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ON THE GEOMETRY OF ROLLING AND INTERPOLATION CURVES ON S^n, SO_n, AND GRASSMANN MANIFOLDS

K. HÜPER and F. SILVA LEITE

Abstract. We present a procedure to generate smooth interpolating curves on submanifolds, which are given in closed form in terms of the coordinates of the embedding space. In contrast to other existing methods, this approach makes the corresponding algorithm easy to implement. The idea is to project the prescribed data on the manifold onto the affine tangent space at a particular point, solve the interpolation problem on this affine subspace, and then project the resulting curve back on the manifold. One of the novelties of this approach is the use of rolling mappings. The manifold is required to roll on the affine subspace like a rigid body, so that the motion is described by the action of the Euclidean group on the embedding space. The interpolation problem requires a combination of a pullback/push forward with rolling and unrolling. The rolling procedure by itself highlights interesting properties and gives rise to a new, but simple, concept of geometric polynomial curves on manifolds. This paper is an extension of our previous work, where mainly the 2-sphere case was studied in detail. The present paper includes results for the n-sphere, orthogonal group SO_n, and real Grassmann manifolds. In particular, we present the kinematic equations for rolling these manifolds along curves without slip or twist, and derive from them formulas for the parallel transport of vectors along curves on the manifold.

1. Introduction

Many engineering applications call for efficient methods to generate smooth interpolating curves on non-Euclidean spaces. This is the case, e.g., in path planning for mechanical systems whose configuration spaces have components which are Lie groups or symmetric spaces. Interpolation over a spherical surface also has immediate applications in the manufacturing

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industry. Several methods to generate interpolating curves on Riemannian manifolds are available in the literature. They correspond to appropriate generalizations of classical methods which have been around for many years. Without being exhaustive, we mention the variational approach to splines on manifolds [3, 6, 22], which can also be reformulated via a Hamiltonian formalism; the geometric approach that corresponds to the generalized De Casteljau algorithm [5, 23]; and the analytic approach undertaken in [10]. These generalized methods posed interesting new mathematical problems and many challenges regarding implementation as well. Even for the most simple cases, such as the 3-dimensional rotation group and the 2-sphere, explicit solutions are extremely hard to obtain. Here, following our previous work [15, 16], we present a method to generate interpolating curves on smooth submanifolds, which is based on a rolling and unwrapping technique. Detailed examples considered in this paper are the sphere S^n, orthogonal group SO_n, and Grassmann manifold of all k-dimensional subspaces of $\mathbb{R}^n$. The solution of the interpolating problem obtained by this method is given explicitly in terms of the coordinates of the embedding space. Moreover, since our solution curves are given in a closed form, they are easily implemented. Some of the ideas contained here were inspired by the work of Jupp and Kent [17] for the 2-sphere. The rolling of a manifold on its affine tangent space at a given point plays an important role here. The kinematic equations for the rolling of our favorite manifolds are derived. While for the sphere these equations are already known in the literature, for other manifolds, like the orthogonal group or Grassmann manifolds, the authors are not aware of any work were the kinematic equations are derived. Properties of the rolling curves are studied in connection with geometric splines, which, in turn, can be formulated as solutions of certain optimal control problems. This brings some insight to explore further optimality properties of the interpolating curves generated by the presented algorithm. This comes in contact with the optimal control problems for rolling bodies studied in [1, 30]. Some of the ideas presented here have been implemented on an experimental robot arm platform, see [25, 26]; for numerical experiments related to SO_3, see [27].

This paper is organized as follows. In Sec. 2, the main problem is stated. Section 3 includes the abstract definition of a rolling mapping and the kinematic equations for rolling S^n, SO_n, and Grassmann manifolds are derived. Rolling along straight lines results in formulas for the geodesics on these manifolds. In addition, the relation to parallel transport including explicit formulas for special cases are considered. The connections between rolling mappings and geometric splines are discussed in Sec. 4. This includes a short review of the variational approach to geometric splines, the relation to constrained variational problems and geodesic curvature. In Sec. 5 we present a procedure for solving interpolation problems explicitly. This includes as an example the 2-sphere followed by a few numerical experiments and plots.

2. Statement of the problem

Let M be a smooth k-dimensional manifold embedded into $\mathbb{R}^n$ (Whitney's theorem guarantees this for suitable n) so that, for all $p \in M$, the corresponding affine tangent space can also be considered as an affine subspace of $\mathbb{R}^n$.

Problem 2.1. Find a C^2-smooth curve

$$\gamma : [0, \tau] \to M \tag{2.1}$$

satisfying

$$\gamma(t_i) = p_i, \qquad 1 \leq i \leq k-1, \tag{2.2}$$

for a given set of distinct points $p_i \in M$ and fixed times t_i, where

$$0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = \tau, \tag{2.3}$$

and, in addition,

$$\begin{aligned} &\gamma(0) = p_0, \ \gamma(\tau) = p_k, \\ &\dot{\gamma}(0) = \xi_0 \in T_{p_0}M, \ \dot{\gamma}(\tau) = \xi_k \in T_{p_k}M, \end{aligned} \tag{2.4}$$

where ξ_0 and ξ_k are given tangent vectors to M at p_0 and p_k, respectively.

3. Rolling mappings

Rolling mappings play an important role in this paper. Here we are interested in rolling mappings that describe how a compact manifold M rolls without slipping or twisting on its affine tangent space V at a point $p_0 \in M$. (Both M and V are submanifolds of $\mathbb{R}^n$.) Since this is a rigid-body motion, it can be described by the usual action of the Euclidean group $\mathrm{SE}_n = \mathrm{SO}_n \ltimes \mathbb{R}^n$ on $\mathbb{R}^n$, through rotations and translations. We represent elements of the Euclidean group as pairs (R, s), $R \in \mathrm{SO}_n$, $s \in \mathbb{R}^n$, so that the group operations are defined by

$$(R_2, s_2) \circ (R_1, s_1) := (R_2 R_1, R_2 s_1 + s_2), \quad (R, s)^{-1} = (R^\top, -R^\top s).$$

The group SE_n acts on points of $\mathbb{R}^n$ in the usual way via $(R, s) \circ p = R \circ p + s$, and this action induces a linear mapping between $T_p\mathbb{R}^n$ and $T_{Rp+s}\mathbb{R}^n$, sending every ξ to $R\xi$. For each $p \in \mathbb{R}^n$, this action defines a mapping

$$\sigma_p : \mathrm{SE}_n \to V, \quad (R, s) \mapsto Rp + s, \tag{3.1}$$

whose derivative $D\sigma_p$, at the group identity $(I, 0)$, is computed as

$$D\sigma_p(I, 0) : \mathfrak{se}_n \to V, \quad (A, v) \mapsto Ap + v, \tag{3.2}$$

where

$$\mathfrak{se}_n = \{(A, v) \mid A \in \mathfrak{so}_n, v \in \mathbb{R}^n\} \tag{3.3}$$

is the Lie algebra of SE_n.

The general definition of a rolling mapping [24] can easily be adapted to the present situation as follows.

Definition 3.1. A smooth mapping

$$h : [0, \tau] \to \mathrm{SE}_n = \mathrm{SO}_n \ltimes \mathbb{R}^n, \quad t \mapsto h(t) = (R(t), s(t)) \tag{3.4}$$

where

$$R : [0, \tau] \to \mathrm{SO}_n, \quad s : [0, \tau] \mapsto \mathbb{R}^n \tag{3.5}$$

satisfying the following properties 1–3 for each $t \in [0, \tau]$ is called a *rolling of M on V without slipping or twisting*.

1. (The rolling condition.) There exists a smooth rolling curve on M, $\alpha : [0, \tau] \to M$ such that for all $t \in [0, \tau]$
   (a) $h(t) \circ \alpha(t) \in V$,
   (b) $T_{h(t)\circ\alpha(t)}(h(t) \circ M) = T_{h(t)\circ\alpha(t)}V$.
   The curve $\alpha_{\mathrm{dev}} : [0, \tau] \to V$ defined by $\alpha_{\mathrm{dev}}(t) = h(t) \circ \alpha(t)$ is called the *development of α on V*.
2. (The no-slip condition.)
$$\left(\dot{h}(t) \circ h(t)^{-1}\right) \circ \alpha_{\mathrm{dev}}(t) = 0$$
   for all $t \in [0, \tau]$.
3. (The no-twist condition.) For all $t \in [0, \tau]$, the following conditions hold:
   (a) (tangential part)
$$\left(\dot{h}(t) \circ h(t)^{-1}\right) \circ T_{\alpha_{\mathrm{dev}}(t)}V \subset (T_{\alpha_{\mathrm{dev}}(t)}V)^{\perp};$$
   (b) (normal part)
$$\left(\dot{h}(t) \circ h(t)^{-1}\right) \circ (T_{\alpha_{\mathrm{dev}}(t)}V)^{\perp} \subset T_{\alpha_{\mathrm{dev}}(t)}V.$$

Remark 3.1. Note that in [24, pp. 376], Definition 3.1 appears with a different notation. Our choice, which will be more convenient in the following sections, needs some clarification. Let $x \in \mathbb{R}^n$ be a point and $\eta \in \mathbb{R}^n$ be a vector, i.e., there exists a smooth curve $y \in (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ such that $\dot{y}(0) = \eta$. Then

$$\dot{h}(t) \circ x = \frac{d}{d\sigma}(h(\sigma) \circ x)\bigg|_{\sigma=t}, \tag{3.6}$$

$$\left(\dot{h}(t) \circ h^{-1}(t)\right) \circ x = \frac{d}{d\sigma}((h(\sigma) \circ h^{-1}(t)) \circ x)\bigg|_{\sigma=t}, \tag{3.7}$$

$$\left(\dot{h}(t) \circ h^{-1}(t)\right) \circ \eta = \frac{d}{d\sigma}((\dot{h}(t) \circ h^{-1}(t)) \circ y(\sigma))\bigg|_{\sigma=0}. \tag{3.8}$$

Remark 3.2. In [24, pp. 381] it is proven that given any piecewise-smooth rolling or development curve, Definition 3.1 ensures the existence and uniqueness of the corresponding rolling mapping.

Using Definition 3.1, any rolling motion of M on V is completely defined by the action of some rolling mapping $h(t)$ satisfying $h(0) = \mathrm{id}$: the "position" of M at the time t is the submanifold $h(t) \circ M$, which is tangent to V at the point $\alpha_{\mathrm{dev}}(t)$, and $\alpha_{\mathrm{dev}}(t)$ is the curve traced by the point of contact of $h(t) \circ M$ on V.

3.1. **The rolling of the n-dimensional sphere.** The n-dimensional sphere S^n is naturally embedded into $\mathbb{R}^{n+1}$ and, therefore, is its affine tangent space at any point. Assume that S^n is rolling (without slipping or twisting) over the affine tangent space at $p_0 \in S^n$ denoted by

$$V := T_{p_0}^{\mathrm{aff}} S^n := \{x \in \mathbb{R}^{n+1} \mid x = p_0 + \Omega p_0,\ \Omega \in \mathfrak{so}_{n+1}\} \tag{3.9}$$

with the rolling curve $t \mapsto \alpha(t)$ satisfying $\alpha(0) = p_0$. The sphere S^n considered as a rigid body in $\mathbb{R}^{n+1}$ rotates in $\mathbb{R}^{n+1}$ so that the proper subspace which is instantaneously left-invariant under the rotation is parallel to V and perpendicular to $\dot{\alpha}_{\mathrm{dev}}(t)$. Simultaneously, the center of S^n imitates the development of α on V on the proper n-dimensional subspace of $\mathbb{R}^{n+1}$ parallel to V. This explains why the kinematic equations for such a motion are

$$\begin{aligned} \dot{s}(t) &= u(t), \\ \dot{R}(t) &= R(t)\left(u(t)p_0^\top - p_0 u^\top(t)\right), \end{aligned} \tag{3.10}$$

where the the control function $u : \mathbb{R} \to \mathbb{R}^{n+1}$, rotational part of the motion $R : \mathbb{R} \to \mathrm{SO}_{n+1}$, coordinate functions $s : \mathbb{R} \to \mathbb{R}^{n+1}$ of the development of the center of S^n, and initial values $R(0) = I_n$ and $s(0) = 0$. The function $s(t)$ lies for all t in the tangent space $T_{p_0}S^n$ considered as a proper n-dimensional subspace of $\mathbb{R}^{n+1}$. Consequently, the control u has to solve $p_0^\top u(t) = 0$ for all t. Equations (3.10) are in accordance with [18, p. 467], where $p_0 = [0, \ldots, 0, -1]^\top$ is used. Actually, (3.10) is easily obtained from the equations from [18] by a rotation of coordinates. Choosing a control function corresponds to fixing a rolling curve on the sphere. For example, if the control function $u(t) = u_0$ is constant, this implies that $R(t)$ is such a one-parameter subgroup of SO_{n+1} that the rolling curve is a geodesic, namely, a great circle, on S^n.

In the sequel, it turns out to be convenient to introduce the skew-symmetric

$$A : \mathbb{R} \to \mathfrak{so}_{n+1}, \quad A(t) := u(t)p_0^\top - p_0 u^\top(t). \tag{3.11}$$

Let A be as in (3.11). One easily proves by induction the following lemma.

Lemma 3.1. *For all $k \in \mathbb{N}$ and all $t \in \mathbb{R}$,*

$$A^{2k-1}(t) = \left(-u^\top(t)u(t)\right)^{k-1} A(t) \tag{3.12}$$

holds, where

$$A^2(t) = -u(t)u^\top(t) - u^\top(t)u(t)p_0p_0^\top. \tag{3.13}$$

Now we show how to construct a rolling mapping from kinematic equations (3.10).

Theorem 3.1. *If R and s are the solution of kinematic equations* (3.10)*, corresponding to a particular choice of the control function and satisfying $R(0) = I$, $s(0) = 0$, then $t \mapsto h(t) = (R^\top(t), s(t)) \in \mathrm{SE}_{n+1}$ is a rolling mapping, in the sense of Definition* 3.1*.*

Proof. Clearly, $\alpha(t) = R(t)p_0$ is the rolling curve. Therefore,

$$\alpha_{\mathrm{dev}}(t) = h(t) \circ \alpha(t) = R^\top(t)\alpha(t) + s(t) = p_0 + s(t) \in V.$$

Condition 1b in Definition 3.1 also holds since the submanifolds S^n and V have exactly one point of contact during the motion. Thus, we can say that

$$\alpha_{\mathrm{dev}}(t) = p_0 + s(t) \tag{3.14}$$

is the *rolling condition* for the sphere.

In order to prove the no-slip condition, we first note that, using (3.7) with α_{dev} instead of x, we have

$$\left(\dot{h}(t) \circ h(t)^{-1}\right) \circ \alpha_{\mathrm{dev}}(t) = \left(\dot{R}^\top(t) \circ R(t)\right) \circ (\alpha_{\mathrm{dev}}(t) - s(t)) + \dot{s}(t), \tag{3.15}$$

where the composition $\circ$ denotes simply matrix multiplication.

Now, using the kinematic equations above, the identity $\dot{s} = A(t)p_0$, which is also easy to derive from the kinematic equations, and the identity $\alpha_{\mathrm{dev}}(t) = p_0 + s(t)$, we obtain that the no-slip condition

$$\left(\dot{h}(t) \circ h(t)^{-1}\right) \circ \alpha_{\mathrm{dev}}(t) = 0 \tag{3.16}$$

is satisfied.

Consequently, for the case of a sphere, the *no-slip condition* is equivalent to

$$\dot{s}(t) = A(t)p_0. \tag{3.17}$$

Finally, the no-twist conditions follow from (3.8) and the next three observations:

$$\begin{gathered} T_{\alpha_{\mathrm{dev}}(t)}V = \{w \in \mathbb{R}^{n+1} : w^\top p_0 = 0\} \cong V, \\ (T_{\alpha_{\mathrm{dev}}(t)}V)^\perp = \mathrm{span}(p_0), \quad \dot{R}^\top(t)R(t) = -A(t). \end{gathered} \tag{3.18}$$

The theorem is proved. ☐

3.2. **The rolling of the rotation group SO_n.** In contrast to the 2-sphere, we now loose 3-dimensional geometric intuition. For this reason, we first construct a rolling mapping and then derive the kinematic equations for the motion of the rotation group as a rigid body rolling (without slip or twist) over its affine tangent space at a point.

First, we define the group action for such kind of the motion. The following statements are easily verified. The Lie group $\mathrm{SO}_n \times \mathrm{SO}_n$ acts transitively on SO_n via equivalence

$$\sigma : (\mathrm{SO}_n \times \mathrm{SO}_n) \times \mathrm{SO}_n \to \mathrm{SO}_n, \quad ((U,W),R) \mapsto URW^\top. \tag{3.19}$$

Moreover, the group $G = \mathrm{SO}_n \times \mathrm{SO}_n \ltimes \mathbb{R}^{n\times n}$ acts on $\mathbb{R}^{n\times n}$ via

$$G \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}, \quad ((U,W,X),Z) \mapsto UZW^\top + X, \tag{3.20}$$

where G acts on itself via

$$(U_2,W_2,X_2) \circ (U_1,W_1,X_1) := \left(U_2U_1, W_2W_1, U_2X_1W_2^\top + X_2\right), \tag{3.21}$$

with the inverse

$$(U,W,X)^{-1} = \left(U^\top, W^\top, -U^\top XW\right). \tag{3.22}$$

Now, let P_0 be an arbitrary point in SO_n and $\alpha : [0,\tau] \to \mathrm{SO}_n$, $\alpha(t) = U(t)P_0W(t)^\top$ be a curve on SO_n starting from P_0 at $t=0$ (the transitive action σ guarantees that any curve on SO_n has this form). We will show that, under some restrictions, the mapping

$$\begin{aligned} h : [0,\tau] &\to G = \mathrm{SO}_n \times \mathrm{SO}_n \ltimes \mathbb{R}^{n\times n}, \\ t &\mapsto h(t) = \left(U^\top(t), W^\top(t), X(t)\right) \end{aligned} \tag{3.23}$$

is a rolling mapping of the rotation group over $V := T_{P_0}^{\mathrm{aff}} \mathrm{SO}_n \cong T_{P_0} \mathrm{SO}_n$, along

$$\alpha(t) = U(t)P_0W(t)^\top, \tag{3.24}$$

with the development

$$\alpha_{\mathrm{dev}}(t) = h(t) \circ \alpha(t) = U^\top(t)\alpha(t)W(t) + X(t) = P_0 + X(t). \tag{3.25}$$

A possible way to understand how G, as a closed subgroup of SE_{n^2}, behaves inside SE_{n^2}, is to use the Kronecker product and vec-notation. The vec-isomorphism, i.e., "stacking columns," is defined as

$$\mathrm{vec} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}, \quad [z_1,\ldots,z_n] = Z \mapsto \mathrm{vec}\, Z := \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \tag{3.26}$$

where

$$\mathrm{vec}(UXW^\top) = (W \otimes U)\,\mathrm{vec}\, X \tag{3.27}$$

for any $U, X, W \in \mathbb{R}^{n\times n}$. Let

$$\theta : G \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}, \quad ((U,W,X),Z) \mapsto UZW^\top + X, \tag{3.28}$$

and let

$$\varphi : G \to \varphi(G) \subset \mathrm{SE}_{n^2} = \mathrm{SO}_{n^2} \ltimes \mathbb{R}^{n^2}, \quad (U, W, X) \mapsto (W \otimes U, \operatorname{vec} X) \tag{3.29}$$

with the induced group action

$$\begin{gathered} \overline{\theta} : \varphi(G) \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}, \\ ((W \otimes U, \operatorname{vec} X), \operatorname{vec} Z) \mapsto (W \otimes U) \operatorname{vec} Z + \operatorname{vec} X. \end{gathered} \tag{3.30}$$

Introducing

$$\phi : \mathrm{SE}_{n^2} \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}, \quad ((E, x), z) \mapsto Ez + x, \tag{3.31}$$

we immediately obtain that $\overline{\theta} = \phi|_{\varphi(G) \times \mathbb{R}^{n^2}}$, where

$$\overline{\theta}(g, x) = \operatorname{vec}(\theta(\varphi^{-1}(g), \operatorname{vec}^{-1}(x))) \tag{3.32}$$

for $g \in \varphi(G)$ and $x \in \mathbb{R}^{n^2}$.

Before proceeding to derive the rolling mapping h, we rewrite relations (3.6)–(3.8), so that they can be used in the present situation, i.e., when $h(t) = \big(U^\top(t), W^\top(t), X(t)\big)$ and, consequently,

$$\begin{gathered} \dot{h}(t) = \left(\dot{U}^\top(t), \dot{W}^\top(t), \dot{X}(t)\right), \\ (h(t))^{-1} = \big(U(t), W(t), -U(t)X(t)W^\top(t)\big). \end{gathered} \tag{3.33}$$

The following formulas, valid for a point $Y \in \mathbb{R}^{n \times n}$ and a vector $\eta \in \mathbb{R}^{n \times n}$, are easily obtained:

$$\dot{h} \circ Y = \dot{U}^\top Y W + U^\top Y \dot{W} + \dot{X}, \tag{3.34}$$

$$\left(\dot{h} \circ h^{-1}\right) \circ Y = \dot{U}^\top U (Y - X) + (Y - X) W^\top \dot{W} + \dot{X}, \tag{3.35}$$

$$\left(\dot{h} \circ h^{-1}\right) \circ \eta = \dot{U}^\top U \eta + \eta W^\top \dot{W}. \tag{3.36}$$

According to Definition 3.1, h defined by (3.23) must satisfy the no-slip condition

$$\left(\dot{h}(t) \circ h^{-1}(t)\right) \circ \alpha_{\mathrm{dev}}(t) = 0 \quad \text{for all } t. \tag{3.37}$$

Therefore, according to (3.34), we can write

$$\begin{aligned} \left(\dot{h} \circ h^{-1}\right) \circ \alpha_{\mathrm{dev}} = 0 \iff \dot{h} \circ \alpha = 0 \iff \dot{h} \circ (U P_0 W^\top) = 0 \\ \iff \dot{U}^\top U P_0 W^\top W + U^\top U P_0 W^\top \dot{W} + \dot{X} = 0 \\ \iff \dot{U}^\top U P_0 + P_0 W^\top \dot{W} + \dot{X} = 0. \end{aligned} \tag{3.38}$$

If we set

$$\dot{U}^\top U =: -\frac{\Omega_U}{2} \in \mathfrak{so}_n, \quad W^\top \dot{W} =: -\frac{\Omega_W}{2} \in \mathfrak{so}_n, \tag{3.39}$$

the no-slip condition takes the form

$$\dot{X} = \frac{\Omega_U}{2} P_0 + P_0 \frac{\Omega_W}{2}. \tag{3.40}$$

Now, the no-twist conditions

$$\left(\dot{h} \circ h^{-1}\right) \circ T_{\alpha_{\mathrm{dev}}} V \subset \left(T_{\alpha_{\mathrm{dev}}} V\right)^{\perp}, \tag{3.41}$$

$$\left(\dot{h} \circ h^{-1}\right) \circ \left(T_{\alpha_{\mathrm{dev}}} V\right)^{\perp} \subset T_{\alpha_{\mathrm{dev}}} V \tag{3.42}$$

must also hold. That is, for all $\xi \in T_{\alpha_{\mathrm{dev}}} V$

$$\left(\left(\dot{U}^{\top}, \dot{W}^{\top}, \dot{X}\right) \circ \left(U, W, -UXW^{\top}\right)\right) \circ \xi = \dot{U}^{\top} U \xi + \xi W^{\top} \dot{W} \in \left(T_{\alpha_{\mathrm{dev}}} V\right)^{\perp}, \tag{3.43}$$

and, similarly, for all $\eta \in \left(T_{\alpha_{\mathrm{dev}}} V\right)^{\perp}$,

$$\left(\left(\dot{U}^{\top}, \dot{W}^{\top}, \dot{X}\right) \circ \left(U, W, -UXW^{\top}\right)\right) \circ \eta = \dot{U}^{\top} U \eta + \eta W^{\top} \dot{W} \in T_{\alpha_{\mathrm{dev}}} V. \tag{3.44}$$

Note that any vector $\xi \in T_{\alpha_{\mathrm{dev}}} V$ is of the form $\xi = P_0 \Psi$ for some $\Psi \in \mathfrak{so}_n$. Similarly, any vector $\eta \in \left(T_{\alpha_{\mathrm{dev}}} V\right)^{\perp}$ is of the form $\eta = P_0 S$ for some symmetric matrix S. Consequently, the tangential part of the no-twist condition is equivalent to requiring that the matrix ${P_0}^{\top}(\dot{U}^{\top} U P_0 \Psi + P_0 \Psi W^{\top} \dot{W})$ is symmetric for all $\Psi \in \mathfrak{so}_n$, while the normal part requires that the matrix ${P_0}^{\top}(\dot{U}^{\top} U P_0 S + P_0 S W^{\top} \dot{W})$ be skew-symmetric for all $S = S^{\top}$. After some simple calculations, we conclude that the tangential condition reduces to

$$\left[{P_0}^{\top} \Omega_U P_0 - \Omega_W, \Psi\right] = 0 \quad \text{for all } \Psi \in \mathfrak{so}_n,$$

which is equivalent to ${P_0}^{\top} \Omega_U P_0 = \Omega_W$. Hence, by (3.39), the tangential condition can be written as $\dot{U}^{\top} U P_0 = P_0 W^{\top} \dot{W}$. However, it turns out that if this condition holds, the normal condition holds as well. Therefore, the no-twist condition reduces to the single equation

$$\dot{U}^{\top} U P_0 = P_0 W^{\top} \dot{W} \iff \Omega_U P_0 = P_0 \Omega_W \quad \text{(no-twist condition)}. \tag{3.45}$$

By introducing the control function

$$t \mapsto \Omega(t) := \Omega_U(t) = P_0 \Omega_W(t) {P_0}^{\top}, \tag{3.46}$$

the kinematic equations for the rolling of SO_n are now easily derived from the no-slip and no-twist conditions:

$$\dot{X}(t) = \Omega(t) P_0, \quad \dot{U}(t) = \frac{1}{2} U(t) \Omega(t), \quad \dot{W}(t) = -\frac{1}{2} W(t) {P_0}^{\top} \Omega(t) P_0 \tag{3.47}$$

with the initial conditions $X(0) = 0$ and $U(0) = W(0) = I$. The skew-symmetric matrix function $t \mapsto \Omega(t)$ plays the role of the control function, since the motion is completely defined by the choice of Ω.

Now we can state an analog of Theorem 3.1.

Theorem 3.2. *If (X, U, W) is the solution of kinematic equations* (3.47) *corresponding to a particular choice of the control function Ω and satisfying*

$$(X(0), U(0), W(0)) = (0, I, I),$$

then
$$t \mapsto h(t) = (U^\top(t), W^\top(t), X(t)) \in \mathrm{SE}_{n^2}$$
is a rolling mapping for SO_n *in the sense of Definition* 3.1.

For every $U, W \in \mathrm{SO}_n$, the action σ in (3.19) defines a mapping
$$\sigma_{U,W} : \mathrm{SO}_n \to \mathrm{SO}_n, \quad R \mapsto URW^\top. \tag{3.48}$$
Consequently,
$$\alpha(t) = U(t)P_0W^\top(t) = \sigma_{U,W}(P_0), \tag{3.49}$$
which shows that the rolling curve depends also on the choice of Ω. Since we have a total freedom in the choice of $\Omega(t) \in \mathfrak{so}_n$, we can conclude that all rolling mappings of SO_n are constructed in this way.

Remark 3.3. The statement of Theorem 3.2 seems to be remarkable. It is *a priori* by no way clear, why the rotational part of the rolling mapping acts simply by equivalence. In a forthcoming paper, the authors will show that the situation for arbitrary Stiefel manifolds is much more subtle, unless the manifold is either a sphere or an orthogonal group (see [14]).

Example 3.1. If, e.g., $\Omega(t) = \Omega$ is a constant matrix, then the solution of the kinematic equations with initial conditions $(X(0), U(0), W(0)) = (0, I, I)$ is
$$X(t) = (t\Omega)P_0, \quad U(t) = e^{t\frac{\Omega}{2}}, \quad W(t) = P_0{}^\top e^{-t\frac{\Omega}{2}}P_0, \tag{3.50}$$
and, in this case,
$$\alpha(t) = e^{t\frac{\Omega}{2}}P_0P_0{}^\top e^{t\frac{\Omega}{2}}P_0 = e^{t\Omega}P_0, \tag{3.51}$$
i.e., $t \mapsto \alpha(t)$ is a geodesic on SO_n, passing through P_0 at $t = 0$. Consequently, $\alpha_{\mathrm{dev}}(t) = P_0 + X(t) = P_0 + t\Omega P_0$ is also a geodesic, i.e., a straight line in $T^{\mathrm{aff}}_{P_0}\mathrm{SO}_n$, passing through P_0 at $t = 0$.

3.3. **Rolling Grassmann manifolds.** We consider the Grassmann manifold $G_{k,n}$ of all k-dimensional subspaces of $\mathbb{R}^n$. Since any k-dimensional subspace in $\mathbb{R}^n$ can be uniquely associated with an orthogonal projection $(n \times n)$-matrix $P = P^\top$ of rank k, we use a representation of the Grassmannian as a particular subset of the symmetric matrices Sym_n, i.e.,
$$G_{k,n} := \{P \in \mathrm{Sym}_n \mid P^2 = P,\ \mathrm{rk}(P) = k\}. \tag{3.52}$$
Using this representation of $G_{k,n}$, the tangent space at any point $P_0 \in G_{k,n}$ is given by
$$\begin{aligned} T_{P_0}G_{k,n} &= \{T \in \mathrm{Sym}_n \mid T = P_0T + TP_0\} \\ &= \{T \in \mathrm{Sym}_n \mid T = [P_0, [P_0, Z]], Z \in \mathrm{Sym}_n\}. \end{aligned} \tag{3.53}$$
Note that the second description of the tangent space in (3.53) is proved in [12].

The normal space is

$$
\begin{aligned}
T_{P_0}^{\perp} G_{k,n} &= \{T \in \mathrm{Sym}_n \mid \mathrm{tr}(TZ) = 0 \text{ for all } Z \in T_{P_0} G_{k,n}\} \\
&= \{T \in \mathrm{Sym}_n \mid T = Z - [P_0, [P_0, Z]],\ Z \in \mathrm{Sym}_n\}
\end{aligned}
\tag{3.54}
$$

with respect to the usual Euclidean inner product in Sym_n.

Example 3.2. In particular, if

$$
P_0 = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}
$$

and tangent vectors in (3.53) are partitioned accordingly, a typical element $\xi \in T_{P_0} G_{k,n}$ is represented by

$$
\xi = \begin{bmatrix} 0 & Z \\ Z^\top & 0 \end{bmatrix}, \tag{3.55}
$$

where Z is any real $(k \times (n-k))$-matrix, while a typical element $\eta \in T_{P_0}^{\perp} G_{k,n}$ has the form

$$
\eta = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \tag{3.56}
$$

where S_1 and S_2 are symmetric matrices of orders k and $n-k$, respectively.

Looking at $G_{k,n} \subset \mathrm{Sym}_n$, we now make the necessary computations to derive the kinematic equations for the rolling of the Grassmann manifold.

Note that $\dim(\mathrm{Sym}_n) = n(n+1)/2$. Let

$$
G = \mathrm{SO}_n \ltimes \mathrm{Sym}_n\,. \tag{3.57}
$$

This group acts on Sym_n by the rule

$$
G \times \mathrm{Sym}_n \to \mathrm{Sym}_n, \quad ((\Theta, X), S) \mapsto \Theta S \Theta^\top + X, \tag{3.58}
$$

while G acts on itself by the rule

$$
(\Theta_2, X_2) \circ (\Theta_1, X_1) := \left(\Theta_2\Theta_1, \Theta_2 X_1 \Theta_2^\top + X_2\right), \tag{3.59}
$$

and the inverse is

$$
(\Theta, X)^{-1} = \left(\Theta^\top, -\Theta^\top X \Theta\right). \tag{3.60}
$$

Remark 3.4. By considerations similar to the case of rolling SO_n, one can find how G behaves inside $\mathrm{SE}_{n(n+1)/2}$. We omit the details since the derivation using symmetric tensor products would obscure our presentation considerably.

By the smooth and transitive action of SO_n on $G_{k,n}$, a smooth curve in $G_{k,n}$ through P_0 can be given as

$$
t \mapsto \alpha(t) = \Theta(t) P_0 \Theta^\top(t), \tag{3.61}
$$

where $\Theta(t) \in \mathrm{SO}_n$ should also be smooth. Now our objective is to find conditions under which the mapping

$$
h : [0, \tau] \to \mathrm{SO}_n \times \mathrm{Sym}_n, \quad t \mapsto h(t) = \left(\Theta^\top(t), X(t)\right) \tag{3.62}
$$

is a rolling mapping of the Grassmann manifold $G_{k,n}$ over the affine subspace associated with $T_{P_0}G_{k,n}$ along the curve $[0,\tau] \to \alpha(t) = \Theta(t)P_0\Theta^\top(t)$ with the development

$$\alpha_{\mathrm{dev}}(t) = h(t) \circ \alpha(t) = \Theta^\top(t)\alpha(t)\Theta(t) + X(t) = P_0 + X(t). \tag{3.63}$$

Let $Y \in \mathrm{Sym}_n$ be a point and $\eta \in \mathrm{Sym}_n$ be a vector. Again, since $h = (\Theta^\top, X)$, $\dot{h} = (\dot{\Theta}^\top, \dot{X})$, and $h^{-1} = (\Theta, -\Theta X \Theta^\top)$, relations (3.6)–(3.8) can be easily written as follows:

$$\dot{h} \circ Y = \dot{\Theta}^\top Y \Theta + \Theta^\top Y \dot{\Theta} + \dot{X}, \tag{3.64}$$

$$\left(\dot{h} \circ h^{-1}\right) \circ Y = \dot{\Theta}^\top \Theta (Y - X) + (Y - X)\Theta^\top \dot{\Theta} + \dot{X}, \tag{3.65}$$

$$\left(\dot{h} \circ h^{-1}\right) \circ \eta = \dot{\Theta}^\top \Theta \eta + \eta \Theta^\top \dot{\Theta}. \tag{3.66}$$

Now, for the no-slip condition, we have

$$\left(\dot{h}(t) \circ h(t)^{-1}\right) \circ \alpha_{\mathrm{dev}}(t) = 0 \quad \text{for all } t. \tag{3.67}$$

Thus, by (3.65) with α_{dev} instead of Y, the no-slip condition is equivalent to

$$\dot{\Theta}^\top \Theta P_0 + P_0 \Theta^\top \dot{\Theta} + \dot{X} = 0. \tag{3.68}$$

Since $\dot{\Theta}^\top\Theta$ is skew-symmetric, we obtain

$$\dot{X} = [P_0, \dot{\Theta}^\top \Theta]. \tag{3.69}$$

In the rest of this subsection, we will assume for simplicity that

$$P_0 = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}.$$

The general case will be covered by Theorem 3.4 at the end of this subsection. We define

$$\dot{\Theta}^\top \Theta =: \begin{bmatrix} \Psi_{11} & \Psi \\ -\Psi^\top & \Psi_{22} \end{bmatrix}, \quad \Psi_{11} \in \mathfrak{so}_k,\ \Psi_{22} \in \mathfrak{so}_{n-k},\ \Psi \in \mathbb{R}^{k \times n-k}. \tag{3.70}$$

Let $\eta \in T^\perp_{P_0}G_{k,n}$ as in (3.56). Now, for the normal part of the no-twist conditions, we must have

$$\left(\dot{h}(t) \circ h(t)^{-1}\right) \circ \eta \in T_{P_0}G_{k,n} \quad \text{for all } \eta \in T^\perp_{P_0}G_{k,n}. \tag{3.71}$$

Taking into account (3.66), a few computations lead to some constraints on the matrix Θ. More precisely, we obtain the relations

$$\begin{aligned} [\Psi_{11}, S_1] &= 0 \quad \text{for all } S_1 = S_1^\top, \\ [\Psi_{22}, S_2] &= 0 \quad \text{for all } S_2 = S_2^\top, \end{aligned} \tag{3.72}$$

which, in turn, imply

$$\Psi_{11} = 0, \quad \Psi_{22} = 0. \tag{3.73}$$

Therefore, (3.70) is reduced to

$$\dot{\Theta}^{\top}\Theta = \begin{bmatrix} 0 & \Psi \\ -\Psi^{\top} & 0 \end{bmatrix}. \tag{3.74}$$

With this restriction, the tangential part of the no-twist condition,

$$\left(\dot{h}(t) \circ h(t)^{-1}\right) \circ \xi \in T_{P_0}^{\perp} G_{k,n} \quad \text{for all } \xi \in T_{P_0} G_{k,n}, \tag{3.75}$$

always holds and the no-slip condition (3.69) is reduced to

$$\dot{X} = \begin{bmatrix} 0 & \Psi \\ \Psi^{\top} & 0 \end{bmatrix}. \tag{3.76}$$

Therefore, the kinematic equations for the rolling of $G_{k,n}$ over its affine tangent space at the point

$$P_0 = \begin{bmatrix} I_k & 0 \\ 0 & 0b \end{bmatrix}$$

are given by

$$\begin{aligned} \dot{X}(t) &= \begin{bmatrix} 0 & \Psi(t) \\ \Psi^{\top}(t) & 0 \end{bmatrix}, \\ \dot{\Theta}(t) &= \Theta(t) \begin{bmatrix} 0 & -\Psi(t) \\ \Psi^{\top}(t) & 0 \end{bmatrix} = \Theta(t) \left[\begin{bmatrix} 0 & \Psi(t) \\ \Psi^{\top}(t) & 0 \end{bmatrix}, P_0 \right] \end{aligned} \tag{3.77}$$

with the initial conditions $\Theta(0) = I_n$ and $X(0) = 0_n$.

The matrix function $\Psi : \mathbb{R} \to \mathbb{R}^{k\times n-k}$ plays the role of the control function, since the motion is completely defined by the choice of Ψ. Now we can state an analog of Theorems 3.1 and 3.2.

Theorem 3.3. *If* (X, Θ) *is the solution of the kinematic equations* (3.77) *corresponding to a particular choice of the control function* Ψ *and satisfying* $(X(0), \Theta(0)) = (0, I)$, *then* $t \mapsto h(t) = (\Theta^{\top}(t), X(t)) \in \mathrm{SE}_{n(n+1)/2}$ *is a rolling mapping for the Grassmann manifold* $G_{k,n}$ *in the sense of Definition* 3.1.

For the special situation, where $\Psi(t) = \Psi$ is constant, the solution of the kinematic equations is given by

$$X(t) = t \begin{bmatrix} 0 & \Psi \\ \Psi^{\top} & 0 \end{bmatrix}, \quad \Theta(t) = \exp\left(t \begin{bmatrix} 0 & -\Psi \\ \Psi^{\top} & 0 \end{bmatrix} \right) \tag{3.78}$$

and, in this case,

$$t \mapsto \alpha(t) = \Theta(t) P_0 \Theta^{\top}(t) \tag{3.79}$$

is a geodesic on $G_{k,n}$ passing through P_0 at $t = 0$ with the velocity

$$\dot{\alpha}(0) = \begin{bmatrix} 0 & \Psi \\ \Psi^{\top} & 0 \end{bmatrix}.$$

Consequently,

$$\alpha_{\mathrm{dev}}(t) = P_0 + X(t) = P_0 + t \begin{bmatrix} 0 & \Psi \\ \Psi^\top & 0 \end{bmatrix}$$

is also a geodesic in the affine subspace $T^{\mathrm{aff}}_{P_0} G_{k,n}$ passing through P_0 at $t = 0$.

An explicit formula for the exponential of matrices with this special block structure can be found in [11, p. 351]. Therefore, Θ in (3.78) can be written in the form

$$\Theta(t) = \begin{bmatrix} \left(I_k - BB^\top\right)^{1/2} & -B \\ B^\top & \left(I_{n-k} - B^\top B\right)^{1/2} \end{bmatrix}, \tag{3.80}$$

where

$$B := \Psi \, \frac{\sin\,(\Psi^\top\Psi)^{1/2}}{(\Psi^\top\Psi)^{1/2}} \tag{3.81}$$

is defined by the series expansion.

Note that the Grassmann manifold is an isospectral manifold (see, e.g., [13]). Consequently, if

$$P_0 = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix},$$

then

$$G_{k,n} = \{\widehat{P}_0 = QP_0Q^\top,\ Q \in \mathrm{SO}_n\}.$$

Theorem 3.4. *The kinematic equations to roll $G_{k,n}$ starting from an arbitrary point $\widehat{P}_0 \in G_{k,n}$ along a curve in $T^{\mathrm{aff}}_{\widehat{P}_0} G_{k,n}$ are as follows*:

$$\dot{X}(t) = \xi(t), \quad \dot{\Theta}(t) = \Theta[\xi(t), \widehat{P}_0], \tag{3.82}$$

with the initial conditions $\Theta(0) = I_n$ *and* $X(0) = 0_n$ *and the control function* $\xi : \mathbb{R} \to T_{\widehat{P}_0} G_{k,n}$, *i.e.,* $\xi(t)$ *has to solve the equation*

$$\xi(t) = \xi(t)\widehat{P}_0 + \widehat{P}_0\xi(t)$$

for all t.

Proof. Note that $T \in T_{P_0} G_{k,n}$ if and only if $QTQ^\top \in T_{\widehat{P}_0} G_{k,n}$, where $\widehat{P}_0 = QP_0Q^\top$. Now, a simple calculation shows that $(\widehat{X}, \widehat{\Theta})$ is a solution of the initial-value problem (3.82) if and only if (X, Θ) is a solution of the initial-value problem (3.77), where $\widehat{X} = QXQ^\top$ and $\widehat{\Theta} = Q\Theta Q^\top$. This proves the theorem. $\square$

Corollary 3.1. *For the simple case where $\xi(t) = \xi_0$ is a constant, we obtain*

$$X(t) = t\xi_0, \quad \Theta(t) = e^{t[\xi_0, \widehat{P}_0]} \tag{3.83}$$

as solutions of kinematic equations (3.82).

3.4. **The rolling versus parallel transport.** The parallel transport of a vector Y_0 tangent to a manifold M at a point p_0 along a curve $t \mapsto \alpha(t) \in M$ satisfying $\alpha(0) = p_0$ can be accomplished by rolling (without slip or twist) of $T^{\mathrm{aff}}_{p_0}M$ on M along this curve. Thus, we can apply the results of the previous section to compute parallel vector fields along curves belonging to our favorite manifolds, S^n, SO_n, and $G_{k,n}$. When the curve is a geodesic, we recover known results contained in the literature on the differential geometry.

3.4.1. *The n-sphere S^n.*

Proposition 3.1. *If $t \mapsto h(t) = (R^\top(t), s(t))$ is a rolling mapping for S^n with rolling curve $t \mapsto \alpha(t)$ satisfying $\alpha(0) = p_0$ and $Y_0 \in T_{p_0}S^n$, then*

$$Y(t) = h^{-1}(t) \circ Y_0 = R(t)Y_0 \tag{3.84}$$

defines the parallel vector field along $t \mapsto \alpha(t)$, satisfying $Y(0) = Y_0$.

Proof. The initial condition is trivially satisfied. Clearly, since $\alpha(t) = R(t)p_0$ and $Y_0^\top p_0 = 0$, we have

$$Y(t)^\top \alpha(t) = (R(t)Y_0)^\top R(t)p_0 = Y_0^\top p_0 = 0, \tag{3.85}$$

i.e., $Y(t) \in T_{\alpha(t)}S^n$ for all t. We just need to show that $\nabla_{\dot\alpha(t)}Y(t) = 0$ for all t. But, since $Y_0^\top p_0 = 0$ and $\dot R(t) = R(t)A(t)$, where $A(t) = u(t)p_0^\top - p_0u^\top(t)$ as in (3.11), we have that $A(t)Y_0 = -u(t)^\top Y_0 \cdot p_0$, and, consequently,

$$\begin{aligned}\nabla_{\dot\alpha(t)}Y(t) &= \left(I_{n+1} - \alpha(t)\alpha^\top(t)\right)\dot Y(t)\\ &= \left(I_{n+1} - R(t)p_0p_0^\top R^\top(t)\right)\dot R(t)Y_0 = R(t)A(t)Y_0 - R(t)p_0p_0^\top A(t)Y_0\\ &= -u(t)^\top Y_0 \cdot R(t)p_0 + u(t)^\top Y_0 \cdot R(t)p_0 = 0. \end{aligned}\tag{3.86}$$

The proposition is proved. ☐

Example 3.3. If $A(t) = A = u_0p_0^\top - p_0u_0^\top$ is constant and $\|u_0\| = 1$, then $\alpha(t) = e^{At}p_0$ is the geodesic emanating from p_0 in the direction of the unit vector $Ap_0 = u_0$. Note that, due to the form of the constant matrix A, we have

$$e^{tA} = I + A\sin t + A^2(1 - \cos t), \tag{3.87}$$

which is easily seen by applying Lemma 3.1. As a consequence, the parallel translation of Y_0 along this geodesic is given by

$$\begin{aligned}Y(t) = e^{tA}Y_0 &= Y_0 + AY_0\sin t + A^2Y_0(1 - \cos t)\\ &= Y_0 - p_0u_0^\top Y_0\sin t - u_0u_0^\top Y_0(1 - \cos t)\\ &= Y_0 - u_0^\top Y_0\left(p_0\sin t + u_0(1 - \cos t)\right). \end{aligned}\tag{3.88}$$

3.4.2. *The special orthogonal group* SO_n. Now we obtain similar results for the orthogonal group.

Proposition 3.2. *Let $t \mapsto h(t) = (U^\top(t), W^\top(t), X(t))$ be a rolling mapping for SO_n, with the rolling curve $t \mapsto \alpha(t)$ satisfying $\alpha(0) = P_0$, and let $Y_0P_0 \in T_{P_0}\mathrm{SO}_n$. Then*

$$Y(t) = h^{-1}(t) \circ (Y_0P_0) = U(t)Y_0P_0W^\top(t) \tag{3.89}$$

defines the parallel vector field along $t \mapsto \alpha(t)$ satisfying $Y(0) = Y_0P_0$.

Proof. The initial condition is satisfied since $U(0) = W(0) = I$. The rolling curve is defined by $\alpha(t) = U(t)P_0W^\top(t)$, and the tangent space to SO_n at each point $\alpha(t)$ can be parameterized by $\{U(t)\Psi P_0W^\top(t),\ \Psi \in \mathfrak{so}_n\}$. Similarly, $\{U(t)SP_0W^\top(t),\ S = S^\top \in \mathbb{R}^{n\times n}\}$ parameterizes the normal space at $\alpha(t)$. Thus, since $Y_0 \in \mathfrak{so}_n$, $Y(t)$ defines a vector field along the rolling curve, and to prove that it is indeed parallel along $t \mapsto \alpha(t)$, it suffices to show that $\dot{Y}(t)$ belongs to the normal space to $\alpha(t)$ for all t. Using kinematic equations (3.47), we can write

$$\begin{aligned}\dot{Y}(t) &= \dot{U}(t)Y_0P_0W^\top(t) + U(t)Y_0P_0\dot{W}^\top(t)\\ &= \frac{1}{2}\Big(U(t)\Omega(t)Y_0P_0W^\top(t) + U(t)Y_0P_0P_0^\top\Omega(t)P_0W^\top(t)\Big)\\ &= U(t)\frac{1}{2}\Big(\Omega(t)Y_0 + Y_0\Omega(t)\Big)P_0W^\top(t).\end{aligned} \tag{3.90}$$

Since the matrix $\Omega(t)Y_0 + Y_0\Omega(t)$ is always symmetric, we conclude that $\nabla_{\dot{\alpha}(t)}Y(t) \equiv 0$. ☐

Example 3.4. If the rolling curve is a geodesic, then $U(t) = e^{t\frac{\Omega}{2}}$ and $W(t) = P_0^\top e^{-t\frac{\Omega}{2}}P_0$, and, therefore, the parallel translation of Y_0P_0 along the rolling geodesic is given by

$$Y(t) = e^{t\frac{\Omega}{2}}Y_0P_0P_0^\top e^{t\frac{\Omega}{2}}P_0 = e^{t\frac{\Omega}{2}}Y_0e^{t\frac{\Omega}{2}}P_0. \tag{3.91}$$

Remark 3.5. Formula (3.88) can be found in [29, p. 120] (see also [28, p. 23]). Formula (3.91) appears in [28, p. 22] for the case $P_0 = I_n$.

3.4.3. *The Grassmann manifold* $G_{k,n}$. Finally, for the Grassmann manifold, we obtain the following proposition.

Proposition 3.3. *Let $t \mapsto h(t) = (\Theta^\top(t), X(t))$ be a rolling mapping for $G_{k,n}$ with the rolling curve $t \mapsto \alpha(t) = \Theta(t)P_0\Theta^\top(t)$ satisfying $\alpha(0) = P_0 \in G_{k,n}$. Let $Y_0 \in T_{P_0}G_{k,n}$. Then*

$$Y(t) = h^{-1}(t) \circ Y_0 = \Theta(t)Y_0\Theta^\top(t) \tag{3.92}$$

defines the parallel vector field along $t \mapsto \alpha(t)$ satisfying $Y(0) = Y_0$.

Proof. The initial condition is satisfied since $\Theta(0) = I$. Also, since $Y_0 \in T_{P_0}G_{k,n}$ and $\alpha(t) = \Theta(t)P_0\Theta^\top(t)$, the invariance properties mentioned above guarantee that $Y(t) = \Theta(t)Y_0\Theta^\top(t) \in T_{\alpha(t)}G_{k,n}$, i.e., it is indeed a vector field along $t \mapsto \alpha(t)$. To complete the proof, it suffices to show that $\dot{Y}(t) \in T^\perp_{\alpha(t)}G_{k,n}$. Using kinematic equations (3.82), we can write

$$\begin{aligned}\dot{Y}(t) &= \dot{\Theta}(t)Y_0\Theta^\top(t) + \Theta(t)Y_0\dot{\Theta}^\top(t)\\ &= \Theta(t)[\xi(t), P_0]Y_0\Theta^\top(t) + \Theta(t)Y_0[P_0, \xi(t)]\Theta^\top(t)\\ &= \Theta(t)\big([\xi(t), P_0]Y_0 + Y_0[P_0, \xi(t)]\big)\Theta^\top(t). \end{aligned} \tag{3.93}$$

Again, by the invariance properties, it suffices to analyze the matrix expression in brackets in the last line of (3.93), i.e., $[\xi(t), P_0]Y_0 + Y_0[P_0, \xi(t)]$ in the case where

$$P_0 = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.94}$$

Setting

$$Y_0 =: \begin{bmatrix} 0 & Z_0 \\ Z_0^\top & 0 \end{bmatrix}, \quad \xi(t) =: \begin{bmatrix} 0 & \eta(t) \\ \eta(t)^\top & 0 \end{bmatrix} \tag{3.95}$$

and computing

$$\begin{aligned}&[\xi(t), P_0]Y_0 + Y_0[P_0, \xi(t)]\\ &\quad= \begin{bmatrix} 0 & -\eta(t) \\ \eta(t)^\top & 0 \end{bmatrix}\begin{bmatrix} 0 & Z_0 \\ Z_0^\top & 0 \end{bmatrix} + \begin{bmatrix} 0 & Z_0 \\ Z_0^\top & 0 \end{bmatrix}\begin{bmatrix} 0 & \eta(t) \\ -\eta(t)^\top & 0 \end{bmatrix}\\ &\quad= \begin{bmatrix} -\eta(t)Z_0^\top - Z_0\eta(t)^\top & 0 \\ 0 & \eta(t)^\top Z_0 + Z_0^\top\eta(t) \end{bmatrix}, \end{aligned} \tag{3.96}$$

we conclude that $\dot{Y}(t) \in T^\perp_{\alpha(t)}G_{k,n}$ for all t. The proposition is proved. □

Example 3.5. Let the control function be constant, i.e.,

$$\xi(t) = [P_0, [P_0, X]] = [[X, P_0], P_0]$$

for some constant symmetric matrix X. Then the rolling curve is a geodesic, and the parallel translation of Y_0, along the rolling geodesic is given by

$$\begin{aligned}Y(t) = \Theta(t)Y_0\Theta^\top(t) &= e^{t[[[X,P_0],P_0],P_0]}Y_0e^{-t[[[X,P_0],P_0],P_0]}\\ &= e^{t[X,P_0]}Y_0e^{-t[X,P_0]}. \end{aligned} \tag{3.97}$$

4. Rolling mappings and geometric splines

Since geometric cubic splines on Riemannian manifolds were defined in [22], there has been an increasing interest to the geometry of these curves, and the several ways to compute them have been found. In spite of many interesting results, which can be found, e.g., in [3, 6, 7] and references therein, many questions remain open. Inspired by some ideas contained in [17], we developed in [15] a rolling and unwrapping technique to construct interpolating curves on manifolds. Again, we concentrate on spheres, orthogonal groups, and Grassmann manifolds to clarify the connection between unrolled splines on the Euclidean space and geometric splines on embedded manifolds. To this end, we first show how rolling mappings transform covariant derivatives of vector fields along rolling curves into usual derivatives of vector fields along their developments.

For a submanifold M of an Euclidean space, the covariant derivative of a vector field X along a smooth curve $t \mapsto \alpha(t)$ on M is obtained by projecting the usual derivative onto the tangent space to M at $\alpha(t)$ with respect to the Euclidean inner product. In what follows, if no explicit reference to α is necessary, we use the notation DX/dt to represent $\nabla_{\dot{\alpha}(t)} X$. Similarly, the $(k+1)$st covariant derivative of a vector field along a smooth curve $t \mapsto \alpha(t)$ on M is obtained from the kth covariant derivative by differentiating it as an ordinary vector-valued function of t and then projecting the result on the tangent space at $\alpha(t)$. For the particular case, where $X = \dot{\alpha}(t)$, we write $\dfrac{D^k \alpha}{dt^k}$ instead of $\dfrac{D^{k-1}\dot{\alpha}}{dt^{k-1}}$, to simplify the notation.

In the previous section, we have seen that, while M is rolling, geodesics on M develop as geodesics on V. At this point, we ask the following natural question.

Question 4.1. If a manifold M embedded in an Euclidean space rolls (without slip or twist) on its affine tangent space at a point along a curve which is a geometric spline, is the development of this curve an Euclidean spline?

4.1. **The variational approach to geometric splines revisited.** In order to answer Question 4.1 for the manifolds under study, we first recall the definition of a geometric spline on a manifold M equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$. In the variational approach to cubic splines on manifolds, one looks for curves on M which minimize the following functional:

$$J(\gamma) = \frac{1}{2} \int_0^T \left\langle \frac{D^2\gamma(t)}{dt^2}, \frac{D^2\gamma(t)}{dt^2} \right\rangle dt, \tag{4.1}$$

over the class Ω of $\mathcal{C}^2$-smooth paths γ on M satisfying interpolation conditions (2.2) and the boundary conditions (2.4).

A well-known result is as follows. More details can be found, e.g., in [6, 7, 22].

Theorem 4.1. *A necessary condition for γ to minimize functional* (4.1) *over Ω is the condition that γ satisfies the Euler–Lagrange equation*

$$\frac{D^4\gamma}{dt^4} + \mathcal{R}\left(\frac{D^2\gamma}{dt^2}, \dot{\gamma}\right)\dot{\gamma} \equiv 0 \tag{4.2}$$

on each sub-interval $[t_{i-1}, t_i]$*, where* $\mathcal{R}$ *denotes the curvature tensor associated with the connection which is compatible with the metric.*

Generalizations of geometric cubic splines appeared in [4]. The second derivative in functional (4.1) is replaced by any higher order derivative, say $\dfrac{D^m\gamma(t)}{dt^m}$, and the corresponding Euler–Lagrange equation becomes

$$\frac{D^{2m}\gamma}{dt^{2m}} + \sum_{j=2}^{m}(-1)^j\mathcal{R}\left(\frac{D^{2m-j}\gamma}{dt^{2m-j}}, \frac{D^{j-1}\gamma}{dt^{j-1}}\right)\dot{\gamma} \equiv 0. \tag{4.3}$$

Geometric cubic (polynomial) splines have been defined in the literature as solutions of Eqs. (4.2) or (4.3), respectively.

Remark 4.1. Note that formulas (4.2) and (4.3) were derived in the general context of a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$, while in this paper we work only in the embedding space, i.e., the Riemannian metric is induced by the Euclidean metric. More precisely, consider the following cases.

1. Let $S^n = \{x \in \mathbb{R}^{n+1} \mid x^\top x = 1\}$. The Riemannian metric on S^n
$$\langle\cdot,\cdot\rangle : TS^n \times TS^n \to \mathbb{R} \tag{4.4}$$
is that induced by the Euclidean metric on $\mathbb{R}^{n+1}$, i.e.,
$$\langle\xi_1, \xi_2\rangle := \xi_1^\top\xi_2 \quad \text{for all} \quad \xi_1, \xi_2 \in T_xS^n = \{\xi \in \mathbb{R}^{n+1} \mid \xi^\top x = 0\}. \tag{4.5}$$
The orthogonal projection operator onto the tangent space at $x \in S^n$ with respect to (4.5) is
$$\pi_{T_xS^n} : \mathbb{R}^{n+1} \to T_xS^n, \quad z \mapsto z - \langle x, z\rangle x = (1 - xx^\top)z. \tag{4.6}$$
2. Let $\mathrm{SO}_n = \{R \in \mathbb{R}^{n\times n} \mid R^\top R = I_n,\ \det R = 1\}$. The Riemannian metric on SO_n
$$\langle\cdot,\cdot\rangle : T\,\mathrm{SO}_n \times T\,\mathrm{SO}_n \to \mathbb{R} \tag{4.7}$$
is that induced by the trace form on $\mathbb{R}^{n\times n}$
$$\langle\Omega_1 R, \Omega_2 R\rangle := \frac{1}{2}\operatorname{tr}((\Omega_1 R)^\top \Omega_2 R) = -\frac{1}{2}\operatorname{tr}(\Omega_1\Omega_2), \tag{4.8}$$
where $\Omega_1 R, \Omega_2 R \in T_R\,\mathrm{SO}_n = \mathfrak{so}_n \cdot R$. This also defines an inner product on $\mathfrak{so}_n$, which is nothing but the Killing form conveniently scaled. The orthogonal projection operator onto the tangent space at $R \in \mathrm{SO}_n$ with respect to (4.8) is
$$\pi_{T_R\,\mathrm{SO}_n} : \mathbb{R}^{n\times n} \to T_R\,\mathrm{SO}_n, \quad X \mapsto \frac{X - RX^\top R}{2}. \tag{4.9}$$
3. Let $G_{k,n} := \{P \in \mathrm{Sym}_n \mid P^2 = P, \mathrm{rk}(P) = k\}$. The Riemannian metric on $G_{k,n}$
$$\langle\cdot,\cdot\rangle : TG_{k,n} \times TG_{k,n} \to \mathbb{R} \tag{4.10}$$

is that induced by the inner product on Sym_n

$$\langle K_1, K_2 \rangle := \frac{1}{2}\,\mathrm{tr}(K_1 K_2), \tag{4.11}$$

where $K_1, K_2 \in T_P G_{k,n}$ (i.e., $K_1, K_2 \in \mathrm{Sym}_n$) solve the equation $K = KP + PK$ with $P \in G_{k,n}$. The orthogonal projection operator onto the tangent space at $P \in G_{k,n}$ with respect to (4.11) is

$$\pi_{T_P G_{k,n}} : \mathrm{Sym}_n \to T_P G_{k,n}, \quad X \mapsto [P, [P, X]]. \tag{4.12}$$

Now consider Question 4.1 for spheres and special orthogonal groups.

4.2. **Rolling mappings and splines on spheres.** The following theorem is a generalization of a result for S^2 contained in [17] to n-dimensional spheres.

Theorem 4.2. *Assume that* $t \mapsto h(t) = (R^\top(t), s(t)) \in \mathrm{SE}_n$ *is a rolling mapping of the sphere* S^n *on* V *with the rolling curve* $t \mapsto \alpha(t)$. *Then*

$$R^\top(t)\frac{D^j \alpha}{dt^j}(t) = \alpha_{\mathrm{dev}}^{(j)}(t) \quad \textit{for all } t \textit{ and all } j \in \mathbb{N}. \tag{4.13}$$

Proof. First, we note that, since $\alpha_{\mathrm{dev}}^{(j)}(t) \in T_{p_0} S^n$ for all t, we have that $R(t)\alpha_{\mathrm{dev}}^{(j)}(t) \in T_{R(t)p_0} S^n = T_{\alpha(t)} S^n$ for all t and, therefore, the statement has sense.

We will use induction on k to prove (4.13). For $k = 1$, using the identities $\alpha(t) = R(t)p_0$ and $\dot R(t) = R(t)A(t)$, where $A(t)$ is as in (3.11) and the no-slip condition (3.17), we obtain

$$\dot\alpha(t) = \dot R(t)p_0 = R(t)A(t)p_0 = R(t)\dot s(t) = R(t)\dot\alpha_{\mathrm{dev}}(t).$$

Now, assuming that (4.13) holds for some $j \in \mathbb{N}$, we can write

$$\begin{aligned}\frac{D^{j+1}\alpha(t)}{dt^{j+1}} &= \pi_{T_{\alpha(t)}S^n}\left(\frac{d}{dt}\frac{D^j\alpha(t)}{dt^j}\right)\\ &= \pi_{T_{\alpha(t)}S^n}\left(\dot R(t)\alpha_{\mathrm{dev}}^{(j)}(t) + R(t)\alpha_{\mathrm{dev}}^{(j+1)}(t)\right).\end{aligned} \tag{4.14}$$

Since $R(t)\alpha_{\mathrm{dev}}^{(j+1)}(t)$ already belongs to $T_{\alpha(t)}S^n$, we just have to show that $\dot R(t)\alpha_{\mathrm{dev}}^{(j)}(t)$ is parallel to $\alpha(t)$ for all t. This easily follows from the properties of the matrix function A and no-slip condition (3.17). Indeed,

$$\begin{aligned}\dot R(t)\alpha_{\mathrm{dev}}^{(j)}(t) &= R(t)A(t)\alpha_{\mathrm{dev}}^{(j)}(t) = R(t)A(t)s^{(j)}(t)\\ &= R(t)A(t)A^{(j-1)}(t)p_0 = -u^\top(t)u^{(j-1)}(t)\cdot R(t)p_0\\ &= -u^\top(t)u^{(j-1)}(t)\cdot\alpha(t),\end{aligned} \tag{4.15}$$

with the scalar-valued function $t \mapsto u^\top(t)u^{(j-1)}(t)$ defined by the controls. Since

$$\pi_{T_{\alpha(t)}S^n}(\dot{R}(t)\alpha_{\text{dev}}^{(j)}(t)) = -u^\top(t)u^{(j-1)}(t)\left(I_{n+1} - \alpha(t)\alpha(t)^\top\right)\alpha(t) = 0, \tag{4.16}$$

the theorem is proved. □

An important conclusion can be derived from this theorem.

Corollary 4.1. *Assume that S^n is rolling without slip or twist on its affine tangent space at a point p_0 along a curve $t \mapsto \alpha(t)$. If the development $t \mapsto \alpha_{\text{dev}}(t)$ is an Euclidean cubic spline, then $t \mapsto \alpha(t)$ is a geometric cubic spline on S^n if and only if it is a re-parameterized geodesic.*

Proof. This can be justified by the result in Theorem 4.2. Indeed, $t \mapsto \alpha_{\text{dev}}(t)$ is an Euclidean cubic spline if and only if $\ddddot{\alpha}_{\text{dev}} \equiv 0$ on each sub-interval and, consequently, $\dfrac{D^4\alpha}{dt^4} \equiv 0$ on each sub-interval as well. The curve $\alpha(t)$ is not a cubic spline on S^n except for the case where the curvature term in (4.2) vanishes. This can be seen as follows.

Recall that

$$\begin{aligned} \alpha(t) = R(t)p_0, \quad \dot{\alpha}(t) = \dot{R}(t)p_0 = R(t)u(t), \\ \ddot{\alpha}(t) = \dot{R}(t)u(t) + R(t)\dot{u}(t) = R(t)\left(\dot{u}(t) - u(t)^\top u(t)p_0\right), \\ \alpha(t)\alpha^\top(t) = R(t)p_0p_0^\top R^\top(t), \quad \dot{\alpha}^\top(t)\dot{\alpha}(t) = u(t)^\top u(t), \\ \dot{\alpha}(t)\dot{\alpha}^\top(t) = R(t)u(t)u(t)^\top R^\top(t), \quad u(t)^\top p_0 = 0, \quad \dot{u}(t)^\top p_0 = 0. \end{aligned} \tag{4.17}$$

Therefore, using the following formula, valid for spaces of constant sectional curvature κ, which can be found in [19],

$$\mathcal{R}(Y,Z)W = \kappa\left(\langle W,Z\rangle Y - \langle W,Y\rangle Z\right),$$

we can write

$$\begin{aligned} \mathcal{R}\left(\frac{D^2\alpha(t)}{dt^2}, \dot{\alpha}(t)\right)\dot{\alpha}(t) &= \langle\dot{\alpha}(t), \dot{\alpha}(t)\rangle\frac{D^2\alpha(t)}{dt^2} - \left\langle\dot{\alpha}(t), \frac{D^2\alpha(t)}{dt^2}\right\rangle\dot{\alpha}(t) \\ &= \Big(\dot{\alpha}^\top(t)\dot{\alpha}(t)I_{n+1} - \dot{\alpha}(t)\dot{\alpha}^\top(t)\Big)\Big(I_{n+1} - \alpha(t)\alpha^\top(t)\Big)\ddot{\alpha}(t) \\ &= R(t)\Big(u(t)^\top u(t)I_{n+1} - u(t)u(t)^\top\Big)\dot{u}(t). \end{aligned} \tag{4.18}$$

But the expression

$$(u(t)^\top u(t)I_{n+1} - u(t)u(t)^\top)\dot{u}(t)$$

in the last line of (4.18) is identically zero if and only if $u(t) = f(t)u_0$, where $f : \mathbb{R} \to \mathbb{R}$ is a smooth scalar-valued function and the constant $u_0 \in T_{p_0}S^n$. As a consequence, using $A_0 := u_0p_0^\top - p_0u_0^\top$, we have

$$R(t) = e^{f(t)A_0} \tag{4.19}$$

and using (3.87), we obtain that the rolling curve is

$$\alpha(t) = R(t)p_0 = \Big(I_{n+1} + A_0 \sin f(t) + A_0^2(1-\cos f(t))\Big)p_0$$
$$= u_0 \sin f(t) + p_0 \cos f(t), \quad (4.20)$$

which is the well-known formula for a re-parameterized great circle on S^n. $\square$

4.2.1. *A relation to a constrained variational problem on* SO_{n+1}. In spite of the last result, we can show that, under the assumption of Theorem 4.1 and for $M = S^n$, the variational problem that gives rise to the Euclidean cubic spline α_{dev} is equivalent to a constrained variational problem on SO_{n+1}. Let $R \in \mathrm{SO}_{n+1}$, and let $\pi_{T_R \mathrm{SO}_{n+1}}$ be the projection operator as defined in (4.9).

Let $R(t)$ and $s(t)$ be the solution of kinematic equations (3.10). Using the relations

$$\dot R(t) = R(t)A(t), \quad \ddot R(t) = \dot R(t)A(t) + R(t)\dot A(t)$$

and the orthogonal projection defined by (4.9), we obtain

$$\frac{D^2R(t)}{dt^2} = \pi_{T_R \mathrm{SO}_{n+1}}(\ddot R(t)) = \frac{\ddot R(t) - R(t)\ddot R^\top(t)R(t)}{2} = R(t)\dot A(t) \quad (4.21)$$

and, therefore,

$$\int_0^\tau \left\langle \frac{D^2R(t)}{dt^2}, \frac{D^2R(t)}{dt^2}\right\rangle dt = \int_0^\tau \left\langle \dot A(t), \dot A(t)\right\rangle dt = -\frac{1}{2}\int_0^\tau \operatorname{tr} \dot A^2(t)dt$$
$$= \int_0^\tau \dot u^\top(t)\dot u(t)dt = \int_0^\tau \ddot s^\top(t)\ddot s(t)dt = \int_0^\tau \ddot\alpha_{\mathrm{dev}}^\top(t)\ddot\alpha_{\mathrm{dev}}(t)dt. \quad (4.22)$$

Therefore, the optimization problem on $\mathbb{R}^{n+1}$

$$\min_{\alpha_{\mathrm{dev}}} \int_0^\tau \langle \ddot\alpha_{\mathrm{dev}}(t), \ddot\alpha_{\mathrm{dev}}(t)\rangle\, dt,$$

which gives rise to the cubic spline on $\mathbb{R}^{n+1}$, is equivalent to the following *constrained* variational problem:

$$\min_R \int_0^\tau \left\langle \frac{D^2R(t)}{dt^2}, \frac{D^2R(t)}{dt^2}\right\rangle dt, \quad (4.23)$$

subject to dynamics (3.10), which, in turn, gives rise to a geometric cubic spline on S^n, with *nonholonomic* constraints. This agrees with the case $n = 2$ discussed in [2, p. 365].

This class of problems was studied in [7] and, in particular, the Euler–Lagrange equations for problem (4.23) have been derived. These equations are highly nonlinear but it is now clear that the solution of such equations, acting on p_0, produces a curve α which satisfies $D^4\alpha/dt^4 = 0$.

4.3. **Rolling mappings and splines on** SO_n**.** We obtain an analog of Theorem 4.2 in the case of the orthogonal group, one just needs to use expression (3.49) for the rolling curve and compute covariant derivatives of the velocity vector field, using also kinematic equations (3.47). Let $\alpha(t) = U(t)P_0W^\top(t)$ be as before, let the orthogonal projection operator $\pi_{T_{\alpha(t)}\mathrm{SO}_n}$ be defined by (4.9), and let the control Ω be as in (3.46).

Claim 4.1.

$$\frac{D^j\alpha(t)}{dt^j} = U(t)\Omega^{(j-1)}(t)P_0W^\top(t), \quad \textit{for all } t \textit{ and all } j \in \mathbb{N}. \tag{4.24}$$

Proof. We prove the claim by induction on j. Indeed, for $j = 1$ this equality holds since

$$\frac{D\alpha(t)}{dt} = \dot\alpha(t) = \dot U(t)P_0W^\top(t) + U(t)P_0\dot W^\top(t) = U(t)\Omega(t)P_0W^\top(t). \tag{4.25}$$

Now we assume that (4.24) holds for j. We define

$$S(t) := \frac{1}{2}\left(\Omega(t)\Omega^{(j-1)}(t) + \Omega^{(j-1)}(t)\Omega(t)\right). \tag{4.26}$$

Then (omitting the dependency on t for the convenience),

$$\begin{aligned}
\frac{D^{j+1}\alpha}{dt^{j+1}} &= \pi_{T_{\alpha(t)}\mathrm{SO}_n}\left(\dot U\Omega^{(j-1)}P_0W^\top + U\Omega^{(j-1)}P_0\dot W^\top + U\Omega^{(j)}P_0W^\top\right)\\
&= \pi_{T_{\alpha(t)}\mathrm{SO}_n}\left(USP_0W^\top + U\Omega^{(j)}P_0W^\top\right)\\
&= \frac{1}{2}\Big(USP_0W^\top + U\Omega^{(j)}P_0W^\top - \alpha(USP_0W^\top + U\Omega^{(j)}P_0W^\top)^\top\alpha\Big)\\
&= \frac{1}{2}\Big(USP_0W^\top + U\Omega^{(j)}P_0W^\top\\
&\qquad - UP_0W^\top(WP_0^\top SU^\top - WP_0^\top\Omega^{(j)}U^\top)UP_0W^\top\Big)\\
&= U\Omega^{(j)}P_0W^\top
\end{aligned}$$

which completes the proof of identity (4.24). ☐

Now, since $X^{(j)}(t) = \Omega^{(j-1)}(t)P_0$ and $\alpha_{\mathrm{dev}}(t) = P_0 + X(t)$, the following result is straightforward.

Theorem 4.3. *Assume that* $t \mapsto h(t) = \left(U^\top(t), W^\top(t), X(t)\right) \in \mathrm{SE}_{n^2}$ *is a rolling mapping for* SO_n *with the rolling curve* $t \mapsto \alpha(t)$*. Then*

$$U^\top(t)\frac{D^j\alpha(t)}{dt^j}W(t) = \alpha^{(j)}_{\mathrm{dev}}(t) \tag{4.27}$$

holds for all t and all $j \in \mathbb{N}$. □

Corollary 4.1 was stated for the sphere S^n. It does not hold for the case of SO_n, except for $n = 3$. To see this, according to Theorem 4.3 and the definition of a geometric spline, one just needs to find conditions on the control function $\Omega(t)$ such that the curvature term in (4.2) vanishes.

Corollary 4.2. *For* SO_n*, the vanishing of the curvature term in* (4.2) *is equivalent to*

$$[\Omega, \dot{\Omega}] = 0. \tag{4.28}$$

Proof. Here we use a result contained in [21], which guarantees that every connected and compact Lie group G admits a bi-invariant Riemannian metric such that if Y, Z, and W are right- or left-invariant vector fields on G, then the curvature tensor is

$$\mathcal{R}(Y, Z)W = -\frac{1}{4}[[Y, Z], W]. \tag{4.29}$$

For SO_n, the corresponding metric is precisely the metric induced by the Euclidean metric of the embedding space, which we have used before. Since $\alpha(t) = U(t)P_0W^\top(t)$,

$$\begin{aligned} \dot{\alpha} &= U\Omega P_0 W^\top = U\Omega U^\top \alpha, \\ \frac{D^2\alpha}{dt^2} &= U\dot{\Omega}P_0W^\top = U\dot{\Omega}U^\top\alpha, \end{aligned} \tag{4.30}$$

and, therefore,

$$\begin{aligned} \left[\frac{D^2\alpha}{dt^2}, \dot{\alpha}\right] &= \left[U\dot{\Omega}U^\top, U\Omega U^\top\right]\alpha = U\left[\dot{\Omega}, \Omega\right]U^\top\alpha, \\ \left[\left[\frac{D^2\alpha}{dt^2}, \dot{\alpha}\right], \dot{\alpha}\right] &= U\left[\left[\dot{\Omega}, \Omega\right], \Omega\right]U^\top\alpha. \end{aligned} \tag{4.31}$$

Consequently, the curvature term in (4.2) vanishes if and only if

$$\left[\left[\dot{\Omega}, \Omega\right], \Omega\right] = 0. \tag{4.32}$$

Therefore, to prove the corollary, it suffices to show that (4.32) implies (4.28).

Recall that an inner product on $\mathfrak{so}_n$ has been defined in terms of the trace form (scaled Killing form) and that the endomorphism ad_W defined by $\mathrm{ad}_W Z = [W, Z]$ is skew-symmetric with respect to the Killing form κ, i.e., $\kappa(\mathrm{ad}_W Z, Y) = -\kappa(Z, \mathrm{ad}_W Y)$). Therefore, (4.32) implies

$$\mathrm{tr}([\Omega, [\Omega, \dot{\Omega}]]\dot{\Omega}) = 0, \tag{4.33}$$

which is equivalent to

$$\mathrm{tr}([\Omega, \dot{\Omega}]^2) = 0. \tag{4.34}$$

Finally, by the sign-definiteness of the trace form, we obtain

$$[\Omega, \dot{\Omega}] = 0. \tag{4.35}$$

The corollary is proved. □

For $n = 3$, due to the isomorphism between the Lie algebra $\mathfrak{so}_3$ and $\mathbb{R}^3$ equipped with the cross product, Eq. (4.32) is equivalent to

$$\omega \times (\omega \times \dot{\omega}) = \langle \omega, \dot{\omega} \rangle \omega - \langle \omega, \omega \rangle \dot{\omega} = 0, \tag{4.36}$$

where $\omega \in \mathbb{R}^3$ is the 3-vector function associated with $\Omega \in \mathfrak{so}_3$. Therefore, in this case, the condition $\Omega(t) = g(t)\Xi$, where $\Xi \in \mathfrak{so}_3$ is a constant skew-symmetric matrix and $g : \mathbb{R} \to \mathbb{R}$ is a scalar function, must hold. For $n > 3$, the condition holds for more general constraints. For example, if $\Omega(t)$ belongs to an Abelian Lie subalgebra of $\mathfrak{so}_n$ for all t, condition (4.32) is always satisfied. Clearly, $\Omega(t)$ belongs to an Abelian Lie subalgebra of $\mathfrak{so}_3$ if and only if $\Omega(t) = g(t)\Xi$, since every nontrivial Abelian subalgebra of $\mathfrak{so}_3$ is of dimension one.

4.3.1. *The relation to constrained variational problems.* Under the assumption of Theorem 4.3, the variational problem that gives rise to the Euclidean cubic spline α_{dev} is equivalent to a constrained variational problem on SO_n as well. Indeed, let (X, U, W) be the solution of kinematic equations (3.47). Then

$$\frac{D^2U}{dt^2} = \frac{1}{2} U\dot{\Omega}, \quad \frac{D^2W}{dt^2} = -\frac{1}{2} W P_0^\top \dot{\Omega} P_0, \tag{4.37}$$

and, therefore,

$$\begin{aligned} \int_0^\tau \left\langle \frac{D^2U}{dt^2}, \frac{D^2U}{dt^2} \right\rangle dt &= \int_0^\tau \left\langle \frac{D^2W}{dt^2}, \frac{D^2W}{dt^2} \right\rangle dt \\ &= \frac{1}{4} \int_0^\tau \left\langle \dot{\Omega}, \dot{\Omega} \right\rangle dt = \frac{1}{4} \int_0^\tau \langle \ddot{\alpha}_{\text{dev}}, \ddot{\alpha}_{\text{dev}} \rangle \, dt. \end{aligned} \tag{4.38}$$

Therefore, the optimization problem

$$\min_{\alpha_{\text{dev}}} \int_0^\tau \langle \ddot{\alpha}_{\text{dev}}, \ddot{\alpha}_{\text{dev}} \rangle \, dt\,,$$

which gives rise to the cubic spline on $T_{P_0} \mathrm{SO}_n$, is equivalent to the constrained variational problem on SO_n

$$\min_U \int_0^\tau \left\langle \frac{D^2U}{dt^2}, \frac{D^2U}{dt^2} \right\rangle dt, \tag{4.39}$$

subject to the dynamics

$$\dot{U}(t) = \frac{1}{2}U(t)\Omega(t), \tag{4.40}$$

which, in turn, is equivalent to

$$\min_U \int_0^\tau \left\langle \frac{D^2W}{dt^2}, \frac{D^2W}{dt^2} \right\rangle dt, \tag{4.41}$$

subject to the dynamics

$$\dot{W}(t) = -\frac{1}{2}W(t)P_0^\top \Omega(t)P_0. \tag{4.42}$$

It is again in the class of problems studied in [7]. For $\Omega(t) = \sum_{i=1}^{n} u_i(t)A_{n+1,i}$ as in (3.10), this constrained problem coincides with the constrained problem formulated for the sphere.

4.4. **Rolling mappings on the Grassmann manifold.** An analog of Theorems 4.2 and 4.3 holds also for $G_{k,n}$.

Theorem 4.4. *Assume that $t \mapsto h(t) = \big(\Theta^\top(t), X(t)\big) \in \mathrm{SE}_{n(n+1)/2}$ is a rolling mapping for $G_{k,n}$ with the rolling curve $t \mapsto \alpha(t) = \Theta(t)P_0\Theta^\top(t)$. Then*

$$\Theta^\top(t)\frac{D^j\alpha}{dt^j}(t)\Theta(t) = \alpha_{\mathrm{dev}}^{(j)}(t) \quad \textit{for all } t \textit{ and all } j \in \mathbb{N}. \tag{4.43}$$

Proof. We prove (4.43) by induction on j.

We recall the kinematic equations for $G_{k,n}$ and formulas for α and α_{dev}. For arbitrary $P \in G_{k,n}$, we have

$$\begin{aligned} \alpha(t) &= \Theta(t)P\Theta^\top(t), \\ \dot{\alpha}(t) &= [\dot{\Theta}(t)\Theta^\top(t), \alpha(t)], \\ \alpha_{\mathrm{dev}}(t) &= P + X(t), \end{aligned} \qquad \begin{aligned} \dot{X}(t) &= \xi(t), \\ \dot{\Theta}(t) &= \Theta(t)[\xi(t), P]. \end{aligned} \tag{4.44}$$

For $j = 1$, (4.43) holds, since

$$\begin{aligned} \Theta^\top(t)\frac{D\alpha(t)}{dt}\Theta(t) &= \Theta^\top(t)\, \pi_{T_{\alpha(t)}G_{k,n}}(\dot{\alpha}(t))\, \Theta(t) \\ &= \Theta^\top(t)[\alpha(t), [\alpha(t), \dot{\alpha}(t)]]\, \Theta(t) = [P, [P, [[\xi(t), P], P]]] \\ &= [P, [P, \xi(t)]] = \xi(t) = \dot{X}(t) = \frac{d}{dt}(P + X(t)) = \dot{\alpha}_{\mathrm{dev}}(t). \end{aligned} \tag{4.45}$$

Assume that (4.43) holds for some $j \in \mathbb{N}$. Hence (omitting t-dependencies), we obtain

$$\begin{aligned}
\Theta^\top \frac{D^{j+1}\alpha}{dt^{j+1}}\,\Theta &= \Theta^\top\, \pi_{T_{\alpha(t)}G_{k,n}} \left(\frac{d}{dt}\frac{D^j\alpha}{dt^j}\right)\Theta \\
&= \Theta^\top\, \pi_{T_{\alpha(t)}G_{k,n}} \left(\frac{d}{dt}\left(\Theta\alpha_{\mathrm{dev}}^{(j)}\Theta^\top\right)\right)\Theta \\
&= \Theta^\top\,[\alpha,[\alpha,\dot\Theta\xi^{(j-1)}\Theta^\top + \Theta\xi^{(j)}\Theta^\top + \Theta\xi^{(j-1)}\dot\Theta^\top]]\,\Theta \\
&= [P,[P,[[\xi,P],\xi^{(j-1)}]]] + [P,[P,\xi^{(j)}]] \\
&= [P,[P,[[\xi,P],\xi^{(j-1)}]]] + [P,[P,\alpha_{\mathrm{dev}}^{(j+1)}]] \\
&= [P,[P,[[\xi,P],\xi^{(j-1)}]]] + \alpha_{\mathrm{dev}}^{(j+1)}. \qquad (4.46)
\end{aligned}$$

It remains to show that

$$[P,[P,[[\xi,P],\xi^{(j-1)}]]] = 0 \tag{4.47}$$

which is equivalent to showing that

$$[[\xi,P],\xi^{(j-1)}] \in T_P^\perp G_{k,n}. \tag{4.48}$$

By the orthogonal invariance properties of $G_{k,n}$ considered as a submanifold of Sym_n, it suffices to show that (4.48) holds at the point $P = \left[\begin{smallmatrix} I_k & 0 \\ 0 & 0\end{smallmatrix}\right]$. Using kinematic equations (3.77) for this special case, i.e., exploiting the fact that $\xi, \xi^{(j-1)}$ are of the form

$$\xi(t) = \begin{bmatrix} 0 & \Psi(t) \\ \Psi^\top(t) & 0 \end{bmatrix}, \quad \xi^{(j-1)}(t) = \begin{bmatrix} 0 & \Psi^{(j-1)}(t) \\ (\Psi^{(j-1)})^\top(t) & 0 \end{bmatrix}, \tag{4.49}$$

we obtain

$$[[\xi,P],\xi^{(j-1)}] = \begin{bmatrix} -\Psi(\Psi^{(j-1)})^\top - \Psi^{(j-1)}\Psi^\top & 0 \\ 0 & \Psi^\top\Psi^{(j-1)} + \Psi^{(j-1)})^\top\Psi \end{bmatrix}, \tag{4.50}$$

which is clearly in $T_P^\perp G_{k,n}$ as commutes with $P = \left[\begin{smallmatrix} I_k & 0 \\ 0 & 0\end{smallmatrix}\right]$. The theorem is proved. □

4.5. **The relation to the geodesic curvature.** Finally, a simple observation shows that the rolling motion preserves the geodesic curvature if M is any of our favorite manifolds, i.e., $M \in \{S^n, SO_n, G_{k,n}\}$.

Corollary 4.3. *Assume that M rolls without slip or twist on its tangent space at a point along a curve $t \mapsto \alpha(t)$ with the development $t \mapsto \alpha_{\mathrm{dev}}(t)$. Then both of these curves have the same geodesic curvature.*

Proof. The geodesic curvature (see, e.g., [20]) of a curve $t \mapsto \gamma(t)$ on a differentiable manifold M with the Riemannian metric $\langle\cdot,\cdot\rangle$ is the function

$t \mapsto \kappa(t)$ defined by

$$\kappa(\gamma(t)) = \frac{\left\| \frac{D\dot{\gamma}(t)}{dt} \right\|}{\|\dot{\gamma}(t)\|^2} - \frac{\left\langle \frac{D\dot{\gamma}(t)}{dt}, \dot{\gamma}(t) \right\rangle}{\|\dot{\gamma}(t)\|^3}. \tag{4.51}$$

Since the Euclidean metric is orthogonally invariant, we can use this definition and the result of theorems of this section to conclude immediately that

$$\kappa(\alpha(t)) = \kappa(\alpha_{\text{dev}}(t)). \tag{4.52}$$

□

5. Solving the interpolation problem

For the problem stated in Sec. 2, we propose the following algorithm, which is based on rolling and unwrapping techniques. This approach works for any manifold M embedded into some Euclidean space $\mathbb{R}^N$, so that both M and $V \cong T_{p_0}^{\text{aff}} M$ can be considered as submanifolds of $\mathbb{R}^N$. The resulting curve will be given explicitly in terms of the coordinates of the embedding space.

The algorithm can be described as follows.

Algorithm 5.1. 1. Compute an arbitrary smooth curve

$$\alpha : [0, \tau] \to M \tag{5.1}$$

connecting p_0 with p_K such that

$$\alpha(0) = p_0, \quad \alpha(\tau) = p_k. \tag{5.2}$$

2. Roll M on V, with the rolling curve $\alpha([0, \tau])$ and rolling mapping $h(t)$. This produces a smooth curve $\alpha_{\text{dev}} : [0, \tau] \to V$, which joins the unrolled initial and final points. The rolling conditions ensure that all boundary conditions are mapped to V as follows:

$$\begin{aligned} \alpha(0) = p_0 &\mapsto \alpha_{\text{dev}}(0) = p_0 =: q_0, \\ \alpha(\tau) = p_k &\mapsto \alpha_{\text{dev}}(\tau) =: q_k, \end{aligned} \tag{5.3}$$

$$\xi_0 \mapsto h(0)\xi_0 = \xi_0 =: \eta_0, \ \xi_k \mapsto h(\tau)\xi_k =: \eta_k. \tag{5.4}$$

3. Choose a suitable local diffeomorphism

$$\phi : M \supset \Omega \to V, \quad p_0 \in \Omega \text{ open}, \tag{5.5}$$

satisfying

$$\phi(p_0) = p_0, \quad D\phi(p_0) = \text{id}, \tag{5.6}$$

to unwrap the remaining data $\{p_1, \ldots, p_{k-1}\}$ onto V, so that

$$p_i \mapsto \phi\Big(h(t_i)p_i - \alpha_{\text{dev}}(t_i) + p_0\Big) + \alpha_{\text{dev}}(t_i) - p_0 =: q_i. \tag{5.7}$$

4. Solve Problem 2.1 on V using the mapped data, namely $\{q_0, \dots, q_k; \eta_0, \eta_k\}$, instead of the data $\{p_0, \dots, p_k; \xi_0, \xi_k\}$. This will generate a curve
$$\beta : [0, \tau] \to V \tag{5.8}$$
with the properties
$$\begin{gathered} \beta(0) = p_0 = q_0, \quad \beta(t_i) = q_i, \quad \beta(\tau) = q_k, \\ \dot\beta(0) = \xi_0 = \eta_0, \quad \dot\beta(\tau) = \eta_k. \end{gathered} \tag{5.9}$$
5. Wrap $\beta([0, \tau])$ back onto the manifold giving the solution γ of Problem 2.1 by means of the following formula:
$$\gamma(t) := h(t)^{-1}\Big(\phi^{-1}\big(\beta(t) - \alpha_{\mathrm{dev}}(t) + p_0\big) + \alpha_{\mathrm{dev}}(t) - p_0\Big). \tag{5.10}$$

Theorem 5.1. *The curve $t \mapsto \gamma(t)$ defined by* (5.10) *solves Problem* 2.1 *if $M = S^n$, or $M = \mathrm{SO}_n$, or $M = G_{k,n}$.*

Proof. This result was proved in [15] for the case where M is the sphere S^n. Now, for $M = \mathrm{SO}_n$, we can equivalently write
$$\gamma(t) = U(t)\Big(\phi^{-1}\big(\beta(t) - X(t)\big)\Big)W^\top, \tag{5.11}$$
where U, W, and X are the solutions of kinematic equations (3.47) that satisfy the conditions $U(0) = W(0) = I$ and $X(0) = 0$. Thus, the following calculations follow from putting together all conditions of Sec. 3.2 and taking into account that
$$\begin{aligned} \dot\gamma = \dot U\phi^{-1}\big(\beta - X\big)W^\top + U\phi^{-1}\big(\beta - X\big)\dot W^\top& \\ + U\Big(D\phi^{-1}\big(\beta - X\big) \circ \big(\dot\beta - \dot X\big)\Big)W^\top&, \end{aligned} \tag{5.12}$$
$$\gamma(0) = \phi^{-1}\big(\beta(0)\big) = \phi^{-1}\big(\alpha(0)\big) = \phi^{-1}\big(P_0\big) = P_0, \tag{5.13}$$
$$\begin{aligned} \gamma(\tau) &= U(\tau)\phi^{-1}\big(\beta(\tau) - X(\tau)\big)W^\top(\tau) \\ &= U(\tau)\phi^{-1}\big(\alpha_{\mathrm{dev}}(\tau) - X(\tau)\big)W^\top(\tau) \\ &= U(\tau)\phi^{-1}\big(P_0 + X(\tau) - X(\tau)\big)W^\top(\tau) \\ &= U(\tau)P_0W^\top(\tau) = \alpha(\tau) = p_k, \end{aligned} \tag{5.14}$$
$$\begin{aligned} \gamma(t_i) &= U(t_i)\phi^{-1}\big(\beta(t_i) - X(t_i)\big)W^\top(t_i) \\ &= U(t_i)\big(\phi^{-1}\big(P_0\big)\big)W^\top(t_i) = U(t_i)P_0W^\top(t_i) = \alpha(t_i) = p_i, \end{aligned} \tag{5.15}$$
$$\begin{aligned} \dot\gamma(0) &= \dot U(0)P_0 + P_0\dot W^\top(0) + \dot\beta(0) - \dot X(0) \\ &= \frac{1}{2}\dot X(0) + \frac{1}{2}\dot X(0) + \dot\beta(0) - \dot X(0) = \dot\beta(0) = \xi_0, \end{aligned} \tag{5.16}$$

$$\begin{aligned}\dot\gamma(\tau) &= \dot U(\tau)P_0W^\top(\tau) + U(\tau)P_0\dot W^\top(\tau)\\ &\quad + U(\tau)\Big(\dot\beta(\tau) - \dot X(\tau)\Big)W^\top(\tau) = U(\tau)\Omega(\tau)P_0W^\top(\tau)\\ &\quad + U(\tau)\Big(U^\top(\tau)\xi_k W(\tau) - \Omega(\tau)P_0\Big)W^\top(\tau) = \xi_k. \end{aligned}\tag{5.17}$$

For the Grassmann manifold, we just have to replace both U and W by Θ and use kinematic equations (3.82). The theorem is proved. □

This procedure is now illustrated by an example.

5.1. **Example.** Here we present an example for the two-sphere

$$S^2 = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

rolling on its tangent plane V at the south pole $p_0 = [0, 0, -1]^\top \in S^2$. We want to solve Problem 2.1 for $M = S^2$ using Algorithm 5.1. Two choices have to be made: the rolling curve α and the diffeomorphism ϕ. For the first, the obvious choice is the geodesic that joins p_0 (at $t = 0$) and p_k (at $t = \tau$). In this case, the rolling mapping is given by

$$h(t) = \big(R(t)^\top, s(t)\big) = \big(e^{-t\Omega}, tAp_0\big), \tag{5.18}$$

where Ω is the constant matrix,

$$\Omega = \begin{bmatrix} 0 & 0 & -u_1 \\ 0 & 0 & -u_2 \\ u_1 & u_2 & 0 \end{bmatrix} \in \mathfrak{so}_3. \tag{5.19}$$

The development $\alpha_{\mathrm{dev}}([0,\tau])$ is a straight line segment in V, parameterized by t, starting for $t = 0$ from p_0 as one would expect.

Let us now fix the diffeomorphism $\phi : S^2 \to V$ corresponding to (5.6). Natural candidates are

(i) the stereographic projection with respect to the north pole,
(ii) orthogonal projection, or
(iii) more general, Riemannian normal coordinates.

The stereographic projection of S^2 with respect to the north pole $[0, 0, 1]^\top \in S^2$ is

$$\phi^{\text{stereo}} : S^2 \setminus \{[0,0,1]^\top\} \to V, \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \frac{1}{1 - x_3}\begin{bmatrix} 2x_1 \\ 2x_2 \\ x_3 - 1 \end{bmatrix}, \tag{5.20}$$

with the inverse

$$(\phi^{\text{stereo}})^{-1} : V \to S^2 \setminus \{[0,0,1]^\top\}, \qquad \begin{bmatrix} \xi_1 \\ \xi_2 \\ -1 \end{bmatrix} \mapsto \frac{1}{\xi_1^2 + \xi_2^2 + 4}\begin{bmatrix} 4\xi_1 \\ 4\xi_2 \\ \xi_1^2 + \xi_2^2 - 4 \end{bmatrix}. \tag{5.21}$$

We define the orthogonal projection on the sphere by

$$\phi^{\mathrm{ortho}} : S^2 \setminus \{x \in S^2 \mid x_3 \geq 0\} \to V, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} -x_1/x_3 \\ -x_2/x_3 \\ -1 \end{bmatrix}, \tag{5.22}$$

with the inverse

$$(\phi^{\mathrm{ortho}})^{-1} : V \to S^2 \setminus \{x \in S^2 \mid x_3 \geq 0\}, \quad \begin{bmatrix} \xi_1 \\ \xi_2 \\ -1 \end{bmatrix} \mapsto \frac{1}{\sqrt{\xi_1^2 + \xi_2^2 + 1}} \begin{bmatrix} \xi_1 \\ \xi_2 \\ -1 \end{bmatrix}. \tag{5.23}$$

Obviously, for the south pole $p_0 = [0, 0, -1]^\top$,

$$\phi^{\mathrm{stereo}}(p_0) = \phi^{\mathrm{ortho}}(p_0) = p_0; \tag{5.24}$$

moreover, taking the derivative evaluated at p_0 acting on an arbitrary $h \in T_{p_0}S^2$, we obtain

$$D\phi^{\mathrm{stereo}}(p_0) \cdot h = D\phi^{\mathrm{ortho}}(p_0) \cdot h = h. \tag{5.25}$$

Interpolating the mapped data on V can be done by computing a cubic spline, e.g., by means of the classical De Casteljau algorithm, see [8, 9]).

According to Problem 2.1, we are given five points on S^2 together with five instants of time. Following Algorithm 5.1, we compute the great circle α connecting the initial point (south pole) with the final point. The development α_{dev} is then a straight line segment in the affine tangent plane attached to the south pole. In Fig. 1, the sphere is attached to the tangent plane at p_0 at the time t_0. One can see the cubic spline lying in the tangent plane and the solution curve of the interpolation problem living on the sphere. Figure 2 shows the sphere after rolling along the blue straight line segment. The ray emanating from the midpoint of the sphere clarifies that we have used the orthogonal projection. On the contrary, Figs. 3 and 4 show the result by using the stereographic projection instead of orthogonal projection. The ray emanating from the top of the sphere connects corresponding points on the sphere and tangent plane. The cubic spline and solution curve are both plotted in white. The last picture Fig. 5 allows for a qualitative comparison of the two methods.

6. Appendix

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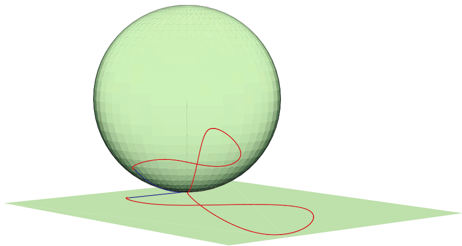

Fig. 1. Wrapping back by the orthogonal projection, see (5.22) and (5.23). The sphere is still at rest at the south pole.

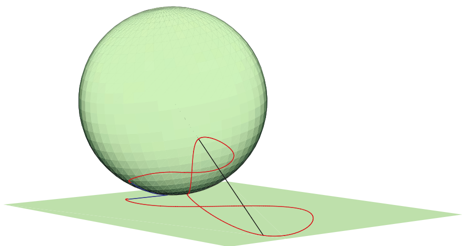

Fig. 2. Wrapping back by the orthogonal projection, see (5.22) and (5.23). The sphere rolls along the straight line segment.

We are grateful to Don Wenura Dissanayake, who produced the five figures during his summer scholar project at the Australian National University, Canberra.

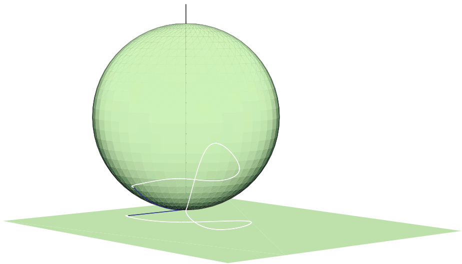

Fig. 3. Wrapping back by the stereographic projection, see (5.20) and (5.21). The sphere is still at rest at the south pole.

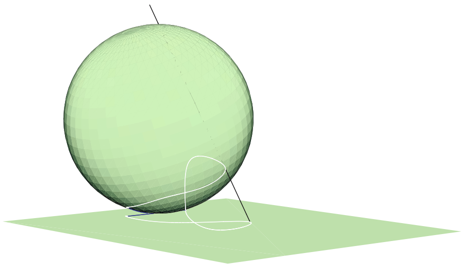

Fig. 4. Wrapping back by the stereographic projection, see (5.20) and (5.21). The sphere rolls along the straight line segment.

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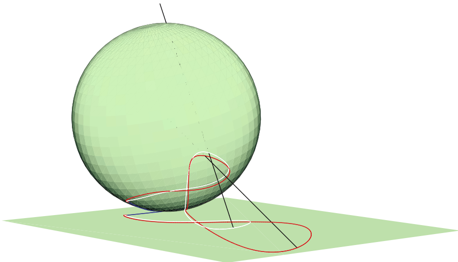

Fig. 5. Comparison of different interpolation curves which both solve the problem.

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Authors' addresses:

K. Hüper
NICTA, Canberra Research Laboratory,
Locked Bag 8001, Canberra ACT 2601, Australia;
Department of Information Engineering,
Research School of Information Sciences and Engineering,
The Australian National University, Canberra ACT 0200, Australia
E-mail: knut.hueper@nicta.com.au

F. Silva Leite
Institute of Systems and Robotics,
University of Coimbra – Pólo II, Pinhal de Marrocos,
3030-290 Coimbra, Portugal;
Department of Mathematics, University of Coimbra,
Largo D. Dinis, 3001-454 Coimbra, Portugal
E-mail: fleite@mat.uc.pt