SLOW VISCOUS FLOW THROUGH REGULAR ARRAYS OF CYLINDERS AND SPHERES

by

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Unless otherwise stated the work is that of the author.

(M.I.TAHIR)
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SUMMARY

Slow viscous flow through regular arrays of cylinders and spheres is studied theoretically and the drag force exerted by the fluid on a cylinder or a sphere forming the regular array is calculated.

The method used essentially consists of comparing a solution of the Stokes equations outside a cylinder or a sphere with a sum of solutions having equal singularities inside every cylinder or sphere of the regular array.

For cylinders, results are given for square and triangular arrays for flow parallel and perpendicular to the axes of cylinders, while for spheres results are given only for a simple cubic array. Results agree well with the corresponding values reported by previous researchers, notably Hasimoto (1959).

1. SLOW VISCOUS FLOW PARALLEL TO REGULAR ARRAYS OF CYLINDERS

1.1 Introduction

The study of the flow of a viscous fluid past a regular array of circular cylinders is important for understanding the operation of many heat and mass transfer equipments.

In this chapter we shall consider square and triangular arrays of cylinders, and calculate the drag force exerted by the fluid on a cylinder forming the array by matching a solution of the Stokes equations outside the cylinder with a sum of solutions having equal singularities inside every cylinder of the array.

This problem was first solved by Emersleben (1925) for a square array. His solution which was based on complex zeta functions appears to be valid only at low values of the volume fraction of the cylinders.

Happel (1959) employed his free-surface model, which he had previously used for the fluid flow relative to arrays of spheres (1958), to the case of flow relative to arrays of cylinders. He developed a mathematical treatment on the basis that two concentric cylinders can serve as the model for fluid flow through an array of cylinders. The inner cylinder consists of one of the cylinders in the array and the outer cylinder of a fluid envelope with zero drag on the surface. The relative volume of fluid to solid in the cell model is taken to be the same as the relative volume of fluid to solid in the array of cylinders. He derived a formula for the Kozeny constant, which is equivalent to a drag force, F, per unit length of the cylinder, given by

$$F = \frac{4\pi\mu U}{\ln(1/\varepsilon)-1.5+2\varepsilon-\frac{1}{2}\varepsilon^2}$$

where ε is the volume fraction of the cylinders, and U is the average speed of the fluid. This formula is moderately accurate for low values of ε.

Sparrow and Loeffler Jr. (1959) applied an analytical method based on truncated trigonometric series to square and equilateral triangular arrays. Their solution was exact on three boundaries of a typical element of a cell of the array and was collocated or fitted at a set of points on the fourth boundary of the element. Their results are graphed and agree well with the formulae of Emersleben and Happel and extend them to higher values of the volume fraction.

We shall use a multipole technique to solve the problem despite Happel and Brenner's (1973, p. 386) claim that the method of reflections cannot be directly applied to problems involving arrays of cylinders because no solution of the creeping motion equations exists for a single cylinder in an unbounded medium. Convergence difficulties concerning the r^2 and $\ell n\ r$ terms can be overcome by using a modification of O'Brien's (1979) method. This will be further discussed in §1.3.3.

1.2 Equations of motion of slow viscous flow

The Navier-Stokes equations for the motion of a viscous incompressible fluid, in the usual notation, are given by (Batchelor (1967), p. 147)

$$\rho \frac{D\underline{V}}{Dt} = \rho\underline{F} - \nabla p + \mu\nabla^2\underline{V} \qquad (1)$$

or

$$\rho \left(\frac{\partial \underline{V}}{\partial t} + \underline{V}.\nabla\underline{V} \right) = \rho\underline{F} - \nabla p + \mu\nabla^2\underline{V}$$

where $\rho\underline{V}.\nabla\underline{V}$ are the inertial terms and $\mu\nabla^2\underline{V}$ the viscous terms.

In general, the body force $\underline{F}$ can be expressed as

$$\underline{F} = -\nabla\Omega$$

and combined with the pressure term, so that

$$\rho \left(\frac{\partial \underline{V}}{\partial t} + \underline{V}.\nabla\underline{V} \right) = -\nabla(p+\rho\Omega) + \mu\nabla^2\underline{V} \qquad (2)$$

If we write p in place of $p+\rho\Omega$, equation (2) becomes

$$\rho\left\{\frac{\partial \underline{V}}{\partial t} + \underline{V}.\nabla\underline{V}\right\} = -\nabla p + \mu\nabla^2\underline{V} \tag{3}$$

The pressure p appearing in equation (3) is termed the dynamic or hydrodynamic pressure and vanishes when the fluid is at rest (Happel and Brenner (1973), p.28).

For steady inertia-free flow, equation (3) reduces to

$$\nabla^2\underline{V} = \frac{1}{\mu}\nabla p \tag{4}$$

which, together with the continuity equation,

$$\nabla.\underline{V} = 0 \tag{5}$$

constitutes the creeping motion or Stokes equations for this situation.

Taking the divergence of equation (4) and using equation (5), we find that

$$\nabla^2 p = 0 \tag{6}$$

i.e. p is harmonic.

If we take the curl of equation (4), we get

$$\nabla^2\underline{\omega} = 0 \tag{7}$$

where $\underline{\omega} = \frac{1}{2}\nabla\times\underline{V}$ is the vorticity vector.

For a two-dimensional flow, we can write from equation (5)

$$\underline{V} = \nabla\times\underline{A}$$

where

$$\underline{A} = (0,0,\psi)$$

and ψ is called the stream function. Then

$$\underline{\omega} = (0,0,-\tfrac{1}{2}\nabla^2\psi)$$

and equation (7) becomes

$$\nabla^4\psi = 0 \tag{8}$$

i.e. ψ is biharmonic.

1.3 Flow parallel to cylinders in a square array

1.3.1 Description

Let solid circular cylinders of radius a be arranged in a square array, parallel to the z-axis, with centres at points having coordinates (pb,qb) where p and q are integers, and b is the distance between the centres of adjacent cylinders. Let the fluid have velocity w parallel to the z-axis. The fluid is then driven by a constant pressure gradient in the z-direction.

The equations of motion (equations (4) and (5) of §1.2) reduce to

$$\nabla^2 w = \frac{1}{\mu}\frac{\partial p}{\partial z}, \quad \frac{\partial w}{\partial z} = 0$$

or

$$\nabla^2 w = -\frac{P}{\mu}, \quad \frac{\partial w}{\partial z} = 0 \tag{1}$$

where $\frac{\partial p}{\partial z} = -P =$ constant, and $w = 0$ on the surface of each cylinder.

In terms of the cylindrical polar coordinates (r,θ,z), the first of equations (1) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = -\frac{P}{\mu}$$

or

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = -\frac{P}{\mu} \tag{2}$$

1.3.2 The first solution

Let

$$w^* = w + \frac{r^2}{4}(P/\mu)$$

Substituting in equation (2) we see that

$$\frac{\partial^2 w^*}{\partial r^2} + \frac{1}{r}\frac{\partial w^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w^*}{\partial \theta^2} = 0 \tag{3}$$

i.e. w^* obeys Laplace's equation.

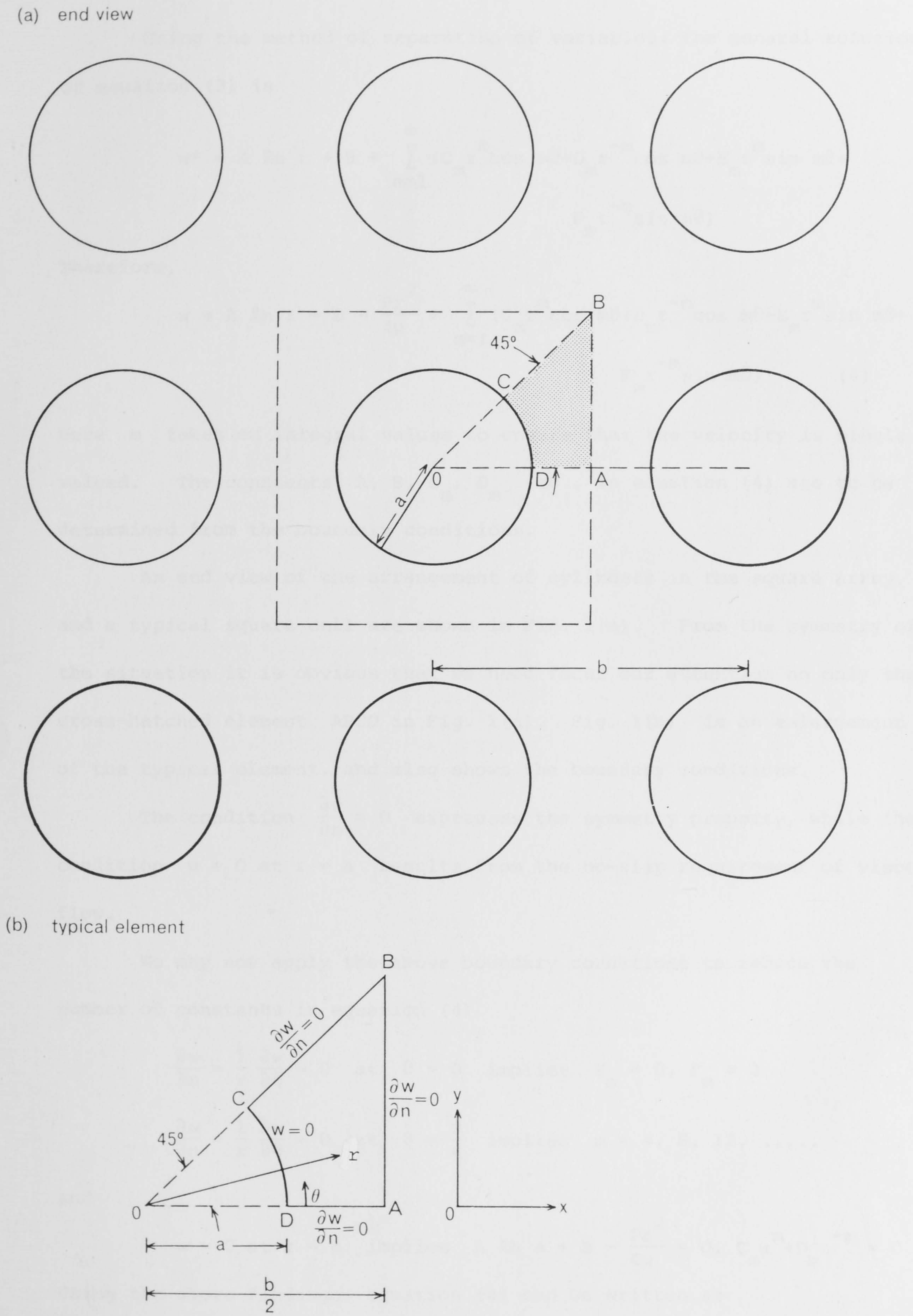

Fig.1 Parallel flow: Cylinders in a square array

Using the method of separation of variables, the general solution of equation (3) is

$$w^* = A\,\ell n\, r + B + \sum_{m=1}^{\infty} (C_m r^m \cos m\theta + D_m r^{-m} \cos m\theta + E_m r^m \sin m\theta + F_m r^{-m} \sin m\theta)$$

Therefore,

$$w = A\,\ell n\, r + B - \frac{Pr^2}{4\mu} + \sum_{m=1}^{\infty} (C_m r^m \cos m\theta + D_m r^{-m} \cos m\theta + E_m r^m \sin m\theta + F_m r^{-m} \sin m\theta) \qquad (4)$$

Here m takes on integral values to ensure that the velocity is single-valued. The constants A, B, C_m, D_m etc., in equation (4) are to be determined from the boundary conditions.

An end view of the arrangement of cylinders in the square array, and a typical square cell are shown in Fig. 1(a). From the symmetry of the situation it is obvious that we need focus our attention on only the cross-hatched element ABCD in Fig. 1(a). Fig. 1(b) is an enlargement of the typical element, and also shows the boundary conditions.

The condition $\frac{\partial w}{\partial n} = 0$ expresses the symmetry property, while the condition $w = 0$ at $r = a$ results from the no-slip requirement of viscous flow.

We may now apply the above boundary conditions to reduce the number of constants in equation (4)

$$\frac{\partial w}{\partial n} = \frac{1}{r}\frac{\partial w}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0 \quad \text{implies} \quad E_m = 0,\ F_m = 0$$

$$\frac{\partial w}{\partial n} = \frac{1}{r}\frac{\partial w}{\partial \theta} = 0 \quad \text{at} \quad \theta = \frac{\pi}{4} \quad \text{implies} \quad m = 4, 8, 12, \ldots..$$

and

$$w = 0 \text{ at } r = a \quad \text{implies} \quad A\,\ell n\, a + B - \frac{Pa^2}{4\mu} = 0,\ C_m a^m + D_m a^{-m} = 0\ .$$

Using the above findings, equation (4) can be written as

$$w = A\,\ell n \frac{r}{a} - \frac{P}{4\mu}(r^2 - a^2) + \sum_{n=1}^{\infty} A_{4n} [(r/a)^{4n} - (a/r)^{4n}] \cos 4n\theta \qquad (5)$$

This solution is unaltered if we change θ to $-\theta$ or increase θ by $\frac{\pi}{2}$.

1.3.3 The second solution

Using a modification of O'Brien's (1979) method, Drummond (1982) has expressed the second solution for w as

$$w = \sum_{p,q}\sum\{A \,\ell n \frac{r_{pq}}{a} - \sum_{n=1}^{\infty} A_{4n} \left(\frac{a}{r_{pq}}\right)^{4n} \cos 4n\theta_{pq}\} - \frac{Pr^2}{4\mu} + I \qquad (6)$$

where (r_{pq}, θ_{pq}) are the polar coordinates of a field point $P(r,\theta)$ referred to the centre of another cylinder, and I is an integral over the surface at infinity which cancels the divergent terms in the solution. Each of the cylinders in the array contributes equally to the singularities in equation (6).

1.3.4 Matching the solutions

Since equations (5) and (6) represent the same function in the domain (allθ, $a \leqslant r < b$), we can write

$$\frac{Pa^2}{4\mu} + \sum_{n=1}^{\infty} A_{4n}(r/a)^{4n} \cos 4n\theta$$

$$= \sum_{p,q\neq 0,0}\sum \{A \,\ell n \frac{r_{pq}}{a} - \sum_{m=1}^{\infty} A_{4m}\left(\frac{a}{r_{pq}}\right)^{4m} \cos 4m\theta_{pq}\} + I \qquad (7)$$

We can match the solutions by using complex variables as follows.

Let $z = re^{i\theta}$, $z_{pq} = r_{pq}e^{i\theta_{pq}}$ and $d_{pq} = (p+iq)b$ where d_{pq} are the complex coordinates of the centres of the cylinders. Then $z_{pq} = z-d_{pq}$ and equation (7) may be written as

$$\frac{Pa^2}{4\mu} + R \sum_{n=1}^{\infty} A_{4n}(z/a)^{4n}$$

$$= R[\sum_{p,q\neq 0,0}\sum \{A \,\ell n \left(1-\frac{z}{d_{pq}}\right) - \sum_{m=1}^{\infty} A_{4m}\left(\frac{a}{z-d_{pq}}\right)^{4m}\} + J] \qquad (8)$$

where $J = I + \sum\sum_{p,q\neq 0,0} A\,\ell n\left(-\frac{d_{pq}}{a}\right)$, and R = real part of.

The right hand side of equation (8) may be expanded as a power series in z and matched term by term to the left hand side. To do this we define a set of constants

$$P_n = \sum\sum_{p,q\neq 0,0} \left(\frac{b}{d_{pq}}\right)^n = \sum\sum_{p,q\neq 0,0} (p+iq)^{-n}$$

For a square grid symmetric in p and q, $P_n = 0$ unless n is a multiple of 4.

Equation (8) can then be written as

$$\frac{Pa^2}{4\mu} + R\sum_{n=1}^{\infty} A_{4n}(z/a)^{4n}$$

$$= R\left[-\sum_{n=1}^{\infty}\left\{\frac{A}{4n}P_{4n}(a/b)^{4n} + \sum_{m=1}^{\infty} A_{4m}\frac{(4n+4m-1)!}{(4m-1)!(4n)!}P_{4m+4n}(a/b)^{4n+4m}\right\}(z/a)^{4n} + J\right]$$

Hence the coefficient of z^{4n} gives

$$A_{4n} = -\frac{A}{4n}P_{4n}(a/b)^{4n} - \sum_{m=1}^{\infty} A_{4m}\frac{(4n+4m-1)!}{(4n)!(4m-1)!}P_{4n+4m}(a/b)^{4n+4m}$$

$$n = 1,2,3,\ \ldots \qquad (9)$$

Equation (9) can be solved step by step to give

$$A_4 = A\left[-\frac{P_4}{4}(a/b)^4 + \frac{7!}{4!4!}P_4P_8(a/b)^{12} + 0(a/b)^{20}\right]$$

$$A_8 = A\left[-\frac{P_8}{8}(a/b)^8 + \frac{11!}{4!8!}P_4P_{12}(a/b)^{16} + 0(a/b)^{24}\right]$$

$$A_{12} = A\left[-\frac{P_{12}}{12}(a/b)^{12} + \frac{15!}{4!12!}P_4P_{16}(a/b)^{20} + 0(a/b)^{28}\right] \qquad (10)$$

and so on.

1.3.5 Determination of P_{4n}

$$P_{4n} = \sum_{p,q\neq 0,0}\sum (p+iq)^{-4n} = \sum_{p,q\neq 0,0}\sum \frac{\cos 4n(\arctan \frac{q}{p})}{(p^2+q^2)^{2n}}$$

For large n these series converge rapidly, and, therefore, P_{12}, P_{16}, P_{20} etc. can be evaluated directly by summing over an octant of the grid and multiplying the edge terms by 4 and the interior terms by 8, and taking sufficient terms to get any desired order of accuracy.

To determine P_4 and P_8 we proceed as follows.

$$P_4 = \sum_{p,q\neq 0,0}\sum \frac{1}{(p+iq)^4}$$

Now $$\sum_{n=-\infty}^{\infty} \frac{1}{(n+iq)^4} = -(\text{residue of } \frac{\pi \cot \pi z}{(n+iq)^4} \text{ at } n = -iq)$$

(Spiegel (1964),p.175)

$$= \frac{\pi^4}{3} \frac{3+2\sinh^2 \pi q}{\sinh^4 \pi q}$$

Summing over q, we get:

$$\sum_{p,q\neq 0,0}\sum \frac{1}{(p+iq)^4} = 2\sum_{p=1}^{\infty} \frac{1}{p^4} + \frac{2\pi^4}{3}\sum_{q=1}^{\infty} \frac{3+2\sinh^2\pi q}{\sinh^4 \pi q}$$

i.e. $$P_4 = 2 \times \frac{\pi^4}{90} + \frac{2\pi^4}{3}\sum_{q=1}^{\infty}\frac{3+2\sinh^2\pi q}{\sinh^4\pi q}$$

(Spiegel (1968),p.108)

or $$P_4 = 3.151211992$$

Similarly

$$P_8 = 2\sum_{p=1}^{\infty}\frac{1}{p^8} + \frac{2\pi^8}{315}\sum_{q=1}^{\infty}\frac{315+420\sinh^2\pi q+126\sinh^4\pi q+4\sinh^6\pi q}{\sinh^8\pi q}$$

or

$$P_8 = 4.255772971$$

The values of P_{4n} are given in Table 1.

1.3.6 Determination of A

For steady flow of fluid through the element ABCD of Fig. 1, the pressure and shear forces must balance. The thrust due to pressure on unit length of the element is $\frac{P(b^2-\pi a^2)}{8}$. The only non-zero shear force is given by

$$\int_0^{\pi/4} \mu \left(\frac{\partial w}{\partial r} \right)_{r=a} a \, d\theta = \left(\mu A - \frac{Pa^2}{2}\right) \frac{\pi}{4}$$

Equating these two forces, we get

$$A = \frac{Pb^2}{2\mu\pi} \qquad (11)$$

Equations (10) and (11) give the unknown constants in equation (5).

Since, by symmetry, the velocity is a maximum at B in Fig. 1 and has a saddle point at A, we can also determine the constant A by putting $\frac{\partial w}{\partial r} = 0$ at either $A(r = \frac{b}{2}, \theta = 0)$ or $B(r = \frac{b}{\sqrt{2}}, \theta = \frac{\pi}{4})$.

1.3.7 The total flux and the drag force

The total flux Q through a square cell of area b^2 around a cylinder is given by

$$Q = 8 \int_0^{\pi/4} d\theta \int_a^{\frac{b}{2}\sec\theta} w \, r \, dr$$

$$= 8\left[\int_0^{\pi/4} d\theta \int_a^{\frac{b}{2}\sec\theta} r\left(A \, \ell n \frac{r}{a}\right) dr - \frac{P}{4\mu} \int_0^{\pi/4} d\theta \int_a^{\frac{b}{2}\sec\theta} (r^2-a^2) r \, dr \right.$$

$$+ \sum_{n=1}^{\infty} A_{4n} \int_0^{\pi/4} d\theta \int_a^{\frac{b}{2}\sec} r(r/a)^{4n} \cos 4n\theta \, dr$$

$$\left. - \sum_{n=1}^{\infty} A_{4n} \int_0^{\pi/4} d\theta \int_a^{\frac{b}{2}\sec} r(a/r)^{4n} \cos 4n\theta \, dr \right]$$

TABLE 1

Values of P_{4n}

n	P_{4n}	n	P_{4n}
1	3.151211992	10	4.000003814
2	4.255772971	11	3.999999046
3	3.938849013	12	4.000000238
4	4.015695033	13	3.999999940
5	3.996096753	14	4.000000014
6	4.000976805	15	3.999999996
7	3.999755875	16	4
8	4.000061036	17	4
9	3.999984741		

TABLE 2

Values of P_{6n}

n	P_{6n}	n	P_{6n}
1	5.863031696	6	6.000000015
2	6.009639971	7	5.999999999
3	5.999718356	8	6
4	6.000011647	9	6
5	5.999999587	10	6

The last two integrals can be evaluated by using the formula

$$\int \cos(n+1)x \cos^{n-1}x\, dx = \frac{1}{n}\cos^n x \sin nx$$

Then Q simplifies to

$$Q = A\left[b^2 \ln\frac{b}{a\sqrt{2}} + b^2\left(\frac{\pi}{4} - \frac{3}{2}\right) + \frac{\pi a^2}{2}\right] + \frac{P}{4\mu}\left(-\frac{b^4}{6} + a^2b^2 - \frac{\pi a^4}{2}\right)$$

$$+ \sum_{n=1}^{\infty} A_{4n}\left[\frac{(-1)^n b^{4n+2}}{2^{2n}(2n+1)(4n+1)a^{4n}} - \frac{(-1)^n 2^{2n} a^{4n}}{(2n-1)(4n-1)b^{4n-2}}\right]$$

Substituting for A and A_{4n} from equations (11) and (10) respectively, and rearranging, we get

$$Q = \frac{Pb^4}{2\pi\mu}\Bigg[\left(\ln\frac{b}{a\sqrt{2}} + \frac{\pi}{6} - \frac{3}{2} + \frac{\pi a^2}{b^2} - \frac{\pi^2 a^4}{4b^4}\right)$$

$$+ 2\left(\frac{P_4}{4\times5\times6\times4} - \frac{P_8}{8\times9\times10\times4^2} + \frac{P_{12}}{12\times13\times14\times4^3} - \ldots\right)$$

$$+ (a/b)^8 \frac{P_4}{3}\left(-1 - \frac{7P_8}{4^2} + \frac{11P_{12}}{4^3} - \frac{15P_{16}}{4^4} + \ldots\right) + 0(a/b)^{16}\Bigg]$$

or, after simplification,

$$Q = \frac{Pb^4}{2\pi\mu}\left[\ln(b/a) - 1.310532926 + \pi(a/b)^2 - \frac{\pi^2}{4}(a/b)^4 - \frac{P_4^2}{4}(a/b)^8 + 0(a/b)^{16}\right] \qquad (12)$$

If $U = \dfrac{\text{flux}}{\text{area of the cell}} = \dfrac{Q}{b^2}$ denotes the superficial velocity parallel to the cylinders, and F is the drag force per unit length on one cylinder given by $F = Pb^2$, then equation (12) can be used to obtain

$$F = \frac{2\pi\mu U}{\ln(b/a) - 1.310532926 + \pi(a/b)^2 - \frac{\pi^2}{4}(a/b)^4 - \frac{P_4^2}{4}(a/b)^8 + 0(a/b)^{16}} \qquad (13)$$

In terms of the volume concentration ε of cylinders given by $\varepsilon = \frac{\pi a^2}{b^2}$, equation (13) becomes

$$F = \frac{4\pi\mu U}{\ln(1/\varepsilon) - 1.476335966 + 2\varepsilon - \frac{1}{2}\varepsilon^2 - \frac{P_4^2}{2\pi^4}\varepsilon^4 + O(\varepsilon^8)} \qquad (14)$$

1.4 Flow parallel to cylinders in an equilateral triangular array

In an equilateral triangular array the solid circular cylinders are arranged parallel to the z-axis, with centres at points $[(p+\frac{q}{2})b, \frac{\sqrt{3}}{2}qb]$, and the fluid is moving with velocity w in the z-direction, driven by a constant pressure gradient -P.

An end view of the arrangement of cylinders in the triangular array, and a typical hexagonal cell are shown in Fig. 2. From the symmetry of the situation, it is obvious that we need focus our attention on only the cross-hatched element in Fig. 2(a). Fig. 2(b) is an enlargement of the typical element, and also shows the boundary conditions. The flow is symmetric about OA, OB and AB.

The equation of motion is the same as before (i.e. equation (2) of §1.3) and its solution satisfying the boundary conditions of Fig. 2 is given by

$$w = A \ln\frac{r}{a} - \frac{P}{4\mu}(r^2 - a^2) + \sum_{n=1}^{\infty} A_{6n}[(r/a)^{6n} - (a/r)^{6n}]\cos 6n\theta \qquad (1)$$

The second solution may be expressed as

$$w = \sum_{p,q}\sum \{ A \ln\frac{r_{pq}}{a} - \sum_{n=1}^{\infty} A_{6n}(a/r_{pq})^{6n}\cos 6n\,\theta_{pq}\} - \frac{Pr^2}{4\mu} + I$$

where the symbols have the same meaning as in §1.3.3.

For matching the two solutions we use the complex variables and define

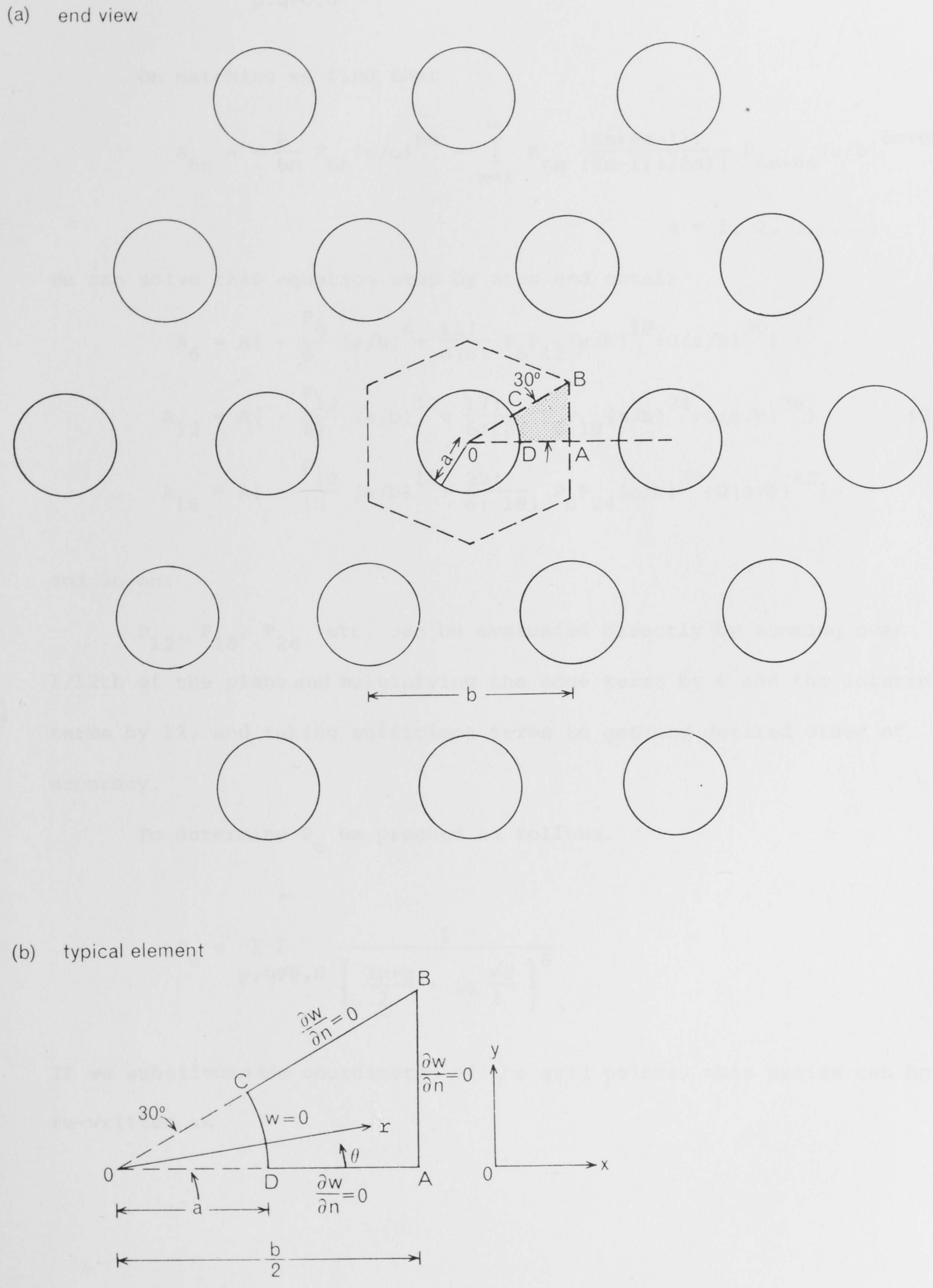

Fig.2 Parallel flow: Cylinders in a triangular array

$$P_{6n} = \sum_{p,q\neq 0,0}\sum \left(p + \frac{q}{2} + iq\frac{\sqrt{3}}{2}\right)^{-6n}$$

On matching we find that

$$A_{6n} = -\frac{A}{6n} P_{6n}(a/b)^{6n} - \sum_{m=1}^{\infty} A_{6m} \frac{(6n+6m-1)!}{(6m-1)!(6n)!} P_{6m+6n}(a/b)^{6n+6m}$$

$$n = 1, 2, 3, \ldots.$$

We can solve this equation step by step and obtain

$$A_6 = A[-\frac{P_6}{6}(a/b)^6 + \frac{11!}{6!6!} P_6 P_{12}(a/b)^{18} + 0(a/b)^{30}]$$

$$A_{12} = A[-\frac{P_{12}}{12}(a/b)^{12} + \frac{17!}{6!12!} P_6 P_{18}(a/b)^{24} + 0(a/b)^{36}] \qquad (2)$$

$$A_{18} = A[-\frac{P_{18}}{18}(a/b)^{18} + \frac{23!}{6!\,18!} P_6 P_{24}(a/b)^{30} + 0(a/b)^{42}]$$

and so on.

P_{12}, P_{18}, P_{24} etc. can be evaluated directly by summing over 1/12th of the plane and multiplying the edge terms by 6 and the interior terms by 12, and taking sufficient terms to get any desired order of accuracy.

To determine P_6 we proceed as follows.

$$P_6 = \sum_{p,q\neq 0,0}\sum \frac{1}{\left(\frac{2p+q}{2} + iq\frac{\sqrt{3}}{2}\right)^6}$$

If we substitute the coordinates of the grid points, this series can be re-written as

$$P_6 = 2 \sum_{n=1}^{\infty} \frac{1}{n^6} + 2 \sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{2n+1}{2} + i\frac{\sqrt{3}}{2}\right)^6} + 2 \sum_{n=-\infty}^{\infty} \frac{1}{(n+i\sqrt{3})^6}$$

$$+ 2 \sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{2n+1}{2} + i\frac{3\sqrt{3}}{2}\right)^6} + 2 \sum_{n=-\infty}^{\infty} \frac{1}{(n+2i\sqrt{3})^6}$$

$$+ 2 \sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{2n+1}{2} + i\frac{5\sqrt{3}}{2}\right)^6} + 2 \sum_{n=-\infty}^{\infty} \frac{1}{(n+3i\sqrt{3})^6}$$

$$+ \; .... \qquad\qquad + \; .... \tag{3}$$

Now

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+i\alpha)^6} = -(\text{residue of } \frac{\pi \cot \pi z}{(z+i\alpha)^6} \text{ at } z = -i\alpha)$$

(Spiegel (1964), p.175)

$$= -\frac{\pi^6}{15}\left(\frac{15+15 \sinh^2 \pi\alpha + 2 \sinh^4 \pi\alpha}{\sinh^6 \pi\alpha}\right) \tag{4}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{1}{\left(\frac{2n+1}{2} + i\beta\right)^6} = (\text{residue of } \frac{\pi \tan \pi z}{(z+i\beta)^6} \text{ at } z = -i\beta)$$

(Spiegel (1964), p.175)

$$= \frac{\pi^6}{15}\left(\frac{15-15 \cosh^2 \pi\beta + 2 \cosh^4 \pi\beta}{\cosh^6 \pi\beta}\right) \tag{5}$$

Putting $\alpha = \sqrt{3}, 2\sqrt{3}, 3\sqrt{3}, ...$ in equation (4) and $\beta = \frac{\sqrt{3}}{2}, 3\frac{\sqrt{3}}{2}, 5\frac{\sqrt{3}}{2}, ...$ in equation (5), and using equation (3), we get

$$P_6 = 5.863031696$$

The values of P_{6n} are given in Table 2.

Applying the force balance condition to the element ABCD of Fig. 2, we get

$$A = \frac{\sqrt{3}\,Pb^2}{4\pi\mu} \tag{6}$$

We can also determine the constant A by putting $\frac{\partial w}{\partial r} = 0$ at either $A(r = \frac{b}{2}, \theta = 0)$ or $B(r = \frac{b}{\sqrt{3}}, \theta = \frac{\pi}{6})$.

Equations (2) and (6) give the unknown constants in equation (1).

The total flux Q through a hexagonal cell of area $\frac{\sqrt{3}\,b^2}{2}$ around a cylinder is given by

$$Q = 12 \int_0^{\pi/6} d\theta \int_a^{\frac{b}{2}\sec\theta} wr\,dr$$

Substituting for w from equation (1) and carrying out the integration, we get

$$Q = 12 \left[A \left\{ \frac{b^2}{8\sqrt{3}} \ln \frac{b}{a\sqrt{3}} - \frac{3b^2}{16\sqrt{3}} + \frac{\pi}{24}\left(\frac{b^2}{2} + a^2\right) \right\} \right.$$

$$- \frac{P}{4\mu}\left(\frac{5b^4}{288\sqrt{3}} - \frac{a^2 b^2}{8\sqrt{3}} + \frac{\pi a^4}{24} \right)$$

$$+ \sum_{n=1}^{\infty} A_{6n} \left\{ \frac{(-1)^n\, b^{6n+2}}{4(6n+1)(6n+2)a^{6n}(\sqrt{3})^{6n+1}} \right.$$

$$\left. \left. - \frac{(-1)^n\, a^{6n}\,(\sqrt{3})^{6n-1}}{4(6n-1)(6n-2)b^{6n-2}} \right\} \right]$$

Using equations (2) and (6) and re-arranging, we find that

$$Q = \frac{3Pb^4}{8\pi\mu}\left[\left(\ln\frac{b}{a} - \tfrac{1}{2}\ln 3 + \frac{13\pi}{36\sqrt{3}} - \frac{3}{2} + \frac{2\pi a^2}{\sqrt{3}\,b^2} - \frac{\pi^2 a^4}{3b^4}\right)\right.$$

$$+ \left(\frac{P_6}{3\times 7\times 8\times 27} - \frac{P_{12}}{6\times 13\times 14\times 27^2} + \frac{P_{18}}{9\times 19\times 20\times 27^3} - \ldots\right)$$

$$+ \frac{3}{40}(a/b)^{12} P_6 \left(-1 - \frac{9\times 10\times 11}{27^2} P_{12} + \frac{15\times 16\times 17}{27^3} P_{18}\right.$$

$$\left.\left. - \frac{21\times 22\times 23}{27^4} P_{24} + \ldots\ldots\right) + O(a/b)^{24}\right]$$

or, after simplification,

$$Q = \frac{3Pb^4}{8\pi\mu}\left[\ln(b/a) - 1.393037947 + \frac{2\pi}{\sqrt{3}}(a/b)^2 - \frac{\pi^2}{3}(a/b)^4 - \frac{P_6^2}{6}(a/b)^{12} + O(a/b)^{24}\right] \quad (7)$$

If $U = \dfrac{2Q}{\sqrt{3}\,b^2}$ is the superficial velocity parallel to the cylinders, and F is the drag force per unit length on a cylinder, given by

$F = \dfrac{\sqrt{3}\,Pb^2}{2}$, then we can use equation (7) to obtain

$$F = \frac{2\pi\mu U}{\ln(b/a) - 1.393037947 + \frac{2\pi}{\sqrt{3}}(a/b)^2 - \frac{\pi^2}{3}(a/b)^4 - \frac{P_6^2}{6}(a/b)^{12} + O(a/b)^{24}} \quad (8)$$

In terms of the volume concentration of cylinders, $\varepsilon = \dfrac{2\pi a^2}{\sqrt{3}\,b^2}$, equation (8) becomes

$$F = \frac{4\pi\mu U}{\ln(1/\varepsilon) - 1.497504971 + 2\varepsilon - \frac{\varepsilon^2}{2} - \frac{9P_6^2}{64\pi^6}\varepsilon^6 + O(\varepsilon^{12})} \quad (9)$$

1.5 Conclusion

The drag calculations are accurate to order ε^4 for the square array and to ε^6 for the triangular array, and may be used from low to moderately high values of ε, particularly in the case of the triangular array, but the errors are not insignificant when the cylinders just touch.

At very low volume concentrations of the cylinders, when $\ln(1/\varepsilon)$ is large, Happel's (1959) free-surface model approximation is good. With zero drag on the outer cell wall, the force balance gives

$$A = \frac{P \times (\text{area of the cell})}{2\pi\mu}$$

independent of the arrangement of the cylinders, and the velocity near a cylinder is

$$w \doteqdot \frac{P \times (\text{area of the cell})}{2\pi\mu} \ln(r/a) - \frac{P}{4\mu}(a^2-r^2)$$

The drag force F on a unit length of a cylinder is then given by

$$F \doteqdot \frac{4\pi\mu U}{\ln(1/\varepsilon) - \kappa}$$

where κ is close to 1.5.

This suggests that the cylinders may be regarded as almost independent of their arrangement and dependent only upon their average volume concentration. In the more refined calculations the value of κ will vary with the arrangement of the cylinders under consideration. Thus a reasonably accurate empirical formula for the drag force would be

$$F = \frac{4\pi\mu U}{\ln(1/\varepsilon) - \kappa + 2\varepsilon - \frac{1}{2}\varepsilon^2}$$

where κ is determined experimentally and is a measure of the regularity of the arrangement.

The method used above for the square and the triagnular arrays may also be applied when the cylinders are arranged in a hexagonal or a rectangular array (Drummond (1982)).

2. SLOW VISCOUS FLOW PERPENDICULAR TO REGULAR ARRAYS OF CYLINDERS

2.1 Introduction

Slow viscous flow past a system of parallel circular cylinders oriented at right angles to the direction of fluid flow is of interest for the theory of filtration through fibrous filters, and has been the subject of many experimental and theoretical studies. A comprehensive review on the subject was made by Davies (1973).

Tamada and Fujikawa (1957) studied, on the basis of Oseen's equations of motion, the steady two-dimensional flow of a viscous fluid passing perpendicularly through an infinite row of equal, parallel and equally spaced circular cylinders, and found that the drag acting on any one of the cylinders in the row was always greater than that acting on the same cylinder when it was immersed alone in an unlimited uniform flow with the same velocity.

Hasimoto (1959) used Fourier series to find solutions of the Stokes equations of motion for the two-dimensional flow past a square array of circular cylinders, and calculated the drag, F, per unit length on one of the cylinders, given by

$$F = \frac{4\pi\mu U}{\ell n(b/a)-1.3105}$$

where U is the mean velocity of the fluid.

He also remarked that, by making use of elliptic functions, he had obtained a formula corresponding to the above in the form

$$F = \frac{4\pi\mu U}{\ell n(b/a)-1.3105+\frac{\pi a^2}{b^2}+0(a^4/b^4)} \qquad (1)$$

Equation (1) can be re-written as

$$F = \frac{8\pi\mu U}{\ell n(1/\varepsilon) - 1.4763 + 2\varepsilon + 0(\varepsilon^2)} \qquad (2)$$

where $\varepsilon = \frac{\pi a^2}{b^2}$ is the volume concentration of the cylinders in the square array.

Happel (1959) applied his free-surface model, discussed in §1.1, to fluid flow perpendicular to cylinders, and obtained the drag force equivalent to

$$F = \frac{4\pi\mu U}{\ell n(b/a) - 1.0724 + 0(a/b)^4}$$

Kuwabara (1959) employed a model identical to that of Happel's, except that on the surface of the outer cylinder vorticity was assumed to be zero, instead of the drag as in Happel's model. He derived a formula for F equivalent to

$$F = \frac{4\pi\mu U}{\ell n(b/a) - 1.3224 + \pi \frac{a^2}{b^2} + 0(a^4/b^4)}$$

Kirsch and Fuchs (1967) set up an experiment to test Happel's and Kuwabara's formulae, and found that Kuwabara's formula fitted the experimental data better than Happel's formula.

More recently, Sangani and Acrivos (1981) extended the earlier analysis of Hasimoto (1959) and derived expressions for the drag force, F, acting on a cylinder in a square and a hexagonal array. For a square array, they obtained

$$F = \frac{8\pi\mu U}{\ell n(1/\varepsilon) - 1.476 + 2\varepsilon - 1.774\varepsilon^2 + 4.076\varepsilon^3 + 0(\varepsilon^4)} \qquad (3)$$

Problems involving transverse flow past regular arrays of cylinders also occur in electric conduction, heat flow and optics. Work in these areas has been done by Lord Rayleigh (1892), Drummond (1971) and Ninham and Sammut (1976).

In the following we shall use our multipole technique to calculate the drag force on a cylinder for transverse flow past a square and a triangular array of cylinders.

2.2 Flow perpendicular to cylinders in a square array

2.2.1 Description

As in §1.3.1, let solid circular cylinders of radius a be arranged in a square array, with their axes parallel to the z-axis, and their centres at points having coordinates (pb,qb) where p and q are integers, and b is the distance between the centres of adjacent cylinders. Let the fluid be driven by a pressure gradient -P in the x-direction and have a mean velocity U in that direction. The fluid velocity is zero on the walls of each cylinder.

A typical quarter cell for the arrangement is shown in Fig. 3(a).

2.2.2 Equations of motion and their solutions

As shown in §1.2, for a two-dimensional flow, the Stokes equations, viz;

$$\nabla^2 \underline{V} = \frac{1}{\mu} \nabla p, \qquad \nabla . \underline{V} = 0$$

are equivalent to

$$\nabla^2 p = 0 \tag{1}$$

and

$$\nabla^4 \psi = 0 \tag{2}$$

where ψ is the stream function related to the velocity components by

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad V_\theta = -\frac{\partial \psi}{\partial r} \tag{3}$$

The equation of continuity $\nabla . \underline{V} = 0$ is satisfied for any choice of ψ .

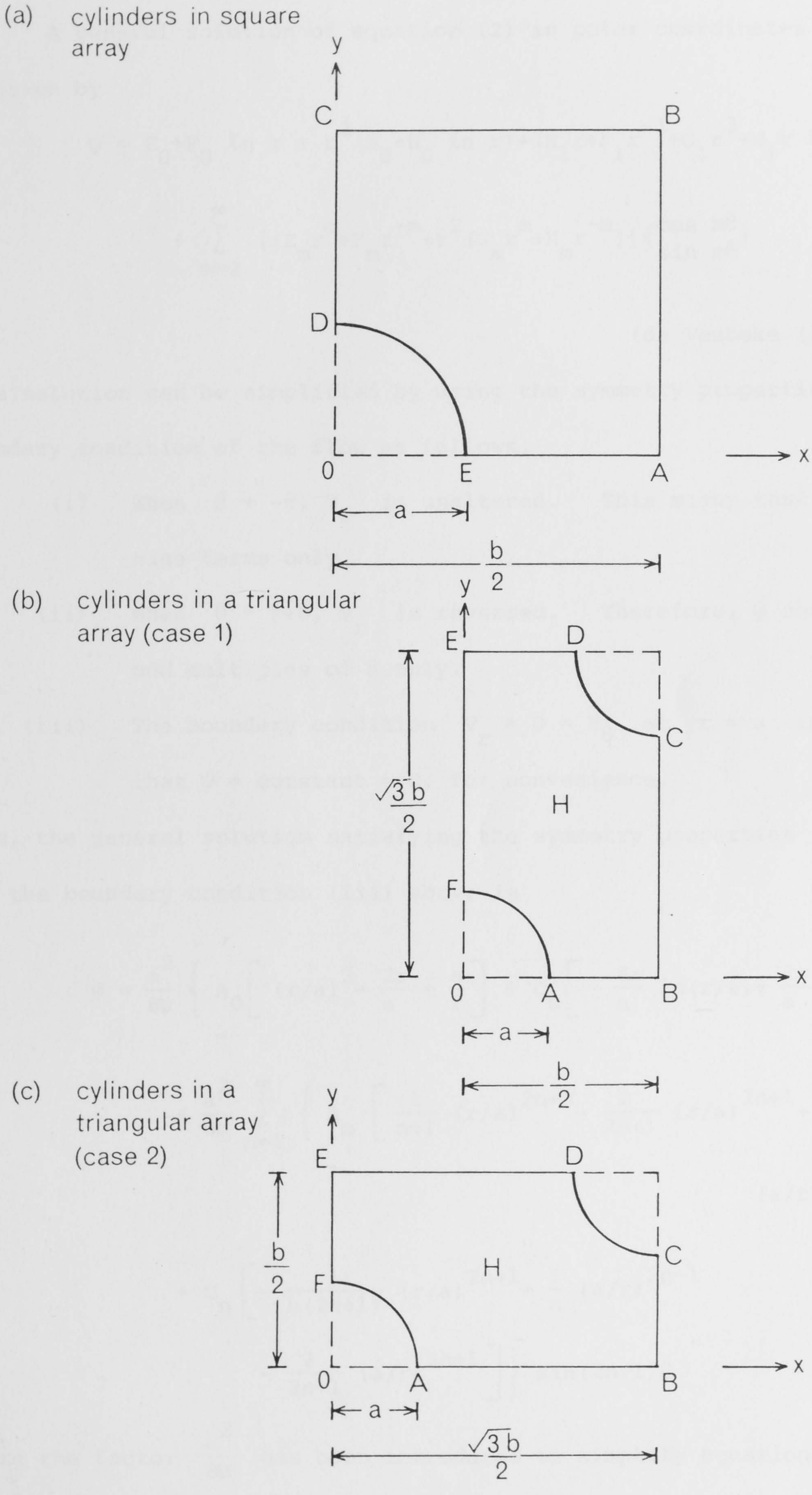

Fig.3 A typical quarter cell for the transverse flow

A general solution of equation (2) in polar coordinates (r,θ) is given by

$$\psi = E_0+F_0 \,\ell n\, r + r^2(G_0+H_0 \,\ell n\, r)+(E_1 r+F_1 r^{-1}+G_1 r^3+H_1 r \,\ell n\, r) \left\{\begin{matrix}\cos\theta \\ \sin\theta\end{matrix}\right\}$$

$$+ \sum_{m=2}^{\infty} [(E_m r^m+F_m r^{-m}+r^2(G_m r^m+H_m r^{-m})] \left\{\begin{matrix}\cos m\theta \\ \sin m\theta\end{matrix}\right\}$$

(de Veubeke (1979),p.242)

This solution can be simplified by using the symmetry properties and the boundary condition of the flow as follows.

(i) When $\theta \to -\theta$, V_r is unaltered. This means that ψ contains sine terms only.

(ii) When $\theta \to \pi+\theta$, V_r is reversed. Therefore, ψ contains odd multiples of θ only.

(iii) The boundary condition $V_r = 0 = V_\theta$ at $r = a$ implies that ψ = constant = 0, for convenience.

Thus, the general solution satisfying the symmetry properties (i) and (ii) and the boundary condition (iii) above is

$$\psi = \frac{a^2}{8\mu}\left\{ A_0\left[(r/a)^3 - \frac{2r}{a} + \frac{a}{r}\right] + C_0\left[- \frac{4r}{a}\,\ell n(r/a) + \frac{2r}{a} - \frac{2a}{r}\right]\right\} \sin\theta$$

$$+ \frac{a^2}{8\mu}\sum_{n=1}^{\infty}\left\{ A_n\left[\frac{1}{n+1}(r/a)^{2n+3} - \frac{2}{2n+1}(r/a)^{2n+1} + \frac{1}{(n+1)(2n+1)}(a/r)^{2n+1}\right]\right.$$

$$\left. + C_n\left[- \frac{1}{n(2n+1)}(r/a)^{2n+1} + \frac{1}{n}(a/r)^{2n-1} - \frac{2}{2n+1}(a/r)^{2n+1}\right]\right\} \sin(2n+1)\theta \qquad (4)$$

where the factor $\frac{a^2}{8\mu}$ has been introduced to simplify equation (8) as we shall see later.

From equations (3) and (4), the velocity components V_r and V_θ are

$$V_r = \frac{a}{8\mu} \{A_0[(r/a)^2-2+(a/r)^2]+C_0[-4 \ln \frac{r}{a} +2-2(a/r)^2]\} \cos \theta$$

$$+ \frac{a}{8\mu} \sum_{n=1}^{\infty} \left\{ A_n \left[\frac{2n+1}{n+1} (r/a)^{2n+2} - 2(r/a)^{2n} + \frac{1}{n+1} (a/r)^{2n+2} \right] \right.$$

$$\left. + C_n \left[\frac{2n+1}{n} (a/r)^{2n} - 2(a/r)^{2n+2} - \frac{1}{n} (r/a)^{2n} \right] \right\} \cos(2n+1)\theta$$

$$V_\theta = \frac{a}{8\mu} \left\{ A_0[-3(r/a)^2+2+(a/r)^2]+C_0[4 \ln \frac{r}{a} +2-2(a/r)^2] \right\} \sin \theta$$

$$+ \frac{a}{8\mu} \sum_{n=1}^{\infty} \left\{ A_n \left[-\frac{2n+3}{n+1} (r/a)^{2n+2} + 2(r/a)^{2n} + \frac{1}{n+1} (a/r)^{2n+2} \right] \right.$$

$$\left. + C_n \left[\frac{2n-1}{n} (a/r)^{2n} - 2(a/r)^{2n+2} + \frac{1}{n} (r/a)^{2n} \right] \right\} \sin(2n+1)\theta$$

(5)

Since the Cartesian components of velocity, u and v are related to V_r and V_θ by

$$u = V_r \cos \theta - V_\theta \sin \theta$$

and

$$v = V_r \sin \theta + V_\theta \cos \theta,$$

we can use equations (5) to determine u and v. Then

$$u = \frac{a}{8\mu} [2A_0(\frac{r^2}{a^2} - 1) - 4 C_0 \ln \frac{r}{a}]$$

$$+ \frac{a}{8\mu} \sum_{n=1}^{\infty} \{2A_n(r/a)^{2n}(\frac{r^2}{a^2} - 1) - \frac{A_{n-1}+C_n}{n} [(r/a)^{2n}-(a/r)^{2n}]$$

$$+ 2C_{n-1}(a/r)^{2n}(\frac{r^2}{a^2} - 1)\} \cos 2n \theta$$

and

$$v = \frac{a}{8\mu} \sum_{n=1}^{\infty} \{2A_n (r/a)^{2n} (1- \frac{r^2}{a^2}) + \frac{C_n - A_{n-1}}{n} [(r/a)^{2n} - (a/r)^{2n}]$$

$$+ 2C_{n-1} (a/r)^{2n} (\frac{r^2}{a^2} - 1)\} \sin 2n\theta \qquad (6)$$

Now

$$\nabla p = \mu \nabla^2 \underline{v}$$

is equivalent to

$$\frac{\partial p}{\partial r} = \mu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} \right)$$

and

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = \mu \left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} \right) \qquad (7)$$

Equations (7), in conjunction with equations (5), yield

$$p = \sum_{n=0}^{\infty} [A_n (r/a)^{2n+1} + C_n (a/r)^{2n+1}] \cos(2n+1)\theta \qquad (8)$$

Equations (4) and (8) provide the first solution of equations (2) and (1) respectively.

As in §1.3.3, using a modification of O'Brien's (1979) method, Drummond (1982) has expressed the second solutions for p and ψ as

$$p = \sum_{p,q}\sum \sum_{n=0}^{\infty} C_n (a/r_{pq})^{2n+1} \cos(2n+1)\theta_{pq} \qquad (9)$$

and

$$\psi = \frac{a^2}{8\mu} \sum_{p,q}\sum \{A_0 (a/r_{pq}) + C_0 [- \frac{4r_{pq}}{a} \ln(r_{pq}/a) - \frac{2a}{r_{pq}}]\} \sin \theta_{pq}$$

$$+ \frac{a^2}{8\mu} \sum_{p,q}\sum \sum_{n=1}^{\infty} \{A_n \frac{1}{(n+1)(2n+1)} (a/r_{pq})^{2n+1} + C_n [\frac{1}{n} (a/r_{pq})^{2n-1}$$

$$- \frac{2}{2n+1} (a/r_{pq})^{2n+1}]\} \sin(2n+1)\theta_{pq} + I$$

(10)

where (r_{pq}, θ_{pq}) are the polar coordinates of a field point $P(r,\theta)$ referred to the centre of another cylinder, and I in equation (10) is an integral over the surface at infinity, which cancels the divergent terms in the solution. Each cylinder of the array contributes equally to the singularities in equations (9) and (10).

If we use complex variables and equate expressions (8) and (9) for p we get

$$\sum_{n=0}^{\infty} [A_n(z/a)^{2n+1}+C_n(a/z)^{2n+1}] = \sum_{n=0}^{\infty} \sum_{p,q}\sum C_n(a/z_{pq})^{2n+1} \qquad (11)$$

This equation is true for all z and not just for the real part of z.

Substituting $z_{pq} = z-d_{pq}$ where $d_{pq} = (p+iq)b$ are the complex coordinates of the centres of the cylinders, and cancelling those singular terms which come from the central cylinder, equation (11) becomes

$$\sum_{n=0}^{\infty} A_n(z/a)^{2n+1} = \sum_{m=0}^{\infty} \sum_{p,q\neq 0,0}\sum C_m [a/(z-d_{pq})]^{2m+1}$$

The right hand side of the above equation may be expanded as a power series in z and matched term by term to the left hand side. The coefficient of z^{2n+1} then gives

$$A_n = -\sum_{m=0}^{\infty} C_m \frac{(2m+2n+1)!}{(2m)!(2n+1)!} P_{2m+2n+2} (a/b)^{2m+2n+2} \qquad (12)$$

$$n = 0, 1, 2, \ldots.$$

where P_n have the same definition as in §1.3.4. For a symmetric grid $P_{2n+1} = 0$.

Similarly, using complex variables and matching expressions (4) and (10) for ψ, we get

$$\frac{a^2}{8\mu} I\{A_0\left(\frac{z^2\bar{z}}{a^3} - \frac{2z}{a}\right) + C_0(2z/a)$$

$$+ \sum_{n=1}^{\infty} A_n \left[\frac{1}{n+1}(z/a)^{2n+1}\frac{z\bar{z}}{a^2} - \frac{2}{2n+1}(z/a)^{2n+1}\right]$$

$$+ \sum_{n=1}^{\infty} C_n \left[-\frac{1}{n(2n+1)}(z/a)^{2n+1}\right]\}$$

$$= \frac{a^2}{8\mu} I \sum_{p,q\neq 0,0}\sum \left\{- A_0 \frac{a}{z_{pq}} + C_0\left(\frac{-2z_{pq}}{a} \ell n \frac{z_{pq}\bar{z}_{pq}}{a^2} + \frac{2a}{z_{pq}}\right)\right.$$

$$- \sum_{n=1}^{\infty} \frac{A_n}{(n+1)(2n+1)} (a/z_{pq})^{2n+1}$$

$$\left.+ \sum_{n=1}^{\infty} C_n \left[-\frac{\bar{z}_{pq}}{an}(a/z_{pq})^{2n} + \frac{2}{2n+1}(a/z_{pq})^{2n+1}\right]\right\} + I$$

where $\bar{z}$ = complex conjugate of z

I = imaginary part of

and we have cancelled those singular terms which come from the central cylinder.

Putting $z_{pq} = z-d_{pq}$, the right hand side of the above equation may be expanded as a power series in z and equated term by term to the left hand side. The coefficient of $z^{2n+2}\bar{z}$ would reproduce equation (12) while the coefficient of z^{2n+1} would give another set of relations which can be written as

$$C_n = - C_0 P_{2n} (a/b)^{2n} - \sum_{m=0}^{\infty} C_m \frac{(2m+2n)!}{(2m)!(2n-1)!} (a/b)^{2m+2n} Q_{2m+2n+2}$$

$$+ \sum_{m=0}^{\infty} \left[C_m \frac{2n}{(2n+1)!} \frac{(2m+2n+2)!}{(2m+1)!} - A_m \frac{(2m+2n+1)!}{(2m+2)!(2n-1)!} \right]$$

$$(a/b)^{2m+2n+2} P_{2m+2n+2} \qquad n = 1,2,3,... \qquad (13)$$

where $$Q_n = \sum_{p,q\neq 0,0}\sum \frac{\bar{d}_{pq} b^{n-2}}{d_{pq}^{n-1}} = \sum_{p,q\neq 0,0}\sum \frac{p^2+q^2}{(p+iq)^n} \qquad (14)$$

For a symmetric grid $Q_{2n+1} = 0$.

Equations (12) and (13) can be solved step by step to give the constants A_n and C_n in terms of C_0. Thus

$$\frac{A_{n-1}}{C_0} (b/a)^{2n} = - P_{2n} + n(2n+1)(P_2+2Q_4)P_{2n+2}(a/b)^4 - 8n(2n+1)P_4 P_{2n+2}(a/b)^6$$

$$+ \left\{ \frac{n(n+1)(2n+1)(2n+3)}{6} (P_4+4Q_6)P_{2n+4} - 3n(2n+1)P_{2n+2} [P_2P_4+4(P_2+2Q_4)Q_6] \right\}$$

$$(a/b)^8 + 0(a/b)^{10} \qquad n = 0,1,2, \ldots \qquad (15)$$

and

$$\frac{C_n}{C_0} (b/a)^{2n} = -(P_{2n}+2nQ_{2n+2}) + 4n(n+1) P_{2n+2}(a/b)^2 + [n(2n+1)P_2P_{2n+2}$$

$$+2n(n+1)(2n+1)(P_2+2Q_4)Q_{2n+4}](a/b)^4 - [16n(n+1)(2n+1)P_4Q_{2n+4}$$

$$+ \frac{4n(n+1)(n+2)(2n+3)}{3} (P_2+2Q_4)P_{2n+4}](a/b)^6$$

$$+ \left\{ \frac{n(n+1)(2n+3)(22n+43)}{2} P_4P_{2n+4} + \frac{n(n+1)(n+2)(2n+1)(2n+3)}{3} \right.$$

$$\times (P_4 + 4Q_6)Q_{2n+6} - 3n(2n+1)(P_2+2Q_4)P_4P_{2n+2}$$

$$\left. -6n(n+1)(2n+1)[P_2P_4+4(P_2+2Q_4)Q_6]Q_{2n+4} \right\} (a/b)^8 + 0(a/b)^{10} \qquad (16)$$

$$n = 1,2,3,.....$$

2.2.3 Determination of P_n and Q_n

$$P_n = \sum_{p,q\neq 0,0}\sum (p+iq)^{-n} = \sum_{p,q\neq 0,0}\sum \frac{\cos n(\arctan \frac{q}{p})}{(p^2+q^2)^{n/2}}$$

and

$$Q_n = \sum_{p,q\neq 0,0}\sum (p^2+q^2)(p+iq)^{-n} = \sum_{p,q\neq 0,0}\sum \frac{\cos n(\arctan \frac{q}{p})}{(p^2+q^2)^{(n/2-1)}}$$

where P_n and Q_n are real, and $P_{2n+1} = 0 = Q_{2n+1}$ for a symmetric grid.

For a square grid (or a triangular grid for that matter) P_2 and Q_4 are not unique as they depend upon the order of summation. Using the method of residues (§1.3.5) we find that

$$P_2 = 2\sum_{n=1}^{\infty} \frac{1}{n^2} - 2\pi^2 \sum_{q=1}^{\infty} \frac{1}{\sinh^2 \pi q}$$

$$= \pi \qquad \text{if we sum over p first}$$

and

$$P_2 = -2\sum_{n=1}^{\infty} \frac{1}{n^2} + 2\pi^2 \sum_{p=1}^{\infty} \frac{1}{\sinh^2 \pi p}$$

$$= -\pi \qquad \text{if we sum over q first}$$

Similarly, the value of Q_4 ranges from -0.94... to 4.08... depending upon whether the grid is first summed at 45° to the axis or whether the sum is taken over p or q first.

Lord Rayleigh (1892) chose $P_2 = -\pi$ in his conductivity claculations and used a physical argument to justify his choice. As we will show in §2.2.5, this difficulty can be avoided by calculating P_2 and Q_4 from the symmetry properties of the pressure and the velocity respectively.

Once we know the values of P_2 and Q_4 from §2.2.5, we can choose the correct order of summation and obtain

$$P_2 = -2 \sum_{n=1}^{\infty} \frac{1}{n^2} + 2\pi^2 \sum_{p=1}^{\infty} \frac{1}{\sinh^2 \pi p}$$

$$= -\pi$$

and

$$Q_4 = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} + 2\pi^2 \sum_{q=1}^{\infty} \frac{2\pi q \coth \pi q - 1}{\sinh^2 \pi q}$$

$$= 4.078451162$$

For a square grid, the other P_n and Q_n are zero if n is not a multiple of 4, and P_4, P_8, P_{12}, ... are the same as for the parallel flow.

Q_8 may be obtained by using the method of residues. Thus

$$Q_8 = \frac{2\pi^6}{945} - \frac{2\pi^6}{45} \sum_{q=1}^{\infty} \frac{45 \sinh \pi q + 45 \sinh^3 \pi q + 6 \sinh^5 \pi q - 2\pi q \cosh \pi q (45 + 30 \sinh^2 \pi q + 2 \sinh^4 \pi q)}{\sinh^7 \pi q}$$

$$= 4.515515440$$

Q_{12}, Q_{16}, Q_{20} etc. can be evaluated by direct summation over the grid. The values of Q_{4n} are given in Table 3.

2.2.4 Determination of C_0

Consider the x-component of the Stokes equations

$$\frac{\partial p}{\partial x} = \mu \nabla^2 u$$

Integrating over the interior of the element ABCDE of Fig. 3(a) and using Green's theorem in the plane, we obtain

$$\int_A^B p\,dy - \int_E^D p\,dy - \int_D^C p\,dy = \mu\left\{ \int_A^B \frac{\partial u}{\partial x}\,dy + \int_E^D \frac{\partial u}{\partial x}\,dy + \int_D^C \frac{\partial u}{\partial x}\,dy \right.$$

$$\left. + \int_C^B \frac{\partial u}{\partial y}\,dx + \int_D^E \frac{\partial u}{\partial y}\,dx + \int_E^A \frac{\partial u}{\partial y}\,dx \right\} \qquad (17)$$

On CD, $p = 0$ and on AB, $p = -\frac{Pb}{2}$. Therefore, $\int_A^B p\,dy = -\frac{Pb^2}{4}$.

Using equation (8), $\int_E^D p\,dy = \int_0^{\pi/2} pa\cos\theta\,d\theta = (A_0+C_0)\,\frac{a\pi}{4}$

By symmetry, $\frac{\partial u}{\partial x} = 0$ on AB and CD

and $\frac{\partial u}{\partial y} = 0$ on CB and EA

The right hand side of equation (17) thus reduces to

$$\mu \int_E^D \left(\frac{\partial u}{\partial x}\,dy - \frac{\partial u}{\partial y}\,dx \right) = -\mu a \int_0^{\pi/2} \frac{\partial u}{\partial r}\,d\theta$$

Using the first of equations (6), the last integral $= -(A_0-C_0)\,\frac{a\pi}{4}$

Hence equation (17) becomes

$$-\frac{Pb^2}{4} - (A_0+C_0)\,\frac{a\pi}{4} = -(A_0-C_0)\,\frac{a\pi}{4}$$

which gives

$$C_0 = -\frac{Pb^2}{2\pi a} \qquad (18)$$

Equations (15), (16) and (18) give the unknown constants in equations (4) and (8).

TABLE 3

Values of Q_{4n}

n	Q_{4n}	n	Q_{4n}
1	4.078451162	10	4.000007629
2	4.515515440	11	3.999998092
3	3.880730845	12	4.000000476
4	4.031540315	13	3.999999988
5	3.992198699	14	4.000000029
6	4.001954101	15	3.999999992
7	3.999511784	16	4.000000001
8	4.000122073	17	3.999999999
9	3.999969482	18	4

TABLE 4

Values of Q_{6n}

n	Q_{6n}	n	Q_{6n}
1	5.656802867	6	6.000000046
2	6.030184464	7	5.999999998
3	5.999179179	8	6
4	6.000035299	9	6
5	5.999998767	10	6

2.2.5 Determination of P_2 and Q_4

As remarked in §2.2.3, P_2 and Q_4 can be calculated by using the symmetry properties of the pressure and the velocity respectively.

By symmetry, the pressure at the point $A(r = \frac{b}{2}, \theta=0)$ in Fig. 3(a) is $-\frac{Pb}{2}$. Equating this to the expression obtained by substituting the coordinates of A in equation (8) and using equations (15) and (16), we get

$$-\frac{Pb}{2} = \frac{2C_0 a}{b} [1 - \frac{P_2}{4} - \sum_{n=1}^{\infty} \frac{P_{4n}}{16^n} + O(a/b)^4]$$

or, using equation (18)

$$P_2 = -2\pi+4-4 \sum_{n=1}^{\infty} \frac{P_{4n}}{16^n}$$

$$= -\pi \qquad \text{to at least 9 decimal places}$$

P_2 can also be calculated by using the pressure at $B(r = \frac{b}{\sqrt{2}}, \theta = \frac{\pi}{4})$, but the series found in this case, $\sum_{n=1}^{\infty} (-1)^n \frac{P_{4n}}{4^n}$, is more slowly convergent.

If we calculate the pressure to higher order in (a/b) and claim that the pressure at A or B is exactly $-\frac{Pb}{2}$, we can find a number of other series satisfied by P_n, such as $\sum_{n=1}^{\infty} (2n-1)(4n-1) \frac{P_{4n}}{16^n} = 1$.

Q_4 may be determined from one of the symmetry properties of the velocity, e.g., the velocity is a maximum at B or it has saddle points at A and C $(r = \frac{b}{2}, \theta = 90°)$. To obtain a more rapidly convergent series we put

$$r \frac{\partial V_r}{\partial r} = 0 \quad \text{at } A$$

and, using equations (5), (15), (16) and (18), obtain

$$Q_4 = 4 - \sum_{n=1}^{\infty} \frac{(2n+1)}{16^n} Q_{4n+4} + \sum_{n=1}^{\infty} \frac{(2n-1)}{4^{1-2n}} P_{4n}$$

$$= 4.078451162$$

We may also calculate Q_4 by putting $v = 0$ at B.

2.2.6 The flux and the drag force

The flux Q through the quarter cell ABCDE of Fig. 3(a) is the flux across DC or AB.

$$Q = \text{flux across DC} = \int_a^{b/2} u \, dy$$

Substituting for u from the first of equations (6) and integrating, we get

$$Q = \frac{ba}{8\mu} \left\{ A_0 \left(\frac{b^2}{8a^2} - 1 + \frac{2a^2}{b^2} \right) - 2C_0 \left(\ln \frac{b}{2a} - \frac{1}{2} + \frac{2a^2}{b^2} \right) \right.$$

$$+ \sum_{n=1}^{\infty} (-1)^n A_n \left[\frac{1}{2(n+1)} (b/2a)^{2n+2} - \frac{1}{2n+1} (b/2a)^{2n} \right.$$

$$\left. + \frac{1}{2(n+1)(2n+1)} (2a/b)^{2n+2} \right]$$

$$+ \sum_{n=1}^{\infty} (-1)^n C_n \left[\frac{-1}{2n(2n+1)} (b/2a)^{2n} + \frac{1}{2n} (2a/b)^{2n} \right.$$

$$\left. \left. - \frac{1}{2n+1} (2a/b)^{2n+2} \right] \right\}$$

Expressing A_n and C_n in terms of C_0 by using equations (15) and (16) and re-arranging, the above equation can be written as

$$Q = -\frac{C_0 ba}{4\mu}\left\{\ell n\frac{b}{2a} - \frac{1}{2} + \frac{P_2}{12} - 2\sum_{n=1}^{\infty}\frac{P_{4n}}{n.16^{n+1}} - 2\sum_{n=1}^{\infty}\frac{P_{4n}}{n(4n+1)16^{n+1}}\right.$$

$$+2\sum_{n=1}^{\infty}\frac{Q_{4n}}{(4n-1)16^{n}} + 2(a/b)^2\left[1 - \frac{P_2}{4} - \sum_{n=1}^{\infty}\frac{P_{4n}}{16^n}\right]$$

$$+(a/b)^4\left[-(P_2+2Q_4)(1+\sum_{n=1}^{\infty}\frac{4n-1}{16^n}P_{4n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{2n+1}{16^n}Q_{4n+4})\right.$$

$$\left. +P_2\left(1-\sum_{n=1}^{\infty}\frac{P_{4n}}{16^n}\right)\right]$$

$$+(a/b)^6[(P_2+2Q_4)(\frac{8}{3} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(2n-1)(4n-1)}{16^{n-1}}P_{4n} - \frac{2}{3}\sum_{n=1}^{\infty}\frac{(n+1)(2n+1)(4n+3)}{(4n+1)16^n}$$

$$\times P_{4n+4})$$

$$+P_4(8+\frac{1}{2}\sum_{n=1}^{\infty}\frac{4n-1}{16^{n-1}}P_{4n} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{2n+1}{16^{n-1}}Q_{4n+4})]$$

$$+(a/b)^8 P_4\left[12P_2\sum_{n=1}^{\infty}\frac{n.P_{4n}}{16^n} - \frac{3P_2}{2}\sum_{n=1}^{\infty}\frac{2n+1}{16^n}Q_{4n+4} + 6Q_4\left(-1+\sum_{n=1}^{\infty}\frac{P_{4n}}{16^n}\right) - 20 - 12P_4\right.$$

$$+\frac{1}{12}\sum_{n=1}^{\infty}\frac{(2n+1)(4n+3)(2n+17)}{16^n}P_{4n+4}$$

$$\left.\left. -\frac{2}{3}\sum_{n=1}^{\infty}\frac{n(2n+1)(4n+1)}{16^n}Q_{4n+4}\right] + 0(a/b)^{10}\right\}$$

Putting the values of C_0, P_n and Q_n, and simplifying, we finally get

$$Q = \frac{Pb^3}{8\pi\mu}[\ell n(b/a)-1.310532926+\pi \frac{a^2}{b^2} - 8.7557339(a/b)^4+63.217216(a/b)^6$$

$$-235.8407556(a/b)^8+0(a/b)^{10}] \qquad (19)$$

Since Q = flux across DC $= \psi_C - \psi_D = \psi_C - 0 = \psi_C$, we can also calculate Q by substituting the coordinates of C in equation (4) and using equations (15), (16) and (18).

If $U = \frac{Q}{b/2}$ is the mean velocity of the fluid, and $F = Pb^2$ is the drag force per unit length on a cylinder, then equation (19) can be used to obtain

$$F = \frac{4\pi\mu U}{\ell n(b/a)-1.310532926+\pi(a/b)^2-8.7557339(a/b)^4+63.217216(a/b)^6 -235.8407556(a/b)^8+0(a/b)^{10}} \qquad (20)$$

In terms of $\varepsilon = \frac{\pi a^2}{b^2}$, the volume concentration of the cylinders, equation (20) can be re-written as

$$F = \frac{8\pi\mu U}{\ell n(1/\varepsilon)-1.476335966+2\varepsilon-1.774282644\varepsilon^2+4.07770443\varepsilon^3-4.842274025\varepsilon^4+0(\varepsilon^5)} \qquad (21)$$

2.3 Flow perpendicular to cylinders in an equilateral triangular array

2.3.1 Case 1: When the flow is in the direction of a nearest neighbour

As in §1.4, the solid circular cylinders are arranged in an equilateral triangular array, with their axes parallel to the z-axis, and their centres at points $[(p+\frac{q}{2})b, \frac{\sqrt{3}}{2}qb]$. Let the fluid be driven by a pressure gradient $-P$ in the direction of a nearest neighbour, which is taken as the x-direction, and have a mean velocity U in that direction. The fluid velocity is zero on the walls of each cylinder.

A typical quarter cell for the arrangement is shown in Fig. 3(b). The equations of motion are the same as for the square array (i.e. equations (1) and (2) of §2.2). Thus, referring to §2.2, the solutions for the pressure, p, and the stream function, ψ, are as given by equations (8) and (4) respectively, where the constants A_n and C_n are related to C_0 by equations (15) and (16).

C_0 can be determined by integrating the x-component of the Stokes equations over the interior of the element ABCDEF of Fig. 3(b). Proceeding on the same lines as in §2.2.4, we get

$$C_0 = \frac{-\sqrt{3}\ Pb^2}{4\pi a} \qquad (1)$$

P_n and Q_n are now defined as

$$P_n = \sum_{p,q\neq 0,0}\sum \left(p+\frac{q}{2}+iq\frac{\sqrt{3}}{2}\right)^{-n}$$

and

$$Q_n = \sum_{p,q\neq 0,0}\sum \frac{\left(p+\frac{q}{2}\right)^2+\left(\frac{\sqrt{3}q}{2}\right)^2}{\left(p+\frac{q}{2}+iq\frac{\sqrt{3}}{2}\right)^n}$$

For a triangular grid, apart from P_2 and Q_4, P_n and Q_n are zero unless n is a multiple of 6, and P_6, P_{12}, P_{18}, ... are the same as for the parallel flow.

Q_6 may be obtained by using the method of residues. Thus

$$Q_6 = \frac{\pi^4}{45} + \frac{2\pi^4}{3}\sum_{q\ \mathrm{even}} \frac{3+2\sinh^2\frac{\pi\sqrt{3}q}{2} - \pi\sqrt{3}q\left(\coth\frac{\pi\sqrt{3}q}{2}\right)\left(3+\sinh^2\frac{\pi\sqrt{3}q}{2}\right)}{\sinh^4\frac{\pi\sqrt{3}q}{2}}$$

$$+ \frac{2\pi^4}{3}\sum_{q\ \mathrm{odd}} \frac{3-2\cosh^2\frac{\pi\sqrt{3}q}{2} - \pi\sqrt{3}q\left(\tanh\frac{\pi\sqrt{3}q}{2}\right)\left(3-\cosh^2\frac{\pi\sqrt{3}q}{2}\right)}{\cosh^4\frac{\pi\sqrt{3}q}{2}}$$

$$= 5.656802867$$

$Q_{12}, Q_{18}, Q_{24}, \ldots\ldots$ can be evaluated by direct summation over the grid. The values of Q_{6n} are listed in Table 4.

P_2 and Q_4 are not unique as they depend upon the order of summation. We can, however, avoid this difficulty by deducing their values from the symmetry of the pressure and the velocity at any of the points $B(r = \frac{b}{2}, \theta=0)$, $H(r = \frac{b}{2}, \theta = \frac{\pi}{3})$ and $E(r = \frac{\sqrt{3}b}{2}, \theta = \frac{\pi}{2})$. B and H could be preferred as they yield more rapidly convergent series.

Following the same procedure as in §2.2.5, if we calculate the pressure at B or H, we get

$$P_2 = -\frac{4\pi}{\sqrt{3}} + 4 - 4\sum_{n=1}^{\infty} \frac{P_{6n}}{64^n}$$

$$= -\frac{2\pi}{\sqrt{3}}$$

Similarly, $\frac{\partial V_r}{\partial r} = 0$ at A gives

$$Q_4 = 4\left(1 + \sum_{n=1}^{\infty} \frac{(3n-1)P_{6n}}{64^n}\right) - \frac{1}{4}\sum_{n=1}^{\infty} \frac{(3n-1)Q_{6n}}{64^{n-1}}$$

$$= \frac{\pi}{\sqrt{3}}$$

Thus $\quad P_2 + 2Q_4 = 0 \qquad (2)$

This relation helps to considerably simplify the equations expressing A_n and C_n in terms of C_0.

Now as we know the values of P_2 and Q_4, we may apply the method of residues and obtain, by choosing the correct order of summation,

$$P_2 = -\frac{\pi^2}{3} - 2\pi^2 \sum_{p\ \mathrm{odd}} \frac{1}{\cosh^2 \frac{\pi\sqrt{3}p}{2}} + 2\pi^2 \sum_{p\ \mathrm{even}} \frac{1}{\sinh^2 \frac{\pi\sqrt{3}p}{2}}$$

$$= -\frac{2\pi}{\sqrt{3}}$$

and

$$Q_4 = \frac{\pi^2}{3} + 2\pi^2 \sum_{q\ \text{odd}} \frac{1 - \pi\sqrt{3}q \tanh \frac{\pi\sqrt{3}q}{2}}{\cosh^2 \frac{\pi\sqrt{3}q}{2}}$$

$$+ 2\pi^2 \sum_{q\ \text{even}} \frac{\pi\sqrt{3}q \coth \frac{\pi\sqrt{3}q}{2} - 1}{\sinh^2 \frac{\pi\sqrt{3}q}{2}}$$

$$= \frac{\pi}{\sqrt{3}}$$

The flux Q through the quarter cell ABCDEF of Fig. 3(b) is the flux across FE or BC and equals $2\psi_H$, where

ψ_H = value of the stream function at H

Thus

$$Q = 2\psi_H$$

Using equations (4), (15) and (16) of §2.2 and equation (2) of §2.3, and re-arranging, the above equation becomes

$$Q = -\frac{\sqrt{3}\, C_0 ba}{4\mu} \left\{ \ln \frac{b}{2a} - \frac{1}{2} + \frac{P_2}{16} - \frac{1}{6} \sum_{n=1}^{\infty} \frac{(3n+1)P_{6n}}{n(6n+1)64^n} \right.$$

$$+ 2\sum_{n=1}^{\infty} \frac{Q_{6n}}{(6n-1)64^n} + 2(a/b)^2 \left[1 - \frac{P_2}{4} - \sum_{n=1}^{\infty} \frac{P_{6n}}{64^n} \right]$$

$$+ P_2 (a/b)^4 \left[1 - \sum_{n=1}^{\infty} \frac{P_{6n}}{64^n} \right] - Q_6 (a/b)^8 \left[8 + \sum_{n=1}^{\infty} \frac{(3n-1)(2n-1)(6n-1)P_{6n}}{64^{2n-1}} \right.$$

$$\left. \left. - \sum_{n=1}^{\infty} \frac{(3n+1)(3n+2)(2n+1)Q_{6n+6}}{64^n} \right] + O(a/b)^{12} \right\}$$

On inserting the values of C_0, P_n and Q_n, and simplifying, we finally get

$$Q = \frac{3Pb^3}{16\pi\mu} [\ln(b/a) - 1.393037946 + \frac{2\pi}{\sqrt{3}} (a/b)^2 - \frac{\pi^2}{3} (a/b)^4 - 63.99883744(a/b)^8 + 0(a/b)^{12}] \tag{3}$$

If $U = \frac{Q}{\sqrt{3}b/2}$ is the mean velocity of the fluid, and

$F = \frac{\sqrt{3}Pb^2}{2}$ is the drag force per unit length on a cylinder, then

$$F = \frac{4\pi\mu U}{\ln(b/a) - 1.393037946 + \frac{2\pi}{\sqrt{3}}(a/b)^2 - \frac{\pi^2}{3}(a/b)^4 - 63.99883744(a/b)^8 + 0(a/b)^{12}} \tag{4}$$

In terms of $\varepsilon = \frac{2\pi a^2}{\sqrt{3}b^2}$, the volume concentration of the cylinders, equation (4) can be re-written as

$$F = \frac{8\pi\mu U}{\ln(1/\varepsilon) - 1.497504970 + 2\varepsilon - \frac{\varepsilon^2}{2} - 0.739137295\varepsilon^4 + 0(\varepsilon^6)} \tag{5}$$

2.3.2 Case 2: When the flow is in the direction bisecting two nearest neighbours

A typical quarter cell for the arrangement is shown in Fig. 3(c).

This problem reduces to the problem of §2.3.1 if we rotate the grid through 30°. The rotation of the grid reverses the signs of P_{12n+6} and Q_{12n+6}, $n = 0,1,2,\ldots$, but leaves P_{12m} and Q_{12m}, $m = 1,2,3,\ldots$ unchanged.

Calculating the pressure at $H(r = \frac{b}{2}, \theta = \frac{\pi}{6})$ in Fig. 3(c), we get

$$P_2 = -\frac{4\pi}{\sqrt{3}} + 4 - 4\sum_{n=1}^{\infty}\frac{P_{6n}}{64^n}$$

$$= -\frac{2\pi}{\sqrt{3}}$$

Similarly, $\frac{\partial v_r}{\partial r} = 0$ at $B(r = \frac{\sqrt{3}b}{2}, \theta = 0)$ gives

$$Q_4 = \frac{4}{3} - \sum_{n=1}^{\infty}(3n-1)(3/4)^{3n-1}P_{6n} + \sum_{n=1}^{\infty}(3n-1)(3/4)^{3n-2}Q_{6n}$$

$$= \frac{\pi}{\sqrt{3}}$$

Thus the values of P_2 and Q_4 are also unaltered.

By integrating the x-component of the Stokes equations over the interior of the element ABCDEF of Fig. 3(c), we get

$$C_0 = -\frac{\sqrt{3}\,Pb^2}{4\pi a}$$

which is the same as before.

The flux Q through the quarter cell in this case is given by

$$Q = \frac{\sqrt{3}\,Pb^3}{16\pi\mu}[\ell n(b/a) - 1.393037946 + \frac{2\pi}{\sqrt{3}}(a/b)^2 - \frac{\pi^2}{3}(a/b)^4$$

$$-63.99883744(a/b)^8 + 0(a/b)^{12}] \qquad (6)$$

A comparison of equations (3) and (6) shows that the two expressions are the same, except for the factor outside the square brackets.

The mean velocity U is now defined as $U = \frac{Q}{b/2}$, but the drag force, F, per unit length, and the volume concentration, ε, of the cylinders have the same definition as in Case 1.

The two expressions for F, one in terms of U and the other in terms of ε, can be obtained by using equation (6) and are exactly the same as given by equaitons (4) and (5).

2.4 Conclusion

The drag calculations are accurate to order ε^4 for both the square and the triangular array, and may be used for low to moderately high values of ε.

For the square array, the drag force given by equations (20) and (21) of §2.2 can be compared with the corresponding results of Hasimoto (1959), Happel (1959), Kuwabara (1959) and Sangani and Acrivos (1981), as summarised in §2.1. There is a complete agreement with the results of Hasimoto, and Sangani and Acrivos.

A comparison of the drag formulas for the parallel and the transverse flow shows that the drag for transverse flow is twice the drag for parallel flow, to an order of ε for the square array and ε^2 for the triangular array.

3. SLOW VISCOUS FLOW PAST A SIMPLE CUBIC ARRAY OF SPHERES

3.1 Introduction

The study of slow viscous flow through a regular array of spheres is important from both the theoretical and the practical point of view.

In this chapter we shall consider a simple cubic array of spheres and by using our multipole technique, calculate the drag force F exerted by the fluid, moving with average speed U, on one sphere forming the array, as a function of the volume concentration ε of the spheres.

Happel (1958) first applied his free-surface model, mentioned earlier in §1.1 and §2.1, to the case of slow viscous flow relative to arrays of spheres. He developed a mathematical treatment on the basis that two concentric spheres can serve as the model for fluid flow through an array of spheres. The inner sphere consists of one of the spheres in the array and the outer sphere of a fluid envelope with zero drag on the surface. The relative volume of fluid to solid in the cell model is taken to be the same as the relative volume of fluid to solid in the array of spheres. He obtained the drag force which for dilute arrays is equivalent to

$$F = \frac{6\pi\mu Ua}{1-1.5\varepsilon^{1/3}+O(\varepsilon^{2/3})}$$

Happel also studied the fluid flow relative to arrays of cylinders on the basis of his free-surface model (§1.1 and §2.1).

Kuwabara (1959) employed a model identical to that of Happel's, except that on the surface of the outer sphere vorticity was assumed to be zero, instead of the drag as in Happel's model. He derived a formula for F equivalent to

$$F = \frac{6\pi\mu Ua}{1-1.62\varepsilon^{1/3}+O(\varepsilon^{2/3})}$$

for dilute arrays.

Hasimoto (1959) used Fourier series to derive periodic fundamental solutions of the Stokes equations of motion and obtained an expression for F for the three cubic arrays (simple, body-centred and face-centred). For the simple cubic array he found that

$$F = \frac{6\pi\mu Ua}{1-1.7601\varepsilon^{1/3}+\varepsilon-1.5593\varepsilon^{2}+O(\varepsilon^{8/3})}$$

He also applied his method to the two-dimensional flow past a square array of circular cylinders (§2.1).

Sangani and Acrivos (1981) modified Hasimoto's method and extended his results to $O(\varepsilon^{10/3})$. For the simple cubic array they obtained

$$F = \frac{6\pi\mu Ua}{1-1.7601\varepsilon^{1/3}+\varepsilon-1.5593\varepsilon^{2}+3.9799\varepsilon^{8/3} - 3.0734\varepsilon^{10/3}+O(\varepsilon^{11/3})}$$

They also calculated the drag force to $O(\varepsilon^{3})$ for flow perpendicular to square and hexagonal arrays of cylinders (§2.1).

Similar transport problems around regular arrays of spheres also occur in electric conduction, heat flow and optics, and have been solved by Lord Rayleigh (1892), Sorensen and Stewart (1974), Ninham and Sammut (1976), McPhedran and McKenzie (1978), and O'Brien (1979).

3.2 Flow past a simple cubic array of spheres

3.2.1 Description

Let us consider the slow viscous flow of a fluid past solid spheres of radius a arranged in a simple cubic array so that the centres of the spheres have coordinates (pb, qb, tb) where p, q and t are integers, and b is the distance between the centres of adjacent spheres. Let the fluid be driven by an average pressure gradient $-P$ in the z-direction and have a mean velocity U in that direction. The no-slip boundary condition requires that the fluid velocity is zero on the surface of each sphere.

The arrangement of the spheres in the simple cubic array and a typical cubic cell are shown in Fig. 4.

3.2.2 Equations of motion and their first solution

The equations of motion (equations (4) and (5) of §1.2) are

$$\nabla^2 \underline{V} = \frac{1}{\mu} \nabla p \qquad (1)$$

and

$$\nabla . \underline{V} = 0 \qquad (2)$$

Taking the divergence of equation (1) and using equation (2), we get

$$\nabla^2 p = 0 \qquad (3)$$

We wish to solve equations (1) and (3) in spherical polar coordinates (r,θ,ϕ). The general solution of equation (3) is

$$p = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (A_{nm} r^n + B_{nm} r^{-n-1}) P_n^m e^{im\phi}$$

(Morse and Feshbach (1953),p.1264)

where

$$P_n^m = P_n^m(\cos \theta)$$

are the associated Legendre polynomials.

If we start with a solution of the form

$$p = A_{nm} r^n P_n^m e^{im\phi}$$

and

$$\underline{V} = (V_r, V_\theta, V_\phi) = r^{n+1} [f(\theta), g(\theta), ih(\theta)] e^{im\phi}$$

and substitute in equation (1), we get

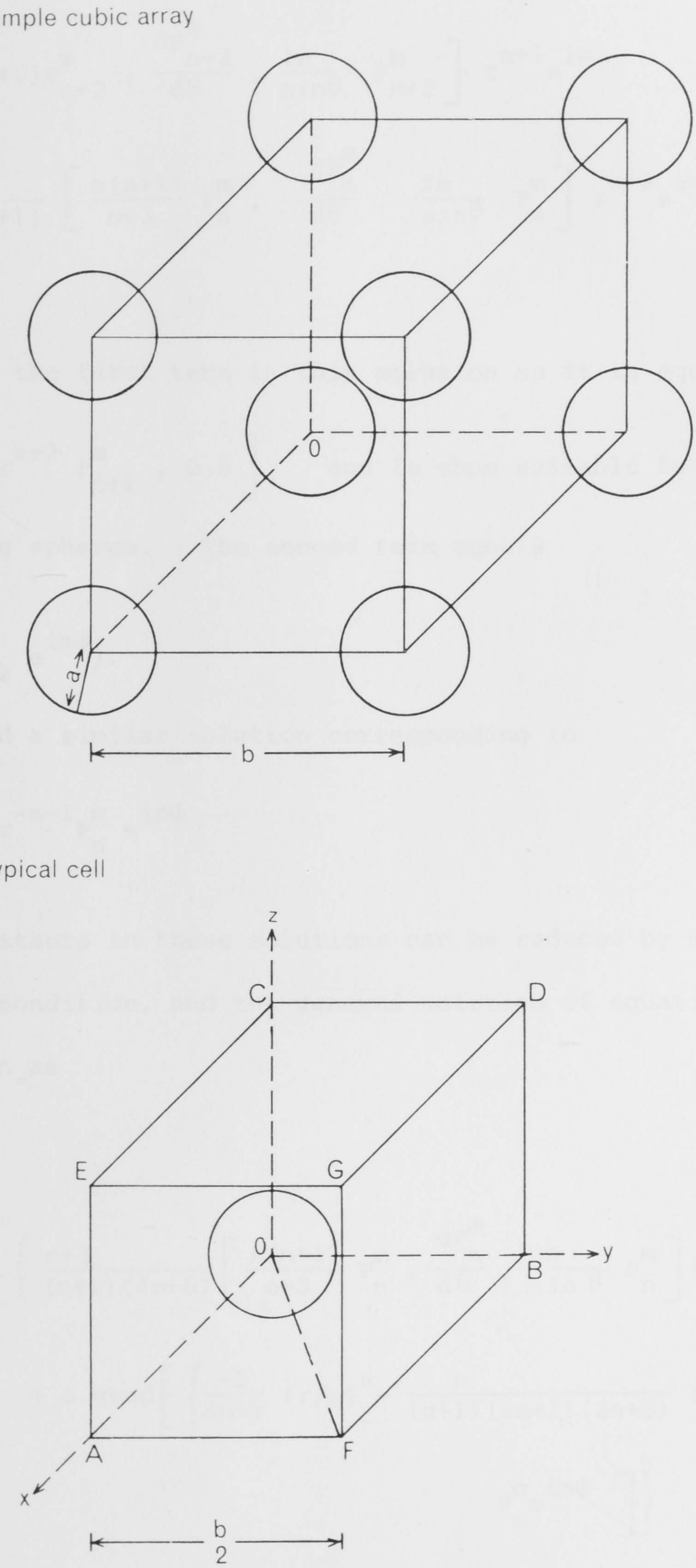

Fig.4 Spheres in a simple cubic array

$$\underline{V} = \frac{C_{nm}}{(n+1)(n+2)}\left[0,\ \frac{m}{\sin\theta}P_{n+1}^m,\ i\frac{dP_{n+1}^m}{d\theta}\right] r^{n+1} e^{im\phi}$$

$$+ \frac{E_{nm}}{n+2}\left[(n+2)P_{n+2}^m,\ \frac{dP_{n+2}^m}{d\theta},\ \frac{im}{\sin\theta}P_{n+2}^m\right] r^{n+1} e^{im\phi}$$

$$+ \frac{A_{nm}(n+3)}{(4n+6)(n+1)}\left[\frac{n(n+1)}{n+3}P_n^m,\ \frac{dP_n^m}{d\theta},\ \frac{im}{\sin\theta}P_n^m\right] r^{n+1} e^{im\phi}$$

We discard the first term in this solution as it is equal to

$\mathrm{Curl}\left[\frac{C_{nm}}{(n+1)(n+2)} r^{n+2} P_{n+1}^m,\ 0,0\right]$ and is thus suitable for the flow involving rotating spheres. The second term equals

$$\mathrm{grad}\ \frac{E_{nm}}{n+2}\left(r^{n+2} P_{n+2}^m e^{im\phi}\right).$$

We can find a similar solution corresponding to

$$p = B_{nm} r^{-n-1} P_n^m e^{im\phi}$$

The number of constants in these solutions can be reduced by using the no-slip boundary condition, and the general solution of equations (1) and (3) can be written as

$$\underline{V} = (V_r, V_\theta, V_\phi)$$

$$= a\sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{nm}\left\{\frac{n+3}{(n+1)(4n+6)}\left[\frac{n(n+1)}{n+3}P_n^m,\ \frac{dP_n^m}{d\theta},\ \frac{im}{\sin\theta}P_n^m\right](r/a)^{n+1}e^{im\phi}\right.$$

$$\left. + a\ \mathrm{grad}\left[\left(\frac{-1}{4n+2}(r/a)^n - \frac{n}{(n+1)(2n+1)(2n+3)}(a/r)^{n+1}\right) P_n^m e^{im\phi}\right]\right\}$$

$$+ a \sum_{n=0}^{\infty} \sum_{m=-n}^{n} D_{nm} \left\{ \frac{n-2}{n(4n-2)} \left[\frac{n(n+1)}{n-2} P_n^m , -\frac{dP_n^m}{d\theta} , -\frac{im}{\sin\theta} P_n^m \right] (a/r)^n e^{im\phi} \right.$$

$$\left. +a \operatorname{grad} \left[\left(-\frac{n+1}{4(2n-1)(2n+1)} (r/a)^n + \frac{1}{4n+2} (a/r)^{n+1} \right) P_n^m e^{im\phi} \right] \right\}$$

and

$$p = \mu \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [A_{nm} (r/a)^n + D_{nm} (a/r)^{n+1}] P_n^m e^{im\phi}$$

We can use the symmetry properties of the pressure to derive restrictions on the range of n and m, and a relationship between the constants A_{nm}, $A_{n,-m}$, and between D_{nm}, $D_{n,-m}$.

(i) Since p changes sign when θ is changed to $\pi-\theta$, n is odd.

(ii) When ϕ is increased by $\frac{\pi}{2}$, p is unaltered. Therefore, m is zero or a multiple of 4.

(iii) Since p is unchanged when ϕ is changed to $-\phi$,

$$A_{nm} P_n^m e^{im\phi} + A_{n,-m} P_n^{-m} e^{-im\phi}$$

$$= [A_{nm} e^{im\phi} + (-1)^m \frac{(n-m)!}{(n+m)!} A_{n,-m} e^{-im\phi}] P_n^m$$

is an even function of ϕ. This requires

$$A_{n,-m} = (-1)^m \frac{(n+m)!}{(n-m)!} A_{nm} \tag{4}$$

Similarly,

$$D_{n,-m} = (-1)^m \frac{(n+m)!}{(n-m)!} D_{nm} \tag{5}$$

Using the above findings (i) and (ii), we can write the velocity components V_r, V_θ, V_ϕ and the pressure p as

$$V_r = a \sum_{n=1}^{\infty} \sum_{m=-n}^{n} A_{nm} [\frac{n}{4n+6} (r/a)^{n+1} - \frac{n}{4n+2} (r/a)^{n-1} + \frac{n}{(2n+1)(2n+3)} (a/r)^{n+2}] P_n^m e^{im\phi}$$

$$+a \sum_{n=1}^{\infty} \sum_{m=-n}^{n} D_{nm} [\frac{n+1}{4n-2} (a/r)^{n} - \frac{n+1}{(2n-1)(2n+1)} (r/a)^{n-1} - \frac{n+1}{4n+2} (a/r)^{n+2}] P_n^m e^{im\phi}$$

$$V_\theta = a \sum_{n=1}^{\infty} \sum_{m=-n}^{n} A_{nm} [\frac{n+3}{(n+1)(4n+6)} (r/a)^{n+1} - \frac{1}{4n+2} (r/a)^{n-1}$$

$$- \frac{n}{(n+1)(2n+1)(2n+3)} (a/r)^{n+2}] \frac{dP_n^m}{d\theta} e^{im\phi}$$

$$+a \sum_{n=1}^{\infty} \sum_{m=-n}^{n} D_{nm} [- \frac{n-2}{n(4n-2)} (a/r)^{n} - \frac{n+1}{n(2n-1)(2n+1)} (r/a)^{n-1}$$

$$+ \frac{1}{4n+2} (a/r)^{n+2}] \frac{dP_n^m}{d\theta} e^{im\phi}$$

$$V_\phi = a \sum_{n=1}^{\infty} \sum_{m=-n}^{n} A_{nm} [\frac{n+3}{(n+1)(4n+6)} (r/a)^{n+1} - \frac{1}{4n+2} (r/a)^{n-1} - \frac{n}{(n+1)(2n+1)(2n+3)}$$

$$(a/r)^{n+2} \frac{im}{\sin\theta} P_n^m e^{im\phi}$$

$$+a \sum_{n=1}^{\infty} \sum_{m=-n}^{n} D_{nm} [- \frac{n-2}{n(4n-2)} (a/r)^{n} - \frac{n+1}{n(2n-1)(2n+1)} (r/a)^{n-1}$$

$$+ \frac{1}{4n+2} (a/r)^{n+2}] \frac{im}{\sin\theta} P_n^m e^{im\phi} \tag{6}$$

and

$$p = \mu \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [A_{nm} (r/a)^{n} + D_{nm} (a/r)^{n+1}] P_n^m e^{im\phi} \tag{7}$$

where n is an odd integer and m is zero or a multiple of 4.

Equations (6) and (7) give the general solution of equations (1) and (3) for the present situation.

Since $V_z = V_r \cos\theta - V_\theta \sin\theta$, using equations (6), we find that

$$V_z = a \sum_{n=1}^{\infty} \sum_{m=-n}^{n} A_{nm} \left\{ \frac{n+m}{2(2n+1)} [(r/a)^{n+1} - (r/a)^{n-1}] P_{n-1}^m \right.$$

$$\left. + \frac{n(n-m+1)}{(n+1)(2n+1)(2n+3)} [(a/r)^{n+2} - (r/a)^{n+1}] P_{n+1}^m \right\} e^{im\phi}$$

$$+ a \sum_{n=1}^{\infty} \sum_{m=-n}^{n} D_{nm} \left\{ \frac{n-m+1}{2(2n+1)} [(a/r)^n - (a/r)^{n+2}] P_{n+1}^m \right.$$

$$\left. + \frac{(n+1)(n+m)}{n(2n-1)(2n+1)} [(a/r)^n - (r/a)^{n-1}] P_{n-1}^m \right\} e^{im\phi} \quad (8)$$

3.2.3 <u>The second solution</u>

As in §1.3.3, using a modification of O'Brien's (1979) method, Drummond (1982) has expressed the second solutions for p and V_z as

$$p = \mu \sum_i \sum_n \sum_m D_{nm} (a/r_i)^{n+1} P_n^m (\cos\theta_i) e^{im\phi_i} \quad (9)$$

and

$$V_z = a \sum_i \sum_n \sum_m A_{nm} \frac{n(n-m+1)}{(n+1)(2n+1)(2n+3)} (a/r_i)^{n+2} P_{n+1}^m (\cos\theta_i) e^{im\phi_i}$$

$$+ a \sum_i \sum_n \sum_m D_{nm} \left\{ \frac{n-m+1}{2(2n+1)} [(a/r_i)^n - (a/r_i)^{n+2}] P_{n+1}^m (\cos\theta_i) \right.$$

$$\left. + \frac{(n+1)(n+m)}{n(2n-1)(2n+1)} (a/r_i)^n P_{n-1}^m (\cos\theta_i) \right\} e^{im\phi_i} \quad (10)$$

where (r_i, θ_i, ϕ_i) are the spherical polar coordinates of a field point $P(r,\theta,\phi)$ referred to the centre of another sphere and $\sum_i$ denotes the sum over all the spheres. Each sphere of the array contributes equally to the singularities in equations (9) and (10).

Equating the two expressions for p and for V_z, and cancelling those singular terms which come from the central sphere, we get

$$\sum_{\ell}\sum_{s} A_{\ell s}(r/a)^{\ell} P_{\ell}^{s}(\cos\theta)e^{is\phi} = \sum_{i}{}' \sum_{n}\sum_{m} D_{nm}(a/r_i)^{n+1} P_{n}^{m}(\cos\theta_i)e^{im\phi_i} \tag{11}$$

and

$$\sum_{\ell}\sum_{s} A_{\ell s}\left\{\frac{\ell+s}{2(2\ell+1)}[(r/a)^{\ell+1}-(r/a)^{\ell-1}]P_{\ell-1}^{m}(\cos\theta) - \frac{\ell(\ell-s+1)}{(\ell+1)(2\ell+1)(2\ell+3)}(r/a)^{\ell+1}P_{\ell+1}^{s}(\cos\theta)\right\}e^{is\phi}$$

$$-\sum_{\ell}\sum_{s} D_{\ell s}\frac{(\ell+1)(\ell+s)}{\ell(2\ell-1)(2\ell+1)}(r/a)^{\ell-1}P_{\ell-1}^{s}(\cos\theta)e^{is\phi}$$

$$= \sum_{i}{}'\sum_{n}\sum_{m} A_{nm}\frac{n(n-m+1)}{(n+1)(2n+1)(2n+3)}(a/r_i)^{n+2}P_{n+1}^{m}(\cos\theta_i)e^{im\phi_i}$$

$$+\sum_{i}{}'\sum_{n}\sum_{m} D_{nm}\left\{\frac{n-m+1}{2(2n+1)}[(a/r_i)^{n}-(a/r_i)^{n+2}]P_{n+1}^{m}(\cos\theta_i) + \frac{(n+1)(n+m)}{n(2n-1)(2n+1)}(a/r_i)^{n}P_{n-1}^{m}(\cos\theta_i)\right\}e^{im\phi_i} \tag{12}$$

where $\sum_{i}{}'$ denotes the sum over all the spheres except the central one.

The two sides of equations (11) and (12) can be matched term by term by using the shift of origin formulae derived below.

3.2.4 The shift of origin formulae

If (d_i, α_i, β_i) are the coordinates of the centre of the ith sphere referred to the centre of the central sphere, then

$$\frac{1}{r_i} = \frac{1}{\sqrt{\{r^2 - 2d_i r[\cos\alpha_i \cos\theta + \sin\alpha_i \sin\theta \cos(\beta_i - \phi)] + d_i^2\}}}$$

Using the generating function for the Legendre polynomials and re-arranging, the above equation may be transformed to

$$\frac{1}{r_i} = \sum_{\ell=0}^{\infty} \sum_{s=-\ell}^{\ell} (-1)^s \frac{r^\ell}{d_i^{\ell+1}} P_\ell^{-s}(\cos\alpha_i) P_\ell^s(\cos\theta) e^{is(\phi-\beta_i)} \qquad (13)$$

Let

$$\bar{\rho} = x - iy = r\sin\theta\, e^{-i\phi}$$

Then

$$\frac{\partial}{\partial\bar{\rho}_i}[r^{-n-1}P_n^m(\cos\theta)e^{im\phi}] = \frac{\partial}{\partial\bar{\rho}}[r^{-n-1}P_n^m(\cos\theta)e^{im\phi}]$$

$$= -\frac{1}{2} r^{-n-2} P_{n+1}^{m+1}(\cos\theta)e^{i(m+1)\phi}$$

and

$$\frac{\partial}{\partial z_i}[r^{-n-1}P_n^m(\cos\theta)e^{im\phi}] = \frac{\partial}{\partial z}[r^{-n-1}P_n^m(\cos\theta)e^{im\phi}] \qquad (14)$$

$$= -(n-m+1)r^{-n-2}P_{n+1}^m(\cos\theta)e^{im\phi}$$

The left hand side of equation (13) is clearly

$\frac{P_0^0(\cos\theta_i)e^{i0\phi_i}}{r_i}$. Differentiating equations (13) n-m and m times with respect to z and $\bar{\rho}$ respectively, we get

$$(\partial/\partial z_i)^{n-m}(\partial/\partial\bar{\rho}_i)^m \left[\frac{P_0^0(\cos\theta_i)e^{i0\phi_i}}{r_i}\right]$$

$$= (\partial/\partial z)^{n-m}(\partial/\partial\bar{\rho})^m \left[\sum_{\ell}\sum_{s}(-1)^s \frac{r^\ell}{d_i^{\ell+1}} P_\ell^{-s}(\cos\alpha_i)P_\ell^s(\cos\theta)e^{is(\phi-\beta_i)}\right]$$

Using equations (14) and simplifying, we find that

$$\frac{P_n^m(\cos\theta_i)e^{im\phi_i}}{r_i^{n+1}} = \sum_{\ell=0}^{\infty}\sum_{s=-\ell}^{\ell}\frac{(-1)^{n+s}(n-m+\ell+s)!}{(n-m)!(\ell+s)!}\frac{r^\ell}{d_i^{n+\ell+1}} P_{n+\ell}^{m-s}(\cos\alpha_i)P_\ell^s(\cos\theta)e^{i[s\phi+(m-s)\beta_i]} \tag{15}$$

This is a shift of origin formula for the harmonics.

We can derive a shift of origin formula for the biharmonics $\frac{P_n^m(\cos\theta_i)e^{im\phi_i}}{r_i^{n-1}}$ by multiplying equation (15) by r_i^2, and simplifying with the help of the following recurrence relations

$$(2n+1)\sin\theta P_n^m(\cos\theta) = (n+m)(n+m-1)P_{n-1}^{m-1}(\cos\theta)-(n-m+1)(n-m+2)P_{n+1}^{m-1}$$

and

$$(2n+1)\cos\theta P_n^m(\cos\theta) = (n-m+1)P_{n+1}^m(\cos\theta)+(n+m)P_{n-1}^m(\cos\theta)$$

(Morse and Feshbach (1953),p.1325-6)

Finally we get

$$\frac{P_n^m(\cos\theta_i)e^{im\phi_i}}{r_i^{n-1}} = \sum_{\ell=0}^{\infty}\sum_{s=-\ell}^{\ell}(-1)^{n+s}\frac{r^\ell}{d_i^{n+\ell-1}}e^{i[s\phi+(m-s)\beta_i]}\times$$

$$\left\{\frac{(n-m+\ell+s)!}{(n-m)!(\ell+s)!}\frac{2n-1}{2n+2\ell-1}P_{n+\ell}^{m-s}(\cos\alpha_i)P_\ell^s(\cos\theta)\right.$$

$$-\frac{(n-m+\ell+s-2)!}{(n-m)!(\ell+s)!}\frac{2n\ell(n+\ell-1)-2\ell(2\ell-1)m^2-2(2\ell-1)(2n-1)sm-2n(2n-1)s^2}{(2\ell-1)(2n+2\ell-1)}$$

$$P_{n+\ell-2}^{m-s}(\cos\alpha_i)P_\ell^s(\cos\theta)$$

$$\left.-\frac{(n-m+\ell+s-2)!}{(n-m)!(\ell+s-2)!}\times\frac{2n-1}{2\ell-1}P_{n+\ell-2}^{m-s}(\cos\alpha_i)P_{\ell-2}^s(\cos\theta)\right\} \qquad (16)$$

3.2.5 Matching the two solutions

Using equation (15) and comparing the coefficient of r^ℓ on both sides of equation (11), we get

$$A_{\ell s} = \sum_i{}' \sum_n \sum_m (-1)^{n+s}\frac{(n-m+\ell+s)!}{(n-m)!(\ell+s)!}D_{nm}\frac{a^{n+\ell+1}}{d_i^{n+\ell+1}}P_{n+\ell}^{m-s}(\cos\alpha_i)e^{i(m-s)\beta_i}$$

or

$$A_{\ell s} = \sum_n \sum_m (-1)^{n+s}\frac{(n-m+\ell+s)!}{(n-m)!(\ell+s)!}D_{nm}U_{n+\ell}^{m-s}(a/b)^{n+\ell+1} \qquad (17)$$

where

$$U_n^m = \sum_i{}' P_n^m(\cos\alpha_i)e^{im\beta_i}(b/d_i)^{n+1} \qquad (18)$$

and we have introduced b, the distance between the centres of adjacent spheres, to make U_n^m dimensionless.

For the simple cubic grid,

$$U_n^{-m} = \frac{(n-m)!}{(n+m)!}U_n^m \qquad (19)$$

Similarly, if we use equations (15) and (16), the coefficient of $r^{\ell+1}$ in equation (12) reproduces equation (17), while the coefficient of $r^{\ell-1}$ gives another set of relations which can be simplifeid with the help of equation (17) and expressed as

$$D_{\ell s} = \sum_n \sum_m (-1)^{n+s} \frac{\ell(2\ell-1)(2\ell+1)}{\ell+1} \frac{(n-m+\ell+s)!}{(n-m)!(\ell+s)!} \left\{ \frac{n}{(n+1)(2n+1)(2n+3)} A_{nm} U^{m-s}_{n+\ell} (a/b)^{n+\ell+1} \right.$$

$$+D_{nm}\left[\frac{1}{2(2n+2\ell-1)} V^{m-s}_{n+\ell}(a/b)^{n+\ell-1} - \frac{n+\ell+1}{(2n+1(2\ell+1)} U^{m-s}_{n+\ell}(a/b)^{n+\ell+1} \right.$$

$$+U^{m-s}_{n+\ell-2}(a/b)^{n+\ell-1}\left(\frac{(n+1)(n^2-m^2)}{n(2n-1)(2n+1)(n-m+\ell+s)(n-m+\ell+s-1)} \right.$$

$$- \frac{(n+1)(\ell-1)(n+\ell-1)-(\ell-1)(2\ell-3)m^2-(2\ell-3)(2n+1)sm-(n+1)(2n+1)s^2}{(2n+1)(2\ell-3)(2n+2\ell-1)(n-m+\ell+s)(n-m+\ell+s-1)}$$

$$\left.\left.\left. - \frac{(\ell-2)(\ell-s-1)(\ell+s-1)}{(\ell-1)(2\ell-1)(2\ell-3)(n-m+\ell+s)(n-m+\ell+s-1)} \right) \right] \right\} \qquad (20)$$

where

$$V^m_n = \sum_i{}' P^m_n(\cos \alpha_i) e^{im\beta_i} (b/d_i)^{n-1} \qquad (21)$$

and is dimensionless.

For the simple cubic grid,

$$V^{-m}_n = \frac{(n-m)!}{(n+m)!} V^m_n \qquad (22)$$

Equations (17) and (20) can be solved step by step to give the constants $A_{\ell s}$ and $D_{\ell s}$ in terms of $D_{1,0}$. Thus

$$A_{\ell s} = (-1)^{1+s} D_{1,0} (\ell+s+1)(a/b)^{\ell+2} \left[U^{-s}_{\ell+1} - \frac{(\ell+s+2)(\ell+s+3)}{4} (5V^0_4 + 3U^0_2)(a/b)^5 U^{-s}_{\ell+3} + 0(a/b)^9 \right] \quad (23)$$

and

$$D_{\ell s} = (-1)^{1+s} D_{1,0} (a/b)^{\ell} \left\{ \frac{\ell(2\ell-1)(\ell+s+1)}{2(\ell+1)} V^{-s}_{\ell+1} + \frac{\ell^2}{\ell+s} \left(1 + \frac{3s^2}{(\ell-1)(\ell+1)} \right) U^{-s}_{\ell-1} \right.$$

$$- (a/b)^2 \frac{\ell(2\ell-1)(\ell+2)(\ell+s+1)}{3(\ell+1)} U^{-s}_{\ell+1}$$

$$+ (a/b)^5 \frac{\ell(2\ell-1)(2\ell+1)(\ell+s+1)}{\ell+1} \left[-\frac{U^0_2}{15} U^{-s}_{\ell+1} + \frac{5V^0_4 + 2U^0_2}{4(2\ell+5)} \left(-\frac{(\ell+s+2)(\ell+s+3)}{2} V^{-s}_{\ell+3} \right. \right.$$

$$\left. \left. \left. + \left(\frac{(\ell+2)(\ell-5)}{5(2\ell-1)} - \frac{(5\ell+2)s^2}{(\ell-1)(2\ell-1)} \right) U^{-s}_{\ell+1} \right) \right] + 0(a/b)^7 \right\} \quad (24)$$

3.2.6 Determination of $D_{1,0}$

Consider the z-component of the flow equation

$$\frac{\partial p}{\partial z} = \mu \nabla^2 v_z$$

Integrating over the typical cubic cell of Fig. 4(b), and using Gauss' divergence theorem, we obtain

$$\iint_{CEGD} p \, dx \, dy - \iint_S p \, dS + \mu \iint_S \frac{\partial v_z}{\partial r} \, dS = 0$$

where S is the surface of an octant of the sphere with centre at O.

Using equations (7) and (8), and taking $p = -\frac{Pb}{2}$ on the face CEGD of the cell, the above equation becomes

$$-\frac{Pb}{2} \times \frac{b^2}{4} - \mu \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} (A_{1,0}+D_{1,0}) \cos\theta \,.\cos\theta \,.a^2 \sin\theta \, d\theta \, d\phi$$

$$+ \mu \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \frac{A_{1,0}-2D_{1,0}}{2} \sin^2\theta . a^2 \sin\theta \, d\theta \, d\phi = 0$$

or , on simplification

$$-\frac{Pb^3}{2} - 2a^2 \mu\pi \, D_{1,0} = 0$$

i.e.

$$D_{1,0} = -\frac{Pb^3}{4\pi\mu a^2} \qquad (25)$$

3.2.7 Determination of U_n^m and V_n^m

U_n^m and V_n^m are defined by equations (18) and (21) respectively. The values of all U_n^m and V_n^m, except U_2^0, V_4^0 and V_4^4, may be calculated by summing the corresponding series.

Using a computer programme, Drummond (1982) obtained a truncated series of sums for U_n^m and V_n^m over successive cubes in the array. These sums were then extrapolated to infinity by using Aitken-Shanks transformation (Greenberg (1978), p. 38). The resulting values are listed in Tables 5 and 6.

By modifying O'Brien's method (1979), Drummond (1982) has shown that U_2^0, V_4^0 and V_4^4 may be evaluated by first summing over a cube of side 2n with centre at 0, and then adding to the sum an appropriate integral over the region outside this cube. The values of U_2^0, V_4^0 and V_4^4 are not unique as they depend upon the choice of this outer integral.

The series for U_2^0, V_4^0 and V_4^4 are

TABLE 5

Values of U_n^m

U_2^0	$\frac{-4\pi}{3}$	U_{12}^8	340666918.7
U_4^0	3.108227	U_{12}^{12}	1.2514587×10^{12}
U_4^4	372.987202	U_{14}^0	1.15363680
U_6^0	0.57332929	U_{14}^4	-35633.53344
U_6^4	-1444.7898	U_{14}^8	-1137422387
U_8^0	3.25929309	U_{14}^{12}	$-1.5696429 \times 10^{13}$
U_8^4	5475.612396	U_{16}^0	2.79235631
U_8^8	8541955.338	U_{16}^4	57065.56931
U_{10}^0	1.00922399	U_{16}^8	3299563700
U_{10}^4	-11190.27558	U_{16}^{12}	1.0684285×10^{14}
U_{10}^8	-68484486.55	U_{16}^{16}	7.6974417×10^{17}
U_{12}^0	2.89125411	U_{18}^0	1.25802437
U_{12}^4	20312.27973		

TABLE 6

Values of V_n^m

V_4^0	4.39948	V_{12}^8	354528438.4
V_4^4	326.875	V_{12}^{12}	1.2383663×10^{12}
V_6^0	0.43265229	V_{14}^0	1.145625473
V_6^4	-1090.28379	V_{14}^4	-35386.07962
V_8^0	3.436246794	V_{14}^8	-1129523662
V_8^4	5772.894614	V_{14}^{12}	$-1.5587427 \times 10^{13}$
V_8^8	9005715.598	V_{16}^0	2.799133885
V_{10}^0	0.9976765397	V_{16}^4	57308.36202
V_{10}^4	-11062.23747	V_{16}^8	3304599032
V_{10}^8	-67700893.32	V_{16}^{12}	1.0693442×10^{14}
V_{12}^0	2.881591468	V_{16}^{16}	7.719138×10^{17}
V_{12}^4	19500.34615		

$$U_2^0 = \sum_i{}' P_2^0(\cos\alpha_i)(b/d_i)^3 = \sum\sum\sum_{x,y,z\neq 0,0,0} \frac{2z^2-x^2-y^2}{2(x^2+y^2+z^2)^{5/2}}$$

$$V_4^0 = \sum_i{}' P_4^0(\cos\alpha_i)(b/d_i)^3 = \sum\sum\sum_{x,y,z\neq 0,0,0} \frac{8z^4-24z^2(x^2+y^2)+3(x^2+y^2)^2}{8(x^2+y^2+z^2)^{7/2}}$$

and

$$V_4^4 = \sum_i{}' P_4^4(\cos\alpha_i)\cos 4\beta_i(b/d_i)^3 = \sum\sum\sum_{x,y,z\neq 0,0,0} \frac{105(x^4-6x^2y^2+y^4)}{(x^2+y^2+z^2)^{7/2}}$$

In the case of U_2^0 the sum over the cube is zero and the outer integral tends to $2\pi/3$ if taken over a tubular region $|x| < n$, $|y| < n$ as $n \to \infty$, and to $-4\pi/3$ if taken over a flat plate $|z| < n$ as $n \to \infty$.

U_2^0 can also be determined from the symmetry of the pressure. If we substitute the coordinates of the point $C(r = \frac{b}{2}, \theta = 0, \phi = 0)$ of Fig. 4(b) in equation (7) and insert the values of $A_{\ell s}$, $D_{\ell s}$ from equations (23) and (24) in terms of $D_{1,0}$, we get

$$-\frac{Pb}{2} = \mu(a/b)^2 D_{1,0}\left[4-U_2^0-2\frac{U_4^0}{4}-3\frac{U_6^0}{4^2}-\ldots.\right]$$

Using equation (25), the above equation becomes

$$2\pi = 4 - U_2^0 - 2\frac{U_4^0}{4} - 3\frac{U_6^0}{4^2} - \ldots.$$

or

$$U_2^0 = 4-2\pi-\left(2\frac{U_4^0}{4} + 3\frac{U_6^0}{4^2} + 4\frac{U_8^0}{4^3} + \ldots\ldots\right)$$

We use Aitken-Shanks transformation (Greenberg (1978), p.38) to accelerate the convergence of the above series and obtain

$$U_2^0 = -\frac{4\pi}{3}$$

correct to 6 decimal places. This value agrees with the value obtained from the integral over the region $|z| < n$ as $n \to \infty$.

For V_4^0 the sum over the cube is 2.35420. If we integrate over a square prism infinitely long in the z-direction, the integral can be related to an integral over a triangular prism repeated 16 times. Thus the outer integral I is given by

$$I = 16 \int_0^n dx \int_0^x dy \int_n^\infty \frac{8z^4 - 24z^2(x^2+y^2) + 3(x^2+y^2)^2}{8(x^2+y^2+z^2)^{7/2}} dz$$

$$= \frac{7}{3\sqrt{3}} - \frac{\pi}{9}$$

Hence

$$V_4^0 = 2.35420 + \frac{7}{3\sqrt{3}} - \frac{\pi}{9} = 3.35228$$

By extending the cube laterally to infinity, we obtain

$$I = 16 \int_0^x dy \int_n^\infty dx \int_0^n \frac{8z^4 - 24z^2(x^2+y^2) + 3(x^2+y^2)^2}{8(x^2+y^2+z^2)^{7/2}} dz$$

$$= \frac{2\pi}{9} + \frac{7}{3\sqrt{3}}$$

Therefore,

$$V_4^0 = 2.35420 + \frac{2\pi}{9} + \frac{7}{3\sqrt{3}} = 4.39948$$

We can also calculate V_4^0 from the symmetry of the velocity. If we put $\frac{\partial v_r}{\partial r} = 0$ at $C(r = \frac{b}{2}, \theta = 0, \phi = 0)$ and use equations (6),(23) and (24), we find that

$$\sum_{n \text{ even}} \left[4 + \frac{n(n^2-n-4)}{2n+3} \frac{U_n^0}{2^n} - \frac{n(n+1)(n+2)}{2n+3} \frac{V_{n+2}^0}{2^n} \right] = 0$$

Substituting the values of U_n^0 and V_n^0, we get

$$V_4^0 = 4.39930$$

We would accept the value obtained by using the integral as it is likely to have less rounding off errors than the value obtained from the symmetry of the velocity. The latter, however, does show once again that the integral over the region $|z| < n$ as $n \to \infty$ yields the correct value.

For V_4^4 the sum over the cube is 282.5035, and the outer integral over the region $|z| < n$ as $n \to \infty$ is given by

$$I = 16\times 105 \int_0^x dy \int_n^\infty dx \int_0^n \frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2+z^2)^{7/2}} dz$$

$$= 112\left(\frac{5}{2\sqrt{3}} - \frac{\pi}{3} \right)$$

Hence

$$V_4^4 = 282.5035 + 112\left(\frac{5}{2\sqrt{3}} - \frac{\pi}{3} \right)$$

$$= 326.875$$

3.2.8 The flux and the drag force

The flux Q through the cube of Fig. 4(a), in the z-direction is

$$Q = 8 \int_0^{\pi/4} d\phi \int_a^{b/(2\cos\phi)} V_z \Big|_{z=0} r\, dr$$

where V_z is given by equation (10) and,

$$P_n^m(0) = (-1)^{n/2} \frac{(n+m)!}{2^n (\frac{n+m}{2})!\ (\frac{n-m}{2})!} \qquad (26)$$

which can be deduced from

$$T_n^m(0) = (-1)^{n/2} \frac{2^m \, \Gamma(m+\frac{n}{2}+\frac{1}{2})}{\sqrt{\pi} \quad (n/2)!} , \quad n = 0,2,4, \ldots.$$

(Morse and Feshbach (1953)p.783)

$T_n^m(z)$ are the Gegenbauer polynomials related to the associated Legendre polynomials $P_n^m(z)$ by

$$T_n^m(z) = (1-z^2)^{-m/2} \, P_{n+m}^m(z)$$

Using equations (10) and (26), and carrying out the integration over r, we obtain

$$Q = 8a^3 \int_0^{\pi/4} \left\{ D_{1,0} \left[-\frac{b^2}{12a^2\cos^2\phi} + \frac{b}{4a\cos\phi} - \frac{a\cos\phi}{3b} \right] \right.$$

$$+ \sum_{n=1}^{\infty} \sum_{m=-[n/2]}^{[n/2]} D_{2n+1,4m} \frac{(-1)^n (2n+4m+1)!}{2^{2n+1}(n-2m)!(n+2m)!(2n+1)} \times$$

$$\left[-2\frac{(b/2a\cos\phi)^{2n+2}}{(4n+1)(4n+3)} + \frac{(2a\cos\phi/b)^{2n-1}}{4n+1} - \frac{(2a\cos\phi/b)^{2n+1}}{4n+3} \right]$$

$$+ \sum_{n=1}^{\infty} \sum_{m=-[\frac{n-1}{2}]}^{[\frac{n-1}{2}]} A_{2n-1,4m} \frac{(-1)^n (2n+4m-1)!}{2^{2n-1}(n-2m-1)!(n+2m-1)!(n)} \times$$

$$\left. \left[-\frac{(b/2a\cos\phi)^{2n+2}}{2(4n+1)} + \frac{(b/2a\cos\phi)^{2n}}{2(4n-1)} - \frac{(2a\cos\phi/b)^{2n-1}}{(4n-1)(4n+1)} \right] \cos 4m\phi \, d\phi \right\}$$

where $[n/2]$ means the integral part of $n/2$.

We define

$$I_{2n}^{4m} = \int_0^{\pi/4} \frac{\cos 4m\phi}{\cos^{2n}\phi} \, d\phi$$

and

$$J_{2n+1}^{4m} = \int_0^{\pi/4} \cos 4m\phi \cos^{2n+1}\phi \, d\phi$$

The values of I_{2n}^{4m} and J_{2n+1}^{4m} used in these calculations have been calculated by Drummond (1982) and are listed in Table 7.

On inserting the integrals I_{2n}^{4m} and J_{2n+1}^{4m} in the expression for Q and re-arranging the terms, we get

$$- \frac{3Q}{2D_{1,0}ab^2} = 1 - \frac{3a}{b}\ln(\sqrt{2}+1)$$

$$+ \frac{3a}{2b} \sum_{n=1}^{\infty} \sum_{m=-[n/2]}^{[n/2]} \frac{(-1)^n (b/a)^{2n+1} \frac{A_{2n-1,4m}}{D_{1,0}}(2n+4m-1)!\,I_{2n+2}^{4m}}{2^{4n-1} n(4n+1)(n-2m-1)!(n+2m-1)!}$$

$$+ \frac{3a}{2b} \sum_{n=1}^{\infty} \sum_{m=-[n/2]}^{[n/2]} \frac{(-1)^n (b/a)^{2n+1} \frac{D_{2n+1,4m}}{D_{1,0}}(2n+4m+1)!\,I_{2n+2}^{4m}}{2^{4n-1} (2n+1)(4n+1)(4n+3)(n-2m)!(n+2m)!}$$

$$+ \frac{3}{2}(a/b)^3 \left[\frac{8}{3\sqrt{2}} + \sum_{n=1}^{\infty} \sum_{m=-[n/2]}^{[n/2]} \frac{(-1)^{n+1} (b/a)^{2n+1} \frac{A_{2n-1,4m}}{D_{1,0}}(2n+4m-1)!\,I_{2n}^{4m}}{2^{4n-4} 2n(4n-1)(n-2m-1)!(n+2m-1)!} \right]$$

$$- \frac{4}{5}(a/b)^6 \frac{(b/a)^3 A_{1,0}}{D_{1,0}} J_1^0 + \frac{6}{5}(a/b)^6 \frac{(b/a)^3 D_{3,0}}{D_{1,0}} J_1^0 + O(a/b)^8$$

Substituting for A_{nm} and D_{nm} in terms of $D_{1,0}$ from equations (23) and (24) on the right hand side of the above expression, and inserting the value of $D_{1,0}$ from equation (25) in the left hand side, we find that

TABLE 7

Values of I^{4m}_{2n} and J^{4m}_{2n+1} used in the flux calculations

I^{0}_{2}	1						
I^{0}_{4}	$\frac{2\times 2}{3}$						
I^{0}_{6}	$\frac{4\times 7}{3\times 5}$	I^{4}_{6}	$-\frac{4}{5}$				
I^{0}_{8}	$\frac{8\times 12}{5\times 7}$	I^{4}_{8}	$-\frac{8\times 4}{3\times 7}$				
I^{0}_{10}	$\frac{16\times 83}{5\times 7\times 9}$	I^{4}_{10}	$-\frac{16\times 11}{7\times 9}$	I^{8}_{10}	$\frac{16}{9}$		
I^{0}_{12}	$\frac{32\times 146}{7\times 9\times 11}$	I^{4}_{12}	$-\frac{32\times 26}{3\times 5\times 11}$	I^{8}_{12}	$\frac{32\times 6}{5\times 11}$		
I^{0}_{14}	$\frac{64\times 1569}{7\times 9\times 11\times 13}$	I^{4}_{14}	$-\frac{64\times 911}{5\times 9\times 11\times 13}$	I^{8}_{14}	$\frac{64\times 15}{11\times 13}$	I^{12}_{14}	$-\frac{64}{13}$
I^{0}_{16}	$\frac{128\times 2856}{9\times 11\times 13\times 15}$	I^{4}_{16}	$-\frac{128\times 1912}{7\times 11\times 13\times 15}$	I^{8}_{16}	$\frac{128\times 136}{7\times 13\times 15}$	I^{12}_{16}	$-\frac{128\times 8}{7\times 15}$
J^{0}_{1}	$1/\sqrt{2}$						

$$
\begin{aligned}
\frac{6\pi\mu a Q}{Pb^5} = 1 - \frac{3a}{2b}\Bigg[& 2\,\ell n(\sqrt{2}+1) + \sum_{n=1}^{\infty}\sum_{m=-[n/2]}^{[n/2]} \frac{(-1)^n (2n+4m+2)!\, I_{2n+2}^{4m}\, V_{2n+2}^{-4m}}{2^{4n+1}(n+1)(4n+3)(n-2m)!(n+2m)!} \\
& + \sum_{n=1}^{\infty}\sum_{m=-[n/2]}^{[n/2]} \frac{(-1)^n (n+2m+1)(n-2m+1)(2n+4m)!\, I_{2n+2}^{4m} U_{2n}^{-4m}}{2^{4n-1}(n+1)(4n+3)(n-2m)!(n+2m)!} \Bigg] \\
& + (a/b)^3 \Bigg[2\sqrt{2} - \frac{1}{2} I_2^0 U_2^0 + \sum_{n=2}^{\infty} \sum_{m=-[\frac{n-1}{2}]}^{[\frac{n-1}{2}]} \frac{(-1)^{n+1}(2n+4m)!\, I_{2n}^{4m} U_{2n}^{-4m}}{2^{4n-2} n (n-2m-1)!(n+2m-1)!} \Bigg] \\
& + (a/b)^6 \Bigg\{ \frac{U_2^0}{5} \Bigg[8J_1^0 + \sum_{n=1}^{\infty}\sum_{m=-[n/2]}^{[n/2]} \frac{(-1)^n (2n+4m+2)!\, I_{2n+2}^{4m} U_{2n+2}^{-4m}}{2^{4n+1}(n+1)(n-2m)!(n+2m)!} \Bigg] \\
& + \frac{3}{5}(5V_4^0 + 2U_2^0) \Bigg[-3J_1^0 + \sum_{n=1}^{\infty}\sum_{m=-[n/2]}^{[n/2]} (-1)^n I_{2n+2}^{4m} \\
& \frac{\frac{5}{4}(2n+4m+4)!\, V_{2n+4}^{-4m} + (5n^2+12n-20m^2+6)(2n+4m+2)!\, U_{2n+2}^{-4m}}{2^{4n+2}(n+1)(4n+7)(n-2m)!(n+2m)!} \Bigg] \Bigg\} + 0(a/b)^8
\end{aligned}
$$

The various series in the above expression can be summed by inserting the values of U_n^m, V_n^m from Tables 5 and 6, and those of I_{2n}^{4m}, J_1^0 from Table 7, and applying Aitken-Shanks transformation (Greenberg (1978), p.38) to accelerate the convergence of the series. Finally, we get

$$
\frac{6\pi\mu a Q}{Pb^5} = 1-2.8373(a/b)+4.1888(a/b)^3-27.3594(a/b)^6+0(a/b)^8 \qquad (27)
$$

If $U = \dfrac{Q}{b^2}$ is the mean velocity of the fluid, and $F = Pb^3$ is the drag force on a sphere, then from equation (27),

$$F = \frac{6\pi\mu Ua}{1-2.8373(a/b)+4.1888(a/b)^3-27.3594(a/b)^6+O(a/b)^8} \qquad (28)$$

which may also be written as

$$F = \frac{6\pi\mu Ua}{1-1.7601\varepsilon^{1/3}+\varepsilon-1.5593\varepsilon^2+O(\varepsilon^{8/3})} \qquad (29)$$

where $\varepsilon = \frac{4\pi a^3}{3b^3}$ is the volume concentration of the spheres.

3.3 Conclusion

The drag calculations are accurate to order ε^2 and may be used for low values of ε. A comparison of the drag force given by equation (29) with the corresponding result of Happel (1958), Kuwabara (1959), and Hasimoto (1959), as quoted in §3.1, shows that there is a complete agreement with Hasimoto's result.

Drummond (1982) has applied the multipole technique to calculate the drag force to $O(\varepsilon^{10/3})$ for the three cubic arrays (simple, body-centred and face-centred). For the simple cubic array his result agrees with that of Sangani and Acrivos (1981).

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