

# Inhomogeneous Strichartz estimates

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**Abstract.** We present abstract inhomogeneous Strichartz estimates for dispersive operators, extending previous work by M. Keel and T. Tao on the one hand, and generalising results of D. Foschi, M. Vilela, M. Nakamura and T. Ozawa on the other hand. It is shown that these abstract estimates imply new inhomogeneous Strichartz estimates for the wave equation and some Schrödinger equations involving potentials.

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## 1 Introduction

This paper is concerned with a priori estimates for dispersive partial differential equations, expressed in norms given by

$$\|G\|_{L^q(\mathbb{R};\mathcal{B})} = \left( \int_{\mathbb{R}} \|G(t)\|_{\mathcal{B}}^q dt \right)^{1/q},$$

where  $\mathcal{B}$  is a Banach space. Consider, for example, the inhomogeneous Schrödinger initial value problem

$$(1) \quad \begin{cases} iu'(t) + \Delta u(t) = F(t) & \forall t \geq 0 \\ u(0) = f, \end{cases}$$

whose formal solution  $u$  is given by

$$u(t) = e^{it\Delta}f - i \int_0^t e^{i(t-s)\Delta}F(s) ds$$

via Duhamel's principle. The seminal paper [18] of R. Strichartz showed that if  $u$  is a solution to (1) in  $n$  spatial dimensions and  $q = \tilde{q} = r = \tilde{r} = 2(n+2)/n$ , then

$$(2) \quad \|u\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^{\tilde{q}'}(\mathbb{R};L^{\tilde{r}'}(\mathbb{R}^n))}$$

whenever  $f \in L^2(\mathbb{R}^n)$  and  $F \in L^{q'}(\mathbb{R}; L^{r'}(\mathbb{R}^n))$ . Since then, various authors (see especially [6], [22], [2] and [13]) have published similar a priori estimates for solutions to Schrödinger’s equation where the time exponent  $q$  and the space exponent  $r$  are unequal. Such estimates have proved fruitful for determining whether various semilinear Schrödinger equations are well-posed (see, for example, [6], [11] and [3]).

By respectively taking  $F$  and  $f$  as 0 in (1), estimate (2) becomes

$$(3) \quad \left\| e^{it\Delta} f \right\|_{L^q(\mathbb{R}; L^r(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$$

and

$$(4) \quad \left\| \int_0^t e^{i(t-s)\Delta} F(s) \, ds \right\|_{L^q(\mathbb{R}; L^r(\mathbb{R}^n))} \lesssim \|F\|_{L^{q'}(\mathbb{R}; L^{r'}(\mathbb{R}^n))}.$$

The first of these estimates is called an *homogeneous Strichartz estimate* and the second is known as an *inhomogeneous Strichartz estimate*. The problem of finding all possible exponent pairs  $(q, r)$  such that (3) is valid has been completely solved. That is, (3) holds if and only if

$$(5) \quad q \in [2, \infty], \quad \frac{1}{q} + \frac{n}{2r} = \frac{n}{4} \quad \text{and} \quad (q, r, n) \neq (2, \infty, 2)$$

(see [13] and the references therein). The corresponding problem for the inhomogeneous Strichartz estimate (4) remains open. It is known that if the exponent pairs  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  satisfy (5) then the inhomogeneous estimate (4) is also valid. However, it was observed by T. Cazenave and F. Weissler [4] and T. Kato [12] that there are exponent pairs  $(q, r)$  for which the inhomogeneous Strichartz estimate holds but the homogeneous estimate fails. Using the techniques introduced in [13], D. Foschi [5] and M. Vilela [21] independently obtained what is currently the largest known range of exponent pairs  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  for which the inhomogeneous Strichartz estimate (4) for the Schrödinger equation is valid.

Similar remarks may be made for the wave equation. The precise set of Lebesgue space-time exponents for which the homogeneous Strichartz estimate for the wave equation is known (see [14], [13, Section 1] and the references therein) while a complete description for the set of exponents for which the inhomogeneous estimate is valid remains open (see the early work of D. Oberlin [17] and J. Harmse [9] and more recent advances by Foschi [5]). It must also be noted that, previous to the work of Foschi, a large set of exponents for inhomogeneous Strichartz-type estimates for solutions of the Klein–Gordon equation were obtained by M. Nakamura and T. Ozawa [16].

In this paper, results of [16], [21] and especially [5] are generalised to the abstract setting introduced in [13], thus enabling us to find new inhomogeneous Strichartz-type estimates for other dispersive equations, including the wave equation and Schrödinger equations with potential. Suppose that  $\mathcal{H}$  is a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ ,  $(\mathcal{B}_0, \mathcal{B}_1)$  is a Banach interpolation couple and  $\sigma > 0$ . Suppose also that for each time  $t$  in  $\mathbb{R}$  we have an operator

$U(t) : \mathcal{H} \rightarrow \mathcal{B}_0^*$ . Its adjoint  $U(t)^*$  is an operator from  $\mathcal{B}_0$  to  $\mathcal{H}$ . We will assume that the family  $\{U(t) : t \in \mathbb{R}\}$  satisfies the *energy estimate*

$$(6) \quad \|U(t)f\|_{\mathcal{B}_0^*} \lesssim \|f\|_{\mathcal{H}} \quad \forall f \in \mathcal{H} \quad \forall t \in \mathbb{R},$$

and the *dispersive estimate*

$$(7) \quad \|U(s)U(t)^*g\|_{\mathcal{B}_1^*} \lesssim |t-s|^{-\sigma} \|g\|_{\mathcal{B}_1} \quad \forall g \in \mathcal{B}_1 \cap \mathcal{B}_0 \quad \forall \text{ real } s \neq t.$$

Using the energy estimate, we consider the operator  $T : \mathcal{H} \rightarrow L^\infty(\mathbb{R}; \mathcal{B}_0^*)$ , defined by the formula

$$Tf(t) = U(t)f \quad \forall f \in \mathcal{H} \quad \forall t \in \mathbb{R}.$$

Its formal adjoint  $T^* : L^1(\mathbb{R}; \mathcal{B}_0) \rightarrow \mathcal{H}$  is given by the  $\mathcal{H}$ -valued integral

$$T^*F = \int_{\mathbb{R}} U(s)^*F(s) \, ds.$$

The composition  $TT^* : L^1(\mathbb{R}; \mathcal{B}_0) \rightarrow L^\infty(\mathbb{R}; \mathcal{B}_0^*)$ , given by

$$TT^*F(t) = \int_{\mathbb{R}} U(t)U(s)^*F(s) \, ds,$$

can be decomposed as the sum of retarded and advanced parts, respectively given by

$$(TT^*)_R F(t) = \int_{s < t} U(t)U(s)^*F(s) \, ds$$

and

$$(TT^*)_A F(t) = \int_{s > t} U(t)U(s)^*F(s) \, ds.$$

In applications (see Sections 7 and 8 for examples),  $\{U(t) : t \in \mathbb{R}\}$  is the evolution family associated to a homogeneous differential equation,  $T$  solves the initial value problem of the homogeneous equation and  $(TT^*)_R$  solves the corresponding inhomogeneous problem with zero initial data. Hence, if  $\mathcal{B}_\theta$  denotes the real interpolation space  $(\mathcal{B}_0, \mathcal{B}_1)_{\theta,2}$  whenever  $\theta \in [0, 1]$ , then corresponding homogeneous and inhomogeneous Strichartz estimates are given by

$$(8) \quad \|Tf\|_{L^q(\mathbb{R}; \mathcal{B}_\theta^*)} \lesssim \|f\|_{\mathcal{H}} \quad \forall f \in \mathcal{H}$$

and

$$(9) \quad \|(TT^*)_R F\|_{L^q(\mathbb{R}; \mathcal{B}_\theta^*)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_\theta)} \quad \forall F \in L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_\theta) \cap L^1(\mathbb{R}; \mathcal{B}_0).$$

The following theorem of M. Keel and T. Tao gives conditions on the exponent pairs  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$  such that these abstract Strichartz estimates hold.

**Definition 1.1.** Suppose that  $\sigma > 0$ . We say that a pair of exponents  $(q, \theta)$  is *sharp  $\sigma$ -admissible* if  $(q, \theta, \sigma) \neq (2, 1, 1)$ ,  $2 \leq q \leq \infty$ ,  $0 \leq \theta \leq 1$  and  $\frac{1}{q} = \frac{\sigma\theta}{2}$ .

**Theorem 1.2** (Keel–Tao, Theorem 10.1 [13]). *Suppose that  $\sigma > 0$ . If  $\{U(t) : t \in \mathbb{R}\}$  satisfies the energy estimate (6) and the dispersive estimate (7) then the Strichartz estimates (8) and (9) hold for all sharp  $\sigma$ -admissible pairs  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$ .*

The range of exponents given by Theorem 1.2 for which the homogeneous estimate (8) is valid cannot be improved. (One can show this by considering the case when  $U(t) = e^{it\Delta}$ ,  $\sigma = n/2$ ,  $\mathcal{H} = L^2(\mathbb{R}^n)$  and  $(\mathcal{B}_0, \mathcal{B}_1) = (L^2(\mathbb{R}^n), L^1(\mathbb{R}^n))$ , which corresponds to the setting of the Schrödinger equation; see, for example, [13, Section 8].) However, as noted in [13], the range of exponents given by Theorem 1.2 for the inhomogeneous estimate (9) is suboptimal. One of the aims of this paper is to extend this range. This has already been achieved by D. Foschi [5, Theorem 1.4] for the special case when  $(\mathcal{B}_0, \mathcal{B}_1) = (L^2(X), L^1(X))$ . The following theorem, introduced after Definition 1.3, generalises Foschi’s result and is the main theorem of our paper.

**Definition 1.3.** Suppose that  $\sigma > 0$ . We say that a pair  $(q, \theta)$  of exponents is  *$\sigma$ -acceptable* if either

$$1 \leq q < \infty, \quad 0 \leq \theta \leq 1, \quad \frac{1}{q} < \sigma\theta$$

or  $(q, \theta) = (\infty, 0)$ .

If  $(\mathcal{B}_0, \mathcal{B}_1)$  is a Banach interpolation couple then write  $\mathcal{B}_{\theta, q}$  for the real interpolation space  $(\mathcal{B}_0, \mathcal{B}_1)_{\theta, q}$ .

**Theorem 1.4.** *Suppose that  $\sigma > 0$  and that  $\{U(t) : t \in \mathbb{R}\}$  satisfies the energy estimate (6) and the dispersive estimate (7). Suppose also that the exponent pairs  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$  are  $\sigma$ -acceptable and satisfy the scaling condition*

$$(10) \quad \frac{1}{q} + \frac{1}{\tilde{q}} = \frac{\sigma}{2}(\theta + \tilde{\theta}).$$

(i) *If*

$$(11) \quad \frac{1}{q} + \frac{1}{\tilde{q}} < 1,$$

$$(12) \quad (\sigma - 1)(1 - \theta) \leq \sigma(1 - \tilde{\theta}), \quad (\sigma - 1)(1 - \tilde{\theta}) \leq \sigma(1 - \theta),$$

and, in the case when  $\sigma = 1$ , we have  $\theta < 1$  and  $\tilde{\theta} < 1$ , then the inhomogeneous Strichartz estimate (9) holds.

(ii) If  $q, \tilde{q} \in (1, \infty)$

$$(13) \quad \frac{1}{q} + \frac{1}{\tilde{q}} = 1$$

and

$$(14) \quad (\sigma - 1)(1 - \theta) < \sigma(1 - \tilde{\theta}), \quad (\sigma - 1)(1 - \tilde{\theta}) < \sigma(1 - \theta)$$

then the inhomogeneous Strichartz estimate

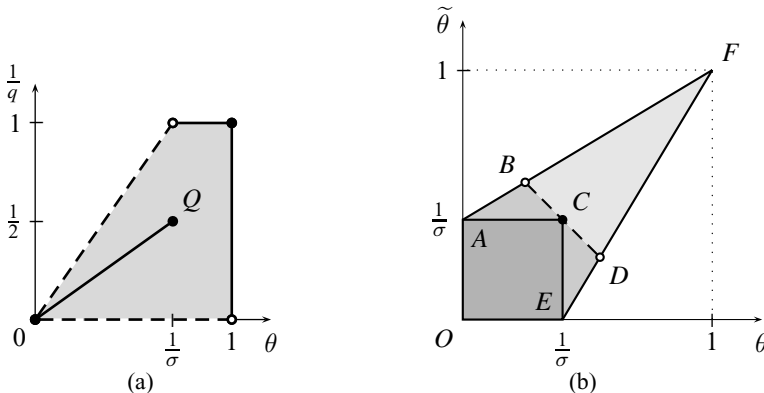
$$(15) \quad \|(TT^*)_R F\|_{L^q(\mathbb{R}; (\mathcal{B}_{\theta, q'})^*)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}, \tilde{q}'})} \quad \forall F \in L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}, \tilde{q}'}) \cap L^1(\mathbb{R}; \mathcal{B}_0)$$

holds.

**Remark 1.5.** Suppose that the scaling condition (10) holds. Then the exponent conditions appearing in (i) and (ii) above are always satisfied if  $\sigma < 1$  or if  $\sigma = 1$ ,  $\theta < 1$  and  $\tilde{\theta} < 1$ .

The closure in the cube  $[0, 1]^4$  of the set of points  $(1/q, \theta, 1/\tilde{q}, \tilde{\theta})$  that satisfy the hypotheses of Theorem 1.4 forms a convex polyhedral solid. Projections of this set onto the  $\theta q^{-1}$ -plane and the  $\theta\tilde{\theta}$ -plane, as well as the improvement of Theorem 1.4 over Theorem 1.2, are shown in Figure 1. In Figure 1 (a), the closed line segment  $OQ$  corresponds to sharp  $\sigma$ -admissible pairs  $(q, \theta)$ , while the shaded region represents the set of  $\sigma$ -acceptable pairs. The region  $AOEDB$  in Figure 1 (b) illustrates pairs  $(\theta, \tilde{\theta})$  that correspond to inhomogeneous Strichartz estimates given by Theorem 1.4. In contrast, the region  $AOEC$  represents pairs  $(\theta, \tilde{\theta})$  that correspond to valid exponents for Theorem 1.2.

The proof of Theorem 1.4 is given in Sections 2 to 5, using techniques introduced in [13] and adapting part of the argument of [5]. Section 2 states some preliminary results that



**Figure 1** Range of exponents for Theorem 1.4 when  $\sigma > 1$ .

are used in later sections. Section 3 presents inhomogeneous Strichartz estimates that are localised in time. In particular, we prove a local estimate corresponding to the point  $F$  in Figure 1 (b), and then interpolate between  $F$  and the square  $AOEC$  (which corresponds to local estimates implied by Theorem 1.2) to obtain local estimates for exponents in the region  $AOEF$ . The global inhomogeneous estimates of Theorem 1.4 are then obtained by decomposing the operator  $(TT^*)_R$  dyadically as a sum of operators (see Section 4) and estimating each term in the sum by a local inhomogeneous estimate. The summability of the local estimates is obtained in Sections 4 and 5 by imposing further conditions on the exponents, from which we deduce global inhomogeneous estimates in the region  $AOEDB$ .

The sharpness of Theorem 1.4 is discussed in Section 6. The rest of the paper is devoted to applications of Theorem 1.4. In Section 7, we indicate how the results of Vilela [21] and Foschi [5] for the Schrödinger equation may be recovered from Theorem 1.4. It is then shown that, for a certain class of potentials, our generalisation allows one to obtain new Strichartz estimates for Schrödinger equations that cannot be deduced from the more specialised theorem of [5]. In Section 8, new inhomogeneous Strichartz-type estimates are obtained for the wave equation by using homogeneous Besov spaces, in the same spirit as those presented by J. Ginibre and G. Velo [7]. Finally, we indicate how one derives Strichartz estimates for the Klein–Gordon equation, thus recovering all the inhomogeneous estimates found by Nakamura and Ozawa [16], as well as giving some new ‘boundary’ estimates.

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## 2 Some preliminaries

In this section we give some basic tools that will be used to prove Theorem 1.4. First we introduce a scaling invariance result that generalises the observation in [13, Section 1]. Second, we present a bilinear formulation of the inhomogeneous Strichartz estimate (9).

**Lemma 2.1.** *If  $\lambda > 0$  then the estimates (6) and (7) are invariant under the scaling*

$$(16) \quad \begin{cases} U(t) & \leftarrow U(t/\lambda) \\ \langle f, g \rangle & \leftarrow \langle f, g \rangle \\ \|f\|_{\mathfrak{B}_0} & \leftarrow \|f\|_{\mathfrak{B}_0} \\ \|f\|_{\mathfrak{B}_1} & \leftarrow \lambda^{\sigma/2} \|f\|_{\mathfrak{B}_1}. \end{cases}$$

The lemma can be easily verified by observing that the scaling (16) induces the scaling

$$\|\phi\|_{\mathfrak{B}_1^*} \leftarrow \lambda^{-\sigma/2} \|\phi\|_{\mathfrak{B}_1^*} \quad \forall \phi \in \mathfrak{B}^*.$$

We note here that if  $\theta \in [0, 1]$  then scaling (16) implies the scaling

$$\|f\|_{\mathcal{B}_\theta} \leftarrow \lambda^{\sigma\theta/2} \|f\|_{\mathcal{B}_\theta}.$$

This may be proved directly from the definition of the real interpolated space  $\mathcal{B}_\theta$  using, say, the K-functional (see [19, Section 4.3] for details).

Following the lead of [13], the Strichartz estimate expressed in (9) as an operator estimate will be re-expressed as a bilinear estimate, thus facilitating flexibility in manipulation and interpolation. When  $\mathcal{B}$  is not the Hilbert space  $\mathcal{H}$ , denote by  $\langle f, g \rangle_{\mathcal{B}}$  the action of a linear functional  $g$  on an element  $f$  of  $\mathcal{B}$ . Suppose that  $F$  and  $G$  are in  $L^1(\mathbb{R}; \mathcal{B}_0)$ . Then

$$\begin{aligned} \langle (TT^*)_R F, G \rangle_{L^\infty(\mathbb{R}; \mathcal{B}_0^*)} &= \int_{\mathbb{R}} \left\langle U(t) \int_{-\infty}^t U(s)^* F(s) ds, G(t) \right\rangle_{\mathcal{B}_0^*} dt \\ &= \iint_{s < t} \langle U(s)^* F(s), U(t)^* G(t) \rangle ds dt. \end{aligned}$$

Now define the bilinear form  $B$  on  $L^1(\mathbb{R}; \mathcal{B}_0) \times L^1(\mathbb{R}; \mathcal{B}_0)$  by

$$(17) \quad B(F, G) = \iint_{s < t} \langle U(s)^* F(s), U(t)^* G(t) \rangle ds dt.$$

It is not hard to prove the following lemma.

**Lemma 2.2.** *Suppose that  $q, \tilde{q} \in [1, \infty]$  and  $\theta, \tilde{\theta} \in [0, 1]$ . Then the inhomogeneous Strichartz estimate (9) is equivalent to the bilinear estimate*

$$(18) \quad |B(F, G)| \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}})} \|G\|_{L^q(\mathbb{R}; \mathcal{B}_\theta)} \\ \forall F \in L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}) \cap L^1(\mathbb{R}; \mathcal{B}_0) \quad \forall G \in L^q(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}) \cap L^1(\mathbb{R}; \mathcal{B}_0),$$

where the bilinear form  $B$  is given by (17).

### 3 Local inhomogeneous Strichartz estimates

Our proof of Theorem 1.4 spans the next three sections, the first two of which closely follow [5, Sections 2 and 3]. The main result of this section gives the existence of localised inhomogeneous Strichartz estimates. The following lemma is a preliminary version of this result.

**Lemma 3.1.** *Suppose that  $\sigma > 0$  and that  $\{U(t) : t \in \mathbb{R}\}$  satisfies the energy estimate (6) and the dispersive estimate (7). Assume also that  $I$  and  $J$  are two time intervals of unit length separated by a distance of scale 1 (that is,  $|I| = |J| = 1$  and  $\text{dist}(I, J) \approx 1$ ). Then the local inhomogeneous Strichartz estimate*

$$(19) \quad \|TT^*F\|_{L^q(J; \mathcal{B}_\theta^*)} \lesssim \|F\|_{L^{\tilde{q}'}(I; \mathcal{B}_{\tilde{\theta}})} \quad \forall F \in L^{\tilde{q}'}(I; \mathcal{B}_{\tilde{\theta}}) \cap L^1(I; \mathcal{B}_0)$$

holds whenever the pairs  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$  satisfy the conditions

$$\begin{aligned}
 (20) \quad & q, \tilde{q} \in [1, \infty], \quad \theta, \tilde{\theta} \in [0, 1], \\
 (21) \quad & (\sigma - 1)(1 - \theta) \leq \sigma(1 - \tilde{\theta}), \quad (\sigma - 1)(1 - \tilde{\theta}) \leq \sigma(1 - \theta), \\
 (22) \quad & \frac{1}{q} \geq \frac{\sigma}{2}(\theta - \tilde{\theta}), \quad \frac{1}{\tilde{q}} \geq \frac{\sigma}{2}(\tilde{\theta} - \theta).
 \end{aligned}$$

If  $\sigma = 1$  then  $\theta$  and  $\tilde{\theta}$  must be strictly less than 1.

This lemma and other results appearing in the ensuing sections are proved using a localised version of Lemma 2.2. Given two intervals  $I$  and  $J$  of  $\mathbb{R}$ , write  $I \times J$  as  $Q$  and define  $B_Q$  by the formula

$$(23) \quad B_Q(F, G) = B(1_I F, 1_J G) = \iint_{(s,t) \in I \times J} \langle U(s)^* F(s), U(t)^* G(t) \rangle \, ds \, dt$$

whenever  $F$  and  $G$  belong to  $L^1(\mathbb{R}; \mathfrak{B}_0)$ . One can easily show that the local inhomogeneous estimate (19) is equivalent to the bilinear estimate

$$\begin{aligned}
 (24) \quad |B_Q(F, G)| &\lesssim \|F\|_{L^{\tilde{q}'}(I; \mathfrak{B}_{\tilde{\theta}})} \|G\|_{L^{q'}(J; \mathfrak{B}_\theta)} \\
 &\quad \forall F \in L^{\tilde{q}'}(I; \mathfrak{B}_{\tilde{\theta}}) \cap L^1(I; \mathfrak{B}_0) \quad \forall G \in L^{q'}(J; \mathfrak{B}_\theta) \cap L^1(J; \mathfrak{B}_0).
 \end{aligned}$$

*Proof of Lemma 3.1.* Suppose that  $I$  and  $J$  are two intervals satisfying the hypothesis of the theorem and write  $I \times J$  as  $Q$ . Let  $\Psi$  denote the set of points  $(1/q, \theta; 1/\tilde{q}, \tilde{\theta})$  in  $[0, 1]^4$  corresponding to the pairs  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$  for which estimate (19), or its bilinear equivalent (24), is valid.

The dispersive estimate (7) implies that

$$\begin{aligned}
 (25) \quad |B_Q(F, G)| &\leq \iint_Q |\langle F(s), U(s)U(t)^* G(t) \rangle_{\mathfrak{B}_1}| \, ds \, dt \\
 &\leq \iint_Q \|F(s)\|_{\mathfrak{B}_1} \|U(s)U(t)^* G(t)\|_{\mathfrak{B}_1^*} \, ds \, dt \\
 &\lesssim \int_J \int_I |t - s|^{-\sigma} \|F(s)\|_{\mathfrak{B}_1} \|G(t)\|_{\mathfrak{B}_1} \, ds \, dt \\
 &\lesssim \|F\|_{L^1(I; \mathfrak{B}_1)} \|G\|_{L^1(J; \mathfrak{B}_1)}.
 \end{aligned}$$

Hence  $(0, 1; 0, 1) \in \Psi$ . On the other hand, the dual

$$\left\| \int_{\mathbb{R}} U(s)^* F(s) \, ds \right\|_{\mathfrak{H}} \lesssim \|F\|_{L^{q'}(\mathbb{R}; \mathfrak{B}_\theta)} \quad \forall F \in L^{q'}(\mathbb{R}; \mathfrak{B}_\theta) \cap L^1(\mathbb{R}; \mathfrak{B}_0)$$

of the homogeneous Strichartz estimate (8) of Theorem 1.2 implies that

$$\begin{aligned}
 (26) \quad |B_Q(F, G)| &\leq \left\| \int_I U(s)^* F(s) \, ds \right\|_{\mathfrak{H}} \left\| \int_J U(t)^* G(t) \, ds \right\|_{\mathfrak{H}} \\
 &\lesssim \|F\|_{L^{\tilde{q}'}(I; \mathfrak{B}_{\tilde{\theta}})} \|G\|_{L^{q'}(J; \mathfrak{B}_\theta)}
 \end{aligned}$$



whenever  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$  are sharp  $\sigma$ -admissible. Complex interpolation between (25) and (26) shows that  $\Psi$  contains the convex hull of the set

$$(27) \quad (0, 1; 0, 1) \cup \left\{ (1/q, \theta; 1/\tilde{q}, \tilde{\theta}) : (q, \theta) \text{ and } (\tilde{q}, \tilde{\theta}) \text{ are } \sigma\text{-admissible pairs} \right\}.$$

Since  $G$  is restricted to a unit time interval, Hölder’s inequality gives

$$\|G\|_{L^{q'}(J; \mathcal{B}_\theta)} = \|1_J G\|_{L^{q'}(J; \mathcal{B}_\theta)} \leq \|1_J\|_{L^{r'}(J)} \|G\|_{L^{p'}(J; \mathcal{B}_\theta)} \lesssim \|G\|_{L^{p'}(J; \mathcal{B}_\theta)}$$

whenever  $1/q' = 1/r' + 1/p'$ . We can always perform this calculation provided that  $p \leq q$ . Similarly, if  $\tilde{p} \leq \tilde{q}$  then

$$\|F\|_{L^{\tilde{q}'}(I; \mathcal{B}_{\tilde{\theta}})} \lesssim \|F\|_{L^{\tilde{p}'}(I; \mathcal{B}_{\tilde{\theta}})}.$$

Hence if  $(1/q, \theta; 1/\tilde{q}, \tilde{\theta}) \in \Psi$  then  $(1/p, \theta; 1/\tilde{p}, \tilde{\theta}) \in \Psi$  whenever  $p \leq q$  and  $\tilde{p} \leq \tilde{q}$ . If we apply this property to the points of the convex hull of (27) then we obtain a set  $\Psi_*$ , contained in  $\Psi$ , that is described precisely by the conditions appearing in Lemma 3.1. Details of this computation are analogous to those of [5, Appendix A] and will be omitted.  $\square$

Recall (see Lemma 2.1) that the energy estimate (6) and dispersive estimate (7) are invariant with respect to the rescaling (16). By applying this scaling to the local inhomogeneous estimate (19), we obtain a version of Lemma 3.1 for intervals  $I$  and  $J$  that are not of unit length. Define  $\beta(q, \theta; \tilde{q}, \tilde{\theta})$  by the formula

$$(28) \quad \beta(q, \theta; \tilde{q}, \tilde{\theta}) = \frac{1}{q} + \frac{1}{\tilde{q}} - \frac{\sigma}{2}(\theta + \tilde{\theta}).$$

**Theorem 3.2.** *Suppose that  $\sigma > 0$ ,  $\lambda > 0$  and  $\{U(t) : t \in \mathbb{R}\}$  satisfies the energy estimate (6) and the untruncated decay estimate (7). Assume also that  $I$  and  $J$  are two time intervals of length  $\lambda$  separated by a distance of scale  $\lambda$  (that is,  $|I| = |J| = \lambda$  and  $\text{dist}(I, J) \approx \lambda$ ). Then the local inhomogeneous Strichartz estimate*

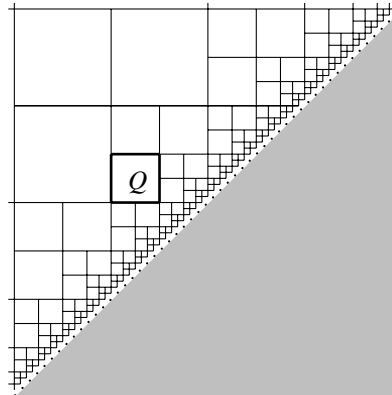
$$(29) \quad \|TT^*F\|_{L^q(J; \mathcal{B}_\theta^*)} \lesssim \lambda^{\beta(q, \theta; \tilde{q}, \tilde{\theta})} \|F\|_{L^{\tilde{q}'}(I; \mathcal{B}_{\tilde{\theta}})} \quad \forall F \in L^{\tilde{q}'}(I; \mathcal{B}_{\tilde{\theta}}) \cap L^1(I; \mathcal{B}_0)$$

holds whenever the pairs  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$  satisfy the conditions appearing in Lemma 3.1.

### 4 Dyadic decompositions

Theorem 3.2 gives spacetime estimates for  $(TT^*)_R$  which are localised in time. In this section we move toward obtaining global spacetime estimates for this operator.

We begin with a few preliminaries. We say that  $\lambda$  is a dyadic number if  $\lambda = 2^k$  for some integer  $k$ . Denote by  $2^{\mathbb{Z}}$  the set of all dyadic numbers. We say that a square in  $\mathbb{R}^2$  is a *dyadic square* if its side length  $\lambda$  is a dyadic number and if the all the coordinates of its vertices are integer multiples of  $\lambda$ . It is well known (see, for example, [8, Appendix J]) that any open set  $\Omega$  in  $\mathbb{R}^2$  can be decomposed as the union of essentially disjoint dyadic squares whose



**Figure 2** Whitney's decomposition for the region  $s < t$ .

lengths are approximately proportional to their distance from the boundary  $\partial\Omega$  of  $\Omega$ . Such a decomposition is known as a *Dyadic Whitney decomposition* of  $\Omega$ .

From now on, let  $\mathcal{Q}$  denote the Dyadic Whitney decomposition illustrated in Figure 2 for the domain  $\Omega$ , where

$$\Omega = \{(s, t) \in \mathbb{R}^2 : s < t\}.$$

For each dyadic number  $\lambda$ , let  $\mathcal{Q}_\lambda$  denote the family contained in  $\mathcal{Q}$  consisting of squares with side length  $\lambda$ . Each square  $Q$  in  $\mathcal{Q}_\lambda$  is the Cartesian product  $I \times J$  of two intervals of  $\mathbb{R}$  and has the property that

$$\lambda = |I| = |J| \approx \text{dist}(Q, \partial\Omega) \approx \text{dist}(I, J).$$

Since the squares  $Q$  in the decomposition of  $\Omega$  are essentially pairwise disjoint, we have the decomposition

$$(30) \quad B = \sum_{\lambda \in 2^{\mathbb{Z}}} \sum_{Q \in \mathcal{Q}_\lambda} B_Q,$$

where  $B$  is given by (17) and  $B_Q$  is given by (23) whenever  $Q = I \times J$ . The bilinear estimate

$$(31) \quad |B_Q(F, G)| \lesssim \lambda^{\beta(q, \theta; \tilde{q}, \tilde{\theta})} \|F\|_{L^{\tilde{q}'}(I; \mathcal{B}_{\tilde{\theta}})} \|G\|_{L^{q'}(J; \mathcal{B}_\theta)}$$

$$\forall F \in L^{\tilde{q}'}(I; \mathcal{B}_{\tilde{\theta}}) \cap L^1(I; \mathcal{B}_0) \quad \forall G \in L^{q'}(J; \mathcal{B}_\theta) \cap L^1(J; \mathcal{B}_0)$$

is equivalent to the scaled local inhomogeneous Strichartz estimate (29). The next proposition will enable us to replace the localised spaces  $L^{\tilde{q}'}(I; \mathcal{B}_{\tilde{\theta}})$  and  $L^{q'}(J; \mathcal{B}_\theta)$  with  $L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}})$  and  $L^{q'}(\mathbb{R}; \mathcal{B}_\theta)$  at the cost of imposing another condition on  $\tilde{q}$  and  $q$ .

**Proposition 4.1.** *Suppose that  $\sigma > 0$ ,  $1/q + 1/\tilde{q} \leq 1$ ,  $\lambda$  is a dyadic number and  $\{U(t) : t \in \mathbb{R}\}$  satisfies the energy estimate (6) and untruncated decay (7). If the pairs  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$  satisfy the conditions appearing in Lemma 3.1 then*

$$(32) \quad \sum_{Q \in \mathcal{Q}_\lambda} |B_Q(F, G)| \lesssim \lambda^{\beta(q, \theta; \tilde{q}, \tilde{\theta})} \|F\|_{L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}})} \|G\|_{L^{q'}(\mathbb{R}; \mathcal{B}_\theta)} \\ \forall F \in L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}) \cap L^1(\mathbb{R}; \mathcal{B}_0) \quad \forall G \in L^{q'}(\mathbb{R}; \mathcal{B}_\theta) \cap L^1(\mathbb{R}; \mathcal{B}_0).$$

The proposition is a consequence of Theorem 3.2 and [5, Lemma 3.2].

### 5 Proof of the global inhomogeneous Strichartz estimates

To obtain the required global bilinear estimate (18), one cannot simply sum (32) over all dyadic numbers  $\lambda$ , since the right-hand side is not summable in  $\lambda$ . To overcome this problem, we perturb the exponents slightly and interpolate to gain summability. It is at this stage that we depart from the approach of Foschi [5, Sections 4 and 5], preferring to use an abstract argument that appeals to real interpolation theory in much the same way as [13, Section 6]. The abstract argument has the advantage of admitting function spaces other than the Lebesgue spaces with the side-benefit of shorter proofs.

We require two facts about real interpolation. The first concerns real interpolation of weighted Lebesgue sequence spaces. Whenever  $s \in \mathbb{R}$  and  $1 < q < \infty$ , let  $\ell_s^q$  denote the space of all scalar-valued sequences  $\{a_j\}_{j \in \mathbb{Z}}$  such that

$$\|\{a_j\}_{j \in \mathbb{Z}}\|_{\ell_s^q} = \left( \sum_{j \in \mathbb{Z}} 2^{js} |a_j|^q \right)^{1/q} < \infty.$$

If  $q = \infty$  then the norm is defined by

$$\|\{a_j\}_{j \in \mathbb{Z}}\|_{\ell_s^\infty} = \sup_{j \in \mathbb{Z}} 2^{js} |a_j|.$$

A special case of [1, Theorem 5.6.1] says that if  $s_0$  and  $s_1$  are two different real numbers and  $0 < \theta < 1$  then

$$(33) \quad (\ell_{s_0}^\infty, \ell_{s_1}^\infty)_{\theta, 1} = \ell_s^1,$$

where  $s = (1 - \theta)s_0 + \theta s_1$ .

The second fact needed is given by the following lemma.

**Lemma 5.1** ([1, pp. 76–77]). *Suppose that  $(\mathcal{A}_0, \mathcal{A}_1)$ ,  $(\mathcal{B}_0, \mathcal{B}_1)$  and  $(\mathcal{C}_0, \mathcal{C}_1)$  are interpolation couples and that the bilinear operator  $S$  acts as a bounded transformation as indicated below:*

$$\begin{aligned} S &: \mathcal{A}_0 \times \mathcal{B}_0 \rightarrow \mathcal{C}_0 \\ S &: \mathcal{A}_0 \times \mathcal{B}_1 \rightarrow \mathcal{C}_1 \\ S &: \mathcal{A}_1 \times \mathcal{B}_0 \rightarrow \mathcal{C}_1. \end{aligned}$$

If  $\theta_0, \theta_1 \in (0, 1)$  and  $p, q, r \in [1, \infty]$  such that  $1 \leq 1/p + 1/q$  and  $\theta_0 + \theta_1 < 1$ , then  $S$  also acts as a bounded transformation in the following way:

$$S : (\mathcal{A}_0, \mathcal{A}_1)_{\theta_0, pr} \times (\mathcal{B}_0, \mathcal{B}_1)_{\theta_1, qr} \rightarrow (\mathcal{C}_0, \mathcal{C}_1)_{\theta_0 + \theta_1, r}.$$

We are now ready to prove the global inhomogeneous Strichartz estimates of Theorem 1.4.

**Lemma 5.2.** *Suppose that  $\sigma > 0$  and that  $\{U(t) : t \in \mathbb{R}\}$  satisfies the energy estimate (6) and the dispersive estimate (7). Then the inhomogeneous Strichartz estimate (9) holds whenever the exponent pairs  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$  satisfy the conditions*

$$\begin{aligned} q, \tilde{q} &\in (1, \infty), & \theta, \tilde{\theta} &\in [0, 1], \\ (\sigma - 1)(1 - \theta) &\leq \sigma(1 - \tilde{\theta}), & (\sigma - 1)(1 - \tilde{\theta}) &\leq \sigma(1 - \theta), \\ \frac{1}{q} &> \frac{\sigma}{2}(\theta - \tilde{\theta}), & \frac{1}{\tilde{q}} &> \frac{\sigma}{2}(\tilde{\theta} - \theta), \\ & & \frac{1}{q} + \frac{1}{\tilde{q}} &< 1 \end{aligned}$$

and

$$(34) \quad \frac{1}{q} + \frac{1}{\tilde{q}} = \frac{\sigma}{2}(\theta + \tilde{\theta}).$$

If  $\sigma = 1$  then we also require that  $\theta < 1$  and  $\tilde{\theta} < 1$ .

*Proof.* Suppose that the exponent pairs  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$  satisfy the conditions appearing in the statement of the theorem. Then there is a positive  $\epsilon$  such that the pairs  $(q_0, \theta)$  and  $(\tilde{q}_0, \tilde{\theta})$  and the pairs  $(q_1, \theta)$  and  $(\tilde{q}_1, \tilde{\theta})$ , defined by

$$\frac{1}{q_0} = \frac{1}{q} - \epsilon, \quad \frac{1}{\tilde{q}_0} = \frac{1}{\tilde{q}} - \epsilon, \quad \frac{1}{q_1} = \frac{1}{q} + 2\epsilon, \quad \frac{1}{\tilde{q}_1} = \frac{1}{\tilde{q}} + 2\epsilon,$$

also satisfy all the conditions appearing in the statement of the theorem except for (34).

Define a function  $\tilde{B}$  on  $L^1(\mathbb{R}; \mathcal{B}_0) \times L^1(\mathbb{R}; \mathcal{B}_0)$  by

$$\tilde{B}(F, G) = \left\{ \sum_{Q \in \mathcal{D}_{2^{-j}}} B_Q(F, G) \right\}_{j \in \mathbb{Z}}.$$

Proposition 4.1 implies that the maps

$$\begin{aligned} \tilde{B} &: L^{\tilde{q}'_0}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}) \times L^{q'_0}(\mathbb{R}; \mathcal{B}_\theta) \rightarrow \ell^\infty_{\beta(q_0, \theta; \tilde{q}_0, \tilde{\theta})} \\ \tilde{B} &: L^{\tilde{q}'_0}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}) \times L^{q'_1}(\mathbb{R}; \mathcal{B}_\theta) \rightarrow \ell^\infty_{\beta(q_1, \theta; \tilde{q}_0, \tilde{\theta})} \\ \tilde{B} &: L^{\tilde{q}'_1}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}) \times L^{q'_0}(\mathbb{R}; \mathcal{B}_\theta) \rightarrow \ell^\infty_{\beta(q_0, \theta; \tilde{q}_1, \tilde{\theta})} \end{aligned}$$

are bounded. Note that  $\beta(q_1, \theta; \tilde{q}_0, \tilde{\theta}) = \beta(q_0, \theta; \tilde{q}_1, \tilde{\theta})$ . So we may apply Lemma 5.1 to obtain the bounded map

$$(35) \quad \tilde{B} : (L^{\tilde{q}'_0}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}), L^{\tilde{q}'_1}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}))_{\eta_0, \tilde{q}'_1} \times (L^{q'_0}(\mathbb{R}; \mathcal{B}_\theta), L^{q'_1}(\mathbb{R}; \mathcal{B}_\theta))_{\eta_1, q'_1} \rightarrow (\ell^\infty_{\beta(q_0, \theta; \tilde{q}_0, \tilde{\theta})}, \ell^\infty_{\beta(q_1, \theta; \tilde{q}_0, \tilde{\theta})})_{\eta, 1}$$

where  $\eta_0 = \eta_1 = \frac{1}{3}$  and  $\eta = \eta_0 + \eta_1$ . It is easy to check that

$$(1 - \eta)\beta(q_0, \theta; \tilde{q}_0, \tilde{\theta}) + \eta\beta(q_1, \theta; \tilde{q}_0, \tilde{\theta}) = \beta(q, \theta; \tilde{q}, \tilde{\theta}) = 0.$$

If we combine this with (33) then (35) simplifies to

$$\tilde{B} : L^{\tilde{q}'_1}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}) \times L^{q'_1}(\mathbb{R}; \mathcal{B}_\theta) \rightarrow \ell^1_0,$$

from which we obtain the bilinear estimate (18). □

In the above proof we first perturbed the time exponents  $q$  and  $\tilde{q}$  in estimate (32) and then interpolated. Successful perturbation required strict inequalities in the conditions appearing in Lemma 3.1 that involved  $q$  and  $\tilde{q}$ . The proof (which we omit) of the next lemma uses the same idea, except that the spatial exponents  $\theta$  and  $\tilde{\theta}$  are perturbed instead. This allows us to recover some boundary cases that the previous lemma excludes.

**Lemma 5.3.** *Suppose that  $\sigma > 0$  and that  $\{U(t) : t \in \mathbb{R}\}$  satisfies the energy estimate (6) and the untruncated decay estimate (7). Then the inhomogeneous Strichartz estimate (15) holds whenever the exponent pairs  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$  satisfy the conditions*

$$(36) \quad \begin{aligned} q, \tilde{q} &\in (1, \infty], & \theta, \tilde{\theta} &\in (0, 1), \\ (\sigma - 1)(1 - \theta) &< \sigma(1 - \tilde{\theta}), & (\sigma - 1)(1 - \tilde{\theta}) &< \sigma(1 - \theta), \\ \frac{1}{q} &> \frac{\sigma}{2}(\theta - \tilde{\theta}), & \frac{1}{\tilde{q}} &> \frac{\sigma}{2}(\tilde{\theta} - \theta), \\ & & \frac{1}{q} + \frac{1}{\tilde{q}} &\leq 1 \end{aligned}$$

and

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{\sigma}{2}(\theta + \tilde{\theta}).$$

The two previous lemmata combine to give Theorem 1.4. For example, suppose that  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$  satisfy the conditions appearing in Theorem 1.4 case (ii). If  $\theta > 0$  and  $\tilde{\theta} > 0$  then  $\sigma$ -acceptability is equivalent to (36) by the scaling condition (10). In this case, Lemma 5.3 shows that the retarded Strichartz estimate (15) holds. On the other hand, if either  $\theta = 0$  or  $\tilde{\theta} = 0$  then  $\sigma$ -acceptability, (10) and (14) imply that both  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$  are sharp  $\sigma$ -admissible. Hence the Strichartz estimate (9) holds by Theorem 1.2. But since  $q \geq 2$  and  $\tilde{q} \geq 2$ , [1, Theorem 3.4.1] gives the continuous embeddings  $\mathcal{B}_{\theta, q'} \subseteq \mathcal{B}_\theta$  and  $\mathcal{B}_{\tilde{\theta}, \tilde{q}'} \subseteq \mathcal{B}_{\tilde{\theta}}$  and thus (9) implies (15).

### 6 The sharpness of the main theorem

In this section we briefly comment on the sharpness of the exponent conditions appearing in Theorem 1.4.

**Proposition 6.1.** *Suppose that  $\sigma > 0$  and that the inhomogeneous Strichartz estimate (9) holds for any  $\{U(t) : t \geq 0\}$  satisfying the energy estimate (6) and the dispersive estimate (7). Then  $(q, \theta)$  and  $(\tilde{q}, \tilde{\theta})$  must be  $\sigma$ -admissible pairs that satisfy the following conditions:*

$$(37) \quad \frac{1}{q} + \frac{1}{\tilde{q}} = \frac{\sigma}{2}(\theta + \tilde{\theta}),$$

$$(38) \quad \frac{1}{q} + \frac{1}{\tilde{q}} \leq 1,$$

$$(39) \quad |\theta - \tilde{\theta}| \leq \frac{1}{\sigma}$$

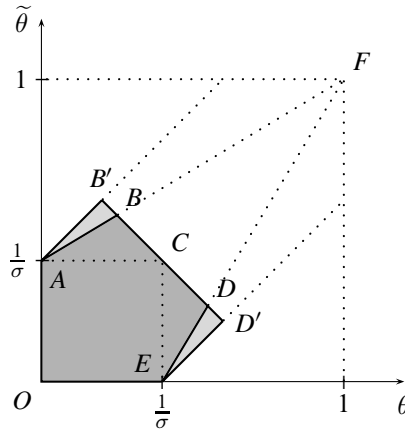
and

$$(40) \quad (\sigma - 1)(1 - \theta) - \frac{2}{q} \leq \sigma(1 - \tilde{\theta}), \quad (\sigma - 1)(1 - \tilde{\theta}) - \frac{2}{q} \leq \sigma(1 - \theta).$$

Moreover, if  $\sigma = 1$  then the inhomogeneous estimate is false when  $\theta = \tilde{\theta} = 1$ .

The proof follows from considerations on the Schrödinger group  $\{e^{it\Delta} : t \in \mathbb{R}\}$  on  $L^2(\mathbb{R}^n)$ ; see [5] or [21] for details.

We note that the difference in the necessary and sufficient conditions for the validity of the inhomogeneous Strichartz estimate (9) essentially lies in two places. First there is the gap between (40) and (12). Second, there is the gap between the range of values for  $\theta$  and  $\tilde{\theta}$  as shown in Figure 3. In particular, the region  $AOEDB$  corresponds to sufficient conditions for  $\theta$  and  $\tilde{\theta}$  while the region  $AOED'B'$  corresponds to necessary conditions. The boundaries of each region are included except the line segment  $BD$  for the sufficient conditions. This discrepancy along  $BD$  is muted somewhat by the validity of the inhomogeneous estimate (15) when  $\frac{\sigma}{2}(\theta + \tilde{\theta}) = 1$ .



**Figure 3** Necessary and sufficient conditions on the exponents  $\theta$  and  $\tilde{\theta}$  for global inhomogeneous Strichartz estimates.

### 7 Application to the Schrödinger equation with potential

In this section we show how Theorem 1.4 is used to obtain Strichartz estimates for various Schrödinger equations. First we consider the standard Schrödinger equation (that is, without potential) and show that Theorem 1.4 recovers the Strichartz estimates obtained by Foschi [5] and Vilela [21]. After this we obtain Strichartz estimates for Schrödinger equations with potential (see Corollary 7.7); these cannot be deduced from the results of [5] and [21].

First we remind readers about the real interpolation of  $L^p$  spaces. Suppose that  $p_0, p_1 \in [1, \infty], p_0 \neq p_1, \min(p_0, p_1) < q \leq \infty$  and  $0 < \theta < 1$ . If  $1/p = (1 - \theta)/p_0 + \theta p_1$  then

$$(L^{p_0}(\mathbb{R}^n), L^{p_1}(\mathbb{R}^n))_{\theta, q} = L^{p, q}(\mathbb{R}^n),$$

where  $L^{p, q}(\mathbb{R}^n)$  denotes the Lorentz space (with exponents  $p$  and  $q$ ) on  $\mathbb{R}^n$  (see [1, Theorem 5.2.1]). Moreover, we have the continuous embedding

$$L^p(\mathbb{R}^n) \subseteq (L^{p_0}(\mathbb{R}^n), L^{p_1}(\mathbb{R}^n))_{\theta, q} = L^{p, q}(\mathbb{R}^n)$$

whenever  $p \leq q$  (see [1, p. 2]).

Suppose that  $n$  is a positive integer. We say that a pair  $(q, r)$  of Lebesgue exponents are *Schrödinger  $n$ -acceptable* if either

$$1 \leq q < \infty, \quad 2 \leq r \leq \infty, \quad \frac{1}{q} < n \left( \frac{1}{2} - \frac{1}{r} \right)$$

or  $(q, r) = (\infty, 2)$ .

**Corollary 7.1** (Foschi [5], Vilela [21]). *Suppose that  $n$  is a positive integer and that the exponent pairs  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  are Schrödinger  $n$ -acceptable, satisfy the scaling condition*

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{n}{2} \left( 1 - \frac{1}{r} - \frac{1}{\tilde{r}} \right)$$

and either the conditions

$$\frac{1}{q} + \frac{1}{\tilde{q}} < 1, \quad \frac{n-2}{r} \leq \frac{n}{\tilde{r}}, \quad \frac{n-2}{\tilde{r}} \leq \frac{n}{r}$$

or the conditions

$$\frac{1}{q} + \frac{1}{\tilde{q}} = 1, \quad \frac{n-2}{r} < \frac{n}{\tilde{r}}, \quad \frac{n-2}{\tilde{r}} < \frac{n}{r}, \quad \frac{1}{r} \leq \frac{1}{q}, \quad \frac{1}{\tilde{r}} \leq \frac{1}{\tilde{q}}.$$

When  $n = 2$  we also require that  $r < \infty$  and  $\tilde{r} < \infty$ . If  $F \in L^{\tilde{q}}(\mathbb{R}; L^{r'}(\mathbb{R}^n))$  and  $u$  is a weak solution of the inhomogeneous Schrödinger equation

$$iu'(t) + \Delta u(t) = F(t), \quad u(0) = 0$$

then

$$(41) \quad \|u\|_{L^q(\mathbb{R}; L^r(\mathbb{R}^n))} \lesssim \|F\|_{L^{\tilde{q}}(\mathbb{R}; L^{r'}(\mathbb{R}^n))}.$$

*Proof.* This is a simple application of Theorem 1.4 when  $\mathcal{H} = L^2(\mathbb{R}^n)$ ,  $(\mathcal{B}_0, \mathcal{B}_1) = (L^2(\mathbb{R}^n), L^1(\mathbb{R}^n))$ ,  $\sigma = n/2$  and  $U(t) = e^{it\Delta}$ . That the energy estimate is satisfied follows from Plancherel’s theorem, while the dispersive estimate follows from a simple bound on the integral representation of  $e^{it\Delta}$  (see, for example, [5, Section 6] for details). To obtain (41) from (15), we use the embedding  $L^{r'}(\mathbb{R}^n) \subseteq L^{r',q'}(\mathbb{R}^n)$  whenever  $r' \leq q'$ .  $\square$

We now show that our generalisation of Foschi’s work [5] allows one to obtain new Strichartz estimates for Schrödinger equations involving certain potentials.

Suppose that  $V : \mathbb{R}^3 \rightarrow \mathbb{R}$  is a real-valued potential on  $\mathbb{R}^3$  with decay

$$(42) \quad |V(x)| \leq C \langle x \rangle^{-\beta} \quad \forall x \in \mathbb{R}^3,$$

where  $\beta > 5/2$  and  $\langle x \rangle = (1 + |x|^2)^{1/2}$ . Consider the Hamiltonian operator  $H$ , given by  $H = -\Delta + V$ , on the Hilbert space  $L^2(\mathbb{R}^3)$  with domain  $W^{2,2}(\mathbb{R}^3)$ , where  $W^{k,p}(X)$  denotes the Sobolev space of order  $k$  in  $L^p(X)$ . Our goal is to obtain spacetime estimates for the solution  $u$  of the inhomogeneous initial value problem

$$(43) \quad \begin{cases} (i\frac{\partial}{\partial t} + H) u(t) = F(t) & \forall t \in [0, \tau], \\ u(0) = f, \end{cases}$$

where  $\tau > 0$  and, for each time  $t$  in  $\mathbb{R}$ ,  $f$  and  $F(t)$  are complex-valued functions on  $\mathbb{R}^3$ .



Hamiltonians that satisfy the above conditions are considered by K. Yajima in [23]. There it mentions that  $H$  is self-adjoint on  $L^2(\mathbb{R}^3)$  with a spectrum consisting of a finite number of nonpositive eigenvalues, each of finite multiplicity, and the absolutely continuous part  $[0, \infty)$ . Denote by  $P_c$  the orthogonal projection from  $L^2(\mathbb{R}^3)$  onto the continuous spectral subspace for  $H$ . Under the general assumption (42), it is known that  $P_c$ , when viewed as an operator on  $L^p(\mathbb{R}^3)$ , is bounded only when  $2/3 < p < 3$ .

Denote by  $\mathcal{H}_\gamma$  the weighted Lebesgue space  $L^2(\mathbb{R}^3, \langle x \rangle^{2\gamma} dx)$ . When  $\gamma \in (1/2, \beta - 1/2)$ , define the null space  $\mathcal{N}$  by

$$\mathcal{N} = \left\{ \phi \in \mathcal{H}_{-\gamma} : \phi(x) + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{V(y)\phi(y)}{|x - y|} dy = 0 \right\}.$$

As noted in [23], the space  $\mathcal{N}$  is finite dimensional and is independent of the choice of  $\gamma$  in the interval  $(1/2, \beta - 1/2)$ . All  $\phi$  belonging to  $\mathcal{N}$  satisfy the stationary Schrödinger equation

$$(44) \quad -\Delta\phi(x) + V(x)\phi(x) = 0,$$

where (44) is to be interpreted in the distributional sense. Conversely, any function  $\phi \in \mathcal{H}_{-3/2}$  which satisfies (44) belongs to  $\mathcal{N}$ . Hence, if 0 is an eigenvalue of  $H$  with associated eigenspace  $\mathcal{E}$ , then  $\mathcal{E}$  is a subspace of  $\mathcal{N}$ .

**Definition 7.2.** We say that  $H$  or  $V$  is of *generic type* if  $\mathcal{N} = \{0\}$  and is of *exceptional type* otherwise. The Hamiltonian  $H$  is of *exceptional type of the first kind* if  $\mathcal{N} \neq \{0\}$  and 0 is not an eigenvalue of  $H$ . It is of *exceptional type of the second kind* if  $\mathcal{E} = \mathcal{N} \neq \{0\}$ . Finally, we say that  $H$  is of *exceptional type of the third kind* if  $\{0\} \subset \mathcal{E} \subset \mathcal{N}$  with strict inclusions.

While most  $V$  are of generic type, examples that are of exceptional type are interesting from a physical point of view. In particular, if  $V$  is of exceptional of the third kind then any function  $\phi$  in  $\mathcal{N} \setminus \mathcal{E}$  is called a *resonance* of  $H$ .

We would like to apply Theorems 1.2 and 1.4 to the case where  $U(t)$  is the operator  $e^{itH}$ , defined by the functional calculus for self-adjoint operators. However, if  $g$  is an eigenfunction of  $H$  with corresponding eigenvalue  $\lambda$ , then

$$(45) \quad U(s)U(t)^*g = e^{i(s-t)H}g = e^{i(s-t)\lambda}g$$

and therefore  $U(s)U(t)^*g$  is stationary. Consequently, the dispersive hypothesis (7) is not satisfied. Fortunately, this is not the case if  $g$  lies in the continuous spectral subspace of  $H$ .

**Theorem 7.3** (K. Yajima [23]). *There exists a positive constant  $C_p$  such that the dispersive estimate*

$$(46) \quad \|e^{itH}P_c g\|_{p'} \leq C_p |t|^{-3(1/p-1/2)} \|g\|_p \quad \forall g \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \quad \forall \text{ real } t \neq 0$$

*is satisfied in the following two cases:*

- (i) if  $H$  is of generic type,  $\beta > 5/2$  and  $1 \leq p \leq 2$ ; and
- (ii) if  $H$  is of exceptional type,  $\beta > 11/2$  and  $3/2 < p \leq 2$ .

**Remark 7.4.** If  $H$  is of exceptional type then (46) cannot hold when  $p = 1$ , otherwise it would contradict the local decay estimate of Jensen–Kato [10] or Murata [15]. Hence one cannot apply the results of Foschi [5] to this situation.

Our immediate goal is to apply Theorems 1.2 and 1.4 to the continuous spectral subspace of  $H$ . If  $u$  is a solution to (43), define  $u_c$  by  $u_c(t) = P_c u(t)$  for all  $t$  in  $[0, \tau]$ . Similarly, let  $P_{pp}$  denote the orthogonal projection onto the pure-point spectral subspace of  $H$  and define  $u_{pp}$  by  $u_{pp}(t) = P_{pp} u(t)$  for all  $t$  in  $[0, \tau]$ . It is clear that  $u = u_{pp} + u_c$ .

The dispersive estimate (46) gives rise to the admissibility conditions

$$(47) \quad \frac{1}{q} + \frac{3}{2r} = \frac{3}{4}, \quad 4 < q \leq \infty; \quad \frac{1}{\tilde{q}} + \frac{3}{2\tilde{r}} = \frac{3}{4}, \quad 4 < \tilde{q} \leq \infty$$

sketched in Figure 4. These correspond to the sharp  $\sigma$ -admissibility conditions in the case when  $\sigma = 3(1/p - 1/2)$ ,  $\mathcal{H} = \mathcal{B}_0 = L^2(\mathbb{R}^3)$ ,  $\mathcal{B}_1 = L^p(\mathbb{R}^3)$  and  $p \rightarrow 3/2$  from above. Note that they also correspond to the Schrödinger admissibility conditions (5) when  $n = 3$ , but with restricted range.

When considering the inhomogeneous problem with zero initial data, the exponent conditions of Theorem 1.4 reduce to the scaling condition

$$(48) \quad \frac{1}{q} + \frac{1}{\tilde{q}} = \frac{3}{2} \left( 1 - \frac{1}{r} - \frac{1}{\tilde{r}} \right)$$

and the acceptability conditions

$$(49) \quad 1 \leq q < \infty, \quad 2 \leq r < 3, \quad \frac{1}{q} < 3 \left( \frac{1}{2} - \frac{1}{r} \right), \quad \text{or } (q, r) = (\infty, 2);$$

$$(50) \quad 1 \leq \tilde{q} < \infty, \quad 2 \leq \tilde{r} < 3, \quad \frac{1}{\tilde{q}} < 3 \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right), \quad \text{or } (\tilde{q}, \tilde{r}) = (\infty, 2).$$

This is because  $\sigma = 3(1/p - 1/2) < 1$ .

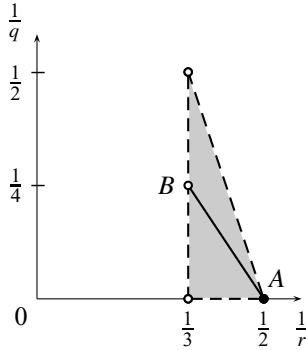
**Corollary 7.5.** Suppose that  $u$  is a (weak) solution to problem (43) for some data  $f$  in  $L^2(\mathbb{R}^3)$ , some source  $F$  and for some time  $\tau$  in  $(0, \infty)$ .

- (i) If  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  satisfy the admissibility condition (47) and  $F$  belongs to  $L^{\tilde{q}'}([0, \tau]; L^{\tilde{r}'}(\mathbb{R}^3))$ , then

$$(51) \quad \|u_c\|_{L^q([0, \tau], L^r(\mathbb{R}^3))} \lesssim \|f\|_{L^2(\mathbb{R}^3)} + \|F\|_{L^{\tilde{q}'}([0, \tau], L^{\tilde{r}'}(\mathbb{R}^3))}.$$

- (ii) If the exponent pairs  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  satisfy conditions (48), (49) and (50),  $f = 0$  and  $F \in L^{\tilde{q}'}([0, \tau]; L^{\tilde{r}'}(\mathbb{R}^3))$ , then

$$\|u_c\|_{L^q([0, \tau], L^r(\mathbb{R}^3))} \lesssim \|F\|_{L^{\tilde{q}'}([0, \tau], L^{\tilde{r}'}(\mathbb{R}^3))}.$$



**Figure 4** The line segment  $AB$  and the shaded region respectively give admissible and acceptable exponents for Strichartz estimates associated to the inhomogeneous initial value problem (43).

*Proof.* Fix  $p$  such that

$$3/2 < p < \min\{r', \tilde{r}'\}.$$

For  $t$  in  $\mathbb{R}$  define  $U(t)$  on  $L^2(\mathbb{R}^3)$  by  $U(t) = 1_{[0,\tau]}(t)e^{itH}P_c$ . If  $g$  belongs to  $L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ , then

$$\|U(s)U(t)^*g\|_{p'} \leq C_p |s - t|^{-3(1/p - 1/2)} \|g\|_p$$

by Theorem 7.3. Therefore  $\{U(t) : t \in \mathbb{R}\}$  satisfies the dispersive estimate (7) when  $\sigma = 3(1/p - 1/2)$ ,  $\mathcal{B}_0 = \mathcal{H} = L^2(\mathbb{R}^3)$  and  $\mathcal{B}_1 = L^p(\mathbb{R}^3)$ . Moreover, since each operator  $e^{-itH}$  on  $L^2(\mathbb{R}^3)$  is unitary and  $P_c$  is an orthogonal projection,  $\{U(t) : t \in \mathbb{R}\}$  also satisfies the energy estimate (6). Now if  $u$  is a weak solution to (43) then

$$(52) \quad u(t) = e^{itH}f - i \int_0^t e^{i(t-s)H}F(s) \, ds$$

by Duhamel’s principle and the functional calculus for self-adjoint operators. Hence  $u_c(t) = Tf(t) - i(TT^*)_R F(t)$ . An application of Theorem 1.2 and Theorem 1.4 gives the required spacetime estimates for  $u_c$  once we observe that

$$L^{r'}(\mathbb{R}^3) \subseteq L^{r',2}(\mathbb{R}^3) = \mathcal{B}_\theta,$$

where  $1/r' = (1 - \theta)/2 + \theta/p$  and the inclusion is continuous. □

To find a spacetime estimate for the solution  $u$  of (43), we now need only analyse the projection of each  $u(t)$  onto the pure point spectral subspace of  $H$ . It is known (see [23, p. 477]) that eigenfunctions of  $H$  with negative eigenvalues decay at least exponentially. Since such eigenfunctions belong to the domain of  $\Delta$ , they are necessarily continuous and consequently also belong to  $L^r(\mathbb{R}^3)$  whenever  $1 \leq r \leq \infty$  by Sobolev embedding. However, if 0 is an eigenvalue then a corresponding eigenfunction  $\phi$  may decay as slowly as  $C\langle x \rangle^{-2}$  when  $|x| \rightarrow \infty$ . Hence, in general,  $\phi$  is a member of  $L^p(\mathbb{R}^3)$  only when  $p > 3/2$ .

Except in the case when the time exponent is  $\infty$ , one cannot hope for a spacetime estimate for  $u_{pp}$  which is global in time due to (45). However, one can still obtain spacetime estimates on finite time intervals.

**Lemma 7.6.** *Suppose that  $\tau > 0$ , that  $q, \tilde{q} \in [1, \infty]$  and that  $r, \tilde{r} \in (3/2, 3)$ . Suppose also that  $f \in L^2(\mathbb{R}^3)$ ,  $F \in L^{\tilde{q}}([0, \tau], L^{\tilde{r}}(\mathbb{R}^3))$  and  $H$  is of exceptional type. If  $u$  is a (weak) solution to problem (43) then*

$$\|u_{pp}\|_{L^q([0, \tau], L^r(\mathbb{R}^3))} \leq C_{r,H} \left( \|P_{pp}f\|_2 + \tau^{1/q+1/\tilde{q}} \|P_{pp}F\|_{L^{\tilde{q}}([0, \tau], L^{\tilde{r}}(\mathbb{R}^3))} \right)$$

where the positive constant  $C_{r,H}$  depends on  $r$  and  $H$  only. If  $q = \tilde{q} = \infty$  then  $\tau^{1/q+1/\tilde{q}}$  is interpreted as 1.

*Proof.* Suppose that  $\{\phi_j : j = 1, \dots, n\}$  is a complete orthonormal set of eigenfunctions for  $H$  on  $L^2(\mathbb{R}^3)$  corresponding to the set  $\{\lambda_j : j = 1, \dots, n\}$  of eigenvalues (counting multiplicities). Write

$$P_{pp}f = \sum_{i=1}^n \alpha_j \phi_j \quad \text{and} \quad P_{pp}F(s) = \sum_{j=1}^n \beta_j(s) \phi_j,$$

where each  $\alpha_j$  and  $\beta_j(s)$  is a complex scalar. By orthogonality and the equivalence of norms in finite dimensional normed spaces, there are positive constants  $C$  and  $C'$  (both independent of  $f, F(s), \{\alpha_j\}$  and  $\{\beta_j(s)\}$ ) such that

$$\sum_{j=1}^n |\alpha_j| \leq C \left( \sum_{j=1}^n |\alpha_j|^2 \right)^{1/2} = C \|P_{pp}f\|_2$$

and

$$\sum_{j=1}^n |\beta_j(s)| \leq C \left( \sum_{j=1}^n |\beta_j(s)|^2 \right)^{1/2} = C \|P_{pp}F(s)\|_2 \leq C' \|P_{pp}F(s)\|_{\tilde{r}}.$$

Following from (52),

$$u_{pp}(t) = \sum_{j=1}^n \alpha_j e^{it\lambda_j} \phi_j - i \int_0^t \sum_{j=1}^n \beta_j(s) e^{i(t-s)\lambda_j} \phi_j \, ds.$$

By taking the  $L^q([0, \tau], L^r(\mathbb{R}^3))$  norm and applying Hölder’s inequality,

$$\begin{aligned} & \|u_{pp}\|_{L^q([0, \tau], L^r(\mathbb{R}^3))} \\ & \leq \sum_{i=j}^n |\alpha_j| \|\phi_j\|_r + \left\| \int_0^t \sum_{j=1}^n |\beta_j(s)| \|\phi_j\|_r \, ds \right\|_{L^q([0, \tau])} \\ & \leq C'' \max_{1 \leq j \leq n} \|\phi_j\|_r \left( \|P_{pp}f\|_2 + \left\| \int_0^t \|P_{pp}F(s)\|_{\tilde{r}} \, ds \right\|_{L^q([0, \tau])} \right) \end{aligned}$$

$$\begin{aligned} &\leq C_{r,H} \left( \|P_{pp} f\|_2 + \tau^{1/q} \|P_{pp} F\|_{L^1([0,\tau];L^{\tilde{r}}(\mathbb{R}^3))} \right) \\ &\leq C_{r,H} \left( \|P_{pp} f\|_2 + \tau^{1/q+1/\tilde{q}} \|P_{pp} F\|_{L^{\tilde{q}'}([0,\tau];L^{\tilde{r}}(\mathbb{R}^3))} \right) \end{aligned}$$

where  $C_{r,H} = C'' \max_{1 \leq j \leq n} \|\phi_j\|_r$ . This completes the proof. □

Combining the lemma with Corollary 7.5 and the fact that  $u = u_c + u_{pp}$  gives the following result.

**Corollary 7.7.** *Suppose that  $H$  is of exceptional type, that  $\tau > 0$  and that  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  satisfy the admissibility conditions (47). If  $f \in L^2(\mathbb{R}^3)$  and  $F \in L^{\tilde{q}'}([0, \tau], L^{\tilde{r}}(\mathbb{R}^3))$  and  $u$  is a (weak) solution to problem (43) then*

$$(53) \quad \|u\|_{L_t^q([0,\tau],L^r(\mathbb{R}^3))} \lesssim \|f\|_{L^2(\mathbb{R}^3)} + (1 + \tau^{1/q+1/\tilde{q}}) \|F\|_{L_t^{\tilde{q}'}([0,\tau],L^{\tilde{r}}(\mathbb{R}^3))}.$$

If  $q = \tilde{q} = \infty$  then  $\tau^{1/q+1/\tilde{q}}$  is interpreted as 1. If  $f = 0$  then the conditions on  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  may be relaxed to satisfying (48), (49) and (50).

### 8 Application to the wave equation

In this section we consider the wave equation. Keel and Tao [13] indicated that Theorem 1.2 could be used to obtain Strichartz estimates (in Besov norms), following a similar approach to [7], but did not show any details. We shall therefore do so here, presenting the results in Corollary 8.3. These estimates coincide with those of [7], except that the so-called ‘endpoint’ estimate is now included (see the point  $Q$  in Figure 5 (a)). Next we apply Theorem 1.4 to the wave equation, thus obtaining a new set of inhomogeneous Strichartz estimates for the wave equation (see Corollary 8.7 and Remark 8.8). Finally, we indicate how a small modification of these arguments enables one to obtain Strichartz estimates for the Klein–Gordon equation (see Corollary 8.9).

This section assumes basic familiarity with homogeneous Sobolev and Besov spaces on  $\mathbb{R}^n$ , which we denote by  $\dot{H}_r^\rho$  and  $\dot{B}_{r,s}^\rho$ . For a brief but sufficient introduction, see [7] or [19, Section 3.4]; for a treatment at greater depth, the reader is referred to [1] and [20]. We here summarise some basic inclusion and interpolation results.

**Lemma 8.1** ([20, Section 2.4]). *Suppose that  $\rho_0, \rho_1 \in \mathbb{R}$ ,  $\rho_0 \neq \rho_1$ ,  $r_0, r_1 \in [1, \infty)$ ,  $r_0 \neq r_1$  and  $\theta \in (0, 1)$ . Then*

$$(54) \quad \dot{B}_{r,2}^\rho \subseteq \left( \dot{B}_{r_0,2}^{\rho_0}, \dot{B}_{r_1,2}^{\rho_1} \right)_{\theta,2}$$

where

$$\rho = (1 - \theta)\rho_0 + \theta\rho_1, \quad \frac{1}{r} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}$$

and the inclusion is continuous.

**Lemma 8.2** ([1, Section 6.5]). *Suppose that  $1 \leq r_2 \leq r_1 \leq \infty$ ,  $1 \leq s \leq \infty$ ,  $\rho_1, \rho_2 \in \mathbb{R}$  and  $\rho_1 - n/r_1 = \rho_2 - n/r_2$ . Then  $B_{r_2, s}^{\rho_2} \subseteq B_{r_1, s}^{\rho_1}$  and*

$$(55) \quad \|u\|_{\dot{B}_{r_1, s}^{\rho_1}} \leq C \|u\|_{\dot{B}_{r_2, s}^{\rho_2}}$$

for some positive constant  $C$ .

The homogeneous Sobolev spaces are related to the homogeneous Besov spaces by the continuous embeddings

$$(56) \quad \dot{B}_{r, 2}^{\rho} \subseteq \dot{H}_r^{\rho} \quad \text{when } 2 \leq r < \infty; \quad \dot{B}_{r, 2}^{\rho} \supseteq \dot{H}_r^{\rho} \quad \text{when } 1 < r \leq 2,$$

whenever  $\rho \in \mathbb{R}$ . When  $r = 2$  it is customary to write  $\dot{H}^{\rho}$  instead of  $\dot{H}_2^{\rho}$ . In this case (56) reduces to  $\dot{H}^{\rho} = \dot{B}_{2, 2}^{\rho}$ .

We are now ready to apply Theorem 1.2 to the wave equation.

**Corollary 8.3.** *Suppose that  $n \geq 1$ , that  $\mu, \rho, \tilde{\rho} \in \mathbb{R}$ , that  $q, \tilde{q} \in [2, \infty]$  and that the following conditions are satisfied:*

$$(57) \quad \begin{aligned} q &\geq 2, & \tilde{q} &\geq 2, \\ \frac{1}{q} &\leq \frac{n-1}{2} \left( \frac{1}{2} - \frac{1}{r} \right), & \frac{1}{\tilde{q}} &\leq \frac{n-1}{2} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right), \\ (q, r, n) &\neq (2, \infty, 3), & (\tilde{q}, \tilde{r}, n) &\neq (2, \infty, 3), \\ \rho + n \left( \frac{1}{2} - \frac{1}{r} \right) - \frac{1}{q} &= \mu = 1 - \left( \tilde{\rho} + n \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) - \frac{1}{\tilde{q}} \right). \end{aligned}$$

Suppose also that  $f \in \dot{H}^{\mu}$ ,  $g \in \dot{H}^{\mu-1}$  and  $F \in L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r}, 2}^{-\tilde{\rho}})$ . If  $u$  is a (weak) solution to the initial value problem

$$(58) \quad \begin{cases} -u''(t) + \Delta u(t) = F(t) \\ u(0) = f \\ u'(0) = g \end{cases}$$

then

$$(59) \quad \|u\|_{L^q(\mathbb{R}; \dot{B}_{r, 2}^{\rho})} \lesssim \|f\|_{\dot{H}^{\mu}} + \|g\|_{\dot{H}^{\mu-1}} + \|F\|_{L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r}, 2}^{-\tilde{\rho}})}.$$

When  $n > 3$ , the darker closed region of Figure 5 (a) represents the range of exponent pairs  $(q, r)$  and  $(\tilde{q}, \tilde{r})$  such that the Strichartz estimate (59) is valid.

**Remark 8.4.** Corollary 8.3 implies Strichartz estimates for spaces more familiar than the Besov spaces. By Besov–Sobolev embedding, estimate (59) still holds when  $\dot{B}_{r, 2}^{\rho}$  is replaced

everywhere by  $\dot{H}_r^\rho$  under the additional assumption that  $r < \infty$  and  $\tilde{r} < \infty$ . In fact, using Sobolev embedding, one can deduce that

$$\|u\|_{L^q(\mathbb{R}; L^r(\mathbb{R}^n))} \lesssim \|f\|_{\dot{H}^\mu} + \|g\|_{\dot{H}^{\mu-1}} + \|F\|_{L^{\tilde{r}'}(\mathbb{R}; L^{r'}(\mathbb{R}^n))}$$

under the additional assumption that  $r < \infty$  and  $\tilde{r} < \infty$ . One may also replace the infinite interval  $\mathbb{R}$  by any finite time interval  $[0, \tau]$  where  $\tau > 0$ . See [13, Corollary 1.3] and [7] for these variations.

We begin with a heuristic argument to indicate how Theorem 1.2 will be applied in this setting. For convenience, write  $\omega$  for the operator  $(-\Delta)^{1/2}$ . The homogeneous problem may be written as

$$v''(t) + \omega^2 v(t) = 0, \quad v(0) = f, \quad v'(0) = g,$$

with solution  $v$  is given by

$$v(t) = \cos(\omega t) f + \omega^{-1} \sin(\omega t) g.$$

The inhomogeneous problem

$$-w''(t) + \Delta w(t) = F(t), \quad w(0) = 0, \quad w'(0) = 0$$

may be solved by Duhamel's principle to give

$$w(t) = \int_{s < t} \omega^{-1} \sin(\omega(t-s)) F(s) ds.$$

Define  $\{U(t) : t \in \mathbb{R}\}$  by  $U(t) = e^{i\omega t}$ . Then the solution  $u$  to problem (58) can be written as

$$\begin{aligned} (60) \quad u(t) &= v(t) + w(t) \\ &= \frac{1}{2}(U(t) + U(-t))f + \omega^{-1} \frac{1}{2i}(U(t) - U(-t))g \\ &\quad + \int_{s < t} \omega^{-1} \frac{1}{2i}(U(t)U(s)^* - U(-t)U(-s)^*)F(s) ds \end{aligned}$$

and it is clear that if we have appropriate Strichartz estimates for the group  $\{U(t) : t \in \mathbb{R}\}$  then (59) will follow. Hence define the operator  $T$  by  $Tf(t) = U(t)f$ , whenever  $f$  belongs to the Hilbert space  $\dot{B}_{2,2}^0$ .

**Lemma 8.5.** *Suppose that  $n \geq 1$  and that the triples  $(q, r, \gamma)$  and  $(\tilde{q}, \tilde{r}, \tilde{\gamma})$  satisfy the conditions*

$$\begin{aligned}
 (61) \quad & q \geq 2, \quad \tilde{q} \geq 2, \\
 (62) \quad & \frac{1}{q} = \frac{n-1}{2} \left( \frac{1}{2} - \frac{1}{r} \right), \quad \frac{1}{\tilde{q}} = \frac{n-1}{2} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right), \\
 (63) \quad & \gamma = \frac{n+1}{2} \left( \frac{1}{2} - \frac{1}{r} \right), \quad \tilde{\gamma} = \frac{n+1}{2} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right), \\
 (64) \quad & (q, r, n) \neq (2, \infty, 3), \quad (\tilde{q}, \tilde{r}, n) \neq (2, \infty, 3).
 \end{aligned}$$

Then the operator  $T$  satisfies the Strichartz estimates

$$(65) \quad \|Tf\|_{L^q(\mathbb{R}; \dot{B}_{r,2}^{-\gamma})} \lesssim \|f\|_{\dot{B}_{2,2}^0} \quad \forall f \in \dot{B}_{2,2}^0$$

and

$$(66) \quad \|(TT^*)RF\|_{L^q(\mathbb{R}; \dot{B}_{r,2}^{-\gamma})} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r},2}^{\tilde{\gamma}})} \quad \forall F \in L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r},2}^{\tilde{\gamma}}).$$

*Proof.* The dispersive estimate

$$\|U(t)f\|_{\dot{B}_{\infty,2}^{-(n+1)/4}} \lesssim |t|^{-(n-1)/2} \|f\|_{\dot{B}_{1,2}^{(n+1)/4}} \quad \forall f \in \dot{B}_{1,2}^{(n+1)/4}$$

is a consequence of a stationary phase estimate (see [7, pp. 62–63] or [19, Section 4.8] for a clear exposition). Moreover, each  $U(t)$  is an isometry on the homogeneous Sobolev space  $\dot{H}^0$  and hence we have the energy estimate

$$\|U(t)f\|_{\dot{B}_{2,2}^0} \lesssim \|f\|_{\dot{B}_{2,2}^0} \quad \forall f \in \dot{B}_{2,2}^0,$$

by (56). If  $\mathcal{H} = \mathcal{B}_0 = \dot{B}_{2,2}^0$  and  $\mathcal{B}_1 = \dot{B}_{1,2}^{(n+1)/4}$  then

$$\dot{B}_{r',2}^\gamma \subseteq \mathcal{B}_\theta = (\mathcal{B}_0, \mathcal{B}_1)_\theta,$$

by (54), where  $1/r' = (1 - \theta)/2 + \theta$  and  $\gamma = (n + 1)\theta/4$ . It is not hard to show from here that Theorem 1.2 proves the lemma. □

*Proof of Corollary 8.3.* It is well known that if  $\mu \in \mathbb{R}$ , then  $\omega^\mu$  is an isomorphism from  $\dot{B}_{r,2}^\gamma$  to  $\dot{B}_{r,2}^{\gamma-\mu}$ . Hence replacing  $f$  with  $\omega^\mu f$  in (65) gives

$$\|Tf\|_{L^q(\mathbb{R}; \dot{B}_{r,2}^{-\gamma+\mu})} \lesssim \|f\|_{\dot{B}_{2,2}^\mu} \quad \forall f \in \dot{B}_{2,2}^\mu.$$

The same trick applied to (66) yields

$$\|(TT^*)RF\|_{L^q(\mathbb{R}; \dot{B}_{r,2}^{-\gamma+\mu})} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r},2}^{\tilde{\gamma}+\mu})} \quad \forall F \in L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r},2}^{\tilde{\gamma}+\mu}).$$



If  $\rho = -\gamma + \mu$  and  $\tilde{\rho} = -(\tilde{\gamma} + \mu - 1)$ , then these estimates combine with (60) and the isomorphism  $\omega^{-1}$  to give

$$(67) \quad \|u\|_{L^q(\mathbb{R}; \dot{B}_{r,2}^\rho)} \lesssim \|f\|_{\dot{H}^\mu} + \|g\|_{\dot{H}^{\mu-1}} + \|F\|_{L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r},2}^{-\tilde{\rho}})}.$$

So far we have imposed the conditions  $\mu \in \mathbb{R}$ , (61), (62), (64) and

$$\rho + \frac{n+1}{2} \left( \frac{1}{2} - \frac{1}{r} \right) = \mu = 1 - \tilde{\rho} - \frac{n+1}{2} \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right).$$

This last condition may be rewritten as

$$\rho + n \left( \frac{1}{2} - \frac{1}{r} \right) - \frac{1}{q} = \mu = 1 - \tilde{\rho} - n \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) + \frac{1}{\tilde{q}}.$$

Now if  $r_1 \geq r$  and  $\rho - n/r = \rho_1 - n/r_1$ , then

$$\|u\|_{L^q(\mathbb{R}; \dot{B}_{r_1,2}^{\rho_1})} \leq C \|u\|_{L^q(\mathbb{R}; \dot{B}_{r,2}^\rho)}$$

by Lemma 8.2. Similarly, if  $\tilde{r}_1 \geq \tilde{r}$  and  $\tilde{\rho} - n/\tilde{r} = \tilde{\rho}_1 - n/\tilde{r}_1$ , then

$$\|F\|_{L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r}_1,2}^{-\tilde{\rho}_1})} \leq C \|F\|_{L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r},2}^{-\tilde{\rho}})}.$$

Applying these estimates to (67) gives

$$(68) \quad \|u\|_{L^q(\mathbb{R}; \dot{B}_{r_1,2}^{\rho_1})} \lesssim \|f\|_{\dot{H}^\mu} + \|g\|_{\dot{H}^{\mu-1}} + \|F\|_{L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r}_1,2}^{-\tilde{\rho}_1})}$$

whenever the conditions

$$\begin{aligned} q &\geq 2, & \tilde{q} &\geq 2, \\ \frac{1}{q} &\leq \frac{n-1}{2} \left( \frac{1}{2} - \frac{1}{r_1} \right), & \frac{1}{\tilde{q}} &\leq \frac{n-1}{2} \left( \frac{1}{2} - \frac{1}{\tilde{r}_1} \right), \\ (q, r_1, n) &\neq (2, \infty, 3), & (\tilde{q}, \tilde{r}_1, n) &\neq (2, \infty, 3), \\ \rho_1 + n \left( \frac{1}{2} - \frac{1}{r_1} \right) - \frac{1}{q} &= \mu = 1 - \tilde{\rho}_1 - n \left( \frac{1}{2} - \frac{1}{\tilde{r}_1} \right) + \frac{1}{\tilde{q}} \end{aligned}$$

are satisfied. These conditions and the Strichartz estimate (68) coincide with those in the statement of Corollary 8.3. □

**Remark 8.6.** One can see from (60) that the derivative  $u'$  can also be expressed in terms of  $T$ ,  $(TT^*)_R$  and  $\omega$ . Thus we have the Strichartz estimate

$$\|u'\|_{L^q(\mathbb{R}; \dot{B}_{r,2}^{\rho-1})} \lesssim \|f\|_{\dot{H}^\mu} + \|g\|_{\dot{H}^{\mu-1}} + \|F\|_{L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r},2}^{-\tilde{\rho}})}$$

whenever the exponents satisfy the conditions of Corollary 8.3.

We now consider the inhomogeneous wave equation with zero initial data. Suppose that  $n$  is a positive integer. We say that a pair  $(q, r)$  of Lebesgue exponents are *wave  $n$ -acceptable* if either

$$1 \leq q < \infty, \quad 2 \leq r \leq \infty, \quad \frac{1}{q} < (n - 1) \left( \frac{1}{2} - \frac{1}{r} \right)$$

or  $(q, r) = (\infty, 2)$ . By using the same method to prove Corollary 8.3, one can show that Theorem 1.4 (i) implies the following result.

**Corollary 8.7.** *Suppose that  $n$  is a positive integer and that the exponent pairs  $(q, r_1)$  and  $(\tilde{q}, \tilde{r}_1)$  are wave  $n$ -acceptable, satisfy the scaling condition*

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{n - 1}{2} \left( 1 - \frac{1}{r_1} - \frac{1}{\tilde{r}_1} \right)$$

and the conditions

$$\frac{1}{q} + \frac{1}{\tilde{q}} < 1, \quad \frac{n - 3}{r_1} \leq \frac{n - 1}{\tilde{r}_1}, \quad \frac{n - 3}{\tilde{r}_1} \leq \frac{n - 1}{r_1}.$$

When  $n = 3$  we also require that  $r_1 < \infty$  and  $\tilde{r}_1 < \infty$ . If  $r \geq r_1, \tilde{r} \geq \tilde{r}_1, \rho \in \mathbb{R}$ ,

$$\rho + n \left( \frac{1}{2} - \frac{1}{r} \right) - \frac{1}{q} = 1 - \left( \tilde{\rho} + n \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) - \frac{1}{\tilde{q}} \right),$$

$F \in L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r}', 2}^{-\tilde{\rho}})$  and  $u$  is a weak solution of the inhomogeneous wave equation

$$(69) \quad -u''(t) + \Delta u(t) = F(t), \quad u(0) = 0, \quad u'(0) = 0,$$

then

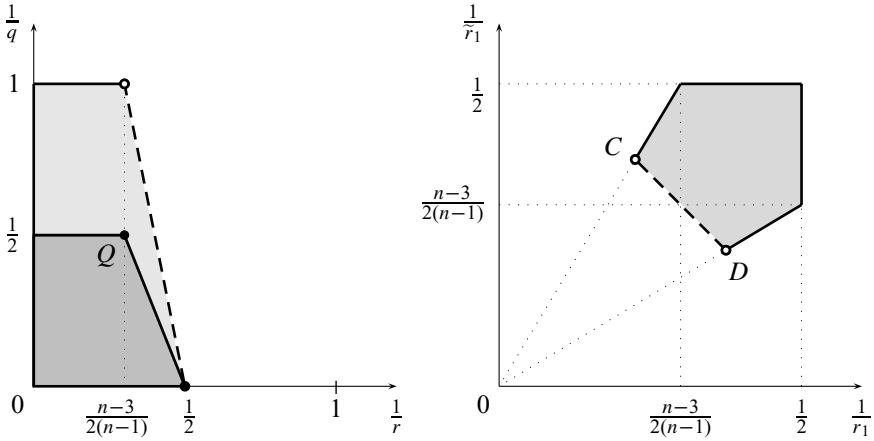
$$(70) \quad \|u\|_{L^q(\mathbb{R}; \dot{B}_{r, 2}^\rho)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r}', 2}^{-\tilde{\rho}})}.$$

Figure 5 shows the range for various exponents appearing in Corollary 8.7. In the first diagram, the dark region represents the range for the homogeneous Strichartz estimate while the union of light and dark regions represents the range for the inhomogeneous Strichartz estimate. In the second diagram, the coordinates of  $C$  and  $D$  are respectively given by

$$\left( \frac{(n-3)^2}{2(n-2)(n-1)}, \frac{n-3}{2(n-2)} \right) \quad \text{and} \quad \left( \frac{n-3}{2(n-2)}, \frac{(n-3)^2}{2(n-2)(n-1)} \right).$$

**Remark 8.8.** An application of Theorem 1.4 (ii) to the inhomogeneous wave equation (69) cannot be simply integrated into the results of Corollary 8.7. Instead, one obtains Strichartz estimates of the form

$$\|u\|_{L^q(\mathbb{R}; \dot{B}_{r, 2 \vee q}^\rho)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R}; \dot{B}_{\tilde{r}', 2 \wedge q}^{-\tilde{\rho}})}$$



**Figure 5** Range of exponents for Corollary 8.7 when  $n > 3$ .

where  $a \vee b$  and  $a \wedge b$  to denote  $\max\{a, b\}$  and  $\min\{a, b\}$  respectively. See [19, Section 5.8] for details.

As a final application of Theorem 1.4 (i), we consider the inhomogeneous Klein–Gordon equation

$$(71) \quad -u''(t) + \Delta u(t) - u = F(t), \quad u(0) = u''(0) = 0, \quad t \geq 0.$$

In a manner analogous to the wave equation, one can show that the weak solution  $u$  of (71) is given by

$$u(t) = \frac{1}{2i} \int_0^t \omega^{-1} (U(t)U(s)^* - U(-t)U(-s)^*) F(s) \, ds,$$

where  $\omega = (1 - \Delta)^{1/2}$  and  $U(t) = e^{it\omega}$ . Estimates for  $U(t)$  are naturally expressed using norms of the inhomogeneous Besov spaces  $B_{r,s}^\rho$ . The dispersive estimate

$$\|U(t)f\|_{B_{\infty,2}^{-\lambda/2}} \lesssim |t|^{-\sigma} \|f\|_{B_{1,2}^{\lambda/2}} \quad \forall f \in B_{1,2}^{\lambda/2},$$

where  $\lambda$  and  $\sigma$  satisfy condition (72) given below, is derived from the method of the stationary phase (see [16, pp. 261–262]). A corresponding energy estimate follows from the unitarity of  $U(t)$  on the Hilbert space  $B_{2,2}^0$ . By using the fact that the operator  $\omega^\mu$  is an isomorphism from  $B_{r,2}^\rho$  to  $B_{r,2}^{\rho-\mu}$  whenever  $\mu \in \mathbb{R}$ , one obtains the following corollary. This improves the range of inhomogeneous Strichartz estimates of Nakamura and Ozawa [16, Proposition 2.1] by relaxing their requirement of strict inequalities in (73).

**Corollary 8.9** (Nakamura–Ozawa). *Suppose that  $n$  is a positive integer,  $0 \leq \eta \leq 1$  and the real numbers  $\lambda$  and  $\sigma$  satisfy*

$$(72) \quad 2\lambda = n + 1 + \eta, \quad n - 1 - \eta \leq 2\sigma \leq n - 1 + \eta, \quad \sigma > 0.$$

*Suppose also that the exponent pairs  $(q, r_1)$  and  $(\tilde{q}, \tilde{r}_1)$  satisfy the acceptability condition*

$$1 \leq q < \infty, \quad 2 \leq r_1 \leq \infty, \quad \frac{1}{q} < 2\sigma \left( \frac{1}{2} - \frac{1}{r_1} \right); \quad \text{or} \quad (q, r_1) = (\infty, 2);$$

$$1 \leq \tilde{q} < \infty, \quad 2 \leq \tilde{r}_1 \leq \infty, \quad \frac{1}{\tilde{q}} < 2\sigma \left( \frac{1}{2} - \frac{1}{\tilde{r}_1} \right); \quad \text{or} \quad (\tilde{q}, \tilde{r}_1) = (\infty, 2);$$

*the scaling condition*

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \sigma \left( 1 - \frac{1}{r_1} - \frac{1}{\tilde{r}_1} \right),$$

*and the conditions*

$$(73) \quad \frac{1}{q} + \frac{1}{\tilde{q}} < 1,$$

$$\frac{\sigma - 1}{r_1} \leq \frac{\sigma}{\tilde{r}_1}, \quad \frac{\sigma - 1}{\tilde{r}_1} \leq \frac{\sigma}{r_1}.$$

*When  $\sigma = 1$  we also require that  $r_1 < \infty$  and  $\tilde{r}_1 < \infty$ . If  $r \geq r_1, \tilde{r} \geq \tilde{r}_1, \rho \in \mathbb{R}$ ,*

$$\rho + n \left( \frac{1}{2} - \frac{1}{r} \right) - \frac{\lambda - n}{\sigma q} = 1 - \left( \tilde{\rho} + n \left( \frac{1}{2} - \frac{1}{\tilde{r}} \right) - \frac{\lambda - n}{\sigma \tilde{q}} \right),$$

*$F \in L^{\tilde{q}'}(\mathbb{R}; B_{\tilde{r}, 2}^{-\tilde{\rho}})$  and  $u$  is a weak solution of the inhomogeneous Klein–Gordon equation (71) then*

$$\|u\|_{L^q(\mathbb{R}; B_{r, 2}^\rho)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R}; B_{\tilde{r}, 2}^{-\tilde{\rho}})}.$$

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