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PAPER

Non-Hermitian trimers: PT-symmetry versus pseudo-Hermiticity

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Abstract

We study a structure composed of three coupled waveguides with gain and loss, a non-Hermitian trimer. We demonstrate that the mode spectrum can be entirely real if the waveguides are placed in a special order and at certain distances between each other. Such structures generally lack a spatial symmetry, in contrast to parity-time symmetric trimers which are known to feature a real spectrum. We also determine a threshold for wave amplification and analyse the scattering properties of such non-conservative systems embedded into a chain of conservative waveguides.

Introduction

One of the postulates of quantum mechanics reads that all physical observables must be described by real variables and thus a system Hamiltonian must be Hermitian [1, 2]. Hermiticity of the Hamiltonian ensures that the system possesses an entirely real eigenspectrum. Interestingly, analogous operators appear in many different contexts beyond conventional quantum mechanics, including optics, where they can be non-Hermitian. Bender *et al* [3] suggested that there exists a class of non-Hermitian Hamiltonians that can possess a real eigenspectrum, if they are parity-time (PT) invariant. Due to an analogy between the Schrödinger equation in quantum mechanics and the equation for slowly varying mode amplitude in optics this phenomenon can be observed in non-conservative optical systems with mutually balanced gain and loss [4–6]. To achieve the balance between gain and loss, the refractive index of the system should satisfy the relation $n(x) = n^*(-x)$, i.e. the active and passive regions of an optical system should be placed symmetrically with respect to each other. In particular, when speaking about PT-symmetric systems, researchers are interested in two features among others: a real spectrum of a non-Hermitian system and a phase transition between a PT-symmetric phase (all eigenvalues are real) and a broken phase (some of the eigenvalues become complex). Such phase transitions are associated with exceptional points [7] in the parameter space, which also appear under more general conditions in non-PT symmetric systems [8–10]. Moreover, it was shown that PT-symmetry is neither a sufficient nor necessary condition to have a real spectrum [11]. Thus, the concept of pseudo-Hermiticity, a condition for a real spectrum of a non-Hermitian system, was introduced.

The main goal of our study is to reveal new possibilities of using general non-Hermitian systems in comparison with PT-symmetric ones. In the present paper we investigate an array of coupled optical weaveguides with gain and loss, and identify several necessary conditions for the spectrum to be real. We then study in detail the case of three coupled waveguides, comparing the features of a general pseudo-Hermitian (PH) trimer to a PT-symmetric trimer. A PH trimer possesses spatially inhomogeneous gain and dissipation, and generally speaking is not PT-symmetric. A PT-symmetric trimer belongs to the class of PH trimers, but hereinafter when referring to PH timer we will imply that it is not PT invariant. We determine conditions under

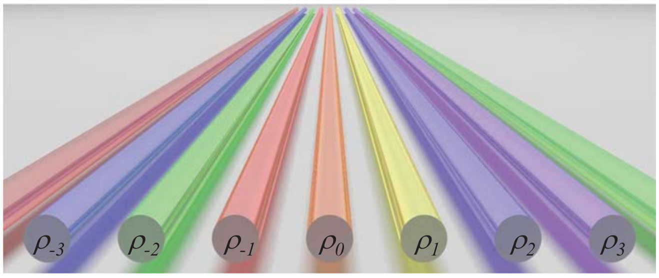

Figure 1. Schematic of an array of coupled waveguides with different gain or dissipation denoted by ρ_j and different colours.

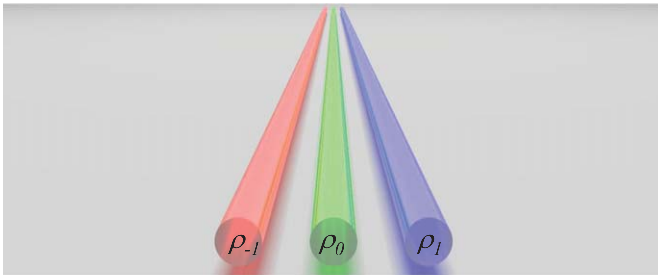

Figure 2. Schematic of a non-Hermitian trimer with gain/loss strength denoted by ρ_{-1}, ρ_0, and ρ_1. $\rho_j > 0$ or $\rho_j < 0$ correspond to loss or gain, respectively.

which PH and PT trimers possess entirely real spectra, and reveal new opportunities of PH structures to flexibly tailor modes' properties, their scattering and amplification.

Chain of non-conservative coupled waveguides

First, we consider the general case of an array of coupled non-conservative optical waveguides, as shown schematically in figure 1. Each waveguide of the chain possesses linear gain or loss, a strength of which is determined by the parameters ρ_j. Waveguides in the chain experience only a nearest-neighbour conservative coupling described by the coefficients C_j. We also assume that all the individual waveguide modes have the same real propagation constants. Light propagation through such a system can be described by the coupled mode equations [4, 5, 12],

$$\mathrm{i}\frac{\mathrm{d}a_j}{\mathrm{d}z} + \mathrm{i}\rho_j a_j + C_j a_{j+1} + C_{j-1}a_{j-1} = 0. \quad (1)$$

Here a_j is the mode amplitude in the jth waveguide, z is the normalised propagation distance. A linear spectrum of the coupled waveguide structure can be determined by seeking solutions of equation (1) in the following form

$$a_j = A_j^{(n)} \exp(\mathrm{i}\beta_n z), \quad (2)$$

where $A_j^{(n)}$ are mode amplitudes corresponding to propagation constant β_n, and n is the eigenmode number.

We now analyse the conditions for the spectrum to be real, i.e. for $\mathrm{Im}(\beta_n) \equiv 0$ for all n. We recall that a trace of any square matrix is equal to the sum of its eigenvalues, and then after representing the eigenmode equation in the matrix form we obtain $\sum_j i\rho_j = \sum_n \beta_n$. If all β_n are real, it follows that

$$\sum_j \rho_j = 0. \quad (3)$$

This balance condition is a necessary one for the spectrum to be real.

We also identify general symmetry properties of the eigenmode spectrum even in absence of PT symmetry. Let $a_j(z)$ be a solution of equation (1). Then it can be shown by simple substitution that $a_j^*(z)(-1)^j$ is also a solution of equation (1). This means that for each propagation constant β_n, $-\beta_n^*$ is also propagation constant. Thus there are two possible cases: (i) $\beta_n = -\beta_n^*$, which leads to $\beta_n = 0$ if we assume that the system has an entirely real spectrum; or (ii) $\beta_n = -\beta_m^*$ for a pair of modes with $n \neq m$. Since the total number of modes is equal to the number of waveguides, it follows that for an odd number of waveguides in the chain there should always exist a mode with zero propagation constant, while all other modes should have a counterpart with an opposite propagation constant. These are the necessary conditions for an entirely real spectrum.

We note that if $\Delta\rho = -\sum_j \rho_j/J \neq 0$, where J is the number of waveguides, it is possible to apply the gauge transformation [12, 13] as follows

$$a_j(z;\{\rho_j\}) = a_j(z;\{\rho_j + \Delta\rho\})\exp(\Delta\rho z). \tag{4}$$

This relation expresses a solution for arbitrary gain/loss (on the left-hand side) through a solution for gain and loss satisfying equation (3). We see that if equation (3) is not satisfied, then spectrum cannot be real, however a more general and practically important situation of all modes having the same spatially averaged gain/loss [13] can be realised when $\mathrm{Im}(\beta_n) \equiv \Delta\rho$ for all n.

General theory of dissipative trimers

We now focus on a particular case with three waveguides in the chain—a non-conservative trimer schematically shown in figure 2. For convenience, we explicitly write down the coupled mode equations according to the general form in equation (1)

$$\begin{aligned}
&\mathrm{i}\frac{\partial a_{-1}}{\partial z} + \mathrm{i}\rho_{-1}a_{-1} + C_{-1}a_0 = 0,\\
&\mathrm{i}\frac{\partial a_0}{\partial z} + \mathrm{i}\rho_0 a_0 + C_{-1}a_{-1} + C_0 a_1 = 0,\\
&\mathrm{i}\frac{\partial a_1}{\partial z} + \mathrm{i}\rho_1 a_1 + C_0 a_0 = 0.
\end{aligned} \tag{5}$$

Since we consider an odd number of waveguides, there must be an eigenmode with zero propagation constant, $\beta = 0$, as a necessary condition for the entire spectrum to be real, as proven in the previous section. The amplitude profile of this mode satisfies

$$\begin{aligned}
&\mathrm{i}\rho_{-1}a_{-1} + C_{-1}a_0 = 0,\\
&\mathrm{i}\rho_0 a_0 + C_{-1}a_{-1} + C_0 a_1 = 0,\\
&\mathrm{i}\rho_1 a_1 + C_0 a_0 = 0.
\end{aligned} \tag{6}$$

This system has a non-trivial solution when the matrix determinant is zero. This provides the following relation for the structure parameters

$$\rho_0 + \frac{C_{-1}^2}{\rho_{-1}} + \frac{C_0^2}{\rho_1} = 0. \tag{7}$$

This condition is necessary (but not sufficient) for the whole spectrum to be real-valued.

Next, we find the eigenmode solutions of equation (5). Substituting anzats (2) into (5) and taking to account equation (7), we find

$$\beta_1 = 0,\ \beta_{2,3} = \pm\sqrt{-\rho_1^2 + C_{-1}^2 + C_0^2 + \rho_{-1}\rho_0}. \tag{8}$$

As expected, one eigenvalue is zero.

We now use condition in equation (3) to express the three loss/gain coefficients through two independent parameters ρ and θ:

$$\rho_{-1} = \rho, \quad \rho_0 = -\theta\rho, \quad \rho_1 = -(1-\theta)\rho. \tag{9}$$

Then we analyse equation (8) and determine that the spectrum of equation (5) is entirely real under the following conditions:

$$C_0 = \sqrt{(1-\theta)(C_{-1}^2 - \theta\rho^2)}, \tag{10}$$

$$|\rho| \leqslant \rho_{\mathrm{crit}} = \sqrt{2-\theta}\,C_{-1}. \tag{11}$$

We notice that since we consider conservative coupling in the present paper, C_{-1} and C_0 are real, then we have an additional restriction on the gain/loss parameter:

$$|\rho| < \rho_{\mathrm{struc}} \equiv C_{-1}/\sqrt{\theta}. \tag{12}$$

From equations (10) and (11) it can be shown that system (5) can have an entirely real spectrum if and only if

$$\theta \leqslant 1. \tag{13}$$

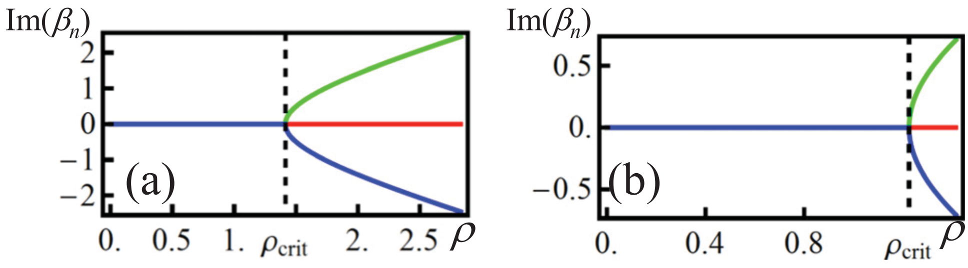

Figure 3. Imaginary part of the propagation constant β_n as a function of gain/loss parameter ρ. Red, green, and blue curves correspond to β_1, β_2, and β_3, respectively. (a) PT-symmetric trimer ($\theta = 0$) for $C_{-1} = 1$, and $C_0 = C_{-1}$; (b) PH trimer ($\theta = 0.5$) for $C_{-1} = 1$, and $C_0(\rho)$ determined by equation (10). Dashed line marks the threshold value of gain/loss parameter ρ_{crit}.

This means that two waveguides of the trimer, which are of the same type (both with gain or loss) should not be separated by a waveguide of other type (loss or gain, respectively). This interesting result stems from the geometric mode symmetry ($\beta_1 = 0$) and specific gain/loss distribution providing energy balance.

Without loss of generality, we consider the case $0 \leqslant \theta < 1$ which corresponds to the right and middle waveguides of the same type and the left waveguide of the opposite type. We will not consider the case $\theta = 1$ as it corresponds to $C_0 = 0$, which reduces the system to an uncoupled dimer and a single waveguide. Note that the particular case of $\theta = 0$ corresponds to the PT-symmetric trimer previously considered in [14–16]. We also note that under the above assumption $\rho_{crit} < \rho_{struc}$.

In what follows, we compare the basic properties of the PT-trimer ($\theta = 0$) with the properties of the PH trimer at $\theta = 0.5$. The latter means that the trimer consists of one lossy waveguide (ρ) and two active waveguides with the same gain ($-\rho/2$, $-\rho/2$).

Under the assumptions formulated above, the relations (8) take the form

$$\beta_1 = 0, \quad (14)$$

$$\beta_{2,3} = \pm\sqrt{(2-\theta)C_{-1}^2 - \rho^2}. \quad (15)$$

It is interesting that the dependence of the propagation constants β_n on the gain/loss strength does not have a qualitative difference for PT-trimers and PH trimers. In figure 3, this dependence is plotted according to equations (8) with C_0 determined by equation (10). Here we should keep in mind that the coupling parameter C_0 does not depend on ρ for the PT-trimer, but it does depend on ρ for the PH timer.

The eigenmode amplitude profiles are

$$A_{-1}^{(1)} = C_{-1}, \quad (16)$$

$$A_0^{(1)} = -\mathrm{i}\rho, \quad (17)$$

$$A_1^{(1)} = -\frac{\sqrt{C_{-1}^2 - \theta\rho^2}}{\sqrt{1-\theta}}, \quad (18)$$

for $\beta_1 = 0$, and

$$A_{-1}^{(2,3)} = C_{-1}, \quad (19)$$

$$A_0^{(2,3)} = -\mathrm{i}\rho \pm \sqrt{(2-\theta)C_{-1}^2 - \rho^2}, \quad (20)$$

$$A_1^{(2,3)} = \sqrt{1-\theta}\,\frac{C_{-1}^2 - \rho^2 \mp \mathrm{i}\rho\sqrt{(2-\theta)C_{-1}^2 - \rho^2}}{\sqrt{C_{-1}^2 - \theta\rho^2}}, \quad (21)$$

for $\beta_{2,3}$.

A relative intensity distribution between the waveguides of the trimer is calculated as $|A_j|^2/(|A_{-1}|^2 + |A_0|^2 + |A_1|^2)$, and it is shown in figure 4. The top row corresponds to the PT-trimer, while the bottom row corresponds to the PH trimer. Figures 4(a) and (d) show how the coupling parameter C_0 changes depending on the gain/loss strength ρ. Panels (b) and (e), and (c) and (f) show the intensity distribution between the waveguides for the propagation constants $\beta_1 = 0$ and $\beta_{2,3}$, respectively. Here red, black, and dashed blue curves represent the light intensity in the left, middle, and right waveguides, respectively. Interestingly, the modes with the zero propagation constant do not depend on the gain/loss strength up to the critical value ρ_{crit}, while for $\beta_{2,3}$ we observe redistribution of the intensity between the waveguides. Note that for the PT-trimer the energy is distributed equally between waveguides with gain and loss, while for the PH-trimer this is not so.

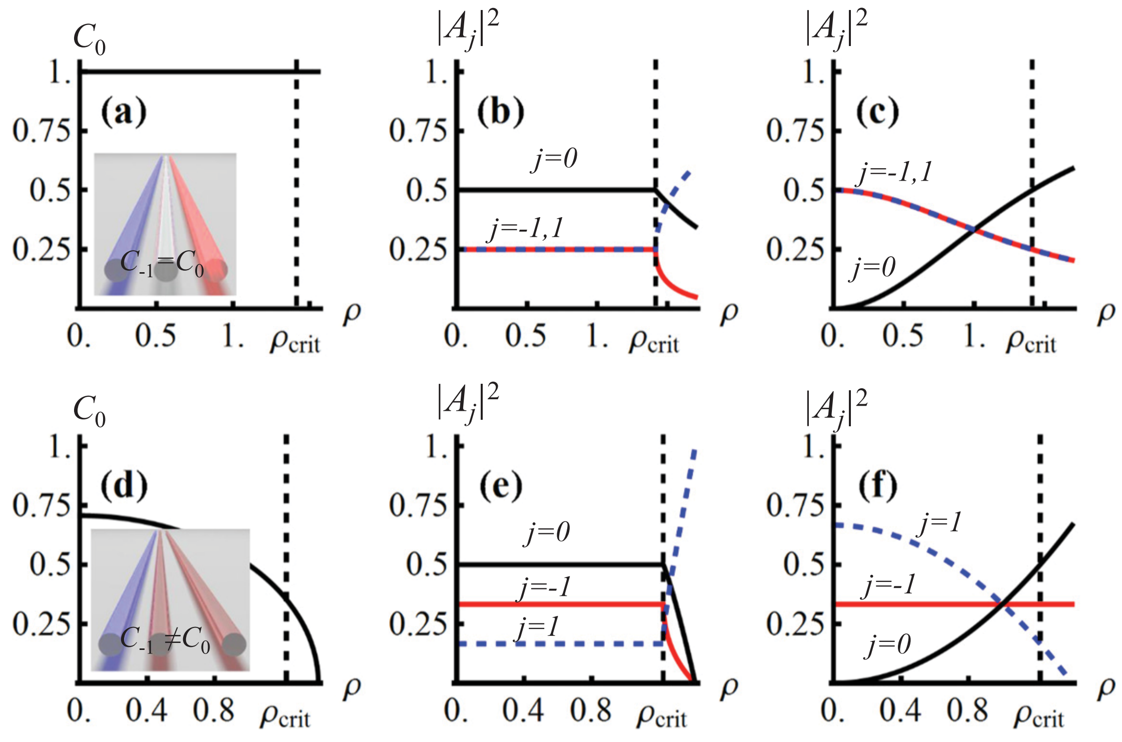

Figure 4. Intensity distribution between the three waveguides depending on the gain/loss strength for the PT trimer ($\theta = 0$)—top row, and the PH trimer ($\theta = 0.5$)—bottom row. Red, black, and dashed blue curves mark the intensity in the left, middle, and right waveguides, respectively. Panels (a) and (d) show the dependence of the coupling parameter C_0; (b) and (e) correspond to the propagation constant β_1; (c) and (f) correspond to β_2 and β_3.

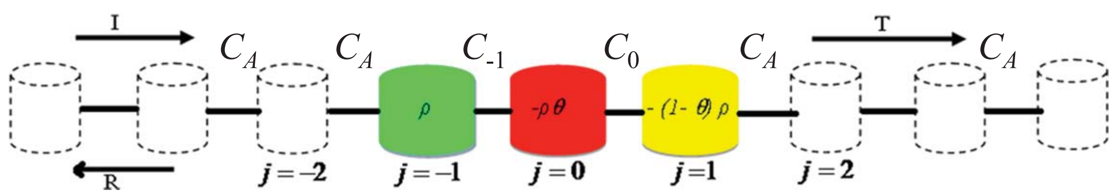

Figure 5. A chain of Hermitian waveguides with an embedded non-Hermitian trimer. Arrows indicate the incident, reflected and transmitted waves in the case when the incident wave comes from the left.

Another observation is that at the critical point ρ_{crit} the propagation constants and full complex amplitude profiles of all three modes coincide.

Thus we reveal that the PH trimer can have an entirely real spectrum as well as a phase transition point denoted as ρ_{crit}, which is usually typical for the PT-symmetric trimer. However, the crucial difference here is that for the PT symmetric trimer the coupling parameter C_0 does not depend on gain/loss parameter ρ, and it is found as $C_0 = C_{-1}$, while for the PH trimer C_0 depends on not only C_{-1}, but on ρ and θ as well. In the next section we investigate the behaviour of a non-Hermitian trimer in a chain of conservative waveguides.

Wave scattering by a dissipative trimer

PT-symmetric elements, incorporated into conservative structures, can demonstrate beneficial effects such as non-reciprocity, signal amplification, suppressed reflection, and invisibility [17–20]. In this section we compare a behaviour of the PT trimer with the PH trimer in terms of light scattering. We consider a non-Hermitian trimer embedded into a long array of Hermitian (conservative) waveguides, and we study wave transmission, reflection, and amplification. Schematic of wave scattering by the defect for the wave propagating from the left is shown in figure 5.

It was previously shown in [21] that when a PT-symmetric system is embedded into a chain of conservative waveguides, the PT symmetry breaking threshold can change. In particular this can lead to spontaneous amplification of modes, even when the isolated system has no growing modes. Therefore, it is important to determine firstly a range of trimer model parameters, when the system does not possess exponentially growing modes (lasing modes). In this regime the PH trimer can be used for active control of propagating signals, i.e. for amplification, filtering, and switching. We note that the balance relation (3) is not a necessary condition for absence of lasing modes in the system due to additional radiation losses through the chain. In what follows, we consider a general case with $\rho_{-1} + \rho_0 + \rho_1 \neq 0$ and introduce the additional gain/loss $\Delta\rho$ for the trimer waveguides. This shifts the spectrum of the isolated trimer by the value $\mathrm{i}\Delta\rho$ according to the gauge transformation (4). When the trimer is embedded into a chain of conservative waveguides, the governing equations take the form

$$\begin{aligned}
&\mathrm{i}\frac{\partial a_j}{\partial z} + C_A a_{j+1} + C_A a_{j-1} = 0, \text{ for } j \neq -1, 0, 1,\\
&\mathrm{i}\frac{\partial a_{-1}}{\partial z} + \mathrm{i}(\rho - \Delta\rho)a_{-1} + C_{-1}a_0 + C_A a_{-2} = 0,\\
&\mathrm{i}\frac{\partial a_0}{\partial z} - \mathrm{i}(\theta\rho + \Delta\rho)a_0 + C_{-1}a_{-1} + C_0 a_1 = 0,\\
&\mathrm{i}\frac{\partial a_1}{\partial z} - \mathrm{i}[(1-\theta)\rho + \Delta\rho]a_1 + C_A a_2 + C_0 a_0 = 0.
\end{aligned} \tag{22}$$

Here C_A is the coupling coefficient between the conservative waveguides.

We seek a solution of equation (22) for wave scattering in the form

$$\begin{aligned}
&a_j = \mathrm{e}^{\mathrm{i}(kj+\beta z)} + R_{\text{left}}\mathrm{e}^{\mathrm{i}(-kj+\beta z)}, \quad j \leqslant -1,\\
&a_j = T\mathrm{e}^{\mathrm{i}(kj+\beta z)}, \quad j \geqslant 1,
\end{aligned} \tag{23}$$

where R_{left} and T are the reflection and transmission coefficients, respectively, k is the wavenumber of an incident wave, and $\beta = 2C_A \cos k$ is the propagation constant far away from the defect. Substituting equations (23) into equation (22), we obtain the following expressions for the scattering coefficients:

$$\begin{aligned}
R_{\text{left}} &= -\mathrm{e}^{-2\mathrm{i}k}\left(1 + \frac{R_1}{D}\right),\\
T &= \frac{2\mathrm{i}\bar{C}_A\bar{C}_0\sin(k)}{D},\\
R_1 &= \bar{C}_A\mathrm{e}^{\mathrm{i}k}(-1 + \mathrm{e}^{2\mathrm{i}k})[\bar{C}_0{}^2 + (2\mathrm{i}\bar{C}_A\cos(k)\\
&\quad + \Delta\rho - \theta\bar{\rho})(\mathrm{i}\bar{C}_1\mathrm{e}^{-\mathrm{i}k} + \delta\rho + (\theta - 1)\bar{\rho})],\\
D &= \mathrm{e}^{\mathrm{i}k}\bar{C}_0{}^2(\bar{C}_1 - \mathrm{i}(\Delta\rho + \bar{\rho})\mathrm{e}^{\mathrm{i}k})\\
&\quad - \mathrm{i}\mathrm{e}^{2\mathrm{i}k}[\mathrm{i}\bar{C}_A\mathrm{e}^{-\mathrm{i}k} + \Delta\rho + (\theta - 1)\bar{\rho}]\\
&\quad \times [1 + (2\mathrm{i}\bar{C}_A\cos(k) + \Delta\rho - \theta\bar{\rho})(\mathrm{i}\bar{C}_A\mathrm{e}^{-\mathrm{i}k} + \Delta\rho + \bar{\rho})],
\end{aligned} \tag{24}$$

where $\bar{\rho} \equiv \rho/C_{-1}$, $\bar{C}_A \equiv C_A/C_{-1}$, $\bar{C}_0 \equiv C_0/C_{-1}$, and $\Delta\bar{\rho} \equiv \Delta\rho/C_{-1}$.

We notice that for the PT trimer (similar to the PT coupler considered in [18]) we can obtain the scattering coefficient for the incident wave approaching the defect from the right by changing the sign of gain/loss strength, i.e. $\rho \to -\rho$ in equations (24). However, for a general case of a non-Hermitian trimer, this is no longer true, due to a broken space symmetry. We determine the scattering coefficient for the incident wave coming from the right as

$$\begin{aligned}
R_{\text{right}} &= \frac{R_2}{D},\\
R_2 &= [\mathrm{i}(\Delta\rho + (\theta - 1)\bar{\rho}) - \bar{C}_A\mathrm{e}^{\mathrm{i}k}]\{1 + \mathrm{e}^{-\mathrm{i}k}[\mathrm{i}\bar{C}_A + \mathrm{e}^{\mathrm{i}k}(\Delta\rho + \bar{\rho})]\\
&\quad \times (2\mathrm{i}\bar{C}_A\cos(k) + \Delta\rho - \theta\bar{\rho})\}\\
&\quad + \mathrm{i}\bar{C}_0{}^2(\mathrm{i}\bar{C}_A\mathrm{e}^{-\mathrm{i}k} + \Delta\rho + \bar{\rho}).
\end{aligned} \tag{25}$$

We emphasise that the transmission coefficients for the right and left propagation of an incident wave coincide due to the reciprocity of transmission for any linear, stationary and non-magnetic medium [19].

We notice that the scattering coefficients have the same denominator D. Let us now fix parameters C_A, C_{-1}, θ, and ρ and consider $\Delta\rho = \Delta\rho_{\min} \equiv \min[\rho, -\rho\theta, -(1-\theta)\rho]$. In this case there is no gain in the system and thus no lasing modes can occur. Next, we gradually increase $\Delta\rho$ until it reaches some critical value $\Delta\rho_{\text{crit}}$, at which denominator D turns to zero for some wavenumber k_{crit}. This means that the wave with the wavenumber k_{crit} is a lasing mode and the energy of the system can grow without any incident light for $\Delta\rho > \Delta\rho_{\text{crit}}$.

In figure 6(a), we plot $\Delta\bar{\rho}_{\text{crit}} \equiv \Delta\rho_{\text{crit}}/C_{-1}$ as a function of $\bar{\rho}$ and θ for $\bar{C}_A = 0.5$. In figure 6(b) we show the corresponding wavenumber k_{crit}. The black dashed line bounds the range of possible structural parameters

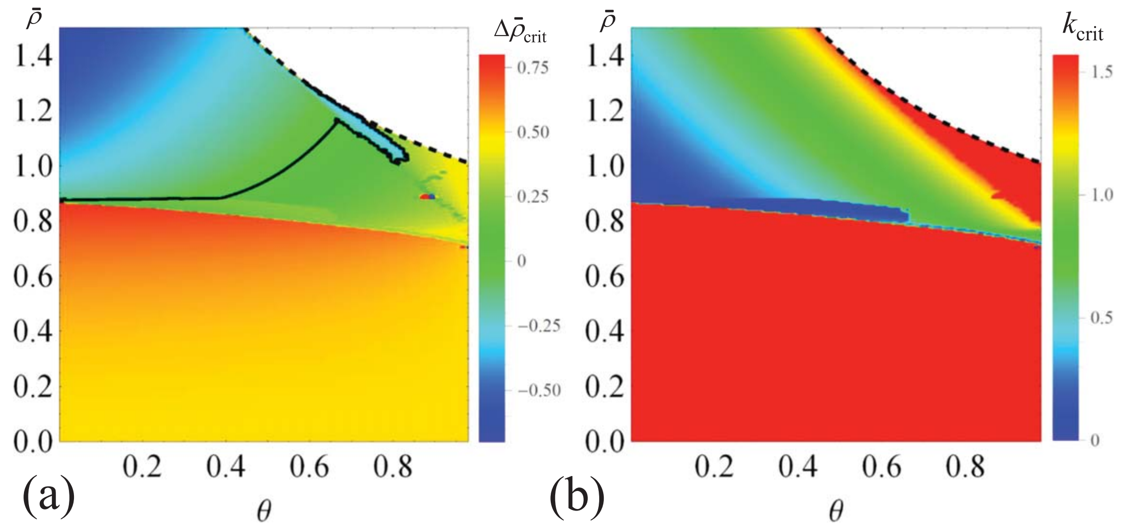

Figure 6. (a) The minimum value of gain/loss imbalance, $\Delta\bar{\rho}_{\rm crit}$ at which a lasing mode occurs in the chain of conservative waveguides with the embedded trimer as a function of $\bar{\rho}$ and θ. (b) A wavenumber of the corresponding lasing mode. Dashed black curve bounds the range of possible structural parameters according to equation (12), and black solid line separates regions where $\Delta\bar{\rho}_{\rm crit} > 0$ and $\Delta\bar{\rho}_{\rm crit} < 0$. Parameter $\bar{C}_A = 0.5$.

according to equation (12) and black solid line separates regions where $\Delta\bar{\rho}_{\rm crit} > 0$ and $\Delta\bar{\rho}_{\rm crit} < 0$. A complex behaviour of $\Delta\bar{\rho}_{\rm crit}$ in the region $\theta > 0.8$ and $\bar{\rho} \approx 0.9$ results from bifurcations of roots of the equation $D = 0$ at particular parameters θ, $\bar{\rho}$ and $\bar{C}_A$.

Remarkably, the value of $\Delta\rho_{\rm crit}$ can be either negative or positive. If $\Delta\rho_{\rm crit} < 0$, then for the range $\Delta\rho_{\rm crit} < \Delta\rho < 0$ the overall gain/loss balance is shifted into loss; however, lasing modes are present in the system. From the other hand, if $\Delta\rho_{\rm crit} > 0$, then for the range $0 < \Delta\rho < \Delta\rho_{\rm crit}$ the overall gain/loss balance is shifted into gain, but no lasing modes are observed.

We simulate numerically the dynamics of input noisy conditions in the chain of conservative waveguides with the PH trimer ($\theta = 0.5$). Although the total number of waveguides in the considered case is 601, we emulate an infinite long chain by introducing perfectly matched layers at the structure boundaries. The noise is introduced only in the trimer waveguides and it is chosen randomly. We simulate light dynamics for several realisations of initial conditions and a representative example for $\bar{C}_A = 0.5$ and $\Delta\rho = 0$ is shown in figure 7. For $\bar{\rho} = 0.85$, which is below the black curve in figure 6(a) and corresponds to $\Delta\rho_{\rm crit} > 0$, we observe that after initial relaxation the total system energy $I = \sum_j |A_j|^2$ is preserved (see figure 7(c)) and the system does not lase. However, if $\bar{\rho} = 1$, then $\Delta\rho_{\rm crit} < 0$ [see figure 6(a)) and the system lases (see figure 7(b)) with growing total energy as shown in figure 7(d).

When operating in the non-lasing regime, we can consider the PH trimer in terms of scattering. Characteristic dependencies for the scattering coefficients are plotted in figures 8(a)–(f) for $\bar{C}_A = 0.5$, $\bar{\rho} = 0.85$. The top and bottom rows represent the cases of the incident wave coming from the left and right, respectively. Panels (a) and (b) are for PT-symmetric trimer ($\theta = 0$), panels (c) and (d)—for the PH trimer ($\theta = 0.5$). For comparison, panels (e) and (f) show the results for the PT-symmetric coupler (coupled gain and loss waveguides (ρ, $-\rho$) with the coupling coefficient C_{-1}, which are embedded into a chain of conservative waveguides with the coupling coefficient C_A), where scattering coefficients are calculated using formulas obtained in [18]. Red and blue curves indicate reflection and transmission coefficients, respectively. We observe that the PH trimer in some cases can be more efficient than the PT-trimer and PT coupler for light amplification. Additionally the resonance position depends not only on gain/loss strength, but also has a non-trivial dependence on the structural parameter θ, and the sign of ρ.

Conclusion and outlook

We have studied the light propagation in complex photonic structures composed of coupled waveguides with arbitrary strength of gain or dissipation. For such structures we identified the mode symmetries which are necessary for the whole spectrum to be real. Next we considered in detail a non-Hermitian trimer and derived the restrictions on the system parameters, which provide an entirely real spectrum. These conditions prescribe how far from each other and in which order these waveguides should be placed to observe the PT-like behaviour in optical systems without PT symmetry. We then identified the threshold conditions for the appearance of

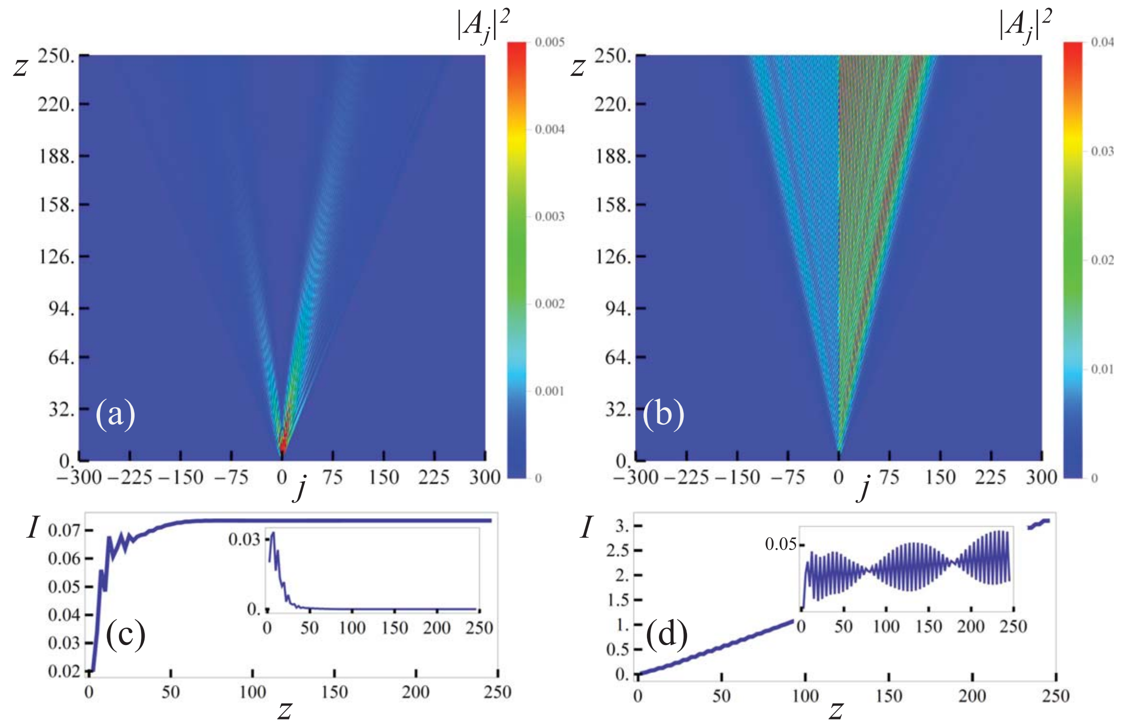

Figure 7. Noise dynamics in the chain of conservative waveguides with the embedded PH trimer for (a) $\bar{\rho} = 0.85$ and (b) $\bar{\rho} = 1$. (c) and (d) evolution the total energy of the system, I, corresponding to (a) and (b), respectively. Insets show evolution of the energy concentrated in the waveguides of the PH trimer only. Parameters are $\theta = 0.5$, $\bar{C}_A = 0.5$, and $\Delta\rho = 0$.

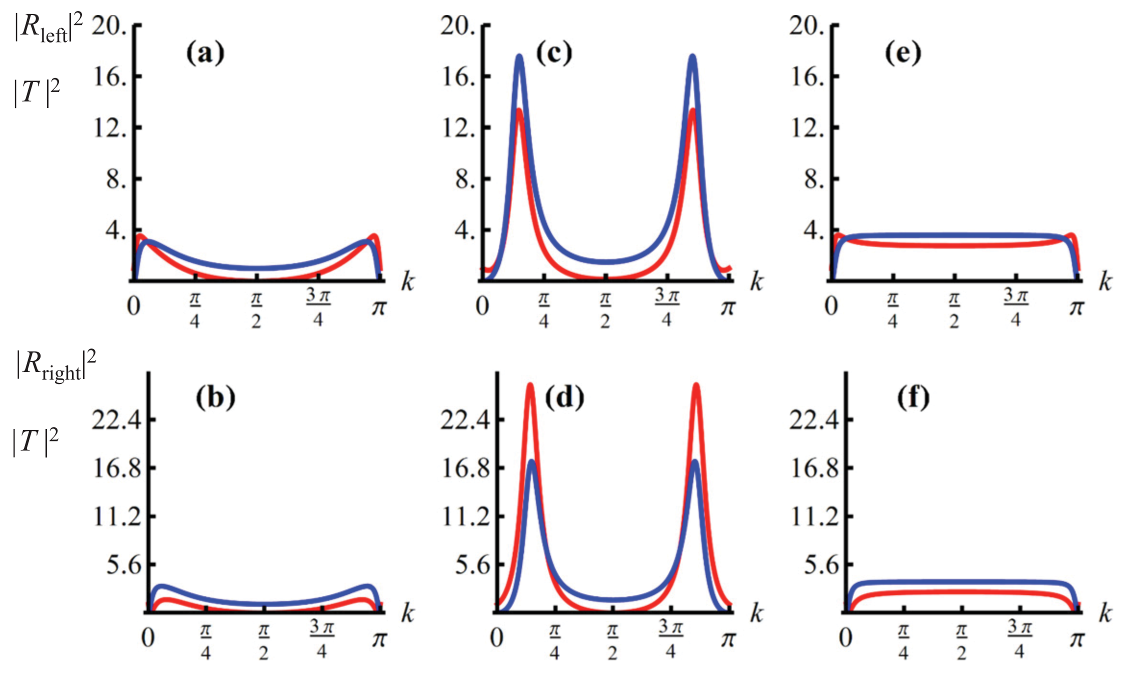

Figure 8. An example of the scattering coefficients for the PT-trimer (left), the PH-trimer (middle) and the PT-coupler (right) for $\bar{C}_A = 0.5$ and $\bar{\rho} = 0.85$. Top—scattering from the left, bottom—from the right. Blue and red curves stand for the transmission and reflection coefficients, respectively.

lasing modes for the PH trimer embedded into a long chain of conservative waveguides. We showed that the PH trimer can be used for light amplification with a higher efficiency than the PT-trimer. Due to the absence of a strict condition on the gain/loss distribution between three waveguides, in contrast to the PT-trimer, PH trimer provides more flexible control of light propagation in the system. Thereby our study reveals new possibilities for using non-Hermitian structures and the presented results suggest practical realisations of PH waveguide arrays. We anticipate that our work can also stimulate following studies to uncover the potential of PH structures for tailoring nonlinear interactions for all-optical applications.

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