^{1/2})/(t+1)\big)^{2n^{1+\epsilon}/3(t+1)}.$$

From this it easily follows that $I_2 = O\bigl(\exp(-c_1 n^{1+\epsilon})\bigr)I_1$ for some $c_1 > 0$. Thus we can suppose that at least $n - n^\epsilon$ of the θ_j lie in $[-2\delta, 2\delta]$. Now define $I_3(r)$ to be the contribution to I of those $\boldsymbol{\theta}$ such that

(i) $3\delta \le |\theta_j| \le \pi$ for r values of j,

(ii) $\theta_j \in [-2\delta, 2\delta]$ for at least $n - n^\epsilon$ values of j, and

(iii) $\theta_j \in A_3 \cup A_{6t+4}$ for any other values of j.

Clearly $I_3(r) = 0$ if $r > n^\epsilon$. If θ_j and θ_k are in classes (i) and (ii), respectively, then $\delta \le |\theta_j - \theta_k| \le \pi + 2\delta$, while if they are both in classes (ii) and (iii), by Lemma 6.2.1, $|T_{jk}(\boldsymbol{\theta})| \le \exp\bigl(-\frac{1}{24}t(t+2)(\theta_j - \theta_k)^2\bigr)$. Using $|T_{jk}(\boldsymbol{\theta})| \le 1$ for the other cases, we find

$$|I_3(r)| \le (2\pi)^r \binom{n}{r} \Bigl(\bigl(t - 1 + (2 + 2\cos(\delta))^{1/2}\bigr)/(t+1)\Bigr)^{2r(n-n^\epsilon)} \times |I_3'(n-r)|, \tag{2.2}$$

where

$$I_3'(m) = \int_{U_m(\pi)} \prod_{1 \le j < k \le m} \exp\bigl(-\tfrac{1}{12}t(t+2)(\theta_j - \theta_k)^2\bigr) d\theta_1 \cdots d\theta_m.$$

We can apply the transformation T defined in Section 4.2.3 (using m in place of n) to easily obtain

$$I_3'(m) \le 2\pi m^{1/2} \Bigl(\frac{12\pi}{t(t+2)m}\Bigr)^{(m-1)/2}.$$

Substituting back into (2.2) we find that

$$\sum_{r=1}^{n^\epsilon} |I_3(r)| \le |I_1| \exp\bigl(-c_2 n + o(n)\bigr)$$

for some $c_2 > 0$. We conclude that only substantial contribution must come from the case $r = 0$.

Next, define $I_4(h)$ to be the contribution to I of those $\boldsymbol{\theta}$ such that

(i) $|\theta_n| \leq 3\delta$,

(ii) $n^{-1/2+\epsilon} < |\theta_j - \theta_n| \leq 6\delta$ for h values of j, and

(iii) $|\theta_j - \theta_n| \leq n^{-1/2+\epsilon}$ for the remaining values of j.

Since $g(\boldsymbol{\theta})$ is invariant under uniform translation of all the θ_j, we see that the contributions to $I_4(h)$ from different values of θ_n are the same. Hence, we have $|I_4(h)| \leq 6\delta |I_4'(h)|$, where $|I_4'(h)|$ is the same integral over $\boldsymbol{\theta}'$ with $\theta_n = 0$. Since we have $|T_{jk}(\boldsymbol{\theta})| \leq \exp\left(-\frac{1}{24}t(t+2)(\theta_j - \theta_k)^2\right)$, apply the transformation T defined in Section 4.2.3 to transform the $\boldsymbol{\theta}'$ to $\boldsymbol{y}$ and the values of $\boldsymbol{\theta}'$ contributing to $I_4'(h)$ for $h \geq 1$ map to a subset of those $\boldsymbol{y}$ such that either $|\sum_{k=1}^{n-1} y_k| > n^{\epsilon}/2$ or $|y_k| > n^{-1/2+\epsilon}/2$ for some k. Since the contribution to

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\Big(-\tfrac{1}{12}t(t+2)n \sum_{k=1}^{n-1} y_k^2\Big)\, d\boldsymbol{y}$$

of those $\boldsymbol{y}$ is $O(n)(12\pi/t(t+2)n)^{(n-1)/2} \exp(-c_3 n^{2\epsilon})$ for some $c_3 > 0$, we conclude that

$$\sum_{h=1}^{n-1} |I_4(h)| \leq O(n) \exp(-c_3 n^{2\epsilon}) |I_1|.$$

The remaining case, $h = 0$, is covered by I_1. Therefore we have completed our proof. ∎

Chapter 7

Conclusions and Future Research

7.1. Introduction.

We have presented a few research results on the asymptotic enumeration of labelled graphs. In this chapter, we give some conclusions and a brief description of potential future research in this area.

7.2. Conclusions.

We obtain an asymptotic formula for the number of the labelled bipartite graphs by degree sequence in Chapter 2. We identified these graphs as one cell of a partiton of a larger class of objects (pairings) and then obtained estimates of the relative sizes of the cells by counting simple perturbations (switchings) which transform one cell into another. Since the total number of pairings is available, the size of each cell can be inferred. The switching argument was first proposed by McKay [18] and then applied to estimate the number of 0–1 matrices with prescribed line sums [21] and to obtain an asymptotic formula for the number of symmetric 0–1 matrices with prescribed row sums [22]. Recently, McKay and Wormald [27] showed how to generate k-regular graphs on n vertices uniformly at random in expected time $O(nk^3)$, provided $k = O(n^{1/3})$, by using a modification of the switching argument. This method will probably be applied to other asymptotic enumeration problems as well as problems relating to uniform generation of graphs.

In Chapters 4–6, we obtain asymptotic formulae for the numbers of labelled tournaments, digraphs and eulerian digraphs with multiple edges. The methods we employ to solve such problems are similar. We first identify the corresponding values as coefficients of their generating functions, and then apply Cauchy's Theorem to express the quantities as n-dimensional integrals. Transformations and statistical arguments are used to purge those terms in the integrands which do not give a major contribution to the integrals. Similar methods were used to obtain asymptotic formulae for regular tournaments, eulerian digraphs and eulerian oriented graphs [17] and general graphs of high degree [25].

7.3. Future research.

7.3.1. Asymptotics of bipartite graphs of high degree.

In Chapter 2, we determine the asymptotic number of bipartite graphs with a given degree sequence for the case where the maximum degree is $o(E^{1/3})$. In particular, if $k = o(n^{1/2})$, the number of regular bipartite graphs of degree k with each part having n vertices is asymptotically

$$\frac{(nk)!}{(k!)^{2n}} \exp\Big(-\frac{(k-1)^2}{2} - \frac{k^3}{6n} + O\big(\frac{k^2}{n}\big)\Big).$$

Both the results we will seek and the method we are proposed to use in the research on asymptotic enumeration of bipartite graphs of high degree are quite different from those in Chapter 2. We will consider the estimation of the number of bipartite graphs with separately labelled parts and degree sequence $s_1, s_2, \ldots, s_n$, $t_1, t_2, \ldots, t_m$ by Cauchy integral, where s_j is approximately a constant times of m for all j and t_k is approximately a constant times of n for all k.

Since the generating function for bipartite graphs by degree sequence is

$$F(x_1, \ldots, x_n, y_1, \ldots, y_m) = \prod_{i,j}(1 + x_i y_j),$$

the number of bipartite graphs with parts X and Y where the degree of x_i is s_i for $1 \le i \le n$ and the degree of y_j is t_j for $1 \le j \le m$ is the coefficient of $x_1^{s_1} \cdots x_n^{s_n} y_1^{t_1} \cdots y_m^{t_m}$. By Cauchy's Theorem, the coefficient is equal to

$$\frac{1}{(2\pi i)^{n+m}} \oint \cdots \oint \frac{F(x_1, \ldots, x_n, y_1, \ldots, y_m)}{x_1^{s_1+1} \cdots x_n^{s_n+1} y_1^{t_1+1} \cdots y_m^{t_m+1}} dx_1 \cdots dx_n dy_1 \cdots dy_m,$$

where each integral is along a closed curve around origin in the anti-clockwise direction. We will estimate the integral to obtain an asymptotic formula for the number of bipartite graphs of high degree.

7.3.2. Asymptotics of eulerian oriented graphs with multiple edges.

By an *eulerian* digraph we mean a digraph in which the in-degree is equal to the out-degree at each vertex and by an *eulerian oriented graph* we mean an eulerian digraph in which at most one of the edges (v, w) or (w, v) are permitted for any distinct v and w. Let $EOG(n, t)$ be the number of labelled loop-free eulerian oriented graphs with n vertices in which the multiplicity of each edge is at most t. Allowing loops would multiply $EOG(n, t)$ by exactly $(t + 1)^n$, since loops do not affect the eulerian property. In Chapters 4–6, we used the saddle-point method to obtain asymptotic formulae for the numbers of labelled tournaments, digraphs and eulerian digraphs with multiple edges. Now we are concerned with the asymptotic value of $EOG(n,t)$. McKay [17] obtained that as $n \to \infty$,

$$EOG(n,1) = \left(\frac{3^{n+1}}{4\pi n}\right)^{(n-1)/2} n^{1/2} e^{-3/8}\left(1 + O(n^{-1/2+\epsilon})\right),$$

for any $\epsilon > 0$. Exact values of $EOG(n,1)$ for $n \leq 15$ were also obtained by McKay [20]. Similarly to Chapter 6, in which we estimated the asymptotic number of eulerian digraphs with multiple edges, we will again apply the saddle-point method to derive an asymptotic formula for $EOG(n,t)$.

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