Interconversion of Prony series for relaxation and creep

R. J. Loy, F. R. de Hoog, and R. S. Anderssen

Citation: Journal of Rheology **59**, 1261 (2015); doi: 10.1122/1.4929398
View online: http://dx.doi.org/10.1122/1.4929398
View Table of Contents: http://scitation.aip.org/content/sor/journal/jor2/59/5?ver=pdfcov
Published by the The Society of Rheology

Interconversion of Prony series for relaxation and creep

R. J. Loy[a)]

Mathematical Sciences Institute, The Australian National University, John Dedman Building 27, Union Lane, Canberra ACT 2601, Australia

F. R. de Hoog[b)] and R. S. Anderssen[c)]

CSIRO Digital Productivity, GPO Box 664, Canberra ACT 2601, Australia

(Received 13 May 2015; final revision received 11 August 2015; published 26 August 2015)

Abstract

Various algorithms have been proposed to solve the interconversion equation of linear viscoelasticity when Prony series are used for the relaxation and creep moduli, $G(t)$ and $J(t)$. With respect to a Prony series for $G(t)$, the key step in recovering the corresponding Prony series for $J(t)$ is the determination of the coefficients $\{j_k\}$ of terms in $J(t)$. Here, the need to solve a poorly conditioned matrix equation for the $\{j_k\}$ is circumvented by deriving elementary and easily evaluated analytic formulae for the $\{j_k\}$ in terms of the derivative $\mathrm{d}\widehat{G}(s)/\mathrm{d}s$ of the Laplace transform $\widehat{G}(s)$ of $G(t)$. © *2015 The Society of Rheology*. [http://dx.doi.org/10.1122/1.4929398]

I. INTRODUCTION

Having obtained experimentally a Prony series representation for the relaxation modulus $G(t)$ of the viscoelastic material being examined, the next step for the practical rheologist is the determination of the creep modulus $J(t)$. The need for knowing both the moduli is discussed in Plazek and Echeverria (2000). Determination of $J(t)$ is achieved by solving the interconversion equation

$$(G * J)(t) = \int_0^t G(t-\tau)J(\tau)\mathrm{d}\tau = (J * G)(t) = t. \tag{1.1}$$

This relation has, of course, been known for some time, see, for example, Gross (1953, p.47), Hopkins and Hamming (1957), Ferry (1980, Chapter 3.E), Giesekus (1994, Sec. 10.10), Anderssen *et al*. (2008b), and Loy and Anderssen (2014).

Traditionally, Prony series representations for $G(t)$ are written

$$G_N(t) = g_0 + \sum_{k=1}^{N} g_k \exp\left(-\frac{t}{\tau_k}\right), \quad g_k \geq 0, \tau_k > 0. \tag{1.2}$$

[a)]Author to whom correspondence should be addressed; electronic mail: rick.loy@anu.edu.au
[b)]Electronic mail: frank.dehoog@csiro.au
[c)]Electronic mail: bob.anderssen@csiro.au

For viscoelastic solids $g_0 > 0$, and the corresponding form (see Sec. II) for $J(t)$ is

$$J_N(t) = j_0 - \sum_{k=1}^{N} j_k \exp\left(-\frac{t}{\lambda_k}\right), \quad j_k \geq 0, \lambda_k > 0. \tag{1.3}$$

The exponents $\{\tau_k\}$ and $\{\lambda_k\}$, $k = 1, 2, \cdots, N$, correspond, respectively, to the relaxation and retardation times, and satisfy

$$0 < \tau_1 < \tau_2 < .... < \tau_N, \quad 0 < \lambda_1 < \lambda_2 < .... < \lambda_N.$$

For viscoelastic liquids $g_0 = 0$, and Eq. (1.3) is replaced by

$$J_N(t) = j_0 t + j_1 - \sum_{k=2}^{N} j_k \exp(-t/\lambda_k). \tag{1.4}$$

See Baumgaertel and Winter (1989) and Mead (1994).

Historically, there are many approaches to solving Eq. (1.1) for either of $J(t)$ or $G(t)$, when the other is given. For example, see Hopkins and Hamming (1957), Knoff and Hopkins (1972), Dooling (1997), Park and Schapery (1999), Nikonov *et al.* (2005), Sorvari and Malinen (2007b), and Luk-Cyr *et al.* (2013).

Such methods apply to more general forms of $G(t)$, not just the Prony series (1.2), and often have a requirement to solve a poorly conditioned matrix equation. To exploit the special form of Prony series, the usual approach is to use Laplace transform techniques.

Recall that the Laplace transform of a bounded or integrable function f on the half-line $[0, \infty)$ is given by

$$\widehat{f}(s) = \int_0^\infty f(t) \exp(-st) \mathrm{d}t \quad (s > 0).$$

It is well known that $\widehat{f}(s)$ extends to a function analytic on some right half plane $\Re(s) > a \geq 0$, and that $f \mapsto \widehat{f}$ maps the convolution product of two functions to the pointwise product of their individual transforms [Widder (1972)]. Consequently, taking the Laplace transform of Eq. (1.1) yields

$$\widehat{G}(s) \cdot \widehat{J}(s) = s^{-2} \quad (s > 0). \tag{1.5}$$

Now for $\alpha \geq 0$, the Laplace transform of $\exp(-\alpha t)$ is $(s + \alpha)^{-1}$, and so for the Prony series (1.2), (1.3), and (1.4), Eq. (1.5) is an algebraic equation, with $\widehat{G_N}(s)$ and $\widehat{J_N}(s)$ rational functions of s. Different procedures have been proposed for determining $\widehat{J_N}(s)$ given $\widehat{G_N}(s)$ in this case. Contributions have been published by Gross (1953), Taylor (1973), Baumgaertel and Winter (1989), Mead (1994), Tschoegl and Emri (1992), and Sorvari and Malinen (2007a).

All of these latter approaches assume, to one extent or another, that given $G_N(t)$ as Eq. (1.2), necessarily $J_N(t)$ is as in Eqs. (1.3) and (1.4), both in overall form and in the number of summands. Our purpose here is to give a completely self-contained and transparent account of getting from $G_N(t)$ to $J_N(t)$ (and conversely) without these assumptions. In particular, simple easily evaluated formulae for the coefficients are given in Table I below. The numerical aspects are reduced to finding the zeros of certain polynomials, for which there is a plethora of sources, for example Traub (1982).

The structure of the paper is as follows. Section II gives some background and basic properties of the rational functions $\widehat{G_N}(s)$ and $\widehat{J_N}(s)$, Sec. III derives the simple and easily

TABLE I. The formulae.

Solid: $g_0 > 0$

$$G_N(t) = g_0 + \sum_{k=1}^{N} g_k \exp(-\tfrac{t}{\tau_k}) \quad J_N(t) = j_0 - \sum_{k=1}^{N} j_k \exp(-\tfrac{t}{\lambda_k})$$

$$G_N(t) \mapsto J_N(t)$$

$$j_k = \left(\sum_{i=0}^{N} \frac{g_i \beta_k^2}{(\alpha_i - \beta_k)^2} \right)^{-1} \quad (1 \le k \le N), \quad j_0 = g_0^{-1}$$

$$J_N(t) \mapsto G_N(t)$$

$$g_k = \left(\sum_{i=0}^{N} \frac{j_i \alpha_k^2}{(\beta_i - \alpha_k)^2} \right)^{-1} \quad (1 \le k \le N), \quad g_0 = j_0^{-1}$$

Liquid: $g_0 = 0$

$$G_N(t) = \sum_{k=1}^{N} g_k \exp(-\tfrac{t}{\tau_k}) \quad J_N(t) = j_0 t + j_1 - \sum_{k=2}^{N} j_k \exp(-t/\lambda_k)$$

$$G_N(t) \mapsto J_N(t)$$

$$j_k = \left(\sum_{i=0}^{N} \frac{g_i \beta_k^2}{(\alpha_i - \beta_k)^2} \right)^{-1} (k \ge 2), \; j_0 = \left(\sum_{k=1}^{N} g_k \alpha_k^{-1} \right)^{-1},$$

$$j_1 = \left(\sum_{i=1}^{N} \frac{g_i}{\alpha_i} \right)^{-2} \left(\sum_{i=1}^{N} \frac{g_i}{\alpha_i^2} \right)$$

$$J_N(t) \mapsto G_N(t)$$

$$g_k = \left(\frac{2j_0}{\alpha_k} - j_1 + \sum_{i=2}^{N} \frac{j_i \alpha_k^2}{(\beta_i - \alpha_k)^2} \right)^{-1} \quad (1 \le k \le N)$$

evaluated formulae, Sec. IV discusses the determination of the zeros of $\widehat{G_N}(s)$ and $\widehat{J_N}(s)$, Sec. V gives a comparison of the new formulae with the results published by Park and Schapery (1999) and Sorvari and Malinen (2007b), and Sec. VI, a brief comparison with some other approaches.

II. BACKGROUND

The fact that, given $G_N(t)$ [resp. $J_N(t)$], then $J_N(t)$ [resp. $G_N(t)$] must have the precise forms (1.2), (1.3), and (1.4) above follows from the work of Whittaker (1918, Theorem 2).

Indeed, it seems that Whittaker (1918), using a resolvent kernel argument, was the first to show how Eq. (1.2) gives rise to Eqs. (1.3) and (1.4), and conversely. His argument, when applied to $\mathrm{d}G_N(t)/\mathrm{d}t$ and $J_N(t)$, shows in particular that the number of exponential terms in each is the same. Crucially, the exponents for $J_N(t)$ are the (reciprocals of) zeros of the rational function $s\widehat{G_N}(s)$, and 0 is one such zero in the case that $g_0 = 0$, which yields the $j_0 t$ term in Eq. (1.4). In addition, Whittaker (1918) observed the "interlacing" properties of the exponents for $\mathrm{d}G_N(t)/\mathrm{d}t$ and $J_N(t)$. In fact, when $g_0 \ge 0$, both $\widehat{G_N}(s)$ and $\widehat{J_N}(s)$ have N poles, and so the interlacing of their exponents takes the form

$$\tau_1 < \lambda_1 < \tau_2 < \cdots < \tau_k < \lambda_k < \tau_{k+1} < \cdots < \lambda_{N-1} < \tau_N < \lambda_N. \tag{2.1}$$

Whereas, when $g_0 = 0$, $\widehat{J_N}(s)$ has $N-1$ nonzero poles, $\widehat{G_N}(s)$ has N, and hence the ordering of the interlacing becomes

$$\tau_1 < \lambda_1 < \cdots < \tau_k < \lambda_k < \tau_{k+1} < \cdots < \lambda_{N-1} < \tau_N. \tag{2.2}$$

The first explicit consideration of determining $J_N(t)$ from $G_N(t)$ available is Gross (1953), who gives the exponents for $J_N(t)$, given $G_N(t)$, and formulae for the coefficients of $J_N(t)$ involving the derivative of $\widehat{G_N}(s)$. The arguments use his somewhat unorthodox calculus of delta functions [Gross and Peltzer (1951); Gross (1987)]. Baumgaertel and Winter (1989) state without proof several formulae which agree with those of Whittaker (1918), though their approach is to use the Laplace transform as we do here. Their source for the interlacing is the lecture notes of Giesekus. These latter are not readily available, though the work of Giesekus (1994) contains interlacing results.

Mead (1994), considering the liquid case only, states the derivative formulae and attributes them to Gross (1953). He gives an argument for interlacing, assuming the same number of exponents without comment.

We have, Eq. (1.5), the "algebraic" form of the interconversion equation

$$\widehat{G_N}(s) \cdot \widehat{J_N}(s) = s^{-2} \quad (s > 0). \tag{2.3}$$

For the specific form (1.2) of $G_N(t)$, it follows that

$$\widehat{G_N}(s) = \frac{g_0}{s} + \sum_{k=1}^{N} \frac{g_k}{s+\alpha_k} = \sum_{k=0}^{N} \frac{g_k}{s+\alpha_k}, \tag{2.4}$$

where $\alpha_k = \tau_k^{-1}$, $\alpha_0 = 0$. This is a rational function on the complex plane with simple poles at $-\alpha_0, \ldots, -\alpha_N$. It follows from Eq. (2.3) that $\widehat{J_N}(s)$ is also a rational function. Furthermore, the zeros of $\widehat{J_N}(s)$ occur at the poles of $\widehat{G_N}(s)$ and are simple. Again, the poles of $\widehat{J_N}(s)$, other than 0, occur at the zeros $\{-\beta_1, \ldots, -\beta_N\}$ of $\widehat{G_N}(s)$, and these are simple zeros, since between the poles of $\widehat{G_N}(s)$

$$\frac{\mathrm{d}\widehat{G_N}(s)}{\mathrm{d}s} = -\sum_{k=0}^{N} \frac{g_k}{(s+\alpha_k)^2} < 0.$$

For a viscoelastic solid, $g_0 > 0$. Letting $s \to 0$ in Eq. (2.3), it follows that $\widehat{J_N}(s)$ has the form $s^{-1}Q(s)$ for a rational function $Q(s)$ with $Q(0) = g_0^{-1}$. For a liquid, with $g_0 = 0$, we have instead that $\widehat{J_N}(s) = s^{-2}Q(s)$ for a rational function $Q(s)$ with $Q(0) = (\sum_{k=1}^{N} g_k \alpha_k^{-1})^{-1}$.

For a known $G_N(t)$, and, hence, the corresponding defining coefficients $\{g_k\}$ and $\{\alpha_k\}$, there are two sets of unknowns $\{j_k\}$ and $\{\beta_k\}$ to be estimated for the determination of $J_N(t)$ from Eqs. (2.3) and (2.4). The direct solution of these equations could involve solving a highly nonlinear system of algebraic equations. Traditionally, the $\{j_k\}$ and the $\{\beta_k\}$ have often been estimated jointly, Nikonov *et al.* (2005), Luk-Cyr *et al.* (2013), and Sorvari and Malinen (2007a, b). This is the reason why such estimation strategies involve matrix inversion or the solution of nonlinear algebraic equations. However, we give a simple procedure to determine the coefficients $\{j_k\}$ of the constant and exponential terms in $J_N(t)$ from the values of $\{g_k\}$, $\{\alpha_k\}$, and $\{\beta_k\}$. Determining the zeros $\{-\beta_k\}$ of $\widehat{G_N}(s)$ is facilitated by the interlacing Eqs. (2.1) and (2.2).

III. NEW ELEMENTARY AND EASILY EVALUATED FORMULA

This approach is different from those previously published in that it directly exploits the deeper algebraic structure within and between $\widehat{J_N}(s)$ and $\widehat{G_N}(s)$. Recall that $\widehat{G_N}(s)$ has

simple poles at $-\alpha_1, \ldots, -\alpha_N$, together with 0 in the case of a viscoelastic solid. It also has simple zeros at $-\beta_1, \ldots -\beta_N$.

Since

$$\widehat{J_N}(s) = \frac{1}{s^2 \widehat{G_N}(s)},$$

the rational function $\widehat{J_N}(s)$ is bounded at infinity, with simple poles at the zeros of $\widehat{G_N}(s)$ together with a double pole at 0 for a viscoelastic liquid, and a simple pole at 0 for a solid. Standard residue calculus [Marsden (1973, Theorem 4.1.1 and Table 4.1)] shows that

$$\widehat{J_N}(s) = \frac{j_0}{s^2} + \frac{j_1}{s} - \sum_{k=\ell}^{N} \frac{j_k}{s+\beta_k}, \tag{3.1}$$

with for $\ell = 2$ in the liquid case, $\ell = 1$ and $j_0 = 0$ in the solid case, where

$$j_k = -\left(\beta_k^2 \frac{\mathrm{d}\widehat{G_N}}{\mathrm{d}s}(-\beta_k)\right)^{-1} \quad (k \geq \ell). \tag{3.2}$$

An identical argument shows that

$$g_k = \left(\alpha_k^2 \frac{\mathrm{d}\widehat{J_N}}{\mathrm{d}s}(-\alpha_k)\right)^{-1} \quad (k \geq 1). \tag{3.3}$$

Note that this latter holds whether the viscoelastic material is solid ($g_0 > 0$) or liquid ($g_0 = 0$).

A. Linear viscoelastic solids

Substituting the expressions (3.1) and (2.4) for $\widehat{J_N}(s)$ and $\widehat{G_N}(s)$ into Eqs. (3.2) and (3.3) gives, for a viscoelastic solid,

$$\begin{aligned} j_k &= \left(\sum_{i=0}^{N} \frac{g_i \beta_k^2}{(\alpha_i - \beta_k)^2}\right)^{-1} \quad (1 \leq k \leq N) \\ g_k &= \left(\sum_{i=0}^{N} \frac{j_i \alpha_k^2}{(\beta_i - \alpha_k)^2}\right)^{-1} \quad (1 \leq k \leq N), \end{aligned} \tag{3.4}$$

together with $g_0 j_0 = 1$.

B. Linear viscoelastic liquids

For a viscoelastic fluid, Eq. (3.1) holds, and $g_0 = 0$. For $k \geq 2$, the same arguments hold as for the solid case. Thus, as before, from Eqs. (3.1) and (3.2),

$$\begin{aligned} j_k &= \left(\sum_{i=0}^{N} \frac{g_i \beta_k^2}{(\alpha_i - \beta_k)^2}\right)^{-1} \quad (2 \leq k \leq N) \\ g_k &= \left(\frac{2j_0}{\alpha_k} - j_1 + \sum_{i=2}^{N} \frac{j_i \alpha_k^2}{(\beta_i - \alpha_k)^2}\right)^{-1} \quad (1 \leq k \leq N). \end{aligned} \tag{3.5}$$

In Sec. II, we determined

$$j_0 = \left(\sum_{k=1}^{N} g_k \alpha_k^{-1} \right)^{-1}.$$

For j_1, this is the residue of $(s^2 \widehat{G_N}(s))^{-1}$ at 0, and so

$$j_1 = \frac{\mathrm{d}}{\mathrm{d}s} \left(\widehat{G_N}(s)^{-1} \right) \Big|_{s=0} = -\frac{\widehat{G_N}'(0)}{\widehat{G_N}(0)^2} = \left(\sum_{i=1}^{N} \frac{g_i}{\alpha_i} \right)^{-2} \left(\sum_{i=1}^{N} \frac{g_i}{\alpha_i^2} \right).$$

C. Liquid case as limit of solid cases as $g_0 \to 0$

The formulae above look distinctly different for the solid and liquid cases. However, one can think of a liquid as the limiting case of a solid as $g_0 \to 0$. In fact, by considering the zero $-\beta_N$ of $\widehat{G_N}(s)$ closest to the origin, we can recover the liquid behavior for small s as $g_0 \to 0$.

From Eq. (3.4),

$$j_N = \left(g_0 + \beta_N^2 \sum_{i=1}^{N} \frac{g_i}{(\alpha_i - \beta_N)^2} \right)^{-1}.$$

Here, $-\beta_N$ is the closest zero of $\widehat{G_N}$ to the origin, so that from Eq. (2.4)

$$\beta_N = g_0 \left(\sum_{i=1}^{N} \frac{g_i}{\alpha_i} \right)^{-1} + O(g_0^2).$$

These give

$$j_N = g_0^{-1} \left(1 - g_0^{-1} \beta_N^2 \sum_{i=1}^{N} \frac{g_i}{\alpha_i^2} + O(g_0^2) \right).$$

However, for fixed $s \neq 0$, and for $g_0 \neq 0$ small, β_N is small, whence

$$(s + \beta_N)^{-1} = s^{-1}(1 - \beta_N s^{-1}) + O(g_0^2).$$

It follows that

$$\begin{aligned} \frac{j_0}{s} - \frac{j_N}{s + \beta_N} &= \frac{1}{g_0 s} \left(1 - \left(1 - \frac{\beta_N^2}{g_0} \sum_{i=1}^{N} \frac{g_i}{\alpha_i^2} + O(g_0^2) \right) \right) \left(1 - \frac{\beta_N}{s} + O(g_0^2) \right) \\ &= \left(g_0^{-2} \beta_N^2 s^{-1} \sum_{i=1}^{N} \frac{g_i}{\alpha_i^2} \right) + g_0^{-1} \beta_N s^{-2} + O(g_0) \\ &= \left(\sum_{i=1}^{N} \frac{g_i}{\alpha_i} \right)^{-2} \left(\sum_{i=1}^{N} \frac{g_i}{\alpha_i^2} \right) s^{-1} + \left(\sum_{i=1}^{N} \frac{g_i}{\alpha_i} \right)^{-1} s^{-2} + O(g_0), \end{aligned}$$

in agreement with the coefficients found earlier.

IV. NUMERICAL ASPECTS OF DETERMINING THE ZEROS

In this new approach, the formulae are exact except for the determination of the zeros of $\widehat{G_N}(s)$ or $\widehat{J_N}(s)$. Since the formula for $\widehat{G_N}(s)$ represents the starting point, the determination of the $\{\beta_k\}$ is simplified by exploiting the interlacing property. In particular, for $\widehat{G_N}(s)$, we have, on an interval $(-\alpha_i, -\alpha_{i+1})$,

$$\frac{\mathrm{d}\widehat{G_N}(s)}{\mathrm{d}s} = -\sum_{k=0}^{N} \frac{g_k}{(s+\alpha_k)^2} < -\frac{g_i}{(\alpha_i - \alpha_{i+1})^2}. \tag{4.1}$$

So the gradient is bounded away from zero, and consequently, numerical methods to determine the zero will converge geometrically.

On the other hand, for $\widehat{J_N}(s)$,

$$\frac{\mathrm{d}\widehat{J_N}(s)}{\mathrm{d}s} = \begin{cases} -\dfrac{2j_0}{s^3} - \dfrac{j_1}{s^2} + \displaystyle\sum_{k=2}^{N} \frac{j_k}{(s+\beta_k)^2} & (g_0 = 0) \\ -\dfrac{j_0}{s^2} + \displaystyle\sum_{k=1}^{N} \frac{j_k}{(s+\beta_k)^2} & (g_0 \neq 0). \end{cases} \tag{4.2}$$

Here, $(\mathrm{d}\widehat{J_N}(s))/\mathrm{d}s$ will change sign twice in the interval $(-\beta_N, 0)$ in the case $g_0 = 0$, and once in the case $g_0 \neq 0$. Furthermore, in both cases, sign changes may occur in other intervals. Thus, the derivative is certainly not bounded away from zero, and so accurately determining the zeros can be expected to be much more difficult. This is a reflection of the result of Anderssen *et al*. (2008a), and also Taylor (1973).

V. COMPARISON OF NEW METHOD WITH PUBLISHED RESULTS

The elementary structure of the formulae in Table I is immediately apparent. Alternative formulae for the $\{j_k\}$ (but not the $\{g_k\}$) are given for both solid and liquid cases in Eqs. (18)–(22), (A2), and (A3) of Baumgaertel and Winter (1989). However, they involve more complicated usage of the $\{\beta_k\}$. In particular, note that, for a liquid, j_0 and j_1 are evaluated above (Sec. III B) in terms of only the $\{g_k\}$ and $\{\alpha_k\}$, which are explicitly known from $G_N(t)$.

Sorvari and Malinen (2007a) checked the accuracy of their direct discretization of the interconversion equation on a test problem using a $G_2(t)$ Prony relaxation modulus with its five parameters listed in Table II. Their algorithm returned, for the corresponding $J_2(t)$, the five parameters as listed under SM. The graphical relationship between $G_2(t)$ and $J_2(t)$ is shown in Fig. 1.

For the evaluation of the new formula (3.2), the retardation times λ_1 and λ_2 were determined using a bisection method to find the zeros of $\widehat{G_2}(s)$. Because of the global

TABLE II. Sorvari and Malinen (SM) versus new algorithm (LdHA).

i	g_i	τ_i	j_i SM	j_i LdHA	λ_i SM	λ_i LdHA
0	1	—	1.00	1.00	—	—
1	0.6	5	0.340	0.33939	29.135	29.1355
2	0.4	20	0.160	0.160198	6.864	6.86448

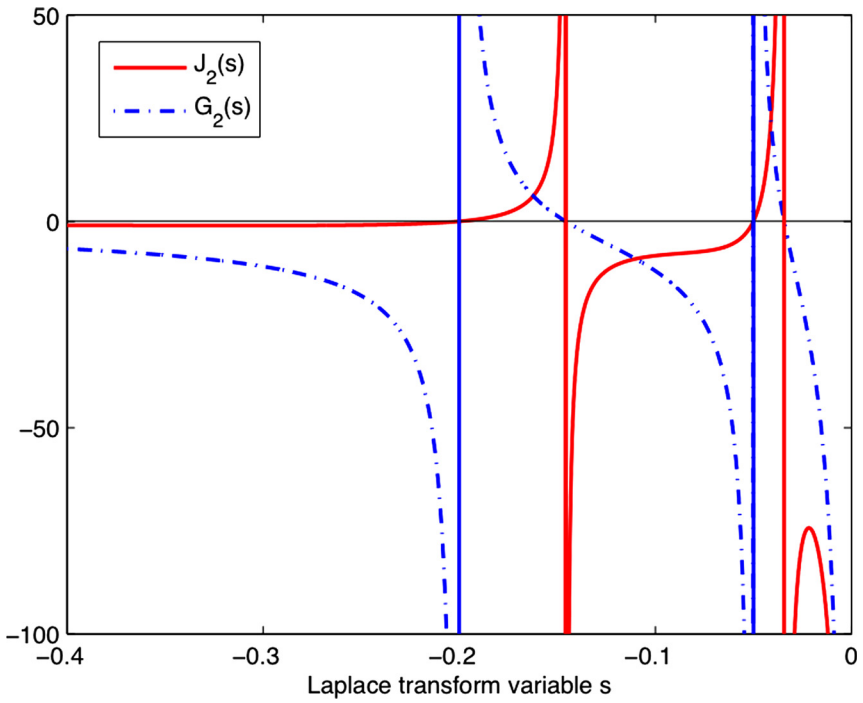

FIG. 1. The interlacing of the poles and zeros of $\widehat{G_2}(s)$ (dotted) and $\widehat{J_2}(s)$ (solid) for the Sorvari and Malinen test case.

monotone structure of the curves forming $\widehat{G_2}(s)$, this simple method converged rapidly. The identity $g_0 j_0 = 1$ was next used to determine j_0. The resulting estimates for λ_1 and λ_2 were then substituted in Eq. (3.4) to determine j_1 and j_2. These values are also listed in Table II under LdHA.

The larger example, for which there are 11 terms in the Prony series, was due to Park and Schapery (1999), and was also considered in Sorvari and Malinen (2007a). The same procedure as above was used with the results given in Table III.

TABLE III. Park and Schapery (PS) versus new algorithm (LdHA).

i	g_i	τ_i	j_i PS(5)	j_i LdHA	λ_i PS	λ_i LdHA
0	2.94×10^{7}	—	4.56×10^{-8}	4.1525×10^{-8}	—	—
1	1.94×10^{9}	2×10^{-2}	4.08×10^{-12}	4.0675×10^{-12}	2.19×10^{-2}	2.1859×10^{-2}
2	2.83×10^{9}	2×10^{-1}	7.37×10^{-12}	7.3049×10^{-12}	2.34×10^{-1}	2.3105×10^{-1}
3	5.54×10^{9}	2×10^{0}	2.25×10^{-11}	2.2456×10^{-11}	2.88×10^{0}	2.8576×10^{0}
4	6.02×10^{9}	2×10^{1}	6.40×10^{-11}	6.4084×10^{-11}	3.80×10^{1}	3.8017×10^{1}
5	3.88×10^{9}	2×10^{2}	2.03×10^{-10}	1.9937×10^{-10}	5.25×10^{2}	5.1445×10^{2}
6	1.56×10^{9}	2×10^{3}	6.86×10^{-10}	6.9284×10^{-10}	6.61×10^{3}	6.5454×10^{3}
7	4.10×10^{8}	2×10^{4}	2.19×10^{-9}	2.1175×10^{-9}	6.03×10^{4}	5.9866×10^{4}
8	1.38×10^{8}	2×10^{5}	6.50×10^{-9}	6.3124×10^{-9}	5.89×10^{5}	5.8858×10^{5}
9	3.68×10^{7}	2×10^{6}	1.37×10^{-8}	1.2293×10^{-8}	4.27×10^{6}	4.2598×10^{6}
10	7.90×10^{6}	2×10^{7}	6.93×10^{-9}	6.8231×10^{-9}	2.57×10^{7}	2.5504×10^{7}
11	9.60×10^{6}	2×10^{8}	1.45×10^{-8}	1.2946×10^{-8}	2.95×10^{8}	2.8963×10^{8}

VI. IMPLEMENTATION

The starting point for the determination of the $\{j_k\}$ is the determination of a Prony series representation (1.2) from a stress strain-rate or oscillatory shear experiment, from which the values of N, $\{g_k\}$ and $\{\alpha_k\}$ are immediately available.

As highlighted in Table I, the $\{j_k\}$ are defined by simple and easily evaluated analytic formulae which depend only on the derivative $\mathrm{d}\widehat{G_N}(s)/\mathrm{d}s$, and the available estimates for the $\{-\beta_k\}$ as the zeros of $\widehat{G_N}(t)$ which, as explained above in Sec. IV, can be easily determined using bisection.

Its great advantage, which characterizes its simplicity, is that it represents a directly evaluated formula which avoids the need of earlier methods [Knoff and Hopkins (1972); Park and Schapery (1999); Sorvari and Malinen (2007a, b)] to construct and solve a matrix equation to obtain estimates for the $\{j_k\}$. For example, for the method proposed by Nikonov *et al*. (2005, Appendix), the determination reduces to solving the matrix equation (6.1) which has an ill-posed Vandermonde type structure

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_N \\ a_1^2 & a_2^2 & \cdots & a_N^2 \\ \cdot & \cdot & \cdots & \cdot \\ a_1^N & a_2^N & \cdots & a_N^N \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ \cdot \\ j_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ f_N \end{bmatrix}, \tag{6.1}$$

where the values of $\{a_k\}$ and $\{f_k\}$ are defined in terms of the values of $\{\tau_k\}$, $\{\lambda_k\}$ and $\{g_k\}$.

The quite severe poor conditioning of Vandermonde matrices has been widely discussed in the numerical analysis literature as it is a commonly occurring matrix structure which arises in practical applications [Bazán (2000); Beckermann (2000)].

ACKNOWLEDGMENTS

Preliminary versions of Figure 1 were produced by the MAFA Function Plotter. The authors wish to thank Daniel Schmidt-Loebe for his help in fine-tuning this online tool. The authors would also like to thank the referees for their suggestions which helped to improve the exposition of the paper.

References

Anderssen, R. S., A. R. Davies, and F. R. de Hoog, "On the sensitivity of interconversion between relaxation and creep," Rheol. Acta **47**, 159–167 (2008a).

Anderssen, R. S., A. R. Davies, and F. R. de Hoog, "On the Volterra integral equation relating creep and relaxation," Inverse Prob. **24**, 035009 (2008b).

Baumgaertel, M., and H. H. Winter, "Determination of discrete relaxation and retardation time spectra from dynamical mechanical data," Rheol. Acta **28**, 511–519 (1989).

Bazán, F. S. V., "Conditioning of rectangular Vandermonde matrices with nodes in the unit disk," SIAM J. Matrix Anal. Appl. **21**, 679–693 (2000).

Beckermann, B., "The condition number of real Vandermonde, Krylov and positive definite Hankel matrices," Numer. Math. **85**, 553–577 (2000).

Dooling, P. J., C. P. Buckley, and S. Hinduja, "An intermediate model method for obtaining a discrete relaxation spectrum from creep data," Rheol. Acta **36**, 472–482 (1997).

Ferry, J. D., *Viscoelastic Properties of Polymers* (Wiley, NY, 1980).

Giesekus, H., *Phänomenolifishe Rheologie. Eine Einführung* (Springer-Verlag, Berlin, Heidelberg, 1994).

Gross, B., *Mathematical Structure of the Theories of Viscoelasticity* (Hermann, Paris, 1953).

Gross, B., "Distribution functions in linear viscoelastic theory," J. Appl. Phys. **62**, 2763–2770 (1987).

Gross, B., and H. Peltzer, "Relations between delta functions," Proc. Roy. Soc. A **210**, 434–437 (1952).

Hopkins, I. L., and R. W. Hamming, "On creep and relaxation," J. Appl. Phys. **28**, 906–909 (1957).

Knoff, W. F., and I. L. Hopkins, "An improved numerical interconversion for creep compliance and relaxation modulus," J. Appl. Polym. Sci. **16**, 2963–2972 (1972).

Loy, R. J., and R. S. Anderssen, "Interconversion relationships for completely monotone functions," SIAM J. Math. Anal. **46**(3), 2008–2032 (2014).

Luk-Cyr, J., T. Crochon, C. Li, and M. Levesque, "Interconversion of linearly viscoelastic material functions expressed as Prony series: a closure," Mech. Time-Dependent Mater. **17**, 53–82 (2013).

Marsden, J. E., *Basic Complex Analysis* (W. H. Freeman, San Francisco, 1973).

Mead, D. W., "Numerical interconversion of linear viscoelastic material functions," J. Rheol. **38**, 1769–1795 (1994).

Nikonov, A., A. R. Davies, and I. Emri, "The determination of creep and relaxation functions from a single experiment," J. Rheol. **49**, 1193–1211 (2005).

Park, S. W., and R. A. Schapery, "Methods of interconversion between linear viscoelastic material functions. part i – a numerical method based on Prony series," Int. J. Solids Struct. **36**, 1653–1675 (1999).

Plazek, D. J., and I. Echeverria, "Dont cry for me Charlie Brown, or with compliance comes comprehension," J. Rheol. **44**, 931–941 (2000).

Sorvari, J., and M. Malinen, "On the direct estimation of creep and relaxation functions," Mech. Time-Depend. Mater. **11**, 143–157 (2007a).

Sorvari, J., and M. Malinen, "Numerical interconversion between linear viscoelastic material functions with regularization," Int. J. Solids Struct. **44**, 1291–1303 (2007b).

Taylor, R. L., "Inversion of Prony series characterization for viscoelastic stress," Int. J. Numer. Methods Eng. **5**, 499–502 (1973).

Traub, J. F., *Iterative Methods for the Solution of Equations* (Chelsea Publishing Company, New York, 1982).

Tschoegl, N. W., and Emri, I., "Generating line spectra from experimental responses. Part III. Interconversion between relaxation and retardation time behaviour," Int. J. Polym. Mater. **18**, 117–127 (1992).

Whittaker, E. T., "On the numerical solution of integral-equations," Proc. Roy. Soc. A **94**, 367–383 (1918).

Widder, D. V., *The Laplace Transform* (Princeton University, Princeton, 1972).