THE HOMOLOGY OF GROUPNETS

by

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ERRATA

A superscript to a page number indicates the number of lines from the top of the page; a subscript the number from the bottom. Titles are included in the line count.

Page number	Erratum
1^{5}	for "Schrier" read "Schreier"
3_{1}	for "$C \xrightarrow{=} C'$" read "$C \xrightarrow{=} C$"
5^{10}	for "p. 46" read "p. 47".
7^{11}	for "its distinct *connected components*" read "the distinct *connected components* of A".
8_{7}	for "Thus a groupnet" read "Thus a connected groupnet".
16^{2}	for "[13]" read "[14]".
23^{11}	for "$F(*, i)$" read "$[F(*, i)]$".
23^{13}	for "$F(*, i)^{-1}$" read "$[F(*, i)^{-1}]$".
23_{7}	for "$F([*^{-1}], 1_i)G([*^{-1}], 1_i)$" read "$G([*^{-1}], 1_i)F([*^{-1}], 1_i)$".
25^{9}	delete sentence beginning "Mitchell ..."
26_{1}	after "*vice versa*" insert "(*cf.* Mitchell [23])".
28_{2}	for "$R \cong R$" read "$R = R$".
29^{11}	for "left R-module" read "left zero R-module".
30^{7}	insert "and the product of generators is a generator" after comma.
33_{3}	for "$S - T$ bimodule" read "$R - T$ bimodule".
38^{5}	for "$\mathrm{Hom}_S(M(i), N)$" read "$\mathrm{Hom}_S(M, N)(i)$".
38^{10}	after "ringnet" insert "and zR is a small monoid".
39_{2}	for "R-module" read "zero R-module".
40^{6}	insert "which admits an R-action with respect to Z" after "Id R".
40_{7}	delete sentence beginning "If M ...".
40	(D2.2.1): delete "zg".

45_{12} for "$C_{n-1}(\partial\partial z)$ " read "$C_{n-2}(\partial\partial z)$ ".

52_{2} for "$f \circ \partial_{n+1}$ " read "$f \circ \partial_{i+1}$ ".

55^{5} for "σ-chain morphisms" read "σ-morphisms"

62^{9} for "*of a standard*" read "*and standard*"

62_{5} for "$((s, ([1], \lambda c)_e), (\mathrm{d}, c))$ " read "$((s, ([1], \lambda c)), (\mathrm{d}_e, c))$ ".

62_{2} for "$((s, ([0], \lambda c)_e), (*\mathrm{d}, c))$ " read "$((s, ([0], \lambda c)), (*\mathrm{d}_e, c))$ ".

68^{9} for "*S-Mod(zM)* " read "*S-Comp(zM)* ".

71^{7-13} replace by

"$$zX_n = \bigcup_{v \in D} \left\{ Z \in zM : (m_v(\lambda z), z) \in Z,\ z \in zX^v_n \right\} \cup \bigcup_{e \in D} \left\{ Z \in zM : (m\lambda_e(\lambda z), z) \in Z,\ z \in zX^{\lambda_e}_{n-1} \right\},$$

$$z : X_n \twoheadrightarrow zX_n \text{ by } \begin{cases} x \longmapsto Z, & \mu(zx) \in Z \\ [x|e] \longmapsto Z, & \mu(zx) \in Z, \end{cases}$$

and

$\lambda : zX_n \to \mathrm{Id}\, G$ by the restriction from zM ".

Delete sentence beginning "A *G*-action ...".

80_{5} for "exact compex" read "exact complex diagram".

81^{8} for "$C^e_1[t] = O[0]$ " read "$C^e_1[t] = O[t]$ ".

81_{5} for "$\varepsilon[0] = -1$ " read "$\varepsilon[0] = 1$ ".

83_{7} for "$1 \le i \le m$ " read "$1 \le i \le n$ ".

86_{5} for "$\to$ *R-Comp(Z)* " read "$\to$ *S-Comp(Z)* ".

89^{4} for "[28, 3.1.1]" read "[28, Ex. 3.1.5]".

92 for "C_{n-1}" in diagram top line read "$\partial_n C_n$ ".

93_{13} for "$[fi]$ " read "$\sigma([*], i)[gi]$ ".

94^{13} for "$(H\lambda x_\gamma, x_\gamma)$ " read "$(B\lambda x_\gamma, x_\gamma)$ ".

95^{7} after "$\lambda x = k$ " insert ", $x \in X_n$ ".

96^{2} for "$N^v \otimes_v C$ " read "$N^v \otimes_v C^v$ ".

96_7	insert new paragraph: "For the next theorem, note that under the conditions of (3.3.3), for each left S-module L , each element h of $\hom_S(zM, zL)$ determines a unique collection $\left\{h_v \in \hom_v(zC^v, z^vL) : v \in D\right\}$ with $h_v(z) = (z, h(\sigma_v(\lambda z), z))$ for z in zC^v , and *vice versa*.
96_1	after "exact" insert "for each Z in $z(N \otimes_S M)$ and h in $\hom_S(zM, zL)$ ".
97	for (D4.3.1) read

$$"\ldots \coprod_{\substack{z\in Z_{\lambda e}\\ e\in D}} H_m(N^{\lambda e} \otimes_{\lambda e} C^{\lambda e})(z) \times \{e\} \xrightarrow{\partial_m} \coprod_{z\in Z_v} H_m(N^v \otimes_v C^v)(z)$$

$$\xrightarrow{i_m} H_m(N \otimes_S M)(Z) \xrightarrow{p_m} \coprod_{\substack{z\in Z_{\lambda e}\\ e\in D}} H_{m-1}(N^{\lambda e} \otimes_{\lambda e} C^{\lambda e})(z) \times \{e\} \to \ldots"$$

and for (D4.3.2) read

$$"\ldots \to H_m(\hom_S(M, L))(h) \xrightarrow{i_m} \prod_{v\in D} H_m(\hom_v(C^v, {}^vL))(h_v)$$

$$\xrightarrow{\delta_m} \prod_{e\in D} H_m(\hom_{\lambda e}(C^{\lambda e}, {}^{\lambda e}L))(h_{\lambda e}) \times \{e\} \to$$

$$\to H_{m+1}(\hom_S(M, L))(h) \to \ldots"$$

with corresponding changes to the proof of (4.3.2) and the statement of (4.3.3).

100_1	insert "because all zero sets are singletons" at the end of the sentence.
103_5	for "left" read "left regular".
104^{11}	For "(D3.3.3)" read "(D 4.3.6)".
107_8	for "wide subgroupnet K_v " read "disjoint union K_v of subgroups".
111	alter (D5.1.1) as for (D4.3.3) and (D5.1.2) as for (D4.3.4) above.
115_3	for "Künnuth" read "Künneth" throughout.
115_2, 116_3	delete "*and M is any right regular flat A-module* ".

116-118	for "*M*" read "*TA*" .
116^{4-11}	the proof should read: "Let C be a left regular free A-resolution of TA . Then $TA \otimes_A C_n$ is a free Z-module and hence a flat Z-module for all n in Z ."
117^2	for "since ..." read "since it is isomorphic to Z ".
117^7	delete "*for any flat right regular* A*-module* M ".
118^6	for "$A \times A$-chain map" read "chain map".
119_1	for "$\tilde{\varepsilon}(\gamma, c) = [\gamma]$ " read "$\tilde{\varepsilon}(\gamma, c) = (\gamma, \varepsilon c)$ ".
120	for "*M*" read "*TB*" .
120^6	delete "and M is a flat regular B-module".
120^{11}	for "and ..." read "and (5.2.8) since $TB \otimes_B {}^iC$ is a free Z-complex and hence flat".
121_1	for "$C(G)$ " read "$C(G)$ ".
124_9	for "nonempty" read "to exist".
125^7	for "left G-module" read "left regular G-module".
125_1	read "*module* $H(A; TA)$, *by means of a* ".
126^1	for "*M*" read "*TA*".
129_5	for (3.3.6.iii)" read "(3.3.6.ii) ".
130^4	for "$\mu_2^v \partial_2[x]$ " read "$\mu_1^v \partial_2[x]$ ".
135_1	for "(5.2.11)" read "(5.2.12)".
149_5	for "$\varepsilon x^2 y$ " read "$\pm x^2 y$ ".
171^{14}	for "47" read "48".

ABSTRACT

In this thesis I use the theory of groupnets (Brandt groupoids) to investigate the homology of mapping cylinder groupnets; that is, groupnets G which are the homotopy colimits of diagrams $(\mathcal{D}, A)$ of groupnets. When the edge morphisms of $(\mathcal{D}, A)$ are all monomorphisms, G is known as a graph product. The principal result of the thesis is the construction of a G-complex with universal properties - the G-mapping cylinder - from a diagram of complexes corresponding to $(\mathcal{D}, A)$, and the subsequent proof that

if G is a graph product and the vertex complexes are all free resolutions of their respective trivial modules, then the G-mapping cylinder is a free resolution of its trivial module.

An extension of the categorical approach to rings and modules is developed in order to provide the general result. The notion of chain homotopy is also extended to a form strongly motivated by the topological definition of homotopy. The mapping cylinder complex determines Mayer-Vietoris sequences for the homology of graph products, which in turn may be used to extend several results on duality groups.

For each group in a certain class of groupnets with cohomological dimension two (including torsion-free one-relator groups and tree products of free groups), the mapping cylinder may be employed to evaluate a comultiplication which gives a coring structure to the integral homology module of the group. This comultiplication is in turn analysed (though not in full generality) to provide further information about the group.

STATEMENT

The results presented in this thesis are my own, except where otherwise stated.

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CONTENTS

INTRODUCTION

This thesis offers a contribution to the theory of homology of groups, approached from the point of view of groupnets (Brandt groupoids). Until the appearance of Higgins' formalisation [13] of the theorems of Kurosh, Grushko, Neilsen and Schrier, the availability of the bridge between topology and combinatorial group theory provided by the groupnet was largely ignored. Possibly it was felt that the advantages of such a formalisation were outweighed by the amount of technical machinery first necessary to make the theory work. However, once this machinery had been assembled most proofs became straightforward, and after the appearance of Higgins' book, renewed interest led to the generalisation of further results by Ordman [27], Crowell and Smythe [7]. Chapter 1 provides the necessary summary of the theory of groupnets.

During the last two decades, as a result of the work of G. Higman, A. Karrass, Hanna Neumann, B.H. Neumann, D. Solitar and many others, much attention has been drawn to groups which are graph products (what Karrass would call treed HNN groups: free products with amalgamation, HNN groups, tree products and the like). The theory of groupnets lends itself neatly to the study of such groups. Of late, interest has been aroused in the comparison of the homology of such groups with that of the subgroups comprising them. This comparison, in the form of a Mayer-Vietoris sequence, was published for free products with amalgamation by Swan [38] in 1969. The principal result of this thesis is the construction of a 'mapping cylinder' complex for the homotopy colimit G of each groupnet diagram $(\mathcal{D}, A)$, given any diagram of complexes corresponding to $(\mathcal{D}, A)$, with the subsequent proof that

if G is a graph product, and each vertex complex is a free resolution of its trivial module, then the G-mapping cylinder is a free resolution of its trivial module.

As a corollary of this result, Mayer-Vietoris sequences with arbitrary coefficients were determined for any graph product in mid-1974. The sequences for an HNN group were found independently by Bieri [1], apparently in late 1973, while more recently Chiswell [3] has published these sequences for the general case. The latter author extends Bieri's method by use of Serre's theory [30] of the fundamental group G of a graph of groups; both proofs construct a short exact sequence of G-modules generated by the cosets of the subgroups comprising G, and then form the corresponding long exact homology sequence, in contrast to this author's method of constructing a G-complex with the required universal properties.

The construction of the mapping cylinder complex enables evaluation of a comultiplication on the integral homology module of certain graph products with cohomological dimension two to be made. This defines a coring structure dual to the ring structure of the cohomology module induced by the cup product. The comultiplication is in fact connected with the lower central series of the group. The canonical form of the comultiplication is determined for several cases with homology modules of low rank; it is hoped that the general solution of the combinatorial problem so raised, is not inaccessible.

In all that follows, it is assumed that the reader is familiar with the category theory in Chapters I to IV and VIII of Mac Lane's book [20]. A general treatment of homological algebra sufficient for the purposes of this thesis may be found in either of the texts of Eilenberg and Steenrod [8] or Northcott [26].

CHAPTER 1

GROUPNETS

This chapter will provide a resumé of the work of Higgins [13], Crowell and Smythe [6, 7], using their terminology. No proofs will be provided. These authors, as well as Ordman [27], have shown that a formalisation of many topological proofs of group-theoretic results (for instance, that of Grushko's Theorem [33]) may be made in purely algebraic terms using Brandt groupoids. The term 'groupoid' here refers to the categorical definition: a small category in which every morphism is an isomorphism; not to the different algebraic notion of a set with a binary operation. Throughout this work the term 'groupnet' is used for four reasons: to avoid ambiguity; to emphasise the presence of the graph underlying any groupnet; to harmonise with more general definitions (of partial product nets and pregroupnets) required for proofs in [6] and [7]; and to allow the distinction to be made between the ringnets of Chapter 2 and the ringoids already known to the literature (see [16], [19, p. 250], [23] and [42]).

A knowledge of the interdependence of categories, groupoids and graphs is assumed (see Higgins [13]). The following notational conventions with respect to a small category $\mathcal{C}$ will be employed throughout the thesis. The object set of $\mathcal{C}$ is denoted $|\mathcal{C}|$ and the statement $f \in \mathcal{C}$ means f is a morphism of $\mathcal{C}$. The hom set $\mathcal{C}(C, C')$ or $\hom_{\mathcal{C}}(C, C')$ is the set of all morphisms f in $\mathcal{C}$ with domain $\operatorname{dom} f = C$ and codomain $\operatorname{cod} f = C'$. An object C may represent its identity morphism at any time. Every diagram in $\mathcal{C}$ denotes the statement that it commutes wherever possible. Finally a morphism written as $C \rightarrowtail C'$ is a monomorphism, one written as $C \twoheadrightarrow C'$ is an epimorphism, and one written as $C \xrightarrow{=} C'$ is the identity.

1.1 DEFINITION. A *partial product net* $A = (A, E(A), \lambda, \rho, \mu)$ consists of two sets A and $E(A)$, two maps $\lambda, \rho : A \to E(A)$ and for some $P \subset A \times A$ a partial product $\mu : P \to A$ satisfying

(i) if $(a, a') \in P$ then $\rho a = \lambda a'$ and

(ii) if $(a, a') \in P$ then $\lambda\mu(a, a') = \lambda a$ and $\rho\mu(a, a') = \rho a'$.

The product $\mu(a, a')$ is usually written aa' and the phrase 'the product aa' is defined' will be taken to mean that $(a, a') \in P$. Elements of $E(A)$ are called the *ends of* A, while if $a \in A$, the elements λa and ρa of $E(A)$ are respectively the *left* and *right end of* a. Generally λ and ρ will denote without further distinction the left and right end maps of any partial product net.

The partial product net A is a *product net* if it further satisfies

(iii) if $(a, a') \in A \times A$ and $\rho a = \lambda a'$ then $(a, a') \in P$.

Thus, any directed graph $D = (E, V)$ consisting of a set of directed edges E with a set of vertices V as its ends is a partial product net, each edge having its initial vertex as left end and terminal vertex as right end, and with no multiplication defined on it.

Conversely, any partial product net A determines a directed graph $D(A)$ having vertices the ends of A and edges the elements a of A directed from λa to ρa. The forgetful functor determined by $A \mapsto D(A)$ ignores the multiplicative structure.

In similar fashion, any small category C is a product net with set of ends $|C|$, each morphism in C having as left end its domain and as right end its codomain. Multiplication is given by composition of morphisms, with $fg = g \circ f$.

1.2 DEFINITION. A *morphism* $f : A \to B$ *of partial product nets* consists of two maps $f : A \to B$ and $E(f) : E(A) \to E(B)$ such that

(i) if aa' is defined in A then $f(a)f(a')$ is defined in

B and $f(aa') = f(a)f(a')$ and

(ii) f preserves ends, that is,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \lambda\downarrow & & \downarrow\lambda \\ E(A) & \xrightarrow[E(f)]{} & E(B) \end{array} \quad \text{and} \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \rho\downarrow & & \downarrow\rho \\ E(A) & \xrightarrow[E(f)]{} & E(B) \end{array} .$$

To facilitate the next definition, the subset $\operatorname{Id} A$ of *identities of* A is distinguished as follows: an element i of A is in $\operatorname{Id} A$ if $ai = a$ and $ia' = a'$ whenever ai and ia' are defined in A.

1.3 DEFINITION. A *(partial) product net with identities* is a (partial) product net A with a set isomorphism $\sigma : E(A) \cong \operatorname{Id} A$ such that for all a in A both $a.\sigma\rho(a)$ and $\sigma\lambda(a).a$ are defined. In such a case the identification (which is unique) is always made, and λ and ρ are retractions (see Higgins [13, p. 46]).

One instance of an associative product net with identities is the set of $n \times m$ integral matrices for all (positive) integers n and m. Each $n \times m$ matrix has left identity I_n and right identity I_m, with the usual matrix multiplication acting as partial product.

A *morphism* $f : A \to B$ of (partial) product nets with identities is a (partial) product net morphism also satisfying

(i) $f(\operatorname{Id} A) \subset \operatorname{Id} B$ and

(ii) $E(f)$ is induced from f *via* σ^{-1}.

The class of associative product nets with identities is precisely the class of small categories; movement from one to the other will be made without comment.

Any subset of a (partial) product net (with identities) which is itself a (partial) product net (with identities) is called a *subnet*; the context will clarify how much structure is involved.

1.4 DEFINITION. A *groupnet* A is an associative product net with identities for which every element has an inverse (necessarily unique);

that is, for every a in A there exists an a^{-1} in A such that $aa^{-1} = \lambda a$ and $a^{-1}a = \rho a$.

The category of groupnets and their morphisms is denoted *Gpnet* and is identifiable as the category of small categories for which every morphism is an isomorphism. The following definitions are required:

(i) An (additively written) groupnet A is *abelian* if whenever $a + a'$ is defined in A then so is $a' + a$ and they are equal. An abelian groupnet is thus a disjoint union of abelian groups whose set of (additive) zeroes is its identity set. The category of abelian groupnets and their morphisms is denoted *Abnet*.

(ii) For any set S the subcategory *Abnet*(S) of *Abnet* has as objects the abelian groupnets which have S as set of zeroes and as morphisms those groupnet morphisms which are the identity when restricted to S. Considered as a set $T(S)$ of trivial abelian groups, S is the null object of *Abnet*(S). Two abelian groupnets A and B over S (that is, objects of *Abnet*(S)) have as binary biproduct $A \oplus B$ the abelian groupnet

$$(A \oplus B)(s) = A(s) \oplus B(s) \quad \forall s \in S$$

with coordinatewise addition. An abelian group structure on $\hom(A, B)$ is given by $(f+g)(a) = f(a) + g(a)$ for a in A. From these remarks it follows that *Abnet*(S) is an abelian category.

Note that the functor $T : Set \to Gpnet$ determined from $S \mapsto T(S)$ is left adjoint to the forgetful functor $U : Gpnet \to Set$ [13, p. 17], where *Set* denotes the category of small sets and set maps. Higgins' simplicial functor Δ is right adjoint [13, p. 17].

(iii) An element a of a groupnet A is a *loop at* i for i in $\mathrm{Id}\, A$ whenever $\rho a = \lambda a = i$. It is obvious that the set of loops at any particular identity forms a group, the *loop group* at that identity. If the only loops in A are the identities themselves, A is said to be *acyclic*.

(iv) Any subnet of a groupnet which is itself a groupnet is called a *subgroupnet*. A subgroupnet B of A is *wide* when $\mathrm{Id}\, A \subset B$. There is always a maximal acyclic subgroupnet of A; clearly it is wide.

(v) When $(i, j) \in \mathrm{Id}\, A \times \mathrm{Id}\, A$ set

$$A(i, j) = \{a \in A : \lambda a = i,\ \rho a = j\} .$$

If $A(i, j)$ is nonempty for every such pair, A is said to be *connected*. Equivalently, A is connected if it is not the disjoint union of two non-empty subgroupnets. Any acyclic connected groupnet is a *tree*. The relation

$$i \sim j \Leftrightarrow A(i, j) \neq \emptyset$$

on $\mathrm{Id}\, A$ is an equivalence relation which partitions $\mathrm{Id}\, A$ into the identities of its distinct *connected components*.

It is apparent that the objects of *Gpnet* with a single identity form a full subcategory which is *Gp*, the category of groups. This extension of *Gp* was required when the concept of the covering space was transferred from topology to group theory. It contains algebraically-determined constructs - homotopies, fibrations and 'unit intervals' - which are either undefined or vacuous in *Gp* yet correspond closely to the topological definitions through the forgetful functor from *Gpnet* to *Graph*, the category of directed graphs.

The following groupnet will be used extensively in later work. Whereas the trivial group is 'the' acyclic connected groupnet with a single identity, the unit interval groupnet is 'the' acyclic connected groupnet with two identities. In Higgins' terminology it is the simplicial groupoid with two vertices and hence the absolute free groupoid of rank one on the graph

$$0 \xrightarrow{\ *\ } 1 .$$

1.5 DEFINITION. The *unit interval groupnet* $I = \{0, 1, *, *^{-1}\}$ has identities

$$\mathrm{Id}\, I = \{0, 1\} ,$$

end maps

$$\lambda 0 = \lambda * = 0 \, , \quad \lambda 1 = \lambda *^{-1} = 1 \, ,$$

$$\rho 0 = \rho *^{-1} = 0 \, , \quad \rho 1 = \rho * = 1 \, ,$$

and partial multiplication

$$0.0 = 0 \, , \quad 0.* = * \, , \quad *.1 = * \, , \quad *.*^{-1} = 0 \, ,$$

$$*^{-1}.0 = *^{-1} \, , \quad *^{-1}.* = 1 \, , \quad 1.1 = 1 \, , \quad 1.*^{-1} = *^{-1} \, .$$

Introduction of this groupnet simplifies the intuitive picture of a groupnet (in terms of its underlying graph).

1.6 EXAMPLE. Let A be a connected groupnet. Construct an isomorphic connected groupnet A^* from A in the following manner. Set $\mathrm{Id}\, A^* = \mathrm{Id}\, A$ and select from it a specific identity, 0 . Let $A^*(0, 0)$ be the group of loops in A at 0 , that is, $A^*(0, 0) = A(0, 0)$. Denote by T a maximal tree in A , so that $T(i, j)$ is a singleton $\{t_{ij}\}$ (say) in A for every ordered pair of identities (i, j) in $\mathrm{Id}\, A$, with $t_{ii} = i$ and $t_{ij}^{-1} = t_{ji}$. For each $j \neq 0$ in $\mathrm{Id}\, A^*$ adjoin a copy I_j of I to $A^*(0, 0) \cup \mathrm{Id}\, A^*$, identifying 0_j with 0 and 1_j with j . The groupnet A^* so formed is connected. Each a in $A(i, j)$ has a unique representation

$$a = t_{i0} a^* t_{0j} \, , \text{ with } a^* \text{ in } A^*(0, 0) \, ,$$

and the map $f : A \to A^*$ given by $f(a) = *_i^{-1} a^* *_j$ is a groupnet isomorphism.

Thus a groupnet essentially consists of a group at a distinguished identity and a set of edges or spines radiating from this identity.

As with groups, a presentation $A \cong \langle X : R \rangle$ can be assigned to a connected groupnet A . Free groupnets and groupnet generators and relations are defined in terms of graphs (see [13, Chs 4,9]) and are the expected analogues.

For example, if A^* is the isomorphic image of the connected groupnet

A as given in (1.6), and

$$A(0, 0) \cong \langle X : R \rangle$$

is a presentation of the loop group in A at identity 0, then

$$A^* \cong \langle X, *_j, j \neq 0, j \in \text{Id } A^* : R \rangle$$

is a presentation of the connected groupnet A^*.

It must be emphasised that a generating set for a groupnet consists of elements of the groupnet and hence may have more than one identity. A presentation of a groupnet is not a group presentation unless every element of the set of generators (and the set of relators) has the same (unique) left and right identity. Thus the presentation

$$I \cong \langle * : \quad \rangle$$

of the free groupnet on a single generator does *not* represent Z, the free group on a single generator.

Category *Gpnet* admits all limits and colimits [13, Chs 7,9]. Two groupnets A and B have as product $A \prod B$, their cartesian set product with $\text{Id}(A \prod B) = \text{Id } A \times \text{Id } B$ and the naturally induced coordinatewise groupnet structure. Their coproduct $A \coprod B$ is the disjoint union $A \vee B$ with $\text{Id}(A \coprod B) = \text{Id } A \vee \text{Id } B$ and groupnet structure induced separately from the two components. Subcategory *Abnet*(S) for a set S admits all products and coproducts, with

$$\left(\prod_\alpha A_\alpha\right)(s) = \prod_\alpha \left(A_\alpha(s)\right) \quad \forall s \in S$$

and

$$\left(\coprod_\alpha A_\alpha\right)(s) = \coprod_\alpha \left(A_\alpha(s)\right) \quad \forall s \in S .$$

The righthand term is that of *Ab*, the category of abelian groups. Usually the finite biproduct is called the *direct sum* and is written

$$\bigoplus_{i=1}^{n} A_i .$$

1.7 DEFINITION. Two groupnet morphisms $f, g : A \to B$ are *homotopic* (written $f \simeq g$) if there is a groupnet morphism

$$F : I \times A \to B$$

for which

(i) $F(0, a) = f(a)\ \forall a \in A$ and

(ii) $F(1, a) = g(a)\ \forall a \in A$.

Such a morphism $F : f \simeq g$ is known as a *homotopy between* f *and* g and is completely determined by f and

$$\{F(*, i) : i \in \mathrm{Id}\, A\}$$

since necessarily

$$g(a) = F(*, \lambda a)^{-1} f(a) F(*, \rho a) .$$

Homotopy of morphisms is an equivalence relation; two morphisms are homotopic precisely when they are naturally equivalent as functors.

If morphisms $f : A \to B$ and $g : B \to A$ exist such that $f \circ g \simeq 1_B$ and $g \circ f \simeq 1_A$ then A and B are *of the same homotopy type* $(A \simeq B)$ with *homotopy equivalence* f and *homotopy inverse* g . For example, a groupnet morphism $f : A \to B$ always determines a *constant homotopy* $\chi(f) : f \simeq f$ given by $\chi(f)(*, i) = f(i)$ for i in $\mathrm{Id}\, A$, and thus it is apparent that isomorphic groupnets have the same homotopy type.

The subgroupnet B of A is a *strong (deformation) retract* of A if there is a homotopy equivalence $f : A \to B$ (called a *retraction*) with homotopy inverse the inclusion morphism $j : B \rightarrowtail A$, such that

(i) $f \circ j = 1_B$ and

(ii) $F : j \circ f \simeq 1_A$ satisfies $F(*, i) = i\ \ \forall i \in \mathrm{Id}\, A$.

As instances, $\mathrm{Id}\, A$ is a strong retract of A *via* either λ or ρ ; while any connected groupnet has as a strong retract the loop group at any selected identity. Thus any groupnet has the homotopy type of a disjoint collection of groups. Clearly equal homotopy type does not imply isomorphism. There is a strong analogy between the distinction of isomorphism from homotopy in *Gpnet* and the distinction of homeomorphism from homotopy equivalence in *Top* , the category of topological spaces. Certainly homotopic topological spaces have homotopic fundamental groupnets [13, Ch. 6, Prop. 13].

At this point, in order to construct a groupnet having universal properties with respect to a particular diagram of groupnets, it is necessary to work in a wider category than *Gpnet* . Thus a *pregroupnet* is a partial product net with identities in which each element has (not necessarily unique) two-sided inverses. Note, however, that a pregroupnet with a single identity is not necessarily a pregroup (Stallings [34, 35]). If A is a pregroupnet and $\equiv$ is a *congruence* (an equivalence relation on a partial product net which preserves left and right identities and products wherever defined) then the set of congruence classes $A/\equiv$ is also a pregroupnet with the product: '$\alpha\alpha^*$ is defined if there is a in α and a^* in α^* such that aa^* is defined, and then $\alpha\alpha^* = [aa^*]$ '. A groupnet A , however, does not always define a groupnet $A/\equiv$.

1.8 DEFINITION. A *universal groupnet for a pregroupnet* A consists of a groupnet $G(A)$ and a morphism $\psi : A \to G(A)$ such that any other morphism from A to a groupnet B factors uniquely through ψ . Such a universal groupnet always exists and is constructed thus: the set

$$S(A) = \{\text{nonempty sequences } a_1, \ldots, a_n : a_i \in A, 1 \le i \le n, \quad \rho(a_i) = \lambda(a_{i+1}), 1 \le i < n\}$$

is a product net with the same ends as A , with end maps

$$\lambda(a_1, \ldots, a_n) = \lambda a_1 , \quad \rho(a_1, \ldots, a_n) = \rho a_n$$

and with juxtaposition of sequences as partial product. A congruence $\equiv$ is generated from the relation $u \backslash v$ on $S(A)$, called *elementary contraction*, which is defined whenever $u = a_1, \ldots, a_n$ and

$$v = a_1, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_n$$

for some $a_i a_{i+1}$ in A .

Then $G(A) = S(A)/\equiv$ is a groupnet in this case, and ψ is the composite

$$A \rightarrowtail S(A) \twoheadrightarrow S(A)/\equiv .$$

The set $\mathrm{Id}\, G(A)$ is identifiable with $\mathrm{Id}\, A$. If A is a groupnet, ψ is an isomorphism. Full exposition of these results occurs in Crowell and Smythe [6].

1.9 DEFINITION. A *groupnet diagram* (C, A) consists of a small category C and a functor $A : C \to \mathit{Gpnet}$. If C is the free category on a directed graph $D = (E, V)$, that is, the category of directed paths in D , then (C, A) is denoted $(\mathcal{D}, A)$ and is considered to be a collection of groupnets $\{A_v\}$ indexed by the vertices v of D , together with a collection of morphisms $\{A_e : A_{\lambda e} \to A_{\rho e}\}$ indexed by the edges e of D . Henceforth this will be the only type of groupnet diagram considered.

1.10 DEFINITION. A *mapping cylinder* $m : (\mathcal{D}, A) \to m(\mathcal{D}, A)$ for a groupnet diagram $(\mathcal{D}, A)$ consists of

(i) a groupnet $m(\mathcal{D}, A)$,

(ii) a morphism $m_v : A_v \to m(\mathcal{D}, A)$ for each v in D ,

(iii) a homotopy $m_e : m_{\lambda e} \simeq m_{\rho e} \circ A_e$ for each e in D

which is

(iv) universal over all collections

$$\{N \in |\mathit{Gpnet}|, \{n_v : A_v \to N, v \in D\}, \{n_e : n_{\lambda e} \simeq n_{\rho e} \circ A_e, e \in D\}\}$$

satisfying conditions (i)-(iii); that is, given such a collection there exists a unique morphism $\chi : m(\mathcal{D}, A) \to N$ such that for all v in D and e in D ,

$$\begin{array}{ccc} & A_v & \\ {\scriptstyle m_v}\swarrow & & \searrow{\scriptstyle n_v} \\ m(\mathcal{D}, A) & \xrightarrow[\chi]{} & N \end{array} \quad \text{and} \quad \begin{array}{ccc} & I \times A_{\lambda e} & \\ {\scriptstyle m_e}\swarrow & & \searrow{\scriptstyle n_e} \\ m(\mathcal{D}, A) & \xrightarrow[\chi]{} & N \end{array} .$$

It is apparent that the mapping cylinder is a looser construction than the colimit $\varinjlim A$ in *Gpnet*, which requires equality rather than homotopy in conditions (iii) and (iv) above. For this reason the mapping cylinder is referred to as the 'homotopy colimit' of the functor A. The colimit object actually appears as a certain double quotient of the mapping cylinder object, with colimit morphisms formed from mapping cylinder morphisms by composition with the canonical quotient morphism (see [7, Th. 6.3]). //

1.11 THEOREM [7, Th. 6.1]. *For any groupnet diagram* $(\mathcal{D}, \mathrm{A})$ *there exists a mapping cylinder which is unique up to a unique isomorphism.*

A sketch of the construction is given. Index copies I_e of I by the edges e of D, and define

$$P = \left(\bigvee_{v \in D} A_v\right) \vee \left(\bigvee_{e \in D} I_e \times A_{\lambda e}\right) .$$

Generate $\equiv$ on P from the relations

$$a \sim (0_e, a) \text{ and } A_e(a) \sim (1_e, a) \quad \forall a \in A_{\lambda e}, \quad e \in D .$$

Then pregroupnet P determines a pregroupnet $P/\equiv$ and the mapping cylinder is, from (1.8),

$$m(\mathcal{D}, \mathrm{A}) = G(P/\equiv) ,$$

$$m_v : A_v \rightarrowtail P \twoheadrightarrow P/\equiv \rightarrow G(P/\equiv) ,$$

and

$$m_e : I \times A_{\lambda e} \rightarrowtail P \twoheadrightarrow P/\equiv \rightarrow G(P/\equiv) .$$

There is a one-to-one correspondence between $\mathrm{Id}\; m(\mathcal{D}, \mathrm{A})$ and $\bigvee_{v \in D} \mathrm{Id}\; A_v$ which will, in future, always be employed. □

A presentation of a mapping cylinder may be given in terms of presentations of the vertex groupnets. Suppose $\langle X_v : R_v \rangle$ is a presentation of the vertex groupnet A_v for v in a connected directed graph D. Then

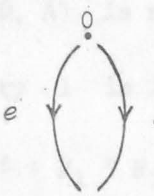
$$\Big\langle \bigvee_{v \in D} m_v(X_v);\ m_e(*,\ i),\ i \in \mathrm{Id}\,A_{\lambda e},\ e \in D :$$

$$\bigvee_{v \in D} m_v(R_v);\ m_e(*,\ \lambda x)^{-1} m_{\lambda e}(x) m_e(*,\ \rho x) = m_{\rho e} A_e(x),\ x \in X_{\lambda e},\ e \in D \Big\rangle$$

is a presentation of $m(\mathcal{D}, \mathsf{A})$. To instance this construction, consider the *trivial groupnet diagram* $(\mathcal{D}, 1)$ on a connected directed graph D having the trivial group at each vertex and the identity morphism on each edge. Its mapping cylinder has a presentation

$$m(\mathcal{D},\ 1) = \langle m_e(*,\ 1),\ e \in D :\ \ \rangle .$$

When A_e is a monomorphism (injection) for each e in D , the mapping cylinder is known as the *graph product* and in this case each m_v for v in D is an embedding [7, Th. 6.2]. In illustration of this case, consider the directed graph

$D =$  ,

and a groupnet diagram $(\mathcal{D}, \mathsf{A})$ in which A_0 is a group and A_e and A_f are monomorphisms. If A_1 is also a group, $m(\mathcal{D}, \mathsf{A})$ has the homotopy type of $\mathrm{HNN}\big(A_1;\ A_e(A_0) \cong A_f(A_0)\big)$, the HNN group with base A_1 and associated subgroups $A_e(A_0)$ and $A_f(A_0)$. If $A_1 = B_0 \vee B_1$ is the disjoint union of two groups and $A_e(A_0) \subset B_0$ while $A_f(A_0) \subset B_1$, then $m(\mathcal{D}, \mathsf{A})$ has the homotopy type of the free product with amalgamation $B_0 *_{A_0} B_1$.

The groups mentioned are the loop groups of their respective mapping cylinder groupnets. In a connected mapping cylinder groupnet G , the loop group is found from a presentation of G by adding further relators to the presentation, corresponding to generators of a maximal tree in G (*cf.* (1.6)). For example, consider the former case above with $A_0 = \langle x_0, x_1 : \ \rangle$, $A_1 = \langle y_0, y_1, y_2 : y_1 = y_2 y_0 \rangle$, $A_e(x_i) = y_i$ and $A_f(x_i) = y_{i+1}$ for $i = 0, 1$. Then

$$m(\mathcal{D}, A) \cong \Big\langle x_0, x_1, y_0, y_1, y_2, *_1, *_2 : \\ y_1 = y_2 y_0, x_0 = *_1 y_0 *_1^{-1} = *_2 y_1 *_2^{-1}, x_1 = *_1 y_1 *_1^{-1} = *_2 y_2 *_2^{-1} \Big\rangle$$

$$\cong \Big\langle y_0, y_1, y_2, t, *_1, *_2 : \\ y_1 = y_2 y_0, t = *_1^{-1} *_2, y_1 = t^{-1} y_0 t, y_2 = t^{-1} y_1 t \Big\rangle$$

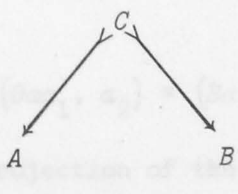
$$\cong \Big\langle y_0, y_1, y_2, t, *_1 : y_1 = y_2 y_0, y_1 = t^{-1} y_0 t, y_2 = t^{-1} y_1 t \Big\rangle$$

where $*_1 = m_e(*, 0)$, $*_2 = m_f(*, 0)$, $\lambda *_i = 0$ and $\rho *_i = 1$ for $i = 1, 2$. A maximal tree in $m(\mathcal{D}, A)$ is represented by $*_1$, so that the loop group of $m(\mathcal{D}, A)$ at identity 1 is isomorphic to

$$m(\mathcal{D}, A)/\langle *_1 \rangle \cong \Big\langle y_i, 0 \le i \le 2, t : y_1 = y_2 y_0, y_{i+1} = t^{-1} y_i t, i = 0, 1 \Big\rangle$$

which is a presentation of the knot group of the trefoil knot, as an HNN group. //

A diagram of groups with monic edge morphisms, together with its graph product, is closely related to a *graph of groups* and its *fundamental group*, as introduced by Bass and Serre [30]. In a graph of groups, each edge corresponds to a pair of amalgamating subgroups, while in a diagram of groups it corresponds to a group monomorphism. The connection is specified thus: $A \overset{C}{\text{——}} B$ in a graph of groups corresponds to



in a diagram of groups, while $f : A \rightarrowtail B$ in a diagram of groups corresponds to $A \xrightarrow{A} B$ with morphisms $1_A : A \rightarrowtail A$ and $f : A \rightarrowtail B$ in a graph of groups. It follows (see Higgins [13]) that the fundamental group of a connected graph of groups is isomorphic to the loop group (at any identity) of the graph product of the corresponding group diagram, and *vice versa*. See Cohen [4], Chiswell [3], Gildenhuys [10] and Cossey and Smythe [5] for examples of the group-theoretic use of these constructions.//

This chapter closes with a description of the covering groupnet corresponding to a given subgroupnet of any groupnet. As in the purely topological approach this has an intimate connection with the problem of finding the homology of a subgroup in terms of that of the group containing it, but the methods used are entirely algebraic.

Morphism $\pi : \tilde{A} \to A$ in *Gpnet* is said to have the *path-lifting property* if whenever $a \in A$ and $i \in \mathrm{Id}\,\tilde{A}$ with $\pi(i) = \lambda a$ then there is $\tilde{a}$ in $\tilde{A}$ such that $\pi(\tilde{a}) = a$ and $\lambda\tilde{a} = i$. Should π also be surjective it is a *fibration*; if further the covering element $\tilde{a}$ is uniquely determined for each a in A (that is, π has the *unique path-lifting property*), then π is a *covering map* and $\tilde{A}$ a *covering groupnet*. //

For a wide subgroupnet B of A there always exists a covering $\pi : \tilde{A} \to A$ for which $\tilde{A} \simeq B$. Denote by A/B the set of right cosets

$$Ba = \{b.a \in A : b \in B\} \text{ for } a \text{ in } A .$$

Then

$$\tilde{A} = \{(Ba, a^*) \in A/B \times A : \rho a = \lambda a^*\}$$

is the covering groupnet, with $\mathrm{Id}\,\tilde{A} = A/B$ under the identification $(Ba, \rho a) \leftrightarrow Ba$, identity maps

$$\lambda(Ba, a^*) = Ba \text{ and } \rho(Ba, a^*) = B(aa^*) ,$$

and partial product

$$(Ba, a_1)(Baa_1, a_2) = (Ba, a_1a_2) .$$

The covering map π is the projection of the second coordinate of $\tilde{A}$. With $\rho : A/B \to \mathrm{Id}\,A$ given by $\rho(Ba) = \rho a$,

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\pi} & A \\ \big\downarrow & & \big\downarrow \lambda \\ A/B & \xrightarrow[\rho]{} & \mathrm{Id}\, A \end{array}$$

is a pullback square in *Set* .

In [7, §8], Crowell and Smythe have shown that the covering $\tilde{G}$ of a graph product $G = m(\mathcal{D}, A)$ corresponding to a wide subgroupnet of G is also a graph product. In fact $\tilde{G} = m(\mathcal{D}, \tilde{A})$ where $\tilde{A}_v$ is a covering of A_v for each v in D, corresponding to certain special wide subgroupnets of A_v. The construction will be detailed later in (5.1).

CHAPTER 2

MODULES OVER RINGNETS

2.1 Ringnets

Groupnets form a wider class than groups, extending the category of small monoids for which every morphism is an isomorphism to the category of small categories for which every morphism is an isomorphism. The analogous extension from unital rings to ringoids; that is, from the category *Rng* of small preadditive monoids to the category *Rngoid* of small preadditive categories has been dealt with in depth in the literature [23]. The category *Rngnet* of unital ringnets, described below, forms an even wider class than *Rngoid*, with subcategory inclusion functors

$$\mathit{Rng} \hookrightarrow \mathit{Rngoid} \hookrightarrow \mathit{Rngnet}$$

which are full as well as faithful.

2.1.1 DEFINITION. A category C is *partially preadditive* if it admits an abelian groupnet structure on hom sets; that is, for any pair of objects (C, C') in $|C|$, $\hom_C(C, C')$ has an abelian groupnet structure, with respect to which the composition map is bilinear. A functor between partially preadditive categories is *partially additive* if it preserves this structure; that is, if it is an abelian groupnet morphism on each hom set.

For instance, *Abnet* is partially preadditive, since for any two abelian groupnets A and B there is the abelian groupnet structure on $\hom(A, B)$ given by

$$\mathrm{Id}\hom(A, B) = \hom_{\mathit{Set}}(\mathrm{Id}\, A, \mathrm{Id}\, B)\ ,$$

$$\hom(A, B)(h) = \{f \in \mathit{Abnet}(A, B) : f|_{\mathrm{Id}A} = h\}$$

for all h in $\mathrm{Id}\hom(A, B)$, and for $f, g \in \hom(A, B)(h)$,

$$(f+g)(a) = f(a) + g(a) \quad \forall a \in A\ .$$

Any preadditive category is at once partially preadditive; any additive functor is at once partially additive. Further, any partially additive functor between preadditive categories is at once additive.

2.1.2 DEFINITION. A *(unital) ringnet* is a small partially preadditive category. The category of all small partially preadditive categories and the covariant partially additive functors between them is called *Rngnet* .

For any pair (i, j) of objects in a ringnet R define

$$zR(i, j) = \mathrm{Id}\, R(i, j)$$

and

$$zR = \bigvee_{(i,j)\in|R|\times|R|} zR(i, j) ;$$

the latter is called the *set of zeroes* of the ringnet R . Bilinearity of composition in R ensures that composition of any morphism in R with a morphism of zR is again a morphism of zR , whence zR is closed under composition in R .

Thus zR is itself a small category, with object class $|zR| = |R|$, composition induced from R and $\hom_{zR}(i, j) = zR(i, j)$ for each pair (i, j) in $|zR|$. For each i in $|zR|$, the identity morphism in $zR(i, i)$ is the zero element of the unique group in $R(i, i)$ containing 1_i , the identity morphism for i in R . The set of identity morphisms in R is known as the *set of identities* Id R and is identifiable with $|R|$. If, for each p in zR , the abelian group in R with zero element p is written $R(p)$, then

$$R = \bigvee_{p\in zR} R(p) .$$

Conversely, any triple $R = (R, zR, \psi)$ is a ringnet if it satisfies conditions (i)-(v) below.

(i) The set of zeroes zR of R is an associative product net with identities.

(ii) The set

$$R = \bigvee_{p \in zR} R(p)$$

is an (additively written) abelian groupnet which has its additive identities equated with the elements of zR.

(iii) Product $\psi : R(p) \otimes_{Z} R(p^*) \to R(pp^*)$, defined whenever $pp^* \in zR$, and abbreviated $\psi(r \otimes r^*) = rr^*$ for r in $R(p)$ and r^* in $R(p^*)$, is an abelian group morphism for all pp^* in zR.

(iv) For each i in Id zR there exists 1_i in $R(i)$ for which $1_i.r = r$ and $r^*.1_i = r^*$ whenever $r \in R(p)$, $r^* \in R(p^*)$ and $i = \lambda p = \rho p^*$.

(v) The product is associative whenever an association is defined. Note that if any finite association is defined in R, all other associations of the same elements are defined in R.

As a ringnet, $|R| = \text{Id } zR$ and $r^* \circ r = rr^*$. In this case, the *zero map* $z : R \to zR$ of the ringnet R is given for r in $R(p)$ by $zr = p$, and $\{1_i : i \in \text{Id } zR\}$ is the set of identities of R.

Consideration of this internal description of a ringnet as a triple determines the following 'internal' description of a morphism in *Rngnet*.

2.1.3 REMARK. A partially additive covariant functor f between two ringnets R and S is called a *ringnet morphism*. It consists of a morphism $zf : zR \to zS$ of product nets with identities, and, for each p in zR, an abelian group morphism $f : R(p) \to S(zf(p))$ satisfying

(i) $f|_{zR} = zf$,

(ii) if $i \in \text{Id } zR$, then $f(1_i) = 1_{zf(i)}$ and

(iii) if $rr^* \in R$, then $f(rr^*) = f(r)f(r^*)$.

In future, movement between the 'internal' and external descriptions of *Rngnet* will be made at will. Context will be indicated by the use of symbols R, zR, and so on, when dealing with the categorical aspects of a particular ringnet, and R, zR, and so on, when dealing with the algebraic aspects.

2.1.4 REMARK. Note that *Rngoid* is the full subcategory of *Rngnet* which has as object class the class of ringnets R for which the small category zR is a *preorder* (that is, each hom set of zR has at most one element). //

Extensive use will be made of the ringnet induced from a groupnet.

2.1.5 EXAMPLE. Let $A \in |Gpnet|$. The *groupringnet* $\mathbb{Z}A$ of A is defined as follows.

(i) Let $z\mathbb{Z}A = \{(i, j) \in \text{Id } A \times \text{Id } A : A(i, j) \neq \emptyset\}$. These zeroes admit a groupnet structure having $\text{Id } z\mathbb{Z}A = \{(i, i) \in \text{Id } A \times \text{Id } A\} \cong \text{Id } A$, $\lambda(i, j) = i$, $\rho(i, j) = j$ and product $(i, j)(j, k) = (i, k)$. (When A is connected, $z\mathbb{Z}A$ is the simplicial groupoid $\Delta(\text{Id } A)$ of Higgins [13, p. 8].) Then,

(ii) $\mathbb{Z}A(i, j)$ is the free abelian group on $\{[a] : a \in A(i, j)\}$ for all (i, j) in $z\mathbb{Z}A$,

(iii) $1_i = [i] \in \mathbb{Z}A(i, i)$ $\forall i \in \text{Id } A$, and

(iv) the partial product on $\mathbb{Z}A$ is extended linearly from that of A.

The ringnet so formed is actually a ringoid and can be identified with Mitchell's $\mathbb{Z}A$ for the small category A [23, p. 11]. Of course, any unital ring R may be used rather than $\mathbb{Z}$ to induce a ringnet from A in a similar manner (*cf.* [16, §2]).

2.1.6 EXAMPLE. For a ringnet R, the discrete category $|zR|$ is a (trivial) abelian groupnet and $\mathbb{Z}(|zR|)$ is the *trivial groupringnet* for R; it is a disjoint union of copies of $\mathbb{Z}$, one for each object of R, and extends the description of $\mathbb{Z}$ as the groupring of the trivial group.

Clearly any groupnet morphism $f : A \to B$ induces a groupringnet morphism $f : \mathbb{Z}A \to \mathbb{Z}B$ by linear extension from the generators. No distinction of terminology will be made between these morphisms.

Tensor products of ringnets are defined pointwise: the *tensor product* $R \otimes S$ of two ringnets R and S is the ringnet with $z(R \otimes S) = zR \times zS$ and $(R \otimes S)(p, q) = R(p) \otimes S(q)$; all actions are defined by coordinate. For preadditive categories this is also the definition in *Rngoid* [23, §2]. Given ringnets R, S and T the following identities hold in *Rngnet*:

$$\mathbb{Z} \otimes R \cong R \ ,$$

$$S \otimes R \cong R \otimes S \ ,$$

$$T \otimes (S \otimes R) \cong (T \otimes S) \otimes R \ ,$$

and for groupnets A and B ,

$$\mathbb{Z}A \otimes \mathbb{Z}B \cong \mathbb{Z}(A \times B) \ .$$

Proofs are straightforward.

As in *Gpnet* , the weakening of similarity conditions in *Rngnet* from isomorphism to homotopy proves very productive. As there, too, the term 'homotopy' is reserved for the internal description of *Rngnet* : it is natural equivalence in the external definition. It is described here in slightly different terms in order to utilise the groupnet I and preserve some visual similarity with the topological definition.

2.1.7 DEFINITION. Two ringnet morphisms $f, g : R \to S$ are *homotopic* (written $f \simeq g$) if there is a ringnet morphism

$$F : f \simeq g : \mathbb{Z}I \otimes R \to S$$

satisfying both

(i) $F([0], r) = f(r)$, $\forall r \in R$, and

(ii) $F([1], r) = g(r)$, $\forall r \in R$.

Such a morphism is called a *homotopy between* f *and* g and is determined entirely by f and

$$\left\{F([*], 1_i), F\left([*^{-1}], 1_i\right) : i \in \text{Id } zR\right\}$$

since $g(r) = F\left([*^{-1}], 1_{\lambda p}\right) f(r) F([*], 1_{\rho p})$ for r in $R(p)$. Ringnets R and S are *homotopic* $(R \simeq S)$ or *of the same homotopy type* precisely when there is a *homotopy equivalence* $f : R \to S$ and a *homotopy inverse* $g : S \to R$ such that $f \circ g \simeq S$ and $g \circ f \simeq R$.

2.1.8 EXAMPLES. (i) There is a *constant homotopy* $\chi(f) : f \simeq f$ for any morphism f.

(ii) A homotopy of groupnet morphisms induces a homotopy of groupringnet morphisms: if $f, g : A \to B$ are groupnet morphisms with a homotopy $F : f \simeq g : I \times A \to B$ then $\overline{F} : f \simeq g : \mathbb{Z}I \otimes \mathbb{Z}A \to \mathbb{Z}B$ is the composed morphism

$$\mathbb{Z}I \otimes \mathbb{Z}A \xrightarrow{\simeq} \mathbb{Z}(I \times A) \xrightarrow{F} \mathbb{Z}B ,$$

so that for all i in $\operatorname{Id} A$,

$$\overline{F}([*], [i]) = F(*, i)$$

and

$$\overline{F}([*^{-1}], [i]) = F(*, i)^{-1} .$$

Morphism $\overline{F}$ is the *induced homotopy* between f and g. Obviously, homotopic groupnets induce homotopic groupringnets.

Homotopy of ringnet morphisms and hence of ringnets is naturally an equivalence relation. For ringnet morphisms $f, g, h : R \to S$ and homotopies $F : f \simeq g$, $G : g \simeq h$, a specific transitive homotopy $FG : f \simeq h$ is generated from

$$FG([*], 1_i) = F([*], 1_i)G([*], 1_i)$$

and

$$FG\left([*^{-1}], 1_i\right) = F\left([*^{-1}], 1_i\right)G\left([*^{-1}], 1_i\right)$$

for all i in $\operatorname{Id} zR$. //

2.1.9 DEFINITION. A *ringnet diagram* $(\mathcal{D}, R)$ consists of a directed graph D and a functor $R : \mathcal{D} \to \mathit{Rngnet}$ where $\mathcal{D}$ is the free category on D. Hence it may be thought of as a collection of ringnets $\{R_v : v \in D\}$ and a collection of ringnet morphisms $\{R_e : R_{\lambda e} \to R_{\rho e} : e \in D\}$.

As an illustration of this definition, any groupnet diagram $(\mathcal{D}, A)$

induces a groupringnet diagram $(\mathcal{D}, \mathbb{Z}A)$ with $(\mathbb{Z}A)_v = \mathbb{Z}(A_v)$ for each vertex v in D and the induced morphism $A_e : \mathbb{Z}A_{\lambda e} \to \mathbb{Z}A_{\rho e}$ for each edge e in D. The trivial groupnet diagram $(\mathcal{D}, 1)$ induces the *trivial ringnet diagram* $(\mathcal{D}, \mathbb{Z})$ in this fashion.

2.1.10 DEFINITION. A *representation* $\sigma : (\mathcal{D}, R) \to \sigma(\mathcal{D}, R)$ *of a ringnet diagram* $(\mathcal{D}, R)$ comprises

(i) a ringnet $\sigma(\mathcal{D}, R)$,

(ii) a ringnet morphism $\sigma_v : R_v \to \sigma(\mathcal{D}, R)$ for each v in D , and

(iii) a ringnet homotopy $\sigma_e : \sigma_{\lambda e} \simeq \sigma_{\rho e} \circ R_e$ for each e in D .

Thus, when $(\mathcal{D}, A)$ is a groupnet diagram with mapping cylinder $m : (\mathcal{D}, A) \to m(\mathcal{D}, A)$ and induced groupringnet diagram $(\mathcal{D}, \mathbb{Z}A)$ there is an *induced representation* (also written 'm') $m : (\mathcal{D}, \mathbb{Z}A) \to m(\mathcal{D}, \mathbb{Z}A)$ having

$$m(\mathcal{D}, \mathbb{Z}A) = \mathbb{Z}(m(\mathcal{D}, A)) ,$$

with groupringnet morphisms m_v and homotopies m_e induced from those of the mapping cylinder. The representation induced from $(\mathcal{D}, 1)$ is called the *trivial representation* of $(\mathcal{D}, \mathbb{Z})$ and is denoted $t : (\mathcal{D}, \mathbb{Z}) \to t(\mathcal{D}, \mathbb{Z})$.

There always exists a homotopy colimit $M : (\mathcal{D}, R) \to M(\mathcal{D}, R)$ for any ringnet diagram $(\mathcal{D}, R)$; that is, a representation of $(\mathcal{D}, R)$ universal with respect to all other representations of $(\mathcal{D}, R)$. It is not required in the theory below so its construction will not be given. It is rather too free an object for present purposes, where attention is directed to induced representations. In general, the homotopy colimit $M(\mathcal{D}, \mathbb{Z}A)$ induced from a groupnet diagram $(\mathcal{D}, A)$ is not isomorphic to the induced representation $\mathbb{Z}(m(\mathcal{D}, A))$.

2.2 Modules

As might be anticipated, the extension of ringoids to ringnets indicates an extension of (unitary) modules over ringoids to (unitary) modules over ringnets. For a unital ring K - in other words a small preadditive monoid - a left (right) unitary K-module is an additive covariant (contravariant) functor $K \to Ab$. For a ringoid C - a small preadditive category - a left (right) unitary C-module is an additive covariant (contravariant) functor $C \to Ab$. (See Latch and Mitchell [16], Lee [17] or Watts [42] for this definition. Mitchell [23, p. 9, p. 17] appears to be in error in his description of right C-modules as covariant functors $C \to Ab$ and left C-modules as contravariant.)

2.2.1 DEFINITION. If R is a ringnet, a *left (right) (unitary) R-module* is a partially additive covariant (contravariant) functor $R \to Abnet$. If A is a groupnet, a $\mathbb{Z}A$-module is referred to as an *A-module*.

Thus, a left R-module M may be thought of as

(i) an abelian groupnet

$$M(i) = \bigvee_{z \in m(i)} M(z)$$

for each i in $|R|$, where $m(i) = \text{Id}\, M(i)$; and

(ii) an abelian groupnet morphism

$$M(r) : M(\text{dom}\, r) \to M(\text{cod}\, r)$$

for each $r \in R$, such that

$$M(r \circ r^*) = M(r) \circ M(r^*) \quad \forall r \circ r^* \in R ,$$

$$M(1_i) = 1_{M(i)} \quad \forall i \in |R| ,$$

and

$$M(r + r^*) = M(r) + M(r^*) \quad \forall r + r^* \in R .$$

If $M(r)(m)$ is abbreviated rm whenever $r \in R$ and $m \in M(\text{dom}\, r)$ then condition (ii) implies that the following equalities hold whenever the left-hand side is defined:

$$r(m+m') = rm + rm' ,$$

$$(r \circ r^*)(m) = r(r^*m) ,$$

$$1_{\mathrm{dom}\, r}(m) = m ,$$

and

$$(r+r^*)(m) = rm + r^*m .$$

Though the abelian groupnets $\{M(i) : i \in |R|\}$ determined by the module M need not be disjoint, in future it will be notationally convenient to assume that they are. This assumption is reasonable, since for any left R-module $M : R \to \mathit{Abnet}$ there is always a naturally isomorphic left R-module $\check{M} : R \to \mathit{Abnet}$ for which these abelian groupnets are pairwise disjoint. Obtain $\check{M}$ from M by replacing $M(z)$ by an (isomorphic) labelled copy $\check{M}(z) = M(z) \times \{i\}$ for each z in $m(i)$ and i in $|R|$. The action of $\check{M}(r)$ for r in R is then given by

$$r(m, \mathrm{dom}\, r) = (rm, \mathrm{cod}\, r) .$$

With this assumption, the identification

$$M = \bigvee_{i \in |R|} M(i)$$

of the functor with an abelian groupnet is made, and

$$zM = \bigvee_{i \in |R|} m(i) ,$$

the identity set of this groupnet, is called the *set of zeroes* of M. It inherits a (left) R-module structure from M. If $m \in M$, the zero of the group containing m is denoted zm. The map $\rho : M \twoheadrightarrow |R|$ given by

$$\rho(m) = i \quad \forall m \in M(i) , \quad i \in |R|$$

is now well-defined and is known as the *right map* of M. A right module is similarly analysed to determine a *left map*.

For the purposes of calculation, it proves much easier to work with an internally defined R-module structure than with the definition given above. Oddly enough, a left R-module M is internally a right R-module M and *vice versa*. The probability of confusion is high and care must be taken in distinguishing context. Generally the symbols M, zM and so on will be reserved for the functorial aspects of a (left) R-module, while M and zM will be reserved for its use as a (right) R-module.

2.2.2 DEFINITION. Let R be a ringnet. A *right R-module* $M = (M, zM, \rho, \psi)$ comprises

(i) a disjoint union

$$M = \bigvee_{z \in zM} M(z)$$

of abelian groups,

(ii) a set map $\rho : zM \twoheadrightarrow \mathrm{Id}\ zR$ which extends by component to M and determines the partition

$$M = \bigvee_{i \in \mathrm{Id} zR} M(i)$$

of M into abelian groupnets

$$M(i) \doteq \bigvee_{\rho z = i} M(z) ,$$

and

(iii) a *right R-action* ψ. That is,

$$\psi : M(z) \otimes_Z R(p) \to M(\rho p)$$

is an abelian groupnet morphism defined whenever $\rho z = \lambda p$ which, when contracted to $\psi(m \otimes r) = mr$, satisfies

$$m.1_{\rho m} = m ,$$

and

$$m(rr^*) = (mr)r^*$$

whenever the left-hand side is defined.

This definition is easily seen to represent the left R-module M with the same underlying groupnet and with R-action $\bar{r}m = mr$.

One virtue of this approach to module theory is that it allows a useful broadening of the definition of a bimodule, which is extensively employed in the succeeding chapters.

2.2.3 DEFINITION. An abelian groupnet M is an R-S *bimodule* if it is both a left R-module and a right S-module such that, for $r \in R$, $s \in S$ and $m \in M$, if either of $(rm)s$ or $r(ms)$ is defined, then both terms are defined and are equal.

The partitions

$$M = \bigvee_{i \in \mathrm{Id}\, zR} M(i) \quad \text{and} \quad M = \bigvee_{j \in \mathrm{Id}\, zS} M(j)$$

determined by the two module structures then determine a further partition

$$M = \bigvee_{(i,j) \in \mathrm{Id}\, zR \times \mathrm{Id}\, zS} M(i, j)$$

of M, where $M(i, j) = M(i) \cap M(j)$. Should $M(i, j) \neq \emptyset$ for every pair (i, j) of $\mathrm{Id}\ zR \times \mathrm{Id}\ zS$ then M may be considered as a bifunctor $S \otimes R^{op} \to \mathit{Abnet}$, partially additive in each argument, covariant in S and contravariant in R. This is not always the case. Any ringnet R is both a left and a right R-module, with

$$R(i, j) = \{R(p) : p \in zR, \lambda p = i, \rho p = j\}$$

for i, j in $\mathrm{Id}\ zR$. If $R = \mathbb{Z}A$ for a disconnected groupnet A, some of these sets will be empty. Under (2.2.3), though, every ringnet R is an R-R bimodule with ringnet multiplication for R-action. Hence, so is zR. The left and right R-module $\mathrm{Id}\ zR$ (hereafter identified with $\mathrm{Id}\ R$) is not generally a bimodule (see p. 33). Any abelian groupnet (and hence any R-module) is a $\mathbb{Z}$-$\mathbb{Z}$ bimodule.

2.2.4 REMARK. If $\sigma : R \to S$ is a partially additive covariant functor and $N : R \to \mathit{Abnet}$ and $M : S \to \mathit{Abnet}$ are both covariant (contravariant) functors, then a natural transformation $\mathfrak{f} : N \dot{\to} M \circ \sigma$ is known 'internally' as a σ-*morphism* $f : N \to M$ of right (left) modules. In this guise (for right modules) it is a groupnet morphism

$$f : N(i) \to M(\sigma(i)) \quad \forall i \in \mathrm{Id}\ R$$

such that $f(nr) = f(n)\sigma(r)$ whenever nr is defined. When $\sigma = 1 : R \cong R$ then f is an *R-morphism*.

2.2.5 NOTATION. The category of partially additive covariant functors and their natural transformations (as in (2.2.4)) is labelled Mod^{ℓ} ; that of contravariant functors is Mod^{r} . Restriction to right (left) R-modules and R-morphisms for a particular ringnet R determines the category

$$R\text{-}Mod^{r} = Abnet^{R} \quad \left(R\text{-}Mod^{l} = Abnet^{R^{op}}\right) .$$

This is not a full subcategory unless the only functorial endomorphism on R is the identity; that is, unless R is a *delta* [23, p. 5].

An R-module Z is a *zero module* if it is a disjoint union of trivial abelian groups; that is, if

$$Z = zZ .$$

Each left R-module Z determines a subcategory $R\text{-}Mod^{l}(Z)$ of $R\text{-}Mod^{l}$ in which

(i) $|R\text{-}Mod^{l}(Z)|$ is the class of all left R-modules M for which $zM = Z$, and

(ii) morphisms are those left R-morphisms which are the identity morphism on the set of zeroes.

The module structure of Z ensures that the left map $\lambda : Z \twoheadrightarrow \text{Id } R$ is common to all objects of $R\text{-}Mod^{l}(Z)$, as is the action of R on the set of zeroes. For each ringnet zero $p \in zR$ and each $z \in Z$ such that $\rho p = \lambda z$, $pz \in Z$. Hence, if $r \in R(p)$ and $m \in M(z)$ for M in $|R\text{-}Mod^{l}(Z)|$, then necessarily $rm \in M(pz)$. Category $R\text{-}Mod^{l}(Z)$ is known as the category of *standard left* R*-modules and* R*-morphisms over* Z , and clearly contains Z as a null object. If $Z = \text{Id } R$, then $R\text{-}Mod^{l}(\text{Id } R)$ is known as the category of *regular left* R*-modules and* R*-morphisms* and is denoted $R\text{-}Modreg^{l}$. A left regular R-module is thus a partially additive contravariant functor $R \to \mathsf{Ab}$ (assuming the image abelian groups are disjoint). Categories $R\text{-}Mod^r(Z)$ and $R\text{-}Modreg^r$ are defined similarly for each right zero module Z, but usually the superscripts l and r are dropped when it is clear which category is intended or when the distinction is unnecessary.

For example, if a ringnet R is a disjoint union of free abelian groups $(R(p)$ is free abelian for each p in $zR)$, and 1_i is a generator of $R(i)$ for each i in $\mathrm{Id}\ zR$, then the trivial groupringnet $\mathbb{Z}(\mathrm{Id}\ zR)$ is regular as either a left or right R-module and is called the *trivial R-module* TR. (When $R = \mathbb{Z}A$ for a groupnet A, TR is written TA.) However, R is not itself regular unless $zR = \mathrm{Id}\ zR$; that is, unless category zR is discrete.

The category $R\text{-}Mod$ is partially preadditive with finite (co)products. Two left R-modules M and N have as coproduct their disjoint union

$$z(M \coprod N) = zM \vee zN ,$$

$$(M \coprod N)(z) = \begin{cases} M(z) , & z \in zM , \\ N(z) , & z \in zN , \end{cases}$$

with componentwise action; and as product their cartesian product

$$z(M \prod N) = zM \times zN ,$$

$$(M \prod N)(u, v) = M(u) \times N(v) ,$$

with action by coordinates. There is no null object in $R\text{-}Mod$.

2.2.6 REMARK. When M is an R-module, the R-module $M \coprod M$ is both a left and a right $\mathbb{Z}I$-module. If the copies of M in $M \coprod M$ are labelled by $\mathrm{Id}\ I$ then a left map $\lambda : zM_0 \vee zM_1 \twoheadrightarrow \mathrm{Id}\ I$ is given by $\lambda z_i = i$ for i in $\mathrm{Id}\ I$. The left I-action is

$$[0]m_0 = m_0 , \quad [1]m_1 = m_1 ,$$

$$[*^{-1}]m_0 = m_1 , \quad [*]m_1 = m_0$$

for m_i in M_i and i in $\mathrm{Id}\ I$. Right action is correspondingly defined.

Since $\lambda = \rho$, $M \coprod M$ is not a bimodule.

2.2.7 LEMMA. *For any ringnet* R *and zero* R*-module* Z *,* $R\text{-}Mod(Z)$ *is an abelian category.*

Proof. (i) Any standard R-morphism is the identity map on zeroes hence all hom sets have an abelian group structure, with $f + g : M \to N$ given by $(f+g)(m) = f(m) + g(m)$ for f, g in $\hom_R(M, N)$, and *zero morphism* $0 : M \to N$ given by $0(m) = zm \in Z$, for m in M . Composition is clearly bilinear over this addition so $R\text{-}Mod\ Z$ is preadditive.

(ii) Any two standard R-modules M and N over Z have a coproduct $M \coprod N$ written $M \oplus N$ with

$$\left(M \coprod N\right)(z) = M(z) \oplus N(z) \quad \forall z \in Z$$

and R-action by coordinate, so $R\text{-}Mod(Z)$ is additive. Its *diagonal morphism* $\Delta : M \to M \oplus M$ and *codiagonal morphism* $\nabla : M \oplus M \to M$ are given for any module M by

$$\Delta(m) = (m, m) \ ,$$

and

$$\nabla(m, m') = m + m' \quad \forall m, m' \in M \ .$$

(iii) Any standard R-morphism $f : M \to N$ has both a kernel $i : \text{Ker}\ f \to M$ and a cokernel $\pi : N \to \text{Coker}\ f$. The submodule $\text{Ker}\ f$ of M is defined by

$$\text{Ker}\ f(z) = \{m \in M(z) : f(m) = z\} \quad \forall z \in Z$$

with R-action restricted from M , and i is the inclusion morphism. The standard R-module $\text{Coker}\ f$ over Z has

$$\text{Coker}\ f(z) = N(z)/fM(z) \quad \forall z \in Z$$

with R-action induced from N , and π is the canonical quotient morphism. Hence $R\text{-}Mod(Z)$ is preabelian. The image and coimage of f are similarly defined pointwise on Z from their corresponding definitions in Ab .

(iv) The parallel map $\overline{f} : \text{Coim}\ f \twoheadrightarrow \text{Im}\ f$ is given as

$$\overline{f}\left(m + \text{Ker}\ f(z)\right) = f(m) \quad \forall m \in M(z) \ , \quad z \in Z \ .$$

If $f(m-m') = z$, $m-m' \in \text{Ker } f(z)$, $zm = zm' = z$ so $m + \text{Ker } f(zm) = m' + \text{Ker } f(zm')$ and $\overline{f}$ is an isomorphism. Hence R-$Mod(Z)$ is abelian [28, 2.3]. □

By applying the argument for Ab pointwise on Z , it follows that any standard R-morphism over Z is mono if and only if it is injective and epi if and only if it is surjective.

2.2.8 LEMMA. *The category R-$Mod(Z)$ admits arbitrary direct products and coproducts.*

Proof. Let $\{M_\alpha : \alpha \in A\}$ be a set of standard (left) R-modules over Z . Set

$$\left(\coprod_\alpha M_\alpha\right)(z) = \coprod_\alpha \left(M_\alpha(z)\right) \quad \forall z \in Z ,$$

and

$$\left(\prod_\alpha M_\alpha\right)(z) = \prod_\alpha \left(M_\alpha(z)\right) \quad \forall z \in Z ,$$

where the terms on the right hand side are defined in Ab .

If $r \in R(p)$, $\sum_{i=1}^{n} m_{\alpha_i} \in \left(\coprod_\alpha M_\alpha\right)(z)$, $\{m_\alpha\} \in \left(\prod_\alpha M_\alpha\right)(z)$ and pz is defined then

$$r\left(\sum_{i=1}^{n} m_{\alpha_i}\right) = \sum_{i=1}^{n} \left(rm_{\alpha_i}\right) \in \left(\coprod_\alpha M_\alpha\right)(pz)$$

and

$$r\{m_\alpha\} = \{rm_\alpha\} \in \left(\prod_\alpha M_\alpha\right)(pz) . \quad \square$$

When the index set A in (2.2.8) is finite, the identification

$$\prod_\alpha M_\alpha \cong \coprod_\alpha M_\alpha$$

is made and the biproduct is called the *direct sum*

$$\bigoplus_\alpha M_\alpha .$$

In particular, this result is true for R-$Modreg$. However, the requirement

$Z = \mathrm{Id}\, R$ is very restrictive: R-*Modreg* does not admit bimodules unless zR is a disjoint union of small monoids. For, if M is a regular R-bimodule, $\lambda = \rho = 1 : \mathrm{Id}\, R = \mathrm{Id}\, R$. If $r \in R(p)$, $r^* \in R(p^*)$, $m \in M(z)$ and $(rm)r^* = r(mr^*)$, then $(pz)p^* = p(zp^*)$, so that $\rho p = z = \lambda p^*$ and $(\lambda p)p^* = p(\rho p^*)$, hence $\lambda p = \rho p = z = \lambda p^* = \rho p^*$. Each p in zR thus satisfies $\mathrm{dom}\, p = \mathrm{cod}\, p$.

2.2.9 DEFINITION. Given a ringnet morphism $\sigma : R \to S$, any left (right) S-module M determines a left (right) R-module ${}^{\sigma}M$ (M^{σ}), which is called the *pullback of* M *along* σ. In *Mod* it is the composed partially additive contravariant (covariant) functor $M \circ \sigma : R \to S \to$ *Abnet*. Thus for a left module,

$$z{}^{\sigma}M = \{(i, z) \in \mathrm{Id}\, R \times zM : \sigma(i) = \lambda z\},$$

$${}^{\sigma}M(i, z) = \{i\} \times M(z),$$

$$\lambda(i, z) = i$$

and

$$r(\rho r, m) = (\lambda r, \sigma(r)m).$$

Clearly the pullback of a regular module is regular since *Abnet* may be replaced by *Ab*.

2.2.10 EXAMPLE. If the groupnet morphism $\sigma : A \to B$ is extended to groupringnets then ${}^{\sigma}(TB) \cong TA$ as A-modules, since

$$z{}^{\sigma}(TB) = \{(i, \sigma i) \in \mathrm{Id}\, A \times \mathrm{Id}\, B\}$$

and

$${}^{\sigma}(TB)(i, \sigma i) = \{i\} \times \mathbb{Z}[\sigma i] \cong \mathbb{Z}.$$

Correspondingly, $(TB)^{\sigma} \cong TA$.

If in (2.2.9), M is an S-T bimodule the obvious induced right T-action gives ${}^{\sigma}M$ an S-T bimodule structure. Composed functor $M \circ \sigma$ determines the identity natural transformation $i(\sigma) : M \circ \sigma \dot{\to} M \circ \sigma$ which in turn determines a canonical σ-morphism $\sigma^* : {}^{\sigma}M \to M$ called (left) *pullback projection* from its position in the *Abnet* diagram

$$\begin{array}{ccc} {}^{\sigma}M & \xrightarrow{\sigma^*} & M \\ \lambda\downarrow & \text{p.b.} & \downarrow\lambda \\ \operatorname{Id} R & \xrightarrow[\sigma]{} & \operatorname{Id} S . \end{array}$$

That is, $\sigma^*(i, m) = m$. Any σ-morphism $f : N \to M$ may then be uniquely factored through σ^* by a morphism $\sigma(f) : N \to {}^{\sigma}M$, evaluated as $\sigma(f)(n) = (\lambda n, fn)$.

The change of rings technique embodied in the pullback is necessary to any investigation of homology. So is the next definition.

2.2.11 DEFINITION. If M is a right R-module and N is a left R-module their *tensor product* $M \otimes_R N$ *over* R is an abelian groupnet with

(i)

$$\operatorname{Id}\left(M \otimes_R N\right) = \{(u, v) \in zM \times zN : \rho u = \lambda v\} / \langle (up, v) \sim (u, pv), \quad p \in zR,\ \lambda p = \rho u,\ \rho p = \lambda v \rangle ,$$

and

(ii)

$$M \otimes_R N(\alpha) = \coprod_{(u,v)\in\alpha} M(u) \otimes_Z N(v) \,/\, \langle (mr, n) = (m, rn), \quad m \in M(u),\ n \in N(v),\ r \in R(p),\ \lambda p = \rho u,\ \rho p = \lambda v \rangle ,$$

so that all the R-action 'available' from each particular zero is divided out.

Should M be an S-R bimodule then $M \otimes_R N$ inherits a left S-module structure from the left map $\lambda : z\left(M \otimes_R N\right) \twoheadrightarrow \operatorname{Id} S$ given by $\lambda(u, v) = \lambda u$, and the S-action $s(m, n) = (sm, n)$. That this inheritance is natural when M is a bifunctor is evident from the adjoint properties of the tensor functor which will be described below. The tensor product associates:

$$L_S \otimes \left({}_S M_R \otimes {}_R N\right) \cong \left(L_S \otimes {}_S M_R\right) \otimes {}_R N \quad \text{in } \mathit{Abnet} \ ;$$

and satisfies $M \otimes_R R \cong M$ under the correspondence $(m, r) \mapsto mr$.

2.2.12 LEMMA. *For a ringnet morphism* $\sigma : R \to S$ *and a right* *S-module* M , $M \otimes_S S^\sigma \cong M^\sigma$ *in* R-*Mod* .

Proof.

$$zM^\sigma = \{(z, i) \in zM \times \mathrm{Id}\, R : \rho z = \sigma i\} .$$

$$z\left(M \otimes_S S^\sigma\right) = \{(z, p, i) \in zM \times zS \times \mathrm{Id}\, R : \rho z = \lambda p, \ \rho p = \sigma i\} / \langle (zs, p, i) \sim (z, sp, i) \rangle .$$

$$M^\sigma(z, i) = M(z) \times \{i\} .$$

$$\left(M \otimes_S S^\sigma\right)(\alpha) = \coprod_{(z,p,i)\in\alpha} M(z) \otimes S(p) \times \{i\} / \langle (ms, s', i) = (m, ss', i) \rangle .$$

The required isomorphism is described by

$$(m, s, i) \mapsto (ms, i)$$

with inverse

$$(m, i) \mapsto (m, \rho m, i) . \qquad \square$$

2.2.13 EXAMPLE. A ringnet R is defined to have: zeroes $zR = \mathbb{Z}$, the additive group of the integers; underlying abelian groupnet $R = \bigvee_{n\in\mathbb{Z}} R(n)$ with $R(0) = \mathbb{Z}$ and $R(n) = \{0\}$ for $n \neq 0$; and tensor multiplication given by multiplication of the integers. That is, for r_j in $R(j)$,

$$\psi : R(n) \otimes R(m) \to R(n+m)$$

is given by $\psi\left(r_n \otimes r'_m\right) = (rr')_{n+m}$. The identity set $\mathrm{Id}\, R$ of R consists of the multiplicative identity $1 \in R(0)$. This ringnet is *not* a ringoid. A *graded group* [31, p. 157] is a set of abelian groups $A = \{A_n : n \in \mathbb{Z}\}$ indexed by the integers. Let $zA = \mathbb{Z}$, $A(n) = A_n$ and $\lambda : zA \twoheadrightarrow \mathrm{Id}\, zR$ be $\lambda(n) = 0$ for n in $\mathbb{Z}$. With

$\phi : R(m) \otimes A(n) \to A(m+n)$ given by $\phi(r_m \otimes a) = ra$ for any pair (m, n), it follows that A is a left R-module. In fact it is an R-R bimodule. If B is another graded group, $A \otimes_R B$ is identifiable with the usual tensor product of these graded groups. For,

$$\begin{aligned} \mathrm{Id}(A \otimes_R B) &= \mathbb{Z} \times \mathbb{Z} \,/\, \langle (m+p, n) \sim (m, p+n),\ m, n, p \in \mathbb{Z} \rangle \\ &= \mathbb{Z} \times \mathbb{Z} \,/\, \langle (m, n) \sim (m+n, 0),\ m, n \in \mathbb{Z} \rangle \\ &\cong \mathbb{Z}, \end{aligned}$$

while

$$\begin{aligned} A \otimes_R B(\alpha) &= \coprod_{(n,m)\in\alpha} A(n) \otimes B(m) \,/\, \langle (ar, b) = (a, rb) \rangle \\ &= \coprod_{(n,m)\in\alpha} A(n) \otimes B(m) \end{aligned}$$

so that

$$A \otimes_R B(n) = \coprod_{i+j=n} A(i) \otimes B(j)$$

for each n in $\mathbb{Z}$. //

For a ringnet R, the *set of connected components* of $\mathrm{Id}\, R$ is a partition of $\mathrm{Id}\, R$ determined by the equivalence relation generated by

$$\lambda p \sim \rho p \quad \forall p \in zR .$$

The tensor product of regular modules M and N has

$$\mathrm{Id}(M \otimes_R N) = \{(i, j) \in \mathrm{Id}\, R \times \mathrm{Id}\, R : i = j\} / \langle (\lambda p, \lambda p) \sim (\rho p, \rho p),\ p \in zR \rangle$$

which is in one-to-one correspondence with the set of connected components of $\mathrm{Id}\, R$. In such cases the terminology

$$\mathrm{Id}(M \otimes_R N) = \mathrm{Id}\, R/\!\sim\, = \{[i_\kappa] : \kappa \in K\}$$

is used, when the connected components are indexed by K.

The tensor product in R-*Mod* is seen to determine a bifunctor $\otimes_R : R\text{-}Mod^r \times R\text{-}Mod^l \to Abnet$ which is covariant in both arguments. When restricted to standard R-modules,

$$\otimes_R : R\text{-}Mod^r(Y) \times R\text{-}Mod^l(Z) \to Abnet(Y \otimes_R Z)$$

is additive.

2.2.14 LEMMA. *Let* N *be a right* R*-module. The functor* $N \otimes_R$ *preserves arbitrary coproducts in* $R\text{-}Mod^l(Z)$.

Proof. In $Abnet(S)$, the canonical morphism

$$\coprod_j A_j \to \prod_j A_j$$

is a monomorphism for each $\{A_j : j \in J\}$. Since both $R\text{-}Mod^l(Z)$ and $Abnet(zN \otimes_R Z)$ are abelian categories admitting all products and coproducts (2.2.8), the canonical morphism

$$\coprod_\alpha (N \otimes_R M_\alpha) \to N \otimes_R \left(\coprod_\alpha M_\alpha\right)$$

is a monomorphism for each collection $\{M_\alpha : \alpha \in A\}$ of standard left R-modules over Z (Popescu [28, Ex. 3.1.5]). It is clearly an epimorphism.□

2.2.15 LEMMA. *Let* $M : R \otimes S^{op} \to Abnet$ *be a bifunctor. Then the functor* $M \otimes_R - : R\text{-}Mod^l \to S\text{-}Mod^l$ *has a right adjoint.*

Proof. Define a functor $\text{Hom}_S(M, -) : S\text{-}Mod^l \to R\text{-}Mod^l$ as follows.

For each i in $|R|$, $M(i) = M(i, -) : S^{op} \to Abnet$ is a right S-module. Let $N : S^{op} \to Abnet$ be a right S-module. A right R-module $\text{Hom}_S(M, N) : R^{op} \to Abnet$ is determined by

$$\text{Hom}_S(M, N)(i) = S\text{-}Mod^r(M(i), N) \quad \forall i \in |R|$$

and for each t in R and f in $S\text{-}Mod^r(M(\text{cod } t), N)$,

$$[\text{Hom}_S(M, N)(t)](f) = f \circ M(t) .$$

Since

$$f \circ M(t)(sm) = f((sm)t) = f(s(mt)) = sf(mt)$$

$$= s.f \circ M(t)(m) ,$$

$f \circ M(t) \in S\text{-}Mod^r(M(\text{dom } t), N)$, as required. This functor is right adjoint to $- \otimes_R M = M \otimes_R -$. The *Set* isomorphism

$$S\text{-}Mod^r(L \otimes_R M, N) \cong R\text{-}Mod^r(L, \mathrm{Hom}_S(M, N))$$

is given for any $L : R^{op} \to$ **Abnet** , $N : S^{op} \to$ **Abnet** , and S-morphism $f : M \otimes_R L \to N$ by the correspondence $f \mapsto f^* : L \dot{\to} \mathrm{Hom}_S(M, N)$, where for each i in $|R|$, $f^*(i) : L(i) \to \mathrm{Hom}_S(M(i), N)$ is the abelian groupnet morphism $f^*(i)(l) = f(- \otimes l)$. □

It is apparent from (2.2.15) that the functor right adjoint to the tensor product is not the usual hom functor, which would be a natural choice. Rather, it is a sort of 'internal hom functor' : if R is a commutative ringnet then the existence of this functor implies R-*Mod* is a closed category (*cf.* Mac Lane [20, VII.7]). For this reason, the straightforward extension from classical theory which has so far been generally employed must be used with caution when dealing with cohomology theory.

The set $\mathrm{hom}_R(M, N)$ of left R-module morphisms has an abelian groupnet structure (*cf.* (2.2.7.i)) distinguished as

(i) $\mathrm{Id}\ \mathrm{hom}_R(M, N) = \mathrm{hom}_R(zM, zN)$, and

(ii) $\mathrm{hom}_R(M, N)(h) = \{f \in \mathrm{hom}_R(M, N) : f|_{zM} = h\}$.

The hom sets thus define a bifunctor $\mathrm{hom}_R : (R\text{-}Mod)^{op} \times (R\text{-}Mod) \to$ **Abnet** , contravariant in the first argument and covariant in the second. When restricted to standard R-modules,

$$\mathrm{hom}_R : (R\text{-}Mod(Y))^{op} \times R\text{-}Mod(Z) \to \mathbf{Abnet}(\mathrm{hom}_R(Y, Z))$$

is additive.

2.2.16 LEMMA. *For a ringnet morphism* $\sigma : R \to S$ *and a left* R*-module* M *,*

$$\mathrm{hom}_S(S^\sigma \otimes_R M, L) \cong \mathrm{hom}_R(M, {}^\sigma L)$$

in **Abnet** *for any left* S*-module* L .

Proof. The map $\phi : \hom_S\left(S^\sigma \otimes_R M, L\right) \to \hom_R\left(M, {}^\sigma L\right)$ determined by $\phi(f)(m) = \left(\lambda m, f\left(\sigma(\lambda m), m\right)\right)$ for each left S-morphism $f : S^\sigma \otimes_R M \to L$ is an abelian groupnet morphism. Its inverse

$$\phi^{-1} : \hom_R\left(M, {}^\sigma L\right) \to \hom_S\left(S^\sigma \otimes_R M, L\right)$$

is defined as

$$\phi^{-1}(g)(s, m) = s.\sigma^* \circ g(m)$$

for each left R-morphism $g : M \to {}^\sigma L$. Here (s, m) denotes the element $\left((s, \lambda m), m\right)$ of $S^\sigma \otimes_R M$, provided that $\sigma(\lambda m) = \rho s$.

2.2.17 REMARK. In comparison with (2.2.14), let $\{M_\alpha : \alpha \in A\}$ be a set of standard left R-modules over Z and let L be any left R-module. Then the canonical isomorphisms

$$\hom_R\left(L, \prod_\alpha M_\alpha\right) \cong \prod_\alpha \hom_R\left(L, M_\alpha\right) \quad [29,\ 7.3.6],$$

and

$$\hom_R\left(\coprod_\alpha M_\alpha, L\right) \cong \prod_\alpha \hom_R\left(M_\alpha, L\right) \quad [29,\ 8.3.4]$$

in *Set* , preserve the abelian groupnet structure of the left hand side. //

The final definitions pertinent to this section are those of projectivity and freedom for standard modules. A standard R-module P over Z is *projective* precisely when in any diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle f} & & \\ M & \xrightarrow[g]{} & N & \xrightarrow[0]{} & Z \end{array}$$

in R-$Mod(Z)$, with an exact row, the morphism f may be factored through g . That is, there exists $\overline{f} : P \to M$ such that $g \circ \overline{f} = f$. For example, the trivial module TR , when defined, is a projective module in Id R-*Modreg*.

2.2.18 DEFINITION. If R is a ringnet, Z is a (left) R-module and X is a set diagram

$$X \xrightarrow{z} zX \xrightarrow{\lambda} \mathrm{Id}\, R \ ,$$

then X is said to *admit an R-action with respect to* Z if there is a set inclusion $zX \hookrightarrow Z$ such that

$$\begin{array}{ccc} zX & \hookrightarrow & Z \\ & \lambda \searrow \quad \swarrow \lambda & \\ & \mathrm{Id}\, R & \end{array} \quad .$$

2.2.19 DEFINITION. Let Z be a (left) zero R-module and $U : R\text{-}Mod(Z) \to Set$ be the forgetful functor. Let X be a set diagram $X \xrightarrow{z} zX \xrightarrow{\lambda} \mathrm{Id}\, R$. A standard R-module M over Z is *free with basis* X if

(i) X is a subset of UM,

(ii) $zX = \{z \in Z : z = zx,\ x \in X\}$ and

(iii) for any N in $|R\text{-}Mod(Z)|$ and Set diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & UN \\ z \downarrow & & \downarrow Uz \\ zX & \xrightarrow{zg} & UzN \\ & \lambda \searrow \quad \swarrow \lambda & \\ & \mathrm{Id}\, R & \end{array} \qquad \text{(D2.2.1)}$$

there is a unique extension $\overline{g} : M \to N$ of g in $R\text{-}Mod(Z)$. If M is free with basis X, X admits an R-action with respect to Z. In the event that $R = \mathbb{Z}A$ for a group A, and Z is the zero A-module $\{0\}$, a standard free A-module with basis X is precisely the 'classical' free A-module with basis X. Free modules may be described internally by the following construction. Suppose $R \in |Rngnet|$ and the Set diagram $Y = Y \xrightarrow{z} zY \xrightarrow{\lambda} \mathrm{Id}\, R$ admits an R-action with respect to the left zero

R-module Z . The standard left R-module FY over Z is defined by

(i) $FY(z) = \coprod_{\substack{p \in zR \\ y \in Y \\ p.zy = z}} R(p) \times \{y\} \quad \forall z \in Z$, and

(ii) R-action $r^*(r, y) = (r^*r, y)$ defined whenever $(r, y) \in FY(z)$ and $\rho r^* = \lambda z \ (= \lambda r)$.

Then FY is free on basis $\overline{\overline{Y}}$, where $\overline{\overline{Y}}$ is defined from the *Set* isomorphism $Y \cong \overline{\overline{Y}}$ given by $y \mapsto (1_{\lambda zy}, y)$. There is an isomorphism from the free R-module M with basis X to the free R-module FX , given by the unique extension of the set diagram isomorphism $X \to \overline{\overline{X}}$. By convention, Z is considered to be the free standard module over Z with empty basis. Clearly R is itself free over zR , on basis

$$\mathrm{Id}\, R \xrightarrow{=} \mathrm{Id}\, R \xrightarrow{=} \mathrm{Id}\, R \ .$$

The structural requirements of each generating diagram X and each set diagram morphism (D2.2.1) imply that each set X may determine more than one free module over Z . Hence a free module in R-*Mod*(Z) is not necessarily a free object of that category [15, II.10]. Despite the non-categorical definition of standard free modules, such objects of R-*Mod*(Z) do have most of the properties associated with free objects [15, II.10]. For example, proof of the following lemma involves straightforward checking, and is only sketched.

2.2.20 LEMMA. *(i) Any free module in* R-Mod(Z) *is a projective object.*

(ii) Any module in R-Mod(Z) *is the epimorphic image of a free module.*

(iii) Any projective object in R-Mod(Z) *is a direct summand of a free module.*

For, each module M in R-*Mod*(Z) determines the set diagram $UM = UM \xrightarrow{Uz} UZ \xrightarrow{\lambda} \mathrm{Id}\, R$ which admits an R-action with respect to Z , and thus defines the free module FUM . There is a short exact sequence

$$\mathrm{Ker}\,\pi \rightarrowtail FUM \xrightarrow{\ \pi\ } \twoheadrightarrow M$$

which splits when M is projective. In this case, since $R\text{-}Mod(Z)$ is abelian, $FUM \cong \mathrm{Ker}\,\pi \oplus M$ [29, 13.2.4]. //

It is possible to generalise the notion of free module to $R\text{-}Mod$.

2.2.21 DEFINITION. Let $U : R\text{-}Mod \to Set$ be the forgetful functor and let $\mathsf{X} = X \twoheadrightarrow zX \to \mathrm{Id}\,R$ be a set diagram. A (left) R-module M is *especially free with basis* X if

(i) X is a subset of UM,

(ii) $zX = \{z \in zM : z = zx,\ x \in X\}$ and zX *freely generates* zM; that is, each z in zM is uniquely of the form $p.zx$ for p in zR and zx in zX, and

(iii) for any N in $|R\text{-}Mod|$ and any set diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & UN \\ \downarrow z & & \downarrow z \\ zX & \xrightarrow{\quad zg \quad} & UzN \\ & {\scriptstyle\lambda}\searrow \quad \swarrow{\scriptstyle\lambda} & \\ & \mathrm{Id}\,R & \end{array} \qquad , \qquad \text{(D2.2.2)}$$

there is a unique extension $\bar{g} : M \to N$ of g in $R\text{-}Mod$.

Certain especially free modules may be described internally by the following construction.

For any Set diagram $\mathsf{Y} = Y \xrightarrow{\ z\ } \twoheadrightarrow zY \xrightarrow{\ \lambda\ } \mathrm{Id}\,R$, the triple $(\hat{F}\mathsf{Y},\ z\hat{F}\mathsf{Y},\ \lambda)$ is defined by

(i) $z\hat{F}\mathsf{Y} = \{(p, z) \in zR \times zY : \rho p = \lambda z\}$,

(ii) $\hat{F}\mathsf{Y}(p, z) = \coprod_{\substack{y \in Y \\ zy = z}} R(p) \times \{y\} \quad \forall (p, z) \in z\hat{F}\mathsf{Y}$, and

(iii) $\lambda : z\hat{F}\mathsf{Y} \to \mathrm{Id}\,R$ is $\lambda(p, z) = \lambda p$.

If λ should be surjective, then $\hat{F}\mathsf{Y}$ is an R-module, with R-action $r^*(r, y) = (r^*r, y)$ defined whenever $(r, y) \in \hat{F}\mathsf{Y}(p, z)$ and $\rho r^* = \lambda r$.

In this case, $\hat{F}Y$ is especially free on the basis $\overline{Y}$, determined from the isomorphism $Y \cong \overline{\overline{Y}}$ in *Set* under the correspondence $y \mapsto (1_{\lambda z y}, y)$. If M is the especially free module on basis X and $\hat{F}X$ is an R-module, there is an isomorphism $M \to \hat{F}X$, given by the unique extension of the set diagram isomorphism $X \to \overline{X}$. Again, R is itself especially free with basis $\mathrm{Id}\, R \xrightarrow{=} \mathrm{Id}\, R \xrightarrow{=} \mathrm{Id}\, R$. If $\hat{F}X$ is the especially free R-module determined by

$$X \twoheadrightarrow zX \to \mathrm{Id}\, R ,$$

the inclusion map $z\overline{\overline{X}} \to z\hat{F}X$ ensures that $\overline{\overline{X}}$ admits an R-action with respect to $z\hat{F}X$. Hence $\hat{F}X$ is free with basis $\overline{\overline{X}}$ in R-*Mod*$(z\hat{F}X)$. //

The closing example of this chapter incorporates several of the above definitions.

2.2.22 LEMMA. *If* $m : B \rightarrowtail A$ *is a monomorphism in* Gpnet *, then the (right) pullback* $\mathbb{Z}A^m$ *of* A *along* $m : \mathbb{Z}B \rightarrowtail \mathbb{Z}A$ *is a free (right) B-module.*

Proof. From (2.2.9), $\mathbb{Z}A^m$ is a right B-module, with

$$z\mathbb{Z}A^m = \{((i, m(j)), j) \in z\mathbb{Z}A \times \mathrm{Id}\, B\} .$$

For convenience, $((i, m(j)), j)$ is contracted to $(i, m(j))$, $\mathbb{Z}A^m(i, m(j)) = \mathbb{Z}A(i, m(j)) \times \{j\}$ is written $\mathbb{Z}A(i, m(j))$, and the induced B-action $a.b$ equals $am(b)$. For each connected component B_κ of B , for κ in K , a distinguished identity i_κ of B_κ is selected. Define for each j in $\mathrm{Id}\, A$, κ in K , and a in $A(j, m(i_\kappa))$, the left coset

$$am(B_\kappa) = \{am(b) : b \in B_\kappa, \lambda b = i_\kappa\} .$$

Choice of a set of coset representatives

$$X = \{a_{\beta(\kappa)_j} \in A(j, m(i_\kappa)) : j \in \mathrm{Id}\, A, \kappa \in K, \beta(\kappa)_j \in Y^\kappa_j\} ,$$

one for each distinct coset

$$a_{\beta(\kappa)_j} m(B_\kappa) ,$$

gives for each a in $A(j, m(l))$ a unique representation

$$a = a_{\beta(\kappa)_j} m(b\langle a\rangle) ,$$

where $l \sim i_\kappa$ in $\mathrm{Id}\, B$, and $b\langle a\rangle \in B(i_\kappa, l)$. Further, $b\langle a.b^*\rangle = b\langle a\rangle b^*$ in B. With

$$zX = \{(j, m(i_\kappa)) \in z\mathbb{Z}A : \kappa \in K\} ,$$

$$z : X \twoheadrightarrow zX \text{ as } z(a_{\beta(\kappa)_j}) = (j, m(i_\kappa)) ,$$

and

$$\rho : zX \to \mathrm{Id}\, B \text{ as } \rho(j, m(i_\kappa)) = i_\kappa ,$$

the triple $(\hat{F}X, z\hat{F}X, \rho)$ of (2.2.21) has

$$z\hat{F}X = \{((j, m(i_\kappa)), (i_\kappa, l)) \in zX \times z\mathbb{Z}B\}$$

and

$$\hat{F}X((j, m(i_\kappa)), (i_\kappa, l)) = \coprod_{\beta(\kappa)_j \in Y_j^\kappa} \{a_{\beta(\kappa)_j}\} \times \mathbb{Z}B(i_\kappa, l) .$$

Since for every l in $\mathrm{Id}\, B$, $l \sim i_\kappa$ for some (unique) κ in K, $\rho : z\hat{F}X \to \mathrm{Id}\, B$ is surjective. Thus $\hat{F}X$ is especially free on basis $\overline{X}$. The isomorphism $\hat{F}X \to \mathbb{Z}A^m$ is given by $(a_{\beta(\kappa)_j}, b) \mapsto a_{\beta(\kappa)_j} m(b)$; its inverse is the map $a \mapsto (a_{\beta(\kappa)_j}, b\langle a\rangle)$. □

CHAPTER 3

THE MAPPING CYLINDER

This chapter describes the construction of a complex for a mapping cylinder groupnet in terms of complexes given for its vertex groupnets and the edge maps between them. It thus paves the way for a comparison of the homology of a mapping cylinder groupnet with that of its vertex groupnets. The emphasis here is on the concrete and algebraic aspects of module theory rather than the abstract and categorical ones.

3.1 Complexes

3.1.1 DEFINITION. If R is a ringnet, a *(left) R-chain complex* (C, ∂) consists of a set $C = \{C_n : n \in Z\}$ of (left) R-modules C_n *in dimension* n and a set $\partial = \{\partial_n : n \in Z\}$ of R-morphisms $\partial_n : C_n \to C_{n-1}$, called the *boundary maps*, such that for all n in Z and z in zC_n,

$$\partial_{n-1}\partial_n = 0 : C_n(z) \to C_{n-1}(\partial\partial z) .$$

Right R-complexes are correspondingly defined, and complex (C, ∂) is an *S-R bicomplex* when C_n is an S-R bimodule and ∂_n is an S-R morphism for all n in Z. Should each boundary map be a zero morphism the complex is known as a *graded module.*

When Z is a zero R-module (C, ∂) is a *standard R-complex over* Z if

(i) $C_n \in |R\text{-}Mod(Z)|$ $\forall n \in Z$, and

(ii) $\partial_n \in R\text{-}Mod(Z)$ $\forall n \in Z$.

In such a case, necessarily $\operatorname{Im} \partial_{n+1}(z) \subset \operatorname{Ker} \partial_n(z)$ for each z in Z. Often, zC will denote the zero set Z common to the modules of (C, ∂). A standard R-complex over $\operatorname{Id} R$ is, of course, *regular.*

Generally, the boundary maps of different complexes will be denoted by the same symbol ∂ , and dimensional subscripts will be suppressed when there is no likelihood of ambiguity or when a statement is true for all dimensions.

Given a ringnet morphism $\sigma : R \to S$ and a (left) S-complex (D, ∂) , then a *σ-chain map (σ-complex morphism)* $f : C \to D$ is a set $f = \{f_n : n \in Z\}$ of σ-morphisms $f_n : C_n \to D_n$ which commute with the boundary maps; that is,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots \\ & & \Big\downarrow f_{n+1} & & \Big\downarrow f_n & & \Big\downarrow f_{n-1} & & \\ \cdots & \longrightarrow & D_{n+1} & \longrightarrow & D_n & \longrightarrow & D_{n-1} & \longrightarrow & \cdots \end{array}$$

If σ is the identity morphism on R , a σ-chain map is an *R-chain map*. An R-chain map $f : C \to D$ is *standard over* Z when both C and D are standard complexes over Z and $f_n \in R\text{-}Mod(Z)(C_n, D_n)$ for each n in Z .

The composition of a σ-chain map with a τ-chain map is a $(\tau \circ \sigma)$-chain map; the category of chain complexes and chain maps with this composition is named *Comp* . Restriction to the (left) R-complexes and R-chain maps for each ringnet R determines the subcategory *R-Comp* . For each zero R-module Z , the standard R-complexes and chain maps over Z in *R-Comp* form the subcategory *R-Comp*(Z) . This last category is abelian and has arbitrary direct products and coproducts, by obvious extension of the respective results (2.2.7, 2.2.8) for *R-Mod*(Z) . Category *R-Compreg* = *R-Comp*(Id R) is the category of *regular* R-complexes and chain maps. A standard R-chain complex is *exact* when $\operatorname{Ker} \partial_n = \operatorname{Im} \partial_{n+1}$ for all n in Z ; that is, when

$$\operatorname{Ker} \partial_n(z) = \operatorname{Im} \partial_{n+1}(z) \quad \forall n \in Z \,, \quad z \in zC \,.$$

3.1.2 EXAMPLES. (i) Any R-module M may be identified with the standard R-complex

$$zM \hookrightarrow M \twoheadrightarrow zM$$

over zM, where M conventionally lies in dimension 0.

(ii) The following 'unit interval' complex $\mathcal{E}$ serves in the homotopy theory for complexes in a fashion comparable to the way ZI, I and $[0, 1]$ serve for ringnets, groupnets and topological spaces respectively. For each ringnet R, consider the complex R in $R\text{-}Comp(zR)$:

$$\mathsf{R} = zR \rightarrowtail \mathsf{R}_1 \xrightarrow[\partial_1]{} \mathsf{R}_0 \twoheadrightarrow zR .$$

Here $\mathsf{R}_1 \cong R$ is the free (right) R-module over zR with basis

$$\{\gamma\} \times \text{Id } R \xrightarrow{z} \text{Id } R \xrightarrow{=} \text{Id } R$$

where $z(\gamma, i) = i$ for i in $\text{Id } R$;

$\mathsf{R}_0 \cong R \oplus R$ is the free (right) R-module over zR with basis

$$\{\alpha, \beta\} \times \text{Id } R \xrightarrow{z} \text{Id } R \xrightarrow{=} \text{Id } R$$

where $z(\alpha, i) = z(\beta, i) = i$ for i in $\text{Id } R$;

and $\partial_1 : \mathsf{R}_1 \to \mathsf{R}_0$ is the *skew-diagonal map*

$$\partial(\gamma, r) = (\alpha, r) - (\beta, r) \text{ for } r \text{ in } R .$$

The *unit* R-*complex* $\mathcal{E}$ in $R\text{-}Comp$ is defined as

$$\mathcal{E} = TI \otimes_Z \mathsf{R} ;$$

that is, the image under the tensor functor $TI \otimes_Z -$ of the $R\text{-}Mod(zR)$-diagram R. It has the form (*cf.* (2.2.6))

$$\mathcal{E}_n = z\mathcal{E} = \{0\} \times zR \vee \{1\} \times zR , \quad n \neq 0, 1 ,$$

$$\mathcal{E}_0(i, p) = \{(i, \alpha)\} \times R(p) \oplus \{(i, \beta)\} \times R(p) ,$$

and

$$\mathcal{E}_1(i, p) = \{(i, \gamma)\} \times R(p) \text{ for } (i, p) \in z\mathcal{E} ,$$

with $\partial_1(i, \gamma, r) = (i, \alpha, r) - (i, \beta, r)$ for i in $\text{Id } I$ and r in R, and inherits the right R-module structure of R. In fact, $\mathcal{E}_1$ is the free right $Z(\text{Id } I) \otimes R$-module over $z\mathcal{E}$ with basis

$$\text{Id } I \times \{\gamma\} \times \text{Id } R \xrightarrow{1\times z} \text{Id } I \times \text{Id } R \xrightarrow{=} \text{Id } I \times \text{Id } R \; ;$$

similarly, $\mathcal{E}_0$ is the free right $Z(\text{Id } I) \otimes R$-module over $z\mathcal{E}$ with basis

$$\text{Id } I \times \{\alpha, \beta\} \times \text{Id } R \xrightarrow{1\times z} \text{Id } I \times \text{Id } R \xrightarrow{=} \text{Id } I \times \text{Id } R \; .$$

The unit R-complex is a left $ZI \otimes R$-complex with left map $1 \times \lambda : z\mathcal{E} = \text{Id } I \times zR \twoheadrightarrow \text{Id } I \times \text{Id } R$ and left action defined by coordinate:

$$([i], r).(\rho i, \xi, r^*) = (\lambda i, \xi, rr^*)$$

for i in I, rr^* in R and ξ in $\{\alpha, \beta, \gamma\}$. In particular,

$$([*], i).(1, \xi, i) = (0, \xi, i)$$

for each i in Id R. With this in mind, we denote the generators of $\mathcal{E}$ as $(1, \alpha, i) = (\mathsf{a}, i)$, $(1, \beta, i) = (\mathsf{b}, i)$, $(1, \gamma, i) = (\mathsf{c}, i)$, $(0, \alpha, i) = (*\mathsf{a}, i)$, $(0, \beta, i) = (*\mathsf{b}, i)$ and $(0, \gamma, i) = (*\mathsf{c}, i)$ for each i in Id R. //

3.1.3 DEFINITIONS. (i) If M is an R-module, a *complex over* M is a standard R-complex (C, ∂) over zM which is *positive* (that is, $C_n = zC$ for $n < 0$), together with a standard R-chain map $\varepsilon : C \to M$. It may thus be thought of as a standard chain complex

$$\ldots \to C_n \to \ldots \to C_1 \to C_0 \xrightarrow{\varepsilon} M \to zM \tag{D3.1.1}$$

with *augmentation map* $\varepsilon : C_0 \to M$.

(ii) A standard R-complex (C, ∂) over Z is a *resolution of the R-module* M, if it is firstly a complex over M, and if secondly the augmented chain complex (D3.1.1) is exact.

(iii) An R-complex is *especially free* if it is especially free (2.2.21) in every dimension. A standard R-complex is *projective (free)* if it is projective (free) in every dimension. A standard resolution (C, ∂) of the R-module M is *projective (free)* if C itself is projective (free). //

For any groupnet A, there exists a free resolution of the trivial A-module TA.

3.1.4 DEFINITION. Let A be a groupnet. The *bar resolution* $\mathcal{B} = \mathcal{B}A$ for A is given by setting

(i) $X_0 = \{[i] : i \in \mathrm{Id}\, A\}$,

$$X_n = \{[a_1 \mid \cdots \mid a_n] : a_i \in A, 1 \le i \le n; \rho a_i = \lambda a_{i+1}, 1 \le i < n\}$$

for $n > 0$;

(ii) $zX_n = \mathrm{Id}\, A \quad \forall n \ge 0$;

(iii) $z : X_n \twoheadrightarrow zX_n$ as

$$z[i] = i \ , \ n = 0 \ ,$$

and

$$z[a_1 \mid \cdots \mid a_n] = \lambda a_1 \quad \forall n > 0 \ ;$$

(iv) $\mathcal{B}_n = FX_n$, the regular free A-module on the set diagram

$$X_n \xrightarrow{z} zX_n \xrightarrow{=} \mathrm{Id}\, A \ ;$$

and

(v) $\partial_n : \mathcal{B}_n \to \mathcal{B}_{n-1}$ as

$$\partial_1[a] = a[\rho a] - [\lambda a] \ ,$$

and

$$\partial_n[a_1 \mid \cdots \mid a_n] = a_1[a_2 \mid \cdots \mid a_n]$$
$$+ \sum_{i=1}^{n-1} (-1)^i [a_1 \mid \cdots \mid a_i a_{i+1} \mid \cdots \mid a_n]$$
$$+ (-1)^n [a_1 \mid \cdots \mid a_{n-1}] \quad \text{for } n \ge 2 .$$

Routine calculation shows $\mathcal{B}$ is a regular positive complex which is free by definition. The augmentation map $\varepsilon : \mathcal{B}_0 \to TA$ is extended by A-action from

$$\varepsilon[i] = 1[i] \quad \text{for } i \text{ in } \mathrm{Id}\, A \ ,$$

and then

$$\varepsilon\partial[a] = \varepsilon(a[\rho a]-[\lambda a]) = a.[\rho a] - [\lambda a] = [\lambda a] - [\lambda a] = 0[\lambda a]$$

as required. Proof that the augmented complex is exact is deferred until (3.2.7).

Any groupnet morphism $f : A \to B$ induces an f-chain map $\mathcal{B}f$ of augmented bar resolutions, *viz.* $\mathcal{B}f_{-1} : TA \to TB$ is linearly extended from

$$\mathcal{B}f_{-1}[i] = [fi] \quad \text{for } i \text{ in } \operatorname{Id} A ,$$

$\mathcal{B}f_0 : \mathcal{B}A_0 \to \mathcal{B}B_0$ is extended to an f-morphism from

$$\mathcal{B}f_0[i] = [fi] \quad \text{for } i \text{ in } \operatorname{Id} A ,$$

and $\mathcal{B}f_n : \mathcal{B}A_n \to \mathcal{B}B_n$ is extended to an f-morphism from

$$\mathcal{B}f_n[a_1 \mid \ldots \mid a_n] = [fa_1 \mid \ldots \mid fa_n] \quad \text{for } [a_1 \mid \ldots \mid a_n]$$

in X_n and $n \geq 1$. //

The technique of changing rings in **Comp** is next investigated: it is the obvious pullback.

3.1.5 DEFINITION. Let $\sigma : R \to S$ be a ringnet morphism and C be a (left) S-complex. The *pullback* ${}^{\sigma}C$ *of* C *along* σ is the (left) R-complex

(i) $({}^{\sigma}C)_n = {}^{\sigma}(C_n)$, and

(ii) $({}^{\sigma}\partial)_n = {}^{\sigma}(\partial_n) : (i, c) \mapsto (i, \partial c)$ for each (i, c) in ${}^{\sigma}C_n$ and n in $\mathbb{Z}$.

Immediately, there is a canonical pullback projection $\sigma^* : {}^{\sigma}C \to C$ which is a σ-chain map, and any σ-chain map $f : D \to C$ factors uniquely (*via* $\sigma(f)$) through σ^* .

3.1.6 LEMMA. *(i) The pullback of a standard (regular) complex is standard (regular).*

(ii) The pullback of a standard exact complex is exact.

(iii) The pullback ${}^{\sigma}C$ *of a resolution* C *of* M *is a resolution of* ${}^{\sigma}M$.

Proof. Let $\sigma : R \to S$ be a ringnet morphism and C be an S-complex.

(i) $C \in |S\text{-}Comp(Z)| \Rightarrow {}^{\sigma}C \in |R\text{-}Comp({}^{\sigma}Z)|$. If $Z = \text{Id}\, S$ then

$${}^{\sigma}Z = \{(i, j) \in \text{Id}\, R \times \text{Id}\, S : \sigma(i) = j\} \cong \text{Id}\, R .$$

(ii) If $(i, c) \in \text{Ker}\, {}^{\sigma}\partial_n(i, z)$, then $zc = z$ and

$${}^{\sigma}\partial_n(i, c) = (i, \partial c) = z(i, c) = (i, z)$$

so there exists c^* in C_{n+1} such that $\partial_{n+1}c^* = c$. Hence ${}^{\sigma}\partial_{n+1}(i, c^*) = (i, c)$.

(iii) Certainly ${}^{\sigma}C$ is a complex over ${}^{\sigma}M$; its augmentation map is the pullback ${}^{\sigma}\varepsilon$ of $\varepsilon : C_0 \to M$. By (ii) the augmented pullback complex is exact. □

3.1.7 COROLLARY. *If* $\sigma : A \to B$ *is a groupnet morphism and* C *is a resolution of* TB *then* ${}^{\sigma}C$ *is a resolution of* TA .

Proof. By (2.2.10), ${}^{\sigma}TB \cong TA$. □ //

3.1.8 DEFINITION. If C and D are respectively right and left standard R-complexes they have a *tensor product* $C \otimes_R D$ which is a standard Z-complex over $zC \otimes_R zD$, and extends the definition of the tensor product of complexes over a ring in an obvious manner. That is,

$$z(C \otimes_R D) = zC \otimes_R zD ,$$

$$(C \otimes_R D)_n(z) = \coprod_{i+j=n} (C_i \otimes_R D_j)(z) \quad \forall z \in z(C \otimes_R D) ,$$

and the boundary map $\partial_n : (C \otimes_R D)_n \to (C \otimes_R D)_{n-1}$ is the tensor extension of

$$\partial_n(c_i \otimes d_j) = (\partial_i c_i) \otimes d_j + (-1)^i c_i \otimes (\partial_j d_j)$$

whenever $i + j = n$.

Clearly, if C here is also a (standard) S-R bicomplex, then $C \otimes_R D$

inherits a left S-complex structure from the action in each dimension on each direct summand.

3.1.9 EXAMPLE. Let (C, ∂) be a standard left R-complex and M be a right R-module. Consideration of (3.1.2.i) determines a standard $\mathbb{Z}$-complex $M \otimes_R C$ with

$$z(M \otimes_R C) = zM \otimes_R zC ,$$

$$(M \otimes_R C)_n = M \otimes_R C_n ,$$

and boundary map $1_M \otimes \partial$. Alternatively, this complex may be considered as the image of C under the covariant additive functor

$$M \otimes_R - : R\text{-}Comp(zC) \to \mathbb{Z}\text{-}Comp(z(M \otimes_R C)) .$$

3.1.10 REMARK. Functor $M \otimes_R -$ is right exact. Since M is a bifunctor $R \otimes \mathbb{Z}^{op} \to Abnet$, it follows from (2.2.15) that $M \otimes_R -$ has a right adjoint. By the dual of [15, II.7.7], the functor preserves cokernels.

3.1.11 DEFINITION. Any two standard (left) R-complexes C and D determine a hom *complex* $\hom_R(C, D)$ which is the standard $\mathbb{Z}$-complex with

$$z \hom_R(C, D) = \hom_R(zC, zD) ,$$

$$\hom_R(C, D)_n(z) = \coprod_{i+j=n} \hom_R(C_i, D_j)(z)$$

for each $z \in z \hom_R(C, D)$. Its boundary map δ is conventionally written with a superscripted dimension to indicate contravariance, and is given by composition on the right. That is, for $f : C_i \to D_j$ and $i + j = n$, $\delta^n : \hom_R(C, D)_n \to \hom_R(C, D)_{n+1}$ is evaluated as $\delta^n f = f \circ \partial_{n+1}$. In this circumstance δ is called the *coboundary map*.

3.1.12 EXAMPLE. For a standard (left) R-complex C and a (left) R-module N, the hom complex $\hom_R(C, N)$ is given, by use of (3.1.2.i), as

$$z \hom_R(C, N) = \hom_R(zC, zN) ,$$

and

$$\hom_R(C, N)_n = \hom_R(C_n, N) ,$$

with coboundary map $\delta^n : f \longmapsto f\partial_{n+1}$ for $f : C_n \to N$. It may also be considered as the image of C under the contravariant additive functor $\hom_R(-, N) : R\text{-}Comp(zC) \to \mathbb{Z}\text{-}Comp(z \hom_R(C, N))$.

3.1.13 REMARK. For any standard R-module N over Z, the functor $\hom_R(-, N)$ is left exact on $R\text{-}Mod(Z)$, since $R\text{-}Mod(Z)$ is abelian (see Popescu [28, 3.2.2]). Moreover, this result holds for any R-module N.

3.1.14 LEMMA. *If C is a left standard R-complex then $\mathcal{E} \otimes_R C$ is a left standard $\mathbb{Z}I \otimes R$-complex.*

Proof. Since $\mathcal{E}$ is a $\mathbb{Z}I \otimes R\text{-}R$ bimodule (3.1.2.ii), the left $\mathbb{Z}I \otimes R$-module structure is inherited by the tensored complex. □

Because $R \otimes_R C_n \cong C_n$ (*cf.* (2.2.11)), the tensor complex $\mathcal{E} \otimes_R C$ will be written henceforth as:

$$z(\mathcal{E} \otimes_R C) = \text{Id } I \times zC ,$$

$$(\mathcal{E} \otimes_R C)_n(1, z) = \{a\} \times C_n(z) \oplus \{b\} \times C_n(z) \oplus \{c\} \times C_{n-1}(z) ,$$

and

$$(\mathcal{E} \otimes_R C)_n(0, z) = \{*a\} \times C_n(z) \oplus \{*b\} \times C_n(z) \oplus \{*c\} \times C_{n-1}(z)$$

for each z in zC. The boundary map is thus

$$\partial_n\{(a, c)+(b, c')+(c, c'')\} = (a, \partial c+c'') + (b, \partial c'-c'') - (c, \partial c'')$$

on $(\mathcal{E} \otimes_R C)_n(1, z)$ and

$$\partial_n\{(*a, c)+(*b, c')+(*c, c'')\} = (*a, \partial c+c'') + (*b, \partial c'-c'') - (*c, \partial c'')$$

on $(\mathcal{E} \otimes_R C)_n(0, z)$, for each z in zC and n in $\mathbb{Z}$. When C is a (classical) chain complex over a ring, $\mathcal{E} \otimes C$ reduces to two copies of the algebraic mapping cylinder of the identity map $C \to C$ (Takasu [39, §1]).

3.2 Homotopy

It is now possible to present two equivalent approaches to the notion of homotopy (the 'deformable equality' of two chain maps between standard complexes). Both definitions are subject to the proviso that the images of the zeroes under the second map are acted on by certain specific ringnet elements to give the images of the zeroes under the first map. This condition is automatically satisfied for regular chain complexes, including complexes over a ring. The first definition - complex homotopy - uses the unit complex in a way comparable to the use of [0, 1], I and ZI in definitions of homotopy in *Top*, *Gpnet* and *Rngnet* respectively. The second - chain homotopy - is an extension of the classical definition for complexes over a ring. Homotopic objects in R-*Comp*(Z) will be shown to have the same homology: it is for this reason such homotopy classes of complexes are defined.

3.2.1 DEFINITION. Let C be a standard R-complex, D be a standard S-complex and suppose $\sigma, \tau : R \to S$ are homotopic ringnet morphisms with homotopy $\nu : \sigma \simeq \tau : ZI \otimes R \to S$. If $f, g : C \to D$ are respectively σ, τ-chain maps such that

$$f(z) = \nu([*], \lambda z)g(z) \quad \forall z \in zC ,$$

then a ν-*complex homotopy* $F : f \simeq g$ *between* f *and* g is a ν-chain map $F : \mathcal{E} \otimes_R C \to D$ satisfying

(i) $F_n(*a, c) = f_n(c) \quad \forall c \in C_n , \quad n \in Z$ and

(ii) $F_n(b, c) = g_n(c) \quad \forall c \in C_n , \quad n \in Z$.

It is thus completely determined by f, g and

$$\{F_n(*c, c) : c \in C_{n-1}, n \in Z\} .$$

3.2.2 DEFINITION. Let C be a standard R-complex, D be a standard S-complex and suppose $\sigma, \tau : R \to S$ are homotopic ringnet morphisms with homotopy $\nu : \sigma \simeq \tau : ZI \otimes R \to S$. If $f, g : C \to D$ are respectively σ, τ-chain maps such that

$$f(z) = \nu([*], \lambda z)g(z) \quad \forall z \in zC ,$$

then a *ν-chain homotopy* $G : f \simeq g$ *between* f *and* g is a set $G = \{G_n : n \in Z\}$ of σ-chain morphisms

$$G_n : C_n \to D_{n+1} ,$$

satisfying

(i) $G_n(z) = f(z) \quad \forall z \in zC$, and

(ii) $(\partial G + G\partial)(c) = f(c) - \nu([*], \lambda c)g(c) \quad \forall c \in C_n , \quad n \in Z$.

For regular complexes C and D , the condition required on the zeroes by (2.2.1) and (2.2.2) is automatically satisfied, since $f|_{\mathrm{Id}R} = \sigma$, $g|_{\mathrm{Id}R} = \tau$, $\lambda\nu([*], i) = \sigma(i)$ and $\rho\nu([*], i) = \tau(i)$ for every i in Id R .

Chain homotopy for complexes over ringnets extends the definition for complexes over a unital ring in a straightforward manner. Suppose $\chi(1) : ZI \otimes K \to K$ is the constant ringnet homotopy determined by the identity *Rng* morphism $1 : K \to K$, so that $\chi(1)([*], 1) = 1$. If $f, g : C \to D$ are chain maps between the necessarily regular K-complexes C and D , then

$$g(c) = \chi(1)([*], 1)g(c) \quad \forall c \in C_n , \quad n \in Z .$$

By definition, f is chain homotopic to g if and only if it is $\chi(1)$-chain homotopic to g .

3.2.3 THEOREM. *Let* C *be a standard* R*-complex,* D *be a standard* S*-complex and suppose* $\sigma, \tau : R \to S$ *are homotopic ringnet morphisms with homotopy* $\nu : \sigma \simeq \tau : ZI \otimes R \to S$. *If* $f, g : C \to D$ *are respectively* σ, τ*-chain maps such that*

$$f(z) = \nu([*], \lambda z)g(z) \quad \forall z \in zC ,$$

then f *is* ν*-complex homotopic to* g *if and only if* f *is* ν*-chain homotopic to* g .

Proof. (i) Suppose $G = \{G_n : n \in \mathbb{Z}\}$ is a ν-chain homotopy between f and g . Generate $F : \mathcal{E} \otimes C \to D$ by ν-action from

$$F(0, z) = f(z) \quad \forall z \in zC ,$$

$$F(1, z) = g(z) \quad \forall z \in zC ,$$

$$F_n(*\mathrm{a}, c) = f_n(c) \quad \forall c \in C_n , \quad n \in \mathbb{Z} ,$$

$$F_n(\mathrm{b}, c) = g_n(c) \quad \forall c \in C_n , \quad n \in \mathbb{Z} ,$$

and

$$F_n(*\mathrm{c}, c) = G_{n-1}(c) \quad \forall c \in C_{n-1} , \quad n \in \mathbb{Z} .$$

Then

$$\begin{aligned} \partial F_n(*\mathrm{c}, c) &= \partial G_{n-1}(c) \\ &= f_{n-1}(c) - \nu([*], \lambda c) g_{n-1}(c) - G_{n-2}\partial(c) \\ &= F_{n-1}(*\mathrm{a}, c) - F_{n-1}(*\mathrm{b}, c) - F_{n-1}(*\mathrm{c}, \partial c) \\ &= F_{n-1}\partial(*\mathrm{c}, c) \quad \forall c \in C_{n-1} , \quad n \in \mathbb{Z} , \end{aligned}$$

and F is a ν-chain map.

(ii) Suppose $F : \mathcal{E} \otimes C \to D$ is a ν-complex homotopy between f and g . Define $G_n : C_n \to D_{n+1}$ to be

$$G_n(c) = F_{n+1}(*\mathrm{c}, c) \quad \forall c \in C_n , \quad n \in \mathbb{Z} .$$

Then

$$\begin{aligned} G_n(rc) = F_{n+1}(*\mathrm{c}, rc) &= F_{n+1}\big(([0], r).(*\mathrm{c}, c)\big) \\ &= \nu([0], r) F_{n+1}(*\mathrm{c}, c) = \sigma(r) G_n(c) , \end{aligned}$$

and G_n is a σ-morphism. Finally, for all c in C_n ,

$$\left(\partial G_n + G_{n-1}\partial\right)(c) = \partial F_{n+1}(*\mathrm{c},\ c) + F_n(*\mathrm{c},\ \partial c)$$

$$= F_n\partial(*\mathrm{c},\ c) + F_n(*\mathrm{c},\ \partial c)$$

$$= F_n(*\mathrm{a},\ c) - F_n(*\mathrm{b},\ c)$$

$$= f_n(c) - \nu([*],\ \lambda c)g_n(c)\ . \qquad \square$$

In future the distinction between complex and chain homotopy will be ignored: chain maps will be ν-*homotopic* and context will determine which definition is in use.

3.2.4 LEMMA. *Homotopy is an equivalence relation on chain maps between standard complexes.*

Proof. Let C be a standard R-complex and D be a standard S-complex. Suppose ρ, σ, $\tau : R \to S$ are ringnet morphisms which are homotopic *via* $\mu : \rho \simeq \sigma : \mathbb{Z}I \otimes R \to S$ and $\nu : \sigma \simeq \tau : \mathbb{Z}I \otimes R \to S$. Suppose f, g, $h : C \to D$ are respectively ρ, σ, τ-chain maps for which

$$f(z) = \mu([*],\ \lambda z)g(z) \quad \text{and} \quad g(z) = \nu([*],\ \lambda z)h(z)$$

for all z in zC.

(i) Since $\chi(\sigma) : \sigma \simeq \sigma : \mathbb{Z}I \otimes R \to S$ is the constant homotopy determined by σ, whence $\chi(\sigma)([*],\ \lambda z) = \sigma(\lambda z)$ for z in zC; it follows that $g \simeq g$ *via* the *induced constant* $\chi(\sigma)$-*homotopy* $\chi(g) : \mathcal{E} \otimes C \to D$ with $\chi(g)(*\mathrm{c},\ c) = g(zc)$. For then,

$$\chi(g)\partial(*\mathrm{c},\ c) = \chi(g)(*\mathrm{a},\ c) - \chi(g)(*\mathrm{b},\ c) - \chi(g)(*\mathrm{c},\ \partial c)$$

$$= g(c) - \chi(\sigma)([*],\ \lambda c)g(c) - g(zc)$$

$$= g(zc)$$

$$= \partial\chi(g)(*\mathrm{c},\ c)\ .$$

(ii) There is a symmetric ringnet homotopy $\overline{\nu} : \tau \simeq \sigma : \mathbb{Z}I \otimes R \to S$, with $\overline{\nu}([*],\ i) = \nu([*^{-1}],\ i)$ for i in $\mathrm{Id}\,R$. Since $h(z) = \overline{\nu}([*],\ \lambda z)g(z)$ for z in zC, the *symmetric* $\overline{\nu}$-*homotopy* $\overline{F} : h \simeq g : \mathcal{E} \otimes C \to D$ determined from the ν-homotopy $F : g \simeq h : \mathcal{E} \otimes C \to D$ is generated from

$$\overline{F}(*\mathrm{c},\ c) = F(\mathrm{c},\ c)\ , \quad \forall c \in C_n\ , \quad n \in \mathbb{Z}\ .$$

(iii) To each μ-homotopy $F : f \simeq g$ and ν-homotopy $G : g \simeq h$ there is a $\mu\nu$-homotopy $H : f \simeq h$. For $f(z) = \mu\nu([*], \lambda z)h(z)$ immediately, and, with

$$H(*c, c) = F(*c, c) + \mu([*], \lambda c)G(*c, c) ;$$

$$\begin{aligned}\partial H(*c, c) &= \partial F(*c, c) + \mu([*], \lambda c)\partial G(*c, c)\\ &= F(*a, c) - F(*b, c) - F(*c, \partial c)\\ &\qquad + \mu([*], \lambda c)\{G(*a, c) - G(*b, c) - G(*c, \partial c)\}\\ &= f(c) - \mu([*], \lambda c)g(c) - F(*c, \partial c)\\ &\qquad + \mu([*], \lambda c)\{g(c) - \nu([*], \lambda c)h(c) - G(*c, \partial c)\}\\ &= f(c) - \mu\nu([*], \lambda c)h(c) - H(*c, \partial c)\\ &= H\partial(*c, c) .\end{aligned}$$

Hence H is the requisite $\mu\nu$-chain map. □

3.2.5 EXAMPLE (Contracting homotopy). Let C be a standard R-complex. A *contracting homotopy* s for C is a $\chi(1)$-homotopy $s : 1 \simeq 0$ between the identity $1 : C \to C$ and the zero chain map $0 : C \to C$. It thus consists of a set $s = \{s_n : n \in \mathbb{Z}\}$ of standard R-morphisms $s_n : C_n \to C_{n+1}$ satisfying $\partial s + s\partial = 1$.

3.2.6 LEMMA. *If a standard* $\mathbb{Z}$*-complex has a contracting homotopy it is exact.*

Proof. Let $s = \{s_n : n \in \mathbb{Z}\}$ be a contracting homotopy for the standard $\mathbb{Z}$-complex C . For each z in zC consider c in $\operatorname{Ker} \partial_n(z)$. Immediately, $(\partial s + s\partial)(c) = \partial s(c) + z = c$, which implies $c \in \operatorname{Im} \partial_{n+1}(z)$. □

3.2.7 EXAMPLE. The augmented bar resolution (3.1.4) of a groupnet is exact. For if B is the bar resolution of TA for a groupnet A , a contracting homotopy s of $\mathbb{Z}$-morphisms may be extended linearly from

$$s_{-1}[i] = [i] \quad \forall i \in \operatorname{Id} A ,$$

$$s_0(a[\rho a]) = [a] \quad \forall a \in A ,$$

and

$s_n(a[a_1 \mid \ldots \mid a_n]) = [a \mid a_1 \mid \ldots \mid a_n] \quad \forall a \in A\,, \quad [a_1 \mid \ldots \mid a_n] \in X_n$

for $n \geq 1$.

Routine calculation shows that $\varepsilon s_{-1} = 1$, $\partial s_0 + s_{-1}\varepsilon = 1$ and, in higher dimensions, $\partial s + s\partial = 1$ (*cf.* [19, IV.5.1]).

There is a partial converse to (3.2.6), namely, that any positive projective exact complex in *R-Mod*(*Z*) has a contracting homotopy (*cf.* [15, IV, Ex.4.1]). Proof is deferred, however, until (4.2.2). Since any free standard *A*-complex is a free standard Id *A*-complex for any groupnet *A* , and *TA* is a projective Id *A*-module, this partial converse immediately implies the next result.

3.2.8 COROLLARY. *For any groupnet* A *, any (regular) free* A*-resolution of* TA *has a contracting homotopy.* □

(This contracting homotopy consists of Id *A*-morphisms.)

3.2.9 DEFINITION. Suppose C is a standard R-chain complex, D is a standard S-chain complex and $R \simeq S$ with homotopy equivalence $\sigma : R \to S$ and homotopy inverse $\tau : S \to R$. Let $\nu : \tau \circ \sigma \simeq 1_R$ and $\mu : \sigma \circ \tau \simeq 1_S$. Then C *has the same homotopy type as* D (or, *is homotopic to* D), written $C \simeq D$, if there is a σ-chain map $f : C \to D$ and a τ-chain map $g : D \to C$ such that

(i) $g \circ f(z) = \nu([*], \lambda z).z \quad \forall z \in zC$,

(ii) $f \circ g(z') = \mu([*], \lambda z').z' \quad \forall z' \in zD$, and

(iii) $g \circ f$ is ν-homotopic to 1_C and $f \circ g$ is μ-homotopic to 1_D .

Homotopy type is clearly an equivalence relation. It will be shown in (4.2.6) that homotopic groupnet morphisms $f, g : A \to B$ with homotopy $\sigma : f \simeq g$ induce σ-homotopic chain maps $Bf \simeq Bg : BA \to BB$ on the augmented bar resolutions.

3.2.10 EXAMPLE. Homotopic groupnets have augmented bar resolutions which have the same homotopy type. For, if $f : A \to B$ and $g : B \to A$ are groupnet morphisms with homotopies $\sigma : g \circ f \simeq 1_A$ and $\tau : f \circ g \simeq 1_B$, then by (4.2.6) there is a σ-homotopy

$$B(g \circ f) \simeq B(1_A)$$

and a τ-homotopy

$$B(f \circ g) \simeq B(1_B) .$$

By definition, $B(g \circ f) = B(g) \circ B(f)$ and $B(1_A) = 1_{BA}$.

3.2.11 LEMMA. *An additive functor* $F : R\text{-Comp}(Y) \to S\text{-Comp}(Z)$ *preserves homotopy.*

Proof. Suppose $G : f \simeq g : C \to D$ where C and D are standard R-complexes over Y, and f and g are standard R-chain maps. Then $G = \{G_n : C_n \to D_{n+1}, n \in Z\}$ satisfies

$$\partial G + G\partial = f - g .$$

Hence

$$\begin{aligned} F(\partial G + G\partial) &= F(\partial G) + F(G\partial) \\ &= F(\partial)F(G) + F(G)F(\partial) \\ &= F(f) - F(g) . \end{aligned}$$

Since $(Ff)(z) = (Fg)(z) = z$ for all z in Z,

$$FG = \{FG_n : FC_n \to FD_{n+1}, n \in Z\}$$

is the required homotopy. This result holds whether F is covariant or contravariant. □

3.3 The Mapping Cylinder

The construction in this section of a manageable free resolution for graph products provides the main tool used in the rest of this work.

3.3.1 DEFINITION. A *complex diagram* $(\mathcal{D}, R, C)$ consists of a directed graph D, a ringnet diagram $(\mathcal{D}, R)$, and a composed covariant functor

$$C \circ R : \mathcal{D} \to Rngnet \to Comp$$

subject to

(i) $C \circ R(v) \in |R_v\text{-}Comp|$ for v in D and

(ii) $C \circ R(e) : C \circ R(\lambda e) \to C \circ R(\rho e)$ is an R_e-chain map for e in D.

It may thus be considered as a collection of complexes $\left\{C^v \in |R_v\text{-}Comp| : v \in D\right\}$ and a collection of chain maps

$$\left\{C^e : C^{\lambda e} \to C^{\rho e} : C^e \text{ is an } R_e\text{-morphism}, \ e \in D\right\}.$$

It is a *standard/regular/projective/free/exact* complex diagram when C^v is a standard/regular/projective/free/exact complex for each v in D, and, in the first two cases, when C^e is a standard/regular chain map for each e in D.

3.3.2 DEFINITION. A $\sigma(\mathcal{D}, R)$*-mapping cylinder* $\mu : (\mathcal{D}, R, C) \to \mu(\mathcal{D}, R, C)$ *for a complex diagram* $(\mathcal{D}, R, C)$ comprises

(i) a representation $\sigma : (\mathcal{D}, R) \to \sigma(\mathcal{D}, R)$ of $(\mathcal{D}, R)$ (see (2.1.10)),

(ii) a $\sigma(\mathcal{D}, R)$-complex $\mu(\mathcal{D}, R, C)$,

(iii) a σ_v-chain map $\mu^v : C^v \to \mu(\mathcal{D}, R, C)$ for v in D, and

(iv) a σ_e-homotopy $\mu^e : \mu^{\lambda e} \simeq \mu^{\rho e} \circ C^e$ for e in D,

which is

(v) universal with respect to all constructions already satisfying conditions (i)-(iv).

Any construction satisfying conditions (i)-(iv) is called a $\sigma(\mathcal{D}, R)$-*representation of* $(\mathcal{D}, R, C)$. Once $\sigma(\mathcal{D}, R)$ has been prescribed, the mapping cylinder may be considered as a homotopy colimit 'with respect to

$\sigma(\mathcal{D}, R)$ '. //

A constructive proof of the existence of mapping cylinders follows. It corresponds to the process in *Top* of adding a handle to the union of vertex spaces for each edge of the directed graph, and identifying its initial boundary with the source complex and its terminal boundary with the sink complex. Hence the name 'mapping cylinder' (*cf.* [31, p. 32]); the mapping cylinder will also be shown to be the algebraic mapping cone of a suitable chain map.

3.3.3 THEOREM. *For any representation* $\sigma : (\mathcal{D}, R) \to S$ *of a standard complex diagram* $(\mathcal{D}, R, C)$ *, there exists a* $\sigma(\mathcal{D}, R)$*-mapping cylinder* $\mu : (\mathcal{D}, R, C) \to M$ *; moreover,* M *is a standard* S*-complex.*

Proof. Let $(\mathcal{D}, R, C)$ be a standard complex diagram and $\sigma : (\mathcal{D}, R) \to S$ be a representation of the ringnet diagram $(\mathcal{D}, R)$. Some simplification of notation is first necessary. For each e in D , $\mathcal{E}_e$ represents a subscripted copy of the unit $R_{\lambda e}$-complex $\mathcal{E}$. Morphism $\sigma_v : R_v \to S$ determines the right pullback S^v of S which is an S-R_v bimodule. Similarly, homotopy $\sigma_e : \mathbb{Z}I \otimes R_{\lambda e} \to S$ determines the right pullback S^e of S which is an S-$\mathbb{Z}I \otimes R_{\lambda e}$ bimodule. Hence the R_v-tensor product $S^v \otimes_v C^v_n$ is a left S-module, as is the $\mathbb{Z}I \otimes R_{\lambda e}$-tensor product $S^e \otimes_e \left(\mathcal{E}_e \otimes C^{\lambda e}\right)_n$. Element $\left((s, \lambda c), c\right)$ of $S^v \otimes_v C^v_n$ - with $\rho s = \sigma_v(\lambda c)$ - will be written (s, c) ; element $\left(\left(s, ([1], \lambda c)_e\right), (\mathrm{d}, c)\right)$ of $S^e \otimes_e \left(\mathcal{E}_e \otimes C^{\lambda e}\right)_n$ - with $\rho s = \sigma_e([1], \lambda c) = \sigma_{\rho e} R_e(\lambda c)$ - will be written (s, d, c, e) for d in $\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$; and similarly element $\left(\left(s, ([0], \lambda c)_e\right), (*\mathrm{d}, c)\right)$, with $\rho s = \sigma_e([0], \lambda c) = \sigma_{\lambda e}(\lambda c)$, will be written $(s, *\mathrm{d}, c, e)$. Thus, for example,

$$(s,\ *\mathrm{d},\ c,\ e) = \left(s\sigma_e([*],\ \lambda c),\ \mathrm{d},\ c,\ e\right) \qquad \text{(D3.3.1)}$$

in this terminology. The construction now proceeds.

(i) For each e in D,

$$z\left(S^e \otimes_e \left(\mathcal{E}_e \otimes C^{\lambda e}\right)\right) \cong z\left(S^{\lambda e} \otimes_{\lambda e} C^{\lambda e}\right)$$

under the identification $(q,\ 0,\ z,\ e) \mapsto (q,\ z)$ for q in zS and z in $zC^{\lambda e}$. Define an abelian groupnet M_n as follows for each n in $\mathbb{Z}$. It has identity set

$$zM_n = \left(\bigvee_{v \in D} z\left(S^v \otimes_v C^v\right)\right) \Big/ \left\langle (q,\ z) \sim \left(q\sigma_e([*],\ \lambda z),\ C^e(z)\right), \right.$$
$$\left. \forall (q,\ z) \in z\left(S^{\lambda e} \otimes_{\lambda e} C^{\lambda e}\right),\ e \in D \right\rangle,$$

written zM for all n in $\mathbb{Z}$.

Set $Z_v = Z \cap z\left(S^v \otimes_v C^v\right)$ for each $\mathcal{S}et$ equivalence class Z of zM, and each v in D. Then let

$$M_n(Z) = \coprod_{\substack{z \in Z_v \\ v \in D}} \left(S^v \otimes_v C^v_n\right)(z) \oplus \coprod_{\substack{z \in Z_{\lambda e} \\ e \in D}} \left(S^e \otimes_e \left(\mathcal{E}_e \otimes C^{\lambda e}\right)_n\right)(z) \Big/$$

$$\left\langle (s,\ c) = (s,\ *\mathrm{a},\ c,\ e),\ \forall (s,\ c) \in S^{\lambda e} \otimes_{\lambda e} C^{\lambda e}_n,\ e \in D, \right.$$
$$\left. \left(s*,\ C^e_n(c)\right) = (s*,\ \mathrm{b},\ c,\ e),\ \forall \left(s*,\ C^e_n(c)\right) \in S^{\rho e} \otimes_{\rho e} C^{\rho e}_n,\ e \in D \right\rangle \qquad \text{(D3.3.2)}$$

for each Z in zM and n in $\mathbb{Z}$. This equation may be simplified by virtue of (D3.3.1) to

$$M_n(Z) \cong \coprod_{\substack{z \in Z_v \\ v \in D}} \left(S^v \otimes_v C^v_n\right)(z) \oplus \coprod_{\substack{z \in Z_{\lambda e} \\ e \in D}} \left(S^{\lambda e} \otimes_{\lambda e} C^{\lambda e}_{n-1}\right)(z) \times \{e\}. \qquad \text{(D3.3.3)}$$

The left S-module structure on each of the zero sets $z\left(S^v \otimes_v C^v\right)$ determines a well-defined left map $\lambda : zM \to \mathrm{Id}\ S$. Similarly, the left S-action on $S^v \otimes_v C^v_n$ and $S^e \otimes_e \left(\mathcal{E}_e \otimes C^{\lambda e}\right)_n$ is compatible with the relations in $M_n(Z)$, so that M_n is a left S-module.

(ii) Boundary map $\partial_n : M_n \to M_{n-1}$ is induced from the boundary maps on the direct summands, as they also are compatible with the relations. That is,

$$\partial(s, c) = (s, \partial c) ,$$

and

$$\partial\big(s, (*)\mathrm{d}, c, e\big) = \big(s, \partial\big((*)\mathrm{d}, c\big), e\big) ,$$

for d in $\{\mathrm{a, b, c}\}$. Under the isomorphism of (D3.3.3) the latter equality reduces to

$$\partial(s, c, e) = (s, c) - \Big(s\sigma_e([*], \lambda c), C^e_{n-1}(c)\Big) - (s, \partial c, e)$$

for all (s, c, e) in $S^{\lambda e} \otimes_{\lambda e} C^{\lambda e}_{n-1} \times \{e\}$.

Routine calculation shows ∂_n is a well-defined S-morphism and that (M, ∂) is a standard S-complex. It remains to prove that (M, ∂) is actually the mapping cylinder; this is more apparent when M is written in the form (D3.3.2).

(iii) Set $\mu^v_n : C^v_n \to M_n$ as

$$\mu^v_n(c) = \big(\sigma_v(\lambda c), c\big) , \quad \forall v \in D , \quad n \in \mathrm{Z} ,$$

so that for r in R_v,

$$\mu^v_n(rc) = \big(\sigma_v(\lambda r), rc\big) = \big(\sigma_v(r), c\big) \quad (\text{in } S^v \otimes_v C^v_n)$$

$$= \sigma_v(r)\mu^v_n(c) .$$

Since $\partial\mu^v(c) = \partial\big(\sigma_v(\lambda c), c\big) = \big(\sigma_v(\lambda c), \partial c\big) = \mu^v\partial(c)$, μ^v is a σ_v-chain map as required.

(iv) Let $\mu^e_n : (\mathcal{E} \otimes C^{\lambda e})_n \to M_n$ be given by

$$\mu^e_n(*\mathrm{d}, c) = \big(\sigma_e([0], \lambda c), *\mathrm{d}, c, e\big)$$

and

$$\mu_n^e(\mathrm{d}, c) = \left(\sigma_e([1], \lambda c), \mathrm{d}, c, e\right)$$

for d in {a, b, c} , e in $\mathcal{D}$ and n in Z . This is a σ_e-chain map. Since

$$\begin{aligned}\mu_n^e(*\mathrm{a}, c) &= \left(\sigma_e([0], \lambda c), *\mathrm{a}, c, e\right)\\ &= \left(\sigma_{\lambda e}(\lambda c), *\mathrm{a}, c, e\right)\\ &= \left(\sigma_{\lambda e}(\lambda c), c\right) = \mu_n^{\lambda e}(c) ,\end{aligned}$$

and

$$\begin{aligned}\mu_n^e(\mathrm{b}, c) &= \left(\sigma_e([1], \lambda c), \mathrm{b}, c, e\right)\\ &= \left(\sigma_{\rho e}R_e(\lambda c), \mathrm{b}, c, e\right)\\ &= \left(\sigma_{\rho e}R_e(\lambda c), C_n^e(c)\right) = \mu_n^{\rho e}C_n^e(c) ,\end{aligned}$$

μ^e is the requisite σ_e-homotopy such that $\mu^{\lambda e} \simeq \mu^{\rho e}C^e$. Thus (M, ∂) is certainly a $\sigma(\mathcal{D}, R)$-representation of $(\mathcal{D}, R, C)$. It must now be shown to be universal.

(v) Assume $\nu : (\mathcal{D}, R, C) \to N$ is any $\sigma(\mathcal{D}, R)$-representation of $(\mathcal{D}, R, C)$, whether N is standard or not. It is necessary to find an S-chain map $\Theta : M \to N$ which satisfies

$$\begin{array}{ccc} & C^v & \\ \mu^v \swarrow & & \searrow \nu^v \\ M & \xrightarrow[\Theta]{} & N \end{array} \quad \forall v \in \mathcal{D} \text{ , and } \quad \begin{array}{ccc} & \mathcal{E}\otimes C^{\lambda e} & \\ \mu^e \swarrow & & \searrow \nu^e \\ M & \xrightarrow[\Theta]{} & N \end{array} \quad \forall e \in \mathcal{D} .$$

For each v in $\mathcal{D}$, n in Z , z in zC^v and $[(q, z)]$ in Z , the zero $q.\nu_n^v(z)$ in zN is well-defined. Hence the S-morphism

$$\Theta_n : M_n(Z) \to N_n\left(q.\nu_n^v(z)\right)$$

given by

$$\Theta_n(s, c) = s.\nu_n^v(c) \quad \forall c \in C_n^v(z) \ , \quad s \in S(q) \ ,$$

and

$$\Theta_n(s, c, e) = s.\nu_n^e(*c, c) \quad \forall c \in C_{n-1}^{\lambda e}(z) \ , \quad s \in S(q)$$

satisfies all these requirements. In fact, it does so uniquely. Thus $\mu : (\mathcal{D}, R, C) \to M$ is a $\sigma(\mathcal{D}, R)$-mapping cylinder for the standard complex diagram $(\mathcal{D}, R, C)$. By definition any two $\sigma(\mathcal{D}, R)$-mapping cylinders of $(\mathcal{D}, R, C)$ are isomorphic as S-complexes, by a uniquely determined isomorphism. The mapping cylinder constructed above (D3.3.2), or its isomorphic form (D3.3.3), will be termed *the* $\sigma(\mathcal{D}, R)$-mapping cylinder of $(\mathcal{D}, R, C)$. □

In fact, there is always a $\sigma(\mathcal{D}, R)$-mapping cylinder for any complex diagram $(\mathcal{D}, R, C)$ and ringnet representation $\sigma : (\mathcal{D}, R) \to \sigma(\mathcal{D}, R)$. It is found by indexing the zeroes of (3.3.3.i) by dimension and then showing the (suitably altered) morphisms of (3.3.3.ii, iii, iv) are well-defined on the zeroes. Proof is no more difficult than above, but the added detail is unnecessary for an understanding of the mapping cylinder construction, and the result is not required below.

3.3.4 LEMMA. *The mapping cylinder is an algebraic mapping cone.*

That is (*cf.* [19, p. 46]); if $\mu : (\mathcal{D}, R, C) \to M$ is the $\sigma(\mathcal{D}, R)$-mapping cylinder of (3.3.3), then there exist S-chain complexes K and K' and an S-chain map $f : K \to K'$ such that

$$M_n = K'_n \oplus K_{n-1} \quad \forall n \in \mathbb{Z} \ ,$$

and

$$\partial(k', k) = (\partial' k' + fk, -\partial k) \ .$$

Proof. For each n in $\mathbb{Z}$, define a standard complex diagram $(\mathcal{D}, R, C_n)$ from $(\mathcal{D}, R, C)$ by

(i) $(C_n)^v = C_n^v$ (*cf.* (3.1.2.i)), and

(ii) $(C_n)^e = C_n^e : C_n^{\lambda e} \to C_n^{\rho e}$.

Let $\kappa^n : (\mathcal{D}, R, C_n) \to K^n$ be the $\sigma(\mathcal{D}, R)$-mapping cylinder of $(\mathcal{D}, R, C_n)$.

Thus, for all n in Z ,

$$zK^n = zM ,$$

$$K_0^n(Z) = \coprod_{\substack{z \in Z_v \\ v \in D}} \left(S^v \otimes_v C_n^v\right)(z) \quad \forall Z \in zM ,$$

$$K_1^n(Z) = \coprod_{\substack{z \in Z_{\lambda e} \\ e \in D}} \left(S^{\lambda e} \otimes_{\lambda e} C_n^{\lambda e}\right)(z) \times \{e\} \quad \forall Z \in zM ,$$

and the boundary map of K^n in dimension 1 is

$$\partial_1^n(s, c, e) = (s, c) - \left(s\sigma_e([*], \lambda c), C_n^e(c)\right) .$$

The S-complexes (K, ∂) and (K', ∂') are

$$K'_n = K_0^n ,$$

$$\partial'_n : (s, c) \mapsto (s, \partial c) ,$$

$$K_n = K_1^n ,$$

and

$$\partial_n : (s, c, e) \mapsto (s, \partial c, e) .$$

Boundary map $\partial_1 = \left\{\partial_1^n : K_n \to K'_n\right\}$ is an S-chain map $K \to K'$ since

$$\partial'\partial_1(s, c, e) = (s, \partial c) - \left(s\sigma_e([*], \lambda c), \partial C_n^e(c)\right)$$

$$= (s, \partial c) - \left(s\sigma_e([*], \lambda c), C_{n-1}^e(\partial c)\right)$$

$$= \partial_1(s, \partial c, e) = \partial_1\partial(s, c, e) .$$

By (D3.3.3),

$$M_n = K'_n \oplus K_{n-1} ,$$

and from (3.3.3.ii),

$\partial\big((s', c'), (s, c, e)\big)$

$$= \Big((s', \partial c')+(s, c)-\big(s\sigma_e([*], \lambda c), C^e(c)\big), -(s, \partial c, e)\Big)$$

$$= \big(\partial'(s', c')+\partial_1(s, c, e), -\partial(s, c, e)\big)$$

as required. □

For the next lemma, replace (K, ∂) in (3.3.4) by (K^+, ∂^+), where $K_n^+ = K_{n-1}$ and $\partial_n^+ = -\partial_{n-1} : K_n^+ \to K_{n-1}^+$ for all n in $\mathbb{Z}$.

3.3.5 LEMMA. *There is a short exact sequence*

$$K' \overset{i}{\rightarrowtail} M \overset{p}{\twoheadrightarrow} K^+$$

in S-*Mod*(zM) .

Proof (*cf*. [19, p. 46]). The injection $i : K' \rightarrowtail M$ is immediately an S-chain map. The projection $p : M \twoheadrightarrow K^+$ of the second coordinate; $p\big((s', c'), (s, c, e)\big) = (s, c, e)$, satisfies

$$\partial^+ p\big((s', c'), (s, c, e)\big) = -(s, \partial c, e)$$

$$= p\big(\partial'(s', c')+\partial_1(s, c, e), -\partial(s, c, e)\big)$$

$$= p\partial\big((s', c'), (s, c, e)\big)$$

and so is an S-chain map. Since S-*Mod*(zM) is abelian, the sequence $K_n' \rightarrowtail M_n \twoheadrightarrow K_n^+$ is short exact for all n in $\mathbb{Z}$ (see, for example, [28, 2.3.5]), as required. □

Obviously, complex M is not a direct sum, so the short exact sequence of (3.3.5) need not split.

The next theorem is essential to any concrete use of the mapping cylinder. For any groupnet diagram $(\mathcal{D}, A)$ and complex diagram $(\mathcal{D}, ZA, C)$, it determines which properties of $(\mathcal{D}, ZA, C)$ are inherited by the $m(\mathcal{D}, A)$-mapping cylinder. Despite the complexity of terminology and detail necessary, it is hoped the arguments used will appear straightforward.

3.3.6 THEOREM. *Let* $(\mathcal{D}, A)$ *be a groupnet diagram with mapping cylinder* $m : (\mathcal{D}, A) \to G$ *and let* $(\mathcal{D}, ZA, C)$ *be a standard complex diagram.*

Let $m : (\mathcal{D}, \mathbb{Z}A) \to \mathbb{Z}G$ *be the induced representation of* $(\mathcal{D}, \mathbb{Z}A)$ *and let* $\mu : (\mathcal{D}, \mathbb{Z}A, C) \to M$ *be the* G*-mapping cylinder of* $(\mathcal{D}, \mathbb{Z}A, C)$. *Then*

(i) *if* $(\mathcal{D}, \mathbb{Z}A, C)$ *is regular,* M *is regular;*

(ii) *if* $(\mathcal{D}, \mathbb{Z}A, C)$ *is free,* M *is free;*

(iii) *if* G *is a graph product and* $(\mathcal{D}, \mathbb{Z}A, C)$ *is exact,* M *is exact; and*

(iv) *if* G *is a graph product and* C^v *is a resolution of* TA_v *for each* v *in* D *, then* M *is a resolution of* TG .

Proof. (i) Complex M is standard by definition, so an isomorphism

$$zM \cong \text{Id}\, G$$

of G-modules is required. Now (*cf.* (2.2.22))

$$z\mathbb{Z}G^v = \{(i, m_v(k)) \in z\mathbb{Z}G : k \in \text{Id}\, A_v\} ,$$

so

$$z\left[\mathbb{Z}G^v \otimes_v C^v\right] = \{(i, m_v(k)) \in z\mathbb{Z}G : k \in \text{Id}\, A_v\} / \langle (i, m_v(k)) \sim (i, m_v(l)) : A_v(k, l) \neq \emptyset \rangle .$$

Thus

$$zM = \{(i, m_v(k)) \in z\mathbb{Z}G : k \in \text{Id}\, A_v,\ v \in D\} / \langle (i, m_v(k)) \sim (i, m_v(l)); A_v(k, l) \neq \emptyset,\ v \in D,\ (i, m_{\lambda e}(j)) \sim (i, m_{\rho e}A_e(j));\ e \in D \rangle .$$

Define $\zeta : zM \to \text{Id}\, G$ from

$$\zeta(i, m_v(k)) = \lambda(i, m_v(k)) = i ;$$

it is a well-defined G-morphism on these zeroes since

$$\zeta(g.(\rho g, m_v(k))) = \zeta(\lambda g, m_v(k)) = \lambda g = g.\rho g ,$$

and is clearly surjective. If $\zeta(i, m_v(j)) = \zeta(i, m_w(k))$, then

$G\big(m_v(j),\ m_w(k)\big) \neq \emptyset$ and there is an element $g = \prod_{l=1}^{n} p_l$ in

$G\big(m_v(j),\ m_w(k)\big)$, where each p_l has one of the forms

$$p_l = \begin{cases} m_{v_l}(a)\ , & a \in A_{v_l}\ ,\quad v_l \in D\ , \\ m_{e_l}([*],\ [q])\ , & q \in \operatorname{Id} A_{\lambda e_l}\ ,\quad e_l \in D\ , \\ m_{e_l}\big([*^{-1}],\ [q]\big)\ , & q \in \operatorname{Id} A_{\lambda e_l}\ ,\quad e_l \in D\ . \end{cases} \tag{D3.3.4}$$

Considered as an element of zZG,

$$\big(i,\ m_w(k)\big) = \big(i,\ m_v(j)\big).\big(\lambda p_1,\ \rho p_1\big)\ \ldots\ \big(\lambda p_n,\ \rho p_n\big)\ ,$$

while as an element of zM,

$$\big(\lambda p_n,\ \rho p_n\big) = \begin{cases} \big(m_{v_n}(\lambda a),\ m_{v_n}(\rho a)\big) \\ \big(m_{\lambda e_n}(q),\ m_{\rho e_n}A_{e_n}(q)\big) \\ \big(m_{\rho e_n}A_{e_n}(q),\ m_{\lambda e_n}(q)\big) \end{cases}$$

$$\sim \begin{cases} \big(m_{v_n}(\lambda a),\ m_{v_n}(\lambda a)\big) \\ \big(m_{\lambda e_n}(q),\ m_{\lambda e_n}(q)\big) \\ \big(m_{\rho e_n}A_{e_n}(q),\ m_{\rho e_n}A_{e_n}(q)\big) \end{cases}$$

$$= \big(\lambda p_n,\ \lambda p_n\big)\ .$$

Hence in zM,

$$\begin{aligned} \big(i,\ m_w(k)\big) &= \big(i,\ \rho p_n\big) \\ &\sim \big(i,\ \lambda p_n\big) = \big(i,\ \rho p_{n-1}\big) \\ &\qquad \sim \big(i,\ \lambda p_1\big) \text{ by induction} \\ &\qquad = \big(i,\ m_v(j)\big)\ , \end{aligned}$$

and ζ is an isomorphism.

(ii) Suppose $(\mathcal{D},\ ZA,\ C)$ is free and that C_n^v is the free left

A_v-module with basis

$$X_n^v = X_n^v \xrightarrow{z} zX_n^v \xrightarrow{\lambda} \mathrm{Id}\, A_v .$$

That is, for each z in zC^v, v in D and n in $\mathbb{Z}$, by (2.2.19),

$$C_n^v(z) = \coprod_{\substack{(\lambda z, \lambda zx) \in z\mathbb{Z}A_v \\ x \in X_n^v \\ (\lambda z, \lambda zx).zx = z}} \mathbb{Z}A_v(\lambda z, \lambda zx) \times \{x\} .$$

Define

$$X_n = \left(\bigvee_{v \in D} X_n^v\right) \vee \left(\bigvee_{e \in D} \left\{[x|e] : x \in X_{n-1}^{\lambda e}\right\}\right),$$

$$zX_n = \left(\bigvee_{v \in D} zX_n^v\right) \vee \left(\bigvee_{e \in D} \left\{[z|e] : z \in zX_{n-1}^{\lambda e}\right\}\right),$$

$$z : X_n \twoheadrightarrow zX_n \text{ by } \begin{cases} x \longmapsto zx \\ [x|e] \longmapsto [zx|e] , \end{cases}$$

and

$$\lambda : zX_n \to \mathrm{Id}\, G \text{ by } \begin{cases} z^v \longmapsto m_v(\lambda z^v) \\ [z|e] \longmapsto m_{\lambda e}(\lambda z) . \end{cases}$$

A G-action with respect to zM is induced on

$$X_n = X_n \xrightarrow{z} zX_n \xrightarrow{\lambda} \mathrm{Id}\, G$$

from the map $zX_n \to zM$ given by $z^v \longmapsto \left(m_v(\lambda z^v), z^v\right)$. The free left G-module FX_n over zM with basis X_n has, for each Z in zM with $\lambda Z = i$,

$$FX_n(Z) = \coprod_{\substack{(i, m_v(\lambda zx)) \in zZG \\ x \in X_n^v \\ ((i, m_v(\lambda zx)), zx) \in Z \\ v \in D}} ZG(i,\, m_v(\lambda zx)) \times \{x\} \oplus$$

$$\coprod_{\substack{(i, m_{\lambda e}(\lambda zx)) \in zZG \\ x \in X_{n-1}^{\lambda e} \\ ((i, m_{\lambda e}(\lambda zx)), zx) \in Z \\ e \in D}} ZG(i,\, m_{\lambda e}(\lambda zx)) \times \{[x|e]\} .$$

The G-morphism $\Phi_n : FX_n \to M_n$ extended from the map $\phi_n : X_n \to M_n$ with

$$\phi_n[x] = \big(m_v(\lambda zx),\ (\lambda zx,\ x)\big) \quad \forall x \in X_n^v ,\quad v \in D ,$$

$$\phi_n[x|e] = \big(m_{\lambda e}(\lambda zx),\ (\lambda zx,\ x),\ e\big) \quad \forall x \in X_{n-1}^{\lambda e} ,\quad e \in D ,$$

has the G-morphism $\psi_n : M_n \to FX_n$, where

$$\psi_n\big(g,\ (a_v,\ x)\big) = \big(gm_v(a_v),\ x\big)$$

and

$$\psi_n\big(g,\ (a_{\lambda e},\ x),\ e\big) = \big(gm_{\lambda e}(a_{\lambda e}),\ [x|e]\big) ,$$

as inverse. Hence M_n is free for n in Z.

When $(\mathcal{D}, ZA, C)$ is a free regular complex diagram, M is a free regular G-complex by (i) and (ii). For each i in $\mathrm{Id}\, G$, it follows that

$$FX_n(i) = \coprod_{\substack{(i, m_v(\lambda zx)) \in zZG \\ x \in X_n^v \\ v \in D}} ZG(i,\, m_v(\lambda zx)) \times \{x\} \oplus$$

$$\coprod_{\substack{(i, m_{\lambda e}(\lambda zx)) \in zZG \\ x \in X_{n-1}^{\lambda e} \\ e \in D}} ZG(i,\, m_{\lambda e}(\lambda zx)) \times \{[x|e]\}$$

in this case.

In the next two sections of the proof it will be assumed that G is a graph product; that is, that A_e is a monomorphism for each e in D.

(iii) Suppose $(\mathcal{D}, \mathbb{Z}A, C)$ is exact. Since the maps $m_v : A_v \to G$ are embeddings [7, Th. 6.2], the right pullback $\mathbb{Z}G^v$ of $\mathbb{Z}G$ along m_v is a free right A_v-module (2.2.22). In the terminology of (2.2.22),

$\mathbb{Z}G^v \cong FX_v$ for each v in D, where

$$X_v = \left\{a_{\beta(\kappa)_j} \in G\left(j, m_v\left(i_\kappa\right)\right) : j \in \text{Id}\, G,\ \kappa \in K_v,\ \beta(\kappa)_j \in Y_j^\kappa\right\},$$

K_v is the set of connected components of A_v, and i_κ is a distinguished identity of the component A_v^κ of A_v for each κ in K_v. Then

$$\mathbb{Z}G^v \otimes_v N \cong FX_v \otimes_v N$$

in $\mathcal{A}bnet$, for any left A_v-module N. But

$$\text{Id}\left(FX_v \otimes_v N\right) \cong \left\{\left(\left(j, m_v\left(i_\kappa\right)\right), z\right) \in zX_v \times zN : \lambda z = i_\kappa,\ \kappa \in K_v\right\}$$

and

$$\left(FX_v \otimes_v N\right)\left(\left(j, m_v\left(i_\kappa\right)\right), z\right) \cong \coprod_{\substack{\beta(\kappa)_j \in Y_j^\kappa \\ \kappa \in K_v}} \left\{a_{\beta(\kappa)_j}\right\} \times N(z) .$$

This implies that the complex $\left(\mathbb{Z}G^v \otimes_v C^v,\ 1 \otimes \partial\right)$ induces the boundary map $1 \times \partial : FX_v \otimes_v C_n^v \to FX_v \otimes_v C_{n-1}^v$ and hence that $\left(FX_v \otimes_v C^v,\ 1 \times \partial\right)$ is exact. Thus, for each v in D, $\mathbb{Z}G^v \otimes_v C^v$ is exact, and for each e in D, $\mathbb{Z}G^{\lambda e} \otimes_{\lambda e} C^{\lambda e} \times \{e\}$ is exact. The exactness of M may now be proved directly. Suppose

$$\sum_{v \in D} \left[\sum_{l=1}^{n_v} \left(g_l,\ c_l\right)\right] + \sum_{e \in D} \left[\sum_{p=1}^{n_e} \left(g_p,\ c_p,\ e\right)\right] \in \text{Ker}\ \partial_n(Z)$$

for some Z in zM (*cf.* (D3.3.3)). Then

$$\sum_{v\in D}\left[\sum_{l=1}^{n_v}(g_l,\ \partial c_l)\right]+\sum_{e\in D}\left[\sum_{p=1}^{n_e}(g_p,\ c_p)\right]$$

$$-\sum_{e\in D}\left[\sum_{p=1}^{n_e}\left(g_p m_e([*],\ \lambda c_p),\ C^e_{n-1}(c_p)\right)\right]-\sum_{e\in D}\left[\sum_{p=1}^{n_e}(g_p,\ \partial c_p,\ e)\right]=Z\ .$$

This implies

(*a*) $\sum_{p=1}^{n_e}(g_p,\ \partial c_p)=0$ in $ZG^{\lambda e}\otimes_{\lambda e}C^{\lambda e}_{n-2}$ for each e in D , and

(*b*)

$$\sum_{l=1}^{n_v}(g_l,\ \partial c_l)+\sum_{\substack{e\in D\\ \lambda e=v}}\left[\sum_{p=1}^{n_e}(g_p,\ c_p)\right]$$

$$-\sum_{\substack{e\in D\\ \rho e=v}}\left[\sum_{p=1}^{n_e}\left(g_p m_e([*],\ \lambda c_p),\ C^e_{n-1}(c_p)\right)\right]$$

$$=0 \text{ in } ZG^v\otimes_v C^v_{n-1} \text{ for each } v \text{ in } D\ .$$

From (*a*) there exists for each e in D , an element

$$g_e=\sum_{q=1}^{k_e}(g_q,\ c_q) \text{ in } ZG^{\lambda e}\otimes_{\lambda e}C^{\lambda e}_n\ ,$$

such that

$$\partial g_e=\sum_{p=1}^{n_e}(g_p,\ c_p)\ .$$

Let

$$g^*_e=\sum_{q=1}^{k_e}\left(g_q m_e([*],\ \lambda c_q),\ C^e_n(c_q)\right) \text{ in } ZG^{\rho e}\otimes_{\rho e}C^{\rho e}_n$$

for each e in D , so that in M ,

$$-\partial\left[\sum_{q=1}^{k_e}(g_q,\ c_q,\ e)\right]=-g_e+g^*_e+\sum_{p=1}^{n_e}(g_p,\ c_p,\ e)\ .$$

The boundary of this equation in M is

$$Z = -\partial g_e + \partial g_e^* + \partial g_e - \sum_{p=1}^{n_e} \left(g_p m_e([*],\ \lambda c_p),\ C^e_{n-1}(c_p)\right),$$

so that (b) may be rewritten as

$$\partial\left\{\sum_{l=1}^{n_v} (g_l,\ c_l) + \sum_{\substack{e \in D \\ \lambda e = v}} g_e - \sum_{\substack{e \in D \\ \rho e = v}} g_e^*\right\} = 0 \quad \text{in} \quad \mathbb{Z}G^v \otimes_v C^v_{n-1}$$

for each v in D. There is thus an element g_v in $\mathbb{Z}G^v \otimes_v C^v_{n+1}$ which maps to this element of the kernel, for each v in D. Hence

$$\partial\left\{\sum_{v \in D} g_v - \sum_{e \in D} \left[\sum_{q=1}^{k_e} (g_q,\ c_q,\ e)\right]\right\}$$

$$= \sum_{v \in D} \left[\sum_{l=1}^{n_v} (g_l,\ c_l)\right] + \sum_{e \in D} \left[\sum_{p=1}^{n_e} (g_p,\ c_p,\ e)\right]$$

in M, as required.

(iv) Suppose C^v is a $TA_v = T_v$-resolution for each v in D, so that for each e in D,

$$\begin{array}{ccc} C_0^{\lambda e} & \xrightarrow{\ \varepsilon\ } & T_{\lambda e} \\ {\scriptstyle C_0^e}\big\downarrow & & \big\downarrow{\scriptstyle A_e} \\ C_0^{\rho e} & \xrightarrow[\ \varepsilon\ ]{} & T_{\rho e} \end{array}$$

because C_0^e is an A_e-morphism and the complexes are regular. Then $(\mathcal{D},\ \mathbb{Z}A,\ C)$ determines an exact regular complex diagram $(\mathcal{D},\ \mathbb{Z}A,\ \mathcal{B})$ which has for each v in D, the augmented complex $C^v \to T_v$ for B^v, and for each e in D, the A_e-chain map

$$B^e_n = C^e_n \quad \forall n \geq 0,$$

and

$$B^e_{-1} = A_e.$$

Denote by M^* the G-mapping cylinder of $(\mathcal{D}, \mathbb{Z}A, \mathcal{B})$. Then the diagram

$$\begin{array}{ccccccccc} \cdots \to M_n^* \to \cdots \to & M_1^* & \xrightarrow{\partial_1^*} & M_0^* & \xrightarrow{\varepsilon^*} & M_{-1}^* & \to \operatorname{Id} G \\ \downarrow = & \downarrow = & & & & & \\ \cdots \to M_n \to \cdots \to & M_1 & \xrightarrow[\partial_1]{} & M_0 & \longrightarrow & \operatorname{Id} G & \end{array}$$

in G-Compreg has an exact top row and a bottom row exact in all dimensions greater than 1 , by (i) and (iii) above. Further, for each i in Id G ,

$$M_0(i) = \coprod_{\substack{(i, m_v(j)) \in \mathbb{Z}\mathbb{Z}G \\ v \in D}} \mathbb{Z}G^v\big(i, m_v(j)\big) \otimes_v C_0^v(j) ,$$

$$M_0^*(i) = M_0(i) \oplus \coprod_{\substack{(i, m_{\lambda e}(j)) \in \mathbb{Z}\mathbb{Z}G \\ e \in D}} \Big[\mathbb{Z}G^{\lambda e}\big(i, m_{\lambda e}(j)\big) \otimes_{\lambda e} T_{\lambda e}(j)\Big] \times \{e\} ,$$

$$M_{-1}^*(i) = \coprod_{\substack{(i, m_v(j)) \in \mathbb{Z}\mathbb{Z}G \\ v \in D}} \mathbb{Z}G^v\big(i, m_v(j)\big) \otimes_v T_v(j) ,$$

and the boundary maps are

$$\partial_1(g, c) = \partial_1^*(g, c) = (g, \partial c) ,$$

$$\partial_1(g, c, e) = (g, c) - \Big[gm_e([*], \lambda c), C_0^e(c)\Big] ,$$

$$\partial_1^*(g, c, e) = (g, c) - \Big[gm_e([*], \lambda c), C_0^e(c)\Big] - (g, \partial c, e) ,$$

$$\varepsilon^*(g, c) = (g, \varepsilon c) ,$$

and

$$\varepsilon^*(g, [j], e) = (g, [j]) - \big(gm_e([*], j), [A_e(j)]\big) .$$

Set

$$N_0(i) = \coprod_{\substack{(i, m_{\lambda e}(j)) \in \mathbb{Z}\mathbb{Z}G \\ e \in D}} \mathbb{Z}G^{\lambda e}\big(i, m_{\lambda e}(j)\big) \otimes_{\lambda e} T_{\lambda e}(j) \times \{e\}$$

for each i in Id G , so that $M_0^*(i) = M_0(i) \oplus N_0(i)$. If $p_0 : M_0^* \to M_0$ is projection of the first coordinate,

$$M_{-1}(i) = M^*_{-1}(i)/\varepsilon^* N_0(i)$$

for all i in Id G, and $p_{-1} : M^*_{-1} \to M_{-1}$ is the canonical quotient map, then

$$\begin{array}{ccccccccccc} \cdots \to & M^*_n & \to \cdots \to & M^*_1 & \to & M^*_0 & \xrightarrow{\varepsilon^*} & M^*_{-1} & \to & \text{Id } G & \\ & \downarrow = & & \downarrow = & & \downarrow p_0 & & \downarrow p_{-1} & & \downarrow = & \\ \cdots \to & M_n & \to \cdots \to & M_1 & \to & M_0 & \xrightarrow[\bar{\varepsilon}]{} & M_{-1} & \to & \text{Id } G, & \end{array} \qquad \text{(D3.3.5)}$$

where $\bar{\varepsilon}(m_0) = p_{-1}\varepsilon^*(m_0, 0)$. But $N_0 \cong \varepsilon^* N_0$ as follows.* As in (iii) above, assume K_v is the set of connected components of A_v for v in D, and i_κ is a distinguished identity of the component A_v^κ of A_v for each κ in K_v. Thus for each e in D and κ in $K_{\lambda e}$ there is a unique identity $i_{e(\kappa)}$ of Id $A_{\rho e}$ such that $i_{e(\kappa)} \sim A_e(i_\kappa)$. There is a groupnet diagram $(\mathcal{D}, \hat{T})$ with trivial vertex groupnet

$$\hat{T}_v = \{i_\kappa : \kappa \in K_v\},$$

for all v in D, and trivial edge morphism

$$\hat{T}_e : i_\kappa \mapsto i_{e(\kappa)}$$

for each κ in $K_{\lambda e}$ and e in D. It has mapping cylinder

$$w(\mathcal{D}, \hat{T}) = \langle w_e(*, i_\kappa), \kappa \in K_{\lambda e}, e \in D : \ \rangle,$$

which is a free groupnet. There is a groupnet morphism $\psi : G \to w(\mathcal{D}, \hat{T})$, with

$$\psi(i) = i_\kappa, \quad i \in \text{Id } A_v, \quad i \sim i_\kappa, \quad \kappa \in K_v, \quad v \in D,$$

$$\psi(m_v(a)) = i_\kappa, \quad a \in A_v^\kappa, \quad \kappa \in K_v, \quad v \in D,$$

and

$$\psi(m_e(*, i)) = w_e(*, i_\kappa), \quad i \in \text{Id } A_{\lambda e}, \quad i \sim i_\kappa, \quad \kappa \in K_{\lambda e}, \quad e \in D.$$

---

* I am indebted to my supervisor, Dr N.F. Smythe, for his formalisation of this proof.

A directed graph D^* is defined from D and the free groups $\mathbb{Z}G^{v}$ to have

$$V^* = \{(gm_v(A_v), i_\kappa) \in G/m_v(A_v) \times K_v, \rho g = m_v(i_\kappa)\} ,$$

and

$$E^* = \{(gm_{\lambda e}(A_{\lambda e}), i_\kappa, e) \in G/m_{\lambda e}(A_{\lambda e}) \times K_{\lambda e} \times E\} ,$$

with

$$\lambda(gm_{\lambda e}(A_{\lambda e}), i_\kappa, e) = (gm_{\lambda e}(A_{\lambda e}), i_\kappa)$$

and

$$\rho(gm_{\lambda e}(A_{\lambda e}), i_\kappa, e) = (gm_e(*, i_\kappa)m_{\rho e}(A_{\rho e}), i_{e(\kappa)}) .$$

From (iii) above, it is possible to write

$$N_0(j) = \coprod_{\substack{(j, m_{\lambda e}(i_\kappa)) \in z\mathbb{Z}G \\ g \in X_{\lambda e} \cap G(j, m_{\lambda e}(i_\kappa)) \\ e \in D}} \{gm_{\lambda e}(A_{\lambda e})\} \times T_{\lambda e}(i_\kappa) \times \{e\} ,$$

if each coset representative g of $G/m_{\lambda e}(A_{\lambda e})$ is replaced by its coset $gm_{\lambda e}(A_{\lambda e})$. If $\tilde{g} \in \operatorname{Ker} \varepsilon^*(j) \cap N_0$, then

$$\tilde{g} = \sum_{e \in D} \sum_{l=1}^{n(e)} p_l(g_l m_{\lambda e}(A_{\lambda e}), i_\kappa, e) ,$$

and

$$\sum_{e \in D} \sum_{l=1}^{n(e)} p_l\{(g_l m_{\lambda e}(A_{\lambda e}), i_\kappa) - (g_l m_e(*, i_\kappa)m_{\rho e}(A_{\rho e}), i_{e(\kappa)})\} = 0$$

in $\varepsilon^* N_0(j)$. Hence $\tilde{g}$ determines a closed edge path in D^* , and

$$(g_l m_{\lambda e}(A_{\lambda e}), i_\kappa) = (g_l m_{e_1}(*^{\pm 1}, i_1) \dots m_{e_r}(*^{\pm 1}, i_r) m_{\rho e}(A_{\rho e}), i_{\kappa'})$$

for a particular l, κ and e . But then $i_\kappa = i_{\kappa'}$ and there exists a in $A^{\kappa}_{\lambda e}$ such that

$$m_{\lambda e}(a) = m_{e_1}(*^{\pm 1}, i_1) \dots m_{e_r}(*^{\pm 1}, i_r) .$$

The image of this equation under ψ in $w(\mathcal{D}, \hat{T})$ is

$$i_\kappa = w_{e_1}\left(*^{\pm 1}, i_1\right) \ldots w_{e_r}\left(*^{\pm 1}, i_r\right).$$

Since $w(\mathcal{D}, T)$ is free, this product is reducible and there is an $s \leq r-1$ such that

$$m_{e_{s+1}}\left(*^{\pm 1}, i_{s+1}\right) = m_{e_s}\left(*^{\pm 1}, i_s\right)^{-1}.$$

The process continues, so that $\tilde{g} = 0$, and since ε^* is an epimorphism, $N_0 \cong \varepsilon^*(N_0)$ as required. Hence the bottom row of (D3.3.5) is exact. But $M_{-1} \cong TG$ in *G-Modreg* under the *G*-morphism $\eta : (g, [j]) \mapsto [\lambda g]$ defined whenever $j \in \mathrm{Id}\, A_v$, $g \in G$ and $m_v(j) = \rho g$. For

$$g = \prod_{l=1}^{n} p_l ,$$

so that by (D3.3.4),

$$(g, [j]) = \begin{cases} \left(\prod_{l=1}^{n-1} p_l . m_{v_n}(a), [\rho a]\right) \\ \left(\prod_{l=1}^{n-1} p_l . m_{e_n}([*], q), \left[A_{e_n}(q)\right]\right) \\ \left(\prod_{l=1}^{n-1} p_l . m_{e_n}([*^{-1}], q), [q]\right) \end{cases}$$

$$= \begin{cases} \left(\prod_{l=1}^{n-1} p_l , [\lambda a]\right) \\ \left(\prod_{l=1}^{n-1} p_l , [q]\right) \\ \left(\prod_{l=1}^{n-1} p_l , \left[A_{e_n}(q)\right]\right) \end{cases} \quad \text{in } M_{-1},$$

$$= (\lambda g, [k])$$

by induction, where $\lambda g = m_w(k)$ for a unique k in $\mathrm{Id}\, A_w$. That is, η is a monomorphism. It is clearly an epimorphism and the proof is complete. □

3.3.7 COROLLARY. *Let* $(\mathcal{D}, A)$ *be a groupnet diagram with* graph product $m : (\mathcal{D}, A) \to G$ *and let* $m : (\mathcal{D}, ZA) \to ZG$ *be the induced representation of* $(\mathcal{D}, ZA)$. *Let* $(\mathcal{D}, ZA, C)$ *be a complex diagram in which* C^v *is a free resolution of the trivial* A_v*-module* T_v *for each* v *in* D . *Then the* G*-mapping cylinder* $\mu : (\mathcal{D}, ZA, C) \to M$ *has for* M *a free* G*-resolution of* TG . *Moreover, if* G^* *is the group of loops at a selected identity* i *of* Id G *- the classical case - then* $M(i)$ *is a free* G^**-resolution of* Z .

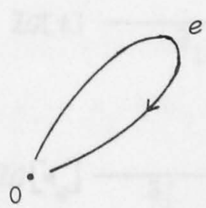
Proof. Suppose T is a maximal tree in the connected component G_i of G containing i . Each element g of $G(i, m_v(j))$, for any j in Id A_v , may be uniquely written in the form

$$g = g^* t_{i,m_v(j)}$$

for g^* in G^* and $t_{i,m_v(j)}$ in T . The classical result follows from (3.3.6) when every free generator x of M_n is replaced by the element $t_{i,\lambda x}[x]$ of M_n , and corresponding adjustments are made to the boundary maps. □

A simple example shows that it is not necessary that G be a graph product for the mapping cylinder M of an exact complex to be exact.

Let

$D =$ 
 ,

$$(\mathcal{D}, A) = \{A_0, A_e : A_0 \to A_0\} ,$$

where $A_0 = \langle t : \ \rangle \cong Z$ and $A_e(t) = 1$, and let $G = m(\mathcal{D}, A)$. Then if

$*_e = m_e([*], [0])$,

$$G = \left\langle t, *_e : *_e^{-1} t *_e = 1 \right\rangle$$

$$= \langle *_e : \ \rangle$$

$$\cong \mathbb{Z} \ \text{ since } \ \lambda *_e = \rho *_e = 0 \ .$$

As $m_0 : t \mapsto 1$ is not mono, G is not a graph product. If C^0 after augmentation is the free A_0-resolution of $\mathbb{Z}$,

$$0 \to \mathbb{Z}A_0[t] \to \mathbb{Z}A_0[0] \to \mathbb{Z} \to 0$$

with $\partial_1[t] = (t-1)[0]$, then $C^e : C^0 \to C^0$ may be extended from $C_0^e[0] = [0]$ and $C_1^e[t] = 0[0]$. The G-mapping cylinder M of $(\mathcal{D}, \mathbb{Z}A, C)$ is a free regular G-complex

$$0 \to \mathbb{Z}G[t|e] \longrightarrow \mathbb{Z}G[t] \oplus \mathbb{Z}G[0|e] \longrightarrow \mathbb{Z}G[0] \longrightarrow 0 \ ,$$

with

$$\partial_2[t|e] = [t] - *_e \mu_1^0 C_1^e[t] - m_0(t-1)[0|e] = [t] \ ,$$

$$\partial_1[t] = m_0(t-1)[0] = 0[0] \ ,$$

and

$$\partial_1[0|e] = \left(1 - *_e\right)[0] \ .$$

It follows that the augmented mapping cylinder is the direct sum of two exact sequences

$$0 \to \mathbb{Z}G[t|e] \xrightarrow[\partial_2]{} \mathbb{Z}G[t] \xrightarrow[\partial_1]{} 0$$

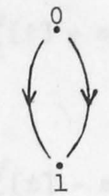
and

$$0 \longrightarrow \mathbb{Z}G[*_e] \xrightarrow[\partial_1']{} \mathbb{Z}G[0] \xrightarrow[\varepsilon]{} \mathbb{Z} \to 0 \ ,$$

where $[*_e] = [0|e]$, $\partial_1'[*_e] = -(*_e - 1)[0]$ and $\varepsilon[0] = -1$.

On the other hand, an equally simple example shows that exactness at each vertex complex is not sufficient to ensure exactness of the mapping cylinder.

Let

$D = e$  f ,

$$(\mathcal{D}, \mathsf{A}) = \{A_0, A_1; A_e, A_f : A_0 \to A_1\} ,$$

where $A_0 = \langle t : \ \rangle \cong \mathbb{Z}$, $A_1 = \langle s : \ \rangle \cong \mathbb{Z}$ and $A_e(t) = A_f(t) = 1$. If $G = m(\mathcal{D}, \mathsf{A})$, $*_e = m_e([*], [0])$ and $*_f = m_f([*], [0])$, then

$$G = \left\langle t, s, *_e, *_f : *_e^{-1} t *_e = 1 = *_f^{-1} t *_f \right\rangle$$
$$= \langle s, *_e, *_f : \ \rangle ,$$

which is a groupnet of the homotopy type of the free group $\left\langle s, *_e^{-1} *_f : \ \right\rangle$ on two generators (*cf.* p. 15). Here $m_0 : t \mapsto 1$ and $m_1 : s \mapsto s$, and G is not a graph product. The augmented free resolutions

$$C^0 = 0 \to \mathbb{Z}A_0[t] \to \mathbb{Z}A_0[0] \to \mathbb{Z} \to 0 ,$$

and

$$C^1 = 0 \to \mathbb{Z}A_1[s] \to \mathbb{Z}A_1[1] \to \mathbb{Z} \to 0$$

with chain maps $C^e, C^f : C^0 \to C^1$ extended from $C_0^e([0]) = C_0^f([0]) = [1]$ and $C_1^e[t] = C_1^f[t] = 0[s]$, determine a complex diagram $(\mathcal{D}, \mathbb{Z}\mathsf{A}, \mathsf{C})$ with augmented free regular G-mapping cylinder

$$M = \mathrm{Id}\ G \to M_2 \to M_1 \to M_0 \to TG \to \mathrm{Id}\ G .$$

Here $\mathrm{Id}\ G = \{0, 1\}$,

$$M_2(1) = \mathbb{Z}G(1, 0)[t|e] \oplus \mathbb{Z}G(1, 0)[t|f] ,$$

$$M_1(1) = \mathbb{Z}G(1, 0)[t] \oplus \mathbb{Z}G(1, 1)[s] \oplus \mathbb{Z}G(1, 0)[0|e] \oplus \mathbb{Z}G(1, 0)[0|f] ,$$

$$M_0(1) = \mathbb{Z}G(1, 0)[0] \oplus \mathbb{Z}G(1, 1)[1] ,$$

and the two-dimensional boundary map is extended from

$$\partial_2[t|e] = [t] - *_e\mu_1^1 C_1^e[t] - m_0(t-1)[0|e] = [t] \ ,$$

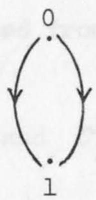
and

$$\partial_2[t|f] = [t] - *_f\mu_1^1 C_1^f[t] - m_0(t-1)[0|f] = [t] \ .$$

Hence $[t|e] - [t|f] \in \text{Ker } \partial_2(0)$, but $\text{Im } \partial_3(0) = 0$ in $M_2(0)$, and the mapping cylinder is not exact.

As an illustration of the case in which the conditions of (3.3.7) are satisfied, the next example is worked through in detail.

3.3.8 EXAMPLE. Let

$D = e$ 

f ,

$$(\mathcal{D}, A) = \{A_0, A_1; A_e, A_f : A_0 \to A_1\} \ ,$$

where A_e and A_f are monomorphisms, $A_0 = \langle x_i, 1 \le i \le n : \ \rangle$ and $A_1 \cong F/R$ is a torsion-free one-relator group (which may be free);

$$A_1 = \langle y_j, 1 \le j \le m : r \rangle .$$

If $G = m(\mathcal{D}, A)$ then

$$G = \Big\langle x_i, y_j, 1 \le i \le m, 1 \le j \le m, *_e, *_f :$$
$$r, *_e^{-1} x_i *_e = A_e(x_i), *_f^{-1} x_i *_f = A_f(x_i), 1 \le i \le n \Big\rangle$$
$$= \Big\langle y_j, 1 \le j \le m, t, *_e, *_f : r, t = *_e^{-1} *_f, t^{-1} A_e(x_i) t = A_f(x_i), 1 \le i \le n \Big\rangle$$

and the loop group G^* at identity 1 is isomorphic to $G/\langle *_e \rangle$;

$$G^* = \Big\langle y_j, 1 \le j \le m, t : r, t^{-1} A_e(x_i) t = A_f(x_i), 1 \le i \le n \Big\rangle .$$

By the Lyndon Identity Theorem it follows [18, §11] that the complex

$$C^1 = 0 \to \mathbb{Z}A_1[r] \to \bigoplus_{j=1}^{m} \mathbb{Z}A_1[y_j] \to \mathbb{Z}A_1[1] \to 0$$

with boundary maps

$$\partial_2[r] = \sum_{j=1}^{m} \frac{\partial r}{\partial y_j}[y_j] ,$$

and

$$\partial_1[y_j] = (y_j-1)[1] ,$$

given in terms of Fox's free differential calculus [9] in $\mathbb{Z}F$, is a free resolution of $\mathbb{Z}$. Together with

$$C^0 = 0 \to \bigoplus_{i=1}^{n} \mathbb{Z}A_0[x_i] \to \mathbb{Z}A_0[0] \to 0 ,$$

and chain maps C^e and C^f extended from $C_0^e[0] = C_0^f[0] = [1]$,

$$C_1^e[x_i] = \sum_{j=1}^{m} \frac{\partial A_e(x_i)}{\partial y_j}[y_j] \quad \text{and} \quad C_1^f[x_i] = \sum_{j=1}^{m} \frac{\partial A_f(x_i)}{\partial y_j}[y_j] ,$$

it forms a complex diagram $(\mathcal{D}, \mathbb{Z}A, C)$ satisfying the requirements of (3.3.7). The G-mapping cylinder M of $(\mathcal{D}, \mathbb{Z}A, C)$ is an exact free resolution of TG , with

$$M_2(1) = \mathbb{Z}G(1, 1)[r] \oplus \left(\bigoplus_{i=1}^{n} \mathbb{Z}G(1, 0)[x_i|e]\right) \oplus \left(\bigoplus_{i=1}^{n} \mathbb{Z}G(1, 0)[x_i|f]\right) ,$$

$$M_1(1) = \left(\bigoplus_{j=1}^{m} \mathbb{Z}G(1, 1)[y_j]\right) \oplus \left(\bigoplus_{i=1}^{n} \mathbb{Z}G(1, 0)[x_i]\right) \oplus \mathbb{Z}G(1, 0)[0|e] \oplus \mathbb{Z}G(1, 0)[0|f] ,$$

$$M_0(1) = \mathbb{Z}G(1, 1)[1] \oplus \mathbb{Z}G(1, 0)[0] ,$$

and boundary maps

$$\partial_2[r] = \sum_{j=1}^{m} \mu_1^1\left(\frac{\partial r}{\partial y_j}\right)[y_j] ,$$

$$\partial_2[x_i|e] = [x_i] - *_e\mu_1^1 C_1^e[x_i] - m_0(x_i-1)[0|e] ,$$

$$\partial_2[x_i|f] = [x_i] - *_f\mu_1^1 C_1^f[x_i] - m_0(x_i-1)[0|f] ,$$

$$\partial_1[y_j] = m_1(y_j-1)[1] ,$$

and

$$\partial_1[x_i] = m_0(x_i - 1)[0] .$$

Since $*_e$ is a maximal tree in G , $M(1)$ is a free G^*-resolution of $\mathbb{Z}$, freely generated as a G^*-module in dimension 2 by

$$\left\{[r],\ *_e^{-1}[x_i|e],\ *_e^{-1}[x_i|f],\ 1 \le i \le n\right\} ;$$

in dimension 1 by

$$\left\{[y_j],\ 1 \le j \le m,\ *_e^{-1}[x_i],\ 1 \le i \le n,\ *_e^{-1}[0|e],\ *_e^{-1}[0|f]\right\} ;$$

and in dimension 0 by $\left\{[1],\ *_e^{-1}[0]\right\}$. //

The essence of Theorem (3.3.6) is that the process of taking mapping cylinders is a well-behaved one which preserves many of the properties of the original complex diagram. In particular, for groups G known to be graph products, the mapping cylinder provides a very simply defined G-free resolution given in terms of the vertex resolutions. The only added structure arises from the diagram itself; as a result the mapping cylinder resolution is much more amenable to computation than the bar resolution BG or the resolution defined by Lyndon [18, §5] from a presentation of G with a complete set of identities.

As was mentioned above, the mapping cylinder is not generally a direct sum of complexes, in contrast with Trotter's *group systems* [40], so its homology cannot be expected to split simply in terms of the homology of its vertex complexes. Chapter 4 examines the relationship between the two.

CHAPTER 4

THE MAYER-VIETORIS SEQUENCE

4.1 Homology

It is possible to measure the deviation from exactness of a chain complex in any abelian category by means of the homology objects of the complex [20, VIII.4].

4.1.1 DEFINITION. Let (C, ∂) be a standard (left) R-complex. The *homology module* $H(C)$ of C is the standard (left) graded R-module $\{H_n(C) : n \in Z\}$ with $zH(C) = zC$ and nth *homology module* $H_n(C)$ defined by

$$H_n(C)(z) = \mathrm{Ker}\ \partial_n(z)/\mathrm{Im}\ \partial_{n+1}(z) \quad \forall z \in zC\ ,\ n \in Z\ .$$

Its R-action is induced from C.

Thus complex C is exact if and only if $H_n(C) = zC$ for each n in Z.

When $\sigma : R \to S$ is any ringnet morphism, D is any standard S-complex and $f : C \to D$ is any σ-chain map, the square

$$\begin{array}{ccc} C_n & \xrightarrow{\partial} & C_{n-1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\ D_n & \xrightarrow[\partial]{} & D_{n-1} \end{array}$$

ensures that f induces a well-defined σ-chain map of graded modules $H(f) : H(C) \to H(D)$, the *induced homology chain map*. Under this definition,

$$H : R\text{-}Comp(Z) \to R\text{-}Comp(Z)$$

is an additive covariant functor, the *homology functor*, on each such abelian category.

Homotopic chain maps induce homology chain maps which differ only by a 'change of base point isomorphism', as the following lemma shows.

4.1.2 LEMMA. *Let* C *be a standard* R*-complex,* D *be a standard* S*-complex, and suppose* $\sigma, \tau : R \to S$ *are homotopic ringnet morphisms with homotopy* $\nu : \sigma \simeq \tau : \mathbb{Z}I \otimes R \to S$. *If* $f, g : C \to D$ *are respectively* σ, τ*-chain maps with*

$$f(z) = \nu([*], \lambda z)g(z) \quad \forall z \in zC ,$$

which are ν*-homotopic, then*

$$H(f) = \nu([*], \lambda -) \circ H(g) : H(C) \to H(D) .$$

Proof. If $F : f \simeq g$, then $(\partial F + F\partial)(c) = f(c) - \nu([*], \lambda c)g(c)$ for all c in C . If $c \in \operatorname{Ker} \partial_n(z)$ then $\partial F(c) = f(c) - \nu([*], \lambda c)g(c)$, so that

$$f(c) + \operatorname{Im} \partial_{n+1}(z) = \nu([*], \lambda c).\big(g(c) + \operatorname{Im} \partial_{n+1}(z)\big) . \qquad \square$$

4.1.3 COROLLARY. *If* $f \simeq g : C \to D$ *in* R-Comp($\mathbb{Z}$) *then*

$$H(f) = H(g) : H(C) \to H(D) . \qquad \square$$

4.1.4 COROLLARY. *Homotopic chain complexes in* R-Comp($\mathbb{Z}$) *have isomorphic homology modules.*

Proof. If $C \simeq D$ in R-Comp($\mathbb{Z}$) with homotopy equivalence $f : C \to D$ and homotopy inverse $g : D \to C$ then $H(f \circ g) = H(1_D)$ and $H(g \circ f) = H(1_C)$ by (4.1.3). Hence $H(f) \circ H(g) = 1_{H(D)}$ and $H(g) \circ H(f) = 1_{H(C)}$. $\square$

4.1.5 LEMMA (Snake Lemma) [20, VIII.4.5]. *Suppose*

$$A \overset{\alpha}{\rightarrowtail} B \overset{\beta}{\twoheadrightarrow} C$$

is a short exact sequence of complexes in R-Comp($\mathbb{Z}$) . *Then there is a set* $w = \{w_n : n \in \mathbb{Z}\}$ *of standard* R*-morphisms* $w_n : \operatorname{Ker} \partial^C_n \to \operatorname{Coker} \partial^A_n$ *such that the sequence*

$$\mathbb{Z} \to \operatorname{Ker} \partial^A_n \to \operatorname{Ker} \partial^B_n \to \operatorname{Ker} \partial^C_n \to \operatorname{Coker} \partial^A_n \to \operatorname{Coker} \partial^B_n \to \operatorname{Coker} \partial^C_n \to \mathbb{Z}$$

with induced morphisms is exact in R-Comp($\mathbb{Z}$) *for each* n *in* $\mathbb{Z}$.

Proof. Category R-Comp($\mathbb{Z}$) is abelian. The morphisms w are constructed as follows: if $c \in \operatorname{Ker} \partial_n^C(z)$ for z in zC then, since β is an epimorphism, there exists b in $B_n(z)$ with $\beta(b) = c$. As $\partial^C\beta(b) = \partial^C(c) = 0 = \beta(\partial^B b)$, the exactness of the middle term ensures the existence of $a\langle c\rangle$ in $A_{n-1}(z)$ for which $\alpha(a\langle c\rangle) = \partial^B b$. Thus the map

$$w_n(c) = a\langle c\rangle + \operatorname{Im} \partial_n^A(z)$$

is well-defined. It is called the *connecting morphism* in dimension n. □

4.1.6 LEMMA (The Long Exact Homology Sequence) [15, IV, Ex.2.4]. *For each short exact sequence of chain complexes*

$$A \overset{\alpha}{\rightarrowtail} B \overset{\beta}{\twoheadrightarrow} C$$

in $R\text{-}Comp(Z)$ *, there is a graded module morphism* $\omega : H(C) \to H(D)$ *of degree* -1 *, such that the long homology sequence*

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\omega_n} H_{n-1}(A) \to \cdots$$

in $R\text{-}Comp(Z)$ *is exact.*

Proof. Since $R\text{-}Comp(Z)$ is abelian the proof is that of [15, IV.2.1] with a suitable change of notation. If w is the connecting morphism of (4.1.5), then

$$\omega_n\left(c + \operatorname{Im} \partial_{n+1}^C(z)\right) = w_n(c) = a\langle c\rangle + \operatorname{Im} \partial_n^A(z)$$

for each z in zC and n in $\mathbb{Z}$, and ω_n is called the nth *invariant boundary*. □

4.1.7 LEMMA. *The functor* H *preserves arbitrary direct products and coproducts in* $R\text{-}Comp(Z)$.

Proof. Let $\{(C^\alpha, \partial^\alpha) : \alpha \in A\}$ be a set of standard R-complexes over Z. Then $\left(\prod_\alpha C^\alpha\right)_n = \prod_\alpha \left(C_n^\alpha\right)$ and the boundary map is $\prod \partial\{c_\alpha\}_A = \left\{\partial^\alpha c_\alpha\right\}_A$. The map

$$H_n\left(\prod_\alpha C^\alpha\right) = \operatorname{Ker}\left(\prod \partial\right)_n \Big/ \operatorname{Im}\left(\prod \partial\right)_{n+1} \to \prod_\alpha H_n(C^\alpha)$$

given by

$$\{c_\alpha\}_A + \operatorname{Im}\left(\prod \partial\right)_{n+1}(z) \longmapsto \left\{c_\alpha + \operatorname{Im}\,\partial^\alpha_{n+1}(z)\right\}_A$$

is the required isomorphism for the product. Popescu [28, 3.1.1] gives the result for the coproduct because H is additive. □

4.1.8 DEFINITION. If A is a groupnet, let M be a right A-module and N be a left A-module. If C is a regular projective left A-resolution of TA, the *homology module* $H_*(A;\, M)$ *of* A *with coefficients in* M is the homology module

$$H(M \otimes_A C) ,$$

and the *cohomology module* $H^*(A;\, N)$ *of* A *with coefficients in* N is the homology module

$$H(\hom_A(C,\, N)) .$$

By convention the induced hom complex is written

$$\cdots \to \hom_A(C,\, N)_n \xrightarrow{\delta^n} \hom_A(C,\, N)_{n+1} \to \cdots$$

so that $H^n(A;\, N) = \operatorname{Ker}\delta^n/\operatorname{Im}\delta^{n-1}$.

Proof that these definitions are independent of the choice of the regular projective A-resolution C of TA is deferred until (4.2.5).

4.2 The Comparison Theorem

In any abelian category it is possible to compare any positive projective complex over one object with any resolution over another object, given a morphism between these objects.

4.2.1 LEMMA (Restricted Comparison Theorem). *In* R-Comp(Z) , *let* X *be a projective complex over* C , *augmented by* $\varepsilon : X \to C$, *and let* X' *be* *a resolution of* C' *, with augmentation map* $\varepsilon' : X' \to C'$ *. For each* *R-morphism* $\gamma : C \to C'$ *there is a chain map* $f : X \to X'$ *such that* $\varepsilon' f = \gamma\varepsilon$ *. Any two such chain maps are* $\chi(1)$*-homotopic.*

Proof. The proof uses only the categorical properties of projectivity and exactness in R-*Comp*(Z) , so that of Mac Lane [19, III.6.1, III.6.2] for the classical case applies *verbatim*. □

The chain map f is said to *lift* γ .

4.2.2 COROLLARY. *A positive projective exact complex in* R-Mod(Z) *has a contracting homotopy.*

Proof. For such a complex P , both the identity $1 : P \to P$ and the zero chain map $0 : P \to P$ lift the R-morphism $1 = 0 : Z \to Z$ in dimension -1 . By (4.2.1), $1 \simeq 0$. □

The availability of a definition of homotopy of chain maps between complexes over different ringnets enables an extension of (4.2.1) to be made. Recall that in any category C an object P is projective if for every diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle f} \\ M & \overset{}{\underset{g}{\twoheadrightarrow}} & N \end{array}$$

in C for which g is an epimorphism, the morphism f factors through g . Further, any epimorphism in R-Mod(Z) is an epimorphism in R-Mod , thus: if $g : M \twoheadrightarrow N$ is in R-Mod(Z) then $g : M(z) \to N(z)$ is a surjection for each z in Z . Hence if $f, h : N \to L$ are morphisms in R-Mod such that $f \circ g = h \circ g$, then $f(z) = h(z)$ for z in Z , and if $n \in N(z)$ there exists $m \in M(z)$ with $g(m) = n$, so that $f(n) = h(n)$.

4.2.3 THEOREM (Comparison Theorem). *Let* C *be a positive complex over* M *in* R-Comp(Y) *, for which* C_n *is projective in* R-Mod *for each* n *, let* D *be a resolution over* N *in* S-Comp(Z) *and let* $\sigma : R \to S$ *be a* *ringnet morphism. Then any* σ*-morphism* $f : M \to N$ *lifts to a* σ*-chain map* $g : C \to D$ *such that* $\varepsilon g = f\varepsilon$ *. Any two such chain maps are* $\chi(\sigma)$*-homotopic.*

Proof. The pullback ${}^{\sigma}D$ is a resolution of ${}^{\sigma}N$ in $R\text{-}Comp({}^{\sigma}Z)$ by (3.1.6.iii). The σ-pullback projection $\sigma^* : {}^{\sigma}D \to D$ and the σ-pullback projection $\sigma^*_{-1} : {}^{\sigma}N \to N$ satisfy

$$\begin{array}{ccc} {}^{\sigma}D_0 & \xrightarrow{{}^{\sigma}\varepsilon} & {}^{\sigma}N \\ {\scriptstyle \sigma^*_0}\downarrow & & \downarrow{\scriptstyle \sigma^*_{-1}} \\ D_0 & \xrightarrow[\varepsilon]{} & N \end{array} \quad .$$

Morphism $f : M \to N$ factors uniquely *via* $\sigma(f) : M \to {}^{\sigma}N$ through σ^*_{-1} . There then exists an R-chain map $\sigma(g) : C \to {}^{\sigma}D$ lifting $\sigma(f)$, for which ${}^{\sigma}\varepsilon\sigma(g) = \sigma(f)\varepsilon$, and such that any two such lifting maps are $\chi(1)$-homotopic. The proof of this result follows the usual course but is sketched here because of the special circumstances involved.

Consider the diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{\varepsilon} & M \\ & & \downarrow{\scriptstyle \sigma(f)} \\ {}^{\sigma}D_0 & \xrightarrow[{}^{\sigma}\varepsilon]{} & {}^{\sigma}N \to {}^{\sigma}Z \end{array}$$

in which ${}^{\sigma}\varepsilon$ is an epimorphism in $R\text{-}Mod({}^{\sigma}Z)$ and hence an epimorphism in $R\text{-}Mod$. Since C_0 is projective in $R\text{-}Mod$, there exists an R-morphism $\sigma(g)_0 : C_0 \to {}^{\sigma}D_0$ lifting $\sigma(f)$. If the R-morphisms $\sigma(g)_k : C_k \to {}^{\sigma}D_k$, commuting with the boundary maps, have been found for $0 \le k \le n\text{-}1$, then $\sigma(g)_n : C_n \to {}^{\sigma}D_n$ is determined as above, since in the diagram

$$\begin{array}{ccc} C_n & \longrightarrow & C_{n-1} \\ & & \big\downarrow \sigma(g)_{n-1} \\ {}^{\sigma}D_n & \xrightarrow[\ {}^{\sigma}\partial_n\ ]{} & \operatorname{Im}{}^{\sigma}\partial_n \to {}^{\sigma}Z\ , \end{array}$$

${}^{\sigma}\partial_n$ is an epimorphism in *R-Mod*$({}^{\sigma}Z)$. Any other morphism $\sigma(h)$ lifting $\sigma(f)$ satisfies $\sigma(g)(y) = \sigma(h)(y)$ for all y in Y ; construction of the $\chi(1)$-homotopy $\sigma(g) \simeq \sigma(h)$ is similar and is not given (*cf.* [19, III.6.2]).

The composite $\sigma^*\sigma(g) = g$ is a σ-chain map satisfying the requirements of the theorem. Any other such chain map h factors uniquely as $h = \sigma^*\sigma(h)$, so that there is a $\chi(1)$-homotopy $\sigma(H) : \sigma(g) \simeq \sigma(h)$. But then the composite σ-morphisms

$$H_n = \sigma^*_{n+1} \circ \sigma(H)_n : C_n \to {}^{\sigma}D_{n+1} \to D_{n+1}$$

comprise a $\chi(\sigma)$-homotopy $g \simeq h$. □

4.2.4 LEMMA (Regular Comparison Theorem). *Let C be a projective complex over M in R-Compreg , let D be a resolution over N in S-Compreg and let $\sigma : R \to S$ be a ringnet morphism. Then any σ-morphism $f : M \to N$ lifts to a σ-chain map $g : C \to D$ with $\varepsilon g = f\varepsilon$ and any two such lifting chain maps are $\chi(\sigma)$-homotopic.*

Proof. Since ${}^{\sigma}D \in$ *R-Compreg* by (3.1.6.i), it is sufficient that C_n be projective in *R-Modreg* for each n , for then the Restricted Comparison Theorem may be applied in the proof of (4.2.3) to give this result. □

4.2.5 COROLLARY. *The definition* (4.1.8) *of the (co)homology modules of a groupnet A with coefficients in a given A-module is independent of the choice of the regular projective A-resolution C of TA .*

Proof. Suppose D is any other regular projective A-resolution of TA . Thus in *A-Compreg* , (4.2.4) implies the existence of A-chain maps $f : C \to D$ and $g : D \to C$ lifting the identity map $TA \to TA$ and hence the existence of $\chi(1)$-homotopies $f \circ g \simeq 1_D$ and $g \circ f \simeq 1_C$. Hence $C \simeq D$.

The additive functors $M \otimes_A -$ and $\hom_A(-, N)$ preserve homotopy in *A-Compreg* (3.2.11), so that $M \otimes_A C \simeq M \otimes_A D$ and $\hom_A(C, N) \simeq \hom_A(D, N)$ in $Z\text{-}Comp(zM \otimes_A \text{Id } A)$ and $Z\text{-}Comp(\hom_A(\text{Id } A, zN))$ respectively. By (4.1.4),

$$H(M \otimes_A C) \cong H(M \otimes_A D) ,$$

and

$$H(\hom_A(C, N)) \cong H(\hom_A(D, N)) . \quad \square$$

4.2.6 COROLLARY. *Homotopic groupnet morphisms induce homotopic chain maps on the augmented bar resolutions.*

Proof. Let $f, g : A \to B$ be homotopic groupnet morphisms with a homotopy $\sigma : f \simeq g$. If $\sigma([*], \lambda-)\mathcal{B}g : \mathcal{B}A \to \mathcal{B}B$ is the f-chain map extended from

$$\sigma([*], \lambda-)\mathcal{B}g_{-1}[i] = \sigma([*], i).[gi] = [fi] \quad \forall i \in \text{Id } A ,$$

$$\sigma([*], \lambda-)\mathcal{B}g_0[i] = [fi] \quad \forall i \in \text{Id } A ,$$

and

$$\sigma([*], \lambda-)\mathcal{B}g_n[a_1 \mid \dots \mid a_n] = \sigma([*], \lambda a_1)[ga_1 \mid \dots \mid ga_n]$$

for all $[a_1 \mid \dots \mid a_n]$ in X_n and $n \geq 1$, then $\mathcal{B}f$ and $\sigma([*], \lambda-)\mathcal{B}g$ both lift the morphism $\mathcal{B}f_{-1} : TA \to TB$. By (4.2.4), the chain maps $\mathcal{B}f$ and $\sigma([*], \lambda-)\mathcal{B}g$ are $\chi(f)$-homotopic, hence $\mathcal{B}f$ and $\mathcal{B}g$ are σ-homotopic. $\square$

4.2.7 THEOREM. *If the groupnet morphism $r : A \to B$ is a retraction, M is any regular right B-module and N is any regular left B-module then*

$$H_*(A; M^r) \cong H_*(B; M) ,$$

and

$$H^*(A; {}^rN) \cong H^*(B; N)$$

in the following two cases:

(i) B *is the loop group of the connected groupnet* A *at a selected identity;*

(ii) A *is the covering groupnet corresponding to the wide subgroupnet* B *of some groupnet* C .

Proof. In both cases, if $k \in \operatorname{Id} B$, $l \in \operatorname{Id} A$ and $r(l) = k$, there exists a unique element $a\langle k, l\rangle = a$ in A such that $\lambda a = k$, $\rho a = l$ and $r(a) = k$. In the former case, if $B = A(0, 0)$ for a selected identity 0 , the spine $*_l$ has this property (1.6). In the latter case, $A = \{(\gamma, c) \in C/B \times C : \rho\gamma = \lambda c\}$, B is a subgroupnet of A under the identification $b \mapsto (B\lambda b, b)$ for all b in B , and if a coset representative x_γ is chosen for each coset γ , then $r(\gamma, c) = x_\gamma c\left(x_{\gamma c}\right)^{-1}$. Hence if $r(\gamma, \rho\gamma) = \lambda x_\gamma = k$, the element $\left(H\lambda x_\gamma, x_\gamma\right)$ of A has the required property.

Let L be any regular left B-module. Then the map $(m, l) \mapsto \big((m, i), (i, l)\big)$, defined whenever $\lambda m = i = \rho l$ in $\operatorname{Id} B$, is an isomorphism

$$M \otimes_B L \cong M^r \otimes_A {}^rL .$$

Any B-morphism $g : L \to N$ determines the pullback A-morphism ${}^rg : {}^rL \to {}^rN$. Any A-morphism $f : {}^rL \to {}^rN$ determines a B-morphism $\bar{f} : L \to N$ where $f(i, l) = (i, \bar{f}l)$ for each i in $\operatorname{Id} B$, since

$$\big(\lambda b, \bar{f}(bl)\big) = f(\lambda b, b.l) = b.f(\rho b, l) = b(\rho b, \bar{f}l) = \big(\lambda b, b\bar{f}(l)\big) .$$

Thus

$$\hom_B(L, N) \cong \hom_A\left({}^rL, {}^rN\right)$$

(compare with the Pullback Lemma [19, V.1.2]).

However, the pullback ${}^r\mathrm{B}B$ of the bar resolution of B is a *projective* A-resolution of TA , as follows. For each pair (k, l) in $\operatorname{Id} B \times \operatorname{Id} A$ such that $r(l) = k$, and each generator x of X_n in $\mathrm{B}B_n$

such that $\lambda x = k$, the element (l, x) of ${}^{r}BB_{n}(l)$ may be uniquely written in the form

$$(l, x) = (\alpha(k, l))^{-1}(k, x) .$$

Hence, if g is an epimorphism in the A-*Compreg* diagram

$$\begin{array}{ccccc} & & {}^{r}BB_{n} & & \\ & & \downarrow{\scriptstyle f} & & \\ P & \underset{g}{\twoheadrightarrow} & Q & \longrightarrow & \mathrm{Id}\, A \, , \end{array}$$

then the lifting map $\bar{g} : {}^{r}BB_{n} \to P$ may be generated by A-action from any set of elements

$$\{p(k, x) \in P : g\big(p(k, x)\big) = f(k, x),\ k \in \mathrm{Id}\, B,\ \lambda x = k\} .$$

That is,

$$H\Big(M^{r} \otimes_{A} {}^{r}BB\Big) \cong H(M \otimes_{B} BB) ,$$

and

$$H\Big(\hom_{A}\big({}^{r}BB, {}^{r}N\big)\Big) \cong H\big(\hom_{B}(BB, N)\big) . \quad \square$$

4.3 Mayer-Vietoris Sequences

This section compares the (co)homology of a mapping cylinder with the (co)homologies of the vertex complexes comprising it, by means of an exact sequence.

4.3.1 LEMMA. *Let* $\sigma : (\mathcal{D}, \mathrm{R}) \to S$ *be a representation of a ringnet diagram and let* $\mu : (\mathcal{D}, \mathrm{R}, \mathrm{C}) \to M$ *be the* $\sigma(\mathcal{D}, \mathrm{R})$*-mapping cylinder of the standard complex diagram* $(\mathcal{D}, \mathrm{R}, \mathrm{C})$ *. For any right* S*-module* N *there is a ringnet representation* $j : (\mathcal{D}, \mathrm{Z}) \to \mathrm{Z}$ *and a standard complex diagram* $(\mathcal{D}, \mathrm{Z}, \mathrm{N} \otimes \mathrm{C})$ *such that the* $j(\mathcal{D}, \mathrm{Z})$*-mapping cylinder of* $(\mathcal{D}, \mathrm{Z}, \mathrm{N} \otimes \mathrm{C})$ *is isomorphic to* $N \otimes_{S} M$ *.*

Proof. The representation $j : (\mathcal{D}, \mathrm{Z}) \to \mathrm{Z}$ consists of the obvious identity morphisms and homotopies. The pullback N^v of N along $\sigma_v : R_v \to S$ determines a standard $\mathcal{Z}$-complex $(N \otimes C)^v = N^v \otimes_v C$.

Morphism $\overline{C}^e = (N \otimes C)^e : (N \otimes C)^{\lambda e} \to (N \otimes C)^{\rho e}$ is given by

$$\overline{C}^e_k(n, c) = \left[n\sigma_e([*], \lambda c),\ C^e_k(c)\right]$$

and is a $\mathcal{Z}1_e$-chain map. (The notational conventions here correspond to those of (3.3.3) with S replaced by N .) If $\eta : (\mathcal{D}, \mathcal{Z}, N \otimes C) \to H$ is the $\mathcal{Z}$-mapping cylinder of $(\mathcal{D}, \mathcal{Z}, N \otimes C)$, then

$$zM = \bigvee_{v \in D} z\left[S^v \otimes_v C^v\right] \Big/ \Big\langle (q, z) \sim \left[q\sigma_e([*], \lambda z),\ C^e(z)\right],\ e \in D \Big\rangle ,$$

$$zH = \bigvee_{v \in D} z\left[N^v \otimes_v C^v\right] \Big/ \Big\langle (p, z) \sim \left[p\sigma_e([*], \lambda z),\ C^e(z)\right],\ e \in D \Big\rangle ,$$

$$M_n(Z) = \coprod_{\substack{z \in Z_v \\ v \in D}} \left[S^v \otimes_v C^v_n\right](z) \oplus \coprod_{\substack{z \in Z_{\lambda e} \\ e \in D}} \left[S^{\lambda e} \otimes_{\lambda e} C^{\lambda e}_{n-1}\right](z) \times \{e\}$$

for each Z in zM , and

$$H_n(Y) = \coprod_{\substack{y \in Y_v \\ v \in D}} \left[N^v \otimes_v C^v_n\right](y) \oplus \coprod_{\substack{y \in Y_{\lambda e} \\ e \in D}} \left[N^{\lambda e} \otimes_{\lambda e} C^{\lambda e}_{n-1}\right](y) \times \{e\}$$

for each Y in zH .

By (2.2.12) and tensor associativity,

$$\left[N^v \otimes_v C^v\right] \cong N \otimes_S \left[S^v \otimes_v C^v\right]$$

for each v in D . The result follows from (2.2.14). □

In other words, the mapping cylinder commutes with tensor products.

4.3.2 THEOREM (Mayer-Vietoris Sequence). *Let $\sigma : (\mathcal{D}, \mathcal{R}) \to S$ be a ringnet representation and $\mu : (\mathcal{D}, \mathcal{R}, \mathcal{C}) \to M$ be the $\sigma(\mathcal{D}, \mathcal{R})$-mapping cylinder of the standard complex diagram $(\mathcal{D}, \mathcal{R}, \mathcal{C})$. Let N be a right S-module and L be a left S-module. There are abelian groupnet morphisms i, p, ∂ and ι, π, δ such that the following homology sequences, respectively, are exact:*

$$\cdots \to \coprod_{e\in D} H_m\left(N^{\lambda e} \otimes_{\lambda e} C^{\lambda e}\right) \times \{e\} \xrightarrow{\partial_m} \coprod_{v\in D} H_m\left(N^{v} \otimes_{v} C^{v}\right) \xrightarrow{i_m} H_m(N \otimes_S M)$$

$$\xrightarrow{p_m} \coprod_{e\in D} H_{m-1}\left(N^{\lambda e} \otimes_{\lambda e} C^{\lambda e}\right) \times \{e\} \to \cdots , \quad \text{(D4.3.1)}$$

$$\cdots \to H_m\left(\hom_S(M, L)\right) \xrightarrow{\iota_m} \prod_{v\in D} H_m\left(\hom_v\left(C^v, {}^vL\right)\right) \xrightarrow{\delta_m}$$

$$\prod_{e\in D} H_m\left(\hom_{\lambda e}\left(C^{\lambda e}, {}^{\lambda e}L\right)\right) \times \{e\} \xrightarrow{\pi_{m+1}} H_{m+1}\left(\hom_S(M, L)\right) \to \cdots . \quad \text{(D4.3.2)}$$

Proof. (D4.3.1). By (4.3.1) there is a Z-mapping cylinder $\eta : (\mathcal{D}, Z, N \otimes C) \to H$ of the standard complex diagram $(\mathcal{D}, Z, N \otimes C)$ such that $H \cong N \otimes_S M$. From (3.3.5) there is a short exact sequence

$$\coprod_{v\in D} \left(N^v \otimes_v C^v_m\right) \xrightarrow{i^*_m} N \otimes_S M_m \xrightarrow{p^*_m} \coprod_{e\in D} \left(N^{\lambda e} \otimes_{\lambda e} C^{\lambda e}_{m-1}\right) \times \{e\}$$

in Z-$Comp(zH)$ for each dimension m. This short exact sequence induces the long exact homology sequence (4.1.6),

$$\cdots \to H_m\left(\coprod_{e\in D} \left(N^{\lambda e} \otimes_{\lambda e} C^{\lambda e}\right) \times \{e\}\right) \xrightarrow{\omega_{m+1}} H_m\left(\coprod_{v\in D} N^v \otimes_v C^v\right) \xrightarrow{H\left(i^*_m\right)} H_m(N \otimes_S M)$$

$$\xrightarrow{H\left(p^*_m\right)} H_{m-1}\left(\coprod_{e\in D} \left(N^{\lambda e} \otimes_{\lambda e} C^{\lambda e}\right) \times \{e\}\right) \to \cdots .$$

Here $\omega_{m+1} = H\left(\partial^m_1\right)$ where by (3.3.4),

$$\partial^m_1\left(n, c_m, e\right) = \left(n, c_m\right) - \left(n\sigma_e([*], \lambda c_m), C^e_m\left(c_m\right)\right) ;$$

the result is found directly from the evaluation of w in (4.1.5).

As the homology functor preserves arbitrary coproducts (4.1.7),

$$H\left(\coprod_{v\in D} N^v \otimes_v C^v\right) \cong \coprod_{v\in D} H\left(N^v \otimes_v C^v\right) ,$$

and

$$H\left(\coprod_{e\in D} \left(N^{\lambda e} \otimes_{\lambda e} C^{\lambda e}\right) \times \{e\}\right) \cong \coprod_{e\in D} H\left(N^{\lambda e} \otimes_{\lambda e} C^{\lambda e}\right) \times \{e\} .$$

The required result is obtained if i_m, p_m and ∂_m is written for the composite of $H(i_m^*)$, $H(p_m^*)$ and $H\left(\partial_1^m\right)$, respectively, with the above isomorphisms.

(D4.3.2). In $S\text{-}Mod(zM)$, $\hom_S(-, L)$ is left exact (3.1.13) so that the short exact sequence

$$\coprod_{v \in D}\left(S^v \otimes_v C_m^v\right) \xrightarrow{\iota_m^*} M_m \xrightarrow{\pi_m^*} \coprod_{e \in D}\left(S^{\lambda e} \otimes_{\lambda e} C_{m-1}^{\lambda e}\right) \times \{e\}$$

of (3.3.5) induces a left exact sequence

$$\hom_S(zM, zL) \to \hom_S\left(\coprod_{e \in D}\left(S^{\lambda e} \otimes_{\lambda e} C_{m-1}^{\lambda e}\right) \times \{e\}, L\right) \xrightarrow{\langle \pi_m^*, L\rangle}$$

$$\hom_S\left(M_m, L\right) \xrightarrow{\langle \iota_m^*, L\rangle} \hom_S\left(\coprod_{v \in D} S^v \otimes_v C_m^v, L\right) .$$

If

$$f \in \hom_S\left(\coprod_{v \in D} S^v \otimes_v C_m^v, L\right)(h) ,$$

then for each Z in zM,

$$f : \coprod_{\substack{z \in Z_v \\ v \in D}}\left(S^v \otimes_v C_m^v\right)(z) \to L\left(h(Z)\right) .$$

Define $g\langle f\rangle : M_m(Z) \to L\left(h(Z)\right)$ for each Z in zM by

$$g\langle f\rangle\left((s, c_m), 0\right) = f\left(s, c_m\right) ,$$

and

$$g\langle f\rangle\left(0, \left(s', c_{m-1}, e\right)\right) = h(Z) ;$$

$g\langle f\rangle$ is an S-morphism. Since $\langle \iota_m^*, L\rangle(g\langle f\rangle) = g\langle f\rangle \circ \iota_m^* = f$, the sequence of hom sets is short exact in $Z\text{-}Comp\left(\hom_S(zM, zL)\right)$ and determines the long exact homology sequence (4.1.6),

$$\ldots \to H_m\left(\hom_S(M, L)\right) \xrightarrow{H\left(\langle \iota_m^*, L\rangle\right)} H_m\left(\hom_S\left(\coprod_{v \in D} S^v \otimes_v C^v, L\right)\right) \xrightarrow{\overline{\omega}_m}$$

$$H_m\left(\hom_S\left(\coprod_{e \in D}\left(S^{\lambda e} \otimes_{\lambda e} C^{\lambda e}\right) \times \{e\}, L\right)\right) \xrightarrow{H\left(\langle \pi_{m+1}^*, L\rangle\right)} H_{m+1}\left(\hom_S(M, L)\right) \to \ldots .$$

Here $\overline{\omega}_m = H\left(\left\langle \overrightarrow{\partial}_1^m, L\right\rangle\right)$ where by (3.3.4),

$$\overrightarrow{\partial}_1^m(s, c_m, e) = (s, c_m) - \left(s\sigma_e([*], \lambda c_m), C_m^e(c_m)\right) ;$$

the result is found directly from the evaluation of $\overline{\omega}$ in (4.1.5). Finally,

$$H\left(\hom_S\left(\coprod_{v\in D} S^v \otimes_v C^v, L\right)\right) \cong H\left(\prod_{v\in D} \hom_S\left(S^v \otimes_v C^v, L\right)\right) \quad (2.2.17)$$

$$\cong H\left(\prod_{v\in D} \hom_v\left(C^v, {}^vL\right)\right) \quad (2.2.16)$$

$$\cong \prod_{v\in D} H\left(\hom_v\left(C^v, {}^vL\right)\right) \quad (4.1.7)$$

and similarly

$$H\left(\hom_S\left(\coprod_{e\in D}\left(S^{\lambda e} \otimes_{\lambda e} C^{\lambda e}\right) \times \{e\}, L\right)\right) \cong \prod_{e\in D} H\left(\hom_{\lambda e}\left(C^{\lambda e}, {}^{\lambda e}L\right)\right) \times \{e\} .$$

The required result is obtained if ι_m, π_m and δ_m is written for the composite of $H(\langle \iota_m^*, L\rangle)$, $H(\langle \pi_m^*, L\rangle)$ and $H\left(\left\langle \overrightarrow{\partial}_1^m, L\right\rangle\right)$, respectively, with the above isomorphisms. □

4.3.3 COROLLARY (Mayer-Vietoris Sequence for Graph Products). *Let* $(\mathcal{D}, \mathrm{A})$ *be a groupnet diagram with graph product* $m : (\mathcal{D}, \mathrm{A}) \to G$. *Let* N *be any right* G*-module and* L *be any left* G*-module. The following (co)homology sequences are exact:*

$$\ldots \to \coprod_{e\in D} H_m\left(A_{\lambda e}; N^{\lambda e}\right) \times \{e\} \xrightarrow{\partial_m} \coprod_{v\in D} H_m\left(A_v; N^v\right) \xrightarrow{i_m} H_m(G; N)$$

$$\xrightarrow{p_m} \coprod_{e\in D} H_{m-1}\left(A_{\lambda e}; N^{\lambda e}\right) \times \{e\} \to \ldots , \quad (D4.3.3)$$

$$\ldots \to H^m(G; L) \xrightarrow{\iota_m} \prod_{v\in D} H^m\left(A_v; {}^vL\right) \xrightarrow{\delta_m} \prod_{e\in D} H^m\left(A_{\lambda e}; {}^{\lambda e}L\right) \times \{e\}$$

$$\xrightarrow{\pi_{m+1}} H^{m+1}(G; L) \to \ldots . \quad (D4.3.4)$$

Proof. There always exists a standard complex diagram $(\mathcal{D}, Z\mathrm{A}, C)$

with C^v a free A_v-resolution of T_v ; for example, $C^v = BA_v$ and C^e the induced A_e-morphism between the bar resolutions, hence there always exists a mapping cylinder complex which is a free G-resolution of TG (3.3.7). □

Any groupnet diagram $(\mathcal{D}, \mathrm{A})$ determines a groupnet diagram $(\mathcal{D}, \mathrm{A}^*)$, called the *derived loop group diagram*, where A_v^* is a set of loop groups of A_v, one for each component of A_v , determined by a retraction $r_v : A_v \to A_v^*$, and where $A_e^* = r_{\rho e} \circ A_e$ (see [7, p. 271]). If A_e is a monomorphism, so is A_e^* . If the mapping cylinder $m(\mathcal{D}, \mathrm{A})$ of $(\mathcal{D}, \mathrm{A})$ is connected, so is the mapping cylinder $m(\mathcal{D}, \mathrm{A}^*)$ of $(\mathcal{D}, \mathrm{A}^*)$, and they have isomorphic loop groups.

4.3.4 COROLLARY (Mayer-Vietoris Sequence for groups with the homotopy type of graph products). *Let* $(\mathcal{D}, \mathrm{A})$ *be a groupnet diagram of connected groupnets with a derived loop group diagram* $(\mathcal{D}, \mathrm{A}^*)$ *, for which each* A_e *is a monomorphism. Assume the graph product* $m : (\mathcal{D}, \mathrm{A}^*) \to G$ *is connected, and let* $r : G \to G^*$ *be a retraction of* G *to its loop group at a selected identity. For any regular right* G^**-module* N *and any regular left* G^**-module* L *the following group (co)homology sequences are exact:*

$$\ldots \to \coprod_{e \in D} H_m(A^*_{\lambda e}; N) \times \{e\} \xrightarrow{\partial_m} \coprod_{v \in D} H_m(A^*_v; N) \xrightarrow{i_m} H_m(G^*; N)$$

$$\xrightarrow{p_m} \coprod_{e \in D} H_{m-1}(A^*_{\lambda e}; N) \times \{e\} \to \ldots , \quad \text{(D4.3.5)}$$

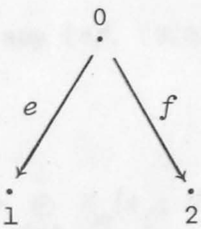
$$\ldots \to H^m(G^*; L) \xrightarrow{\imath_m} \prod_{v \in D} H^m(A^*_v; L) \xrightarrow{\delta_m} \prod_{e \in D} H^m(A^*_{\lambda e}; L) \times \{e\}$$

$$\xrightarrow{\pi_{m+1}} H^{m+1}(G^*; L) \to \ldots . \quad \text{(D4.3.6)}$$

Proof. Since $(N^r)^v \cong N$ and $^v(^rL) \cong L$, the result follows immediately from (4.2.7) and (4.3.3). □

In the examples following, it will be assumed that $(\mathcal{D}, A)$ is a group diagram, its graph product G is connected, and G^* is the loop group of G at a specified identity. The left regular G^*-module L and the right regular G^*-module N are arbitrary. Clearly it is possible to modify $(\mathcal{D}, A)$ without altering G^*, so that all the edge monomorphisms are subgroup inclusion morphisms. This is done in the first example, for the sake of simplicity only.

4.3.5 EXAMPLE. Let

$D =$ 

,

so that $G^* = A_1 *_{A_0} A_2$. The Mayer-Vietoris sequences for G^* are thus

$$\ldots \to H_m(A_0;\, N) \oplus H_m(A_0;\, N) \xrightarrow{\partial_m} \bigoplus_{i=0}^{2} H_m(A_i;\, N) \xrightarrow{i_m} H_m(G^*;\, N)$$

$$\xrightarrow{p_m} H_{m-1}(A_0;\, N) \oplus H_{m-1}(A_0;\, N) \to \ldots ,$$

and

$$\ldots \to H^m(G^*;\, L) \xrightarrow{\iota_m} \bigoplus_{i=0}^{2} H^m(A_i;\, L) \xrightarrow{\delta_m} H^m(A_0;\, L) \oplus H^m(A_0;\, L)$$

$$\xrightarrow{\pi_{m+1}} H^{m+1}(G^*;\, L) \to \ldots ,$$

if the edge labels for the homology of A_0 are dropped. Here

$$\partial = \big((\mathrm{cor}_*, -\mathrm{cor}_*, 0), (\mathrm{cor}_*, 0, -\mathrm{cor}_*)\big) ,$$

$$i = (\mathrm{cor}_*, \mathrm{cor}_*, \mathrm{cor}_*) ,$$

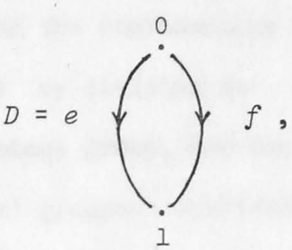
$$\delta = \big((\mathrm{res}^*, \mathrm{res}^*), (-\mathrm{res}^*, 0), (0, -\mathrm{res}^*)\big) ,$$

and

$$\iota = (\mathrm{res}^*, \mathrm{res}^*, \mathrm{res}^*) ,$$

where the maps res^* and cor_* are induced by the inclusions of the respective subgroups.

4.3.6 EXAMPLE. Let



where A_e is a subgroup inclusion morphism, so that

$$G^* = \mathrm{HNN}\big(A_1;\ A_0 \cong A_f(A_0)\big) ,$$

with stable letter $t = *_e^{-1} *_f$, say (*cf.* (3.3.8)). The Mayer-Vietoris sequences for G^* are thus

$$\ldots \to H_m(A_0;\ N) \oplus H_m(A_0;\ N) \xrightarrow{\partial_m} \bigoplus_{i=0}^{1} H_m(A_i;\ N) \xrightarrow{i_m} H_m(G^*;\ N)$$

$$\xrightarrow{p_m} H_{m-1}(A_0;\ N) \oplus H_{m-1}(A_0;\ N) \to \ldots ,$$

and

$$\ldots \to H^m(G^*;\ L) \xrightarrow{\iota_m} \bigoplus_{i=0}^{1} H^m(A_i;\ L) \xrightarrow{\delta_m} H^m(A_0;\ L) \oplus H^m(A_0;\ L)$$

$$\xrightarrow{\pi_{m+1}} H^{m+1}(G^*;\ L) \to \ldots ,$$

if the edge labels for the homology of A_0 are dropped. Here, if $A_2 = A_f(A_0)$,

$$\partial = \big((\mathrm{cor}_{*0},\ -\mathrm{cor}_{*0}),\ (\mathrm{cor}_{*0},\ -\mathrm{cor}_{*2} \circ t_*)\big) ,$$

$$i = (\mathrm{cor}_*,\ \mathrm{cor}_*) ,$$

$$\delta = \big((\mathrm{res}_0^*,\ \mathrm{res}_0^*),\ (-\mathrm{res}_0^*,\ -t^* \circ \mathrm{res}_2^*)\big) ,$$

and

$$\iota = (\mathrm{res}^*,\ \mathrm{res}^*) ,$$

where the maps res^* and cor_* are induced by the inclusions of the indicated subgroups, and conjugation by t in G^* induces the isomorphisms

$$t^* : H^m(A_2;\ L) \cong H^m(A_0;\ L) \quad \text{and} \quad t_* : H_m(A_0;\ N) \cong H_m(A_2;\ N) .$$

If $(\mathcal{D}, A)$ is a group diagram derived from a graph of groups, the Mayer-Vietoris sequences for the (co)homology of the graph of groups are found from those of $(\mathcal{D}, A)$ by dividing out, in each dimension, one copy of each source vertex (co)homology group, and for each such, one of the corresponding pair of 'edge' groups. Specifically, in (4.3.5), the Mayer-Vietoris sequences for the graph of groups

$$A_1 \xrightarrow{A_0} A_2$$

are obtained from those of $(\mathcal{D}, A)$ by dividing out, in dimension m, the short exact sequence

$$H_m(A_0; N) \xrightarrow{\partial_m} \partial_m H_m(A_0; N)$$

in the homology sequence, and the short exact sequence

$$H^m(A_0; L) \xrightarrow{\delta_m} \delta_m H^m(A_0; L)$$

in the cohomology sequence.

These sequences are due to Lyndon and Swan [38, 2.3] in the case of the free product with amalgamation (4.3.5) and to Bieri [1] in the case of the HNN group (4.3.6). Recently Chiswell [3] has independently determined the Mayer-Vietoris sequences for fundamental groups of graphs of groups, and applied them to the calculation of the Euler characteristics of such groups. //

This chapter closes with a generalisation of some results of Bieri and Eckmann on duality groups.

4.3.7 DEFINITION. A connected groupnet A is of *finite cohomological dimension* $\mathrm{cd}(A) \le m$ if $H^k(A; L) = 0$ for every $k > m$ and every left A-module L. It is of *cohomological dimension* m if $\mathrm{cd}(A) \le m$ but $\mathrm{cd}(A) \nleq m-1$. *Homological dimension* $\mathrm{hd}(A)$ is correspondingly defined.

4.3.8 DEFINITION. A connected group A is of *type* (FP) if the trivial A-module $\mathbb{Z}$ admits a finite A-projective resolution. It is of *type* $(\overline{FP})$ if $\mathbb{Z}$ admits a finitely generated A-free resolution.

A group is of type (FP) if and only if it is of type $(\overline{FP})$ and of finite cohomological dimension.

4.3.9 LEMMA. *If* D *is a finite graph,* $(\mathcal{D}, A)$ *is a group diagram, the graph product* $m : (\mathcal{D}, A) \to G$ *is connected, and* G^* *is the loop group of* G *at a specified identity, then*

(i) if A_v *is of type* $(\overline{FP})$ *for all* v *in* D *, so is* G^* *, and*

(ii) if A_v *is of type* (FP) *for all* v *in* D *, so is* G^* *.*

Proof. Both results are immediate consequences of (3.3.7) and (D3.3.3). □

This lemma is also proved by Chiswell [3, Th. 3] and, for amalgamated free products and HNN groups, by Bieri and Eckmann [1, 2].

4.3.10 DEFINITION. A group A is a *duality group of dimension* n if there is a dualising right A-module N and a fundamental class c in $H_n(A; N)$ such that the cap-product $c \cap -$ induces isomorphisms

$H^k(A; L) \cong H_{n-k}(A; N \otimes L)$ for every left A-module L and all k in $\mathbb{Z}$ (see Bieri and Eckmann [2]).

A group A of type (FP) is a duality group of dimension n if and only if $H^k(A; \mathbb{Z}A) = 0$ for all $k \neq n$ and $H^n(A; \mathbb{Z}A)$ is torsion free as an abelian group; or if and only if $H^k(A; M) = 0$ for all $k \neq n$ and all induced A-modules $M = L \otimes \mathbb{Z}A$ [2, §3.1].

4.3.11 THEOREM. *Let* D *be a finite graph,* $(\mathcal{D}, A)$ *be a group diagram, with connected graph product* $m : (\mathcal{D}, A) \to G$ *, and let* G^* *be the loop group of* G *at a specified identity. Assume* A_v *is of type* (FP) *for all* v *in* D *. If* *(i)* A_v *is a duality group of dimension* $n - 1$ *for all* v *in* D *, such that* $v = \lambda e$ *for an* e *in* D *, and*

(ii) A_v *is a duality group of dimension* n *for all other* v *in* D *,*

then G^* *is a duality group of dimension* n *.*

Proof. Since for each v in D , $\mathbb{Z}G^*$ is an induced A_v-module, the Mayer-Vietoris sequence (D4.3.6) with $L = \mathbb{Z}G^*$ ensures immediately that $H^k(G^*; \mathbb{Z}G^*) = 0$ for all $k \neq n$, and determines the short exact sequence

$$0 \to \prod_{e \in D} H^{n-1}(A_{\lambda e}; \mathbb{Z}G^*) \times \{e\} \to H^n(G^*; \mathbb{Z}G^*)$$

$$\to \prod_{\substack{v \in D \\ v \neq \lambda e \\ e \in D}} H^n(A_v; \mathbb{Z}G^*) \to 0 .$$

Then by duality, $H^{n-1}(A_{\lambda e}; \mathbb{Z}G^*) \cong H^{n-1}(A_{\lambda e}; \mathbb{Z}A_{\lambda e}) \otimes_{\lambda e} \mathbb{Z}G^*$ for all e in D and $H^n(A_v; \mathbb{Z}G^*) \cong H^n(A_v; \mathbb{Z}A_v) \otimes_v \mathbb{Z}G^*$ for all v in D such that $v \neq \lambda e$ for any e in D . These groups are torsion free over $\mathbb{Z}$ so that $H^n(G^*; \mathbb{Z}G^*)$ is torsion free. □

4.3.12 LEMMA. *With the conditions of* (4.3.11), *if* A_v *is a duality group of dimension* $n - 1$ *for all* v *in* D *, and if* $\operatorname{cd} G^* \leq n-1$ *, then* G^* *is a duality group of dimension* $n - 1$ *.*

Proof. Any induced G^*-module L is an induced A_v-module for each v in D . From (D4.3.6), $H^k(G^*; L) = 0$ for all $k \neq n, n-1$ and all induced G^*-modules L , and the sequence

$$0 \to H^{n-1}(G^*; L) \to \prod_{v \in D} H^{n-1}(A_v; L) \to \prod_{e \in D} H^{n-1}(A_{\lambda e}; L) \times \{e\}$$

$$\to H^n(G^*; L) \to 0$$

is exact. Finally, $\text{cd}(G^*) \leq n-1$ implies $H^n(G^*; L) = 0$. □

These results extend those of Bieri and Eckmann for free groups with amalgamation [2, Th. 3.2, Th. 3.5.i] and of Bieri for HNN groups of type (*FP*) [1, Th.5.2, Th.5.3.i]. It has recently been shown by Strebel [36, Theorem, §4.4] that all duality groups are necessarily of type (*FP*) , hence this assumption may be removed from (4.3.11) and (4.3.12).

CHAPTER 5

A CLASS OF GROUPNETS WITH COHOMOLOGICAL DIMENSION TWO

5.1 Subgroupnets

As was indicated at the end of Chapter 1, a description of the construction of a mapping cylinder groupnet homotopic to a wide subgroupnet of a mapping cylinder groupnet is given here. There is a corresponding construction of a mapping cylinder complex for a wide subgroupnet which directly determines a Mayer-Vietoris sequence for the wide subgroupnet.

5.1.1 LEMMA [7, §§8, 9]. *If* $m : (\mathcal{D}, \mathrm{A}) \to G$ *is a connected mapping cylinder for the group diagram* $(\mathcal{D}, \mathrm{A})$, H *is a wide subgroupnet of* G *and* $\pi : \tilde{G} \to G$ *is the covering map corresponding to* H , *there exists a groupnet diagram* $(\mathcal{D}, \tilde{\mathrm{A}})$ *such that* $\tilde{m} : (\mathcal{D}, \tilde{\mathrm{A}}) \to \tilde{G}$ *is a mapping cylinder. Moreover, there exists a covering map* $\pi_v : \tilde{A}_v \to A_v$ *corresponding to a certain wide subgroupnet* K_v *of* A_v , *for each* v *in* D ; *and if* G *is a graph product then so is* $\tilde{G}$.

Sketch of Construction. Each $\tilde{A}_v$ is defined from the pullback square

$$\begin{array}{ccc} \tilde{A}_v & \xrightarrow{\pi_v} & A_v \\ {\scriptstyle \tilde{m}_v}\downarrow & & \downarrow{\scriptstyle m_v} \\ \tilde{G} & \xrightarrow[\pi]{} & G \end{array}$$

in *Set* , with groupnet structure induced from the cartesian product $\tilde{G} \times A_v$. The groupnet morphism $\tilde{A}_e : \tilde{A}_{\lambda e} \to \tilde{A}_{\rho e}$ is defined as

$$\tilde{A}_e(\gamma, m_{\lambda e}(a), a) = (\gamma m_e(*, \lambda a), m_{\rho e}A_e(a), A_e(a)) ;$$

it is clearly a monomorphism if A_e is. Groupnet $\tilde{G}$ is a mapping cylinder $\tilde{m} : (\mathcal{D}, \tilde{\mathrm{A}}) \to \tilde{G}$ with groupnet morphisms

$$\tilde{m}_v(\gamma, m_v(a), a) = (\gamma, m_v(a))$$

for all $(\gamma, m_v(a), a)$ in $\tilde{A}_v$ and v in D, and groupnet homotopies

$$\tilde{m}_e(*, (\gamma, m_{\lambda e}(i_{\lambda e}), i_{\lambda e})) = (\gamma, m_e(*, i_{\lambda e}))$$

for all γ in G/H with $\rho\gamma = m_{\lambda e}(i_{\lambda e})$, where i_v represents the identity of A_v. If, for g in G, the double coset $Hgm_v(A_v)$ is defined as

$$\{h.g.m_v(a) \in G : h \in H, a \in A_v\}$$

and the set of double cosets $G/H : A_v$ is denoted by

$$P_v = \{Hgm_v(A_v) : g \in G, \rho g = m_v(i_v)\},$$

then a subgroup K_v^d of A_v is defined for each d in P_v and each v in D in the following manner. A coset representative u_d in d is chosen to satisfy $\rho u = m_v(i_v)$ and $u_d = m_v(i_v)$ if $d = Hm_v(i_v)m_v(A_v)$; the subgroup K_v^d is then

$$K_v^d = \left\{a \in A_v : m_v(a) \in u_d^{-1}Hu_d\right\}.$$

Crowell and Smythe [7, 5.2] show that $\pi_v : \tilde{A}_v \to A_v$ is the covering map corresponding to $K_v = \left\{K_v^d : d \in P_v\right\}$, and thus

$$\tilde{A}_v \cong B_v = \bigvee_{d \in P_v} B_v^d,$$

where

$$B_v^d = \left\{\left(\sigma_v^d, a\right) \in A_v/K_v^d \times A_v : \rho\sigma_v^d = \lambda a\right\}.$$

This isomorphism is determined from a one-to-one correspondence of G/H with

$$\bigvee_{v \in D} \bigvee_{d \in P_v} \left(A_v/K_v^d\right),$$

viz.

$$\sigma_v^d \longmapsto \gamma\left(\sigma_v^d\right) = Hu_d m_v(a)$$

for an a in σ_v^d . If $\gamma \in G/H$ with $\rho\gamma = m_v\left(i_v\right)$ for a unique v in D , then γ determines a unique double coset $d(\gamma) = Hu_{d(\gamma)}m_v\left(A_v\right) = \gamma m_v\left(A_v\right)$, and the inverse map is given by

$$\gamma \longmapsto \sigma(\gamma) = K_v^{d(\gamma)}a$$

for any a in A_v such that there is h in H for which

$hu_{d(\gamma)}m_v(a) \in d(\gamma)$. □

5.1.2 LEMMA. *If B is a wide subgroupnet of the connected groupnet A and $\pi : \tilde{A} \to A$ is the covering corresponding to B , then the pullback of any free A-resolution C of TA along π is a free $\tilde{A}$-resolution of $T\tilde{A}$.*

Proof. By (3.1.7) it is necessary only to prove $^{\pi}C_n$ free for all n in $\mathbb{Z}$. Suppose C_n is the free A-module with basis

$$X_n = X_n \xrightarrow{\ z\ } zX_n \subset \mathrm{Id}\, A$$

so that the module's left map λ equals z , and for each i in $\mathrm{Id}\, A$,

$$C_n(i) = \coprod_{x \in X_n} \mathbb{Z}A(i, \lambda x)[x] .$$

Hence for each γ in A/B ,

$$^{\pi}C_n(\gamma) = \{\gamma\} \times C_n(\rho\gamma) = \coprod_{x \in X_n} \{\gamma\} \times \mathbb{Z}A(\rho\gamma, \lambda x)[x] .$$

With

$$\tilde{X}_n = \{(\gamma, x) \in A/B \times X_n : \rho\gamma = \lambda x\} ,$$

$$z\tilde{X}_n = \{\gamma \in A/B : \rho\gamma = \lambda x, x \in X_n\} ,$$

$$z : \tilde{X}_n \to z\tilde{X}_n \text{ as } z(\gamma, x) = \gamma ,$$

and

$$\lambda : z\tilde{X}_n \to \text{Id}\,\tilde{A} \quad \text{as the inclusion map,}$$

the free $\tilde{A}$-module $F\tilde{X}_n$ with basis $\tilde{X}_n$ is isomorphic to ${}^{\pi}C_n$. As

$$F\tilde{X}_n(\gamma) = \coprod_{(\delta,x)\in\tilde{X}_n} Z\tilde{A}(\gamma, \delta)[(\delta, x)] ,$$

the isomorphism $\phi : {}^{\pi}C_n(\gamma) \to F\tilde{X}_n(\gamma)$ is extended linearly from

$\phi(\gamma, a, x) = (\gamma, a)[(\gamma a, x)]$, and has inverse ψ extended by $\tilde{A}$-action from

$\psi[(\gamma, x)] = (\gamma, \lambda x, x)$. □

5.1.3 THEOREM. *Let* $m : (\mathcal{D}, A) \to G$ *be a connected mapping cylinder for the group diagram* $(\mathcal{D}, A)$, H *be a wide subgroupnet of* G , $\pi : \tilde{G} \to G$ *be the covering map corresponding to* H *and let* $\tilde{m} : (\mathcal{D}, \tilde{A}) \to \tilde{G}$ *be the pullback mapping cylinder. Let* $(\mathcal{D}, ZA, C)$ *be a complex diagram where* C^v *is a free* A_v*-resolution of* T_v *for each* v *in* D *, let* $\mu : (\mathcal{D}, ZA, C) \to M$ *be the* G*-mapping cylinder of* $(\mathcal{D}, ZA, C)$ *, and let* $\tilde{M}$ *be the pullback of* M *along* π *.*

(i) If G *is a graph product,* $\tilde{M}$ *is a free* $\tilde{G}$*-resolution of* $T\tilde{G}$ *.*

(ii) There exists a complex diagram $(\mathcal{D}, Z\tilde{A}, \tilde{C})$ *with* $\tilde{G}$*-mapping cylinder* $\tilde{\mu} : (\mathcal{D}, Z\tilde{A}, \tilde{C}) \to \tilde{M}$ *.*

Proof. (i) If G is a graph product, M is a free G-resolution of TG and (5.1.2) applies.

(ii) By (5.1.2) and the isomorphism $\tilde{A}_v \cong B_v$ of (5.1.1), the pullback $\tilde{C}^v$ of C^v along $\pi_v : \tilde{A}_v \to A_v$ is a free regular $\tilde{A}_v$-resolution of $T\tilde{A}_v = \tilde{T}_v$. For each e in D , $\tilde{C}^e : \tilde{C}^{\lambda e} \to \tilde{C}^{\rho e}$ is extended linearly from

$$\tilde{C}^e_n(\gamma, c) = \left[\gamma m_e(*, \lambda c),\ C^e_n(c)\right]$$

for all c in $C^{\lambda e}_n$ and γ in G/H with $\rho\gamma = m_{\lambda e}(\lambda c)$. The $\tilde{G}$-mapping cylinder $\tilde{\mu} : (\mathcal{D}, Z\tilde{A}, \tilde{C}) \to \tilde{M}$ of the complex diagram $(\mathcal{D}, Z\tilde{A}, \tilde{C})$ so formed, is determined from the pullback squares

$$\begin{array}{ccc} \tilde{C}^v & \xrightarrow{\pi_v^*} & C^v \\ \downarrow \tilde{\mu}^v & \text{p.b.} & \downarrow \mu^v \\ \tilde{M} & \xrightarrow{\pi^*} & M \end{array} \quad \forall v \in D \text{, and} \quad \begin{array}{ccc} \mathcal{E} \otimes \tilde{C}^{\lambda e} & \xrightarrow{\pi_{\lambda e} \otimes \pi_{\lambda e}^*} & \mathcal{E} \otimes C^{\lambda e} \\ \downarrow \tilde{\mu}^e & \text{p.b.} & \downarrow \mu^e \\ \tilde{M} & \xrightarrow{\pi^*} & M \end{array} \quad \forall e \in D .$$

Hence $\tilde{\mu}^v(\gamma, c) = (\gamma, \mu^v(c))$. The required isomorphism

$$\tilde{M}_n(\gamma) \cong \coprod_{\substack{\delta \sim \gamma \\ v \in D}} Z\tilde{G}(\gamma, \delta) \otimes_v \tilde{C}_n^v(\delta) \oplus \coprod_{\substack{\delta \sim \gamma \\ e \in D}} Z\tilde{G}(\gamma, \delta) \otimes_{\lambda e} \tilde{C}_{n-1}^{\lambda e}(\delta)$$

is extended linearly from

$$(\gamma, g, c) \mapsto (\gamma, g) \otimes (\gamma g, c) . \quad \square$$

5.1.4 COROLLARY. *With the conditions of* (5.1.3), *if* G *is a graph product,* N *is any regular right* H*-module,* L *is any regular left* H*-module, and* $r : \tilde{G} \to H$ *is the retraction* $\tilde{G} \simeq H$ *, there are Mayer-Vietoris sequences*

$$\cdots \to \coprod_{e \in D} H_m\left(\tilde{A}_{\lambda e};\ (N^r)^{\lambda e}\right) \times \{e\} \xrightarrow{\partial_m} \coprod_{v \in D} H_m\left(\tilde{A}_v;\ (N^r)^v\right) \xrightarrow{i_m} H_m(H;\ N)$$

$$\xrightarrow{p_m} \coprod_{e \in D} H_{m-1}\left(\tilde{A}_{\lambda e};\ (N^r)^{\lambda e}\right) \times \{e\} \to \cdots , \quad \text{(D5.1.1)}$$

and

$$\cdots \to H^m(H;\ L) \xrightarrow{\iota_m} \prod_{v \in D} H^m\left(\tilde{A}_v;\ {}^v({}^rL)\right) \xrightarrow{\delta_m} \prod_{e \in D} H^m\left(\tilde{A}_{\lambda e};\ {}^{\lambda e}({}^rL)\right) \times \{e\}$$

$$\xrightarrow{\pi_{m+1}} H^{m+1}(H;\ L) \to \cdots . \quad \text{(D5.1.2)}$$

Proof. The Mayer-Vietoris sequences (4.3.3) for the graph product $\tilde{m} : (\mathcal{D}, \tilde{A}) \to \tilde{G}$, with coefficients in N^r and ${}^rL$, determine these sequences by (4.2.7). $\square$

5.1.5 REMARK. If $i : B \to A$ is a subgroupnet inclusion morphism and $\mathcal{B}A$ is the bar resolution for A , then ${}^i\mathcal{B}A$ is a free B-resolution of TB by consideration of (2.2.22) (for the left pullback) and (3.1.7). Tensor associativity and (2.2.12) imply

$$N \otimes_B {}^{i}BA \cong \left(N \otimes_B {}^{i}ZA\right) \otimes_A BA \ .$$

Hence for any subgroupnet B of A and any right B-module N ,

$$H_*(B;\, N) \cong H_*\left(A;\, N \otimes_B {}^{i}ZA\right) \ .$$

Since the hom functor is not generally right adjoint to the tensor functor, the corresponding result for cohomology may not be expected to hold unless A is a group. //

5.2 Diagonal Maps

This section develops the theory of diagonal approximations on complexes. This is then applied to determine a coring structure on the homology module of a groupnet, subject to the restriction that the comultiplication associates only up to isomorphism in dimensions greater than two. It parallels the development of a ring structure on the cohomology module (see for instance Mac Lane [19, VIII, §9]).

5.2.1 **DEFINITION.** For a groupnet A , the *diagonal map* $\Delta : A \to A \times A$ is defined as $\Delta(a) = (a,\, a)$ for a in A . It associates, so that the induced ringnet morphism also does:

$$\begin{array}{ccc} ZA & \xrightarrow{\ \Delta\ } & ZA\otimes ZA \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\ ZA\otimes ZA & \xrightarrow[\Delta\otimes 1]{} & ZA\otimes ZA\otimes ZA \ . \end{array}$$

5.2.2 **LEMMA.** *If C is a free A-resolution of TA , then $C \otimes_Z C$ is a free $A \times A$-resolution of $TA \otimes_Z TA$.*

Proof. If C_n is the regular free A-module with basis X_n , then it is the regular free Id A-module with basis X_n^* , where

$$X_n^* = \{ax : a \in A,\ x \in X_n,\ \rho a = \lambda x\} \ .$$

Since $T(\mathrm{Id}\ A)$, which has the same underlying abelian groupnet as TA , is projective (in fact, free) as an Id A-module, the augmented complex C^+ of C with augmentation ε has a contracting homotopy s of Id A-morphisms (4.2.2). A set

$$\bar{s} = \{\bar{s}_n : (C \otimes_Z C)^+_n \to (C \otimes_Z C)^+_{n+1},\ n \geq -1\}$$

of Id $A \times$ Id A-morphisms is defined on the augmented tensor complex by

$$\bar{s}_{-1} = s_{-1} \otimes s_{-1} ,$$

and

$$\bar{s}_n = \sum_{k=0}^{n} s_k \otimes 1_{n-k} + s_{-1}\varepsilon \otimes s_n \quad \forall n \geq 0 ,$$

where $C \otimes_Z C$ is augmented by $\bar{\varepsilon} = \varepsilon \otimes \varepsilon : C_0 \otimes C_0 \to TA \otimes TA$. Then

$$\bar{\varepsilon}\bar{s}_{-1} = 1 , \quad \partial_1\bar{s}_0 + \bar{s}_{-1}\bar{\varepsilon} = 1 ,$$

and for $n \geq 1$,

$$\partial_{n+1}\bar{s}_n + \bar{s}_{n-1}\partial_n = \partial_{n+1}\Big(\sum_{k=0}^{n} s_k \otimes 1_{n-k} + s_{-1}\varepsilon \otimes s_n\Big)$$

$$+ \bar{s}_{n-1}\Big(\sum_{k=0}^{n}\big(\partial_k \otimes 1_{n-k} + (-1)^k 1_k \otimes \partial_{n-k}\big)\Big)$$

$$= \sum_{k=1}^{n} (\partial_{k+1}s_k + s_{k-1}\partial_k) \otimes 1_{n-k} + \partial_1 s_0 \otimes 1_n$$

$$+ s_{-1}\varepsilon \otimes \partial_{n+1}s_n + s_{-1}\varepsilon \otimes s_{n-1}\partial_n$$

$$= \sum_{k=0}^{n} 1_k \otimes 1_{n-k} = 1 ,$$

hence $\bar{s}$ is a contracting homotopy of $(C \otimes_Z C)^+$. By (3.2.6) the augmented complex is exact. A simple computation shows that $(C \otimes_Z C)_n$ is isomorphic to the free regular $A \times A$-module on basis Y_n , where

$$Y_n = \bigvee_{k=0}^{n} X_k \times X_{n-k} . \qquad \square$$

5.2.3 COROLLARY. *If* C *is a free* A*-resolution of* TA *for a groupnet* A *, there exists a* Δ*-chain map* $\omega : C \to C \otimes_Z C$ *, commuting with the induced* *morphism* $\Delta : TA \to TA \otimes_Z TA$. *Moreover, any two such chain maps are* $\chi(\Delta)$-*homotopic.*

Proof. This is an immediate consequence of the regular comparison theorem (4.2.4). □

Any Δ-chain map $\omega : C \to C \otimes_Z C$ lifting $\Delta : TA \to TA \otimes_Z TA$ which satisfies

$$\omega(c) = c \otimes c \quad \forall c \in C_0$$

is called a *diagonal approximation*; clearly such maps exist.

5.2.4 LEMMA. *The diagonal map* $\Delta : A \to A \times A$ *induces a homology map* $H_*(\Delta) : H_*(A; M) \to H_*(A \times A; M \otimes_Z M)$ *and a cohomology map* $H^*(\Delta) : H^*(A \times A; L) \to H^*(A; L)$ *for any right* A*-module* M *and any left* $A \times A$*-module* L

Proof. Let C be any free A-resolution of TA , and let $\omega : C \to C \otimes_Z C$ be any Δ-chain map lifting $\Delta : TA \to TA \otimes_Z TA$. If $\overline{\Delta} : M \to M \otimes_Z M$ is the composed morphism

$$M \xrightarrow{\Delta} M \oplus M \twoheadrightarrow {}^{\Delta}(M \otimes_Z M) \xrightarrow{\Delta^*} M \otimes_Z M \ ,$$

with leftmost map the diagonal in $A\text{-}Mod(zM)$ (2.2.7.ii), then $\overline{\Delta} \otimes \omega : M \otimes_A C \to (M \otimes_Z M) \otimes_{A\times A} (C \otimes_Z C)$ determines the homology map and $\langle \omega, L \rangle : \hom_{A\times A}(C \otimes_Z C, L) \to \hom_A(C, L)$ determines the cohomology map. □

5.2.5 LEMMA. *If* $r : A \to A^*$ *is the retraction of the connected groupnet* A *onto its loop group and* $j : A^* \to A$ *is the inclusion map then*

(i) if $M \in |A^*\text{-}Modreg|$ *then* $M \cong {}^{j}L$ *for an* L *in* $|A\text{-}Modreg|$,

(ii) if $N \in |A\text{-}Modreg|$ *then* $N \cong {}^{r}K$ *for a* K *in* $|A^*\text{-}Modreg|$ *, and*

(iii) if N *is a left regular* A*-module and* P *is a right regular* A*-module then*

$$P^j \otimes_{A^*} {}^jN \cong P \otimes_A N \ .$$

Proof. (i) Since $r \circ j = 1_{A^*}$, $M = {}^j({}^rM)$.

(ii) If i is the selected identity of A onto which A is retracted, then $N(i)$ is an A^*-module under the subgroupnet action of A^* in A , and $N \cong {}^rN(i)$. The isomorphism is given by

$$n \longmapsto (\lambda n, *_{\lambda n}.n) \ \forall n \in N \ .$$

(iii) (Compare with (4.2.7.i).) The isomorphism $P \otimes_A N \to P^j \otimes_{A^*} {}^jN$ is given by

$$(p, n) \longmapsto \left(\left(p*_j^{-1}, i\right), \left(i, *_j n\right)\right)$$

for all (p, n) in $P \otimes_A N$ such that $\rho p = j = \lambda n$.

5.2.6 DEFINITION. A right regular A-module M for a connected groupnet A is *flat* if, whenever $f : C \to B$ is a monomorphism in A-*Modreg* , $1 \otimes f : M \otimes_A C \to M \otimes_A B$ is a monomorphism in $\mathbb{Z}$-*Modreg* .

This definition extends that for groups [19, V.8.6] in a straight-forward manner. If $r : A \to A^*$ is the retraction of A onto its loop group A^*, and $j : A^* \to A$ is the inclusion morphism, then (4.2.7.i) and (5.2.5) imply that M is flat in A^*-*Modreg* if and only if M^r is flat in A-*Modreg* ; conversely, N is flat in A-*Modreg* if and only if N^j is flat in A^*-*Modreg* .

5.2.7 REMARK. Regular projective A-modules are flat for a connected groupnet A . The proof of Hilton and Stammbach [15, III.7.4] suffices by virtue of (2.2.20.iii), (2.2.14), and the isomorphism $\mathbb{Z}A \otimes_A M \cong M$.

5.2.8 THEOREM (The Künnuth Formula). *If A is a connected groupnet and M is any right regular flat A-module there is a split short exact sequence*

$$0 \to [H(A;M) \otimes H(A;M)]_n \xrightarrow{\bar{p}} H_n(A \times A; M \otimes_{\mathbb{Z}} M)$$

$$\to [\mathrm{Tor}_1(H(A;M), H(A;M))]_{n-1} \to 0 \quad \text{(D5.2.1)}$$

for each n *in* $\mathbb{Z}$ *(although the splitting is not natural).*

Proof. Let C be a left regular free A-resolution of TA and let $j : A^* \to A$ be the inclusion of the loop group. Since M^j is flat in A^*-*Modreg* and ${}^jC$ is a free A^*-resolution of TA^* (compare with (5.1.5)), if $x : D \rightarrowtail B$ is an abelian group monomorphism, then $1 \otimes x : {}^jC_n \otimes_{\mathbb{Z}} D \to {}^jC_n \otimes_{\mathbb{Z}} B$ is an A^*-monomorphism, and so $1 \otimes (1 \otimes x) : M^j \otimes_{A^*} \left({}^jC_n \otimes_{\mathbb{Z}} D\right) \to M^j \otimes_{A^*} \left({}^jC_n \otimes_{\mathbb{Z}} B\right)$ is an abelian group monomorphism. Hence $M \otimes_A C_n$ is a flat $\mathbb{Z}$-module for all n in $\mathbb{Z}$ by (5.2.5.iii). It is thus torsionfree, and the Künnuth Formula for abelian groups [19, V.10.4], together with the 'middle four interchange':

$$(M \otimes_A C_p) \otimes_{\mathbb{Z}} (M \otimes_A C_q) \cong (M \otimes_{\mathbb{Z}} M) \otimes_{A \times A} (C_p \otimes_{\mathbb{Z}} C_q)$$

for all nonnegative integers p and q, leads to the required result. □

The torsion product in (D5.2.1) is of the $\mathbb{Z}$-complex $H(A;M)$ with itself. The map p is a specific example of the *external homology product*, defined for any standard right R-complex K and any standard left R-complex L to be the map

$$p : H(K) \otimes_R H(L) \to H(K \otimes_R L)$$

given by tensor extension of

$$p([u] \otimes [v]) = [u \otimes v] \quad \forall [u] \in H_k(K), \quad [v] \in H_l(L).$$

The map $\bar{p}$ is the composition of p with the 'middle four interchange' isomorphism.

5.2.9 COROLLARY. *If* A *is a connected groupnet and* M *is any right regular flat* A*-module then there is a natural isomorphism*

$$[H(A;M) \otimes_{\mathbb{Z}} H(A;M)]_n \cong H_n(A \times A; M \otimes_{\mathbb{Z}} M), \quad 0 \le n \le 2.$$

Proof. The right $\mathbb{Z}$-module $H_0(A;\, M) \cong H_0(A^*;\, M^j) \cong M^j \otimes_{A^*} \mathbb{Z}$ is flat, since if $x : C \rightarrowtail B$ is an abelian group monomorphism so is $1 \otimes x : M^j \otimes_{A^*} C \rightarrowtail M^j \otimes_{A^*} B$. As $H_{-1}(A;\, M) = 0$, it follows from [19, V.8.6] that

$$[\mathrm{Tor}_1(H(A;\, M),\, H(A;\, M))]_{n-1} = 0\ , \quad 0 \le n \le 2\ . \qquad \square$$

5.2.10 COROLLARY. *The diagonal map* $\Delta : A \to A \times A$ *for a connected groupnet* A *induces* $\Omega : H_*(A;\, M) \to H_*(A;\, M) \otimes_{\mathbb{Z}} H_*(A;\, M)$ *for any flat right regular* A*-module* M. *In dimensions* 0, 1 *and* 2 *it is unique; in higher dimensions it is unique to within the splitting isomorphism of the Künnuth Formula.* $\square$

Such a homology map is called a *diagonal comultiplication*. It induces a coring structure on the homology module $H_*(A;\, M)$ which associates to within the splitting isomorphism of (5.2.7) by virtue of the regular comparison theorem.

5.2.11 REMARK. The diagonal map $\Delta : A \to A \times A$ for a groupnet A induces a cohomology map $H^*(A;\, L) \otimes_{\mathbb{Z}} H^*(A;\, L) \to H^*(A;\, L \otimes_{\mathbb{Z}} L)$ for any left A-module L. It is the composite map

$$H^*(A;\, L) \otimes_{\mathbb{Z}} H^*(A;\, L) \xrightarrow{p} H^*(\hom_A(C,\, L) \otimes_{\mathbb{Z}} \hom_A(C,\, L)) \xrightarrow{H^*(\eta)}$$
$$H^*(A \times A;\, L \otimes_{\mathbb{Z}} L) \xrightarrow{H^*(\Delta)} H^*(A;\, L \otimes_{\mathbb{Z}} L)\ ,$$

where p is the external homology product and $H^*(\eta)$ is the homology map induced from $\eta : \hom_A(C,\, L) \otimes_{\mathbb{Z}} \hom_A(C,\, L) \to \hom_{A\times A}(C \otimes_{\mathbb{Z}} C,\, L \otimes_{\mathbb{Z}} L)$, with $[\eta(f \otimes g)](c \otimes c^*) = f(c) \otimes g(c^*)$, for a projective A-resolution C of TA. When L is regular, $H^*(A;\, L \otimes_{\mathbb{Z}} L)$ may be replaced by $H^*\left(A;\, {}^{\Delta}(L \otimes_{\mathbb{Z}} L)\right)$; if, further, $L = TA$, the diagonal map induces the *cup product*

$$\cup : H^*(A;\, TA) \otimes_{\mathbb{Z}} H^*(A;\, TA) \to H^*(A;\, TA)\ .$$

As in the case when A is a group, the cup product induces a ring structure on the cohomology module $H^*(A; TA)$. //

Let C be a free A-resolution of TA . The *switch map* $S_* : C \otimes C \to C \otimes C$ is given by

$$S_n(c_k \otimes c_{n-k}) = (-1)^{k(n-k)} c_{n-k} \otimes c_k$$

and is an $A \times A$-chain map satisfying $S_0(c_0 \otimes c_0) = c_0 \otimes c_0$ on $C_0 \otimes C_0$. If $\omega : C \to C \otimes C$ is any diagonal approximation lifting $\Delta : TA \to TA \otimes TA$ then so is $S\omega : C \to C \otimes C$. Consequently,

$$H(\overline{\Delta} \otimes \omega) = H(\overline{\Delta} \otimes S\omega) : H(A;\ M) \to H\big(A \times A;\ M \otimes_Z M\big)$$

by (5.2.3) and (4.1.2), and the next lemma follows.

5.2.12 LEMMA. *The graded coring $H_*(A; M)$ is commutative to within the splitting isomorphism of the Künnuth Formula.* □

In other words, if $q : H_*\big(A \times A; M \otimes_Z M\big) \to H_*(A; M) \otimes H_*(A; M)$ is the map given by the splitting isomorphism, then the induced diagram

$$\begin{array}{ccc} H_*(A;M) & \xrightarrow{\ \Omega\ } & H_*(A;M)\otimes H_*(A;M) \\ \Big\downarrow = & & \Big\downarrow q\circ H_*(1\otimes S)\circ\overline{p} \\ H_*(A;M) & \xrightarrow[\ \Omega\ ]{} & H_*(A;M)\otimes H_*(A;M) \end{array}$$

holds for $\Omega = q \circ H_*(\Delta)$. Hence, for h in $H_n(A; M)$ and any summand $w_k \otimes w^*_{n-k}$ of $\Omega(h)$, if both k and $n - k$ are odd, the element $-\big(w^*_{n-k} \otimes w_k\big)$ must also be a summand of $\Omega(h)$; if one of them is even the element $+\big(w^*_{n-k} \otimes w_k\big)$ must be a summand of $\Omega(h)$. A map with this property for some tensor product $C \otimes C$ of complexes is called *antisymmetric*.

As usual with proofs involving the comparison theorem, the existence of a required chain map is comparatively easy to demonstrate but its construction is often difficult. The next example gives a diagonal approximation for the bar resolution (*cf*. [19, VIII.9, Ex. 1]); this construction is, of course, extremely cumbersome to manipulate on the homology level. Explicit construction of a simpler diagonal approximation in low dimensions for any mapping cylinder complex is deferred until Chapter 6.

5.2.13 EXAMPLE. A diagonal approximation $\omega : BA \to BA \otimes BA$ on the bar resolution of a connected groupnet A is given by

$$\omega_0[i] = [i] \otimes [i] \quad \forall i \in \operatorname{Id} A\,,$$

and

$$\omega_n[a_1 \mid \dots \mid a_n] = [\lambda a_1] \otimes [a_1 \mid \dots \mid a_n]$$
$$+ \sum_{k=1}^{n-1} [a_1 \mid \dots \mid a_k] \otimes a_1 \dots a_k [a_{k+1} \mid \dots \mid a_n]$$
$$+ [a_1 \mid \dots \mid a_n] \otimes a_1 \dots a_n [\rho a_n]$$

for $[a_1 \mid \dots \mid a_n] \in X_n$ and $n \geq 1$. Routine calculation determines that this is a Δ-chain map.

The section closes with the construction of a diagonal comultiplication for the homology module of a subgroupnet from that of the connected groupnet containing it.

5.2.14 LEMMA. *Let B be a wide subgroupnet of the connected groupnet A and let $\pi : \tilde{A} \to A$ be the covering corresponding to B. If C is a free A-resolution of TA, any Δ-chain map $\omega : C \to C \otimes C$ lifting $\Delta : TA \to TA \otimes TA$ induces a $\tilde{\Delta}$-chain map $\tilde{\omega} : {}^{\pi}C \to {}^{\pi}C \otimes {}^{\pi}C$ lifting $\tilde{\Delta} : T\tilde{A} \to T\tilde{A} \otimes T\tilde{A}$, which is a diagonal approximation if ω is.*

Proof. The square

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{\ \pi\ } & A \\ {\scriptstyle\tilde{\Delta}}\downarrow & & \downarrow{\scriptstyle\Delta} \\ \tilde{A}\times\tilde{A} & \xrightarrow[\ \pi\times\pi\ ]{} & A\times A \end{array}$$

is defined in *Gpnet* ; its bottom row is the covering corresponding to the wide subgroupnet $B \times B$ of $A \times A$. By (5.1.2), ${}^{\pi}C$ is a free $\tilde{A}$-resolution of $T\tilde{A}$ with augmentation $\tilde{\varepsilon}(\gamma, c) = [\gamma]$. If

$$\omega_n(c) = \sum_{k=0}^{n} c_k \otimes c^*_{n-k}$$

for an element c of C_n, then

$$\tilde{\omega}_n(\gamma, c) = \sum_{k=0}^{n} (\gamma, c_k) \otimes (\gamma, c^*_{n-k}) ,$$

defined for each γ in A/B with $\rho\gamma = \lambda c$, satisfies the requirements of the lemma. □

If B is a wide subgroupnet of the connected groupnet A and M is a flat regular B-module, the diagonal comultiplication on $H_*(B; M)$ may be induced from that of A in two ways. Firstly, it may be found directly as the diagonal comultiplication

$$\Omega : H\left(A; M \otimes_B {}^{i}\mathbb{Z}A\right) \to H\left(A; M \otimes_B {}^{i}\mathbb{Z}A\right) \otimes H\left(A; M \otimes_B {}^{i}\mathbb{Z}A\right)$$

by (5.1.5) and (5.2.10) since $M \otimes_B {}^{i}\mathbb{Z}A$ is a flat regular right A-module.

Secondly, if $\omega : C \to C \otimes C$ is any diagonal approximation of $\Delta : TA \to TA \times TA$, then by (4.2.7.ii) and (5.2.14) it may be induced from the map

$$\bar{\tilde{\Delta}} \otimes \tilde{\omega} : M^r \otimes_{\tilde{A}} {}^{\pi}C \to (M^r \otimes M^r) \otimes_{\tilde{A}\times\tilde{A}} ({}^{\pi}C \otimes {}^{\pi}C)$$

of (5.2.4). The latter method is preferable when information on the cosets A/B is available.

5.3 A Class of Groupnets

The class of groupnets described here has elements in common with the class $C = \bigcup_m C_m$ of groups of Waldhausen [41, p. 158] and the class $\bigcup_n \mathrm{HNN}^n$ of groupnets of Crowell and Smythe [7, §10]. Its nice properties arise from the well-behaved nature of the mapping cylinder.

5.3.1 DEFINITION. A class of groupnets A is said to be *admissible* if it is closed under homotopy type, disjoint unions and the taking of wide subgroupnets. That is,

(i) if $A \in \mathsf{A}$ and $A' \simeq A$ then $A' \in \mathsf{A}$,

(ii) if $A \in \mathsf{A}$ and $A' \leq A$ then $A' \in \mathsf{A}$, and

(iii) if $\{A_\alpha\} \subset \mathsf{A}$ then $\bigvee_\alpha A_\alpha \in \mathsf{A}$.

If T is the class of acyclic groupnets and F is the class of groupnets of the homotopy type of disjoint unions of free groups, both T and F are admissible classes.

5.3.2 DEFINITION. Let D be the class of directed graphs and let A be a class of groupnets. The class $\mathsf{D}(\mathsf{A})$ is defined to consist of those groupnets G such that

(i) there is a graph product $m(\mathcal{D}, \mathsf{A}) \simeq G$ with D in D,

(ii) for each e in D, $A_{\lambda e}$ is a disjoint union of free groups, and

(iii) for each v in D such that $v \neq \lambda e$ for any e in D, $A_v \in \mathsf{A}$.

5.3.3 LEMMA. *If* A *is an admissible class, so is* $\mathsf{D}(\mathsf{A})$.

Proof. Closure under homotopy type and disjoint unions is immediate from the definitions. If $G \in \mathsf{D}(\mathsf{A})$ and H is a wide subgroupnet of G, by (5.1.1) there exists a graph product $\tilde{m}(\mathcal{D}, \tilde{\mathsf{A}}) = \tilde{G} \simeq H$, where $\tilde{G}$ is the covering of G corresponding to H. Each $\tilde{A}_v$ has the homotopy type of a disjoint union $K_v = \bigvee_{d \in P_v} K_v^d$ of subgroups of A_v, hence if $v \neq \lambda e$ for any e in D, $\tilde{A}_v \in \mathsf{A}$. Since $K_{\lambda e}$ is a disjoint union of free groups, and is necessarily a strong deformation retract of $\tilde{A}_{\lambda e}$, [7, Th. 8.4] implies that each $\tilde{A}_{\lambda e}$ may be replaced by $K_{\lambda e}$ without altering the homotopy type of the mapping cylinder. Thus H is homotopic to a graph product of the required kind. □

Class $C(G)$ of [7] differs from the class $\mathsf{D}(\mathsf{A})$ both in its use of a class of small categories C rather than the class of directed graphs D and in its requirement that *all* vertex groupnets (*cf.* (5.3.2.ii, iii)) should belong to the class *G* .

5.3.4 DEFINITION. The class G of groupnets is defined inductively to be:

$$G_0 = T ,$$

$$G_n = D(G_{n-1}) , \quad \forall n \geq 1 ,$$

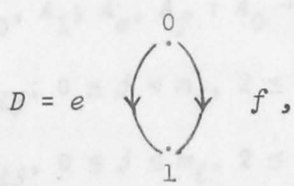
and

$$G = \bigcup_n G_n .$$

Trivially $G_{n-1} \subset G_n$, since for any $G \in G_{n-1}$ the groupnet diagram $(\mathcal{D}, A)$ with $D = \{\cdot\}$ (the directed graph with one vertex and no edges) and $A_\bullet = G$, has mapping cylinder $m(\mathcal{D}, A) = G$.

As with the class $\bigcup_n \mathrm{HNN}^n$ of [7, §10], G_1 contains the free groupnets and G_2 contains the free products of free groups amalgamated over subgroups.

5.3.5 LEMMA [24]. *If* G *is a one-relator group without torsion, then* $G \in G$. *In fact,* G *has the homotopy type of a subgroupnet of a graph product* $m(\mathcal{D}, A)$ *, where*



and A_1 *is a torsionfree one-relator group whose defining relator has shorter length than that of* G .

Proof. This observation, originally due to Moldavanski [24], employs the standard embedding of the Freiheitssatz [21, Th. 4.10]. Suppose $G = \langle x_1, \ldots, x_n : r \rangle$ is a one-relator group without torsion, where the relator r involves all the generators $(n \geq 2)$, and is not a proper power, and is cyclically reduced. Assume G is not free. Let $\sigma_i(r)$ denote the exponent sum of generator x_i in r and assume initially that $\sigma_1(r) = 0$. Let $p = \sigma_1^+(r)$ be the sum of positive exponents of x_1 in r , and replace r by $r^+ = x_1^{-p} r x_1^p$. Though r^+ is no longer cyclically reduced, the proof stated in [22, Th. 1] may be followed, except that where these authors would define a new generator by $x_1^j x_i x_1^{-j}$, we will define a new generator for every occurrence of an element $x_1^{-j} x_i x_1^j$ (for $i \neq 1$). With this slight alteration, necessary to give the result in the form of (5.3.5), it is possible to write

$$G = \left\langle x_1, \ldots, x_n, t_{ij}, 0 \le j \le m_i, 2 \le i \le n : r^+, t_{ij} = x_1^{-j} x_i x_1^j, \quad 0 \le j \le m_i, 2 \le i \le n \right\rangle$$

$$= \left\langle x_1, t_{ij}, 0 \le j \le m_i, 2 \le i \le n : r^*, t_{i,j+1} = x_1^{-1} t_{ij} x_1, \quad 0 \le j < m_i, 2 \le i \le n \right\rangle$$

where the replacement of r by r^+ ensures that j is non-negative, and where r^* is a word in the generators t_{ij} which is of length strictly less than that of r . Then $G \simeq m(\mathcal{D}, \mathsf{A})$ where

$$\mathsf{A} = \langle A_0, A_1; A_e, A_f : A_0 \to A_1 \rangle ,$$

$$A_0 = \langle y_{ij}, 0 \le j < m_i, 2 \le i \le n : \ \rangle ,$$

$$A_1 = \langle t_{ij}, 0 \le j \le m_i, 2 \le i \le n : r^* \rangle ,$$

$$A_e(y_{ij}) = t_{ij} , \quad 0 \le j < m_i , \quad 2 \le i \le n ,$$

and

$$A_f(y_{ij}) = t_{i,j+1} , \quad 0 \le j < m_i , \quad 2 \le i \le n .$$

Since A_0 is a free group, A_1 is a torsion-free one-relator group whose relator has length less than r , and A_e and A_f are both group monomorphisms, G has the homotopy type of a graph product of the specified form (*cf.* (3.3.8)). If no generator of G has exponent sum zero in r , G is a subgroup of the torsion-free one-relator group

$G^* = \left\langle z_1, z_2, x_3, \ldots, x_n : r\left(z_1^{\beta}, z_2 z_1^{-\alpha}, x_3, \ldots, x_n\right)\right\rangle$, where $\sigma_1(r) = \alpha$ and $\sigma_2(r) = \beta$. Since z_1 has exponent sum zero, and any added length of the relator in G^* is due solely to appearances of z_1 , G has the homotopy type of a subgroupnet of a graph product of the specified form.

Then either A_1 is free, in which case $G \in \mathrm{G}_2$, or the process may be repeated a finite number of times only. □

5.3.6 DEFINITION. If $(\mathcal{D}, \mathrm{A})$ is a groupnet diagram with connected vertex groupnets,

$$n^E = \sup\{\mathrm{cd}\, A_{\lambda e} : e \in D\} \leq \infty$$

and

$$n^V = \sup\{\mathrm{cd}\, A_v : v \in D,\ v \neq \lambda e \quad \forall e \in D\} \leq \infty ,$$

where if either sum is infinite it implies that an element of the respective set does not have finite cohomological dimension. The numbers n_E and n_V are correspondingly defined in terms of homological dimension. Both n^V and n_V are assumed nonempty.

5.3.7 LEMMA. *If* $(\mathcal{D}, \mathrm{A})$ *is a groupnet diagram with connected vertex groupnets and connected graph product* $m : (\mathcal{D}, \mathrm{A}) \to G$ *, then*

$$\mathrm{cd}\, G = n^V \quad \text{if} \quad n^E < n^V ,$$

and

$$n^V \leq \mathrm{cd}\, G \leq n^V + 1 \quad \text{if} \quad n^E = n^V .$$

Moreover, the corresponding result holds for the homological dimensions hd .

Proof. Since the maps A_e for e in D and m_v for v in D are group monomorphisms, $n^E \leq n^V \leq \text{cd } G$ [11, 8.1.2] and n^V may be assumed finite. Then by (D4.3.4), for all $m > n^V$ there is a short exact sequence

$$0 \to \coprod_{e \in D} H^m\left(A_{\lambda e}; {}^{\lambda e}L\right) \times \{e\} \to H^{m+1}(G; L) \to 0$$

and a short right exact sequence

$$\coprod_{v \in D} H^{n^V}\left(A_v; {}^{v}L\right) \to \coprod_{e \in D} H^{n^V}\left(A_{\lambda e}; {}^{\lambda e}L\right) \times \{e\} \to H^{n^V+1}(G; L) \to 0$$

for any left G-module L. The result follows immediately. □

Lemma (5.3.7) is proved by Bieri [1, 4.1] for HNN groups of the homotopy type of G, and partially proved by Gildenhuys [10, Th. 2] in the case when D is a tree. The result $\text{cd } G \leq 1 + \sup\left(n^E, n^V\right)$ for the fundamental group G of a graph of groups is also obtained by Chiswell [3, Corollary, Th. 2].

5.3.8 COROLLARY. *If* $A \in \mathsf{G}$, *then* $\text{cd } A \leq 2$ *and* $\text{hd } A \leq 2$.

Proof. The proof proceeds by induction. Any acyclic groupnet has the homotopy type of a disjoint collection of trivial groups, so for any A in G_0, TA admits a free A-resolution of length 0 and $\text{cd } A = 0$. Consequently, for any A in G_1, $\text{cd } A \leq 1$ and TA admits a free A-resolution of length ≤ 1. Suppose that for any A in G_k, where $0 \leq k \leq n-1$, $\text{cd } A \leq 2$. Let G in G_n be homotopic to $m(\mathcal{D}, \mathsf{A})$. As $n^E \leq 1$ and $n^V \leq 2$, $\text{cd } G \leq 2$ immediately. □

Note that for any groupnet G in G, TG admits a free G-resolution of length ≤ 2, by (3.3.7) and consideration of the inductive process of (5.3.8).

5.3.9 COROLLARY. *For any connected groupnet* A *in* G, *the diagonal map* $\Delta : A \to A \times A$ *induces a commutative coring structure on the homology module* $H(A\,; M)$ *for any flat right regular* A*-module* M, *by means of a* unique *induced diagonal comultiplication*

$$\Omega : H(A;\, M) \to H(A;\, M) \otimes H(A;\, M)\ . \qquad \square$$

The elements of G are comparatively easy to manipulate; as an example, a reasonably simple diagonal approximation for any element of G is constructed in the next chapter.

CHAPTER 6

THE DIAGONAL COMULTIPLICATION AND ITS INVARIANTS

6.1 Diagonal Approximations for Mapping Cylinders

As was mentioned in the previous chapter, it is not usually easy to construct explicit chain maps between free resolutions using the pullback methods of the comparison theorem. However, given the simple form of a groupnet in G , a diagonal approximation may be detailed for its mapping cylinder complex which extends those of the associated vertex complexes.

6.1.1 LEMMA (Existence). *Let* $(\mathcal{D}, A)$ *be a groupnet diagram with connected mapping cylinder* $m : (\mathcal{D}, A) \to G$ *and let* $(\mathcal{D}, ZA, C)$ *be a complex diagram. If*

(i) C^v *is a free* A_v*-resolution of* T_v *for all* v *in* D *,*

(ii) the G-mapping cylinder $\mu : (\mathcal{D}, ZA, C) \to M$ *is a free G-resolution of* TG *, and*

(iii) $\omega^v : C^v \to C^v \otimes C^v$ *is a diagonal approximation for each* v *in* D *, then*

there exists a diagonal approximation $\omega : M \to M \otimes M$ *extending the* ω^v .

Proof. For each i in Id G and n in Z ,

$$M_n(i) = \coprod_{\substack{j \in \mathrm{Id} A_v \\ v \in D}} ZG(i, m_v(j)) \otimes C_n^v(j) \oplus \coprod_{\substack{j \in \mathrm{Id} A_{\lambda e} \\ e \in D}} ZG(i, m_{\lambda e}(j)) \otimes C_{n-1}^{\lambda e}(j) \times \{e\} .$$

Let $\Delta_v : A_v \to A_v \times A_v$ be the diagonal map for each v in D and $\Delta : G \to G \times G$ be the diagonal map for the mapping cylinder groupnet.

Define, for each $n \geq 0$, v in D and c in C_n^v ,

$$\omega_n\left(m_v(\lambda c),\ c\right) = \left(\mu^v \otimes \mu^v\right)_n \omega_n^v(c) .$$

This map may be extended by Δ-action to form a Δ-chain map on the n-dimensional direct summand of M_n . Hence ω_0 is defined on M_0 and $\omega_0(m) = m \otimes m$, as required. Suppose that for $0 \le k \le n-1$, a Δ-chain map $\xi : M_k \to (M \otimes M)_k$ exists which extends the vertex diagonal approximations in these dimensions; that is,

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & M_n & \longrightarrow & M_{n-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & TG & \longrightarrow & \mathrm{Id}\ G \\
 & & & \downarrow \xi_{n-1} & & & & \downarrow \xi_0 & & \downarrow \Delta & & \\
\cdots \to & (M \otimes M)_n & \to & (M \otimes M)_{n-1} & \to & \cdots & \to & (M \otimes M)_0 & \to & TG \otimes TG & \to & \mathrm{Id}\ G \times \mathrm{Id}\ G .
\end{array}$$

For each basis element x of $C^{\lambda e}_{n-1}$, $\partial\xi_{n-1}\partial\left(m_{\lambda e}(\lambda x),\ x\right) = 0$; since the bottom row is exact there exists an element $\xi_n\left(m_{\lambda e}(\lambda x),\ x\right)$ in $(M \otimes M)_n$ completing the square. The Δ-morphism ξ_n may then be freely extended by Δ-action to the $(n-1)$-dimensional direct summand of M_n . □

The following theorem allows explicit description of a diagonal approximation for any connected groupnet G in G provided its structure as an element of G is known. In it, an element $(g,\ x)$ of a free G-module with basis X will be written $g[x]$ (*cf.* (3.3.8)).

6.1.2 THEOREM. *Let* $(\mathcal{D},\ \mathsf{A})$ *be a groupnet diagram with connected graph product* $m : (\mathcal{D},\ \mathsf{A}) \to G$. *Let* $(\mathcal{D},\ \mathsf{ZA},\ \mathsf{C})$ *be a complex diagram where* C^v *is a free* A_v*-resolution of* T_v *and* C^v_n *has basis* X^v_n *for all* n *in* Z *and* v *in* D . *Suppose the following four conditions are satisfied by* $(\mathcal{D},\ \mathsf{ZA},\ \mathsf{C})$:

(i) for all v *in* D , $C^v_3 = \mathrm{Id}\ A_v$;

(ii) for all e *in* D , $A_{\lambda e}$ *is a disjoint collection of free groups and* $C^{\lambda e}_2 = \mathrm{Id}\ A_{\lambda e}$;

(iii) for all v in D and x in X_1^v, there exist Rx and Lx in X_0^v, and $r(x)$ and $l(x)$ in A_v, such that

$$\partial[x] = r(x)[Rx] - l(x)[Lx] ,$$

provided that $l(x) = \lambda(Lx)$ if $v = \lambda e$ for an e in D ; and

(iv) for all e in D and x in $X_0^{\lambda e}$, there exist $e(x)$ in $X_0^{\rho e}$ and $A_e\langle x\rangle$ in $A_{\rho e}\big(A_e(\lambda x), \lambda e(x)\big)$ such that

$$C_0^e[x] = A_e\langle x\rangle[e(x)] .$$

If $\mu : (\mathcal{D}, ZA, C) \to M$ is the G-mapping cylinder of $(\mathcal{D}, ZA, C)$, there are diagonal approximations for the C^v which extend to a diagonal approximation for M.

Proof. A diagonal approximation $\omega^v : C^v \to C^v \otimes C^v$ may be extended freely by Δ_v-action and the comparison theorem from

$$\omega_0^v[z] = [z] \otimes [z] , \quad \forall z \in X_0^v ,$$

and

$$\omega_1^v[y] = l(y)[Ly] \otimes [y] + [y] \otimes r(y)[Ry] , \quad \forall y \in X_1^v ,$$

for any v in D. The mapping cylinder is (3.3.6.iii)

$$M_0(i) = \coprod_{\substack{z \in X_0^v \\ v \in D}} ZG\big(i, m_v(\lambda z)\big)[z] ,$$

$$M_1(i) = \coprod_{\substack{y \in X_1^v \\ v \in D}} ZG\big(i, m_v(\lambda y)\big)[y] \oplus \coprod_{\substack{z \in X_0^{\lambda e} \\ e \in D}} ZG\big(i, m_{\lambda e}(\lambda z)\big)[z|e] ,$$

and

$$M_2(i) = \coprod_{\substack{x \in X_2^v \\ v \in D}} ZG\big(i, m_v(\lambda x)\big)[x] \oplus \coprod_{\substack{y \in X_1^{\lambda e} \\ e \in D}} ZG\big(i, m_{\lambda e}(\lambda y)\big)[y|e] ,$$

for i in Id G , with boundary maps

$$\partial_1[y] = m_v\big(r(y)\big)[Ry] - m_v\big(l(y)\big)[Ly] \qquad \forall y \in X_1^v ,\ v \in D ,$$

$$\partial_1[z|e] = [z] - m_e(*, \lambda z)m_{\rho e}(A_e\langle z\rangle)[e(z)] \qquad \forall z \in X_0^{\lambda e} ,\ e \in D ,$$

$$\partial_2[x] = \mu_2^v\partial_2[x] \qquad \forall x \in X_2^v ,\ v \in D ,$$

and

$$\partial_2[y|e] = [y] - m_e(*, \lambda y)\mu_1^{\rho e}C_1^e[y] - m_{\lambda e}\big(r(y)\big)[Ry|e]$$
$$+ m_{\lambda e}\big(l(y)\big)[Ly|e] \quad \forall y \in X_1^{\lambda e} ,\ e \in D .$$

Note that the mapping cylinder itself 'satisfies' condition (i) above, and its boundary map in dimension 1 'satisfies' condition (iii). The diagonal approximation $\omega : M \to M \otimes M$ is partially determined (6.1.1) by

$$\omega_0[z] = [z] \otimes [z] \quad \forall z \in X_0^v ,\ v \in D ,$$

$$\omega_1[y] = m_v\big(l(y)\big)[Ly] \otimes [y] + [y] \otimes m_v\big(r(y)\big)[Ry] \quad \forall y \in X_1^v ,\ v \in D ,$$

and

$$\omega_2[x] = (\mu \otimes \mu)_2\omega_2^v[x] \quad \forall x \in X_2^v ,\ v \in D .$$

Routine calculation shows that the evaluation

$$\omega_1[z|e] = \{m_e(*, \lambda z)m_{\rho e}\big(A_e\langle z\rangle\big)[e(z)]\} \otimes [z|e] + [z|e] \otimes [z]$$

for all z in $X_0^{\lambda e}$ and e in D , completes the chain map ω in dimension 1 . In fact ω_1 is derived from the dimension 1 boundary map of M in the same way as ω_1^v is derived from the dimension 1 boundary map of C^v for any v in D . It remains to evaluate $\omega_2[y|e]$ for y in $X_1^{\lambda e}$ and e in D . Define

$$\Gamma[y|e] = m_e(*, \lambda y)m_{\rho e}\big(A_e\langle Ly\rangle\big)[e(Ly)] \otimes [y|e]$$

$$- m_e(*, \lambda y)\mu_1^{\rho e}C_1^e[y] \otimes m_{\lambda e}\big(r(y)\big)[Ry|e]$$

$$+ [Ly|e] \otimes [y] + [y|e] \otimes m_{\lambda e}\big(r(y)\big)[Ry] ,$$

so that

$$\Delta\Big(m_e\big(*^{-1}, \lambda y\big)\Big).(\omega\partial-\partial\Gamma)[y|e] = m_{\rho e}\big(A_e\langle Ly\rangle\big)[e(Ly)] \otimes \mu_1^{\rho e}C_1^e[y]$$

$$+ \mu_1^{\rho e}C_1^e[y] \otimes m_{\rho e}\big(A_e\big(r(y)\big)A_e\langle Ry\rangle\big)[e(Ry)] - \big(\mu^{\rho e} \otimes \mu^{\rho e}\big)_1\omega_1^{\rho e}C_1^e[y] .$$

Since $C^{\rho e}$ is exact, for each e in D, y in $X_1^{\lambda e}$, and t in $X_1^{\rho e}$, there exists a set map

$$D(e, y, t) : A_{\rho e}\big(A_e(\lambda y), \lambda t\big) \to C_1^{\rho e}\big(A_e(\lambda y)\big)$$

such that

$$\partial_1^{\rho e}D(e, y, t)(a) = al(t)[Lt] - A_e\langle Ly\rangle[e(Ly)]$$

for all a in $A_{\rho e}\big(A_e(\lambda y), \lambda t\big)$. If

$$C_1^e[y] = \sum_t \sum_{a_t} n\big(a_t\big)a_t[t] ,$$

define

$$D[y|e] = \Delta\big(m_e(*, \lambda y)\big) \sum_t \sum_{a_t} n\big(a_t\big)\{D(e, y, t)\big(a_t\big) \otimes m_{\rho e}\big(a_t\big)[t]\} ,$$

so that

$$\Delta\Big(m_e\big(*^{-1}, \lambda y\big)\Big).(\omega\partial-\partial\Gamma+\partial D)[y|e] =$$

$$\sum_t \sum_{a_t} n\big(a_t\big)\Big\{m_{\rho e}\big(a_t\big)[t] \otimes m_{\rho e}\big(A_e\big(r(y)\big)A_e\langle Ry\rangle\big)[e(Ry)]$$

$$- \Delta\big(m_{\rho e}\big(a_t\big)\big)\big[[t] \otimes m_{\rho e}\big(r(t)\big)[Rt]\big] - D(e, y, t)\big(a_t\big) \otimes m_{\rho e}\big(a_t\big)\partial_1^{\rho e}[t]\Big\} ,$$

which is a cycle of $M_1 \otimes M_0$. As $M \otimes M$ is exact, there exists $B[y|e]$ in $M_2 \otimes M_0$ such that $\partial B[y|e] = (\omega\partial-\partial\Gamma+\partial D)[y|e]$. Thus ω_2 may be freely extended by Δ-action from

$$\omega_2[y|e] = (\Gamma - D + B)[y|e] ,$$

where the *differential term* $D[y|e] \in M_1 \otimes M_1$ and the *boundary term* $B[y|e] \in M_2 \otimes M_0$. □

The boundary term always disappears at the homology level.

NOTE. For any group A generated by X , such that A is the epimorphic image $\phi : F \to A$ of the free group generated by X , the term $\sum_x \frac{\partial a}{\partial x}$ is understood to mean the image under ϕ of the Fox derivative of a in $\mathbb{Z}F$. If the term $C_1^e[y]$ above is a 'derivative'; that is,

$$C_1^e[y] = \sum_t \frac{\partial A_e(y)}{\partial t}[t] ,$$

then it is possible to set $D(e, y, t)(a_t) = \sum_{t^*} \frac{\partial a_t}{\partial t^*}[t^*]$, and in this case a straightforward, if tedious, calculation shows that $(\omega\partial - \partial\Gamma + \partial D)[y|e] = 0$, so $B[y|e]$ may be equated with zero.

6.1.3 EXAMPLE. Consider Example (3.3.8) where $G = m(\mathcal{D}, \mathrm{A})$ is the torsion-free one-relator group of (5.3.5). Then

$$A_0 = \langle y_{ij},\ 0 \le j < m_i,\ 2 \le i \le n :\ \rangle ,$$

$$A_1 = \langle t_{ij},\ 0 \le j \le m_i,\ 2 \le i \le n : r \rangle ,$$

$$A_e(y_{ij}) = t_{ij} \quad \text{and} \quad A_f(y_{ij}) = t_{i,j+1} \quad \text{for } 0 \le j < m_i ,\quad 2 \le i \le n ,$$

and

$$C_1^e[y_{ij}] = [t_{ij}] \quad \text{and} \quad C_1^f[y_{ij}] = [t_{i,j+1}] \quad \text{for } 0 \le j < m_i ,\quad 2 \le i \le n .$$

Consequently, $\partial_1^{\rho e} D(e, y, t)(1) = \partial_1^{\rho e} D(f, y, t)(1) = 0[1]$ for all generators y of A_0 and t of A_1 . Hence $\omega_2[y_{ij}|e] = \Gamma[y_{ij}|e]$, and

$$\omega_2[y_{ij}|e] = m_e(*, 0)[1] \otimes [y_{ij}|e] + [y_{ij}|e] \otimes y_{ij}[0]$$
$$+ [0|e] \otimes [y_{ij}] - m_e(*, 0)[t_{ij}] \otimes y_{ij}[0|e] ;$$

$$\omega_2[y_{ij}|f] = m_f(*, 0)[1] \otimes [y_{ij}|f] + [y_{ij}|f] \otimes y_{ij}[0] + [0|f] \otimes [y_{ij}] - m_f(*, 0)[t_{i,j+1}] \otimes y_{ij}[0|f] .$$

As an element of G, y_{ij} is equal to either $m_e(*, 0)t_{ij}m_e(*^{-1}, 0)$ or $m_f(*, 0)t_{i,j+1}m_f(*^{-1}, 0)$. The homology of G with trivial coefficients is found as follows. The tensored complex has

$$TG \otimes_G M_2 = Z[r] \oplus \sum_{i=2}^{n} \sum_{j=0}^{m_i-1} Z[y_{ij}|e] \oplus \sum_{i=2}^{n} \sum_{j=0}^{m_i-1} Z[y_{ij}|f] ,$$

$$TG \otimes_G M_1 = \sum_{i=2}^{n} \sum_{j=0}^{m_i} Z[t_{ij}] \oplus \sum_{i=2}^{n} \sum_{j=0}^{m_i-1} Z[y_{ij}] \oplus Z[0|e] \oplus Z[0|f] ,$$

$$TG \otimes_G M_0 = Z[1] \oplus Z[0] ,$$

and boundary maps $\partial_2[r] = \sum_{i=2}^{n} \sum_{j=0}^{m_i} \sigma_{ij}(r)[t_{ij}]$, where $\sigma_{ij}(r)$ denotes the exponent sum of generator t_{ij} in r,

$$\partial_2[y_{ij}|e] = [y_{ij}] - [t_{ij}] , \quad \partial_2[y_{ij}|f] = [y_{ij}] - [t_{i,j+1}] ,$$

$$\partial_1[t_{ij}] = \partial_1[y_{ij}] = 0 ,$$

and

$$\partial_1[0|e] = \partial_1[0|f] = [0] - [1] .$$

Suppose $\sum_{i=2}^{n} \sum_{j=0}^{m_i-1} \varepsilon_{ij}([y_{ij}|e]-[y_{ij}|f]) + l[r] \in \operatorname{Ker} \partial_2$. That is,

$$\varepsilon_{i,0} = l\sigma_{i,0}(r) , \quad 2 \le i \le n ,$$

$$\varepsilon_{i,j} = \varepsilon_{i,j-1} + l\sigma_{ij}(r) , \quad 1 \le j \le m_i-1 , \quad 2 \le i \le n ,$$

and

$$\varepsilon_{i,m_i-1} = -l\sigma_{i,m_i} ,$$

so that $l \sum_{j=0}^{m_i} \sigma_{ij}(r) = 0$, $2 \le i \le n$.

Either $\sum_{j=0}^{m_i} \sigma_{ij}(r) \neq 0$ for some $2 \leq i \leq n$ (in other words, the relator of G is not a commutator), in which case $H_2(G; TG) = 0$; or $\sum_{j=0}^{m_i} \sigma_{ij}(r) = 0$ for all $2 \leq i \leq n$, in which case $H_2(G; TG) \cong \mathbb{Z}$, with generating element

$$\mathsf{g} = [r] + \sum_{i=2}^{n} \sum_{j=0}^{m_i-1} \left\{ \sum_{p=0}^{j} \sigma_{ip}(r) \right\} \left([y_{ij}|e] - [y_{ij}|f] \right) .$$

In the latter case, suppose $(\overline{\Delta} \otimes \omega^1)_2 : \mathbb{Z} \otimes C_2^1 \to [(\mathbb{Z} \otimes C^1) \otimes (\mathbb{Z} \otimes C^1)]_2$ is given by

$$(\overline{\Delta} \otimes \omega^1)_2[r] = [1] \otimes [r] + [r] \otimes [1] + \sum_{i=2}^{n} \sum_{j=0}^{m_i} \sum_{k=2}^{n} \sum_{l=0}^{m_i} \varepsilon(i, j, k, l)[t_{ij}] \otimes [t_{kl}] .$$

Then, since $H_1(G; TG) \cong \mathbb{Z}([0|e]-[0|f]) \oplus \sum_{i=2}^{n} \mathbb{Z}[t_{i0}]$ and

$$H_0(G; TG) \cong \mathbb{Z}[1] ,$$

$$\Omega_2(\mathsf{g}) = [1] \otimes \mathsf{g} + \mathsf{g} \otimes [1] + ([0|e]-[0|f]) \otimes \mathsf{t} - \mathsf{t} \otimes ([0|e]-[0|f]) + \sum_{i=2}^{n} \sum_{k=2}^{n} \left\{ \sum_{j=0}^{m_i} \sum_{l=0}^{m_i} \varepsilon(i, j, k, l) \right\} [t_{i0}] \otimes [t_{k0}] ,$$

where

$$\mathsf{t} = \sum_{i=2}^{n} \left\{ \sum_{j=0}^{m_i-1} \sum_{p=0}^{j} \sigma_{ip}(r) \right\} [t_{i0}] . \qquad \square$$

Hence in the case of a one-relator group, the extending part of the diagonal approximation for the mapping cylinder is particularly simple, and the diagonal comultiplication for G depends essentially on any diagonal approximation for the embedded one-relator group A_1 .

6.2 The Problem of Invariance

It is hoped to use the diagonal comultiplication to give a finer classification of groups in G than is given by their homology modules with integral coefficients. Thus it is necessary to determine the invariants of such maps; that is, to abstract that information contained in the diagonal comultiplication which is independent of any choice of basis for the homology module. I have not as yet been able to proceed far in this determination.

The diagonal comultiplication is related to the lower central series of the group G, as is the low-dimensional homology (*cf.* Stallings [32]), and appears in particular to provide information about G_2/G_3. It is hoped the diagonal comultiplication may be refined to provide a quick and computable method of abstracting this information from a presentation of G.

The work below assumes $H_1(G; TG)$ is torsion-free; if $H_1(G; TG)$ has torsion, the results hold for the torsion-free part.

Let $A = Z^n$ and $B = Z^m$ be two finite direct sums of copies of the integers and suppose A and B are freely generated as abelian groups by the sets $\{x_i : 1 \leq i \leq n\}$ and $\{y_j : 1 \leq j \leq m\}$ respectively. Here A is intended to represent $H_2(G; TG)$ and B, the torsion-free part of $H_1(G; TG)$. Let $\Delta : A \rightarrow B \otimes_Z B$ be an antisymmetric group morphism, so that if

$$\Delta(x_i) = \sum_{j=1}^{m} \sum_{k=1}^{m} \delta(i, j, k) y_j \otimes y_k , \quad 1 \leq i \leq n ,$$

then the *associated set*

$$D = \{D_i = (\delta(i, j, k)) : 1 \leq i \leq n\}$$

of n $m \times m$ integral matrices, consists only of skew-symmetric matrices (*cf.* (5.2.11)).

If $S(n, m)$ denotes the set of all such antisymmetric group morphisms $A \to B \otimes B$, a relation $\sim$ on $S(n, m)$ is defined by

$$\Delta \sim \Gamma \iff \exists \alpha : A \cong A,\ \beta : B \cong B : \begin{array}{ccc} A & \xrightarrow{\Delta} & B\otimes B \\ \alpha\downarrow & & \downarrow\beta\otimes\beta \\ A & \xrightarrow[\Gamma]{} & B\otimes B \end{array}, \qquad \text{(D6.2.1)}$$

in which case Γ is said to be *similar to* Δ *in* $S(n, m)$ *via* (α, β). This relation is obviously an equivalence relation and partitions $S(n, m)$ into *similarity classes.*

The invariants of a similarity class thus represent the 'basis-free' information carried by the elements of the class.

6.2.1 PROBLEM. What is the form of a canonical representative of each similarity class of $S(n, m)$ for each n and m in $\mathbb{Z}$? Alternatively, how may a complete set of similarity invariants be found for each element of $S(n, m)$?

Clearly the cases $S(0, m)$ and $S(n, 0)$ for all $n, m \geq 0$ are trivial, as the only possible map is the zero morphism, in its own similarity class. Similarly, the only map in $S(n, 1)$ is the zero map, by antisymmetry. Henceforward it will be assumed that $S(n, m)$ has $n \geq 1$ and $m \geq 2$. The remainder of this section is devoted to a few general considerations.

6.2.2 PROPOSITION. *The zero morphism* $0 : A \to B \otimes B$ *is the only element of its similarity class in* $S(n, m)$.

Proof. If $0 \sim \Delta$ *via* (α, β) , $\Delta = \beta \otimes \beta 0 \alpha^{-1} = 0$. □

All antisymmetric maps are in future assumed nonzero.

6.2.3 REMARK. The *rank* $r\Delta = \text{rank}(\text{Im } \Delta)$ and *nullity* $\text{null } \Delta = \text{rank}(\text{Ker } \Delta)$ of an element Δ of $S(n, m)$ are invariants of its similarity class.

6.2.4 DEFINITION. If $\Delta \in S(n, m)$ then the *greatest common divisor* $g\Delta$ of Δ is the integer

$$g\Delta = \text{g.c.d.}\left(\delta(i, j, k) : 1 \le j, k \le m, 1 \le i \le n\right) .$$

Since Δ is antisymmetric,

$$g\Delta = \text{g.c.d.}\left(\delta(i, j, k) : 2 \le k \le m, 1 \le j < k, 1 \le i \le n\right) .$$

6.2.5 LEMMA. *The greatest common divisor is an invariant of each similarity class.*

Proof. Suppose $\Delta \sim \Gamma$ in $S(n, m)$ *via* (α, β), and let $\overline{\alpha} = \left(\alpha_{ij}\right)$ and $\overline{\beta} = \left(\beta_{ij}\right)$ represent the matrices associated with α and β respectively. This notation will be used throughout the chapter. Let $\overline{\alpha}^{-1} = \left(\sigma_{ij}\right)$ and $\overline{\beta}^{-1} = \left(\tau_{ij}\right)$. Since $\Gamma = \beta \otimes \beta\Delta\alpha^{-1}$ and $\Delta = \beta^{-1} \otimes \beta^{-1}\Gamma\alpha$,

$$\Gamma\left(x_i\right) = \sum_{p=1}^{m} \sum_{q=1}^{m} \left(\sum_{j=1}^{n} \sum_{k=1}^{m} \sum_{l=1}^{m} \sigma_{ij}\delta(j, k, l)\beta_{kp}\beta_{lq}\right)y_p \otimes y_q , \quad 1 \le i \le n ,$$

so that

$$\gamma(i, p, q) = \sum_{k=1}^{m} \sum_{l=1}^{m} \beta_{kp}\beta_{lq}\left(\sum_{j=1}^{n} \sigma_{ij}\delta(j, k, l)\right) ,$$

$$1 \le p, q \le m , \quad 1 \le i \le n , \quad \text{(D6.2.2)}$$

and similarly

$$\delta(i, p, q) = \sum_{k=1}^{m} \sum_{l=1}^{m} \tau_{kp}\tau_{lq}\left(\sum_{j=1}^{n} \alpha_{ij}\gamma(j, k, l)\right) ,$$

$$1 \le p, q \le m , \quad 1 \le i \le n . \quad \text{(D6.2.3)}$$

Hence $g\Delta = g\Gamma$. □

Incidentally, since Δ is antisymmetric, it follows from (D6.2.2) that any map Γ similar to Δ is necessarily antisymmetric.

6.2.6 DEFINITION. Let $\Delta \in S(n, m)$ with associated set of matrices $D = \left\{D_i : 1 \le i \le n\right\}$. The ith *common divisor* $g_i\Delta$ of Δ for $1 \le i \le n$ is defined as

$$g_i\Delta = \text{g.c.d.}\left(\delta(i, j, k) : 1 \le j, k \le m\right) ,$$

so that $g\Delta = \text{g.c.d.}\{g_i\Delta : 1 \le i \le n\}$.

6.2.7 REMARK. If $\Delta \sim \Gamma$ *via* $(1, \beta)$ in $S(n, m)$, then

$$g_i\Delta = g_i\Gamma , \quad 1 \le i \le n .$$

This is an immediate corollary of (6.2.5), since, with $\overline{\alpha} = I_n$, (D6.2.2) and (D6.2.3) become

$$\gamma(i, p, q) = \sum_{k=1}^{m} \sum_{l=1}^{m} \beta_{kp}\beta_{lq}\delta(i, k, l)$$

and

$$\delta(i, p, q) = \sum_{k=1}^{m} \sum_{l=1}^{m} \tau_{kp}\tau_{lq}\gamma(i, k, l)$$

for $1 \le p, q \le m$ and $1 \le i \le n$, or in terms of the associated matrices,

$$G_i = \overline{\beta}^T D_i \overline{\beta} , \quad 1 \le i \le n .$$

6.2.8 LEMMA. *If* $\Delta \in S(n, m)$, $\overline{\Delta} \in S(n-p, m)$ *for a* p *in* $\mathbb{Z}$, $1 \le p \le n-1$, *and*

$$\overline{\Delta}(x_i) = \Delta(x_i) , \quad 1 \le i \le n-p ,$$

any $\overline{\Gamma}$ *similar to* $\overline{\Delta}$ *may be extended to a* Γ *similar to* Δ *for which* $g_i\Gamma = g_i\Delta$, $n-p+1 \le i \le n$.

Proof. Let $\overline{\Delta} \sim \overline{\Gamma}$ *via* $(\check{\alpha}, \beta)$ in $S(n-p, m)$, and define α from $\check{\alpha}$ by

$$\alpha(x_i) = \check{\alpha}(x_i) , \quad 1 \le i \le n-p ,$$

$$\alpha(x_i) = x_i , \qquad n-p+1 \le i \le n .$$

In matrix terms, $\overline{\alpha} = \overline{\check{\alpha}} \dotplus I_p$. If $\Gamma \in S(n, m)$ is the morphism $\beta \otimes \beta\Delta\alpha^{-1} : A \to B \otimes B$, then

$$\Gamma(x_i) = \begin{cases} \beta \otimes \beta\overline{\Delta}(\check{\alpha})^{-1}(x_i) , & 1 \le i \le n\text{-}p , \\ \beta \otimes \beta\Delta(x_i) , & n\text{-}p\text{+}1 \le i \le n , \end{cases}$$

$$= \begin{cases} \overline{\Gamma}(x_i) , & 1 \le i \le n\text{-}p , \\ \beta \otimes \beta\Delta(x_i) , & n\text{-}p\text{+}1 \le i \le n ; \end{cases}$$

$\Delta \sim \Gamma$ by definition, and $g_i\Delta = g_i\Gamma$ for $n\text{-}p\text{+}1 \le i \le n$, by comparison with (6.2.7). □

6.2.9 REMARK. Clearly a similar result holds for any subset of $(n\text{-}p)$ elements of $\{x_i : 1 \le i \le n\}$ on application of a suitable permutation of $\{i : 1 \le i \le n\}$. If it should happen that $\overline{\beta} = I_m$ in (6.2.8), then

$$\gamma(i, k, l) = \sum_{j=1}^{n-p} \check{\sigma}_{ij}\delta(j, k, l) , \quad 1 \le i \le n\text{-}p ,$$

$$\gamma(i, k, l) = \delta(i, k, l) , \qquad n\text{-}p\text{+}1 \le i \le n ;$$

if, further, for a pair (k, l) with $1 \le k, l \le m$, it should happen that $\delta(i, k, l) = 0$ for $1 \le i \le n$, then similarly $\gamma(i, k, l) = 0$ for $1 \le i \le n$.

These last two results form the basis of inductive proofs on n in $S(n, m)$, used in the succeeding section.

6.3 Some Solutions

The table below summarises the number and general form of those similarity invariants of an antisymmetric morphism $\Delta : A \to B \otimes B$ which are ascertained in this section.

<table>
<tr><th>$n \backslash m$</th><th>2</th><th>3</th><th>4</th></tr>
<tr><td>1</td><td colspan="3">$g\Delta,\ \left(\prod_{i=1}^{j} p_i\right) g\Delta,\ 1 \le j \le s\text{-}1,\ r\Delta = 2s;\ p_i > 0,\ 1 \le i \le s\text{-}1$</td></tr>
<tr><td>2</td><td rowspan="3">$g\Delta$</td><td>$g\Delta,\ p(g\Delta),\ p \ge 0$</td><td>$g\Delta,\ p(g\Delta),\ q(g\Delta)$
$a(g\Delta),\ ba(g\Delta);\ a,\ p,\ q \ge 0$
?</td></tr>
<tr><td>3</td><td rowspan="2">$g\Delta,\ p(g\Delta),\ qp(g\Delta),$
$q,\ p \ge 0$</td><td>?</td></tr>
<tr><td></td><td>?</td></tr>
</table>

TABLE 6.1. Similarity Invariants

Before proof of these results is given, some simplified notation is required. If $\Delta \sim \Gamma$ *via* (α, β) in $S(n, m)$, then, for $1 \le i \le n$,

$$\sum_{j=1}^{n} \alpha_{ij}\gamma(j, p, q) = \sum_{k=1}^{m} \sum_{l=1}^{m} \beta_{kp}\,\delta(i, k, l)\beta_{lq}\ , \quad 1 \le p,\ q \le m\ .$$

If the matrix associated with $\Gamma\alpha(x_i)$ is denoted

$$[\overline{\alpha}|\gamma]_i = \left(\sum_{j=1}^{n} \alpha_{ij}\gamma(j, p, q)\right)\ , \quad 1 \le i \le n\ ,$$

then

$$[\overline{\alpha}|\gamma]_i = \overline{\beta}^T D_i \overline{\beta}\ , \quad 1 \le i \le n\ .$$

The skew-symmetric matrix D_i will be written

$$D_i = \langle \delta(i, 1, 2), \ldots, \delta(i, 1, m), \delta(i, 2, 3), \ldots, \delta(i, m\text{-}1, m) \rangle\ , \quad 1 \le i \le n\ ,$$

and the associated set of matrices D will be written

$$D = \langle \delta(1, 1, 2), \ldots, \delta(1, m\text{-}1, m); \ldots; \delta(n, 1, 2), \ldots, \delta(n, m\text{-}1, m) \rangle\ .$$

For example, if $m = 3$ and $n = 2$,

$$D = \langle \delta(1, 1, 2), \delta(1, 1, 3), \delta(1, 2, 3);\ \delta(2, 1, 2), \delta(2, 1, 3), \delta(2, 2, 3) \rangle\ .$$

If M is a $p \times p$ integral matrix, then

$$M\begin{pmatrix} j_1, \ldots, j_k \\ i_1, \ldots, i_k \end{pmatrix}$$

is the $k \times k$ minor of M composed of the intersections of rows i_1 to i_k, $1 \le i_1 < \ldots < i_k \le p$, with columns j_1 to j_k, $1 \le j_1 < \ldots < j_k \le p$.

6.3.1 DEFINITION. If M is a $p \times p$ integral matrix, the kth *determinantal divisor* $d_k(M)$ of M for $1 \le k \le p$ is defined to be

$$d_k(M) = \text{g.c.d.}\left\{ \left| M\begin{pmatrix} j_1, \ldots, j_k \\ i_1, \ldots, i_k \end{pmatrix} \right| : 1 \le i_1 < \ldots < i_k \le p,\ 1 \le j_1 < \ldots < j_k \le p \right\}.$$

Hence, for $M = (\mu_{ij})$, $d_1(M) = \text{g.c.d.}(\mu_{ij} : 1 \le i, j \le p)$, and if $d_k(M) \ne 0$, $d_k(M) | d_{k+1}(M)$. For convenience, 1 is defined to be $d_0(M)$. The kth *(positive) invariant factor* $s_k(M)$ of M for $1 \le k \le p$ is

$$s_k(M) = \begin{cases} d_k(M)/d_{k-1}(M), & d_{k-1}(M) \ne 0, \\ 0, & d_{k-1}(M) = 0. \end{cases}$$

If $s_k(M) \ne 0$, $s_k(M) | s_{k+1}(M)$.

6.3.2 LEMMA. *If $\Delta \in S(1, m)$ for any m in Z, there exists a unique element $I(\Delta) \sim \Delta$ in $S(1, m)$ such that*

$$I(D) = \begin{bmatrix} 0 & g\Delta \\ -g\Delta & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & p_1(g\Delta) \\ -p_1(g\Delta) & 0 \end{bmatrix} \dotplus \ldots \dotplus \begin{bmatrix} 0 & \left(\prod_{j=1}^{s-1} p_j\right) g\Delta \\ -\left(\prod_{j=1}^{s-1} p_j\right) g\Delta & 0 \end{bmatrix} \dotplus 0_{m-2s},$$

where $r\Delta = 2s$ *and* $p_j > 0$, $1 \le j \le s-1$.

Proof. As the only isomorphisms $Z \cong Z$ are $\alpha = \pm 1$, it follows that $\Delta \sim \Gamma$ in $S(1, m)$ if and only if there exist $\varepsilon = \pm 1$ and $\beta : B \cong B$ such that

$$G = \overline{\beta}^T (\varepsilon D) \overline{\beta} \ ;$$

or, in matrix terms, if and only if G is congruent to εD . By Newman [25, Th. IV.1, Th. IV.2], $r\Delta = 2s$ for some integer s , and D is congruent to a unique matrix

$$I(D) = \begin{bmatrix} 0 & h_1 \\ -h_1 & 0 \end{bmatrix} \dot{+} \begin{bmatrix} 0 & h_2 \\ -h_2 & 0 \end{bmatrix} \dot{+} \dots \dot{+} \begin{bmatrix} 0 & h_s \\ -h_s & 0 \end{bmatrix} \dot{+} 0_{m-2s} ,$$

where $h_i | h_{i+1}$ for $1 \le i \le s-1$ and the elements

$$h_1, h_1, h_2, h_2, \dots, h_s, h_s$$

are the positive invariant factors of D . Since D and $-D$ have the same positive invariant factors and $h_1 = s_1(D) = d_1(D) = g\Delta$, the result follows with $h_i = p_{i-1} h_{i-1}$ for $2 \le i \le s$. Matrix $I(D)$ is the *skew-normal form* of D .

The next two theorems rely on an inductive proof: their difficulty lies only in the proof of their respective results for low integer values.

6.3.3 THEOREM. *If* $\Delta \in S(n, 2)$ *for any* n *in* Z *, there exists a unique element* $I(\Delta) \sim \Delta$ *in* $S(n, 2)$ *such that*

$$I(D)_1 = \begin{bmatrix} 0 & g\Delta \\ -g\Delta & 0 \end{bmatrix} \text{ and } I(D)_i = 0_2 , \quad 2 \le i \le n .$$

Proof. It is necessary only to prove the existence of such an element: its subsequent uniqueness is quite apparent. When $n = 1$, the result holds from (6.3.2); in fact, each similarity class contains precisely one anti-symmetric morphism together with its negative, since then $g\Delta = g(-\Delta) = |\delta(1, 1, 2)|$. Suppose the result holds for elements of $S(k, 2)$, $1 \le k \le n-1$, and let $\Delta \in S(n, 2)$. Define $\overline{\Delta} \in S(n-1, 2)$ as

$$\overline{\Delta}(x_i) = \Delta(x_i)\ , \quad 1 \le i \le n\text{-}1\ ,$$

with canonical form $\overline{\Delta} \sim I(\overline{\Delta})$ *via* $(\check{\alpha}, \beta)$ in $S(n\text{-}1, 2)$. That is,

$g\overline{\Delta}$ = g.c.d.$(g_i\Delta : 1 \le i \le n\text{-}1)$, $g\Delta = (g\overline{\Delta}, g_n\Delta)$,

$$I(\overline{D})_1 = \begin{bmatrix} 0 & g\overline{\Delta} \\ -g\overline{\Delta} & 0 \end{bmatrix} \text{ and } I(\overline{D})_i = 0_2\ , \quad 2 \le i \le n\text{-}1\ .$$

By (6.2.8) there exists $\Delta \sim \Gamma$ in $S(n, 2)$ with

$$G_1 = \begin{bmatrix} 0 & g\overline{\Delta} \\ -g\overline{\Delta} & 0 \end{bmatrix},\ G_i = 0_2\ , \quad 2 \le i \le n\text{-}1\ ,$$

and for some $\varepsilon = \pm 1$,

$$G_n = \begin{bmatrix} 0 & \varepsilon g_n\Delta \\ -\varepsilon g_n\Delta & 0 \end{bmatrix}.$$

Of course if $g_n\Delta = 0$ the result is immediate. For $g_n\Delta \ne 0$, there exist integers x and y such that

$$g\overline{\Delta}x + \varepsilon g_n\Delta y = g\Delta\ .$$

If

$$\bar{\check{\alpha}} = \begin{bmatrix} \dfrac{g\overline{\Delta}}{g\Delta} & -y \\ \dfrac{\varepsilon g_n\Delta}{g\Delta} & x \end{bmatrix},\ \overline{\beta} = I_2\ ,$$

$\overline{\alpha}$ is extended from $\bar{\check{\alpha}}$ (6.2.9), and $I(\Delta) = \beta \otimes \beta\Gamma\alpha^{-1}$, then

$$I(\Delta)(x_i) = \sum_{j=1}^{n} \sigma_{ij}\gamma(j, 1, 2)\{y_1 \otimes y_2 - y_2 \otimes y_1\}$$

$$= \begin{cases} g\Delta\ , & i = 1\ , \\ 0\ , & 2 \le i \le n\ , \end{cases}$$

as required. □

6.3.4 THEOREM. *If* $\Delta \in S(n, 3)$ *for any* n *in* $\mathbb{Z}$, *there exists a unique element* $I(\Delta) \sim \Delta$ *in* $S(n, 3)$ *such that*

$$I(D)_1 = \begin{bmatrix} 0 & g\Delta & 0 \\ -g\Delta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I(D)_2 = \begin{bmatrix} 0 & 0 & p(g\Delta) \\ 0 & 0 & 0 \\ -p(g\Delta) & 0 & 0 \end{bmatrix} \quad (n \geq 2),$$

$$I(D)_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & qp(g\Delta) \\ 0 & -qp(g\Delta) & 0 \end{bmatrix} \quad (n \geq 3), \text{ and } I(D)_i = 0 \text{ for } 4 \leq i \leq n,$$

where $p, q \geq 0$.

Proof. The proof is inductive for $n > 3$, but the cases $n = 1, 2$, and 3 must be determined individually. When $n = 1$ the result is immediate from (6.3.2), as $r\Delta = 2$.

Case $n = 2$. Both $g_1\Delta$ and $g_2\Delta$ may be assumed nonzero by (6.3.2) and (6.2.9). There exists $\Delta \sim \Gamma$ in $S(2, 3)$ such that

$$G = \langle g_1\Delta, 0, 0; a, b, c \rangle, \quad (a, b, c) = g_2\Delta .$$

Step *a*. Either $(b, c) = 0$ or there exist integers x and y such that $xb + yc = (b, c)$. In either case,

$$\langle g_1\Delta, 0, 0; a, b, c \rangle \sim \langle g_1\Delta, 0, 0; a, (b, c), 0 \rangle ;$$

the similarity is *via* (α, β) where $\overline{\alpha} = I_2$ and

$$\overline{\beta} = \begin{bmatrix} x & \frac{-c}{(b,c)} & 0 \\ y & \frac{b}{(b,c)} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in the latter case.

Step *b*. If $(b, c) \neq 0$, it is possible to choose integers u and v such that

(i) $ua + v(b, c) = g_2\Delta$, and

(ii) $(u, g_2\Delta) = 1$,

since by Dirichlet's Theorem (see, for instance, [12]), the general solution for u is infinitely often prime. With such a choice, set $\overline{\alpha} = I_2$, and

$$\overline{\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & u & \dfrac{-(b,c)}{g_2\Delta} \\ 0 & v & \dfrac{a}{g_2\Delta} \end{bmatrix},$$

so that

$$\langle g_1\Delta,\ 0,\ 0;\ a,\ (b,\ c),\ 0\rangle \sim \left\langle ug_1\Delta,\ \frac{-(b,c)g_1\Delta}{g_2\Delta},\ 0;\ g_2\Delta,\ 0,\ 0\right\rangle$$

via (α, β) . When $(b, c) = 0$ this similarity holds with $u = 1$.

Step *c*. Let w and z be integers for which

$$(ug_1\Delta)w + g_2\Delta z = g\Delta .$$

Then

$$\left\langle ug_1\Delta,\ \frac{-(b,c)g_1\Delta}{g_2\Delta},\ 0;\ g_2\Delta,\ 0,\ 0\right\rangle \sim \left\langle g\Delta,\ \frac{-w(b,c)g_1\Delta}{g_2\Delta},\ 0;\ 0,\ \frac{(b,c)g_1\Delta}{g\Delta},\ 0\right\rangle$$

via (α, β) , where $\overline{\beta} = I_3$ and

$$\overline{\alpha} = \begin{bmatrix} \dfrac{ug_1\Delta}{g\Delta} & -z \\ \dfrac{g_2\Delta}{g\Delta} & w \end{bmatrix}.$$

Step *d*. Since $\dfrac{w(b,c)g_1\Delta}{g_2\Delta}$ is a multiple of $g\Delta$,

$$\left\langle g\Delta,\ \frac{-w(b,c)g_1\Delta}{g_2\Delta},\ 0;\ 0,\ \frac{(b,c)g_1\Delta}{g\Delta},\ 0\right\rangle \sim \left\langle g\Delta,\ 0,\ 0;\ 0,\ \frac{(b,c)g_1\Delta}{g\Delta},\ 0\right\rangle$$

via (α, β) , where $\overline{\alpha} = I_2$ and

$$\overline{\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \dfrac{w(b,c)g_1\Delta}{g_2\Delta g\Delta} \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence $D \sim I(D) = \langle g\Delta,\ 0,\ 0;\ 0,\ p(g\Delta),\ 0\rangle$ as required, where

$p = \dfrac{(b,c)g_1\Delta}{(g\Delta)^2}$. It remains to show that this form is unique.

Step e. Suppose $\Delta \sim \Gamma$ *via* (α, β) in $S(2, 3)$, where

$$D = \langle g, 0, 0; 0, pg, 0 \rangle , \quad G = \langle g, 0, 0; 0, qg, 0 \rangle$$

and $g, p, q > 0$. The following simultaneous equations hold:

$$\alpha_{11} = \{\beta_{11}\beta_{22}-\beta_{21}\beta_{12}\} , \tag{D6.3.1}$$

$$q\alpha_{12} = \{\beta_{11}\beta_{23}-\beta_{21}\beta_{13}\} , \tag{D6.3.2}$$

$$0 = \{\beta_{12}\beta_{23}-\beta_{22}\beta_{13}\} , \tag{D6.3.3}$$

$$\alpha_{21} = p\{\beta_{11}\beta_{32}-\beta_{31}\beta_{12}\} , \tag{D6.3.4}$$

$$q\alpha_{22} = p\{\beta_{11}\beta_{33}-\beta_{31}\beta_{13}\} , \tag{D6.3.5}$$

$$0 = p\{\beta_{12}\beta_{33}-\beta_{32}\beta_{13}\} . \tag{D6.3.6}$$

If $\det \overline{\beta} = \varepsilon_\beta$ and $\det \overline{\alpha} = \varepsilon_\alpha$, (D6.3.3) and (D6.3.6) imply

$$\beta_{11}\{\beta_{22}\beta_{33}-\beta_{32}\beta_{23}\} = \varepsilon_\beta . \tag{D6.3.7}$$

If $\beta_{12} \neq 0$, then $\beta_{23} = (\beta_{22}\beta_{13})/\beta_{12}$ and $\beta_{33} = (\beta_{32}\beta_{13})/\beta_{12}$, in contradiction to (D6.3.7). Hence $\beta_{12} = 0$; consequently, $\beta_{13} = 0$. Since $\beta_{11} = \pm 1$, the remaining equations ensure that $q\varepsilon_\alpha = p\beta_{11}\varepsilon_\beta$ and thus $q = p$.

Case $n = 3$. All of $g_1\Delta$, $g_2\Delta$, and $g_3\Delta$ may be assumed nonzero by (6.2.9), (6.3.2) and Case $n = 2$ above. With $g = (g_1\Delta, g_2\Delta)$, Case $n = 2$ and (6.2.8) together imply the existence of $\Delta \sim \Gamma$ in $S(3, 3)$ with

$$G = \langle g, 0, 0; 0, pg, 0; a, b, c \rangle , \quad (a, b, c) = g_3\Delta ,$$

and $p \geq 0$. The same process applied to the pair (G_1, G_3) determines a similar map with associated set

$$\langle g\Delta, 0, 0; e, f, h; 0, qg\Delta, 0 \rangle ; \quad (e, f, h) = pg .$$

Since e is a multiple of $g\Delta$,

$$D \sim L = \langle g\Delta, 0, 0; 0, f, h; 0, qg\Delta, 0 \rangle$$

via (α, β) , where $\overline{\beta} = I_3$ and

$$\alpha = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-e}{g\Delta} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Repetition of Case $n = 2$ for the pair (L_2, L_3) shows that

$$\langle qg\Delta, 0, 0; f, h, 0 \rangle \sim \langle g^*, 0, 0; 0, p^*g^*, 0 \rangle$$

in $S(2, 3)$ *via* $(\check{\alpha}, \beta)$ where $g^* = (qg\Delta, f, h)$,

$$\bar{\check{\alpha}} = \begin{bmatrix} \frac{uqg\Delta}{g^*} & -z \\ \frac{(f,h)}{g^*} & w \end{bmatrix} , \ \overline{\beta} = \begin{bmatrix} x & 0 & 0 \\ 0 & xu & k \\ 0 & v & l \end{bmatrix} ,$$

and

$$hx = |h| ,$$

$$fu + |h|v = (f, h) ,$$

$$(u, f, h) = 1 ,$$

$$(uqg\Delta)w + (f, h)z = g^* ,$$

$$k = (hz)/g^* ,$$

and

$$l = \left(v|h|q(g\Delta)w+fg^*\right)/\left((f, h)g^*\right) .$$

If α is extended from $\check{\alpha}$ by (6.2.9), then

$$\langle 0, 0, g\Delta; f, h, 0; qg\Delta, 0, 0 \rangle \sim \langle 0, 0, xg\Delta; 0, p^*g^*, 0; g^*, 0, 0 \rangle$$

via (α, β) in $S(3, 3)$, where g^* is a multiple of $g\Delta$ and $p^* \geq 0$.

That is, for a suitable permutation β ,

$$D \sim \langle xg\Delta, 0, 0; 0, 0, qp(g\Delta); 0, p(g\Delta), 0 \rangle ,$$

and for $\beta = I_3$ and

$$\alpha = \begin{bmatrix} x & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} ,$$

$$D \sim I(D) = \langle g\Delta, 0, 0; 0, p(g\Delta), 0; 0, 0, qp(g\Delta) \rangle$$

as required. Since the isomorphisms by which $I(\Delta)$ is similar to Δ need not be unique, it is still necessary to prove that this form is unique.

Suppose $\Delta \sim \Gamma$ in $S(3, 3)$, where

$D = \langle g, 0, 0; 0, ug, 0; 0, 0, vug \rangle$, $G = \langle g, 0, 0; 0, xg, 0; 0, 0, yxg \rangle$,

and g, u, v, x and y are all assumed positive. The following simultaneous equations hold:

$$\alpha_{11} = \{\beta_{11}\beta_{22}-\beta_{21}\beta_{12}\},$$
$$x\alpha_{12} = \{\beta_{11}\beta_{23}-\beta_{21}\beta_{13}\},$$
$$yx\alpha_{13} = \{\beta_{12}\beta_{23}-\beta_{22}\beta_{13}\},$$
$$\alpha_{21} = u\{\beta_{11}\beta_{32}-\beta_{31}\beta_{12}\},$$
$$x\alpha_{22} = u\{\beta_{11}\beta_{33}-\beta_{31}\beta_{13}\},$$
$$yx\alpha_{23} = u\{\beta_{12}\beta_{33}-\beta_{32}\beta_{13}\},$$
$$\alpha_{31} = vu\{\beta_{21}\beta_{32}-\beta_{31}\beta_{22}\},$$
$$x\alpha_{32} = vu\{\beta_{21}\beta_{33}-\beta_{31}\beta_{23}\},$$
$$yx\alpha_{33} = vu\{\beta_{22}\beta_{33}-\beta_{32}\beta_{23}\}. \quad \text{(D6.3.8)}$$

If $\det \overline{\alpha} = \varepsilon_\alpha$ and $\det \overline{\beta} = \varepsilon_\beta$, the evaluation of ε_α through these equations implies $\varepsilon_\alpha x^2 y = \varepsilon_\beta u^2 v$, and hence that $x^2y = u^2v$ and $\varepsilon_\alpha = \varepsilon_\beta$.

With $x = \overline{x}(x, u)$, $u = \overline{u}(x, u)$, $y = s\overline{u}^2$, $v = s\overline{x}^2$, $(s \in Z)$, and $\alpha_{33} = \alpha^*_{33}\overline{x}$, the third, sixth and ninth equations of (D6.3.8) imply

$$\varepsilon_\beta = \overline{u}\left(\alpha_{13}\beta_{31}s\overline{u}x-\alpha_{23}\beta_{21}s\overline{x}+\alpha^*_{33}\beta_{11}\right).$$

That is, $\overline{u} = 1$. The converse argument with $\Gamma \sim \Delta$ ensures that $\overline{x} = 1$. It follows that $x = u$ and $y = v$.

Case $n > 3$. Assume the inductive hypothesis holds for $3 \le k \le n-1$. Let $\Delta \in S(n, 3)$ with $g_i\Delta \ne 0$ for $1 \le i \le n$. There exists $\Delta \sim \Gamma$ in $S(n, 3)$ such that

$G = \langle g\overline{\Delta}, 0, 0; 0, p(g\overline{\Delta}), 0; 0, 0, qp(g\overline{\Delta}); 0, 0, 0; \ldots; 0, 0, 0; a, b, c \rangle$,

where $g\overline{\Delta} = \text{g.c.d.}\left(g_i\Delta : 1 \le i \le n-1\right)$, $(a, b, c) = g_n\Delta$, and $p, q \ge 0$.

Then

$G \sim \langle g\overline{\Delta}, 0, 0; 0, p(g\overline{\Delta}), 0; a, b, c; 0, 0, qp(g\overline{\Delta}); 0, 0, 0; \ldots; 0, 0, 0\rangle$

by a suitable permutation of the generators of A,

$\sim \langle g\Delta, 0, 0; 0, r(g\Delta), 0; 0, 0, sr(g\Delta); e, f, h; 0, 0, 0; \ldots; 0, 0, 0\rangle$

where $(e, f, h) = qp(g\overline{\Delta})$ and $r, s \geq 0$, on application of Case $k = 3$ and (6.2.8) to the first three matrices,

$\sim \langle g\Delta, 0, 0; 0, r(g\Delta), 0; 0, 0, sr(g\Delta); 0, f, h; 0, 0, 0; \ldots; 0, 0, 0\rangle$

since e is a multiple of $g\Delta$,

$\sim \langle g\Delta, 0, 0; 0, u(g\Delta), 0; x, y, z; 0, 0, wu(g\Delta); 0, 0, 0; \ldots; 0, 0, 0\rangle$

where $(x, y, z) = sr(g\Delta)$ and $u, w \geq 0$, on application of Case $k = 3$ to the first, second and fourth matrices,

$\sim \langle g\Delta, 0, 0; 0, u(g\Delta), 0; 0, 0, z; 0, 0, wu(g\Delta); 0, 0, 0; \ldots; 0, 0, 0\rangle$

since $g\Delta | x$ and $ug\Delta | y$,

$\sim \langle g\Delta, 0, 0; 0, u(g\Delta), 0; 0, 0, vu(g\Delta); 0, 0, 0; \ldots; 0, 0, 0\rangle = I(D)$,

on application of (6.2.9) and Case $n = 2$ to the third and fourth matrices, as required.

Again it is necessary to prove an element of this form is unique. If $\langle g, 0, 0; 0, ug, 0; 0, 0, vug; 0, 0, 0; \ldots; 0, 0, 0\rangle \sim$

$\langle g, 0, 0; 0, xg, 0; 0, 0, yxg; 0, 0, 0; \ldots; 0, 0, 0\rangle$

via (α, β) in $S(n, 3)$ for $g, u, v, x, y > 0$, then the simultaneous equations (D6.3.8) hold, together with the equations

$$\alpha_{i1} = 0, \quad 4 \leq i \leq n,$$

$$x\alpha_{i2} = 0, \quad 4 \leq i \leq n,$$

$$yx\alpha_{i3} = 0, \quad 4 \leq i \leq n. \qquad \text{(D6.3.9)}$$

If $\overline{\alpha}^* = (\alpha_{ij} : 1 \leq i, j \leq 3)$ then $\det \overline{\alpha}^* = \pm 1$, because $\alpha_{ij} = 0$ for $1 \leq j \leq 3$ and $4 \leq i \leq n$ by (D6.3.9). As for Case $n = 3$, $\varepsilon x^2 y = \varepsilon_\beta u^2 v$ and the resultant analysis carries over *verbatim*. Hence $I(\Delta)$ is unique. □

By methods akin to those used in (6.3.4) it is possible to determine that each element Δ of $S(2, 4)$ is similar to a map Γ with associated set

$$G = \langle g\Delta, 0, 0, 0, 0, p(g\Delta); 0, a(g\Delta), 0, 0, ba(g\Delta), q(g\Delta) \rangle ,$$

$$p, q, a \geq 0 . \quad (D6.3.10)$$

6.3.5 CONJECTURE. In $S(2, 4)$, each element Δ is similar to a unique element of the form (D6.3.10).

6.4 Application of the Diagonal Comultiplication

The final section of this work examines the way in which those invariants of the diagonal comultiplication so far evaluated, can distinguish between elements of G with the same integral homology. Two examples are investigated in detail and the chapter closes with a tabulation of these and several other examples.

The diagonal comultiplication for a group G apparently measures the torsion subgroup of the abelian group G_2/G_3. If G is an element of G, its diagonal comultiplication may be built up by successive applications of Theorem (6.1.2) from the diagonal approximations of elements in G_1 (*cf.* (6.1.3)). The first such stage is explicitly calculated below.

6.4.1 NOTATION. Let w be a word in the free group $F = \langle x_i, 1 \leq i \leq n : \ \rangle$. For each pair of generators (x_i, x_j) of F, the symbol $\langle w; x_i, x_j \rangle$ denotes the integer $\varepsilon\left[\dfrac{\partial^2 w}{\partial x_i \partial x_j}\right]$, where $\varepsilon : \mathbb{Z}F \to \mathbb{Z}$ is the augmentation map of the group ring $\mathbb{Z}F$. For each generator x_i of F, the symbol $\langle w, x_i \rangle$, previously written as $\sigma_i(w)$, denotes the integer $\varepsilon\left[\dfrac{\partial w}{\partial x_i}\right]$. That is, for $i \neq j$, $\langle w; x_i, x_j \rangle$ is the exponent sum in w of occurrences of x_i preceding each occurrence of x_j^{+1}, minus the exponent sum of occurrences of x_i preceding each occurrence of x_j^{-1}. For example,

$$\left\langle x_1 x_2^{-2} x_1^2 x_2 ;\ x_1,\ x_2 \right\rangle = (-1) + (-1) + 3 = 1 .$$

By induction on the length of w, it may be shown that

$$\langle w;\ x_i,\ x_i \rangle = \tfrac{1}{2}\langle w, x_i\rangle \big(\langle w, x_i\rangle - 1\big)$$

and

$$\langle w;\ x_i,\ x_j \rangle + \langle w;\ x_j,\ x_i \rangle = \langle w, x_i\rangle \langle w, x_j\rangle .$$

Suppose $G = m(\mathcal{D}, \mathrm{A})$ is a connected groupnet in G_2 for which A_v is a free group with basis X_v for each v in D. If C^v is the standard A_v-free resolution of $\mathbb{Z}$ with length 1,

$$C^v = 0 \to \bigoplus_{x \in X_v} \mathbb{Z}A_v[x] \to \mathbb{Z}A_v[v] \to 0 ,$$

and, as before, $C_0^e[\lambda e] = [\rho e]$ and

$$C_1^e[x] = \sum_{y \in X_{\rho e}} \frac{\partial A_e(x)}{\partial y}[y]$$

for all x in $X_{\lambda e}$, then the next result is a corollary of (6.1.2) and the Note following it.

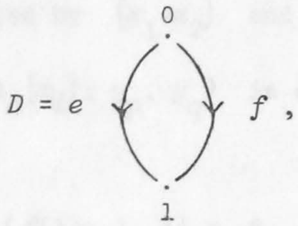
6.4.2 COROLLARY. *If M is the G-mapping cylinder of this complex $(\mathcal{D}, \mathbb{Z}\mathrm{A}, C)$, then the tensored diagonal approximation* $\overline{\Delta} \otimes \omega_2 : TG \otimes M_2 \to [(TG \otimes M) \otimes (TG \otimes M)]_2$ *is*

$$\begin{aligned}\overline{\Delta} \otimes \omega_2[x|e] = {} & [\rho e] \otimes [x|e] + [x|e] \otimes [\lambda e] \\ & + [\lambda e|e] \otimes [x] - \sum_{y \in X_{\rho e}} \langle A_e(x), y\rangle [y] \otimes [\lambda e|e] \\ & - \sum_{y \in X_{\rho e}} \sum_{z \in X_{\rho e}} \langle A_e(x);\ z,\ y\rangle [z] \otimes [y] ,\end{aligned}$$

for all e in D and x in $X_{\lambda e}$. □

Once a basis for $H_2(G;\ TG)$ has been found, the diagonal comultiplication may be calculated from (6.4.2). //

For the remainder of this chapter, all the connected graph product groupnets dealt with will be of the specific form $G = m(\mathcal{D}, A)$, where



A_e and A_f are monomorphisms, and A_0 and A_1 are finitely generated free groups, with $A_0 = \langle x_i,\ 1 \le i \le k :\ \rangle$ and $A_1 = \langle y_j,\ 1 \le j \le l :\ \rangle$. The loop group of G at identity 1 is thus

$$G^* = \left\langle t,\ y_j,\ 1 \le j \le l \ :\ t^{-1}A_e(x_i)t = A_f(x_i),\ 1 \le i \le k \right\rangle .$$

If A_i^* is the commutator quotient group of A_i for $i = 0, 1$, then A_e and A_f induce the abelian group morphisms $A_e^*,\ A_f^* : A_0^* \to A_1^*$ respectively. As the tensored boundary map $\overline{\Delta} \otimes \partial_2 : TG \otimes M_2 \to TG \otimes M_1$ is

$$\overline{\Delta} \otimes \partial_2[x_i|g] = [x_i] - \sum_{j=1}^{l} \langle A_g(x_i),\ y_j\rangle [y_j]$$

for g in $\{e, f\}$, it follows that

$$\sum_{i=1}^{k} p_i\big([x_i|e]-[x_i|f]\big) \in H_2(G;\ TG) \iff \sum_{i=1}^{k} p_i x_i \in \operatorname{Ker}\big(A_e^*-A_f^*\big) .$$

This isomorphism is convenient for use in the choice of basis elements for $H_2(G;\ TG)$. In similar fashion, $H_1(G;\ TG)$ is the quotient of the free abelian group with basis $\{[0|e]-[0|f],\ [y_j],\ 1 \le j \le l\}$ by the group $\operatorname{Im}\big(A_e^*-A_f^*\big)$.

6.4.3 EXAMPLE. Let $k = 2$, $l = 3$, and

$$A_e(x_1) = y_3 \quad ,\ A_e(x_2) = (y_1y_2)^4 y_3^{-1} y_1^2 y_2^2 ,$$

$$A_f(x_1) = y_2^4 y_1^4 ,\ A_f(x_2) = y_2^2 y_1^2 .$$

The loop group of G at 1 is the one-relator group

$$G^* = \left\langle t,\ y_j,\ 1 \le j \le 3 \ :\ t^{-1}y_3t = y_2^4y_1^4,\ t^{-1}\left(y_1y_2\right)^4y_3^{-1}y_1^2y_2^2t = y_2^2y_1^2\right\rangle ,$$

while $H_2(G;\ TG)$ is generated by $\left(x_1+x_2\right)$ and $\operatorname{Im}\left(A_e^*-A_f^*\right)$ is generated by $\left(4\left[y_1\right]+4\left[y_2\right]-\left[y_3\right]\right)$. If $\langle A_e\left(x_i\right);\ y_p,\ y_q\rangle$ is contracted to $\langle e(i);\ p,\ q\rangle$ then

$$\begin{aligned}
&\langle e(1);\ p,\ q\rangle = 0\ ;\ \langle f(1);\ 1,\ 1\rangle = 6\ ,\ \langle f(1);\ 2,\ 1\rangle = 16\ ,\\
&\langle f(1);\ 3,\ 1\rangle = 0\ ,\ \langle f(1);\ 1,\ 2\rangle = 0\ ,\ \langle f(1);\ 2,\ 2\rangle = 6\ ,\\
&\langle f(1);\ 3,\ 2\rangle = 0\ ,\ \langle f(1);\ p,\ 3\rangle = 0\ ;\ \langle e(2);\ 1,\ 1\rangle = 15\ ,\\
&\langle e(2);\ 2,\ 1\rangle = 14\ ,\ \langle e(2);\ 3,\ 1\rangle = -2\ ,\ \langle e(2);\ 1,\ 2\rangle = 22\ ,\\
&\langle e(2);\ 2,\ 2\rangle = 15\ ,\ \langle e(2);\ 3,\ 2\rangle = -2\ ,\ \langle e(2);\ 1,\ 3\rangle = -4\ ,\\
&\langle e(2);\ 2,\ 3\rangle = -4\ ,\ \langle e(2);\ 3,\ 3\rangle = 1\ ;\ \langle f(2);\ 1,\ 1\rangle = 1\ ,\\
&\langle f(2);\ 2,\ 1\rangle = 4\ ,\ \langle f(2);\ 3,\ 1\rangle = 0\ ,\ \langle f(2);\ 1,\ 2\rangle = 0\ ,\\
&\langle f(2);\ 2,\ 2\rangle = 1\ ,\ \langle f(2);\ 3,\ 2\rangle = 0\ ,\ \langle f(2);\ p,\ 3\rangle = 0\ .
\end{aligned}$$

If Ω_2 is restricted to its image in $H_1(G;\ TG) \otimes H_1(G;\ TG)$, then

$$\begin{aligned}
\Omega_2\left(x_1+x_2\right) = {} & 6\{([0|e]-[0|f]) \otimes \left[y_1\right]-\left[y_1\right] \otimes ([0|e]-[0|f])\}\\
& + 6\{([0|e]-[0|f]) \otimes \left[y_2\right]-\left[y_2\right] \otimes ([0|e]-[0|f])\}\\
& - 14\{\left[y_1\right] \otimes \left[y_2\right] - \left[y_2\right] \otimes \left[y_1\right]\} .
\end{aligned}$$

The matrix associated with Ω_2 is thus

$$W = \begin{bmatrix} 0 & 6 & 6 \\ -6 & 0 & -14 \\ -6 & 14 & 0 \end{bmatrix} ,$$

which is similar to

$$I(W) = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

since $(6,\ 14) = 2$. The invariant 2 of Ω_2 measures the torsion of G_2^*/G_3^* : simple computation shows that G_2^*/G_3^* is generated by the cosets of $\{\left[t,\ y_1\right],\ \left[t,\ y_2\right],\ \left[y_1,\ y_2\right]\}$ subject to the relation

$$6\left[t, y_1\right] + 6\left[t, y_2\right] - 14\left[y_1, y_2\right] \equiv 0 .$$

Thus $G_2^*/G_3^* \cong Z \oplus Z \oplus Z_2$.

6.4.4 EXAMPLE. Let $k = 2$, $l = 2$, and

$$A_e(x_1) = y_1^3 \quad , \quad A_e(x_2) = y_2^4 y_1^2 ,$$

$$A_f(x_1) = y_1 y_2 y_1^2 y_2^{-1} \ , \ A_f(x_2) = y_2^2 y_1^2 y_2^2 .$$

The loop group of G at 1 is

$$G^* = \left\langle t, y_1, y_2 : t^{-1} y_1^3 t = y_1 y_2 y_1^2 y_2^{-1}, t^{-1} y_2^4 y_1^2 t = y_2^2 y_1^2 y_2^2 \right\rangle ,$$

while $H_2(G; TG)$ is generated by $\{x_1, x_2\}$ and $\mathrm{Im}(A_e^* - A_f^*)$ is trivial.

Since

$$\langle e(1); 1, 1\rangle = 3 , \ \langle e(1); 2, 1\rangle = 0 , \ \langle e(1); p, 2\rangle = 0 ;$$

$$\langle f(1); 1, 1\rangle = 3 , \ \langle f(1); 2, 1\rangle = 2 , \ \langle f(1); 1, 2\rangle = -2 ,$$

$$\langle f(1); 2, 2\rangle = 0 ; \ \langle e(2); 1, 1\rangle = 1 , \ \langle e(2); 2, 1\rangle = 8 ,$$

$$\langle e(2); 1, 2\rangle = 0 , \ \langle e(2); 2, 2\rangle = 6 ; \ \langle f(2); 1, 1\rangle = 1 ,$$

$$\langle f(2); 2, 1\rangle = 4 , \ \langle f(2); 1, 2\rangle = 4 , \ \langle f(2); 2, 2\rangle = 6 ;$$

if Ω_2 is restricted to its image in $H_1(G; TG) \otimes H_1(G; TG)$ then

$$\Omega_2(x_1) = 3\{([0|e]-[0|f]) \otimes [y_1] - [y_1] \otimes ([0|e]-[0|f])\} - 2\{[y_1] \otimes [y_2] - [y_2] \otimes [y_1]\},$$

and

$$\Omega_2(x_2) = 2\{([0|e]-[0|f]) \otimes [y_1] - [y_1] \otimes ([0|e]-[0|f])\} + 4\{([0|e]-[0|f]) \otimes [y_2] - [y_2] \otimes ([0|e]-[0|f])\} + 4\{[y_1] \otimes [y_2] - [y_2] \otimes [y_1]\} .$$

The matrices associated with Ω_2 are thus

$$W_1 = \begin{bmatrix} 0 & 3 & 0 \\ -3 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix} \text{ and } W_2 = \begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & 4 \\ -4 & -4 & 0 \end{bmatrix} .$$

By (6.3.4),

$$\langle 3, 0, -2; 2, 4, 4\rangle \sim \langle 1, 0, 0; 6, -16, 4\rangle$$
$$\sim \langle 1, 0, 0; 6, 4, 0\rangle$$
$$\sim \langle 1, 0, 0; 0, 4, 0\rangle .$$

Here the invariant 4 measures the torsion of G_2^*/G_3^* : this group is generated by the cosets of $\{[t, y_1], [t, y_2], [y_1, y_2]\}$ subject to the relations

$$3[t, y_1] - 2[y_1, y_2] \equiv 0 ,$$
$$2[t, y_1] + 4[t, y_2] + 4[y_1, y_2] \equiv 0 .$$

Hence $G_2^*/G_3^* \cong \mathbb{Z} \oplus \mathbb{Z}_4$. //

The tables below summarise this information for several other groups, according to the rank of their respective second homology modules. In these tables, the integer m is the rank of the free abelian group $H_1(G; TG)$, and it may be checked that in all the examples given,

$$\mathrm{r}\left(G_2^*/G_3^*\right) = \tfrac{1}{2}m(m-1) - \mathrm{r}H_2(G; TG) .$$

6.4.5 DEFINITION. If B is a finitely generated free abelian group of rank m , the *symmetric difference* $B\nabla B$ of B is the rank $\binom{m}{2}$ free abelian subgroup of $B \otimes B$ generated by

$$\{x \otimes y - y \otimes x : y, x \in B\} .$$

Any basis $\{b_i, 1 \le i \le m\}$ of B determines a basis

$$\{b_i \otimes b_j - b_j \otimes b_i : 1 \le i < j \le m\}$$

of $B\nabla B$. There is thus an abelian group isomorphism between the symmetric difference and a subgroup of the *wedge product*

$$B \wedge B = B \otimes B/\{x \otimes y \sim -y \otimes x : x, y \in B\} ,$$

evaluated as

$$b_i \otimes b_j - b_j \otimes b_i \mapsto b_i \wedge b_j . \qquad //$$

As the diagonal comultiplication is skew-symmetric, it may be considered as a morphism $\Omega_2 : H_2(G; TG) \to H_1(G; TG) \wedge H_1(G; TG)$. Examination of the

G^*	k	l	m	A_e	A_f	Basis $H_2(G;\ TG)$	Basis $\mathrm{Im}(A_e^*-A_f^*)$	W	$I(W)$
$\langle t,y_1 : t^{-1}y_1^2t=y_1^3\rangle$	1	1	1	y_1^2	y_1^3	$\emptyset$	-	-	-
$\langle t,y_1 : t^{-2}y_1t^2=[t,y_1]^{-1}\rangle$	2	2	1	y_1 y_2	y_2 $y_2y_1^{-1}$	$\emptyset$	-	-	-
$\langle t,y_1 : [t^{-1},y_1^p][t,y_1^{-1}]=1\rangle$	2	2	2	$y_1^py_2$ y_1	$y_1^py_2$ y_2	x_1	y_1-y_2	$\langle p+1\rangle$	$\langle \lvert p+1\rvert\rangle$
$\langle t,y_1,y_2,y_3 : t^{-1}y_2^{-2}y_3^{-2}t=y_1^5,$ $t^{-1}y_1^3y_3^2y_2^{-1}t=y_1^{-2}y_2^{-3}\rangle$	2	3	3	$y_2^{-2}y_3^{-2}$ $y_1^3y_3^2y_2^{-1}$	y_1^5 $y_1^{-2}y_2^{-3}$	x_1+x_2	$5y_1+2y_2$ $+2y_3$	$\langle 3,-3,0\rangle$	$\langle 3,0,0\rangle$
$\langle t,y_1,y_2,y_3 :$ $t^{-1}y_1y_2^2y_1^{-1}t=y_3^{-2}y_2y_1,$ $t^{-1}y_1^2y_2^4y_1^{-2}t=y_3^{-2}y_2y_1^2y_3^{-2}y_2\rangle$	2	3	3	$y_1y_2^2y_1^{-1}$ $y_1^2y_2^4y_1^{-2}$	$y_3^{-2}y_2y_1$ $y_3^{-2}y_2y_1^2y_3^{-2}y_2$	$-2x_1+x_2$	$-y_1+y_2$ $+2y_3$	$\langle 0,0,4\rangle$	$\langle 4,0,0\rangle$
$\langle t,y_1,y_2 :$ $t^{-1}(y_1y_2)^4ty_1^{-4}y_2^{-4}t^{-1}y_1^2y_2^2t=y_2^2y_1^2\rangle$	2	3	3	y_3 $(y_1y_2)^4y_3^{-1}y_1^2y_2^2$	$y_2^4y_1^4$ $y_2^2y_1^2$	x_1+x_2	$4y_1+4y_2$ $-y_3$	$\langle 6,6,-14\rangle$	$\langle 2,0,0\rangle$
$\langle t,y_1,y_2 : [t^2,y_2^py_1^p]=1\rangle$	2	3	3	y_3 $y_2^py_1^p$	$y_1^{-p}y_2^{-p}$ y_3^{-1}	x_1-x_2	py_1+py_2 $+y_3$	$\langle -2p,-2p,0\rangle$	$\langle \lvert 2p\rvert,0,0\rangle$
$\langle t,y_1,y_2,y_3 : t^{-1}y_2y_3^{-2}t=y_1^py_3,$ $t^{-1}y_1^{-1}y_3y_2^{-1}y_1y_3t=y_1^{1-p}y_3^{-1}y_1^{-1}\rangle$	2	3	3	$y_2y_3^{-2}$ $y_1^{-1}y_3y_2^{-1}y_1y_3$	$y_1^py_3$ $y_1^{1-p}y_3^{-1}y_1^{-1}$	x_1+x_2	py_1-y_2 $+py_3$	$\langle 0,0,2p-3\rangle$	$\langle \lvert 2p-3\rvert,0,0\rangle$
$\langle t,y_j,\ 2\le j\le 2p :$ $[t,y_2]\prod_{q=2}^{p}[y_{2q-1},y_{2q}]=1\rangle$	1	$2p-1$	$2p$	y_2	$y_2\prod_{q=2}^{p}[y_{2q-1},y_{2q}]$	x_1	$\emptyset$	$\dotplus_{q=1}^{p}\begin{bmatrix}0&1\\-1&0\end{bmatrix}$	$\dotplus_{q=1}^{p}\begin{bmatrix}0&1\\-1&0\end{bmatrix}$

TABLE 6.2. Rank $H_2(G;\ TG) = n \le 1$

G^*	k	l	m	Basis $H_2(G;TG)$	Basis $\mathrm{Im}(A_e^*-A_f^*)$	W	$I(W)$
$\langle t,y_1,y_2 : t^{-1}y_1^3t=y_1y_2y_1^2y_2^{-1},\ t^{-1}y_2^4y_1^2t=y_2^2y_1^2y_2^2\rangle$	2	2	3	x_1,x_2	$\emptyset$	$\langle 3,0,-2;2,4,4\rangle$	$\langle 1,0,0;0,4,0\rangle$
$\langle t,y_1,y_2 : t^{-1}y_1^4t=y_1^2y_2y_1^2y_2^{-1},\ t^{-1}y_2^4y_1^2t=y_2^2y_1^2y_2^2\rangle$	2	2	3	x_1,x_2	$\emptyset$	$\langle 4,0,-2;2,4,4\rangle$	$\langle 2,0,0;0,2,0\rangle$
$\langle t,y_1,y_2,y_3 : t^{-1}y_1^{-2}y_3y_1^2y_2^{-1}t=y_2^{-1}y_3,$ $t^{-1}y_2^5t=y_3^{-3}y_1,\ t^{-1}y_1y_2^2t=y_1^2y_2^{-1}y_1^{-1}y_3^{-2}y_2^{-2}y_3^{-1}\rangle$	3	3	3	$x_1,$ x_2-x_3	$-y_1+5y_2$ $+3y_3$	$\langle -1,1,10;3,0,1\rangle$	$\langle 1,0,0;0,1,0\rangle$
$\langle t,y_1,y_2,y_3 : t^{-1}y_1t=y_1,\ t^{-1}y_2t=y_3^{-2}y_1^2,$ $t^{-1}y_1^{-1}y_3^2y_1t=y_1y_2^{-1}y_1,\ t^{-1}y_2^2y_3^4t=y_3^2y_2^2y_3^2\rangle$	4	3	3	$x_1,x_4,$ x_2-x_3	$2y_1-y_2$ $-2y_3$	$\langle 1,0,0;2,-4,0;4,0,-8\rangle$	$\langle 1,0,0;0,4,0;0,0,8\rangle$
$\langle t,y_1,y_2,y_3 : t^{-1}y_1^2t=y_1^2,\ t^{-1}y_2t=y_3^{-2}y_1^2,$ $t^{-1}y_1^{-1}y_3^2y_1t=y_1y_2^{-1}y_1,\ t^{-1}y_2^2y_3^4t=y_3^2y_2^2y_3^2\rangle$	4	3	3	$x_1,x_4,$ x_2-x_3	$2y_1-y_2$ $-2y_3$	$\langle 2,0,0;2,-4,0;4,0,-8\rangle$	$\langle 2,0,0;0,4,0;0,0,8\rangle$
$\langle t,y_1,y_2,y_3 : t^{-1}y_1t=y_1,\ t^{-1}y_2t=y_3^{-2}y_1^2,$ $t^{-1}y_1^{-1}y_3^2y_1t=y_1y_2^{-1}y_1,\ t^{-1}y_2^2y_3^6t=y_3^3y_2^2y_3^3\rangle$	4	3	3	$x_1,x_4,$ x_2-x_3	$2y_1-y_2$ $-2y_3$	$\langle 1,0,0;2,-4,0;4,2,-12\rangle$	$\langle 1,0,0;0,2,0;0,0,24\rangle$
$\langle t,y_1,y_2,y_3 : t^{-1}y_1^2t=y_1^2,\ t^{-1}y_2t=y_3^{-2}y_1^2,$ $t^{-1}y_1^{-1}y_3^2y_1t=y_1y_2^{-1}y_1,\ t^{-1}y_2^2y_3^6t=y_3^3y_2^2y_3^3\rangle$	4	3	3	$x_1,x_4,$ x_2-x_3	$2y_1-y_2$ $-2y_3$	$\langle 2,0,0;2,-4,0;4,2,-12\rangle$	$\langle 2,0,0;0,2,0;0,0,24\rangle$

TABLE 6.3. $2 \le \text{Rank } H_2(G;\ TG) = n \le 3$

examples above, and comparison with the dual case (see Sullivan [37]), strongly suggest that the sequence

$$H_2(G;\, TG) \xrightarrow{\Omega} \wedge_2 H_1(G;\, TG) \xrightarrow{[\ ,\ ]} G_2^*/G_3^* \to 0$$

is exact, modulo torsion in $H_1(G;\, TG)$. Here the skew-symmetric bilinear map $[\ ,\ ]$ is defined by $[\ ,\ ](x \wedge y) = [x^{-1}, y^{-1}]$. However, the diagonal comultiplication provides more information than is implied by the isomorphism $\operatorname{Coker} \Omega \cong G_2^*/G_3^*$, as is illustrated below. If $m \le 3$, every element of $H_1(G;\, TG) \nabla H_1(G;\, TG)$ may be written as a simple skew product (that is, in the form $x \nabla y$ for some pair (x, y) in $H_1(G;\, TG)$), but this is not so for higher m. For any linearly independent set of elements $\{x_i : 1 \le i \le 4\}$ of $H_1(G;\, TG)$, for instance, the element $x_1 \nabla x_2 + x_3 \nabla x_4$ cannot be written in this form. Let G^* be the final group of Table 6.2, with $p = 2$, so that G_2^*/G_3^* is free abelian of rank $5 = \frac{1}{2}(4.3) - 1$, and $\Omega_2(x_1) = t \nabla y_2 + y_3 \nabla y_4$. Hence rank $W = 4$. Let $H = m(\mathcal{D}, \mathcal{B})$ be the mapping cylinder groupnet for $B_0 = \langle x_1 : \ \rangle$, $B_1 = \langle y_2, y_3, y_4 : \ \rangle$, $B_e(x_1) = y_2$ and $B_f(x_1) = y_2[y_2, y_3][y_2, y_4]$. That is, $H^* = \langle t, y_2, y_3, y_4 : [t, y_2][y_2, y_3][y_2, y_4] = 1 \rangle$, $H_2(H;\, TH)$ is generated by x_1, $rH_1(H;\, TH) = 4$ and $H_2^*/H_3^* \cong \mathbb{Z}^5$. The diagonal comultiplication Δ for H is given by

$$\Delta_2(x_1) = t \nabla y_2 + y_2 \nabla y_3 + y_2 \nabla y_4 ,$$

with associated matrix $D = \langle 1, 0, 0, 1, 1, 0 \rangle$ and $I(D) = \langle 1, 0, 0, 0, 0, 0 \rangle$. Hence rank $D = 2$ and Δ is not similar to Ω. The similarity invariants of Ω thus reflect more of the structure of G^* than is given by G_2^*/G_3^*.

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INDEX OF NOTATION

The following notational conventions are also employed: $C \hookrightarrow C'$ denotes a subset inclusion, $C \xrightarrow{\sim} C'$ denotes an isomorphism, 'p.b.' denotes a pullback square, $\square$ denotes the end of a proof, and // denotes the end of the discussion of a topic.

INDEX OF DEFINITIONS