PRODUCT INTEGRATION TECHNIQUES
FOR THE NUMERICAL SOLUTION OF
INTEGRAL EQUATIONS

by

F.R. de Hoog

A thesis submitted to the
Australian National University
for the degree of Doctor of Philosophy

August 1973

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

This work was carried out at the Computer Centre, Australian National University. I gratefully acknowledge the financial assistance of a Commonwealth Postgraduate Award during this period. I wish to thank the head of the centre, Mike Osborne, for his general support throughout this time.

In particular, I would like to thank Bob Anderssen, not only for his constructive criticism of the draft and final version of this thesis but also for many stimulating discussions which on a number of occasions have led to and helped crystalise ideas. This work has greatly benefited from the collaboration with Richard Weiss which is mentioned in the preface.

I also wish to thank the typist Mrs Barbara Geary whose patience and skill are clearly evident in the thesis. I am also grateful to Greg Woodford for his careful reading, checking and correcting of the final typescript.

PREFACE

Much of the work in this thesis was done in collaboration with Richard Weiss. In particular, chapters 2, 3 and 4 are based on results established during this collaboration.

The publication details of this thesis are de Hoog (1973), and de Hoog and Weiss (1972 e, f, g). In many cases the text of these papers have been followed closely.

Elsewhere in the thesis, unless another source is acknowledged, the work described is my own.

F. de Hoog

ABSTRACT

This thesis examines the use of product integration for the construction of numerical schemes for integral equations. After a brief discussion of the literature in chapter 1, the product integration technique, based on piecewise interpolating polynomials is described in chapter 2 for integrals of the form

$$I_g(f) = \int_0^1 f(s)g(s)ds$$

where $f(t)$ is 'smooth' and $g(t)$ is absolutely integrable. Euler Maclaurin sum formula are derived for the cases when $g(t)$ is 'smooth' and when $g(t)$ has a finite number of algebraic and logarithmic singularities. The application of these Euler Maclaurin expansions to numerical schemes for integral equations is briefly discussed.

In chapter 3, the smoothness of the solution of the second kind Volterra equation

$$y(t) = f_1(t) + \sqrt{t}\, f_2(t) + \int_0^t \frac{g(t,s,y(s))}{\sqrt{t-s}}\, ds \ , \quad 0 \le t \le T \qquad (*)$$

is examined. It is shown that

$$y(t) = u(t) + \sqrt{t}\, v(t) \ ,$$

where, under appropriate smoothness conditions on $f_1(t)$, $f_2(t)$ and $g(t, s, y)$, $u(t)$ and $v(t)$ are smooth functions which are the components of a system of equations of the form

$$\left.\begin{aligned} u(t) &= f_1(t) + \int_0^t \frac{\sqrt{s}}{\sqrt{t-s}}\, g_1\big(t,\ s,\ u(s),\ v(s)\big)\,ds \\ v(t) &= f_2(t) + \frac{1}{\sqrt{t}} \int_0^t \frac{g_2\big(t,s,u(s),v(s)\big)}{\sqrt{t-s}}\,ds \end{aligned}\right\} \qquad (**)$$

The smoothness results for $g(t)$, established in chapter 3, are used in chapter 4 to construct efficient numerical schemes for (*). These schemes are based on the product integration technique described in chapter 2 and makes use of (**) in a neighbourhood of the origon. Convergence and stability of these schemes are examined.

In chapter 5, the product integration technique described in chapter 2, is used to construct numerical schemes for the first kind Fredholm equation

$$\int_a^b k(t,\ s)y(s)ds = g(t)\ ,\quad a \le t \le b\ .$$

Convergence results for various choices of $k(t,\ s)$ and $g(t)$ are obtained.

CHAPTER 1

INTRODUCTION

1.1 Introduction

Many problems in mathematical physics can be reduced to finding the solutions of integral equations of the form

$$\lambda y(t) = f(t) + \int_a^t K\big(t, s, y(s)\big)ds \ , \quad a \le t \le T \qquad (1.1.1)$$

and

$$\lambda y(t) = f(t) + \int_a^b k(t, s)y(s)ds \ , \quad a \le t \le b \ . \qquad (1.1.2)$$

Equations of the form (1.1.1) are called Volterra equations whilst equations of the form (1.1.2) are called Fredholm equations. If $\lambda \neq 0$, the equations are of the second kind and are of the first kind otherwise.

The usual method of solving second kind equations is to discretize (1.1.1) or (1.1.2) at a number of grid points and then to replace the integral by a quadrature formula based on those points (see, for instance, Linz (1967), de Hoog and Weiss (1972 a) and Atkinson (1971)). This reduces the problem to that of finding the solution of a set of equations (non-linear for (1.1.1) and linear for (1.1.2)). In order that such schemes be efficient, it is necessary that the quadrature rules used yield 'good' approximations to the integral terms. Thus, special care must be taken when dealing with equations where a singularity occurs in the integrand or one of its derivatives. Clearly, this requires that the smoothness of the solution $y(t)$ should be known, since the most efficient quadrature rules are those which take the exact nature of the singularity into account.

For first kind equations, however, numerical schemes as described above do not in general converge. This has been illustrated by Linz (1967) and Jennings (1972) for linear first kind Volterra and Fredholm equations, respectively. However, convergent schemes for such Volterra equations have been constructed and analysed by Linz (1967), Noble (1964), Hung (1970), Kobazasi (1966), Jones (1961) and de Hoog and Weiss (1972 b, c) when the kernel $K(t, s, y(s))$ is smooth and linear and by Weiss and Anderssen (1972) and Weiss (1972) when the kernel is linear and has an algebraic singularity at $t = s$. Numerical schemes for first kind Fredholm equation with singular kernels have been constructed by Christiansen (1971) and Noble (1971). Although convergence for these schemes has been demonstrated numerically, no analytic convergence results have yet been obtained as far as the author is aware.

It should be noted, however, that numerical solutions of integral equations can sometimes be obtained by indirect means. Examples of this are the numerical schemes based on the inversion formulae for Abel type equations (see, for instance, Minerbo and Levy (1969) and Anderssen and Weiss (1973)) and numerical schemes based on regularized forms of first kind Fredholm equations (see, for instance, Wahba (1970) and Jennings (1972)).

In this thesis, we examine the use of product integration to obtain direct finite difference schemes for second kind Volterra and first kind Fredholm equations where the kernels are singular. We attempt to show that this technique is particularly suitable for integral equations when the smoothness of the solution $y(t)$, as well as the singularity in the kernel can be taken into account. The product integration technique was first applied by Young (1954) to integral equations and has subsequently received considerable attention in this context by Noble (1964), Atkinson (1967), Linz (1967), Hung (1970), Noble and Tavernini (1971), Weiss and Anderssen (1972), Weiss (1972), de Hoog and Weiss (1972 d) and Anderssen, de Hoog and Weiss (1973).

1.2 Thesis Outline

In chapter 2, the product integration schemes are described. Quadrature formulae based on piecewise polynomial interpolation are constructed and generalised Euler Maclaurin sum formulae are derived for the case when the integrand has a finite number of algebraic or logarithmic singularities.

In chapter 3, the smoothness of the solution of the second kind Volterra equation

$$y(t) = f_1(t) + \sqrt{t}\, f_2(t) + \int_0^t \frac{g\{t,s,y(s)\}}{\sqrt{t-s}}\, ds \qquad (1.2.2)$$

is examined. A particular case of this equation gives the solution of a heat conduction problem. It is found that

$$y(t) = u(t) + \sqrt{t}\, v(t)$$

where $u(t)$ and $v(t)$ satisfy a system of coupled Volterra equations.

The results of chapter 3 are then used in chapter 4 to construct a number of numerical schemes for equation (1.1.2) based on the product integration technique described in chapter 2. Convergence results and asymptotic expansions for the numerical solution are obtained using the results of chapter 2. Stability of the numerical schemes is then examined via these asymptotic results.

In chapter 5, numerical schemes are constructed for the first kind Fredholm equation

$$g(t) = \int_a^b \{k(t, s)p(t, s)+q(t, s)\}y(s)ds \ , \quad a \le t \le b \quad (1.2.2)$$

where $k(t, s)$ has a singularity at $t = s$ and $p(t, s)$ and $q(t, s)$ are smooth, periodic functions. It is indicated how equations of the form (1.2.2) arise naturally in the solution of Laplace's equation, conformal mapping problems and scattering problems. The numerical schemes considered are the product integration analogues of the midpoint, trapezoidal and Simpson schemes. Convergence results for various choices of $k(t, s)$, $p(t, s)$ and $q(t, s)$ are derived.

CHAPTER 2

PRODUCT INTEGRATION

2.1 Introduction

A widely used technique for the evaluation of integrals of the form

$$I_g(f) = \int_0^1 f(s)g(s)ds \tag{2.1.1}$$

where $f(t)$ is 'smooth' and $g(t)$ is absolutely integrable is product integration. In this technique, $I_g(f)$ is replaced by $I_g(\tilde{f})$ where $\tilde{f}(t)$ is some approximation to $f(t)$ such that $I_g(\tilde{f})$ can be calculated in a simple manner.

The class of methods we shall examine are the product integration rules in which $f(t)$ is approximated by piecewise polynomial interpolation. Although the convergence of these schemes follows immediately from standard results on Lagrangian interpolation, the correct rate of convergence for specific choices of $g(t)$ has not been investigated.

In this chapter, we derive generalised Euler Maclaurin sum formulae for schemes where $g(t)$ may have a finite number of algebraic and logarithmic singularities. We then indicate how such expansions can be used to obtain accurate convergence rates for integral equations with weakly singular kernels.

In section 2.2, we introduce the quadrature rules under consideration and prove a basic lemma. Euler Maclaurin sum formulae are established for 'smooth' and 'weakly singular' $g(t)$ in sections 2.3 and 2.4 respectively. These results are then applied in section 2.5 to obtain accurate convergence results for second kind Fredholm and Volterra equations.

2.2 The Product Integration Rule

Let

$$0 \le u_1 < u_2 < \dots < u_n \le 1$$

be a fixed set of points and define

$$\omega(t) = \prod_{k=1}^{n} (t-u_k) ,$$

the Lagrangian polynomials

$$L_k(t) = \omega(t)/\omega'(u_k)(t-u_k) , \quad k = 1, \dots, n$$

and the grids

$$t_l = lh , \quad l = 0, \dots, m ; \quad h = 1/m$$

and

$$t_{lk} = t_l + u_k h , \quad k = 1, \dots, n ; \quad l = 0, \dots, m-1 . \qquad (2.2.1)$$

On

$$t_l \le t < t_{l+1} , \quad l = 0, \dots, m-1 ,$$

the approximation to $f(t)$ is

$$\tilde{f}(t) = \sum_{k=1}^{n} L_k\left(\frac{t-t_l}{h}\right) f(t_{lk})$$

and hence

$$I_g(\tilde{f}) = \sum_{l=0}^{m-1} \int_{t_l}^{t_{l+1}} g(s)\tilde{f}(s)ds$$

$$= \sum_{l=0}^{m-1} \sum_{k=1}^{n} f(t_{lk}) \int_{t_l}^{t_{l+1}} g(s)L_k\left(\frac{s-t_l}{h}\right) ds$$

$$= \sum_{l=0}^{m-1} \sum_{k=1}^{n} hf(t_{lk}) \int_0^1 g(t_l+sh)L_k(s)ds . \qquad (2.2.2)$$

This is the mn point quadrature rule with which we are concerned. The weights

$$\int_0^1 g\left(t_l+sh\right)L_k(s)ds \ , \quad k = 1, \ldots, n \ ; \quad l = 0, \ldots, m-1$$

are calculated analytically. The error functional for the rule is

$$\begin{aligned} E_g(f) &= I_g(\tilde{f}) - I_g(f) \\ &= I_g(\tilde{f}-f) \end{aligned} \qquad (2.2.3)$$

and an expression for it is obtained in the following lemma.

Lemma 2.2.1. *If* $f(t) \in C^{p+1}[0, 1]$, $p \geq n$, *then*

$$E_g(f) = \sum_{r=0}^{p-n} h^{n+r} \int_0^1 \omega_r(s)h \sum_{l=0}^{m-1} g\left(t_l+sh\right)f^{(n+r)}\left(t_l+sh\right)ds$$
$$+ O\left(h^{p+1}\right) \ , \qquad (2.2.4)$$

where

$$\omega_r(t) = \omega(t)p_r(t) \qquad (2.2.5)$$

and $p_r(t)$ *is a polynomial of degree* r .

Proof. It is clear that

$$E_g(f) = h \sum_{l=0}^{m-1} \int_0^1 g\left(t_l+sh\right)\left\{\tilde{f}\left(t_l+sh\right)-f\left(t_l+sh\right)\right\}ds \ . \qquad (2.2.6)$$

For $0 \leq s \leq 1$, it follows from (2.2.1) and Taylor's theorem that

$$f\left(t_{lk}\right) = f\left(t_l+sh-\left(s-u_k\right)h\right)$$
$$= \sum_{r=0}^{p} h^r \frac{(-1)^r\left(s-u_k\right)^r}{r!} f^{(r)}\left(t_l+sh\right) + O\left(h^{p+1}\right) ,$$
$$k = 1, \ldots, n \ ; \quad l = 0, \ldots, m-1 \ .$$

Hence

$$\tilde{f}(t_l+sh) - f(t_l+sh)$$

$$= \sum_{k=1}^{n} \{f(t_{lk})-f(t_l+sh)\}L_k(s)$$

$$= \sum_{r=1}^{p} h^r \frac{(-1)^r}{r!} f^{(r)}(t_l+sh) \sum_{k=1}^{n} (s-u_k)^r L_k(s) + O(h^{p+1}) ,$$

$$l = 0, \ldots, m-1 . \quad (2.2.7)$$

Since

$$\sum_{k=1}^{n} (s-u_k)^r L_k(s) = \omega(s) \sum_{k=1}^{n} \frac{(s-u_k)^{r-1}}{\omega'(u_k)} \quad (2.2.8)$$

and

$$\sum_{k=1}^{n} \frac{u_k^q}{\omega'(u_k)} = 0 , \quad q = 1, \ldots, n-2 ,$$

it follows that

$$\sum_{k=1}^{n} (s-u_k)^r L_k(s) = 0 , \quad r = 0, \ldots, n-1 . \quad (2.2.9)$$

For $r \geq n-1$,

$$\sum_{k=1}^{n} \frac{(s-u_k)^r}{\omega'(u_k)} = \sum_{q=0}^{r} \binom{r}{q} (-1)^q s^{r-q} \sum_{k=1}^{n} \frac{u_k^q}{\omega'(u_k)}$$

$$= \sum_{q=n-1}^{r} \sum_{k=1}^{n} \binom{r}{q} (-1)^q \frac{u_k^q}{\omega'(u_k)} s^{r-q} . \quad (2.2.10)$$

Substitution of (2.2.8), (2.2.9) and (2.2.10) into (2.2.7) yields

$$\tilde{f}(t_l+sh) - f(t_l+sh) = \sum_{r=0}^{p-n} h^{n+r} f^{(n+r)}(t_l+sh)\omega(s)p_r(s) + O(h^{p+1}) ,$$

$$l = 0, \ldots, m-1 , \quad (2.2.11)$$

where

$$p_r(s) = \frac{(-1)^r}{(n+r)!} \sum_{q=0}^{r} \sum_{k=1}^{n} \binom{n+r-1}{n+q-1} (-1)^{q-1} \frac{u_k^{n+q-1}}{\omega'(u_k)} s^{r-q} . \quad (2.2.12)$$

The result follows on substitution (2.2.11) into (2.2.6). #

REMARK. Clearly, $\omega_r(t)$, $r = 0, \ldots, p-n$, also depend on u_k , $k = 1, \ldots, n$. In addition, it should be noted that lemma 2.2.1 is valid for any absolutely integrable $g(t)$.

For fixed s , $0 \leq s \leq 1$, the sum

$$h \sum_{l=0}^{m-1} g\left(t_l+sh\right)f^{(n+r)}\left(t_l+sh\right) \tag{2.2.13}$$

is a generalized Euler approximation to $\int_0^1 g(s)f^{(n+r)}(s)ds$.
Summation formulae for (2.2.13) have been investigated by Lyness and Ninham (1967) and the application of their results to (2.2.4) is the basis of sections 2.3 and 2.4.

2.3 Smooth $g(t)$

Let $f(t) \in C^{p+1}[0, 1]$, $p \geq n$ and $g(t) \in C^{p-n+1}[0, 1]$. Applying the Euler Maclaurin sum formula to $g(t)f^{(n+r)}(t)$, we find

$$h \sum_{l=0}^{m-1} g\left(t_l+xh\right)f^{(n+r)}\left(t_l+xh\right) = \int_0^1 g(t)f^{(n+r)}(t)dt$$

$$+ \sum_{q=0}^{p-n-r-1} \frac{h^{q+1}B_{q+1}(x)}{(q+1)!} \int_0^1 \frac{d^{q+1}}{dt^{q+1}}\left[g(t)f^{(n+r)}(t)\right]dt + O\left(h^{p-n-r+1}\right) ,$$

$$r = 0, \ldots, p-n \tag{2.3.1}$$

where $B_q(x)$, $q = 1, 2, \ldots$, are the Bernoulli polynomials.

Substituting (2.3.1) into (2.2.4) and collecting powers of h , we obtain

$$E_g(f) = h^n \int_0^1 \omega_0(s)ds \cdot \int_0^1 g(t)f^{(n)}(t)dt$$

$$+ \sum_{r=0}^{p-n-1} h^{n+r+1}\left\{\int_0^1 \omega_{r+1}(s)ds \cdot \int_0^1 g(t)f^{(n+r+1)}(t)dt\right.$$

$$+ \sum_{l=0}^{r} \frac{1}{(r-l+1)!}\int_0^1 \omega_l(s)B_{r-l+1}(s)ds \times$$

$$\left.\int_0^1 \frac{d^{r-l+1}}{dt^{r-l+1}}\left[g(t)f^{(n+l)}(t)\right]dt\right\} + O(h^{p+1}) . \quad (2.3.2)$$

The above equation is a generalised Euler Maclaurin expansion for the error functional.

If u_k , $k = 1, \ldots, n$, are chosen such that

$$\int_0^1 s^r\omega(s)ds = 0 \ , \quad r = 0, 1, \ldots, q < n \qquad (2.3.3)$$

it is clear from (2.2.5) that

$$\int_0^1 s^l\omega_r(s)ds = 0 \ , \quad r = 0, 1, \ldots, q \ ; \quad l = 0, \ldots, q-r \ ,$$

and hence the first $q + 1$ terms in (2.3.2) vanish. This may be expected since for $g(t) = 1$, (2.3.2) reduces to the Euler Maclaurin sum formula for the corresponding composite interpolatory quadrature rule (see for instance, Baker and Hodgson (1971)).

In the case that $g(t) = 1$ and a symmetric rule is used, the coefficients of the odd powers of h are zero and so the expansion is in integer powers of h^2 . For a general $g(t)$, however, the rule is not symmetric and so this does not happen.

2.4 Singular $g(t)$

In this section, we shall consider the case where $g(t)$ has a finite number of algebraic and logarithmic singularities.

Firstly we shall establish an Euler Maclaurin sum formula when

$$g(t) = t^{\beta}(1-t)^{\omega}|t-v_k|^{\gamma}\operatorname{sgn}(t-v_i)|t-v_i|^{\delta} . \tag{2.4.1}$$

As in section 2.3, expansions for sums of the form

$$h\sum_{l=0}^{m-1} g(t_l+xh)z(t_l+xh)$$

where $z(t)$ is a smooth function are required. Such expansions have been derived by Lyness and Ninham (1967) who use Lighthill's procedure to obtain asymptotic expansions for the integral terms in Poisson's summation formula

$$h\sum_{l=0}^{m-1} g(t_l+xh)z(t_l+xh) - \int_0^1 g(s)z(s)ds$$

$$= \sum_{q=-\infty}^{+\infty}{}' (-1)^q \exp\{-\pi i(2x-1)q\} \int_0^1 g(s)z(s)\exp(2\pi iqms)ds$$

$$= \sum_{q=-\infty}^{+\infty}{}' \exp(-2\pi iqx) \int_0^1 g(s)z(s)\exp\left(\frac{2\pi iqs}{h}\right)ds , \tag{2.4.2}$$

where the prime on the summation sign indicates that the term corresponding to $q = 0$ has been deleted.

Applying the results of Lyness and Ninham (1967, Eq. 8.1) to $g(t)f^{(n+r)}(t)$ we find that

$$h\sum_{l=0}^{m-1} g(t_l+xh)f^{(n+r)}(t_l+xh) = \int_0^1 g(s)f^{(n+r)}(s)ds$$

$$+ \sum_{q=0}^{p-n-r} \frac{h^{q+1}}{q!}\Big\{h^{\beta}\tilde{\zeta}(-\beta-q,\ x)\psi_{0r}^{(q)}(0)$$

$$+ h^{\omega}(-1)^q\tilde{\zeta}(-\omega-q,\ 1-x)\psi_{1r}^{(q)}(1)$$

$$+ h^{\gamma}\left[\tilde{\zeta}(-\gamma-q,\ x-mv_k)+(-1)^q\tilde{\zeta}(-\gamma-q,\ mv_k-x)\right]\psi_{2r}^{(q)}(v_k)$$

$$+ h^{\delta}\left[\tilde{\zeta}(-\delta-q,\ x-mv_i)-(-1)^q\tilde{\zeta}(-\delta-q,\ mv_i-x)\right]\psi_{3r}^{(q)}(v_i)\Big\}$$

$$+ O(h^{p-n-r+1}) ,\quad r = 0, \ldots, p-n , \tag{2.4.3}$$

where

$$\psi_{0r}(t) = f^{(n+r)}(t)(1-t)^{\omega}|t-v_k|^{\gamma}\operatorname{sgn}(t-v_i)|t-v_i|^{\delta} ,$$

$$\psi_{1r}(t) = f^{(n+r)}(t)t^{\beta}|t-v_k|^{\gamma}\operatorname{sgn}(t-v_i)|t-v_i|^{\delta} ,$$

$$\psi_{2r}(t) = f^{(n+r)}(t)t^{\beta}(1-t)^{\omega}\operatorname{sgn}(t-v_i)|t-v_i|^{\delta} ,$$

$$\psi_{3r}(t) = f^{(n+r)}(t)t^{\beta}(1-t)^{\omega}|t-v_k|^{\gamma}$$

and $\tilde{\zeta}(\alpha, x)$ is the periodic generalised zeta function. The periodic generalised zeta function is defined by

$$\tilde{\zeta}(\alpha, x) = \zeta(\alpha, \tilde{x}) \ , \quad x - \tilde{x} = \text{integer}, \quad 0 < \tilde{x} \le 1$$

where $\zeta(\alpha, x)$ is the generalised Riemann zeta function (see, for instance, Whittaker and Watson (1958)).

Substitution of (2.4.3) into (2.2.4) yields

$$\begin{aligned}
E_g(f) = {} & \sum_{r=0}^{p-n} h^{n+r}\int_0^1 \omega_r(s)ds \int_0^1 g(s)f^{(n+r)}(s)ds \\
& + \sum_{r=0}^{p-n} h^{n+r+\beta+1} \sum_{l=0}^{r} \frac{\psi_{0l}^{(r-l)}(0)}{(r-l)!}\int_0^1 \omega_l(s)\tilde{\zeta}(-\beta-r+l, s)ds \\
& + \sum_{r=0}^{p-n} h^{n+r+\omega+1} \sum_{l=0}^{r} \frac{(-1)^{r-l}\psi_{1l}^{(r-l)}(1)}{(r-l)!}\int_0^1 \omega_l(s)\tilde{\zeta}(-\omega-r+l, 1-s)ds \\
& + \sum_{r=0}^{p-n} h^{n+r+\gamma+1} \sum_{l=0}^{r} \frac{\psi_{2l}^{(r-l)}(v_k)}{(r-l)!}\int_0^1 \omega_l(s)\Big[\tilde{\zeta}(-\gamma-r+l, s-mv_k) \\
& \qquad + (-1)^{r-l}\tilde{\zeta}(-\gamma-r+l, mv_k-s)\Big]ds \\
& + \sum_{r=0}^{p-n} h^{n+r+\delta+1} \sum_{l=0}^{r} \frac{\psi_{3l}^{(r-l)}(v_i)}{(r-l)!}\int_0^1 \omega_l(s)\Big[\tilde{\zeta}(-\delta-r+l, s-mv_i) \\
& \qquad - (-1)^{r-l}\tilde{\zeta}(-\delta-r+l, mv_i-s)\Big]ds + O(h^{p+1}) \ .
\end{aligned} \qquad (2.4.4)$$

This is the desired Euler Maclaurin expansion for $g(t)$ given by (2.4.1). For the important case of end point singularities (i.e.

$g(t) = t^{\beta}(1-t)^{\omega}$) terms of the form $\int_0^1 \omega_l(s)\tilde{\zeta}(\alpha, s)ds$ and

$\int_0^1 \omega_l(s)\tilde{\zeta}(\alpha, 1-s)ds$ can be reduced to sums of ordinary zeta functions

by the relations

$$\int_0^1 \zeta(\alpha, s)ds = 0 \;, \quad \alpha < 1$$

and

$$\int_0^1 s^r\zeta(\alpha, s)ds = \frac{1}{1-\alpha}\left[\zeta(\alpha-1)-r\int_0^1 s^{r-1}\zeta(\alpha-1, s)ds\right] ,$$

$$r = 1, 2, \ldots , \quad \alpha < 1 .$$

If u_k , $k = 1, \ldots, n$, are chosen such that

$$\int_0^1 \omega(s)ds = 0 \tag{2.4.5}$$

the first term in (2.4.4) is deleted. However, from (2.4.4), conditions for higher order convergence depend on $g(t)$, and therefore (2.3.3) does not in general lead to higher order convergence.

To illustrate this, we take $g(t) = t^{-\frac{1}{2}}$ and determine the conditions necessary for optimal convergence in the cases $n = 2$ and $n = 3$.

If $n = 2$ we require (2.4.5) and

$$\int_0^1 \omega(s)\tilde{\zeta}(\tfrac{1}{2}, s)ds = 0 . \tag{2.4.6}$$

Numerical calculation yields

$$u_1 = .1182506123 \;, \quad u_2 = .7182932992 . \tag{2.4.7}$$

For $n = 3$ we require (2.4.5), (2.4.6) and

$$\int_0^1 s\omega(s)ds = 0 \ .$$

Numerical calculation yields

$$u_1 = .04456270208 \ , \quad u_2 = .3909749362 \ , \quad u_3 = .8537066313 \ . \qquad (2.4.8)$$

The quadrature formula with the points given by (2.4.7) and (2.4.8) has been applied to

$$I_g(f) = \int_0^1 \frac{\sqrt{2-x}}{\sqrt{x}}\,dx = 1 + \pi/2 \ .$$

Numerical results for various stepsizes are tabulated in table 2.1. The order of convergence can be seen to be three and four and a half, respectively.

Table 2.1

Stepsize h	$n=2$ $E_g(f)$	$n=3$ $E_g(f)$
0.2	6.008 E-6	3.025 E-9
0.1	7.004 E-7	1.505 E-10
0.05	8.287 E-8	6.956 E-12
0.025	9.933 E-9	3.013 E-13

The extension of (2.4.4) to a $g(t)$ which includes terms of the form $\ell n t$, $\ell n(1-t)$, $\ell n|t-v_k|$ and $\mathrm{sgn}(t-v_i)\ell n|t-v_i|$ can be made by differentiation with respect to β, ω, γ and δ , respectively. To illustrate this, we consider the case when

$$g(t) = \ell n|t-v_k| = \frac{\partial}{\partial\gamma}\left(|t-v_k|^{\gamma}\right)\Big|_{\gamma=0} \ , \quad 0 < v_k < 1 \ .$$

Putting β, ω and δ to zero, differentiating (2.4.4) with respect to γ and then putting $\gamma = 0$, we find that

$$E_g(f) = h^n \int_0^1 \omega_0(s)ds \int_0^1 g(s)f^{(n)}(s)ds$$

$$+ \sum_{r=0}^{p-n-1} h^{n+r+1} \left\{\int_0^1 \omega_{r+1}(s)ds \int_0^1 g(s)f^{(n+r+1)}(s)ds\right.$$

$$+ \sum_{l=0}^{r} \frac{1}{(r-l)!} \left[\frac{d^{r-l}}{dt^{r-l}} \left(g(t)f^{(n+l)}(t)\right)\right|_{t=0} \int_0^1 \omega_l(s)\tilde{\zeta}(-r+l,\ s)ds$$

$$\left.\left. + (-1)^{r-l} \frac{d^{r-l}}{dt^{r-l}} \left(g(t)f^{(n+l)}(t)\right)\right|_{t=1} \int_0^1 \omega_l(s)\tilde{\zeta}(-r+l,\ 1-s)ds\right]\right\}$$

$$+ \sum_{r=0}^{p-n} f^{(n+r)}(v_k)h^{n+r+1}\left\{\ell nh \sum_{l=0}^{r} \frac{1}{(r-l)!} \int_0^1 \omega_l(s)\left(\tilde{\zeta}(-r+l,\ s-mv_k)\right.\right.$$

$$\left. + (-1)^{r-l}\tilde{\zeta}(-r+l,\ mv_k-s)\right)ds + \sum_{l=0}^{r} \frac{1}{(r-l)!} \int_0^1 \omega_l(s)\left(\tilde{\zeta}'(-r+l,\ s-mv_k)\right.$$

$$\left.\left. + (-1)^{r-l}\tilde{\zeta}'(-r+l,\ mv_k-s)\right)ds\right\} + O\left(h^{p+1}\right) , \qquad (2.4.9)$$

where

$$\tilde{\zeta}'(\alpha,\ s) = \frac{-\partial}{\partial\alpha}\tilde{\zeta}(\alpha,\ s) .$$

This expansion can be simplified slightly by substitution of the relations

$$\zeta(-q,\ s) = -B_{q+1}(s)/(q+1) , \quad q = 0,\ 1,\ 2,\ \ldots .$$

Again, if (2.4.5) holds, the first term in (2.4.9) is deleted.

2.5 The Application to Integral Equations

Atkinson (1967) considers the numerical solution of linear Fredholm integral equations of the second kind with singular kernels,

$$y(t) = G(t) + \lambda \int_0^1 K(t,\ s)y(s)ds , \quad 0 \le t \le 1 , \qquad (2.5.1)$$

where

$$K(t, s) = \sum_{k=1}^{r} P_k(t, s)Q_k(t, s) \ , \quad r \geq 1 \ , \qquad (2.5.2)$$

and $P_k(t, s)$, $Q_k(t, s)$, $k = 1, \ldots, r$, satisfy

(i) $Q_k(t, s)$ is continuous on $0 \leq s$, $t \leq 1$;

(ii) $\int_0^1 |P_k(t, s)| ds$ is bounded, and

(iii) $\lim_{|t_1-t_2| \to 0} \int_0^1 |P_k(t_1, s) - P_k(t_2, s)| ds = 0$ uniformly in t_1 and t_2 .

Important cases of $P_k(t, s)$ are

$$|t-s|^{\gamma} \ , \quad |v-s|^{\gamma} \ , \quad 0 > \gamma > -1 \ , \quad \ln|t-s| \ , \quad \ln|v-s| \ , \quad 0 \leq v \leq 1 \ . \qquad (2.5.3)$$

For illustrative purposes it is sufficient to consider the case

$$K(t, s) = P(t, s)Q(t, s) \ .$$

The application of product integration to the integral term in (2.5.1) yields the numerical scheme

$$Y_{ij} = G(t_{ij}) + \lambda \sum_{l=0}^{m-1} \sum_{k=1}^{n} W_{lk}(t_{ij})Q(t_{ij}, t_{lk})Y_{lk} \ ,$$
$$j = 1, \ldots, n \ ; \quad i = 0, \ldots, m-1 \ , \qquad (2.5.4)$$

where

$$W_{lk}(t) = \int_{t_l}^{t_{l+1}} P(t, s)L_k\left(\frac{s-t_l}{h}\right) ds$$

and Y_{ij} denotes the numerical approximation to $y(t_{ij})$. Atkinson has shown that if λ is not an eigenvalue of (2.5.1), then (2.5.4) has a unique solution for sufficiently small h and

$$\max_{\substack{j=1,\ldots,n \\ i=0,\ldots,m-1}} |y(t_{ij}) - Y_{ij}| = O(E)$$

where

$$E = \max_{\substack{j=1,\dots,n \\ i=0,\dots,m-1}} \left| \sum_{l=0}^{m-1} \sum_{k=1}^{n} W_{lk}(t_{ij}) Q(t_{ij}, t_{lk}) y(t_{ij}) - \int_0^1 K(t_{ij}, s) y(s) ds \right| . \quad (2.5.5)$$

We shall indicate how the results of section 2.4 can be extended to obtain accurate convergence estimates for (2.5.5). It will be assumed that $Q(t, s)y(s)$ is $p+1$ times continuously differentiable with respect to s.

The direct application of the results of section 2.4 yields the following estimates for E:

$$\text{(i)} \quad E = O\left(h^n \int_0^1 \omega(s) ds\right) + O\left(h^{n+1+\gamma}\right) \quad \text{for } P(t, s) = |v-s|^\gamma$$

and

$$\text{(ii)} \quad E = O\left(h^n \int_0^1 \omega(s) ds\right) + O\left(h^{n+1} \ell n h\right) \quad \text{for } P(t, s) = \ell n |v-s| .$$

However for the case $P(t, s) = |t-s|^\gamma$ or $\ell n|t-s|$, (2.4.4) and (2.4.9) are no longer valid, since the singularity can be at $s = t_{ij}$, $j = 1, \dots, n$; $i = 0, \dots, m-1$, and thus depends on h. The extension of section 2.4 to these cases is obtained in the following way. First, the integral terms in (2.4.2) are rewritten as

$$\int_0^1 g(s)z(s)\exp\left(\frac{2\pi iqs}{h}\right) ds = t_{ij} \int_0^1 g(t_{ij}s) z(t_{ij}s) \exp\left(\frac{2\pi iqt_{ij}s}{h}\right) ds$$

$$+ (1-t_{ij})\exp\left(\frac{2\pi iqt_{ij}}{h}\right) \int_0^1 g\left((1-t_{ij})s+t_{ij}\right) z\left((1-t_{ij})s+t_{ij}\right) \times$$

$$\exp\left(\frac{2\pi iq(1-t_{ij})s}{h}\right) ds , \quad j = 1, \dots, n ; \quad i = 0, \dots, m-1 . \quad (2.5.6)$$

For

$$g(t) = |t_{ij}-t|^{\gamma} \;, \quad 0 < t_{ij} < 1 \;, \qquad (2.5.7)$$

equation (2.5.6) becomes

$$\int_0^1 g(s)z(s)\exp\left(\frac{2\pi iqs}{h}\right) = t_{ij}^{1+\gamma} \int_0^1 (1-s)^{\gamma} z\left(t_{ij}s\right)\exp\left(\frac{2\pi iqs}{\tilde{h}}\right)ds$$

$$+ \left(1-t_{ij}\right)^{1+\gamma}\exp\left(\frac{2\pi iq}{\tilde{h}}\right)\int_0^1 s^{\gamma} z\left(\left(1-t_{ij}\right)s+t_{ij}\right)\exp\left(\frac{2\pi iqs}{\hat{h}}\right)ds \;,$$

$$0 < t_{ij} < 1 \;, \qquad (2.5.8)$$

where

$$\tilde{h} = h/t_{ij} \;; \quad \hat{h} = h/\left(1-t_{ij}\right) \;.$$

The singularities of the integrands on the right hand side of (2.5.8) are now end point singularities independent of t_{ij} and so asymptotic expansions in $\tilde{h}$ and $\hat{h}$ respectively for the corresponding integrals can be calculated in a similar way to Lyness and Ninham (1967) by Lighthill's procedure.

Define

$$G_1\left(t_{ij}, \tilde{\tau}\right) = \int_0^1 (1-s)^{\gamma} z\left(t_{ij}s\right)\exp(2\pi i\tilde{\tau}s)ds$$

and

$$G_2\left(t_{ij}, \hat{\tau}\right) = \int_0^1 s^{\gamma} z\left(\left(1-t_{ij}\right)s+t_{ij}\right)\exp(2\pi i\hat{\tau}s)ds$$

where

$$\tilde{\tau} = q/\tilde{h} \;; \quad \hat{\tau} = q/\hat{h} \;; \quad q = 0, 1, 2, \ldots \;.$$

Clearly $G_1\left(t_{ij}, \tau\right)$ and $G_2\left(t_{ij}, \tau\right)$ are the Fourier transforms of the generalized functions

$$\phi_1\left(t_{ij}, s\right) = (1-s)^{\gamma} z\left(t_{ij}s\right)H(s)H(1-s)$$

and

$$\phi_2(t_{ij}, s) = s^\gamma z((1-t_{ij})s+t_{ij})H(s)H(1-s)$$

where H is the Heaviside step function defined by

$$H(s) = \begin{cases} 1 & s \geq 0 \\ 0 & s < 0 \end{cases} .$$

For $k \geq 0$, let

$$\psi_1(t_{ij}, s) = (1-s)^\gamma z(t_{ij}s) ,$$

$$\psi_2(t_{ij}, s) = s^\gamma z((1-t_{ij})s+t_{ij}) ,$$

$$R_1(t_{ij}, s) = \sum_{q=0}^{k} \frac{1}{q!} \frac{\partial^q \psi_1}{\partial s^q} (t_{ij}, 0) s^q H(s) ,$$

$$R_2(t_{ij}, s) = \sum_{q=0}^{k} \frac{(-t_{ij})^q}{q!} z^{(q)}(t_{ij})(1-s)^{q+\gamma} H(1-s) ,$$

$$R_3(t_{ij}, s) = \sum_{q=0}^{k} \frac{(1-t_{ij})^q}{q!} z^{(q)}(t_{ij}) s^{q+\gamma} H(s)$$

and

$$R_4(t_{ij}, s) = \sum_{q=0}^{k} \frac{(-1)^q}{q!} \frac{\partial^q \psi_2}{\partial s^q} (t_{ij}, 1)(1-s)^q H(1-s) .$$

Then it follows from Lighthill's theorem that

$$G_1(t_{ij}, \tilde{\tau}) = \int_{-\infty}^{+\infty} \{R_1(t_{ij}, s)+R_2(t_{ij}, s)\}\exp(2\pi i\tilde{\tau}s)ds + O(|\tilde{\tau}|^{-k-1}) \tag{2.5.9}$$

and

$$G_2(t_{ij}, \hat{\tau}) = \int_{-\infty}^{+\infty} \{R_3(t_{ij}, s)+R_4(t_{ij}, s)\}\exp(2\pi i\hat{\tau}s)ds + O(|\hat{\tau}|^{-k-1}) . \tag{2.5.10}$$

The generalized Fourier transforms in (2.5.9) and (2.5.10) can be evaluated by standard integrals (see, for instance, Lyness and Ninham (1967, Eq. (6.14))). Substituting the resulting asymptotic expansions for $G_1(t_{ij}, q/\tilde{h})$ and $G_2(t_{ij}, q/\hat{h})$, $q = 0, 1, \ldots$, into (2.4.2) we obtain in the same way as Lyness and Ninham (1967)

$$h \sum_{l=0}^{m-1} |t_l+xh-t_{ij}|^{\gamma} z(t_l+xh) = \int_0^1 |s-t_{ij}|^{\gamma} z(s)ds$$

$$+ \sum_{q=0}^{k} \frac{h^{q+1}}{q!} \left\{ \tilde{\zeta}(-q, x) \frac{d^q}{dt^q} \left(|t-t_{ij}|^{\gamma} z(t) \right) \Big|_{t=0} \right.$$

$$\left. + (-1)^q \tilde{\zeta}(-q, 1-x) \frac{d^q}{dt^q} \left(|t-t_{ij}|^{\gamma} z(t) \right) \Big|_{t=1} \right\}$$

$$+ \sum_{q=0}^{k} \frac{h^{q+1+\gamma}}{q!} \left\{ \tilde{\zeta}(-\gamma-q, x-u_j) + (-1)^q \tilde{\zeta}(-\gamma-q, 1+u_j-x) \right\} z^{(q)}(t_{ij})$$

$$+ O\left(t_{ij}^{1+\gamma} \tilde{h}^{k+1} \right) + O\left((1-t_{ij})^{1+\gamma} \hat{h}^{k+1} \right) , \quad 0 < t_{ij} < 1 , \quad k \geq 0 .$$

Hence it is easy to verify that for $g(t)$ defined by (2.5.7), equation (2.4.4) remains valid if the order term is replaced by

$$O\left(h^{p+1} / t_{ij}^{p-n-\gamma} \right) + O\left(h^{p+1} / (1-t_{ij})^{p-n-\gamma} \right) .$$

In a similar way it can be shown that for $g(t) = \ln|t-t_{ij}|$ the order terms in (2.4.9) have to be replaced by

$$O\left(\ln(t_{ij}) h^{p+1} / t_{ij}^{p-n} \right) + O\left(\ln(1-t_{ij}) h^{p+1} / (1-t_{ij})^{p-n} \right) .$$

We thus obtain the estimates

(iii) $E = O\left(h^n \int_0^1 \omega(s)ds \right) + O\left(h^{n+1+\gamma} \right)$ for $P(t, s) = |t-s|^{\gamma}$,

and

(iv) $E = O\left(h^n \int_0^1 \omega(s)ds \right) + O\left(h^{n+1} \ln h \right)$ for $P(t, s) = \ln|t-s|$.

As an example consider the equation

$$y(t) = 1 + \int_0^{\pi} \sum_{k=1}^{4} P_k(t, s) Q_k(t, s) y(s) ds , \quad 0 \leq t \leq \pi ,$$

where

$$Q_1(t, s) = \left\{\frac{\sin\left(\frac{t-s}{2}\right)}{\left(\frac{t-s}{2}\right)}\right\} + \ell n\left\{\frac{\sin\left(\frac{t+s}{2}\right)}{(t+s)(2\pi-t-s)}\right\} ,$$

$$P_2(t, s) = \ell n|t-s| \ , \quad P_3(t, s) = \ell n(2\pi-t-s) \ ,$$

$$P_4(t, s) = \ell n(t+s) \ , \quad P_1 = Q_2 = Q_3 = Q_4 = 1 \ ,$$

which has the solution,

$$y(t) = 1/(1+\pi\ell n2) \ .$$

Atkinson has applied the product Simpson rule $\left\{u_1 = 0,\ u_2 = \tfrac{1}{2},\ u_3 = 1,\ \int_0^1 \omega(s)ds = 0\right\}$ to this equation. Although the rate of convergence was observed to be approximately $O(h^4)$, only $O(h^3)$ convergence was established. The above estimates yield $O(h^4 \ell nh)$ convergence.

The above analysis, can also be extended to Volterra integral equations with singular kernels. In particular, the important case

$$y(t) = f(t) + \int_0^t \frac{g(t,s,y(s))}{\sqrt{t-s}}\, ds \ , \quad t \geq 0$$

will be treated in some detail in chapter 4.

CHAPTER 3

SMOOTHNESS OF A SECOND KIND VOLTERRA EQUATION

3.1 Introduction

As noted in chapter 1, finite difference schemes for the second kind Volterra equation

$$y(t) = f(t) + \int_0^t K\big(t, s, y(s)\big)ds \ , \quad 0 \le t \le T \qquad (3.1.1)$$

are usually constructed by discretizing equation (3.1.1) at a number of grid points and then replacing the resultant integrals by quadrature formulae. For example, let

$$t_i = ih \ , \quad i = 0, \ldots, N \ ; \quad h = T/N \ .$$

Then, discretizing (3.1.1) at these points yields

$$y(t_i) = f(t_i) + \int_0^{t_i} K\big(t_i, s, y(s)\big)ds \ .$$

In order that the quadrature formulae used to approximate the integrals $\int_0^{t_i} K\big(t_i, s, y(s)\big)ds$ be efficient, it is necessary to know the smoothness of the kernel $K\big(t, s, y(s)\big)$, or in particular the smoothness of the solution $y(t)$.

Equation (3.1.1) with

$$K(t, s, y) = \frac{g(t,s,y)}{\sqrt{t-s}} \qquad (3.1.2)$$

where $g(t, s, y)$ is 'smooth', has received considerable attention (see Chambre (1959), Levinson (1960) and Keller and Olmstead (1971)). This equation arises for instance in the heat conduction problem

$$\left.\begin{aligned} &\frac{\partial \omega(x,t)}{\partial t} = \frac{\partial^2 \omega(x,t)}{\partial x^2}, \quad x > 0, \quad t > 0 \\ &\omega(x, 0) = 0 \\ &\frac{\partial \omega(0,t)}{\partial x} = G\big(\omega(0, t) - f(t)\big), \quad t > 0 \end{aligned}\right\} \qquad (3.1.3)$$

where $\Phi(y)$ and $f(t)$ are assummed to be continuous. In this case, it can be verified (see Mann and Wolf (1951)) that

$$F(t) = \omega(0, t) - f(t)$$

satisfies

$$F(t) = -f(t) - \frac{1}{\sqrt{\pi}} \int_0^t \frac{G\big(F(s)\big)}{\sqrt{t-s}} ds . \qquad (3.1.4)$$

The solution can then be found using the relation

$$\omega(x, t) = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{G\big(F(s)\big)}{\sqrt{t-s}} \exp\{-x^2/4(t-s)\} ds .$$

The smoothness of the solution of equation (3.1.1) with the kernel given by (3.1.2) has been investigated by Miller and Feldstein (1971) who show under suitable smoothness conditions on $g(t, s, y)$ and $f(t)$ that

$$y'(t) = O\big(t^{-\frac{1}{2}}\big) \quad \text{as} \quad t \to 0 .$$

In this chapter, we extend this result and show that if

$$f(t) = f_1(t) + \sqrt{t}\, f_2(t) ,$$

then

$$y(t) = u(t) + \sqrt{t}\, v(t)$$

where $u(t)$ and $v(t)$ are smooth under suitable smoothness conditions on $f_1(t)$, $f_2(t)$ and $g(t, s, y)$ and satisfy the system of equations

$$\left.\begin{aligned} u(t) &= f_1(t) + \int_0^t \frac{\sqrt{s}}{\sqrt{t-s}}\, g_1\big(t,\, s,\, u(s),\, v(s)\big)\, ds \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le t \le T \\ v(t) &= f_2(t) + \frac{1}{\sqrt{t}} \int_0^t \frac{g_2\big(t, s, u(s), v(s)\big)}{\sqrt{t-s}}\, ds \end{aligned}\right\} \qquad (3.1.5)$$

where

$$g_1(t,\, s,\, u,\, v) = \frac{g(t, s, u+\sqrt{s}v) - g(t, s, u-\sqrt{s}v)}{2\sqrt{s}}$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.1.6)$

$$g_2(t,\, s,\, u,\, v) = \frac{g(t, s, u+\sqrt{s}v) + g(t, s, u-\sqrt{s}v)}{2}$$

The system (3.1.6) provides an alternative for the numerical computation of $y(t)$ in a neighbourhood of the origin. This will be examined in chapter 4.

In section 2.2, we establish a number of basic lemmas. The equivalence of (1.1.1) with the kernel (3.1.2) and (3.1.5) and the smoothness of $u(t)$ and $v(t)$ are examined in section 3.3.

3.2 Preliminaries

In this section, we shall establish a number of lemmas which will be required in the subsequent analysis.

LEMMA 3.2.1. *Let* $f_1(t)$, $f_2(t)$ *and* $g(t, s, y)$ *be continuous with respect to* t *and* s *on* $0 \le s \le t \le T$ *and globally Lipschitz continuous with respect to* y . *Then,* (1.1.1) *with the kernel given by* (1.1.2) *has a unique continuous solution* $y(t)$ *on* $[0, T]$.

Proof. The result follows from the usual contraction mapping and translation argument on $C[0, T]$. #

LEMMA 3.2.2. *Let*

(i) $f_1(t)$ *and* $f_2(t)$ *be continuous on* $0 \le t \le T$,

(ii) $g_1(t, s, u, v)$ *and* $g_2(t, s, u, v)$ *be continuous with respect to* t, s, u *and* v *on* $0 \le s \le t \le T$, $-\infty < u, v < \infty$ *and*

(iii)

$$\left.\begin{aligned} |g_1(t, s, u, v_1) - g_1(t, s, u, v_2)| &\le L|v_1 - v_2| \\ |g_1(t, s, u_1, v) - g_1(t, s, u_2, v)| &\le \frac{L}{\sqrt{s}}|u_1 - u_2| \\ |g_2(t, s, u, v_1) - g_2(t, s, u, v_2)| &\le L\sqrt{s}\,|v_1 - v_2| \\ |g_2(t, s, u_1, v) - g_2(t, s, u_2, v)| &\le L|u_1 - u_2| \end{aligned}\right\} \quad (3.2.1)$$

for some constant L *and all* u, u_1, u_2, v, v_1 *and* v_2.

Then the system of equations

$$\left.\begin{aligned} u(t) &= f_1(t) + \frac{1}{t^r}\int_0^t \frac{s^{r+\frac{1}{2}}}{\sqrt{t-s}} g_1(t, s, u(s), v(s))\,ds, \\ &\qquad\qquad 0 \le t \le T, \quad r = 0, 1, \ldots \\ v(t) &= f_2(t) + \frac{1}{t^{r+\frac{1}{2}}}\int_0^t \frac{s^r}{\sqrt{t-s}} g_2(t, s, u(s), v(s))\,ds, \end{aligned}\right\} \quad (3.2.2)$$

has a unique solution $u(t), v(t) \in C[0, T]$.

Proof. Let K be a positive constant such that

$$\frac{2KL}{t^{r+\frac{1}{2}}}\int_0^t \frac{s^r}{\sqrt{t-s}}\,ds < 1, \quad 0 \le t \le T.$$

On defining

$$x(t) = Kv(t),$$

the result follows by the application of a contraction mapping and translation argument on $C[0, T]$ to the corresponding system of equations for $u(t)$ and $x(t)$. #

DEFINITION. The resolvent $R(t)$ associated with a given kernel function $a(t) \in L^1(0, T)$ is defined as the unique L^1 solution of the linear equation

$$R(t) = a(t) + \int_0^t a(t-s)R(s)ds \ , \quad 0 \le t \le T \ .$$

Remark. If the kernel function $a(t)$ is non-negative a.e., then the resolvent $R(t)$ is non-negative a.e. (see Miller and Feldstein (1971)).

LEMMA 3.2.3 (Tricomi (1957), Chapter 1). *If $X(t)$ is the solution of the linear equation*

$$X(t) = f(t) + \int_0^t a(t-s)X(s)ds \ , \quad 0 \le t \le T \ ,$$

then

$$X(t) = f(t) + \int_0^t R(t-s)f(s)ds \ , \quad 0 \le t \le T \ .$$

LEMMA 3.2.4. *Let $f_1(t), f_2(t) \in C[0, T]$ and $u(t), v(t)$ be the solution of the system*

$$\left.\begin{aligned} u(t) &= f_1(t) + \frac{L}{t^r}\int_0^t \frac{s^r}{\sqrt{t-s}}\left(u(s)+\sqrt{s}v(s)\right)ds \ , \\ &\qquad\qquad 0 \le t \le T \ , \quad r = 0, 1, \ldots \\ v(t) &= f_2(t) + \frac{L}{t^{r+\frac{1}{2}}}\int_0^t \frac{s^r}{\sqrt{t-s}}\left(u(s)+\sqrt{s}v(s)\right)ds \ , \end{aligned}\right\} \qquad (3.2.3)$$

then,

$$\left.\begin{aligned} u(t) &= f_1(t) + \frac{1}{2t^r}\int_0^t R(t-s)s^r\left(f_1(s)+\sqrt{s}f_2(s)\right)ds \ , \\ &\qquad\qquad 0 \le t \le T \ , \quad r = 0, 1, \ldots \\ v(t) &= f_2(t) + \frac{1}{2t^{r+\frac{1}{2}}}\int_0^t R(t-s)s^r\left(f_1(s)+\sqrt{s}f_2(s)\right)ds \ , \end{aligned}\right\} \qquad (3.2.4)$$

where $R(t)$ is the resolvent associated with the kernel function

$a(t) = 2Lt^{-\frac{1}{2}}$.

Proof. Clearly from (3.2.3),

$$u(t) - \sqrt{t}\, v(t) = f_1(t) - \sqrt{t}\, f_2(t) . \tag{3.2.5}$$

Let

$$x(t) = t^r\{u(t)+\sqrt{t}v(t)\} .$$

Then from (3.2.3), $x(t)$ satisfies

$$x(t) = t^r\{f_1(t)+\sqrt{t}f_2(t)\} + 2L\int_0^t \frac{x(s)}{\sqrt{t-s}}\,ds , \quad 0 \le t \le T$$

and applying lemma 3.2.3 we find that

$$x(t) = t^r\{f_1(t)+\sqrt{t}f_2(t)\} + \int_0^t R(t-s)s^r\{f_1(s)+\sqrt{s}f_2(s)\}ds . \tag{3.2.6}$$

The result for $0 < t \le T$ follows from (3.2.5) and (3.2.6). From lemma 3.2.2, $u(t), v(t) \in C[0, T]$ and the result follows. #

LEMMA 3.2.5. *Let $u_1(t)$, $u_2(t)$, $v_1(t)$ and $v_2(t)$ be the unique continuous solutions of the systems*

$$\left.\begin{aligned} u_1(t) &= f_1(t) + \frac{1}{t^r}\int_0^t \frac{s^{r+\frac{1}{2}}}{\sqrt{t-s}}\, g_1\{t, s, u_1(s), v_1(s)\}ds \\ v_1(t) &= f_2(t) + \frac{1}{t^{r+\frac{1}{2}}}\int_0^t \frac{s^r}{\sqrt{t-s}}\, g_2\{t, s, u_1(s), v_1(s)\}ds \end{aligned}\right\} \tag{3.2.7}$$

and

$$\left.\begin{aligned} u_2(t) &= q_1(t) + \frac{1}{t^r}\int_0^t \frac{s^{r+\frac{1}{2}}}{\sqrt{t-s}}\, k_1\{t, s, u_2(s), v_2(s)\}ds \\ v_2(t) &= q_2(t) + \frac{1}{t^{r+\frac{1}{2}}}\int_0^t \frac{s^r}{\sqrt{t-s}}\, k_2\{t, s, u_2(s), v_2(s)\}ds \end{aligned}\right\} \tag{3.2.8}$$

where $f_1(t)$, $f_2(t)$, $q_1(t)$, $q_2(t) \in C[0, T]$, $g_1(t, s, u, v)$, $q_2(t, s, u, v)$, $k_1(t, s, u, v)$ and $k_2(t, s, u, v)$ are continuous with respect to t, s, u and v on $0 \le s \le t \le T$, $-\infty < u$, $v < \infty$ and $g_1(t, s, u, v)$ and $g_2(t, s, u, v)$ satisfy (3.2.1).

Then

$$|u_1(t)-u_2(t)| \le |\tilde{f}_1(t)| + \frac{1}{2t^r}\int_0^t R(t-s)s^r\big(|\tilde{f}_1(s)|+\sqrt{s}|\tilde{f}_2(s)|\big)ds$$

and

$$|v_1(t)-v_2(t)| \le |\tilde{f}_2(t)| + \frac{1}{2t^{r+\frac{1}{2}}}\int_0^t R(t-s)s^r\big(|\tilde{f}_1(s)|+\sqrt{s}|\tilde{f}_2(s)|\big)ds$$

where

$$\tilde{f}_1(t) = f_1(t) - q_1(t) +$$

$$\frac{1}{t^r}\int_0^t \frac{s^{r+\frac{1}{2}}}{\sqrt{t-s}}\big(g_1(t,\ s,\ u_2(s),\ v_2(s))-k_1(t,\ s,\ u_2(s),\ v_2(s))\big)ds\ ,$$

$$\tilde{f}_2(t) = f_2(t) - q_2(t) +$$

$$\frac{1}{t^{r+\frac{1}{2}}}\int_0^t \frac{s^r}{\sqrt{t-s}}\big(g_2(t,\ s,\ u_2(s),\ v_2(s))-k_2(t,\ s,\ u_2(s),\ v_2(s))\big)ds\ ,$$

and $R(t)$ *is the resolvent associated with the kernel* $2Lt^{-\frac{1}{2}}$.

Proof. Define

$$z(t) = u_1(t) - u_2(t)\ ,$$

$$w(t) = v_1(t) - v_2(t)\ ,$$

$$C_1(t,\ s) = \begin{cases} \dfrac{g_1(t,s,u_1(s),v_1(s))-g_1(t,s,u_1(s),v_2(s))}{w(s)} & ;\ w(s) \ne 0\ , \\ 0 & ;\ w(s) = 0\ , \end{cases}$$

$$D_1(t,\ s) = \begin{cases} \dfrac{\sqrt{s}\big(g_1(t,s,u_1(s),v_2(s))-g_1(t,s,u_2(s),v_2(s))\big)}{z(s)} & ;\ z(s) \ne 0\ , \\ 0 & ;\ z(s) = 0\ , \end{cases}$$

$$C_2(t,\ s) = \begin{cases} \dfrac{g_2(t,s,u_1(s),v_1(s))-g_2(t,s,u_1(s),v_2(s))}{\sqrt{s}\ w(s)} & ;\ \sqrt{s}\ w(s) \ne 0\ , \\ 0 & ;\ \sqrt{s}\ w(s) = 0\ , \end{cases}$$

$$D_2(t,\, s) = \begin{cases} \dfrac{g_2\big(t,s,u_1(s),v_2(s)\big)-g_2\big(t,s,u_2(s),v_2(s)\big)}{z(s)} & ;\ z(s) \neq 0\ , \\ 0 & ;\ z(s) = 0\ . \end{cases}$$

Clearly from (3.2.1),

$$|C_1(t,\, s)|\, ,\ |C_2(t,\, s)|\, ,\ |D_1(t,\, s)|\, ,\ |D_2(t,\, s)| \le L\ . \tag{3.2.9}$$

Subtraction of (3.2.8) from (3.2.7) yields

$$z(t) = \tilde{f}_1(t) + \frac{1}{t^r}\int_0^t \frac{s^r}{\sqrt{t-s}}\big(\sqrt{s}C_1(t,\, s)w(s)+D_1(t,\, s)z(s)\big)ds\ ,$$

$$w(t) = \tilde{f}_2(t) + \frac{1}{t^{r+\frac{1}{2}}}\int_0^t \frac{s^r}{\sqrt{t-s}}\big(\sqrt{s}C_2(t,\, s)w(s)+D_2(t,\, s)z(s)\big)ds\ ,$$

and it follows from (3.2.9) that

$$|z(t)| \le |\tilde{f}_1(t)| + \frac{L}{t^r}\int_0^t \frac{s^r}{\sqrt{t-s}}\big(\sqrt{s}|w(s)|+|z(s)|\big)ds\ ,$$

$$|w(t)| \le |\tilde{f}_2(t)| + \frac{L}{t^{r+\frac{1}{2}}}\int_0^t \frac{s^r}{\sqrt{t-s}}\big(\sqrt{s}|w(s)|+|z(s)|\big)ds\ .$$

Let $x_1(t),\, x_2(t) \in C[0,\, T]$ be two non-negative functions such that

$$|z(t)| = |\tilde{f}_1(t)| - x_1(t) + \frac{L}{t^r}\int_0^t \frac{s^r}{\sqrt{t-s}}\big(\sqrt{s}|w(s)|+|z(s)|\big)ds\ ,$$

$$|w(t)| = |\tilde{f}_2(t)| - x_2(t) + \frac{L}{t^{r+\frac{1}{2}}}\int_0^t \frac{s^r}{\sqrt{t-s}}\big(\sqrt{s}|w(s)|+|z(s)|\big)ds\ .$$

Then from lemma 3.2.4,

$$|z(t)| = |\tilde{f}_1(t)| - x_1(t) + \frac{1}{2t^r}\int_0^t R(t-s)s^r\big\{|\tilde{f}_1(s)|-x_1(s)+\sqrt{s}\big(|f_2(s)|-x_2(s)\big)\big\}ds\ ,$$

$$|w(t)| = |\tilde{f}_2(t)| - x_2(t) + \frac{1}{2t^{r+\frac{1}{2}}}\int_0^t R(t-s)s^r\big\{|\tilde{f}_1(s)|-x_1(s)+\sqrt{s}\big(|\tilde{f}_2(s)|-x_2(s)\big)\big\}ds\ .$$

The result follows since $R(t)$, $x_1(t)$ and $x_2(t)$ are non-negative. #

LEMMA 3.2.6. *Let* $f(t), g(t) \in C^{2n}[-b, b]$, $b > 0$ *and define*

$$F(t) = \sqrt{t}\,\{f(\sqrt{t})-f(-\sqrt{t})\}$$

and

$$G(t) = g(\sqrt{t}) + g(-\sqrt{t}) .$$

Then, $F(t), G(t) \in C^n[0, b^2]$.

Proof. The result is clearly true for $n = 0$. Assume the result is true for $n = r$ and consider the case $n = r + 1$.

Clearly,

$$F'(t) = \frac{1}{2\sqrt{t}}\{f(\sqrt{t})-f(-\sqrt{t})\} + \tfrac{1}{2}\{f'(\sqrt{t})+f'(-\sqrt{t})\} .$$

From Taylor's theorem with integral remainder,

$$\frac{1}{2\sqrt{t}}\{f(\sqrt{t})-f(-\sqrt{t})\} = f'(0) + \frac{\sqrt{t}}{2}\left\{\int_0^1 (1-s)f''(s\sqrt{t})ds - \int_0^1 (1-s)f''(-s\sqrt{t})ds\right\} .$$

Hence,

$$F'(t) = f'(0) + \{\tilde{g}(\sqrt{t})+\tilde{g}(-\sqrt{t})\} + \sqrt{t}\{\tilde{f}(\sqrt{t})-\tilde{f}(-\sqrt{t})\}$$

where $\tilde{f}(t)$, $\tilde{g}(t)$ satisfy the hypothesis with $n = r$. It follows that $F(t) \in C^{r+1}[0, b^2]$. Similarly, $G(t) \in C^{r+1}[0, T]$ and the result follows by induction. #

COROLLARY 3.2.1. *Let* $g(t, s, y)$ *be* n *times continuously differentiable with respect to* t *and* s *on* $0 \le s \le t \le T$ *and* $2n$ *times continuously differentiable with respect to* y *for* $-\infty < y < \infty$. *Then* $sg_1(t, s, u, v)$ *and* $g_2(t, s, u, v)$ *where* $g_1(t, s, u, v)$ *and* $g_2(t, s, u, v)$ *are defined by* (3.1.6) *are* n *times continuously differentiable with respect to* t, s, u *and* v *on* $0 \le s \le t \le T$, $-\infty < u$, $v < \infty$.

3.3 Smoothness Results

We first consider the relation between $y(t)$ defined by (3.1.1) with the kernel given by (3.1.2) and $u(t)$ and $v(t)$ defined by (3.1.5).

THEOREM 3.3.1. *If*

(i) $f_1(t), f_2(t) \in C[0,T]$ *and*

(ii) $g(t, s, y)$ *is continuous with respect to* t *and* s *on* $0 \le s \le t \le T$ *and globally Lipschitz continuous with respect to* y ,

then (3.1.5) *and* (3.1.1), *with*

$$f(t) = f_1(t) + \sqrt{t}\, f_2(t)$$

and kernel given by (3.1.2), *have unique continuous solutions* $u(t)$, $v(t)$ *and* $y(t)$. *Furthermore,*

$$y(t) = u(t) + \sqrt{t}\, v(t) .$$

Proof. Existence, continuity and uniqueness of $u(t)$, $v(t)$ and $y(t)$ follows from lemmas 3.2.2 and 3.2.1. The result follows since $u(t) + \sqrt{t}\, v(t)$ satisfies (3.1.1) with the kernel given by (3.1.2). #

We shall now examine the smoothness of $u(t)$ and $v(t)$. Since inductive arguments will be used, it is convenient to consider the more general problem

$$\left.\begin{aligned} u(t) &= f_1(t) + \frac{1}{t^r}\int_0^t \frac{s^{r+\frac{1}{2}} g_1\big(t,s,u(s),v(s)\big)}{\sqrt{t-s}}\, ds \\ & \qquad\qquad\qquad\qquad r = 0,\, 1,\, \ldots \\ v(t) &= f_2(t) + \frac{1}{t^{r+\frac{1}{2}}}\int_0^t \frac{s^r g_2\big(t,s,u(s),v(s)\big)}{\sqrt{t-s}}\, ds \end{aligned}\right\} , \quad (3.3.1)$$

or equivalently,

$$\left.\begin{aligned} u(t) &= f_1(t) + \int_0^1 \frac{ts^{r+\frac{1}{2}} g_1\big(t, ts, u(ts), v(ts)\big)}{\sqrt{1-s}}\, ds \\ & \qquad\qquad\qquad\qquad r = 0, 1, \ldots \\ v(t) &= f_2(t) + \int_0^1 \frac{s^r g_2\big(t, ts, u(ts), v(ts)\big)}{\sqrt{1-s}}\, ds \end{aligned}\right\} . \quad (3.3.2)$$

Formally differentiating (3.3.2), we obtain

$$u'(t) = F_1(t) + \frac{1}{t^{r+1}} \int_0^t \frac{s^{r+\frac{3}{2}}}{\sqrt{t-s}} \left\{ \frac{\partial g_1}{\partial u}\big(t, s, u(s), v(s)\big) u'(s) + \frac{\partial g_1}{\partial v}\big(t, s, u(s), v(s)\big) v'(s) \right\} ds ,$$

$$v'(t) = F_2(t) + \frac{1}{t^{r+\frac{3}{2}}} \int_0^t \frac{s^{r+1}}{\sqrt{t-s}} \left\{ \frac{\partial g_2}{\partial u}\big(t, s, u(s), v(s)\big) u'(s) + \frac{\partial g_2}{\partial v}\big(t, s, u(s), v(s)\big) v'(s) \right\} ds ,$$

where

$$F_1(t) = f_1'(t) + \int_0^1 \frac{s^{r+\frac{1}{2}}}{\sqrt{1-s}} \left\{ g_1\big(t, ts, u(ts), v(ts)\big) + t \frac{\partial g_1}{\partial t}\big(t, ts, u(ts), v(ts)\big) + ts \frac{\partial g_1}{\partial s}\big(t, ts, u(ts), v(ts)\big) \right\} ds ,$$

and

$$F_2(t) = f_2'(t) + \int_0^1 \frac{s^r}{\sqrt{1-s}} \left\{ \frac{\partial g_2}{\partial t}\big(t, s, u(ts), v(ts)\big) + s \frac{\partial g_2}{\partial s}\big(t, ts, u(ts), v(ts)\big) \right\} ds .$$

We now consider the system

$$\left.\begin{aligned} U(t) &= F_1(t) + \frac{1}{t^{r+1}} \int_0^t \frac{s^{r+\frac{3}{2}}}{\sqrt{t-s}} \left\{ \frac{\partial g_1}{\partial u} \big(t,\ s,\ u(s),\ v(s)\big) U(s) \right. \\ &\qquad \left. + \frac{\partial g_1}{\partial v} \big(t,\ s,\ u(s),\ v(s)\big) V(s) \right\} ds\ , \\ V(t) &= F_2(t) + \frac{1}{t^{r+\frac{3}{2}}} \int_0^t \frac{s^{r+1}}{\sqrt{t-s}} \left\{ \frac{\partial g_2}{\partial u} \big(t,\ s,\ u(s),\ v(s)\big) U(s) \right. \\ &\qquad \left. + \frac{\partial g_2}{\partial v} \big(t,\ s,\ u(s),\ v(s)\big) V(s) \right\} ds\ . \end{aligned}\right\} \quad (3.3.3)$$

In the following lemma, we prove that under appropriate conditions on $f_1(t)$, $f_2(t)$, $g_1(t, s, u, v)$ and $g_2(t, s, u, v)$, (3.3.3) has a unique continuous solution on $[0, T]$ which coincides with $u'(t)$ and $v'(t)$ where $u(t)$ and $v(t)$ are the solution of (3.3.1).

LEMMA 3.3.1. *Let*

(i) $f_1(t)$, $f_2(t)$, $sg_1(t, s, u, v)$ *and* $g_2(t, s, u, v)$ *be continuously differentiable with respect to* t *and* t, s, u *and* v *respectively for* $0 \le s \le t \le T$ *and* $-\infty < u$, $v < \infty$, *and*

(ii)

$$\left|\frac{\partial g_1}{\partial u}\right| < \frac{L}{\sqrt{s}}, \quad \left|\frac{\partial g_1}{\partial v}\right| < L\ , \quad \left|\frac{\partial g_2}{\partial u}\right| < L\ , \quad \left|\frac{\partial g_2}{\partial v}\right| < \sqrt{s}\,L\ , \quad L = \text{const.}$$

Then the solution of (3.3.1) *is continuously differentiable and satisfies*

$$u'(t) = U(t)$$

and

$$v'(t) = V(t)$$

where $U(t)$ *and* $V(t)$ *is the unique continuous solution of* (3.3.3).

Proof. It follows from lemma 3.2.2 that (3.3.3) has a unique continuous solution. Using the same argument as in Miller and Feldstien (1971) (theorem 1), we may assume that $sg_1(t, s, u, v)$

and $g_2(t, s, u, v)$ have compact support.

Let δ be a real number in the range $0 < \delta < T/2$. For $0 < h \le \delta$ and $0 < t \le T-\overline{\delta}$, $\overline{\delta} = \delta/T$, define

$$z(t, h) = \frac{u\{t(1+h)\}-u(t)}{th}, \quad t > 0,$$

and

$$w(t, h) = \frac{v\{t(1+h)\}-v(t)}{th}, \quad t > 0.$$

Then

$$z(t, h) = \frac{f_1\{t(1+h)\}-f_1(t)}{th} + \frac{1}{th}\int_0^1 \frac{s^{r+\frac{1}{2}}t}{\sqrt{1-s}}\{(1+h)g_1\{t(1+h),$$

$$ts(1+h), u\{ts(1+h)\}, v\{ts(1+h)\}\}-g_1\{t, ts, u(ts), v(ts)\}\}ds.$$

By the mean value theorem,

$$(1+h)g_1\{t(1+h), ts(1+h), u\{ts(1+h)\}, v\{ts(1+h)\}\}$$
$$- g_1\{t, ts, u(ts), v(ts)\}$$

$$= hg_1\{t(1+h), ts(1+h), u\{ts(1+h)\}, v\{ts(1+h)\}\}$$

$$+ th\frac{\partial g_1}{\partial t}\{t+\Theta(th), ts(1+h), u\{ts(1+h)\}, v\{ts(1+h)\}\}$$

$$+ tsh\frac{\partial g_1}{\partial s}\{t, ts+\eta(tsh), u\{ts(1+h)\}, v\{ts(1+h)\}\}$$

$$+ \{u\{ts(1+h)\}-u(ts)\}\frac{\partial g_1}{\partial u}\{t, ts, \hat{u}(ts), v\{ts(1+h)\}\}$$

$$+ \{v\{ts(1+h)\}-v(ts)\}\frac{\partial g_1}{\partial v}\{t, ts, u(ts), \hat{v}(ts)\},$$

where $0 < \Theta(th) < th$, $0 < \eta(tsh) < tsh$, $\hat{u}(ts)$ lies between $u(ts)$ and $u\{ts(1+h)\}$ and $\hat{v}(ts)$ lies between $v(ts)$ and $v\{ts(1+h)\}$.

Hence

$$z(t, h) = F_1(t, h) + \frac{1}{t^{r+1}} \int_0^t \frac{s^{r+\frac{3}{2}}}{\sqrt{t-s}} \left\{ \frac{\partial g_1}{\partial u} \big(t, s, \hat{u}(s), v(s+sh)\big) z(s, h) \right.$$

$$\left. + \frac{\partial g_1}{\partial v} \big(t, s, u(s), \hat{v}(s)\big) w(s, h) \right\} ds , \quad (3.3.4)$$

where

$$F_1(t, h) = \frac{f_1\big(t(1+h)\big) - f_1(t)}{th} + \int_0^1 \frac{s^{r+\frac{1}{2}}}{\sqrt{1-s}} \, g_1\big(t(1+h), ts(1+h),$$

$$u\big(ts(1+h)\big), v\big(ts(1+h)\big)\big) + t \frac{\partial g_1}{\partial t} \big(t+\Theta(th), ts(1+h),$$

$$u\big(ts(1+h)\big), v\big(ts(1+h)\big)\big) + ts \frac{\partial g_1}{\partial s} \big(t, ts+\eta(tsh),$$

$$u\big(ts(1+h)\big), v\big(ts(1+h)\big)\big) \Big\} ds .$$

Similarly

$$w(t, h) = F_2(t, h) + \frac{1}{t^{r+\frac{3}{2}}} \int_0^t \frac{s^{r+1}}{\sqrt{t-s}} \left\{ \frac{\partial g_2}{\partial u} \big(t, s, \tilde{u}(s), v(s+h)\big) z(s, h) \right.$$

$$\left. + \frac{\partial g_2}{\partial v} \big(t, s, u(s), \tilde{v}(s)\big) w(s, h) \right\} ds , \quad (3.3.5)$$

where

$$F_2(t, h) = \frac{f_2\big(t(1+h)\big) - f_2(t)}{th} + \int_0^t \frac{s^{r+1}}{\sqrt{t-s}} \left\{ \frac{\partial g_2}{\partial t} \big(t+\alpha(th), \right.$$

$$ts(1+h), u\big(ts(1+h)\big), v\big(ts(1+h)\big)\big) + \frac{\partial g_2}{\partial s} \big(t, ts+\beta(tsh),$$

$$\left. u\big(ts(1+h)\big), v\big(ts(1+h)\big)\big) \right\} ds ,$$

$0 < \alpha(th) < th$, $0 < \beta(tsh) < tsh$, $\tilde{u}(s)$ lies between $u(s)$ and $u(s+sh)$, and $\tilde{v}(s)$ lies between $v(s)$ and $v(s+sh)$. On defining

$$z(0, h) = \lim_{t \to 0} F_1(t, h)$$

$$= F_1(0, h)$$

and

$$w(0, h) = F_2(0, h) + \frac{(-\frac{1}{2})!(r+1)!}{(r+\frac{3}{2})!} \frac{\partial g_2}{\partial u} \{0, 0, u(0), v(0)\} z(0, h) ,$$

it follows from the application of lemma 3.2.4 to the system (3.3.4) - (3.3.5) that $z(t, h)$ and $w(t, h)$ are continuous on $0 \le t \le T-\overline{\delta}$. Define

$$Q_1(t, h) = F_1(t, h) - F_1(t) + \frac{1}{t^{r+1}} \int_0^t \frac{s^{r+\frac{3}{2}}}{\sqrt{t-s}}$$

$$\left\{\left[\frac{\partial g_1}{\partial u} \{t, s, \hat{u}(s), v(s+sh)\} - \frac{\partial g_1}{\partial u} \{t, s, u(s), v(s)\}\right] U(s)\right.$$

$$\left. + \left[\frac{\partial g_1}{\partial v} \{t, s, u(s), \tilde{v}(s)\} - \frac{\partial g_1}{\partial v} \{t, s, u(s), v(s)\}\right] V(s)\right\} ds ,$$

and

$$Q_2(t, h) = F_2(t, h) - F_2(t) + \frac{1}{t^{r+\frac{3}{2}}} \int_0^t \frac{s^{r+1}}{\sqrt{t-s}}$$

$$\left\{\left[\frac{\partial g_2}{\partial u} \{t, s, \tilde{u}(s), v(s+sh)\} - \frac{\partial g_2}{\partial u} \{t, s, u(s), v(s)\}\right] U(s)\right.$$

$$\left. + \left[\frac{\partial g_2}{\partial v} \{t, s, u(s), \tilde{v}(s)\} - \frac{\partial g_2}{\partial v} \{t, s, u(s), v(s)\}\right] V(s)\right\} ds .$$

Let ε be a positive real number. Since $sg_1(t, s, u, v)$ and $g_2(t, s, u, v)$ have compact support, there exists an h_0 such that for $0 < h < h_0$,

$$\left.\begin{array}{l} |Q_1(t, h)| \le \varepsilon \\ \text{and} \\ |Q_2(t, h)| \le \varepsilon \end{array}\right\} , \quad 0 \le t \le T-\overline{\delta} . \qquad (3.3.6)$$

Hence, the application of lemma 3.2.5 to the systems (3.3.3) and (3.3.4)-(3.3.5) and the subsequent use of (3.3.6) yields

$$|z(t,h)-U(t)| \le |Q_1(t,h)| + \frac{1}{2t^{r+1}}\int_0^t R(t-s)s^{r+1}\{|Q_1(s,h)| + \sqrt{s}|Q_2(s,h)|\}ds$$

$$\le \varepsilon\left(1+\frac{1}{2t^{r+1}}\int_0^t R(t-s)s^{r+1}(1+\sqrt{s})ds\right)$$

and

$$|w(t,h)-V(t)| \le |Q_2(t,h)| + \frac{1}{2t^{r+\frac{3}{2}}}\int_0^t R(t-s)s^{r+1}\{|Q_1(s,h)| + \sqrt{s}|Q_2(s,h)|\}ds$$

$$\le \varepsilon\left(1+\frac{1}{2t^{r+\frac{3}{2}}}\int_0^t R(t-s)s^{r+1}(1+\sqrt{s})ds\right)$$

for $0 \le t \le T-\overline{\delta}$, where $R(t)$ is the resolvent associated with the kernel $2Lt^{-\frac{1}{2}}$. From Miller and Feldstien (1971) (lemmas 2 and 4),

$$R(t) \le \frac{C}{\sqrt{t}} \text{ a.e. on } 0 \le t \le T\ , \quad C = \text{const.}$$

and hence

$$\left.\begin{aligned} |z(t,h)-U(t)| &\le D\varepsilon \\ |w(t,h)-V(t)| &\le D\varepsilon \end{aligned}\right\}\ , \quad 0 \le t \le T-\delta\ , \quad D = \text{const}\ .$$

Thus,

$$z(t,h) \to U(t)$$

and

$$w(t,h) \to V(t)$$

uniformly on $0 \le t \le T-\delta$. Since δ is arbitrary, $U(t)$ and $V(t)$ are the continuous right derivatives of $u(t)$ and $v(t)$ respectively on $0 \le t < T$. In addition, from the uniform convergence to $U(t)$ and $V(t)$, respectively, it follows that for any interval $I = \{t, \overline{\delta} \le t \le T-\overline{\delta}\}$ the sets $\{z(\cdot, h) : 0 < h < \delta\}$ and

$\{w(\cdot, h) : 0 < h < \delta\}$ are equicontinuous and hence

$$\lim_{h\to 0} z(t, h) = \lim_{h\to 0} z(t-h, h) = U(t) ,$$

$$\lim_{h\to 0} w(t, h) = \lim_{h\to 0} w(t-h, h) = V(t)$$

uniformly on I. This implies that $U(t)$ and $V(t)$ are the left hand derivatives of $u(t)$ and $v(t)$ respectively on I. A simple argument shows that $U(t)$ and $V(t)$ are the left derivatives of $u(t)$ and $v(t)$ respectively at $t = T$. This completes the proof. #

We can now prove the principal result of this chapter.

THEOREM 3.3.2. *Let*

(i) $f_1(t), f_2(t) \in C^n[0, T]$,

(ii) $g(t, s, y)$ *be* n *times continuously differentiable with respect to* t *and* s *on* $0 \le s \le t \le T$ *and* $2n$ *times continuously differentiable with respect to* y *on* $-\infty < y < \infty$, *and*

(iii) $g(t, s, y)$ *be globally Lipschitz continuous with respect to* y .

Then, the solution $u(t), v(t)$ *of* (3.1.5) *with* $g_1(t, s, u, v)$ *and* $g_2(t, s, u, v)$ *defined by* (3.1.6) *is* n *times continuously differentiable. Furthermore,* $u^{(m)}(t)$ *and* $v^{(m)}(t)$, $m = 0, \ldots, n$ *is the solution of the equations obtained by formally differentiating* (3.1.5) m *times.*

Proof. From corollary 3.2.1, it follows that $sg_1(t, s, u, v)$ and $g_2(t, s, u, v)$ are n times continuously differentiable with respect to t, s, u and v on $0 \le s \le t \le T$, $-\infty < u , v < \infty$. The result follows by induction and lemma 3.3.1. #

The above result can be extended, when $f_1(t), f_2(t)$ and

$g(t, s, y)$ are analytic.

THEOREM 3.3.3. *Let*

(i) $f_1(t)$ *and* $f_2(t)$ *be real analytic in a neighbourhood of* $0 \le t \le T$,

(ii) $g(t, s, y)$ *be real analytic in an open set containing all real ordered triples* (t, s, y) , $0 \le s \le t \le T$, $-\infty \le y \le \infty$ *and*

(iii) equation (3.1.5) *have a unique continuous solution* $u(t)$, $v(t)$ *in an open set containing the interval* $0 \le t \le T$.

Then $u(t)$ *and* $v(t)$ *are analytic in an open set containing* $0 \le t \le T$.

Proof. It follows from *(ii)* and (3.1.6) that $g_1(t, s, u, v)$ and $g_2(t, s, u, v)$ are real analytic in an open set containing the real ordered quadruples (t, s, u, v) , $0 \le s \le t \le T, -\infty < u, v < \infty$.

For $\varepsilon > 0$ define

$D(\varepsilon) = \{z : -\varepsilon \le \text{Re}z \le T+\varepsilon \ , \ |\text{Im}z| \le \varepsilon\}$,

$P = \text{Max}\{|f_1(z)|, |f_2(z)| ; z \in D(\varepsilon)\}$,

$E(\varepsilon) = \{(z, zs, y) : z \in D(\varepsilon), 0 \le s \le 1, |y| \le P+1\}$,

$F(\varepsilon) = \{(z, zs, p, q) : z \in D(\varepsilon), 0 \le s \le 1, |p|, |q| \le P+1\}$,

$$M = \text{Max}\left\{|g(z, zs, y)|, \left|\frac{\partial g}{\partial y}(z, zs, y)\right| : z \in D(\varepsilon), \quad 0 \le s \le 1, |y| \le 2P+2\right\}$$

and

$G(\varepsilon) = \{z : -\varepsilon \le \text{Re}z \le \varepsilon, |\text{Im}z| \le \varepsilon/2\}$.

Choose ε such that $f_1(z)$ and $f_2(z)$ are analytic on $D(\varepsilon)$ and $g_1(z, w, u, v)$ and $g_2(z, w, u, v)$ are analytic on $F(\varepsilon)$. Let

$H(\varepsilon)$ denote the set of all functions ϕ, real analytic in the interior of $G(\varepsilon)$, continuous on $G(\varepsilon)$ and satisfying $|\phi(z)| \le P+1$ for $z \in G(\varepsilon)$. Given ϕ, ψ in $H(\varepsilon)$, define

$$\left.\begin{aligned} R_1(\phi, \psi)(z) &= f_1(z) + \int_0^1 z \frac{\sqrt{s}}{\sqrt{1-s}} g_1\left(z, sz, \phi(sz), \frac{\psi(sz)}{K}\right) ds \\ R_2(\phi, \psi)(z) &= Kf_2(z) + K \int_0^1 \frac{1}{\sqrt{1-s}} g_2\left(z, sz, \phi(sz), \frac{\psi(sz)}{K}\right) ds \end{aligned}\right\} \quad (3.3.7)$$

where $K = \sqrt{\varepsilon}$. As in lemma 3.2.2, K is introduced to obtain a contraction mapping. From (3.1.6) it follows that

$$|R_1(\phi, \psi)(z)| \le P + (\sqrt{\varepsilon}M)/2 ,$$

$$|R_2(\phi, \psi)(z)| \le \sqrt{\varepsilon}\ (P+M/2) .$$

Hence, if $\varepsilon < \varepsilon_0$, $\varepsilon_0 = \min\{1, 1/M^2\}$, then (3.3.7) is a mapping from $H \times H$ into itself. It can easily be verified that (3.3.7) is a contraction mapping and consequently $u(z)$ and $v(z)$ are real analytic in the interior of $G(\varepsilon)$.

This result can be extended in the following way. If $u(t)$ and $v(t)$ are real analytic in a neighbourhood of $0 \le t \le \tau+\delta$, $\tau, \delta > 0$, then (3.1.5) is rewritten as

$$u(\tau+t) = \tilde{f}_1(t) + \int_0^t \frac{\sqrt{s+\tau}}{\sqrt{t-s}} g_1\{\tau+t, \tau+s, u(\tau+s), v(\tau+s)\} ds$$

$$, \quad 0 \le t \le T-\tau ,$$

$$v(\tau+t) = \tilde{f}_2(t) + \frac{1}{\sqrt{\tau+t}} \int_0^t \frac{1}{\sqrt{t-s}} g_2\{\tau+t, \tau+s, u(\tau+s), v(\tau+s)\} ds$$

where

$$\tilde{f}_1(t) = f_1(\tau+t) + \int_0^\tau \frac{\sqrt{s}}{\sqrt{\tau+t-s}} g_1\{\tau+t, s, u(s), v(s)\} ds$$

and

$$\tilde{f}_2(t) = f_2(\tau+t) + \frac{1}{\sqrt{\tau+t}} \int_0^\tau \frac{1}{\sqrt{\tau+t-s}} g_2\big(\tau+t,\ s,\ u(s),\ v(s)\big) ds \ .$$

Clearly $\tilde{f}_1(t)$ and $\tilde{f}_2(t)$ are real analytic in the interior of $R(\tilde{\varepsilon}) = \{z : 0 \le \text{Re}z \le \tilde{\varepsilon},\ |\text{Im}z| \le \tilde{\varepsilon}/2\}$ for some $\tilde{\varepsilon} > 0$. From *(iii)*, (3.3.7) can be replaced by

$$\left.\begin{aligned} \tilde{R}_1(\phi,\ \psi)(z) &= u(\tau+z) + z^{\frac{1}{2}} \int_0^1 \frac{\sqrt{sz+\tau}}{\sqrt{1-s}} \Big\{ g_1\big(\tau+z,\ \tau+sz,\\ &\qquad \phi(\tau+sz),\ \psi(\tau+sz)\big) - g_1\big(\tau+z,\ \tau+sz,\ u(\tau+sz),\ v(\tau+sz)\big)\Big\} ds \\ \tilde{R}_2(\phi,\ \psi)(z) &= v(\tau+z) + \frac{z^{\frac{1}{2}}}{\sqrt{\tau+z}} \int_0^1 \frac{1}{\sqrt{1-s}} \Big\{ g_2\big(\tau+z,\ \tau+sz,\\ &\qquad \phi(\tau+sz),\ \psi(\tau+sz)\big) - g_2\big(\tau+z,\ \tau+sz,\ u(\tau+sz),\ v(\tau+sz)\big)\Big\} ds \ . \end{aligned}\right\} \quad (3.3.8)$$

As previously, $u(\tau+t)$ and $v(\tau+t)$ can be shown to be analytic in the interior of $R(\tilde{\varepsilon})$ for $\tilde{\varepsilon}$ sufficiently small. If δ is chosen such that $\tilde{\varepsilon} - 2\delta > 0$, this process can be continued with $\tilde{\tau} = \tau + \tilde{\varepsilon} - \delta$.

Since $u(z)$ and $v(z)$ are bounded for $z \in D(\varepsilon)$, it follows from the form of (3.3.8) that the interval $[\varepsilon,\ T]$ can be covered by a finite number of applications of the above process. #

CHAPTER 4

NUMERICAL SCHEMES FOR SECOND KIND VOLTERRA EQUATIONS WITH SINGULAR KERNELS

4.1 Introduction

In chapter 3, we have shown that, under appropriate smoothness assumptions on $f_1(t)$, $f_2(t)$ and $g(t, s, y)$, the solution of the second kind Volterra equation

$$y(t) = f_1(t) + \sqrt{t}\, f_2(t) + \int_0^t \frac{g\big(t, s, y(s)\big)}{\sqrt{t-s}}\, ds \ , \quad 0 \le t \le T \qquad (4.1.1)$$

has the form

$$y(t) = u(t) + \sqrt{t}\, v(t)$$

where $u(t)$ and $v(t)$ are the components of the solution of the system (3.1.5). Since, from theorem 3.3.2, $u(t)$ and $v(t)$ are smooth, derivatives with respect to s of $g\big(t, s, y(s)\big)$ may become unbounded in a neighbourhood of the origin. This indicates that quadrature rules for integrals of the form

$$I\big(t_i\big) = \int_0^{t_i} \frac{g\big(t_i, s, y(s)\big)}{\sqrt{t_i - s}}\, ds \ , \quad i = 1, \ldots, N \ ,$$

and hence the associated numerical schemes for (4.1.1) will often yield low order convergence, if they are based on polynomial or piecewise polynomial approximations to $g\big(t_i, s, y(s)\big)$, $i = 1, \ldots, m$.

However, from theorem 3.3.2, $u(t)$ and $v(t)$ and hence $g_1\big(t, s, u(s), v(s)\big)$ and $g_2\big(t, s, u(s), v(s)\big)$ are smooth and this suggests that the system (3.1.5) be used to obtain numerical approximations to $y(t)$. This approach however has two drawbacks:

(i) The computation involved in solving a system of equations is necessarily greater than solving a single scalar equation.

(ii) The system (3.1.5) is unstable in the sense that while the solution $y(t)$ may not grow rapidly, the individual components $u(t)$ and $v(t)$ may grow exponentially. To illustrate this, we consider

$$y(t) = 1 - \int_0^t \frac{y(s)}{\sqrt{t-s}}\, ds \ ,$$

which, by a Laplace transform argument, has the solution

$$y(t) = \exp(\pi t) erfc(\sqrt{\pi t})$$

where

$$erfc(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty \exp\left(-s^2\right) ds \ .$$

The solution of (3.1.5) is given by

$$u(t) = \frac{\exp(\pi t)}{2} \{erfc(\sqrt{\pi t}) + erfc(-\sqrt{\pi t})\}$$

and

$$v(t) = \frac{\exp(\pi t)}{2\sqrt{t}} \{erfc(\sqrt{\pi t}) - erfc(-\sqrt{\pi t})\}$$

which grow exponentially.

In this chapter, we suggest numerical schemes for (4.1.1) which are based on (3.1.5) only in a neighbourhood of the origin and on (4.1.1) for the rest of the interval. In this way, it is possible to take advantage of the smoothness properties of $u(t)$ and $v(t)$ without letting the instability of the system (3.1.5) become dominant and also minimises the extra computation involved in solving the system of equations. As an example, schemes based on the product integration analogues of Simpson's rule are treated in detail. An alternative scheme which can be applied when the derivatives of

$g(t, s, y)$ are known explicitly is also suggested.

In section 4.2, some notation is introduced and a number of lemmas required for the subsequent analysis are established. The numerical schemes are given in section 4.3 and their convergence is examined in section 4.4. Asymptotic results and a numerical example for the 'Simpson' schemes are given in sections 4.5 and 4.6, respectively. The alternative scheme is outlined in section 4.7.

4.2 Preliminaries

Since in this chapter we only consider the product integration analogues of Simpson's rule $\{u_1 = 0,\ u_2 = \tfrac{1}{2}$ and $u_3 = 1\}$, it is convenient to simplify slightly the notation used in chapter 2. In particular, we define

$$t_i = ih\ ,\quad i = 0, \ldots, m\ ;\quad h = T/m\ ,$$

$$\omega(t) = \prod_{k=0}^{2} (t-k)\ ,$$

$$l_k(t) = \omega(t)H(t)H(2-t)/\{\omega'(k)(t-k)\}\ ,\quad k = 0, 1, 2\ ,$$

$$W(t) = \prod_{k=0}^{3} (t-k)$$

and

$$L_k(t) = W(t)H(t)H(3-t)/\{W'(k)(t-k)\}\ ,\quad k = 0, \ldots, 3$$

where $H(t)$ is the Heaviside step function defined by

$$H(t) = \begin{cases} 1\ , & t \geq 0 \\ 0\ , & t < 0\ . \end{cases}$$

The following lemmas will be required in sections 4.4 and 4.5.

LEMMA 4.2.1. *There exists a constant* K *such that*

$$\sum_{l=1}^{i-1} \frac{1}{l^{\frac{3}{2}}(i-l)^{\frac{1}{2}}} \le \frac{K}{(i-1)^{\frac{1}{2}}} , \quad i \ge 2 .$$

Proof.

$$\sum_{l=1}^{i-1} \frac{1}{l^{\frac{3}{2}}(i-l)^{\frac{1}{2}}} \le \frac{1}{(i-1)^{\frac{1}{2}}} + \frac{1}{(i-1)^{\frac{3}{2}}} + \int_1^{i-2} \frac{1}{s^{\frac{3}{2}}(i-1-s)^{\frac{1}{2}}} ds$$

$$\le \frac{3}{(i-1)^{\frac{1}{2}}} . \qquad \#$$

LEMMA 4.2.2. *If* $f(t)$ *is continuously differentiable,* $0 \le u \le 1$, $0 < s < 1$ *and* $u \ne s$, *then*

$$h \sum_{l=0}^{i-1} \frac{f\left(t_l+sh\right)\sqrt{t_l+sh}}{\sqrt{t_i+uh-t_l-sh}} = \int_0^{t_i} \frac{f(s)\sqrt{s}}{\sqrt{t_i-s}} ds + h^{\frac{1}{2}} \sqrt{t_i}\, f\left(t_i\right)\alpha(u, s) + O\left(\frac{h}{\sqrt{t_i}}\right) ,$$

$$i = 1, \ldots, m \quad (4.2.1)$$

and

$$h \sum_{l=0}^{i-1} \frac{f\left(t_l+sh\right)}{\sqrt{t_i+uh-t_l-sh}} = \int_0^{t_i} \frac{f(s)}{\sqrt{t_i-s}} ds + h^{\frac{1}{2}} f\left(t_i\right)\alpha(u, s) + O\left(\frac{h}{\sqrt{t_i}}\right) ,$$

$$i = 1, \ldots, m \quad (4.2.2)$$

where

$$\alpha(u, s) = \begin{cases} \tilde{\zeta}(\tfrac{1}{2}, 1-s+u) , & s > u \\ \tilde{\zeta}(\tfrac{1}{2}, u-s) - \dfrac{1}{\sqrt{u-s}} , & u > s \end{cases}$$

and $\tilde{\zeta}(-s, a)$ *is the generalised periodic zeta function.*

Proof. Since

$$\int_0^t f(s)g(s)ds = t \int_0^1 f(ts)g(ts)ds ,$$

the lemma follows immediately from the generalized Euler Maclaurin sum formula for integrands with algebraic singularities given in Lyness and Ninham (1967) (Equation (2.4.3)) for $u = 0$ or 1 .

For $s > u$, the result for (4.2.1) follows in the same way by the generalised Euler Maclaurin sum formula, since it is easily verified that

$$h\sum_{l=0}^{i-1} \frac{f(t_l+sh)\sqrt{t_l+sh}}{\sqrt{t_i+uh-t_l-sh}} = h\sum_{l=0}^{i-1} \frac{f(t_l+(s-u)h)\sqrt{t_l+(s-u)h}}{\sqrt{t_i-t_l-(s-u)h}} + O(h) .$$

If $u > s$, then

$$h\sum_{l=0}^{i-1} \frac{f(t_l+sh)\sqrt{t_l+sh}}{\sqrt{t_i+uh-t_l-sh}} = h\sum_{l=0}^{i} \frac{f(t_l+(1-u+s)h)\sqrt{t_l+(1-u+s)h}}{\sqrt{t_{i+1}-t_l-(1-u+s)h}}$$

$$- h^{\frac{1}{2}} \frac{f(t_i)\sqrt{t_i}}{\sqrt{u-s}} + O(h) .$$

Again, the use of the Euler Maclaurin sum formula yields the required result.

The result expressed by equation (4.2.2) can be established in a similar way. #

If the hypothesis on $f(t)$ in lemma 4.2.2 is weakened, we can obtain:

LEMMA 4.2.3. *If* $f(t)$ *is continuous,*

$$f(t+h) - f(t) = O\left(h^{\frac{1}{2}}\right) , \quad 0 \le t \le T ,$$

$0 \le u \le 1$, $0 < s < 1$ *and* $u \ne s$, *then*

$$h\sum_{l=0}^{i-1} \frac{f(t_l+sh)\sqrt{t_l+sh}}{\sqrt{t_i+uh-t_l-sh}} = \int_0^{t_i} \frac{f(s)\sqrt{s}}{\sqrt{t_i-s}}\,ds + O\left(h^{\frac{1}{2}}\right) , \quad i = 1, \ldots, m$$

and

$$h\sum_{l=0}^{i-1} \frac{f(t_l+sh)}{\sqrt{t_i+uh-t_l-sh}} = \int_0^{t_i} \frac{f(s)}{\sqrt{t_i-s}}\,ds + O\left(h^{\frac{1}{2}}\right) , \quad i = 1, \ldots, m .$$

Proof. Clearly

$$\left|\sum_{l=0}^{i-1} h \frac{f(t_l+sh)\sqrt{t_l+sh}}{\sqrt{t_i+uh-t_l-sh}} - \int_0^{t_i} \frac{f(s)\sqrt{s}}{\sqrt{t_i-s}}\,ds\right|$$

$$\le \sum_{l=0}^{i-1} h\left|\frac{f(t_l+sh)\sqrt{t_l+sh}}{\sqrt{t_i+uh-t_l-sh}} - \int_0^1 \frac{f(t_l+xh)\sqrt{t_l+xh}}{\sqrt{t_i-t_l-xh}}\,dx\right|$$

$$= \sum_{l=0}^{i-1} h\left|f(t_l+sh)\sqrt{t_l+sh}\right|\left|\frac{1}{\sqrt{t_i+uh-t_l-sh}} - \int_0^1 \frac{dx}{\sqrt{t_i-t_l-xh}}\right| + O\left(h^{\frac{1}{2}}\right)$$

$$\le C\sum_{l=0}^{i-1} h\left|\frac{1}{\sqrt{t_i+uh-t_l-sh}} - \int_0^1 \frac{dx}{\sqrt{t_i-t_l-xh}}\right| + O\left(h^{\frac{1}{2}}\right)$$

where

$$C = \max_{0\le t\le T} \left|\sqrt{t}\, f(t)\right| .$$

The term which is summed can be shown to be $O\left(h^{\frac{1}{2}}\right)$ by the generalised Euler Maclaurin sum formula. The second part of the lemma follows in a similar way. #

4.3 Numerical Schemes

First, we shall describe the finite difference schemes proposed by Linz (1969) for equations of the form

$$y(t) = f(t) + \int_0^t g\{t, s, y(s)\}p(t, s)ds \ , \quad 0 \le t \le T \quad (4.3.1)$$

where $g\{t, s, y(s)\}$ is 'smooth' with respect to s and $p(t, s)$ may contain integrable singularities or singularities in its derivatives. We note that (4.3.1) may be a system of equations.

Discretising (4.3.1) we obtain

$$y(t_i) = f(t_i) + \int_0^{t_i} g\{t_i, s, y(s)\}p(t_i, s)ds \ ,$$

$$i = 0, \ldots, m . \quad (4.3.2)$$

As mentioned previously, replacing the integral terms in (4.3.2) by quadrature rules will yield a numerical scheme for (4.3.1). Since $p(t_i, s)$ may be singular, suitable quadrature rules are those based on product integration. In particular let

$$\tilde{g}(t_i, s, y(s)) = \sum_{l=0}^{i} C_{il}(s) g(t_i, t_l, y(t_l)) , \quad i = q, \ldots, m$$

be the approximation to $g(t_i, s, y(s))$ on $0 \le s \le t_i$. Product integration then yields the quadrature rule

$$\int_0^{t_i} g(t_i, s, y(s)) p(t_i, s) ds \simeq \sum_{l=0}^{i} B_{il} g(t_i, t_l, y(t_l)) ,$$

$$i = q, \ldots, m ,$$

where

$$B_{il} = \int_0^{t_i} C_{il}(s) p(t_i, s) ds .$$

This leads to the numerical scheme

$$y_i = f(t_i) + \sum_{l=0}^{i} B_{il} g(t_i, t_l, y_l) , \quad i = q, \ldots, m , \qquad (4.3.3)$$

where y_i denotes the numerical approximation to $y(t_i)$. The scheme (4.3.3) requires q starting values $y_0, \ldots, y_{q-1}$ which must be determined independently. If $B_{ii} \ne 0$ the scheme is implicit and explicit otherwise. Implicit schemes require the solution of a nonlinear equation at each step.

The concept of a repetition factor as introduced by Linz (1967) for second kind Volterra equations with 'smooth' kernels (i.e. $p(t, s) = 1$) can be generalized in the following way to include the scheme (4.3.3).

DEFINITION. The scheme (4.3.3) has a *repetition factor* p if

p is the smallest integer such that

$$C_{il}(t) = C_{i+p,l}(t) \ , \quad 0 \le t \le t_i \ ,$$

for $l = 0, \ldots, i-r$, where r is a fixed integer independent of i .

In the remainder of this chapter, we shall consider schemes which are obtained in the following way.

If i is even, $g\left(t_i, s, y(s)\right)$, $i \ge 2$, is approximated by a piecewise quadratic interpolating to $g\left(t_i, s, y(s)\right)$ at the points t_{2l}, t_{2l+1} and t_{2l+2} . If i is odd this can be extended as follows:

(a) $g\left(t_i, s, y(s)\right)$ is approximated by a piecewise quadratic on $\left[0, t_{i-3}\right]$ and a cubic interpolating to $g\left(t_i, s, y(s)\right)$ at $t_{i-3}, t_{i-2}, t_{i-1}$ and t_i on $\left[t_{i-3}, t_i\right]$, or

(b) On $[0, 3h]$, $g\left(t_i, s, y(s)\right)$ is approximated by a cubic interpolating at $0, h, 2h$ and $3h$ and on $\left[3h, t_i\right]$ by a piecewise quadratic interpolating at t_{2l+1}, t_{2l+2} and t_{2l+3} . That is,

(a)

$$C_{2i,0}(s) = l_0\left(\frac{s}{h}\right) ,$$

$$C_{2i,2l}(s) = l_2\left(\frac{s-t_{2l-2}}{h}\right) + l_0\left(\frac{s-t_{2l}}{h}\right) , \quad l = 1, \ldots, i-1 ,$$

$$C_{2i,2l+1}(s) = l_1\left(\frac{s-t_{2l}}{h}\right) , \quad l = 0, \ldots, i-1 ,$$

$$C_{2i,2i}(s) = l_2\left(\frac{s-t_{2i-2}}{h}\right) ,$$

$$C_{2i+1,l}(s) = C_{2i,l}(s) , \quad l = 0, \ldots, 2i-3 ,$$

$$C_{2i+1,2i-2}(s) = l_2\left(\frac{s-t_{2i-4}}{h}\right) + L_0\left(\frac{s-t_{2i-2}}{h}\right) ,$$

$$C_{2i+1,2i-2+r}(s) = L_r\left(\frac{s-t_{2i-2}}{h}\right) , \quad r = 1, 2, 3 ,$$

(b)

$$\hat{C}_{2i,l}(s) = C_{2i,l}(s) , \quad l = 1, \ldots, i-2 ,$$

$$\hat{C}_{2i+1,r}(s) = L_r\left(\frac{s}{h}\right) , \quad r = 0, 1, 2 ,$$

$$\hat{C}_{2i+1,3}(s) = L_3\left(\frac{s}{h}\right) + l_0\left(\frac{s-3h}{h}\right) ,$$

$$\hat{C}_{2i+1,2l}(s) = l_1\left(\frac{s-t_{2l-1}}{h}\right) , \quad l = 2, \ldots, i ,$$

$$\hat{C}_{2i+1,2l+1}(s) = l_2\left(\frac{s-t_{2l-1}}{h}\right) + l_0\left(\frac{s-t_{2l+1}}{h}\right) , \quad l = 2, \ldots, i-1 ,$$

$$\hat{C}_{2i+1,2i+1} = l_2\left(\frac{s-t_{2i-1}}{h}\right) .$$

The methods (a) and (b) have a repetition factor of one and two respectively and can be thought of as generalizations of the schemes Simpson #1 and #2 for Volterra integral equations of the second kind with 'smooth' kernels investigated by Linz (1967) and Noble (1964). Clearly the above can be generalized to piecewise polynomial interpolation of higher order. However the present schemes contain most of the features of the more general class and most of the analysis given in the sequel generalizes easily. A similar analysis can also be used to obtain corresponding results for the block by block methods suggested by Linz (1969).

We now apply the above to obtain numerical schemes for the solution of (4.1.1). As suggested in section 4.1, approximations to $u(t)$ and $v(t)$ are obtained on $0 \le t \le a$, $a > 0$, via (3.1.5). For $a \le t \le T$, (4.1.1) can be rewritten as

$$y(t+a) = F(t+a) + \int_0^t \frac{g\big(t+a,s+a,y(s+a)\big)}{\sqrt{t-s}}\,ds\ , \quad 0 \le t \le T-a \qquad (4.3.4)$$

where

$$F(t+a) = f_1(t+a) + \sqrt{t+a}\, f_2(t+a) + \int_0^a \frac{g\big(t+a,s,y(s)\big)}{\sqrt{t+a-s}}\,ds\ .$$

The term $F(t+a)$ is approximated by applying product integration to each of the terms on the right hand side of

$$\int_0^a \frac{g\big(t+a,s,y(s)\big)}{\sqrt{t+a-s}}\,ds = \int_0^a \frac{\sqrt{s}g_1\big(t+a,s,u(s),v(s)\big)}{\sqrt{t+a-s}}\,ds$$

$$+ \int_0^a \frac{g_2\big(t+a,s,u(s),v(s)\big)}{\sqrt{t+a-s}}\,ds\ ,$$

where $g_1(t,\ s,\ u,\ v)$ and $g_2(t,\ s,\ u,\ v)$ are given by (3.1.6).

For example, if $a = t_{2n}$, then

$$\int_0^a \frac{\sqrt{s}g_1\big(t_i+a,s,u(s),v(s)\big)}{\sqrt{t_i+a-s}}\,ds \simeq \sum_{l=0}^{2n} Q_{2n+i,l}\,g_1\big(t_i+a,\ t_l,\ u_l,\ v_l\big)$$

and

$$\int_0^a \frac{g_2\big(t_i+a,s,u(s),v(s)\big)}{\sqrt{t_i+a-s}}\,ds \simeq \sum_{l=0}^{2n} P_{2n+i,l}\,g_2\big(t_i+a,\ t_l,\ u_l,\ v_l\big)$$

where

$$Q_{2n+i,l} = \int_0^a \frac{\sqrt{s}C_{2n+i,l}(s)}{\sqrt{t_i+a-s}}\,ds\ , \quad l = 0,\ \dots,\ 2n-1\ ,$$

$$Q_{2n+i,2n} = \int_0^a \frac{\sqrt{s}l_2\left(\frac{s-t_{2n-2}}{h}\right)}{\sqrt{t_i+a-s}}\,ds$$

$$P_{2n+i,l} = \int_0^a \frac{C_{2n+i,l}(s)}{\sqrt{t_i+a-s}}\,ds\ , \quad l = 0,\ \dots,\ 2n-1\ ,$$

$$P_{2n+i,2n} = \int_0^a \frac{l_2\left(\frac{s-t_{2n-2}}{h}\right)}{\sqrt{t_i+a-s}}\, ds$$

and u_l, v_l , $l = 0, \ldots, 2n$ are approximations to $u(t_l)$ and $v(t_l)$ respectively. Using this approximation for $F(t+a)$, a scheme applied to (4.3.4) is then used for the calculation of $y(t)$ on $a \le t \le T$.

For the analysis of these composite methods it is sufficient to investigate the schemes for (3.1.5) and (4.3.4) separately. Although (4.3.4) has a 'smooth' solution, it is of the form (4.1.1). Hence, for notational convenience, we shall consider the schemes applied to (4.1.1) rather than (4.3.4) and assume that the solution is 'smooth'.

The finite difference methods for (4.1.1) corresponding to (a) and (b) are

$$y_i = f_1(t_i) + \sqrt{t_i}\, f_2(t_i) + \sum_{l=0}^{i} W_{il}\, g(t_i, t_l, y_l) ,$$

$$i = 2, \ldots, \qquad (4.3.5a)$$

and

$$\hat{y}_i = f_1(t_i) + \sqrt{t_i}\, f_2(t_i) + \sum_{l=0}^{i} \hat{W}_{il}\, g(t_i, t_l, y_l) ,$$

$$i = 2, \ldots, m \qquad (4.3.5b)$$

where

$$W_{il} = \int_0^{t_i} \frac{C_{il}(s)}{\sqrt{t_i-s}}\, ds ,$$

$$\hat{W}_{il} = \int_0^{t_i} \frac{\hat{C}_{il}(s)}{\sqrt{t_i-s}}\, ds .$$

and y_i, $\hat{y}_i$ are the approximations to $y(t_i)$. Similarly, the schemes for (3.1.5) are

$$\left.\begin{aligned} u_i &= f_1(t_i) + \sum_{l=0}^{i} X_{il}\, g_1(t_i, t_l, u_l, v_l) \\ v_i &= f_2(t_i) + \frac{1}{\sqrt{t_i}} \sum_{l=0}^{i} W_{il}\, g_2(t_i, t_l, u_l, v_l) \end{aligned}\right\} i = 2, \ldots, m \quad (4.3.6a)$$

and

$$\left.\begin{aligned} \hat{u}_i &= f_1(t_i) + \sum_{l=0}^{i} \hat{X}_{il}\, g_1(t_i, t_l, \hat{u}_l, \hat{v}_l) \\ \hat{v}_i &= f_2(t_i) + \frac{1}{\sqrt{t_i}} \sum_{l=0}^{i} \hat{W}_{il}\, g_2(t_i, t_l, \hat{u}_l, \hat{v}_l) \end{aligned}\right\} i = 2, \ldots, m \quad (4.3.6b)$$

where

$$X_{il} = \int_0^{t_i} \frac{\sqrt{s}\, C_{il}(s)}{\sqrt{t_i - s}}\, ds ,$$

and

$$\hat{X}_{il} = \int_0^{t_i} \frac{\sqrt{s}\, \hat{C}_{il}(s)}{\sqrt{t_i - s}}\, ds .$$

Note that the schemes (4.3.5 a, b) and (4.3.6 a, b) require starting values at $t = 0$ and $t = h$.

4.4 Convergence Results

In the sequel, we shall assume that $g(t, s, y(s))$, $g_1(t, s, u(s), v(s))$ and $g_2(t, s, u(s), v(s))$ are sufficiently smooth. Subtracting (4.1.1) from (4.3.5a), (3.1.5) from (4.3.6a) and defining

$$\alpha_i = x_i - x(t_i) ,$$

$$\beta_i = u_i - u(t_i) ,$$

$$\gamma_i = (v_i - v(t_i))\sqrt{t_i} ,$$

we obtain

$$\alpha_i = \sum_{l=0}^{i} W_{il}\big(g(t_i, t_l, y_l) - g(t_i, t_l, y(t_l))\big) + P_i$$

$$i = 2, \ldots, I , \quad (4.4.1)$$

and

$$\left.\begin{aligned} \beta_i &= \sum_{l=0}^{i} X_{il}\big(g_1(t_i, t_l, u_l, v_l) - g_1(t_i, t_l, u(t_l), v(t_l))\big) + Q_i \\ & \qquad\qquad\qquad\qquad\qquad\qquad i = 2, \ldots, m , \\ \gamma_i &= \sum_{l=0}^{i} W_{il}\big(g_2(t_i, t_l, u_l, v_l) - g_2(t_i, t_l, u(t_l), v(t_l))\big) + R_i \end{aligned}\right\} (4.4.2)$$

where

$$P_i = \sum_{l=0}^{i} W_{il} g(t_i, t_l, y(t_l)) - \int_0^{t_i} \frac{g(t_i, s, y(s))}{\sqrt{t_i - s}}\, ds ,$$

$$Q_i = \sum_{l=0}^{i} X_{il} g_1(t_i, t_l, u(t_l), v(t_l))$$

$$- \int_0^{t_i} \frac{\sqrt{s}}{\sqrt{t_i - s}}\, g_1(t_i, s, u(s), v(s))\, ds ,$$

$$R_i = \sum_{l=0}^{i} W_{il} g_2(t_i, t_l, u(t_l), v(t_l)) - \int_0^{t_i} \frac{g_2(t_i, s, u(s), v(s))}{\sqrt{t_i - s}}\, ds .$$

Corresponding equations for (4.3.5b) and (4.3.6b) are obtained by replacing $\alpha_i, W_{i,l}, P_i, \ldots$ by $\hat{\alpha}_i, \hat{W}_{i,l}, \hat{P}_i, \ldots$.

The following lemma examines the asymptotic behaviour of P_i, Q_i, R_i and $\hat{P}_i, \hat{Q}_i, \hat{R}_i$.

LEMMA 4.4.1. *There exist continuously differentiable functions* $\phi_r(t)$, $\psi_r(t)$ *and* $\Theta_r(t)$, $r = 0, 1$ *such that for* $i \geq 1$,

$$P_{2i+r} = h^{\frac{7}{2}}\phi_r(t_{2i+r}) + O\left(\frac{h^4}{\sqrt{t_{2i+r}}}\right) ,$$

$$Q_{2i+r} = h^{\frac{7}{2}}\sqrt{t_{2i+r}}\,\psi_r(t_{2i+r}) + O\left(\frac{h^4}{\sqrt{t_{2i+r}}}\right) ,$$

$$R_{2i+r} = h^{\frac{7}{2}}\Theta_r(t_{2i+r}) + O\left(\frac{h^4}{\sqrt{t_{2i+r}}}\right) .$$

Corresponding relations for $\hat{P}_{2i+r}$, $\hat{Q}_{2i+r}$ *and* $\hat{R}_{2i+r}$, $r = 0, 1$ *with* $\phi_r(t)$, $\psi_r(t)$ *and* $\Theta_r(t)$ *replaced by* $\hat{\phi}_r(t)$, $\hat{\psi}_r(t)$ *and* $\hat{\Theta}_r(t)$ *respectively are also valid.*

Proof. For simplicity, denote $g(t, s, y(s))$ by $g(t, s)$. By definition,

$$\sum_{l=0}^{2i} W_{2i,l}\,g(t_{2i}, t_l) = \sum_{l=0}^{i-1}\sum_{k=0}^{2} g(t_{2i}, t_{2l+k})\int_{t_{2l}}^{t_{2l+2}} \frac{l_k\left(\frac{s-t_{2l}}{h}\right)}{\sqrt{t_{2i}-s}}\,ds .$$

Applying lemma 2.2.1 with h replaced by $2h$, $u_1 = 0$, $u_2 = \frac{1}{2}$ and $u_3 = 1$, we find that

$$\sum_{l=0}^{i-1}\sum_{k=0}^{2} g(t_{2i}, t_{2l+k})\int_{t_{2l}}^{t_{2l+2}} \frac{l_k\left(\frac{s-t_{2l}}{h}\right)}{\sqrt{t_{2i}-s}}\,ds - \int_0^{t_{2i}} \frac{g(t_{2i},s)}{\sqrt{t_{2i}-s}}\,ds$$

$$= \text{const}\; h^3\int_0^1\left\{\omega(2s)2h\sum_{l=0}^{i-1}\frac{g^{(3)}(t_{2i},t_{2l}+2sh)}{\sqrt{t_{2i}-t_{2l}-2sh}}\right\}ds + O(h^4) ,$$

where

$$g^{(3)}(t, s) = \frac{\partial^3 g(t,s)}{\partial s^3}$$

The result for P_{2i} now follows by the application of lemma 4.2.2, with $u = 0$, to the right hand side of the above equation, since

$$\int_0^1 \omega(2s) = 0 \ .$$

To obtain the required asymptotic estimate for P_{2i+1} , we first note that Lagrangian interpolation yields the estimate

$$\sum_{k=0}^{3} g\left(t_{2i+1}, t_{2i+k-2}\right) \int_{t_{2i-2}}^{t_{2i+1}} \frac{L_k\left(\frac{s-t_{2i-2}}{h}\right)}{\sqrt{t_{2i+1}-s}}\, ds$$

$$= \int_{t_{2i-2}}^{t_{2i+1}} \frac{g\left(t_{2i+1}, s\right)}{\sqrt{t_{2i+1}-s}}\, ds + O\left(h^4\right) \ .$$

Hence, we find that

$$\sum_{l=0}^{2i+1} W_{2i+1,l}\, g\left(t_{2i+1}, t_l\right) = \sum_{l=0}^{i-2} \sum_{k=0}^{2} g\left(t_{2i+1}, t_{2l+k}\right) \int_{t_{2l}}^{t_{2l+2}} \frac{l_k\left(\frac{s-t_{2l}}{h}\right)}{\sqrt{t_{2i+1}-s}}\, ds$$

$$+ \int_{t_{2i-2}}^{t_{2i+1}} \frac{g\left(t_{2i+1}, s\right)}{\sqrt{t_{2i+1}-s}}\, ds + O\left(h^4\right) \ .$$

Proceeding in the same way as for P_{2i} , but using lemma 4.2.2, with $u_1 = 1$, yields the result.

The other estimates follow in the same way. #

Remark. Note that the proof of lemma 4.4.1 is very similar to the method used in sections 2.2 and 2.3 to establish Euler Maclaurin sum formulae. The only difficulty is that which arises when i is odd.

We shall now prove some convergence results for the schemes (4.3.5 a, b).

THEOREM 4.4.1. *Let* $g(t, s, y)$ *be globally Lipschitz continuous with respect to* y *and*

$$x_r - x(t_r) = \delta_r \,, \quad \hat{x}_r - x(t_r) = \hat{\delta}_r \,, \quad \delta_r = O(\delta) \,, \quad \hat{\delta}_r = O(\hat{\delta}) \,,$$

$$r = 0, 1 \,.$$

Then there exist constants $K, \hat{K}$ *such that*

$$|\alpha_i| \le K\left(h^{\frac{7}{2}} + \frac{h\delta}{\sqrt{t_i}}\right) \,, \quad i = 2, \ldots, m \,,$$

and

$$|\hat{\alpha}_i| \le \hat{K}\left(h^{\frac{7}{2}} + \frac{h\hat{\delta}}{\sqrt{t_i}}\right) \,, \quad i = 2, \ldots, m \,.$$

Proof. Since the arguments for the scheme (4.3.5a) can be extended to (4.3.5b), only the scheme (4.3.5a) will be considered. Let $\tilde{y}_i$, $i = 2, \ldots, m$, be the solution of (4.3.5a) obtained with exact starting values, i.e.

$$\tilde{y}_i = W_{i,0}K(t_i, 0, y(0)) + W_{i,1}K(t_i, h, y(h))$$
$$+ \sum_{l=2}^{i} W_{i,l}K(t_i, t_l, \tilde{y}_l) \,, \quad i = 2, \ldots, m \,. \tag{4.4.3}$$

Clearly

$$\alpha_i = (y_i - \tilde{y}_i) + (\tilde{y}_i - y(t_i)) \,. \tag{4.4.4}$$

We now estimate the terms in brackets separately. Subtraction of (4.3.5a) from (4.4.3) yields

$$\tilde{y}_i - y_i = W_{i,0}\big(g(t_i, 0, y(0)) - g(t_i, 0, y_0)\big)$$
$$+ W_{i,1}\big(g(t_i, h, y(h)) - g(t_i, h, y_1)\big)$$
$$+ \sum_{l=2}^{i} W_{i,l}\big(g(t_i, t_l, \tilde{y}_l) - g(t_i, t_l, y_l)\big) \,, \quad i = 2, \ldots, m \,. \tag{4.4.5}$$

Let C be a constant such that

$$|g(t_i, rh, y(rh)) - g(t_i, rh, y_r)| \le \sqrt{C}\,\delta, \quad r = 0, 1; \; i = 2, \ldots, m \,,$$

and

$$|W_{i,r}| \le \frac{\sqrt{C}h}{\sqrt{t_i}}\ , \quad r = 0, 1\ ; \quad i = 2, \ldots, m\ . \qquad (4.4.6)$$

Then, defining

$$\tilde{\alpha}_i = |(\tilde{y}_i - y_i)\sqrt{t_i}|\ , \quad i = 2, \ldots, m\ ,$$

multiplying (4.4.5) by $\sqrt{t_i}$, taking absolute values and applying the Lipschitz inequality for $g(t, s, y)$, we find that

$$\tilde{\alpha}_i \le \sqrt{t_i}\, L \sum_{l=2}^{i} \frac{|W_{i,l}|}{\sqrt{t_l}} \tilde{\alpha}_l + Ch\delta\ , \quad i = 2, \ldots, m\ ,$$

where L is the Lipschitz constant of $g(t, s, y)$ with respect to y . Since $\sum_{l=2}^{i} |W_{i,l}|/\sqrt{t_l}$, $i = 2, \ldots, m$, are uniformly bounded, it follows in the same way as in Linz (1969) that

$$\tilde{\alpha}_i \le Kh\delta\ , \quad i = 2, \ldots, m\ .$$

To estimate $|\tilde{y}_i - y(t_i)|$, we examine (4.4.1) with $y_r = y(t_r)$, $r = 0, 1$. Using the Lipschitz continuity of $g(t, s, y)$ and lemma 4.4.1, we obtain

$$|\tilde{y}_i - y(t_i)| \le L \sum_{l=2}^{i} |W_{i,l}||\tilde{y}_l - y(t_l)| + O(h^{\frac{7}{2}})\ , \quad i = 2, \ldots, m\ .$$

Hence, using the arguments in Linz (1969), we obtain

$$|\tilde{y}_i - y(t_i)| \le Kh^{\frac{7}{2}}\ , \quad i = 2, \ldots, m\ .$$

The result follows. #

A similar convergence result for the schemes (4.3.6 a, b) can also be obtained.

THEOREM 4.4.2. *Let $g(t, s, y)$ and $\frac{\partial}{\partial y} g(t, s, y)$ be globally Lipschitz continuous with respect to y ,*

$$u_r - u(t_r) = \delta_r\ , \quad v_r - v(t_r) = \eta_r\ ; \quad \delta_r, \eta_r = O(\sigma)\ , \quad r = 0, 1$$

and

$$\hat{u}_r - u(t_r) = \hat{\delta}_r \,, \quad \hat{v}_r - v(t_r) = \hat{\eta}_r \;; \quad \hat{\delta}_r, \hat{\eta}_r = O(\hat{\sigma}) \,, \quad r = 0, 1 \,.$$

Then there exist constants K, $\hat{K}$ *such that*

$$|\beta_i|, \; |\gamma_i| \le K\left(h^{\frac{7}{2}} + \frac{\sigma h}{\sqrt{t_i}}\right), \quad i = 2, \ldots, I ,$$

and

$$|\hat{\beta}_i|, \; |\hat{\gamma}_i| \le K\left(h^{\frac{7}{2}} + \frac{\sigma h}{\sqrt{t_i}}\right), \quad i = 2, \ldots, I .$$

Proof. Again, we shall only examine the scheme (4.3.6 a) as the analysis extends simply to (4.3.6 b). As in theorem 4.4.1, we write

$$\beta_i = (u_i - \tilde{u}_i) + (\tilde{u}_i - u(t_i))$$

and

$$\gamma_i = \left\{(v_i - \tilde{v}_i) + (\tilde{v}_i - v(t_i))\right\}\sqrt{t_i}$$

where $\tilde{u}_i$ and $\tilde{v}_i$ are the components of the solution of (4.3.6 a) obtained with exact starting values.

Let C be a constant such that

$$|g_p(t_i, rh, u(rh), v(rh)) - g_p(t_i, rh, u_r, v_r)| \le \sqrt{C}\,\delta ,$$

$$p = 1, 2 \;; \; r = 0, 1 \;; \; i = 2, \ldots, m ,$$

$$|X_{ir}| \le \frac{\sqrt{C}h^{\frac{3}{2}}}{\sqrt{t_i}}, \quad r = 0, 1 \;; \; i = 2, \ldots, m$$

and

$$|W_{ir}| \le \frac{\sqrt{C}h}{\sqrt{t_i}}, \quad r = 0, 1 \;; \; i = 2, \ldots, m .$$

Then, defining

$$\tilde{\beta}_i = |(\tilde{u}_i - u_i)\sqrt{t_i}| , \quad i = 2, \ldots, m$$

and

$$\tilde{\gamma} = |(\tilde{v}_i - v_i)t_i| , \quad i = 2, \ldots, m$$

we obtain in a similar way to theorem 4.4.1,

$$\tilde{\beta}_i \le \sqrt{t_i}\, L \sum_{l=2}^{m} \frac{|X_{il}|}{t_l} \left(\tilde{\beta}_l + \tilde{\gamma}_l\right) + Ch\sigma \ , \quad i = 2, \ldots, m \ ,$$

$$\tilde{\gamma}_i \le \sqrt{t_i}\, L \sum_{l=2}^{m} \frac{|W_{il}|}{\sqrt{t_l}} \left(\tilde{\beta}_l + \tilde{\gamma}_l\right) + Ch\sigma \ , \quad i = 2, \ldots, m \ .$$

Hence,

$$\left(\tilde{\beta}_i + \tilde{\gamma}_i\right) \le \sqrt{t_i}\, L \sum_{l=2}^{m} \left\{\frac{|X_{il}|}{t_l} + \frac{|W_{il}|}{\sqrt{t_l}}\right\} \left(\tilde{\beta}_l + \tilde{\gamma}_l\right) + 2Ch\sigma \ , \quad i = 2, \ldots, m \ .$$

Since $\sum_{l=2}^{m} \left\{\frac{|X_{il}|}{t_l} + \frac{|W_{il}|}{\sqrt{t_l}}\right\}$ is uniformly bounded, it follows in the same way as in Linz (1969) that

$$\tilde{\beta}_i + \tilde{\gamma}_i \le Kh\sigma \ , \quad i = 2, \ldots, m \ .$$

The estimates for $|\tilde{u}_i - u(t_i)|$ and $|\tilde{v}_i - v(t_i)|$ follow in a similar way. #

4.5 Asymptotic Expansions and Numerical Stability

Suppose that in (4.3.1) a perturbation $\delta f(t)$ in $f(t)$ causes a change $\delta y(t)$ in $y(t)$, i.e.

$$y(t) + \delta y(t) = f(t) + \delta f(t) + \int_0^1 p(t, s) g\left(t, s, y(s) + \delta y(s)\right) ds \ .$$

Then, neglecting the $O\left(\delta y^2\right)$ term, we obtain

$$\delta y(t) = \delta f(t) + \int_0^t p(t, s) \frac{\partial g}{\partial y} \left(t, s, y(s)\right) \delta y(s) ds \ . \qquad (4.5.1)$$

The linearized equation (4.5.1) characterizes the sensitivity of (4.3.1) with respect to a perturbation in $f(t)$. This sensitivity must be reflected in the growth of the discretization error and the propagation of rounding errors in finite difference schemes for (4.3.1). Hence, the best we can expect for a finite difference method is that the leading term in the asymptotic expansion of the error grows in a similar way to the solution of (4.5.1). In this case, the scheme will be called *numerically stable*.

We shall first derive an asymptotic error estimate for the scheme (4.3.5 a). To study the effect of rounding errors, we shall consider the propagation of the starting errors $y_r - y(t_r) = \delta_r$, $\delta_r = O(\delta)$, $r = 0, 1$. Without loss of generality, we take the starting errors to be non-zero.

To simplify the notation, we introduce

$$G(t, s) = \frac{\partial g}{\partial y}\left(t, s, y(s)\right)$$

and

$$z_r(t) = \frac{g\left(t, 0, y_r\right) - g\left(t, 0, y(t_r)\right)}{\delta_r}, \quad r = 0, 1 . \tag{4.5.2}$$

The following theorem examines the asymptotic behaviour of the numerical solution.

THEOREM 4.5.1. *Let* $\zeta_r(t)$, $r = 0, 1$, *be the solution of the system*

$$\zeta_r(t) = \phi_r(t) + \int_0^t \frac{G(t,s)}{\sqrt{t-s}}\left(\tfrac{1}{3}\zeta_0(s) + \tfrac{2}{3}\zeta_1(s)\right)ds , \quad r = 0, 1 \tag{4.5.3}$$

and let $x(t)$ *be the solution of*

$$x(t) = \delta_0 F_0(t) + \delta_1 F_1(t) + \int_0^t \frac{G(t,s)}{\sqrt{t-s}} x(s)ds$$

where

$$F_r(t) = a_r \int_0^t \frac{G(t,s)z_r(s)}{\sqrt{t-s}} ds$$

and $a_0 = \frac{1}{3}$, $a_1 = \frac{4}{3}$. *Then*

$$\alpha_{2i+r} = h^{\frac{7}{2}}\zeta_r(t_{2i+r}) + \delta_0 z_0(t_{2i+r})W_{2i+r,0} + \delta_1 z_1(t_{2i+r})W_{2i+r,1}$$

$$+ hx(t_{2i+r}) + O\left(\frac{h^4}{\sqrt{t_{2i+r}}}\right) + O\left(\frac{h^{\frac{3}{2}}\delta}{\sqrt{t_{2i+r}}}\right) , \quad r = 0, 1 ; \quad i = 1, \ldots, m/2 .$$

Proof. Applying Taylor's theorem to (4.4.1) and using theorem 4.4.1 and lemma 4.4.1, we obtain

$$\alpha_{2i+r} = \sum_{l=2}^{2i+r} W_{2i+r,l}G(t_{2i+r}, t_l)\alpha_l + h^{\frac{7}{2}}\phi_r(t_{2i+r})$$

$$+ W_{2i+r,0}\delta_0 z_0(t_{2i+r}) + W_{2i+r,1}\delta_1 z_1(t_{2i+r})$$

$$+ O\left(\frac{h^4}{\sqrt{t_{2i+r}}}\right) + O\left(\frac{h^{\frac{3}{2}}\delta}{\sqrt{t_{2i+r}}}\right) , \quad r = 0, 1 ; \quad i = 1, \ldots, m/2 . \quad (4.5.4)$$

From the linearity of (4.5.4), we can write

$$\alpha_i = p_i + q_i , \quad i = 2, \ldots, m$$

where

$$p_{2i+r} = \sum_{l=2}^{2i+r} W_{2i+r,l}G(t_{2i+r}, t_l)p_l + h^{\frac{7}{2}}\phi_r(t_{2i+r}) + O\left(\frac{h^4}{\sqrt{t_{2i+r}}}\right) ,$$

$$r = 0, 1 ; \quad i = 1, \ldots, m/2 \quad (4.5.5)$$

and

$$q_{2i+r} = \sum_{l=2}^{2i+r} W_{2i+r,l}G(t_{2i+r}, t_l)q_l + W_{2i+r,0}\delta_0 z_0(t_{2i+r})$$

$$+ W_{2i+r,1}\delta_1 z_1(t_{2i+r}) + O\left(\frac{h^{\frac{3}{2}}\delta}{\sqrt{t_{2i+r}}}\right) ,$$

$$r = 0, 1 ; \quad i = 1, \ldots, m/2 . \quad (4.5.6)$$

Let

$$q_i = W_{i,0}\delta_0 z_0(t_i) + W_{i,1}\delta_1 z_1(t_i) + c_i , \quad i = 2, \ldots, m .$$

Substitution into (4.5.6) yields

$$c_i = \sum_{l=2}^{i} W_{i,l} G(t_i, t_l) c_l$$

$$+ \sum_{l=2}^{i} W_{i,l} G(t_i, t_l)\{\delta_0 W_{l,0} z_0(t_l) + \delta_1 W_{l,1} z_1(t_l)\} + O\left(\frac{h^{\frac{3}{2}}\delta}{\sqrt{t_i}}\right),$$

$$i = 2, \ldots, m \; . \quad (4.5.7)$$

Clearly,

$$\sum_{l=2}^{2i+r} W_{2i+r,l} G(t_{2i+r}, t_l) W_{l,0} z_0(t_l)$$

$$= \int_0^1 \left\{ 2h \sum_{l=1}^{i-1} \frac{1}{\sqrt{t_{2i+r} - t_{2l} - 2sh}} \sum_{k=0}^{2} l_k(2s) G(t_{2i+r}, t_{2l+k}) \right.$$

$$\left. z_0(t_{2l+k}) W_{2l+k,0} \right\} ds + O\left(\frac{h^{\frac{3}{2}}}{\sqrt{t_{2i+r}}}\right), \quad i = 1, \ldots, m/2 .$$

Since

$$W_{i,r} = \int_0^{2h} \frac{l_r\left(\frac{t}{h}\right)}{\sqrt{t_i - t}} \, dt$$

$$= \int_0^{2h} \frac{l_r\left(\frac{t}{h}\right)}{\sqrt{t_i + 2hs}} \, dt + \int_0^{2h} l_r\left(\frac{t}{h}\right) \left\{ \frac{1}{\sqrt{t_i - t}} - \frac{1}{\sqrt{t_i + 2hs}} \right\} dt$$

$$= \frac{a_r h}{\sqrt{t_i + 2hs}} + O\left(\frac{h}{i^{\frac{3}{2}}}\right), \quad r = 0, 1 \; ; \quad 0 \le s \le 1 \; ; \quad i \ge 2 \quad (4.5.8)$$

it follows that

$$\sum_{l=2}^{2i+r} W_{2i+r,l} G(t_{2i+r}, t_l) W_{l,0} z_0(t_l)$$

$$= \int_0^{2h} l_0\left(\frac{t}{h}\right) dt \int_0^1 2h \sum_{l=1}^{i-1} \left\{ \frac{1}{\sqrt{t_{2i+r} - t_{2l} - 2sh}\sqrt{t_{2l} + 2sh}} \sum_{k=0}^{2} l_k(2s) \right.$$

$$\left. G(t_{2i+r}, t_{2l+k}) z_0(t_{2l+k}) ds + O\left(\frac{h}{(i-l)^{\frac{1}{2}} l^{\frac{3}{2}}}\right) \right\} + O\left(\frac{h^{\frac{3}{2}}}{\sqrt{t_{2i+r}}}\right),$$

$$= a_0 \int_0^1 2h^2 \sum_{l=1}^{i-1} \left\{ \frac{G(t_{2i+r}, t_{2l}+2sh)z_0(t_{2l}+2sh)}{\sqrt{t_{2i+r}-t_{2l}-2sh}\sqrt{t_{2l}+2sh}} ds + O(h^3) \right.$$

$$\left. + O\left(\frac{h}{(i-l)^{\frac{1}{2}} l^{\frac{3}{2}}}\right) \right\} + O\left(\frac{h^{\frac{3}{2}}}{\sqrt{t_{2i+r}}}\right),$$

$$r = 0, 1 \; ; \; i = 1, \ldots, m/2 \; . \quad (4.5.9)$$

Applying the Euler Maclaurin sum formula for integrands with algebraic singularities and lemma 4.2.1 to the right hand side of (4.5.9), we obtain

$$\sum_{l=2}^{i} W_{i,l} G(t_i, t_l) W_{l,0} z_0(t_l) = hF_0(t_i) + O\left(\frac{h^{\frac{3}{2}}}{\sqrt{t_i}}\right),$$

$$i = 2, \ldots, m \; . \quad (4.5.10)$$

Similarly,

$$\sum_{l=2}^{i} W_{i,l} G(t_i, t_l) W_{l,1} z_1(t_l) = hF_1(t_i) + O\left(\frac{h^{\frac{3}{2}}}{\sqrt{t_i}}\right),$$

$$i = 2, \ldots, m \; . \quad (4.5.11)$$

Since $x(t)$ satisfies an equation of the form (4.1.1) it follows from the results in chapter 3 that

$$|x(t+h)-x(t)| = O(h^{\frac{1}{2}}) \; , \quad 0 \le t \le T-h \; .$$

Hence, using lemma 4.2.3,

$$\sum_{l=2}^{2i+r} W_{2i+r,l} G(t_{2i+r}, t_l) x(t_l)$$

$$= \int_0^1 \left\{ 2h \sum_{l=1}^{i-1} \frac{1}{\sqrt{t_{2i+r}-t_{2l}-2sh}} \sum_{k=0}^{2} l_k(2s) G(t_{2i+r}, t_{2l+k}) x(t_{2l+k}) \right\} ds$$

$$+ O(h^{\frac{1}{2}})$$

$$= \int_0^{t_{2i+r}} \frac{G(t_{2i+r}, s) x(s)}{\sqrt{t_{2i+r}-s}} ds + O(h^{\frac{1}{2}}),$$

$$r = 0, 1 \; ; \; i = 1, \ldots, m/2 \; . \quad (4.5.12)$$

Let $\tau_i = c_i - hx(t_i)$, $i = 2, \ldots, m$. Then, subtraction of (4.5.12) from (4.5.7) and the use of (4.5.10) and (4.5.11) yield

$$\tau_i = \sum_{l=2}^{i} W_{il} G(t_i, t_l)\tau_l + O\left(\frac{h^{\frac{3}{2}}\delta}{\sqrt{t_i}}\right) , \quad i = 2, \ldots, m .$$

Hence, in the same way as in theorem 4.4.1,

$$\tau_i = O\left(\frac{h^{\frac{3}{2}}\delta}{\sqrt{t_i}}\right) , \quad i = 2, \ldots, m .$$

To estimate p_i , (4.5.5) is rewritten as

$$p_{2i+r} = \sum_{l=1}^{i-1} W_{2i+r,2l} G(t_{2i+r}, t_{2l}) p_{2l} + \sum_{l=1}^{i-1+r} W_{2i+r,2l+1} G(t_{2i+r}, t_{2l+1}) p_{2l+1}$$

$$+ h^{\frac{7}{2}}\phi_r(t_{2i+r}) + O\left(\frac{h^4}{\sqrt{t_{2i+r}}}\right) , \quad r = 0, 1 \; ; \; i = 1, \ldots, m/2 .$$

In a similar way to (4.5.8), we obtain

$$W_{i,2l+r} = \frac{ha_r}{\sqrt{t_i - t_{2l-2} - 2sh}} + O\left(\frac{h^{\frac{1}{2}}}{(i+2-2l)^{\frac{3}{2}}}\right)$$

$$0 < s < 1 \; ; \; r = 0, 1 \; ; \quad 2l = 2, \ldots, i-3r; \; i = 2, \ldots, m .$$

An analogous analysis to that used for the estimation of c_i yields

$$p_{2i+r} = h^{\frac{7}{2}}\zeta_r(t_{2i+r}) + O\left(\frac{h^4}{\sqrt{t_{2i+r}}}\right) , \quad r = 0, 1 \; ; \; i = 1, \ldots, m/2 .$$

Hence, the result follows. #

In the following theorem, an asymptotic error estimate for the scheme (4.3.5 b) is given.

THEOREM 4.5.2. *Let* $\hat{z}_r(t)$ *be defined analogously to* (4.5.2). *Let* $\hat{\zeta}_r(t)$, $r = 0, 1$, *be the solution of*

$$\hat{\zeta}_r(t) = \hat{\phi}_r(t) + \int_0^t \frac{G(t,s)}{\sqrt{t-s}} \left(a_r\hat{\zeta}_0(s) + b_r\hat{\zeta}_1(s)\right) ds , \quad r = 0, 1 \quad (4.5.13)$$

and $\hat{x}_r(t)$, $r = 0, 1$, *be the solution of*

$$\hat{x}_r(t) = \hat{\delta}_0 c_r F_0(t) + \hat{\delta}_1 d_r F_1(t) + \int_0^t \frac{G(t,s)}{\sqrt{t-s}} \{a_r \hat{x}_0(s) + b_r \hat{x}_1(s)\} ds \ , \qquad r = 0, 1$$

where

$$\hat{F}_r(t) = \int_0^t \frac{G(t,s)\hat{z}_r(s)}{\sqrt{s}\sqrt{t-s}} ds \ , \quad r = 0, 1$$

and $a_0 = b_1 = \frac{1}{3}$, $a_1 = b_0 = \frac{2}{3}$, $c_0 = 26/72$, $c_1 = 25/72$,

$d_0 = 86/72$, $d_1 = 75/72$. *Then*

$$\hat{\alpha}_{2i+r} = h^{\frac{7}{2}}\hat{\zeta}_r(t_{2i+r}) + \hat{\delta}_0 z_0(t_{2i+r})\hat{W}_{2i+r,0} + \delta_1 z_1(t_{2i+r})\hat{W}_{2i+r,1}$$

$$+ h\hat{x}_r(t_{2i+r}) + O\left(\frac{h^4}{\sqrt{t_{2i+r}}}\right) + O\left(\frac{h^{\frac{3}{2}}\hat{\delta}}{\sqrt{t_{2i+r}}}\right) , \quad r = 0, 1 ; \ i = 1, \ldots, m/2 .$$

Proof. The proof proceeds as for theorem 4.5.1. The only difference is that the repetition factor is two and hence estimates such as (4.5.8) are no longer valid for all i but depend on whether i is even or odd. The ramifications of this are that the error due to the inexact starting values no longer varies continuously with t_i . #

From (4.5.3), we obtain

$$\zeta_0(t) - \zeta_1(t) = \phi_0(t) - \phi_1(t) ,$$

$$\tfrac{1}{3}\zeta_0(t) + \tfrac{2}{3}\zeta_1(t) = \lambda(t) ,$$

where

$$\lambda(t) = \tfrac{1}{3}\phi_0(t) + \tfrac{2}{3}\phi_1(t) + \int_0^t \frac{G(t,s)}{\sqrt{t-s}} \lambda(s) ds .$$

Hence, $\zeta_r(t)$, $r = 0, 1$, and $x(t)$ are obtained as the solutions of equations of the form (4.5.1), which implies that (4.3.5 a) is a numerically stable scheme for (4.1.1).

From (4.5.13),

$$\hat{\zeta}_0(t) + \hat{\zeta}_1(t) = \hat{\lambda}_1(t) \ ,$$

$$\hat{\zeta}_0(t) - \hat{\zeta}_1(t) = \hat{\lambda}_2(t)$$

where

$$\hat{\lambda}_1(t) = \hat{\phi}_0(t) + \hat{\phi}_1(t) + \int_0^t \frac{G(t,s)}{\sqrt{t-s}} \hat{\lambda}_1(s)ds \ ,$$

$$\hat{\lambda}_2(t) = \hat{\phi}_0(t) - \hat{\phi}_1(t) - \tfrac{1}{3} \int_0^t \frac{G(t,s)}{\sqrt{t-s}} \hat{\lambda}_2(s)ds \ .$$

Hence, $\hat{\zeta}_r(t)$, $r = 0, 1$ (and, clearly, also $\hat{x}_r(t)$, $r = 0, 1$) may have unstable growth for a stable equation (4.1.1). For instance, if $G(t, s) = -1$, it follows from a Laplace transform that $\hat{\lambda}_2(t)$ behaves like $\exp\left(\sqrt{\frac{\pi}{3}}\, t\right)$. Thus the scheme (4.3.5 b) is not numerically stable.

It is clear that the preceeding arguments can be generalized to show that all methods based on piecewise polynomial interpolation and having a repetition factor of one are numerically stable. On the other hand, numerical instability may occur in methods with a repetition factor greater than one. For Volterra integral equations of the second kind with smooth kernels, this result has been established by Linz (1967) and Noble (1969).

Asymptotic error estimates similar to those given in theorems 4.5.1 and 4.5.2, can be obtained for the schemes (4.3.6 a, b). However, since the system (3.1.5) is unstable, for practical purposes the schemes (4.3.6 a, b) can not be regarded as being numerically stable with respect to equation (4.1.1). As indicated previously, it is therefore necessary to terminate the schemes (4.3.6 a, b) before the instability becomes dominant.

4.6 Numerical Example

The investigation of the radiation of heat from a semi-infinite solid having a constant heat source leads to the equation (see for instance, Keller and Olmstead (1971)),

$$x(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{1-x^4(s)}{\sqrt{t-s}} \, ds \ .$$

The composite scheme (4.3.6 a) on $0 \le t \le a$ and (4.3.5 a) on $a \le t \le 1$ was applied to this problem with various step sizes. Starting values $u(0)$ and $v(0)$ were obtained via (3.1.5), i.e.,

$$u(0) = g_1(0) \ ,$$

$$v(0) = g_2(0) + 2g\{0, 0, u(0)\} \ .$$

The other starting values $\{u_1, u_2, v_1$ and $v_2\}$ were obtained by applying one step of the block-by-block method suggested by Linz (1969) to (3.1.5). All the resultant nonlinear equations were solved by a Newton Raphson iteration.

The numerical results for various stepsizes h are tabulated in table 4.1.

4.7 An Alternative Formulation

From chapter 3, the solution of (4.1.1) has the form

$$y(t) = u(t) + \sqrt{t}\, v(t)$$

where $u(t)$ and $v(t)$ are the components of the solution of (3.1.5). The derivatives of $y(t)$ will therefore, in general, become unbounded in a neighbourhood of the origin. This difficulty, however, can be avoided, if we extract a truncated Taylor series expansion (about the origin) of $v(t)$ multiplied by $\sqrt{t}$ from $y(t)$. In particular, if $u(t)$ and $v(t)$ are suitably smooth, then

Table 4.1

t	a = 0.5 h = 0.1	a = 0.45 h = 0.05	a = 0.475 h = 0.025	a = 0.4875 h = 0.0125
0.1	3.53715229E-1	3.53823239E-1	3.53818931E-1	3.53818448E-1
0.2	4.88883032E-1	4.88809522E-1	4.88802420E-1	4.88801735E-1
0.3	5.78790512E-1	5.78796717E-1	5.78790942E-1	5.78790440E-1
0.4	6.42585689E-1	6.42542771E-1	6.42539428E-1	6.42539175E-1
0.5	6.89183436E-1	6.89216386E-1	6.89214888E-1	6.89214802E-1
0.6	7.24398475E-1	7.24383572E-1	7.24383127E-1	7.24383123E-1
0.7	7.51593944E-1	7.51600536E-1	7.51600597E-1	7.51600626E-1
0.8	7.73190081E-1	7.73187020E-1	7.73187284E-1	7.73187322E-1
0.9	7.90688141E-1	7.90685735E-1	7.90686053E-1	7.90686089E-1
1.0	8.05144191E-1	8.05144999E-1	8.05145307E-1	8.05145339E-1

$$\tilde{u}(t) = y(t) - \sqrt{t} \sum_{k=0}^{p} \frac{t^k v^{(k)}(0)}{k!}$$

$$= y(t) - \sqrt{t}\, \tilde{v}(t)$$

is p times continuously differentiable. As (theorem 3.3.2) $u^{(k)}(t)$ and $v^{(k)}(t)$, $k = 0, \ldots, p$ are the components of the solution of the system of equations obtained when (3.1.5) is differentiated k times, we can obtain $u^{(k)}(0)$ and $v^{(k)}(0)$, if the derivatives of $f_1(t)$, $f_2(t)$ and $g(t, s, y)$ are known analytically, by letting t tend to zero in these equations. When $p = 2$, for example, we obtain

$$u(0) = f_1(0) ,$$

$$v(0) = f_2(0) + 2g\{0, 0, u(0)\} ,$$

$$u'(0) = f_1'(0) + 2 \frac{\partial g}{\partial y} \{0, 0, u(0)\}$$

and

$$v'(0) = f_2'(0) + 2 \frac{\partial g}{\partial y} \{0, 0, u(0)\} + \tfrac{4}{3}u'(0) \frac{\partial g}{\partial y} \{0, 0, u(0)\} .$$

Since $\tilde{u}(t)$ is smooth, $g_1\{t, s, \tilde{u}(s), \tilde{v}(s)\}$ and $g_2\{t, s, \tilde{u}(s), \tilde{v}(s)\}$ will be smooth, if $g(t, s, y)$ is suitably smooth. Hence, if (4.1.1) is rewritten as

$$\tilde{u}(t) = f_1(t) + \sqrt{t}\, \{f_2(t) - \tilde{v}(t)\} + \int_0^t \frac{\sqrt{s} g_1\{t,s,\tilde{u}(s),\tilde{v}(s)\} + g_2\{t,s,\tilde{u}(s),\tilde{v}(s)\}}{\sqrt{t-s}}\, ds ,$$

product integration schemes {based on piecewise polynomial interpolation to $g_1\{t, s, \tilde{u}(s), \tilde{v}(s)\}$ and $g_2\{t, s, \tilde{u}(s), \tilde{v}(s)\}$ } for the above equation should yield reasonable results.

The scheme corresponding to the Simpson scheme (a) is

$$\tilde{u}_i = f_1(t_i) + \sqrt{t_i}\,\left(f_2(t_i) - \tilde{v}(t_i)\right) + \sum_{l=0}^{i} \{X_{il} g_1(t_i,\, t_l,\, \tilde{u}_l,\, \tilde{v}(t_l))$$

$$+ W_{il} g_2(t_i,\, t_l,\, \tilde{u}_l,\, \tilde{v}(t_l))\}\ , \quad i = 2,\, \ldots,\, m\ .$$

The scheme corresponding to the Simpson scheme (b) is obtained by replacing X_{il} and W_{il} by $\hat{X}_{il}$ and $\hat{W}_{il}$ respectively.

Convergence results corresponding to theorem 4.4.1 and asymptotic results corresponding to theorems 4.5.1 and 4.5.2 can easily be obtained for these schemes.

CHAPTER 5

THE NUMERICAL SOLUTION OF FIRST KIND FREDHOM EQUATIONS WITH WEAKLY SINGULAR PERIODIC KERNELS

5.1 Introduction

The numerical solution of a Fredholm equation of the first kind

$$Ky = \int_a^b k(t, s)y(s)ds = f(t) \ , \quad a \le t \le b \qquad (5.1.1)$$

poses a number of theoretical and practical difficulties. Some of these may be illustrated by considering the case when $k(t, s)$ is symmetric, square integrable and has a complete set of orthonormal eigenfunctions $\phi_r(t)$ with associated eigenvalues λ_r , $r = 0, 1, 2, \ldots$ with $|\lambda_r| \ge |\lambda_{r+1}|$. In this case,

$$k(t, s) = \sum_{r=0}^{\infty} \lambda_r \phi_r(t)\phi_r(s) \ , \text{ a.e.}$$

and so a perturbation

$$\delta f(t) = \sum_{r=0}^{\infty} \delta_r \phi_r(t) \ ,$$

in $f(t)$ causes a perturbation

$$\delta y(t) = \sum_{r=0}^{\infty} \frac{\delta_r}{\lambda_r} \phi_r(t)$$

in the solution. Since the operator K is compact, λ_r tends to zero as r increases. As the components $\delta_r \phi_r(t)$ in $\delta f(t)$ are amplified by a factor $1/\lambda_r$ in $\delta y(t)$, the rate at which the eigenvalues tend to zero will characterize the sensitivity of (5.1.1) to perturbations in $g(t)$.

It must be expected that the sensitivity inherent in (5.1.1) will manifest itself in the approximate solutions obtained by the application of numerical schemes (such as finite difference schemes) directly to (5.1.1). Hence, direct methods will fail to yield satisfactory results, if the eigenvalues tend to zero too quickly or if the data $f(t)$ contains highly oscillatory errors. In these cases, schemes should be based on a regularized form of (5.1.1) (see Tikhonov (1963 a), (1963 b)) where the solution will depend continuously on the data.

The behaviour of eigenvalues of Fredholm operators has received extensive investigation. As a general rule, the smoother the kernel, the faster the eigenvalues tend to zero. Consequently, it may be expected that direct methods will be unsuitable unless the kernel or one of its derivatives is singular or discontinuous. In fact, many Fredholm equations which arise in practical applications have kernels which are singular at $t = s$. A typical example is

$$k(t,\ s) = \log\left(\sqrt{\{x(t)-x(s)\}^2+\{w(t)-w(s)\}^2}\right) \qquad (5.1.2)$$

where $\{x(t),\ w(t)\}$ is the parametric equation of a plane, closed, smooth and simple curve. Details about applications where this kernel arises are given in section 5.2.

A number of methods for the solution of first kind Fredholm equations with singular kernels have been proposed, (see for instance Noble (1971) and Christiansen (1971)). Convergence of these methods has been observed, but, as far as the author is aware, no convergence results have been established.

In this chapter, we examine the product integration analogues of the mid-point (*cf*. Noble (1971)), trapezoidal and Simpson schemes for the equation

$$\int_a^b \{k(t, s)q(t, s)+p(t, s)\}y(s)ds = f(t) \ , \quad a \le t \le b \quad (5.1.3)$$

where $q(t, s)$ and $p(t, s)$ are periodic and 'smooth' and $k(t, s)$ is periodic and has a singularity at $t = s$. In particular, we shall examine the case when the eigenfunctions of the operator associated with the kernel $k(t, s)$ are

$$\left.\begin{array}{ll} \text{(i)} & \cos \dfrac{2\pi rt}{(b-a)} \ , \ \sin \dfrac{2\pi rt}{(b-a)} \\ \text{(ii)} & \cos \dfrac{\pi rt}{(b-a)} \ , \ \text{or} \\ \text{(iii)} & \sin \dfrac{\pi rt}{(b-a)} \end{array}\right\} \ r = 0, 1, 2, \ldots .$$

It will be assumed in the sequel that (5.1.3) has a unique, periodic and 'smooth' solution.

In section 5.2, a number of applications for the Fredholm equations under consideration are given. Numerical schemes, based on product integration, for (5.1.3) are derived in section 5.3 and the convergence of these schemes is then investigated in section 5.4. A numerical example is given in section 5.5.

5.2 Some Applications of First Kind Fredholm Equations

Let $u(x, w)$ satisfy Laplace's equation

$$u_{xx} + u_{ww} = 0 \ , \quad (x, w) \in S \qquad (5.2.1)$$

subject to the Dirichlet boundary condition

$$u(x, w) = f(x, w) \ , \quad (x, w) \in C \qquad (5.2.2)$$

where S is the region enclosed by the plane, closed smooth and simple curve C given by

$$x = x(t) \ , \quad 0 \le t \le 2\pi$$

and

$$w = w(t) \ , \quad 0 \leq t \leq 2\pi \ .$$

It is well known (see, for instance, Greenberg (1971, Ex. 6.3)) that

$$u(x, w) = - \int_0^{2\pi} \left\{\frac{\partial u}{\partial n}\big(x(s), y(s)\big) - \frac{\partial v}{\partial n}\big(x(s), y(s)\big)\right\}$$

$$\left\{\frac{\big(x^2(s)+y^2(s)\big)^2+4\big(x(s)x'(s)+y(s)y'(s)\big)^2}{x^2(s)+y^2(s)}\right\}U\big(x, y, x(s), y(s)\big)ds \quad (5.2.3)$$

where $v(x, w)$ satisfies (5.2.2) and (5.2.1) for $(x, w) \notin S$, $\frac{\partial}{\partial n}$ denotes the normal derivative at the boundary C and

$$U(x, w, \xi, \eta) = \frac{1}{4\pi} \ln\{(x-\xi)^2+(w-\eta)^2\} \ .$$

For $(x, w) \in C$, equation (5.2.3) reduces to

$$f\big(x(t), w(t)\big) = \int_0^{2\pi} k(t, s)y(s)ds$$

where $k(t, s)$ is given by (5.1.2).

Other problems which may be reduced to integral equations with logarithmic kernels are given by Christiansen (1971) and are summarised below:

(i) The solution of the reduced wave equation in two dimensions. This yields equations (see Noble (1962)) with kernels which can be expressed as a Hankel function of order zero and thus has a logarithmic singularity.

(ii) Electrostatic and low frequency electromagnetic problems (see Mei and van Bladel (1963)).

(iii) The computation of conformal mappings (see Symm (1966), (1967) and Hayes, Kahaner and Kellner (1972)).

(iv) Electromagnetic scattering problems (see Tanner and Andreasen (1967)).

(v) The propagation of acoustic and classic waves (see Banaugh and Goldsmith (1963 a, b)).

5.3 Numerical Schemes

In this section, we shall construct finite difference schemes for (5.1.3) which are based on the product integration analogues of the midpoint, trapezoidal and Simpson rules.

Let

$$h = \frac{b-a}{m}$$

and again introduce grids

$$t_i = a + ih \ , \quad i = 0, \ldots, m$$

and

$$t_{i-\frac{1}{2}} = a + (i-\tfrac{1}{2})h \ , \quad i = 1, \ldots, m \ .$$

Discretization of (5.1.3) on these grids yields

$$\int_a^b \{k(t_i, s)q(t_i, s)+p(t_i, s)\}y(s)ds = f(t_i) \ ,$$

$$i = 0, \ldots, m-1 \quad (5.3.1)$$

and

$$\int_a^b \{k(t_{i-\frac{1}{2}}, s)q(t_{i-\frac{1}{2}}, s)+p(t_{i-\frac{1}{2}}, s)\}y(s)ds = f(t_{i-\frac{1}{2}}) \ ,$$

$$i = 1, \ldots, m \ . \quad (5.3.2)$$

In order to obtain numerical schemes, the left hand side of (5.3.1) or (5.3.2) is replaced by a quadrature formula. However, since $k(t, s)$ is singular at $t = s$, quadrature formulae based on approximating $k(t_i, s)p(t_i, s)y(s)$ and $k(t_{i-\frac{1}{2}}, s)p(t_{i-\frac{1}{2}}, s)y(s)$ by a polynomial or piecewise polynomial will generally yield poor results. More suitable formulae are obtained by product integration where $p(t_i, s)y(s)$ or $p(t_{i-\frac{1}{2}}, s)y(s)$ is replaced by an approximation $P(t_i, s)$ or $P(t_{i-\frac{1}{2}}, s)$. The quadrature weights

$$\int_a^b k(t_i, s)P(t_i, s)ds \ , \quad i = 0, \ldots, m\text{-}1$$

or

$$\int_a^b k(t_{i-\frac{1}{2}}, s)P(t_{i-\frac{1}{2}}, s)ds \ , \quad i = 1, \ldots, m$$

are evaluated analytically.

For the product integration of the midpoint scheme, $p(t_{i-\frac{1}{2}}, s)y(s)$ is approximated by a piecewise constant; viz.

$$p(t_{i-\frac{1}{2}}, s)y(s) \simeq P(t_{i-\frac{1}{2}}, s)$$

$$= \sum_{l=1}^{m} \psi_l(t)p(t_{i-\frac{1}{2}}, t_{l-\frac{1}{2}})y(t_{l-\frac{1}{2}}) \ , \quad i = 1, \ldots, m$$

where

$$\psi_l(t) = H(t-t_{l-1})H(t_l-t)$$

and $H(t)$ is the heaviside step function. This leads to the quadrature formula

$$\int_a^b k(t_{i-\frac{1}{2}}, s)p(t_{i-\frac{1}{2}}, s)y(s)ds \simeq \sum_{l=1}^{n} a_{il}p(t_{i-\frac{1}{2}}, t_{l-\frac{1}{2}})y(t_{l-\frac{1}{2}}) \ ,$$

$$i = 1, \ldots, m \quad (5.3.3)$$

where

$$a_{il} = \int_a^b k(t_{i-\frac{1}{2}}, s)\psi_l(s)ds \ .$$

Since the term $q(t_{i-\frac{1}{2}}, s)y(s)$ is 'smooth', a suitable approximation for the second term on the left hand side of (5.3.2) is

$$\int_a^b q\left(t_{i-\frac{1}{2}}, s\right)y(s)ds \simeq h \sum_{l=1}^{n} q\left(t_{i-\frac{1}{2}}, t_{l-\frac{1}{2}}\right)y\left(t_{l-\frac{1}{2}}\right) ,$$

$$i = 1, \ldots, m . \quad (5.3.4)$$

Replacing the integral terms in the left hand side of (5.3.2) by (5.3.3) and (5.3.4) yields the scheme

$$\sum_{l=1}^{n} \left\{a_{il}p\left(t_{i-\frac{1}{2}}, t_{l-\frac{1}{2}}\right)+hq\left(t_{i-\frac{1}{2}}, t_{l-\frac{1}{2}}\right)\right\}y_l = f\left(t_{i-\frac{1}{2}}\right) ,$$

$$i = 1, \ldots, m \quad (5.3.5)$$

where y_l , $l = 1, \ldots, m$ denotes the numerical approximation to $y\left(t_{l-\frac{1}{2}}\right)$.

In the trapezoidal scheme, $p\left(t_i, s\right)y(s)$ is approximated by a piecewise linear function. Since $p(t, s)$ and $y(t)$ are periodic, we use the approximation

$$p\left(t_i, s\right)y(s) \simeq P\left(t_i, s\right)$$

$$= \sum_{l=1}^{n} \chi_l(t)p\left(t_i, t_{l-1}\right)y\left(t_{l-1}\right)$$

where

$$\chi_1(t) = \frac{(h-t)}{h} H(h-t) + \frac{(t+h-b)}{h} H(b-h-t) ,$$

$$\chi_l(t) = \frac{\left(t-t_{l-2}\right)}{h} H\left(t-t_{l-2}\right)H\left(t_{l-1}-t\right) + \frac{\left(t_l-t\right)}{h} H\left(t-t_{l-1}\right)H\left(t_l-t\right) ,$$

$$l = 2, \ldots, m$$

and $H(t)$ is the Heaviside step function.

In a similar way to the midpoint scheme we obtain

$$\sum_{l=1}^{m} \left\{a_{il}p\left(t_{i-1}, t_{l-1}\right)+hq\left(t_{i-1}, t_{l-1}\right)\right\}y_l = f\left(t_{i-1}\right) ,$$

$$i = 1, \ldots, m \quad (5.3.6)$$

where

$$a_{il} = \int_a^b k(t_{i-1}, s)\chi_l(s)ds$$

and y_l denotes the numerical approximation to $y(t_{l-1})$.

For the Simpson scheme, $p(t_i, s)y(s)$ is approximated by a piecewise quadratic. In particular, we assume that m is even and that

$$p(t_i, s)y(s) \simeq P(t_i, s)$$

$$= \sum_{l=1}^{m} \Lambda_l(t)p(t_i, t_{l-1})y(t_{l-1})$$

where

$$\Lambda_1(t) = l_0\left(\frac{t}{h}\right)H(2h-t) + l_2\left(\frac{t+2h-b}{h}\right)H(b-2h-t) ,$$

$$\Lambda_{2k}(t) = l_1\left(\frac{t-t_{2k-2}}{h}\right)H(t-t_{2k-2})H(t_{2k}-t) , \quad k = 1, \ldots, m/2 ,$$

$$\Lambda_{2k+1}(t) = l_2\left(\frac{t-t_{2k-2}}{h}\right)H(t-t_{2k-2})H(t_{2k}-t)$$

$$+ l_0\left(\frac{t-t_{2k}}{h}\right)H(t-t_{2k})H(t_{2k+2}-t) , \quad k = 1, \ldots, m/2-1$$

where

$$l_r(t) = \prod_{\substack{j \neq r \\ j=0}}^{2} \frac{(t-j)}{(r-j)} , \quad r = 0, 1, 2 .$$

This leads to the scheme

$$\sum_{l=1}^{n} \{a_{il}p(t_{i-1}, t_{l-1}) + h\gamma_l q(t_{i-1}, t_{l-1})\}y_l = f(t_{i-1}) ,$$

$$i = 1, \ldots, m \quad (5.3.7)$$

where

$$a_{il} = \int_a^b k(t_{i-1}, s)\Lambda_l(s)ds \ ,$$

$$\gamma_{2l} = 4/3 \ ,$$

$$\gamma_{2l-1} = 2/3$$

and y_l denotes the numerical approximation to $y(t_{l-1})$.

REMARK. The numerical schemes in this section, are very similar to those proposed by Atkinson (1967) for second kind Fredholm equations.

5.4 Convergence Results

Initially, we shall examine the midpoint scheme (5.4.5) for the equation

$$\int_0^{2\pi} k(t-s)y(s)ds = g(t) \ , \quad 0 \leq t \leq 2\pi \tag{5.4.1}$$

where $k(t)$ is square integrable, periodic with period 2π and even. In this case it is easy to verify that

$$\begin{aligned} k(t-s) &= \sum_{r=0}^{\infty} \lambda_r \cos r(t-s) \\ &= \sum_{r=0}^{\infty} \lambda_r \{\sin rt \sin rs + \cos rt \cos rs\} \ , \quad \text{a.e.} \end{aligned}$$

where

$$\lambda_0 = \frac{1}{2\pi}\int_0^{2\pi} k(s)ds$$

and

$$\lambda_r = \frac{1}{\pi}\int_0^{2\pi} k(s)\cos rs ds \ , \quad r > 0 \ .$$

The results for this problem are summarized in the following theorem.

THEOREM 5.4.1. *If* $y(t)$ *is two times continuously differentiable, and* λ_r *satisfies*

$$|\lambda_r| \geq K_1 r^{-q} \ , \quad K_1 > 0$$

and

$$\lambda_r = ar^{-q} + O\left(r^{-1-q}\right) \ , \quad 0 < q < 3/2 \ ,$$

then the midpoint scheme (5.3.5) *applied to* (5.4.1) *is convergent in the sense that*

$$\|\underset{\sim}{\varepsilon}\|_2 \leq K_2 h^{\frac{3}{2}-q} \ , \quad K_2 = const. \ ,$$

where

$$\|\underset{\sim}{\varepsilon}\|_2 = \left(\sum_{l=1}^{n} \varepsilon_l^2\right)^{\frac{1}{2}}$$

and

$$\varepsilon_l = y\left(t_{l-\frac{1}{2}}\right) - y_l \ .$$

Proof. From section 5.3, with $a = 0$ and $b = 2\pi$, we obtain

$$a_{ij} = \int_0^{2\pi} k\left(t_{i-\frac{1}{2}}-s\right)\psi_j(s)ds$$

$$= \sum_{r=0}^{\infty} \lambda_r \left\{\sin rt_{i-\frac{1}{2}} \int_{t_{j-1}}^{t_j} \sin rs\,ds + \cos rt_{i-\frac{1}{2}} \int_{t_{j-1}}^{t_j} \cos rs\,ds\right\}$$

$$= \lambda_0 h + \sum_{r=1}^{\infty} \frac{2\sin\left(\frac{rh}{2}\right)\lambda_r}{r} \cos r\left(t_i-t_j\right) \ ,$$

$$i = 1, \ldots, n \ ; \quad j = 1, \ldots, m \ .$$

Since

$$\cos\left[(np+r)\left(t_i-t_j\right)\right] = \cos r\left(t_i-t_j\right) \ , \quad p = 0, 1, \ldots$$

it follows that

$$a_{ij} = \sum_{r=0}^{m-1} d_r \cos r\left(t_i-t_j\right) \qquad (5.4.2)$$

where

$$d_0 = \lambda_0 h$$

and

$$d_r = 2\sin\left(\frac{rh}{2}\right) \sum_{p=0}^{\infty} \frac{(-1)^p \lambda_{np+r}}{np+r} , \quad r = 1, \ldots, m . \tag{5.4.3}$$

Define the $m \times m$ matrix

$$A = (a_{ij})$$

and the $m \times 1$ vectors

$$\mathbf{y} = (y_1, \ldots, y_m)^T ,$$

$$\mathbf{z} = \left(y\left(\frac{h}{2}\right), \ldots, y\left(2\pi - \frac{h}{2}\right)\right)^T ,$$

$$\mathbf{f} = \left(f\left(\frac{h}{2}\right), \ldots, f\left(2\pi - \frac{h}{2}\right)\right)^T ,$$

and

$$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)^T .$$

Then (5.3.5) can be written in matrix form as

$$A\mathbf{y} = \mathbf{f} .$$

In addition, $\mathbf{z}$ satisfies an equation of the form

$$A\mathbf{z} = \mathbf{g} + \mathbf{r}$$

and so it follows that

$$A\varepsilon = \mathbf{r} , \tag{5.4.4}$$

where

$$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)^T .$$

From (5.4.2), it can easily be verified that A is a circulent matrix (see, for instance, Varga (1962)) and consequently

$$A = EPE^{-1}$$

where

$$E = (e_{lr}) = \left(m^{-\frac{1}{2}}\exp(ilt_r)\right) , \quad (5.4.5)$$

$$E^{-1} = (e_{lr}^{*}) = \left(m^{-\frac{1}{2}}\exp(-ilt_r)\right) \quad (5.4.6)$$

and

$$P = (\mathrm{diag}(p_l)) \quad (5.4.7)$$

with

$$p_1 = md_0$$

and

$$p_r = \frac{m}{2}(d_{r-1}+d_{m-r+1}) , \quad r = 2, \ldots, m . \quad (5.4.8)$$

Taking norms in (5.4.4) yields

$$\|A\varepsilon\|_2 = \|\mathbf{r}\|_2$$

and since

$$\|A\varepsilon\|_2 \geq \frac{\|\varepsilon\|_2}{\|A^{-1}\|_2} = p\|\varepsilon\|_2$$

where

$$p = \min_{1\leq l\leq m} \{|p_l|\} , \quad (5.4.9)$$

it follows that

$$\|\varepsilon\|_2 \leq \frac{\|\mathbf{r}\|_2}{p} \quad \text{if} \quad p \neq 0 .$$

From the conditions on λ_r and (5.4.3), we obtain

$$|p_r| \geq K_3\{r^{-q}+(m+1-r)^{-q}\} , \quad K_3 = \text{const.} ,$$

if m is sufficiently large, and hence

$$\|\varepsilon\|_2 \leq K_3 h^{-q}\|\mathbf{r}\|_2 . \quad (5.4.10)$$

It now remains to estimate $\|\mathbf{r}\|_2$. Clearly

$$r_i = \sum_{l=0}^{m-1} \int_{t_l}^{t_{l+1}} k\left(t_{i-\frac{1}{2}}-s\right)\left\{y(s)-y\left(t_{l+\frac{1}{2}}\right)\right\}ds$$

$$= \sum_{l=0}^{m-1} \int_{t_l}^{t_{l+1}} \left(s-t_{l+\frac{1}{2}}\right)k\left(t_{i-\frac{1}{2}}-s\right)y'(s)ds + O\left(h^2\right)$$

$$= h\int_0^1 \left(s-\tfrac{1}{2}\right)h \sum_{l=0}^{m-1} k\left(t_{i-\frac{1}{2}}-t_l-sh\right)y'\left(t_l+sh\right)ds + O\left(h^2\right) .$$

As in chapter 2, we obtain the Poisson summation formula,

$$h\sum_{l=0}^{m-1} k\left(t_{i-\frac{1}{2}}-t_l-sh\right)y'\left(t_l+sh\right) - \int_0^{2\pi} k\left(t_{i-\frac{1}{2}}-x\right)y'(x)dx$$

$$= \sum_{n=1}^{\infty}\left\{b_{nm}\left(t_{i-\frac{1}{2}}\right)\cos 2\pi ns + c_{nm}\left(t_{i-\frac{1}{2}}\right)\sin 2\pi ns\right\} , \quad i = 1, \ldots, m ,$$

where

$$b_n(t) = 2\int_0^{2\pi} k(t-x)y'(x)\cos nx dx , \quad n = 1, 2, \ldots$$

and

$$c_n(t) = 2\int_0^{2\pi} k(t-x)y'(x)\sin nx dx , \quad n = 1, 2, \ldots .$$

Since

$$\int_0^1 \left(s-\tfrac{1}{2}\right)\cos 2\pi ns ds = 0 , \quad n = 0, 1, \ldots$$

it follows that

$$r_i = h\sum_{n=1}^{\infty} \int_0^1 \left(s-\tfrac{1}{2}\right)\sin 2\pi ns ds c_{nm}\left(t_{i-\frac{1}{2}}\right) + O\left(h^2\right)$$

$$= \frac{-h}{2\pi}\sum_{n=1}^{\infty} \frac{c_{nm}\left(t_{i-\frac{1}{2}}\right)}{n} + O\left(h^2\right) .$$

Assuming that the only singularities of $k(t)$ are at -2π, 0 and 2π , we find that

$$c_n(t) = 2y'(t)\int_0^{2\pi} k(t-s)\sin nsds + 2\int_0^{2\pi} k(t-s)\left(y'(s)-y'(t)\right)\sin nsds \ .$$

Since $k(t-s)\left(y'(s)-y'(t)\right)$ is Lipschitz continuous, we obtain

$$\int_0^{2\pi} k(t-s)\left(y'(t)-y'(s)\right)\sin nsds = O\left(\frac{1}{n}\right)$$

and hence

$$c_n(t) = 2y'(t)\sin nt\int_0^{2\pi} k(s)\cos nsds + O\left(\frac{1}{n}\right)$$
$$= 2\pi y'(t)\sin nt\lambda_n + O\left(\frac{1}{n}\right) .$$

Consequently,

$$r_i = -hy'\left(t_{i-\frac{1}{2}}\right)\sum_{n=1}^{\infty}\frac{\sin nmt_{i-\frac{1}{2}}\lambda_{nm}}{n} + O\left(h^2\right)$$
$$= O\left(h^2\right)$$

since $\sin nmt_{i-\frac{1}{2}} = 0$, and hence

$$\|r\|_2 \le Ch^{\frac{3}{2}} \ , \quad C = \text{const.} \tag{5.4.11}$$

The result now follows on substitution of (5.4.11) into (5.4.10). #

To illustrate this result, consider the case when

$$k(t) = \log\left(\sin\left|\frac{t}{2}\right|\right) ,$$

which arises when the curve in (5.1.2) is a circle with radius $\frac{1}{2}$ and centre at the origin. Then,

$$\lambda_0 = -\log 2 \ ,$$

$$\lambda_r = -1/r \ , \quad r = 1, 2, \ldots$$

and hence,

$$\|\varepsilon\|_2 \le Kh^{\frac{1}{2}} \ , \quad K = \text{const.}$$

The above result for the scheme (5.3.5) can easily be extended to equations of the form

$$\int_0^{2\pi} \{k(t-s)+p(t,\ s)\}y(s)ds = g(t)\ ,\quad 0 \le t \le 2\pi\ , \tag{5.4.12}$$

where $p(t, s)$ is periodic and 'smooth'. In this case we obtain an error equation of the form

$$(A+hB)\varepsilon = r \tag{5.4.13}$$

where A is defined as previously and

$$B = (b_{ij}) = (p(t_{i-\frac{1}{2}},\ t_{j-\frac{1}{2}}))\ .$$

Multiplication of (5.4.13) by A^{-1} yields

$$(I+hA^{-1}B)\varepsilon = A^{-1}r\ . \tag{5.4.14}$$

Let $f(t, s)$ be the 'smooth' solution of the equation

$$\int_0^{2\pi} k(t-\tau)f(\tau,\ s)d\tau = p(t,\ s),\quad 0 \le t\ ,\quad s \le 2\pi\ . \tag{5.4.15}$$

For example, if

$$k(t) = \ell n|\sin t/2|$$

we find from the Fourier representation of $k(t)$ and $p(t, s)$ that

$$f(t,\ s) = K\int_0^{2\pi} \cot\left(\frac{t-\tau}{2}\right) \frac{\partial}{\partial\tau} p(\tau,\ s)d\tau + c\int_0^{2\pi} p(\tau,\ s)d\tau\ .$$

Then it follows from theorem 5.4.1 that

$$A^{-1}B = F + h^b C\ ,\quad b = 3/2 - q\ ,$$

where

$$F = (f_{ij}) = (f(t_{i-\frac{1}{2}},\ t_{j-\frac{1}{2}}))$$

and

$$C = (c_{ij})\ ,\quad c_{ij} = O(1)\ .$$

If -1 is not an eigenvalue of the operator

$$Fy = \int_0^{2\pi} f(t,\ s)y(s)ds\ ,$$

it follows from the theory on the numerical solution of second kind

Fredholm equations (see, for instance, Atkinson (1971)) that the matrix $I + hF$ has a bounded inverse if h is sufficiently small.

Thus multiplication of (5.4.14) by $(I+hF)^{-1}$ and taking norms yields

$$\|\left(I+h^{1+b}(I+hF)^{-1}C\right)\varepsilon\|_2 = \|(I+hF)^{-1}A^{-1}\mathbf{r}\|_2$$

$$\leq \frac{K\|\mathbf{r}\|_2}{p}, \quad K = \text{const.},$$

where p is defined by (5.4.9).

Since

$$\|\left(I+h^{1+b}(I+hF)^{-1}C\right)\varepsilon\|_2 \geq \left(1+h^b K_1\right)\|\ \|_2, \quad K_1 = \text{const.},$$

it follows that

$$\|\varepsilon\|_2 \leq \frac{K_2\|\mathbf{r}\|_2}{p}.$$

Hence, we obtain

COROLLARY 5.4.1. *Let the hypothesis of theorem* 5.4.1 *be satisfied and in addition let* (5.4.15) *have a unique continuous solution. Then if* -1 *is not an eigenvalue of the operator* F, *the scheme* (5.3.5) *applied to* (5.4.12) *is convergent and*

$$\|\varepsilon\|_2 \leq Kh^{\frac{3}{2}-q}, \quad 0 < q < 3/2\ ;\ K = \mathit{const}.$$

REMARK. Equation (5.1.2) can be written in the above form with

$$k(t-s) = \log\left(\sin\left|\frac{t-s}{2}\right|\right)$$

and

$$p(t, s) = \log\left\{\frac{\sqrt{\left(x(t)-x(s)\right)^2+\left(w(t)-w(s)\right)^2}}{\sin\left|\frac{t-s}{2}\right|}\right\}.$$

Convergence for the scheme (5.3.5) can also be established for equations of the form

$$\int_0^{2\pi} k(t-s)q(t,\ s)y(s)ds = f(t)\ ,\quad 0 \le t \le 2\pi\ . \qquad (5.4.16)$$

LEMMA 5.4.1. *Let the solution of* (5.4.16) *be twice continuously differentiable,* $p_i > 0$ *(or* < 0*)* , $i = 1, \ldots, n$ *{where* p_i *are defined by* (5.4.8)*} and*

$$p = \min_{1\le i\le n} \{|p_i|\} \ge Kh^q\ ,\quad 0 < q < 3/2\ ,\quad K = const.$$

If, in addition, $q(t, s)$ *satisfies*

(i) $q(t, s)$ *is periodic and symmetric,*

(ii) $q(t, t) > 0$ *(or* < 0*), and*

(iii) $\int_0^{2\pi}\int_0^{2\pi} q(t,\ s)q(t)q(s)dsdt > 0$ *(or* < 0*) for all non zero functions* $q(t)$,

then the scheme (5.3.5) *applied to* (5.4.16) *is convergent and*

$$\|\varepsilon\|_2 \le Kh^{\frac{3}{2}-q}\ ,\qquad K = const.$$

Proof. Under the above hypothesis, there exists a set of functions $\phi_r(t)$, $r = 0, 1, \ldots$, such that

$$q(t,\ s) = \sum_{r=0}^{\infty} \beta_r\phi_r(t)\phi_r(s)\ ,\quad \beta_r > 0\quad \{\text{or } \beta_r < 0\}\ .$$

The error equation for this case is

$$A\varepsilon = r \qquad (5.4.17)$$

where

$$A = \left(q\left(t_{i-\frac{1}{2}},\ t_{j-\frac{1}{2}}\right)a_{ij}\right)$$

$$= \left(\sum_{r=0}^{\infty} \beta_r\phi_r\left(t_{i-\frac{1}{2}}\right)a_{ij}\phi_r\left(t_{j-\frac{1}{2}}\right)\right)$$

and a_{ij} is given by (5.4.2). Multiplication of (5.4.17) by ε^T yields

$$\varepsilon^T A = \varepsilon^T \mathbf{r} .$$

Defining

$$\varepsilon_r = \left(\phi_r(h/2)\varepsilon_1, \ldots, \phi_r(2\pi-h/2)\varepsilon_m\right) , \quad r = 0, 1, \ldots ,$$

it can easily be verified that

$$\varepsilon^T A\varepsilon = \sum_{r=0}^{\infty} \beta_r \varepsilon_r^T EPE^{-1}\varepsilon$$

where E, E^{-1} and P are the $m \times m$ matrices defined by (5.4.5), (5.4.6) and (5.4.7), respectively. Since

$$\left|\varepsilon_r^T EPE^{-1}\varepsilon\right| \geq p\|\varepsilon_r\|_2^2$$

it follows that

$$|\varepsilon^T A\varepsilon| \geq p \sum_{r=0}^{\infty} |\beta_r| \|\varepsilon_r\|_2^2$$

$$= p \sum_{l=1}^{m} \varepsilon_l^2 \sum_{r=0}^{\infty} \phi_r^2(t_{l-\frac{1}{2}}) |\beta_r|$$

$$= p \sum_{l=1}^{m} \varepsilon_l^2 |q(t_{l-\frac{1}{2}}, t_{l-\frac{1}{2}})|$$

$$\geq pq\|\varepsilon\|_2^2$$

where

$$q = \inf_{0\leq t\leq 2\pi} |q(t, t)| .$$

Hence

$$\|\varepsilon\|_2 \leq \frac{\|\mathbf{r}\|_2}{pq}$$

and the result follows as previously. #

REMARK. Using a combination of the arguments in corollary 5.4.1 and lemma 5.4.1 we can obtain corresponding results for

$$\int_0^{2\pi} \{k(t-s)q(t, s)+p(t, s)\}y(s)ds = g(t) , \quad 0 \leq t \leq 2\pi .$$

The above analysis can easily be extended to

$$\int_0^{\pi} k(t,\,s)y(s)ds = g(t)\ ,\quad 0 \le t \le \pi\ , \qquad (5.4.18)$$

where

$$k(t,\,s) = \sum_{r=0}^{\infty} \lambda_r \cos rt \cos rs\ ,\quad \text{a.e.}$$

As in theorem 5.4.1, we obtain an error equation,

$$\sum_{j=1}^{m} \sum_{r=0}^{2m-1} d_r \cos rt_{i-\frac{1}{2}} \cos rt_{j-\frac{1}{2}} \varepsilon_j = r_i\ ,\quad i = 1,\,\ldots,\,m\ ,$$

where d_r , $r = 0,\,\ldots,\,2m-1$, are given by (5.4.3) with m replaced by $2m$ and $h = \frac{\pi}{m}$. Defining the $2m$ dimensional vectors $\hat{\varepsilon}$ and $\hat{r}$ where

$$\left.\begin{aligned} \hat{\varepsilon}_i &= \varepsilon_i \\ \hat{\varepsilon}_{m+i} &= \varepsilon_{m+1-i} \\ \hat{r}_i &= r_i \\ \hat{r}_{m+i} &= r_{m+1-i} \end{aligned}\right\} i = 1,\,\ldots,\,m$$

we find that

$$\sum_{j=1}^{2m} \sum_{r=0}^{2m-1} d_r \cos rt_{i-\frac{1}{2}} \cos rt_{j-\frac{1}{2}} \hat{\varepsilon}_j = \hat{r}_i\ ,\quad i = 1,\,\ldots,\,2m$$

and

$$\sum_{j=1}^{2m} \sum_{r=0}^{2m-1} d_r \sin rt_{i-\frac{1}{2}} \sin rt_{j-\frac{1}{2}} \hat{\varepsilon}_j = 0\ ,\quad i = 1,\,\ldots,\,2m\ ,$$

and hence

$$\sum_{j=1}^{2m} \sum_{r=0}^{2m-1} d_r \cos r\left(t_i - t_j\right)\hat{\varepsilon}_j = \hat{r}_i\ ,\quad i = 1,\,\ldots,\,2m\ .$$

Convergence now follows as previously. In a similar way we can establish convergence for (5.4.18) when

$$k(t, s) = \sum_{r=1}^{\infty} \lambda_r \sin rt \sin rs \quad , \quad \text{a.e.}$$

The results of corollary 5.4.1 and lemma 5.4.1 can easily be verified for these kernels.

We shall now indicate how the above results can be extended to the trapezoidal (5.3.6) and Simpson (5.3.7) schemes.

Firstly, we shall examine the trapezoidal scheme (5.3.6) for equations of the form (5.4.1). In this case, we obtain

$$a_{ij} = \int_0^{2\pi} k\left(t_{i-1}-s\right)\chi_j(s)ds$$

$$= \sum_{r=0}^{\infty} \lambda_r\left\{\sin rt_{i-1} \int_0^{2\pi} \chi_j(s)\sin rs\, ds + \cos rt_{i-1} \int_0^{2\pi} \chi_j(s)\cos rs\, ds\right\}$$

$$= \lambda_0 h + \sum_{r=1}^{\infty} \frac{4\sin^2\left(\frac{rh}{2}\right)\lambda_r}{hr^2} \cos r\left(t_i-t_j\right)$$

$$= \sum_{r=0}^{m-1} d_r \cos r\left(t_i-t_j\right) \quad , \quad i, j = 1, \ldots, m \; , \qquad (5.4.19)$$

where

$$d_0 = \lambda_0 h$$

and

$$d_r = \frac{4\sin^2\left(\frac{rh}{2}\right)}{h} \sum_{p=0}^{\infty} \frac{\lambda_{mp+r}}{(mp+r)^2} \; , \quad r = 1, \ldots, m-1 \; .$$

Since (5.4.19) is of the same form as (5.4.2), the results follow in the same way as in theorem 5.4.1. In particular, it can be shown that under the hypothesis of theorem 5.4.1 that,

$$\|\varepsilon\| \le Kh^{\frac{3}{2}-q} \; , \quad K = \text{const.} \; ,$$

where

$$\varepsilon = \left(\varepsilon_1, \ldots, \varepsilon_m\right)^T$$

with

$$\varepsilon_l = y(t_{l-1}) - y_l \, , \; l = 1, \ldots, m \, .$$

Clearly, the analysis easily extends to the results given in corollary 5.4.1 and lemma 5.4.1. It can also be verified that the results remain valid when the eigenfunctions are $\cos rt$ or $\sin rt$, $r = 0, 1, \ldots$.

The application of the Simpson scheme (5.3.7) to (5.4.1) yields

$$a_{ij} = \int_0^{2\pi} k(t_{i-1}-s)\Lambda_j(s)ds$$

$$= \sum_{r=0}^{\infty} \lambda_r \left\{ \sin rt_{i-1} \int_0^{2\pi} \Lambda_j(s)\sin rs\, ds + \int_0^{2\pi} \Lambda_j(s)\cos rs\, ds \right\}$$

$$= \sum_{r=0}^{\infty} \lambda_r \alpha_{rj} \cos r(t_i - t_j) \qquad (5.4.20)$$

where

$$\alpha_{0,2l-1} = 2h/3 \, , \quad l = 1, \ldots, m/2$$

$$\alpha_{0,2l} = 4h/3 \, , \quad l = 1, \ldots, m/2$$

$$\alpha_{r,2l-1} = \frac{h\left(3+\cos(2rh)\right) - \frac{2}{r}\sin(2rh)}{r^2h^2} \, ,$$

$$l = 1, \ldots, m/2 \; ; \quad r = 1, 2, \ldots$$

and

$$\alpha_{r,2l} = -\frac{4h\cos(rh) - \frac{4}{r}\sin(rh)}{r^2h^2} \, , \quad l = 1, \ldots, m/2 \; ; \quad r = 1, 2, \ldots \, .$$

It follows that

$$a_{ij} = \sum_{r=0}^{m-1} d_{rj} \cos r(t_i - t_j)$$

where

$$d_{rj} = \sum_{p=0}^{\infty} \lambda_{mp+r} \alpha_{mp+r,j} \, .$$

As previously, the error equation for (5.3.7) has the form

$$A\varepsilon = r$$

where

$$A = \{a_{ij}\}$$

and hence

$$\|\varepsilon\|_2 \le \|A^{-1}\|_2 \|r\|_2 \ . \tag{5.4.21}$$

From (5.4.20), it follows that

$$A = EP_1E^{-1}U_1 + EP_2E^{-1}U_2$$

where E and E^{-1} are given by (5.4.5) and (5.4.6), P_1 and P_2 are given by (5.4.7) with d_r replaced by d_{r1} and d_{r2}, respectively,

$$U_1 = \{\text{diag}\{u_{i1}\}\}$$

with

$$u_{i1} = \begin{cases} 1 & i \text{ odd} \\ 0 & i \text{ even} \end{cases}$$

and

$$U_2 = I - U_1 \ .$$

Hence

$$AA^T = E\left(P_1E^{-1}U_1EP_1 + P_2E^{-1}U_2EP_2\right)E^{-1}$$

and since

$$E^{-1}U_1E = \tfrac{1}{2}\left[\begin{array}{c|c} I_{m/2} & -I_{m/2} \\ \hline -I_{m/2} & I_{m/2} \end{array}\right]$$

where $I_{m/2}$ denotes the $m/2 \times m/2$ identity, it follows that

$$AA^T = EWE^{-1}$$

where

$$W = \begin{bmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{bmatrix}$$

with

$$V_{11} = \left(\mathrm{diag}\left(\tfrac{1}{2}\left(p_{i1}^2 + p_{i2}^2\right)\right)\right)$$

$$V_{12} = \left(\mathrm{diag}\left(\tfrac{1}{2}\left(-p_{i1}p_{m/2+i,1} + p_{i2}p_{m/2+i,2}\right)\right)\right)$$

and

$$V_{22} = \left(\mathrm{diag}\left(\tfrac{1}{2}\left(p_{m/2+i,1}^2 + p_{m/2+i,2}^2\right)\right)\right) .$$

It can easily be verified that the eigenvalues of $(AA^T)^{-1}$ are

$$\rho_r = \frac{a_r \mp \sqrt{a_r^2 - 4b_r^2}}{2b_r^2} , \quad r = 1, \ldots, [m/4]+1$$

where

$$a_r = \tfrac{1}{2}\left(p_{r1}^2 + p_{r2}^2 + p_{m/2+r,1}^2 + p_{m/2+r,2}^2\right)$$

and

$$b_r = \tfrac{1}{2}\left(p_{r1}p_{m/2+r,2} + p_{r2}p_{m/2+r,1}\right) .$$

Since

$$p_{r+1,1} = \left(3+\cos(2rh)\right)\sigma_{r+1} - \sin(2rh)\tau_{r+1}$$

and

$$p_{r+1,2} = -4\cos(rh)\sigma_{r+1} + 4\sin(rh)\tau_{r+1}$$

where

$$\sigma_{r+1} = \frac{\pi}{h^2} \sum_{q=0}^{\infty} \left\{\frac{\lambda_{mq+r}}{(mq+r)^2} + \frac{\lambda_{mq+m-r}}{(mq+m-r)^2}\right\}$$

and

$$\tau_{r+1} = \frac{\pi}{h^3} \sum_{q=0}^{\infty} \left\{\frac{\lambda_{mq+r}}{(mq+r)^3} - \frac{\lambda_{mq+m-r}}{(mq+m-r)^3}\right\}$$

it follows that

$$b_{r+1} = \sin(rh)\{\tau_{r+1}\sigma_{m/2+r+1} - \tau_{m/2+r+1}\sigma_{r+1}\} \ , \quad r = 1, \ldots, [m/4] \ .$$

If

$$\lambda_r = O\left(r^{-q}\right) \ , \quad 0 < q < 5/2 \ ,$$

a similar argument to that for the midpoint scheme (5.2.5) yields

$$\|\mathrm{r}\|_2 \le Kh^{\frac{7}{2}} \ , \quad K = \text{const.} \qquad (5.4.22)$$

Thus, if

$$|\rho_r| \ge Kh^{-2q}$$

(for instance if $\lambda_r = ar^{-q}$), then substitution of (5.4.22) into (5.4.21) gives

$$\|\varepsilon\|_2 \le Kh^{\frac{7}{2}-q} \ , \quad K = \text{const.}$$

The above result does not generalize immediately to lemma 5.4.1 since p_{i1} and p_{i2} are different. However, all the other results for the midpoint scheme (5.3.5) can be extended to the Simpson scheme (5.3.7).

In many cases, the analytic calculation of the elements

$$a_{ij} = \int_0^{2\pi} k\left(t_{i-\frac{1}{2}}, s\right)\psi_j(s)ds \ ,$$

$$a_{ij} = \int_0^{2\pi} k\left(t_i, s\right)\chi_j(s)ds$$

or

$$a_{ij} = \int_0^{2\pi} k\left(t_i, s\right)\Lambda_j(s)ds$$

is difficult or even impossible. However, it is often possible (for instance, by extraction of singularities) to obtain approximations of the form

$$\tilde{a}_{ij} = a_{ij} + O\left(h^n\right) \ , \quad n > q+\tfrac{3}{2}$$

To illustrate this we consider the case when

$$\begin{aligned} k(t,\ s) &= \log\left(\sin\left|\frac{t-s}{2}\right|\right) \\ &= \log(|t-s|) + \log\left(\frac{\sin\left|\frac{t-s}{2}\right|}{|t-s|}\right) . \end{aligned}$$

For $|t-s| < \pi$, suitable coefficients can be obtained by applying product integration to the term $\log(|t-s|)$ and ordinary quadrature to the term $\log\left(\frac{\sin\left|\frac{t-s}{2}\right|}{|t-s|}\right)$. If we use $\tilde{a}_{ij}$ instead of a_{ij}, the error equation has the form

$$\begin{aligned} A\varepsilon &= \left(A+h^n Q\right)\tilde{\varepsilon} \\ &= \mathrm{r} + h^{n-1}\mathrm{b} \end{aligned}$$

where

$$Q = \left(q_{ij}\right) \ , \ q_{ij} = O(1)$$

and

$$\mathrm{b} = \left(b_1,\ \ldots,\ b_m\right)^T \ , \ b_i = O(1) \ .$$

This gives

$$\|\varepsilon\|_2 \le \frac{\|A^{-1}\|_2\left\{\|\mathrm{r}\|_2 + Kh^{n-\frac{3}{2}}\right\}}{1-h^n\|A^{-1}\|_2\|Q\|_2} , \quad K = \text{const.}$$

Convergence can now be established as previously.

5.5 Numerical Results

In order to demonstrate the convergence, the schemes were applied to

$$\int_0^{2\pi} \left\{ \log\left| \sin\left| \frac{t-s}{2} \right| \right| + \sin(t-s) \right\} y(s)ds = \pi(\sin t - \cos t) \ , \quad 0 \le t \le 2\pi$$

which has the solution

$$y(t) = \cos t \ .$$

The coefficients a_{ij} were calculated as suggested at the end of section 5.4. In table 5.1, $\|\varepsilon\|_2$ and $\|\varepsilon\|_\infty$ where ε is defined as previously, are tabulated for the three schemes with various step-sizes h. It can be seen that the convergence is slightly better than the order $\frac{1}{2}$ (midpoint (5.3.5) and trapezoidal (5.3.6), schemes) and 5/2 (Simpson's (5.3.7) scheme) predicted. This was also found to be the case for a number of other examples.

TABLE 5.1

		$h=\pi/4$	$h=\pi/8$	$h=\pi/16$	$h=\pi/32$
Midpoint	$\|\varepsilon\|_\infty$	1.173E-2	3.736E-3	1.026E-3	2.672E-4
	$\|\varepsilon\|_2$	2.531E-2	1.078E-2	4.125E-3	1.513E-3
Trapezoidal	$\|\varepsilon\|_\infty$	3.193E-2	8.452E-3	2.166E-3	5.480E-4
	$\|\varepsilon\|_2$	6.387E-2	2.391E-2	8.664E-3	3.100E-3
Simpson	$\|\varepsilon\|_\infty$	1.865E-2	1.582E-3	1.291E-4	1.134E-5
	$\|\varepsilon\|_2$	2.806E-2	3.374E-3	3.912E-4	4.903E-5

REMARKS. The foregoing analysis and numerical example indicates that the Simpson method will in general yield superior results to the midpoint and trapezoidal schemes. However, it must be remembered that the calculation of

$$A = (a_{ij})$$

is more difficult and that this may offset the superior accuracy for a given stepsize.

It is clear from section 5.4 that when the schemes are applied to (5.4.1), premultiplication and postmultiplication of A by E^{-1} and E respectively will lead to a diagonal matrix (midpoint, (5.3.5) and trapezoidal (5.3.6)) or a matrix of the same form as (5.4.22). Hence, for this case, the use of fast Fourier transform techniques will yield effecient inversion methods for the resulting system of linear equations. This will also be the case for (5.4.18).

REFERENCES

Anderssen, R.S., de Hoog, F.R. and Weiss, R. (1973). On the numerical solution of Brownian motion processes. J. Appl. Prob. - In press.

Anderssen, R.S. and Weiss, R. (1973). A stable procedure for the inversion of Abel's equation. To be published.

Atkinson, K.E. (1967). The numerical solution of Fredholm integral equations of the second kind. SIAM J. Numer. Anal. 4, pp 337-348.

Atkinson, K.E. (1971). A survey of numerical methods for the solution of Fredholm integral equations of the second kind. Proc. Symposium 'Numerical Solutions of Integral Equations with Physical Applications', Fall National Meeting of SIAM, Madison, Wisconsin.

Baker, C.T.H. and Hodgson, G.S. (1971). Asymptotic expansions for integration formulas in one or more dimensions. SIAM J. Numer. Anal., 8, pp 473-480.

Banaugh, R.P. and Goldsmith, W. (1963 a). Diffraction of steady acoustic waves by surfaces of arbitrary shapes. J. Acoust. Soc. Amer., 35, pp 1590-1601.

Banaugh, R.P. and Goldsmith, W. (1963 b). Diffraction of steady elastic waves by surfaces of arbitrary shape. J. Appl. Mech., 30, pp 589-597.

Chambre, P.L. (1960). Nonlinear heat transfer problem. J. Appl. Phys., 30, pp 1683-1688.

Christiansen, S. (1971). Numerical solution of an integral equation with a logarithmic kernel. BIT, 11, pp 276-287.

de Hoog, F.R. and Weiss, R. (1972 a). Implicit Runge-Kutta methods for second kind Volterra integral equations. To be published.

de Hoog, F.R. and Weiss, R. (1972 b). On the solution of Volterra integral equations of the first kind. Num. Math. - to appear.

de Hoog, F.R. and Weiss, R. (1972 c). High order methods for Volterra integral equations of the first kind. SIAM J. Numer. Anal. - to appear.

de Hoog, F.R. and Weiss, R. (1972 d). The numerical solution of first kind Volterra integral equations with weakly singular kernels. 'Proceedings of the Fifth Australian Computer Conference', pp 413-418.

de Hoog, F.R. and Weiss, R. (1972 e). Asymptotic expansions for product integration. Math. Comp. - to appear.

de Hoog, F.R. and Weiss, R. (1972 f). On the solution of a Volterra integral equation with a weakly singular kernel. SIAM J. Math. Anal. - to appear.

de Hoog, F.R. and Weiss, R. (1972 g). High order methods for a class of Volterra integral equations with weakly singular kernels. To be published.

de Hoog, F.R. (1972). On the numerical solution of first kind Fredholm equations with weakly singular kernels. SIAM J. Numer. Anal. - to appear.

Greenberg, M.D. (1971). 'Applications of Green's Functions in Science and Engineering'. Prentice-Hall, New Jersey.

Hayes, J.K., Kahamer, D.K. and Kellner, R.G. (1972). An improved method for numerical conformal mapping. Math. Comp., 26, pp 327-334.

Hung, H.S. (1970). The numerical solution of differential and integral equations by spline functions. MRC Technical Summary Report #1053. University of Wisconsin, Madison.

Jennings, L.S. (1972). Orthogonal Transformations and Improperly Posed Problems. PhD Thesis - Australian National University, Canberra.

Jones, J.G. (1961). On the numerical solution of convolution integral equations and systems of such equations. Math. Comp., 15, pp 131-142.

Keller, J.B. and Olmstead, W.E. (1971). Temperature of a nonlinearly radiating semi-infinite solid. Quart. Appl. Math., 29, pp 559-566.

Kobayasi, M. (1966). On the numerical solution of the Volterra integral equation of the second kind by linear multistep methods. Rep. Stat. Appl. Res. JUSE, 13, pp 1-21.

Kobayasi, M. (1967). On the numerical solution of the Volterra integral equations of the first kind by trapezoidal rule. Rep. Stat. Appl. Res., JUSE, 14, pp 1-14.

Levinson, N. (1960). A nonlinear Volterra equation arising in the theory of super-fluidity. J. Math. Anal. Appl., 1, pp 1-11.

Linz, P. (1967). The numerical solution of Volterra integral equations by finite difference methods. MRC Technical Summary Report #825, University of Wisconsin, Madison.

Linz, P. (1969). Numerical methods for Volterra integral equations with singular kernels. SIAM J. Numer. Anal., 6, pp 365-374.

Lyness, J.N. and Ninham, B.W. (1967). Numerical quadrature and asymptotic expansions. Math. Comp., 21, pp 162-178.

Mann, W.R. and Wolf, F. (1951). Heat transfer between solids and gases under nonlinear boundary conditions. Quart. Appl. Math., 9, pp 163-184.

Mei, K. and Van Bladel, J. (1963). Low-frequency scattering by rectangular cylinders. IEEE Trans. Antennas and Propagation, AP-11, pp 52-56.

Miller, R.K. and Feldstein, A. (1971). Smoothness of solutions of Volterra integral equations with weakly singular kernels. SIAM J. Math. Anal., 2, pp 242-258.

Minerbo, G.N. and Levy, M.E. (1969). Inversion of Abel's integral equation by means of orthogonal polynomials. SIAM J. Numer. Anal., 6, pp 598-616.

Noble, B. (1962). Integral equation perturbation methods in low-frequency diffraction; in 'Electromagnetic Waves'. Editor, R.E. Langer. The University of Wisconsin Press, Madison.

Noble, B. (1964). The numerical solution of nonlinear integral equations and related topics; in "Nonlinear Integral Equations", Editor, P.M. Anselone. University of Wisconsin Press, Madison.

Noble, B. (1971). Some applications of the numerical solution of integral equations to boundary value problems, in 'Conference on Applications of Numerical Analysis', Lecture Notes in Mathematics, No. 228, Springer-Verlag, Berlin.

Symm, G.T. (1966). An integral equation method in conformal mapping. Num. Math., 9, pp 250-258.

Symm, G.T. (1967). Numerical mapping of exterior domains. Num. Math., 10, pp 437-445.

Tanner, R.L. and Andreasen, M.G. (1967). Numerical solution of electromagnetic problems. IEEE Spectrum, 4, pp 53-61.

Tikhonov, A.N. (1963 a). Improperly posed problems and the method of regularization. Dokl. Akad. Nauk. SSSR, 151, pp 501-504; English translation, Soviet Math. Dokl., 4, pp 1035-1038.

Tikhonov, A.N. (1963 b). On the regularization of improperly posed problems. Dokl. Akad. Nauk. SSSR, 153, pp 49-52; English translation, Soviet Math. Dokl., 4, pp 1624-1627.

Tricomi, F.G. (1957). 'Integral Equations'. Interscience Publishers Inc., New York.

Varga, R.S. (1962). 'Matrix Iterative Analysis'. Prentice Hall, New Jersey.

Wahba, G. (1970). Uniform convergence rates for certain approximate solutions to first kind integral equations. TR-218, Stats. Dept., University of Wisconsin, Madison.

Weiss, R. (1972). Product integration for the generalised Abel equation. Math. Comp., 26, pp 177-190.

Weiss, R. and Anderssen, R.S. (1972). A product integration method for a class of singular first kind Volterra equations. Num. Math., 18, pp 442-456.

Whittaker, E.T. and Watson, G.N. (1958). 'A course of Modern Analysis'. Cambridge University Press, New York.

Young, A. (1954). The application of approximate product integration to the numerical solution of integral equations. Proc. Roy. Soc. London (A), 225, pp 561-573.