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A. Ballester-Bolinches, John Cossey, S.F. Kamornikov, H. Meng

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On two questions from the Kourovka Notebook

A. Ballester-Bolinches[*] John Cossey[†] S. F. Kamornikov[‡]
H. Meng [§]

Abstract

The aim of this paper is to give answers to some questions concerning intersections of system normalisers and prefrattini subgroups of finite soluble groups raised by the third author, Shemetkov and Vasil'ev in the Kourovka Notebook [10]. Our approach depends on results on regular orbits and it can be also used to extend a result of Mann [9] concerning intersections of injectors associated to Fitting classes.

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1 Introduction

The first part of this paper has as its main theme the study of intersections of normalisers and prefrattini subgroups of finite soluble groups associated to saturated formations, and provides answers to two questions raised by the third author and Shemetkov and Vasil'ev in the Kourovka Notebook [10]. Hence all groups to be considered here are finite, and we assume this without further comment.

*Departament de Matemàtiques, Universitat de València, 46100 Burjassot, València, Spain, email: Adolfo.Ballester@uv.es

†Australian National University, Canberra, ACT 2601, Australia, email: John.Cossey@anu.edu.au

‡Department of Mathematics, Francisk Skorina Gomel State University, Gomel 246019, Belarus, email: sfkamornikov@mail.ru

§Departament de Matemàtiques, Universitat de València, 46100 Burjassot, València, Spain, email: hangyangmenges@gmail.com

In order to understand and motivate what is to follow it is convenient to use theorems of Dolfi, Passman and Zenkov as a model. Dolfi [3] proved that if π is a set of primes, the largest normal π-subgroup $\mathrm{O}_\pi(G)$ of a π-soluble group is the intersection of three G-conjugates of a given Hall π-subgroup H of G. This result extends earlier theorems of Passman [12] (case $|\pi| = 1$) and Zenkov [13](case H nilpotent). On the other hand, as Mann pointed out in [9], the results of Passman imply that the Fitting subgroup $\mathrm{F}(G)$ of a soluble group G is the intersection of three G-conjugates of a nilpotent injector H of G.

Bearing in mind the above results and the important role played by the system normalisers and prefrattini subgroups in the structural study of soluble groups, the following questions turn out to be natural and interesting:

Question 1. [10, Kamornikov, Problem 17.55] Does there exist an absolute constant k such that the Frattini subgroup $\Phi(G)$ of a soluble group G is the intersection of k G-conjugates of any prefrattini subgroup H of G?

Question 2. [10, Shemetkov and Vasil'ev, Problem 17.39] Is there a positive integer k such that the hypercentre of any finite soluble group coincides with the intersection of k system normalisers of that group? What is the least number with this property?

Recall that a formation is a class of groups $\mathfrak{F}$ which is closed under taking epimorphic images and such that every group G has an smallest normal subgroup with quotient in $\mathfrak{F}$. This subgroup is called the $\mathfrak{F}$-*residual* of G and denoted by $G^{\mathfrak{F}}$. A maximal subgroup M of a group G containing $G^{\mathfrak{F}}$ is called $\mathfrak{F}$-*normal* in G; otherwise, M is said to be $\mathfrak{F}$-*abnormal*.

We say that $\mathfrak{F}$ is *saturated* if it is closed under Frattini extensions. In such case, by a well-known theorem of Gaschütz-Lubeseder-Schmid [2, Theorem IV.4.6], there exists a collection of formations $\mathrm{F}(p) \subseteq \mathfrak{F}$, one for each prime p, such that $\mathfrak{F}$ coincides with the class of all groups G such that if H/K is a chief factor of G, then $G/\mathrm{C}_G(H/K) \in \mathrm{F}(p)$ for all primes p dividing $|H/K|$. In this case, we say that H/K is $\mathfrak{F}$-*central* in G and $\mathfrak{F}$ is *locally defined* by the $\mathrm{F}(p)$. H/K is called $\mathfrak{F}$-*eccentric* if it is not $\mathfrak{F}$-central.

Note that a chief factor H/K supplemented by a maximal subgroup M is $\mathfrak{F}$-central in G if and only if M is $\mathfrak{F}$-normal in G.

Every group G has a largest normal subgroup such that every chief factor of G below it is $\mathfrak{F}$-central in G. This subgroup is called the $\mathfrak{F}$-*hypercentre* of G and it is denoted by $\mathrm{Z}_{\mathfrak{F}}(G)$ (see [2, Section IV.6]).

Our first main theorem subsumes the main result of [8] and gives a complete answer to a general version of Question 1. It provides a beautiful description of the intersection of four (in some cases three) $\mathfrak{F}$-prefrattini subgroups of a soluble group introduced by Hawkes in [6], where $\mathfrak{F}$ is a saturated formation. If $\mathfrak{F}$ is the trivial formation, then these subgroups coincide with the prefrattini subgroups formulated by Gaschütz in [5].

In order to state it, we consider convenient to give some definitions.

Definition 3. A 3-tuple (G, X, Y) is said to be a *k-conjugate system* if G is a group, X, Y are subgroups of G with $Y = \mathrm{Core}_G(X)$, and there exist k elements $g_1, ..., g_k$ such that $Y = \bigcap_{i=1}^{k} X^{g_i}$.

Let Σ be a Hall system of the soluble group G (see [2, Chapter I, Section 1.4]). Let S^p be the p-complement of G contained in Σ, and denote by $W^p(G)$ the intersection of all $\mathfrak{F}$-abnormal maximal subgroups of G containing S^p ($W^p(G) = G$, if the set of all $\mathfrak{F}$-abnormal maximal subgroups of G containing S^p is empty). Then $W(G, \Sigma, \mathfrak{F}) = \bigcap_{p \in \pi(G)} W^p(G)$ is called the $\mathfrak{F}$-*prefrattini subgroup* of G associated to Σ. The $\mathfrak{F}$-prefrattini subgroups of G form a characteristic class of G-conjugate subgroups (see [1, Section 4.3] for an exhaustive study of $\mathfrak{F}$-prefrattini subgroups).

According to [1, Proposition 4.3.17], the intersection $\mathrm{L}_{\mathfrak{F}}(G)$ of all $\mathfrak{F}$-abnormal maximal subgroups of a soluble group G is the core of every $\mathfrak{F}$-prefrattini subgroup of G and $\mathrm{L}_{\mathfrak{F}}(G)/\Phi(G) = \mathrm{Z}_{\mathfrak{F}}(G/\Phi(G))$ for every group G. In fact, we have:

Theorem A. *Let $\mathfrak{F}$ be a saturated formation and let H be an $\mathfrak{F}$-prefrattini subgroup of a soluble group G. Then $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a 4-conjugate system. Furthermore, if either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups, then $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a 3-conjugate system.*

Recall that a group X is S_n-free if the symmetric group of degree n does not appear as a quotient of any subgroup of X.

If $\mathfrak{F} = \mathfrak{N}$, the formation of all nilpotent groups, then $\mathrm{L}_{\mathfrak{F}}(G) = \mathrm{L}(G)$ is the intersection of all self-normalising maximal subgroups of G. It is a characteristic nilpotent subgroup of G that was introduced by Gaschütz in [4]. If $\mathfrak{F}$ is the trivial formation, then $\mathrm{L}_{\mathfrak{F}}(G) = \Phi(G)$, the Frattini subgroup of G. Hence:

Corollary 4 ([7])**.** *If G is soluble and H is an $\mathfrak{N}$-prefrattini subgroup of G, then $(G, H, \mathrm{L}(G))$ is a 3-conjugate system.*

Corollary 5 ([8])**.** *If G is soluble and H is a prefrattini subgroup of G, then $(G, H, \Phi(G))$ is a 3-conjugate system.*

To describe our second main result, we shall give a review of the definition of the $\mathfrak{F}$-normalisers of a soluble group.

Let $\mathrm{F}(p)$ be a particular family of formations locally defining $\mathfrak{F}$ and such that $\mathrm{F}(p) \subseteq \mathfrak{F}$ for all primes p. Let $\pi = \{p : \mathrm{F}(p) \neq \emptyset\}$. For an arbitrary soluble group G and a Hall system Σ of G, choose for any prime p, the p-complement $K^p = S^p \cap G^{\mathrm{F}(p)}$ of the $\mathrm{F}(p)$-residual $G^{\mathrm{F}(p)}$ of G, where S^p is the p-complement of G in Σ. Then $D_{\mathfrak{F}}(\Sigma) = G_\pi \cap (\bigcap_{p\in\pi} \mathrm{N}_G(K^p))$, where G_π is the Hall π-subgroup of G in Σ, is the $\mathfrak{F}$-*normaliser* of G associated to Σ. The $\mathfrak{F}$-normalisers of G are a characteristic class of G-conjugate subgroups. There were introduced by Carter and Hawkes and coincide with the classical system normalisers of Hall when $\mathfrak{F}$ is the formation of all nilpotent groups (see [2, Sections V.2 and V.3] for details).

According to [1, Proposition 4.2.6], if D is an $\mathfrak{F}$-normaliser of G, then $\mathrm{Core}_G(D) = \mathrm{Z}_{\mathfrak{F}}(G)$. We prove:

Theorem B. *Let $\mathfrak{F}$ be a saturated formation and let D be an $\mathfrak{F}$-normaliser of a soluble group G such that $\Phi(G) = 1$. Then $(G, D, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 4-conjugate system. Furthermore, if either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups, then $(G, D, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 3-conjugate system.*

Recall that if $\mathfrak{F} = \mathfrak{N}$ is the formation of all nilpotent groups, then the $\mathfrak{N}$-normalisers of a soluble group G are exactly the system normalisers of G and $\mathrm{Z}_{\mathfrak{N}}(G) = \mathrm{Z}_\infty(G)$ is the hypercentre of G. Therefore the answer of Question 2 for groups with trivial Frattini subgroup is contained in the following:

Corollary 6. *Let G be a soluble group with $\Phi(G) = 1$. If D is a system normaliser of G, then $(G, D, \mathrm{Z}_\infty(G))$ is a 3-conjugate system.*

Our next example shows that $(G, D, \mathrm{Z}_\infty(G))$ is not a 2-conjugate system in general.

Example 7. Let D be the dihedral group of order 8. Then D has an irreducible and faithful module V of dimension 2 over the field of 3-elements such that $C_D(v) \neq 1$ for all $v \in V$. Let $G = V \rtimes D$ be the corresponding semidirect product. Then D is a system normaliser of G and $Z_\infty(G) = 1$. By [2, Lemma A.16.3], $D \cap D^v = C_D(v) \neq 1$ for all $v \in V$. Hence $(G, D, Z_\infty(G))$ is not a 2-conjugate system.

Our third main theorem has Mann's result as starting point and analyses the intersections of injectors associated to Fitting classes of soluble groups. A class of groups $\mathfrak{F}$ is said to be a *Fitting class* if $\mathfrak{F}$ is a class under taking subnormal subgroups and such that every group G has a largest normal $\mathfrak{F}$-subgroup called $\mathfrak{F}$-*radical* and denoted by $G_{\mathfrak{F}}$. Every soluble group G has a conjugacy class of subgroups, called $\mathfrak{F}$-*injectors*, which are defined to be those subgroups I of G such that if S is a subnormal subgroup of G, then

$I \cap S$ is $\mathfrak{F}$-maximal subgroup of S ([2, Theorem IX.1.4]). Note that, in this case, $\mathrm{Core}_G(I) = G_{\mathfrak{F}}$. We prove:

Theorem C. *Let $\mathfrak{F}$ be a Fitting class and let I be an $\mathfrak{F}$-injector of a soluble group G. Then $(G, I, G_{\mathfrak{F}})$ is a 4-conjugate system. Furthermore, if either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups, then $(G, I, G_{\mathfrak{F}})$ is a 3-conjugate system.*

Corollary 8 ([9])**.** *If I is a nilpotent injector of a soluble group G, then $(G, I, \mathrm{F}(G))$ is a 3-conjugate system.*

2 Background results

The notation and terminology agree with the books [1, 2], and we refer the reader to them for results on formations.

In the sequel, $\mathfrak{F}$ will be a saturated formation.

We begin with an elementary observation which will be used throughout the paper.

Lemma 9 ([2, Lemma A.16.3])**.** *Let $G = NH$ be a semidirect product of a normal subgroup N with a subgroup H.*
(a) If $n \in N$, then $H \cap H^n = C_H(n)$,
(b) $Core_G(H) = C_H(N)$.

The elementary properties of the subgroup $\mathrm{L}_{\mathfrak{F}}(G)$ are collected in the following.

Lemma 10. *If N is a normal subgroup of a group G, then the following conditions hold:*

1. $\mathrm{L}_{\mathfrak{F}}(G)N/N \leq \mathrm{L}_{\mathfrak{F}}(G/N)$.
2. *If $N \leq \mathrm{L}_{\mathfrak{F}}(G)$, then* $\mathrm{L}_{\mathfrak{F}}(G/N) = \mathrm{L}_{\mathfrak{F}}(G)/N$.
3. $\mathrm{L}_{\mathfrak{F}}(G/\mathrm{L}_{\mathfrak{F}}(G)) = 1$.

The set all $\mathfrak{F}$-prefrattini subgroups of a group G is denoted by $\mathbf{Pref}_{\mathfrak{F}}(G)$.

We begin by recalling some known properties about $\mathfrak{F}$-prefrattini subgroups.

Recall that a subgroup X of a group G *covers* the section A/B of G if $A \leq XB$ and *avoids* A/B if $X \cap A \leq B$.

Lemma 11 ([1, 6])**.** *Let G be a soluble group and N a normal subgroup of G.*

1. $\mathbf{Pref}_{\mathfrak{F}}(G)$ *is a* G*-conjugacy class of subgroups of* G.

2. $\mathbf{Pref}_{\mathfrak{F}}(G/N) = \{HN/N : H \in \mathbf{Pref}_{\mathfrak{F}}(G)\}$.

3. *If* $H \in \mathbf{Pref}_{\mathfrak{F}}(G)$, *then* H *avoids every complemented* $\mathfrak{F}$*-eccentric chief factor of* G *and covers the rest.*

Our next result turns out to be crucial in the proof of Theorem A.

Lemma 12. *Let* N *be a minimal normal subgroup of a soluble group* G. *Assume that* M *is an* $\mathfrak{F}$*-abnormal maximal subgroup of* G *complementing* N *in* G. *Then* $\mathbf{Pref}_{\mathfrak{F}}(G) = \bigcup_{g\in G} \mathbf{Pref}_{\mathfrak{F}}(M^g)$.

Proof. Since $\mathfrak{F}$-prefrattini subgroups of G are conjugate in G, it suffices to show that $\mathbf{Pref}_{\mathfrak{F}}(M) \subseteq \mathbf{Pref}_{\mathfrak{F}}(G)$.

Let $H = W(M, \Sigma_M, \mathfrak{F})$ be the $\mathfrak{F}$-prefrattini subgroup of M associated to the Hall system Σ_M of M. Let p be the prime dividing the order of N and let P be the Sylow p-subgroup of M in Σ. Then $\Sigma = \Sigma_M \cup \{PN\}$ is a Hall system of G.

Let $1 = A_0 \leq A_1 \leq ... \leq A_n = M$ be a chief series of M, and let $\{A_i/A_{i-1} | i \in I\}$ be the set of all complemented $\mathfrak{F}$-eccentric chief factors in this series. By [1, Proposition 4.3.6], $H = W(\Sigma) = \bigcap_{i\in I} M_i$, where M_i is a maximal subgroup of M, complementing A_i/A_{i-1} in G, into which the Hall system Σ_M reduces, $i \in I$. Consider the following chief series of G:

$$1 \leq N = A_0N \leq A_1N \leq ... \leq A_nN = MN = G$$

Then $A_iN/A_{i-1}N$ is a complemented $\mathfrak{F}$-eccentric chief factor of G if and only if A_i/A_{i-1} is a complemented $\mathfrak{F}$-eccentric chief factor of M. Moreover, N is an $\mathfrak{F}$-eccentric chief factor of G which is complemented by M, and Σ reduces into M. Thus $\{N, A_i/A_{i-1} | i \in I\}$ is the set of all complemented $\mathfrak{F}$-eccentric chief factors in the above chief series.

On the other hand, M_iN is a maximal subgroup of G complementing $A_iN/A_{i-1}N$ in G and Σ reduces into M_iN for all $i \in I$. Applying [1, Proposition 4.3.6], $M \cap (\bigcap_{i\in I} M_iN) = \bigcap_{i\in I} M_i(M \cap N) = \bigcap_{i\in I} M_i = H$ is the $\mathfrak{F}$-prefrattini subgroup of G associated to Σ.

□

Remark 13. *Under the hypotheses of Lemma 12,* $(H \cap H^m)N = HN \cap H^mN$ *for all* $m \in M$.

Proof. $HN \cap H^mN = (H \cap H^mN)N = (H \cap M \cap H^mN)N$ and $M \cap H^mN = H^m(M \cap N) = H^m$. □

Lemma 14. *Let N be a minimal normal subgroup of a soluble group G. Assume that M is an $\mathfrak{F}$-abnormal maximal subgroup of G complementing N in G. Then $\mathrm{L}_{\mathfrak{F}}(G) = C_{\mathrm{L}_{\mathfrak{F}}(M)}(N)$.*

Proof. By Lemma 12, we have:

$$\begin{aligned} \mathrm{L}_{\mathfrak{F}}(G) &= \bigcap\{H : H \in \mathbf{Pref}_{\mathfrak{F}}(G)\} \\ &= \bigcap_{g\in G}\bigcap\{H : H \in \mathbf{Pref}_{\mathfrak{F}}(M^g)\} \\ &= \bigcap_{g\in G} \mathrm{L}_{\mathfrak{F}}(M)^g = Core_G(\mathrm{L}_{\mathfrak{F}}(M)). \end{aligned}$$

Since $\mathrm{L}_{\mathfrak{F}}(G)\cap N \leq M\cap N = 1$, we have $\mathrm{L}_{\mathfrak{F}}(G) \leq C_{\mathrm{L}_{\mathfrak{F}}(M)}(N)$. On the other hand, since $C_{\mathrm{L}_{\mathfrak{F}}(M)}(N)$ is normalised by M and centralised by N, we have that $C_{\mathrm{L}_{\mathfrak{F}}(M)}(N)$ is normal in G and hence $C_{\mathrm{L}_{\mathfrak{F}}(M)}(N) \leq Core_G(\mathrm{L}_{\mathfrak{F}}(M)) = \mathrm{L}_{\mathfrak{F}}(G)$. □

The following facts about S_3-free groups are quite useful.

Lemma 15 (see [11, Lemma 1])**.** *Let G be a soluble group and let H be a Hall $\{2,3\}$-subgroup of G. Then G is S_3-free if and only if H is 3-nilpotent.*

Lemma 16 (see [11, Lemma 2])**.** *Let G be a soluble group with $\mathrm{O}_{2'}(G) = 1$. Then G is S_3-free if and only if G is S_4-free.*

Corollary 17. *Let G be a soluble S_3-free group such that $\mathrm{O}_{3'}(G) = 1$. Then G is of odd order.*

Proof. Let H be a Hall $\{2,3\}$-subgroup of G and let X be a Hall $3'$-subgroup of G. Then $H \cap X$ is a Sylow 2-subgroup of G and $G = HX$ by [2, Lemma A.1.6]. Hence $H \cap X \trianglelefteq H$ by Lemma 15. Therefore

$$(H \cap X)^G = (H \cap X)^{HX} = (H \cap X)^X \leq X.$$

This implies that $(H\cap X)^G$ is a $3'$-subgroup of G and so $H\cap X \leqslant (H\cap X)^G \leqslant \mathrm{O}_{3'}(G) = 1$. Thus G is of odd order. □

Lemma 18. *Let G be a group and $L, K \trianglelefteq G$ such that $K \leqslant \Phi(G)$. If L/K is a soluble S_3-free group, then L is a soluble S_3-free group.*

Proof. Assume that (G, L, K) satisfies the hypotheses but L is not a soluble S_3-free group. Choose such counterexample (G, L, K) such that $|G|+|L|+|K|$ is minimal. Let H be a Hall $\{2,3\}$-subgroup of L.

Write $X = \mathrm{O}_{3'}(L)$. Then $X \trianglelefteq G$. Denote with bars the images in $\overline{G} = G/X$. We have that $(\overline{G}, \overline{L}, \overline{K})$ satisfies the hypotheses of the lemma. Hence, if $X \neq 1$, it follows that $\overline{L}$ is a soluble S_3-free group. Since $\overline{H}$ is a Hall $\{2,3\}$-subgroup of $\overline{L}$, we can apply Lemma 15 to conclude that $\overline{H} = HX/X \cong H/H \cap X$ is 3-nilpotent. Then $H/\mathrm{O}_{3'}(H)$ is 3-nilpotent since $H \cap X \leqslant \mathrm{O}_{3'}(H)$. Thus H is 3-nilpotent and so L is S_3-free by Lemma 15, which is a contradiction. Consequently, $X = 1$.

Since $\mathrm{O}_{3'}(K) \leqslant \mathrm{O}_{3'}(L) = 1$ and K is nilpotent, we have that K is a 3-group. Let $T/K = \mathrm{O}_{3'}(L/K)$. Then $T \trianglelefteq G$. By [2, Theorem A.11.3], $T = KT_1$, where T_1 is a Hall $3'$-subgroup of T. The Frattini Argument for Hall subgroups implies that $G = \mathrm{N}_G(T_1)T = \mathrm{N}_G(T_1)K = \mathrm{N}_G(T_1)$ since $K \leqslant \Phi(G)$. Thus $T_1 \trianglelefteq G$ and so $T_1 \leqslant \mathrm{O}_{3'}(L) = 1$. Hence $\mathrm{O}_{3'}(L/K) = 1$. By Corollary 17, L/K is of odd order. Thus the order of L is odd and so it is S_3-free. This final contradiction proves the lemma. □

Combining Lemmas 15 and 18, we have:

Corollary 19. *The class of all soluble S_3-free groups is a subgroup-closed saturated formation.*

Recall that a *regular orbit* of the action of the group G on a set X is an orbit with $|G|$ elements. Clearly G has a regular orbit on X if and only if there exists $x \in X$ such that $C_G(x) = 1$. Considering the natural action of G on $X \times X$, we have that G has a regular orbit on $X \times X$ if and only if there exist $x, y \in X$ such that $C_G(x) \cap C_G(y) = 1$.

The proofs of our main theorems strongly depend on the following results.

Lemma 20 ([3, Theorem 1.4])**.** *Let G be a soluble group and V a finite faithful G-module. If V is completely reducible (possibly of mixed characteristic), then there exist $v_1, v_2, v_3 \in V$ such that $C_G(v_1) \cap C_G(v_2) \cap C_G(v_3) = 1$.*

Lemma 21 (see [11, Theorem A])**.** *Suppose that G is a soluble group and V is a finite, faithful and completely reducible G-module (possibly of mixed characteristic). Let H be a subgroup of G such that the semidirect product VH is S_4-free. Then H has at least two regular orbits on $V \bigoplus V$.*

3 Proof of Theorem A

Assume we are trying to prove a result of the following type: Let G be a soluble group and let H be an $\mathfrak{F}$-prefrattini subgroup of G. Then $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a k-conjugate system.

Assume the statement is false. Thus there would exist a counterexample G of minimal order. Let H be an $\mathfrak{F}$-prefrattini subgroup of G such that $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is not a k-conjugate system. Then:

(i) $\mathrm{L}_{\mathfrak{F}}(G) = 1$. In particular, $\Phi(G) = 1$.

For suppose that X is a minimal normal subgroup of G contained in $\mathrm{L}_{\mathfrak{F}}(G)$. Then H/X is an $\mathfrak{F}$-prefrattini subgroup of G by Lemma 11. Therefore, because $|G/X| < |G|$, it follows that $(G/X, H/X, \mathrm{L}_{\mathfrak{F}}(G/X))$ is a k-conjugate system. Since $\mathrm{L}_{\mathfrak{F}}(G/X) = \mathrm{L}_{\mathfrak{F}}(G)/X$ by Lemma 10, we have that $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a k-conjugate system, giving a contradiction. Thus Statement (i) must hold.

Also (ii) There exists a minimal normal subgroup N and an $\mathfrak{F}$-abnormal maximal subgroup M containing H of G such that $G = MN$ and $M \cap N = 1$ and $(M, H, \mathrm{L}_{\mathfrak{F}}(M))$ is a k-conjugate system.

Let N be the minimal normal subgroup of G. Then N is a p-group for some prime p. By Statement (i), N is not contained in $\mathrm{L}_{\mathfrak{F}}(G) = 1$ and so there exists an $\mathfrak{F}$-abnormal maximal subgroup of M such that $G = NM$ and $N \cap M = 1$. By Lemma 12, we may assume that H is an $\mathfrak{F}$-prefrattini subgroup of M. Again by choice of G, $(M, H, \mathrm{L}_{\mathfrak{F}}(M))$ is a k-conjugate system and therefore there exist $m_1, ...m_k \in M$ such that $\bigcap_{i=1}^{k} H^{m_i} = \mathrm{L}_{\mathfrak{F}}(M)$.

(iii) Assume that N is a p-group for some prime p and $L = \mathrm{L}_{\mathfrak{F}}(M)$. Then N is a faithful completely reducible L-module over $\mathrm{GF}(p)$, the finite field of p-elements.

Clearly N is an irreducible M-module over $\mathrm{GF}(p)$. By [2, Theorem B.7.3], N is a completely reducible L-module. By Lemma 14 and Statement (i), $\mathrm{C}_L(N) = 1$ and so N is faithful for L.

Let $T = LN$. Then $\mathrm{Core}_T(L) = 1$. Moreover:

(iv) $(T, L, 1)$ is not a k-conjugate system.

Assume that $(T, \mathrm{L}, 1)$ is a k-conjugate system. Let $n_1, ..., n_k \in N$ such that $\bigcap_{i=1}^{k} L^{n_i} = 1$. We consider the subgroup $D = \bigcap_{i=1}^{k} H^{m_i n_i}$. Then

$$D \leq \bigcap_{i=1}^{k} H^{m_i n_i} N = \bigcap_{i=1}^{k} H^{m_i} N = (\bigcap_{i=1}^{k} H^{m_i}) N = LN$$

by Remark 13. Then

$$\begin{aligned} D = D \cap LN &= \bigcap_{i=1}^{k} H^{m_i n_i} \cap LN \\ &= \bigcap_{i=1}^{k} (H^{m_i} \cap LN)^{n_i} = \bigcap_{i=1}^{k} L^{n_i} = 1 = \mathrm{L}_{\mathfrak{F}}(G). \end{aligned}$$

Therefore $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a k-conjugate system, against our supposition.

Theorem 22. *Let H be an $\mathfrak{F}$-prefrattini subgroup of a soluble group G. Then $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a 4-conjugate system.*

Proof. Assume that the result is not true and let G be a counterexample of minimal order such that $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is not a 4-conjugate system. Then Statements (i)-(iv) hold for $k = 4$. By Statement (iii), N is a faithful completely reducible L-module over $\mathrm{GF}(p)$ for some prime p. By Lemma 20, there exist $v_1, v_2, v_3 \in N$ such that $C_L(v_1) \cap C_L(v_2) \cap C_L(v_3) = 1$. It implies that $L \cap L^{v_1} \cap L^{v_2} \cap L^{v_3} = 1$ by Lemma 9. Thus $(T, \mathrm{L}, 1)$ is a 4-conjugate system, contrary to Step (iv). □

Theorem 23. *Let H be an $\mathfrak{F}$-prefrattini subgroup of a soluble group G. Assume that either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups. Then $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is a 3-conjugate system.*

Proof. Suppose, arguing by contradiction, that $(G, H, \mathrm{L}_{\mathfrak{F}}(G))$ is not a 3-conjugate system. Let us choose G a counterexample of least order. Then Statements (i)-(iv) hold for $k = 3$. By Statement (iii), $L \cap N = 1$ and N is a faithful completely reducible L-module over $\mathrm{GF}(p)$ for some prime p. If G is S_4-free, then LN is S_4-free. Assume that $\mathfrak{F}$ is composed of S_3-free groups. Recall that $L = \mathrm{L}_{\mathfrak{F}}(M)$, by [1, Proposition 4.3.17], $L/\Phi(M) = \mathrm{Z}_{\mathfrak{F}}(M/\Phi(M))$. Let $\mathfrak{X}$ be the class of all soluble S_3-free groups. By Corollary 19, $\mathfrak{X}$ is a subgroup-closed saturated formation. Since $\mathfrak{F} \subseteq \mathfrak{X}$ by hypothesis, it follows that $\mathrm{Z}_{\mathfrak{F}}(M/\Phi(M)) \leqslant \mathrm{Z}_{\mathfrak{X}}(M/\Phi(M))$. By [2, Theorem IV.6.15], $\mathrm{Z}_{\mathfrak{X}}(M/\Phi(M)) \in \mathfrak{X}$. Thus $L/\Phi(M) = \mathrm{Z}_{\mathfrak{F}}(M/\Phi(M))$ is S_3-free. Then, by Lemma 18, L is S_3-free. If p is odd, then LN is S_4-free and if $p = 2$, then LN is S_4-free by Lemma 16. In both cases, we can apply Lemma 21 to conclude that there exist $v_1, v_2 \in N$ such that $C_L(v_1) \cap C_L(v_2) = 1$. Thus, by Lemma 9, $(T, L, 1)$ is a 3-conjugate system, contrary to Statement (iv). □

4 Proof of Theorem B

The proof of Theorem B depends on a nice result about factorisations of prefrattini subgroups proved in [6, Theorem 4.1] (see [1, Theorem 4.3.32]).

Lemma 24. *If D is an $\mathfrak{F}$-normaliser and W is a prefrattini subgroup of a soluble group G, both associated to the Hall system Σ of G, then D and W permute and DW is the $\mathfrak{F}$-prefrattini subgroup of G associated to Σ.*

Theorem 25. *Let D be an $\mathfrak{F}$-normaliser of a soluble group G. If $\Phi(G) = 1$, then $(G, D, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 4-conjugate system.*

Proof. Let D be the $\mathfrak{F}$-normaliser of G associated to the Hall system Σ. Assume that H is the $\mathfrak{F}$-prefrattini subgroup of G associated to Σ. Then, by Lemma 24, we have $D \leq H$. Since $\Phi(G) = 1$, it follows by [1, Proposition 4.3.17] that $\mathrm{L}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$. By Theorem A, we have that $(G, H, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 4-conjugate system. Hence

$$\begin{aligned} \mathrm{Z}_{\mathfrak{F}}(G) &\leq D \cap D^x \cap D^y \cap D^z \\ &\leq H \cap H^x \cap H^y \cap H^z \\ &= \mathrm{Z}_{\mathfrak{F}}(G). \end{aligned}$$

Thus $(G, D, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 4-conjugate system. □

Theorem 26. *Let D be an $\mathfrak{F}$-normaliser of a soluble subgroup G such that $\Phi(G) = 1$. Assume that either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups. Then $(G, D, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 3-conjugate system.*

Proof. Assume that Σ is the Hall system of G to which D is associated. Let H be the $\mathfrak{F}$-prefrattini subgroup of G associated to Σ. By Theorem A, $(G, H, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 3-conjugate system. Since $D \leq H$ by Lemma 24 and $\mathrm{L}_{\mathfrak{F}}(G) = \mathrm{Z}_{\mathfrak{F}}(G)$ by [1, Proposition 4.3.17], it follows that $(G, D, \mathrm{Z}_{\mathfrak{F}}(G))$ is a 3-conjugate system. □

5 Proof of Theorem C

Let $R = \mathrm{Core}_G(I) = G_{\mathfrak{F}}$. We prove that (G, I, R) is a 4-conjugate system by induction on the order of G. Let F be the normal subgroup of G such that $F/R = \mathrm{F}(G/R)$, the Fitting subgroup of G/R. Clearly, $F \cap I$ is contained in R. Hence $F \cap I = R$. On the other hand, by [2, Theorem IX.1.5], I is an $\mathfrak{F}$-injector of FI. Thus $R \leq S = (FI)_{\mathfrak{F}}$ is contained in I. Assume that R is a proper subgroup of S and let N/R be a minimal normal subgroup of FI/R contained in S/R. Then N belongs to $\mathfrak{F}$ and so N is contained in R. This is a contradiction yields $S = R$. If FI were a proper subgroup of G, (FI, I, R) would be a 4-conjugate system. Hence (G, I, R) would be a 4-conjugate system and the result would follow. Therefore we may assume that $G = FI$. Let M be the normal subgroup of G such that $M/R = \Phi(G/R)$. Then $G/M = (IM/M)(F/M)$. Applying [2, Theorem A.10.6], $F/M = \mathrm{Soc}(G/M)$ is a self-centralising normal subgroup of G/M. In particular, F/M is a completely reducible G/M-module (possibly of mixed characteristic). By Lemma 20, there exist $v_1M, v_2M, v_3M \in F/M$ such that $C_{IM/M}(v_1M) \cap C_{IM/M}(v_2M) \cap C_{IM/M}(v_3M) = 1$. It implies that $I \cap I^{v_1} \cap I^{v_2} \cap I^{v_3} \leq R$ by Lemma 9. Thus (G, I, R) is a 4-conjugate system.

Assume that either G is S_4-free or $\mathfrak{F}$ is composed of S_3-free groups. If G is S_4-free, then $G/M = (IM/M)(F/M)$ is S_4-free. By Lemma 21, there exist $v_1M, v_2M \in F/M$ such that $C_{IM/M}(v_1M) \cap C_{IM/M}(v_2M) = 1$.

Suppose that $\mathfrak{F}$ is composed of S_3-free groups. Denote with bars the images in $\overline{G} = G/M = \overline{IF}$. Since $\overline{I} \in \mathfrak{F}$, $\overline{I}$ is S_3-free. Let A be the Hall $2'$-subgroup of $\overline{F}$. It follows that $\overline{I}A$ is S_4-free. Let B be the Sylow 2-subgroup of $\overline{F}$. By Lemma 16, $\overline{I}B/\mathrm{C}_{\overline{I}}(B)$ is S_4-free. Then we can apply Lemma 21 to conclude that there exist $a_1M, a_2M \in A$ and $b_1M, b_2M \in B$ such that $C_{\overline{I}}(a_1M) \cap C_{\overline{I}}(a_2M) \subseteq \mathrm{C}_{\overline{I}}(A)$ and $C_{\overline{I}}(b_1M) \cap C_{\overline{I}}(b_2M) \subseteq \mathrm{C}_{\overline{I}}(B)$. Let $v_i = a_i + b_i$, $i = 1, 2$. Then $C_{\overline{I}}(v_1M) \cap C_{\overline{I}}(v_2M) \subseteq \mathrm{C}_{\overline{I}}(A) \cap \mathrm{C}_{\overline{I}}(B) = \mathrm{C}_{\overline{I}}(\overline{F}) = 1$.

In both cases, we have that (G, I, R) is a 3-conjugate system by Lemma 9. This completes the proof of the theorem.

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