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A Distributed Control Law with Guaranteed Convergence Rate for Identically Coupled Linear Systems

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***Abstract*— This paper investigates the stabilization and optimization problems for a group of identically linear agents with undirected interaction topology. It is shown that a distributed control law based on local measurements and relative information exchanged from neighboring agents can be designed for each agent to enable the agent states to be stabilized. Furthermore, due to the use of a parametric Lyapunov approach, the designed distributed control law guarantees not only optimization performance at a network level but also a convergence rate for the group of agents. Finally, a simulation example is provided to demonstrate the advantage as well as the effectiveness of the proposed method.**

I. Introduction

Over the last few years, control of multiagent systems has received considerable attention, motivated by recent advances in computation and communication, especially problems such as synchronization, consensus, flocking, and formation control, see e.g. [1]–[9]. For more details and developments, see the survey papers [10] on formation control, [11] on consensus and references therein.

Multiagent systems with agents interacting with each other over an interconnection topology are often studied from the perspective of graph theory [12]. As for the system control of multiagent systems, there are three possible schemes including centralised, decentralised, and distributed control. The literature tends to favour distributed control [13]. Recently there has been progress in addressing research efforts on optimal control for a group of dynamical systems in [14], which aims at studying how the stability of a multiagent system is related to the robustness of local controllers and the network topology. The necessary and sufficient conditions for stability of an interconnected system of identical vehicles in terms of the eigenvalues of the graph Laplacian was provided in [3]. A distributed control law was further investigated in [15] under the frame of linear quadratic regulator (LQR), in which different methods were used for the stability analysis. Furthermore, a linear matrix inequality (LMI) method was proposed in [16] to find a distributed control law with guaranteed LQR cost for identical dynamically coupled linear systems. Note here that the widely considered problem was posed as a LQR cost function and the associated minimization problem. A similar LQR cost function was used but in [14] the solution depended on the maximum vertex degree while in [16] it is tied to the total number of agents. However, none of the papers analyzed the convergence rate.

In this paper, we also investigate the stabilization and optimization problems for a group of identically linear agents interconnected over an information network, exchanging relative measurements. The main difference is that a parametric Lyapunov design method is presented to stabilize the network. It is shown that the distributed control law not only guarantees optimization performance at a network level but also derives a convergence rate for the group of agents.

The paper is structured as follows. Section 2 contains all the preliminary notions, including a brief summary of some relevant results in graph and matrix theory. In Section 3, a LQR cost function and the corresponding optimization problem are investigated. Section 4 presents the stabilizing distributed controller design procedure using the local LQR cost function, and a convergence rate is analytically obtained by solving the corresponding optimization problem. Its effectiveness is illustrated by a simulation example in Section 5. Some concluding remarks are given in Section 6.

II. Notation and Preliminaries

A. Notation

The notation used in this paper is standard. The superscript "T" stands for matrix transposition. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. The notation $P > 0$ (≥ 0) means that P is a real symmetric positive (semi-positive) definite matrix. I_n and 0 represent, respectively, the n dimension identity matrix and zero matrix. The set of real numbers is denoted by $\mathbb{R}$. The set of real-valued vectors of length m is given by $\mathbb{R}^m$. The set of arbitrary real-valued $m \times n$ matrices is given by $\mathbb{R}^{m\times n}$.

A concise review of the relationship between the eigenvalues of a Laplacian matrix and the topology of the associated graph are quoted in this section. Please refer to [17] for further reading on graph theory. We list a collection of properties associated with undirected graph Laplacians and adjacency matrices, which will be used in subsequent sections of the paper. A graph $\mathcal{G}$ is defined as $\mathcal{G} = (\mathcal{V}, \mathcal{A})$, where $\mathcal{V}$ is the set of nodes (or vertices) $\mathcal{V} = \{1, \dots, N\}$ and $\mathcal{A} \subseteq \mathcal{V}^2$ the set of edges (i, j) with $i \in \mathcal{V}$, $j \in \mathcal{V}$. Let $\mathcal{A}(\mathcal{G}) = [a_{ij}]$ denotes the adjacency matrix of the graph $\mathcal{G}$. If i and j are adjacent nodes of the graph, $a_{ij} = 1$, and $a_{ij} = 0$

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otherwise. We focus on undirected graphs, for which the adjacent matrix $\mathcal{A}(\mathcal{G})$ is symmetric. Let define the Laplacian matrix of a graph $\mathcal{G}$ as $\mathcal{L} = \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$, where $\mathcal{D}(\mathcal{G})$ is the diagonal matrix of vertex degrees. Eigenvalues of Laplacian matrices have been widely studied by graph theorists. Their properties are strongly related to the structural properties of their associated graphs. Every Laplacian matrix is a singular matrix. The real part of each nonzero eigenvalue of $\mathcal{L}$ is strictly positive. For undirected graphs, $\mathcal{L}$ is symmetric, positive semidefinite matrix, which has only real eigenvalues. The smallest eigenvalue of $\mathcal{L}$ is exactly zero and the corresponding eigenvector is given by $1_N = [1, \ldots, 1]^T$. Moreover, the rank of $\mathcal{L}$ is $n-1$ if and only if $\mathcal{L}$ is connected.

Let $\otimes$ denotes the *Kronecker product*. We recall some properties for Kronecker product which will be used throughout the paper. For matrices A, B, C and D with compatible dimensions:

1) $(\alpha A) \otimes B = A \otimes (\alpha B)$, where α is a constant;
2) $(A + B) \otimes C = A \otimes B + B \otimes C$;
3) $(A \otimes B)(C \otimes D) = (AC \otimes BD)$;
4) $(A \otimes B)^T = A^T \otimes B^T$;
5) Let A be an $m \times m$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_m$, and let B be an $n \times n$ matrix with eigenvalues $\mu_1, \ldots, \mu_n$, then the mn eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$, where $i = 1, \ldots, m; j = 1, \ldots, n$.

B. Preliminaries

Consider a linear system

$$\left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \end{array} \right. , \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are, respectively, the system matrix and control matrix. Define a cost function

$$J = \int_0^\infty \left(x(t)^T Q x(t) + u(t)^T R u(t) \right) dt, \tag{2}$$

where $Q = C^T C \geq 0$ and $R > 0$. We first recall the following well-known result.

Lemma 1: [18] Consider the linear system (1) and the cost function (2). Assume that the pair (A, B) is stabilizable and the pair (A, C) is detectable. Then, J is minimized with

$$u(t) = -R^{-1} B^T P x(t),$$

where P is the unique positive definite solution to the following algebraic Riccati equantion (ARE)

$$A^T P + PA + Q - PBR^{-1} B^T P = 0.$$

We assume that there exists a positive scalar γ such that $P(\gamma)$ is the solution to the above ARE, and we choose

$$Q = \gamma P(\gamma).$$

In this case, it becomes

$$A^T P(\gamma) + P(\gamma) A - P(\gamma) BR^{-1} B^T P(\gamma) = -\gamma P(\gamma), \tag{3}$$

which leads to the following lemma.

Lemma 2: Consider the linear system (1) and the pair (A, B) is controllable. Assume that there exists a positive scalar γ such that $\gamma > -2 \min \{\mathrm{Re}(\lambda(A))\}$, and $P(\gamma)$ is the solution to the ARE (3). Then,

$$u^*(t) = -R^{-1} B^T P(\gamma) x(t) \tag{4}$$

stabilize the linear system (1), and

1) $u^*(t)$ is the solution to the following optimization problem

$$\inf_{u(t)} \left\{ \int_0^\infty e^{\gamma t} u(t)^T R u(t) dt \right\}, \text{ s.t. } \lim_{t \to \infty} e^{\frac{\gamma}{2} t} x(t) = 0. \tag{5}$$

2) $u^*(t)$ can minimize the following LQR cost function

$$J = \int_0^\infty \left(x(t)^T \gamma P(\gamma) x(t) + u(t)^T R u(t) \right) dt. \tag{6}$$

Proof: See Appendix. ■

Remark 1: It is noted that the control laws in (4) is the solution to the optimization problem (5). Meanwhile, it minimizes the LQR cost function (6). The term $\lim_{t \to \infty} e^{\frac{\gamma}{2} t} x(t) = 0$ in (5) indicates that the convergence rate of the closed-loop system is faster than $e^{-\frac{\gamma}{2} t}$, and (5) corresponds to the minimal energy control with guaranteed convergence rate (MECGCR) problem [19].

III. Problem Statement

A. Linear Dynamic Model

Consider a collection of N identical agents, the state-space representation of the i-th agent's dynamic is given by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{7}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^m$ represent the states and the distributed control inputs. The matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, and it is assumed that the pair (A, B) is controllable. In this paper, each agent is assumed to have access to its own state and relative external measurements with respect to the neighbouring agents which it can sense or interact with. Such a system can be represented as a graph with N vertices, and each vertex represents an agent. The existence of relative sensing and communication between two agents is represented by an edge in the graph. The signals representing the relative external measurements are assumed to have the form

$$z_i(t) = \sum_{j \in \mathcal{J}_i} (x_i(t) - x_j(t)) \tag{8}$$

for $i = 1 \ldots N$. The set $\mathcal{J}_i \subset \{1, 2, \ldots N\} / \{i\}$ denotes the external agents, for which the i-th agent has information. The signals $z_i(t)$ represent the sum of the external measurements relative to the other dynamics which the i-th agent can sense. At a network of N agents, the state-space representation of the network is given by

$$\dot{X}(t) = (I_N \otimes A) X(t) + (I_N \otimes B) U(t), \tag{9}$$

where

$$X(t) = \left[\begin{array}{ccc} x_1(t) & \ldots & x_N(t) \end{array} \right]^T,$$

$$U(t) = \left[\begin{array}{ccc} u_1(t) & \ldots & u_N(t) \end{array} \right]^T.$$

At a network level, (8) can be expressed as

$$Z(t) = (\mathcal{L} \otimes I_N)X(t),$$

where

$$Z(t) = \left[\begin{array}{ccc} z_1(t) & \ldots & z_N(t) \end{array} \right]^T.$$

It is assumed that each linear dynamical system has information about at least one other system which ensures $rank(\mathcal{L}) = N - 1$.

B. Problem Definition

In this paper, we are interested in the design of state feedback control laws

$$u_i(t) = Fx_i(t) + \Gamma F z_i(t) \tag{10}$$

for $i = 1, \ldots, N$, where $F \in \mathbb{R}^{m\times n}$ and $\Gamma = fI_m \in \mathbb{R}^{m\times m}$, f is a scalar. That is, for a positive scalar β, design a set of control laws (10) such that the LQR cost function

$$J = \int_0^\infty (X(t)^T ((I_N \otimes \beta P(\beta)) + (\mathcal{L} \otimes Q)) X(t) + U(t)^T (I_N \otimes R) U(t))dt \tag{11}$$

where $R = R^T \in \mathbb{R}^{m\times m} > 0, Q = Q^T \in \mathbb{R}^{n\times n} \geq 0$, and $P(\beta)$ is the unique positive definite solution to the following ARE

$$A^T P + PA - PBR^{-1}B^T P = -\beta P, \tag{12}$$

is minimized. The procedure to solve this problem is composed from two steps, which are given as below:

1). A decoupled LQR problem: consider the gain matrix F in (10), find a control law

$$u_i(t) = Fx_i(t), \tag{13}$$

for $i = 1, \ldots, N$, which stabilizes (7) subject to minimizing a LQR cost function

$$J_i = \int_0^\infty \left(x_i(t)^T \beta P(\beta) x_i(t) + u_i(t)^T R u_i(t) \right) dt \tag{14}$$

for $i = 1, \ldots, N$, where $\beta, P(\beta)$ and R are associated with the LQR cost function given in (11). In this step, the agents are identical and decoupled, and no interactions among the agents are considered.

2). A coupled LQR problem: once the matrix F has been designed, choose a matrix $\Gamma \in \mathbb{R}^{m\times m}$ such that all the N systems (7) are stabilized by the distributed control laws (10). The state-space control law at the network level is represented by

$$U(t) = (I_N \otimes F) X(t) + (\mathcal{L} \otimes \Gamma F)X(t), \tag{15}$$

and the objective is to minimize (11).

Remark 2: According to Lemma 2, the control law (13) solves an optimization problem

$$\inf_{u(t)} \left\{ \int_0^\infty e^{\beta t} u_i(t)^T R u_i(t) dt \right\}, \text{ s.t. } \lim_{t\to\infty} e^{\frac{\beta}{2}t} x_i(t) = 0. \tag{16}$$

for $i = 1, \ldots, N$, and the optimization problem (16) can be viewed as a MECGCR problem. Meanwhile, the control laws (10) are the solution to the optimization problem

$$\inf_{U(t)} \left\{ \int_0^\infty e^{\gamma t} U(t)^T R U(t) dt \right\}, \text{ s.t. } \lim_{t\to\infty} e^{\frac{\gamma}{2}t} X(t) = 0, \tag{17}$$

where γ is a positive scalar.

Remark 3: Similar LQR cost functions (11) are also considered in [14] and [16]. However, the approaches to solve the associated LQR problem are different. In [14], the suboptimal distributed LQR problem is posed as a single LQR problem exploiting the properties of the graph associated with the communication topology, but not on the total number of nodes in the network. In [16], the suboptimal distributed LQR problem is solved systematically in two steps, but the second step creates an optimization problem which depends on the number of nodes of the graph, which has to solve the optimization problem in terms of LMI conditions as many as the the number of nodes. In this paper, we use a parametric Lyapunov approach to solve the LQR problem with guaranteed convergence rate.

In the first step, we directly solve a classic decoupled LQR problem to obtain a set of gains F. The second step is solved to obtain the matrix Γ. Then, the controllers are obtained. In what follows, we will design each of these steps in detail.

IV. Distributed Control Design

A. Solution to a decoupled LQR problem

Let $\beta > 0$ be such that

$$\beta > -2 \min \{\mathrm{Re}(\lambda(A))\},$$

where $\mathrm{Re}(\lambda(A))$ denotes the set of the real parts of the eigenvalues of A. Then, the feedback gain is

$$F = -R^{-1}B^T P(\beta), \tag{18}$$

where $P(\beta) = W^{-1}(\beta)$, and $W^{-1}(\beta)$ is the unique positive definite solution to the following Lyapunov matrix equation

$$W(A + \frac{\beta}{2}I_n)^T + (A + \frac{\beta}{2}I_n)W = BR^{-1}B.$$

In view of Lemma 2, the feedback gain (18) minimizes the LQR cost function (14) and solves the optimization problem (16).

B. Solution to a coupled LQR problem

In this step, we introduce a change of coordinates. It is known that $\mathcal{L}$ is semipositive symmetric. By spectral decomposition, $\mathcal{L} = V\Lambda V^T$, where $V \in \mathbb{R}^{N\times N}$ is an orthogonal matrix formed from the eigenvectors of $\mathcal{L}$, and $\Lambda = diag(\lambda_1, \ldots, \lambda_N)$ is the matrix of the eigenvalues of $\mathcal{L}$. Consider an orthogonal transformation

$$X \longmapsto (V \otimes I_N)X = \bar{X},$$
$$U \longmapsto (V \otimes I_N)U = \bar{U}.$$

In this new coordinates, the LQR performance (11) can be expressed as

$$J=\int_0^{\infty}\bar{X}(t)^T\left(\left(I_N\otimes\beta P(\beta)\right)+\left(\Lambda\otimes Q\right)\right)\bar{X}(t)$$
$$+\bar{U}(t)^T\left(I_N\otimes R\right)\bar{U}(t)dt,$$

where we have defined $\bar{X}(t)=\left[\bar{x}_1(t),\cdots,\bar{x}_N(t)\right]^T$, $\bar{U}(t)=\left[\bar{u}_1(t),\cdots,\bar{u}_N(t)\right]^T$, and (9) can be represented at a network level in the transformed coordinates as

$$\dot{\bar{x}}_i(t)=(A+BF+\lambda_i B\Gamma F)\bar{x}_i(t), \tag{19}$$

$$\bar{u}_i(t)=\left(F+\lambda_i\Gamma F\right)\bar{x}_i(t), \tag{20}$$

for $i=1,\ldots,N$. Since Λ is a diagonal matrix, the LQR performance can be rewritten as

$$J=\sum_{i=1}^{N}\int_0^{\infty}(\bar{x}_i(t)^T\left(\beta P(\beta)+\lambda_i Q\right)\bar{x}_i(t)$$
$$+\bar{u}_i(t)^T R\bar{u}_i(t))dt. \tag{21}$$

Since the matrix F is fixed in Step 1, The optimization of the cost function (21) is further constrained by choosing a proper matrix $\Gamma\in\mathbb{R}^{m\times m}$ in (19) to ensure that a linear feedback controller results. A direct result of this constraint is that the optimization is now dependent upon the initial state of the system. It is assumed that the initial state is $\bar{x}_i(t_0)$. So the state trajectory, $\bar{x}_i(t)$, is a direct function of the initial state as stated

$$\bar{x}_i(t)=e^{(A+BF+\lambda_i B\Gamma F)t}\bar{x}_i(t_0)=\Phi_i(t)\bar{x}_i(t_0).$$

In view of (19), (20), the cost function (21) is written to reflect the dependence of the state trajectory on its initial state.

$$J=\bar{x}_i(t_0)^T(\sum_{i=1}^{N}\int_0^{\infty}(\Phi_i(t)^T(\beta P+\lambda_i Q$$
$$+\left(F+\lambda_i\Gamma F\right)^T R\left(F+\lambda_i\Gamma F\right))\Phi_i(t))dt)\bar{x}_i(t_0).$$

By definition, the matrices, Q and P, are symmetric, the cost function is equivalent to

$$J=trace\left(\sum_{i=1}^{N}P_i\bar{x}_i(t_0)\bar{x}_i(t_0)^T\right),$$

where

$$P_i=\int_0^{\infty}\Phi_i(t)^T(\beta P(\beta)+\lambda_i Q$$
$$+\left(F+\lambda_i\Gamma F\right)^T R\left(F+\lambda_i\Gamma F\right))\Phi_i(t)dt. \tag{22}$$

Here, P_i is the solution to the well known Lyapunov stability equation

$$(A+BF+\lambda_i B\Gamma F)^T P_i+P_i(A+BF+\lambda_i B\Gamma F)$$
$$=-\left(\beta P(\beta)+\lambda_i Q\right)-\left(F+\lambda_i\Gamma F\right)^T R\left(F+\lambda_i\Gamma F\right) \tag{23}$$

Provided that the initial state of the system represents an unknown, the mean value of the minimization of the cost function is sought over all possible $\bar{x}_i(t_0)$. This is equivalent to simply minimizing

$$\bar{J}=trace\left(\sum_{i=1}^{N}P_i\right) \tag{24}$$

subject to (23).

To solve the problem (24), let $\gamma>0$ be such that

$$\gamma<2\beta+2\min\left\{\mathrm{Re}\left(\lambda(A)\right)\right\}.$$

Then $P_i(\gamma)$ is the unique positive solution to the following Lyapunov equation

$$(A+\frac{\gamma}{2}I_n+BF+\lambda_i B\Gamma F)^T P_i(\gamma)$$
$$+P_i(\gamma)(A+\frac{\gamma}{2}I_n+BF+\lambda_i B\Gamma F)$$
$$=-\left(F+\lambda_i\Gamma F\right)^T R\left(F+\lambda_i\Gamma F\right)$$

for $i=1,\ldots,N$. The LQR cost function (21) is equivalent to

$$J=\sum_{i=1}^{N}\int_0^{\infty}\left(\bar{x}_i(t)^T\gamma P_i(\gamma)\bar{x}_i(t)+\bar{u}_i(t)^T R\bar{u}_i(t)\right)dt,$$

Meanwhile, the control law (20) solves the following optimization problem

$$\sum_{i=1}^{N}\inf_{\bar{u}_i(t)}\left\{\int_0^{\infty}e^{\gamma t}\bar{u}_i(t)^T R\bar{u}_i(t)dt\right\},\ \text{s.t.}\ \lim_{t\to\infty}e^{\frac{\gamma}{2}t}\bar{x}_i(t)=0.$$

which is equivalent to the optimization problem (17).

In what follows, for the sake of convenience, the constraints can be written as

$$A_i^T P_i(\gamma)+P_i(\gamma)A_i+Q_i=0,$$

for $i=1,\ldots,N$, where

$$A_i=A+BF+\lambda_i B\Gamma F+\frac{\gamma}{2}I_n,$$

$$Q_i=\left(F+\lambda_i\Gamma F\right)^T R\left(F+\lambda_i\Gamma F\right).$$

By definition, A_i and Q_i are parametric matrices, the above constrain is a Lyapunov matrix equation and $P_i(\gamma)$ is the solution to the Lyapunov equation. The minimization of the LQR cost function (24), minimize $trace\left(\sum_{i=1}^{N}P_i\right)$, is stated in terms of parameters γ and f. The design matrix Γ can be obtained in terms of the parameter γ.

Remark 4: From the above steps, it is known that we should solve the the optimization problem in terms of parametric Lyapunov equations as many as the number of nodes. For large scale systems, this would be a little computational cost. However, it requires less computation than the LMI conditions in [16].

V. Numerical Example

Consider a group of three agents moving in a plane with the dynamic of each described by a double integrator in each of the directions x and y. The system is represented by

$$\dot{\zeta}_i = A\zeta_i + Bu_i,$$

where ζ_i represents the position and the velocity states of the i-th agent. The plant matrix and input matrix are given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Suppose that a communication network is given by Fig. 1, which is represented by the Laplacian matrix

$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

In the first step, for any $\beta > 0$, a control law (18) is obtained by solving a Lyapunov equation, and the solution to the ARE (12)

$$P(\beta) = \begin{bmatrix} \beta^3 & 0 & \beta^2 & 0 \\ 0 & \beta^3 & 0 & \beta^2 \\ \beta^2 & 0 & 2\beta & 0 \\ 0 & \beta^2 & 0 & 2\beta \end{bmatrix}.$$

Then, the control gain matrix is given by

$$\begin{aligned} F &= -R^{-1}B^T P(\beta) \\ &= \begin{bmatrix} -\beta^2 & 0 & -2\beta & 0 \\ 0 & -\beta^2 & 0 & -2\beta \end{bmatrix}. \end{aligned}$$

For any positive β, let $\beta = 0.07 > 0$, the above control gain matrix can be shown as

$$F = \begin{bmatrix} -0.0049 & 0 & -0.14 & 0 \\ 0 & -0.0049 & 0 & -0.14 \end{bmatrix}.$$

Since the LQR cost function (14) and the optimization problem (16) have been solved, we will exploit the second step of the design procedure. We assume that the three agents have an interconnection topology as Fig. 1. A distributed control law as in (10) is designed by using the constraints, for $i = 1, 2, 3$. Following the design procedure in the second step, we obtain that, for any positive γ such that $\gamma < 2\beta + 2\min\{\mathrm{Re}(\lambda(A))\}$, let $\gamma = 0.05 < 2\beta$, the matrix Γ for the control gain is given by

$$\Gamma = 0.30 I_2.$$

Then, the control laws in the form of (10) such that

$$\begin{aligned} u_i(t) = &\begin{bmatrix} -0.0049 I_2 & -0.14 I_2 \end{bmatrix} x_i(t) \\ &+ \begin{bmatrix} -0.00147 I_2 & -0.0042 I_2 \end{bmatrix} z_i(t), \end{aligned}$$

for $i = 1, 2, 3$, can solve the LQR cost function (14) and the optimization problem (17). That is, the control law can stabilize (7) and guarantee convergence rate of the agents faster than $e^{-0.025t}$.

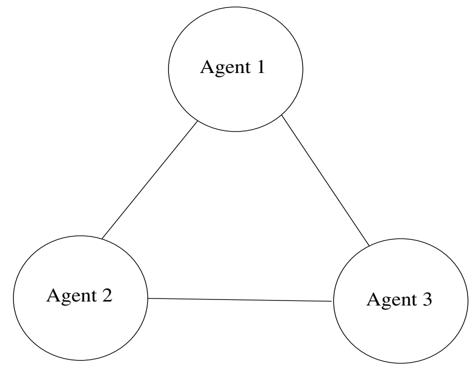

Fig. 1. An undirected communication topology

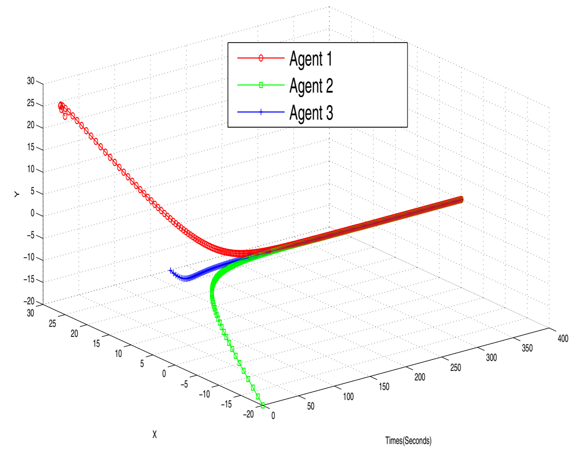

Fig. 2. Trajectories of three agents with random initial conditions

To show that this control law can achieve stabilization and guarantee convergence rate, we simulate the group of agents with the given control law. Shown in Fig. 2 is the simulation result for random initial conditions. It is shown that, for random initial conditions, the control laws enable the states of each agent to be stabilized.

VI. Conclusions

In this paper, a distributed control law for identically coupled linear systems was proposed via a parametric Lyapunov approach, and a two-step design procedure was present. All the agents were interconnected over an information network, and were supposed to be stabilized under the distributed control law, which not only guaranteed optimization performance at a network level but also analytically resulted in a convergence rate for the group of agents.

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APPENDIX

Proof of Lemma 2.

Proof: 1) In view of (4), the closed-loop system matrix $A_c(\gamma) = A - BR^{-1}B^T P(\gamma)$ satisfies

$$\begin{aligned}&P(\gamma)A_c(\gamma) + A_c(\gamma)^T P(\gamma)\\ &= -\gamma P(\gamma) - P(\gamma)BR^{-1}B^T P(\gamma)\\ &< 0.\end{aligned}$$

It is implies that the linear system is stable under the control law (4). Suppose

$$\bar{x}(t) = e^{\frac{\gamma}{2}t}x(t),\ \bar{u}(t) = e^{\frac{\gamma}{2}t}x(t),$$

and we have

$$\begin{aligned}\dot{\bar{x}}(t) &= e^{\frac{\gamma}{2}t}\dot{x}(t) + \frac{\gamma}{2}e^{\frac{\gamma}{2}t}x(t)\\ &= \left(A + \frac{\gamma}{2}\right)\bar{x}(t) + B\bar{u}(t)\\ &\triangleq \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t).\end{aligned}$$

Then, the optimization problem (5) is equivalent to

$$\inf_{\bar{u}(t)}\left\{\int_0^\infty e^{\gamma t}\bar{u}(t)^T R\bar{u}(t)dt\right\},\ \text{s.t.}\ \lim_{t\to\infty} e^{\frac{\gamma}{2}t}\bar{x}(t) = 0,$$

and the solution to this problem is

$$u^*(t) = -R^{-1}\bar{B}^T\bar{P}(\gamma)\bar{x}(t), \tag{25}$$

where $\bar{P}(\gamma)$ is the solution to the following ARE:

$$\bar{A}^T\bar{P}(\gamma) + \bar{P}(\gamma)\bar{A} - \bar{P}(\gamma)\bar{B}R^{-1}\bar{B}^T\bar{P}(\gamma) = 0. \tag{26}$$

Suppose that $P(\gamma) = \bar{P}(\gamma)$, (25) and (26) are, separately, equivalent to (4) and (3). According to [20], $\bar{P}(\gamma)$ is the unique solution to (26).

2) It is known [18] that there exists a control law such that $u^*(t) = -R^{-1}B^T Sx(t)$ minimizes the LQR cost function (6), where S is the unique positive solution to the following ARE:

$$A^T S + SA + \gamma S - SBR^{-1}B^T S = 0. \tag{27}$$

In what follows, we will prove that $S = P(\gamma)$. Define $\triangle S = S - P(\gamma)$, and in view of (27) and (3), we obtain

$$\begin{aligned}0 &= A^T \triangle S + \triangle SA - SBR^{-1}B^T S\\ &\quad + P(\gamma)BR^{-1}B^T P(\gamma)\\ &= A_c(\gamma)^T \triangle S + \triangle SA_c(\gamma) - \triangle SBR^{-1}B^T \triangle S.\end{aligned}$$

According to Item 1, we know that $A_c(\gamma) < 0$. Then, we can obtain $\triangle S \le 0$. That is, $S \le P(\gamma)$. Meanwhile, we define $\triangle P = P(\gamma) - S$, and in view of (27) and (3), we obtain

$$\begin{aligned}0 &= (A - BR^{-1}B^T S)^T \triangle P\\ &\quad + \triangle P(A - BR^{-1}B^T S) - \triangle PBR^{-1}B^T \triangle P.\end{aligned}$$

Then, we have $\triangle P \le 0$, that is, $P(\gamma) \le S$. Thus, we obtain $S = P(\gamma)$, which completes the proof. ∎