52nd IEEE Conference on Decision and Control
December 10-13, 2013. Florence, Italy

Stability Analysis and Near Optimal Gain Tuning of an Attitude Estimator on the Special Orthogonal Group

Alireza Khosravian

Abstract— **In this paper, we investigate the effect of non-ideal sensor measurements on the performance of a nonlinear attitude estimator. Bounded deterministic disturbances are considered in both vector measurements and angular velocity reading. We derive conditions under which the practical stability of estimator is guaranteed and present an upper-bound for the steady state attitude estimation error. We also consider the problem of minimizing the ultimate bound of estimation error by proper choice of constant observer gains according to the properties of vector measurements. A key contribution of the paper is to propose a set of Linear Matrix Inequalities which result in the near optimal observer gains in a sense that they minimize the ultimate bound of estimation error. Meanwhile a case study, some interesting results of the optimization procedure are also discussed which provide a greater insight into the problem.**

I. Introduction

The orientation of a rigid body with respect to a known reference frame is usually called its attitude. Attitude estimation is known to be a classical problem with a long history still holding a forefront position as the subject of intensive research [6]. The focus on developing low cost sensors for commercial navigation systems has lead to a strong interest in employing advanced observers to reduce the effect of measurement noise on the performance of such systems. Kalman filtering techniques [13] are probably the most popular methods of attitude estimation and they have been applied successfully for many years. Nevertheless, Kalman Filters are difficult to be applied robustly to some applications involving low quality sensors. Inherent nonlinearity of the attitude dynamics and non-Gaussian sensor noise can potentially lead to poor behavior of such filters in certain practical situations. On the other hand, more sophisticated stochastic filtering techniques, such as particle filters and unscented filters [5], place too much computational load on the low cost processing systems associated with the commercial applications. Consequently, several nonlinear attitude observers have been proposed in recent literature to tackle those issues. One interesting approach for deriving nonlinear attitude observers is to represent the attitude as an element of a Lie group (such as the special orthogonal group $\mathsf{SO}(3)$) rather than considering Euclidean formulation of attitude (such as Euler angles representation). Using this approach, complexity of the nonlinear attitude dynamics is encoded to the structure of the Lie group itself rather than the dynamics equations of motion. In fact, recent publications provide significant guidelines for the design of nonlinear attitude observers by pointing out the topological issues that hinder global stabilization on non-Euclidean spaces such as $\mathsf{SO}(3)$ [1], [3], [17]. Following those guidelines, continuous-time nonlinear estimators on $\mathsf{SO}(3)$ were designed in [14], [18] which ensured almost globally asymptotic estimation of attitude and gyro bias. In these references, the observers were derived assuming that perfect attitude was available by batch preprocessing of vector readings. A number of methods were developed which directly utilized the sensor reading in construction of attitude estimator which potentially can lead to a better error characterization of the estimated attitude. For example, [4], [15], [19], [23] used gyro together with two or more vector measurements and asymptotically estimated the attitude and gyro bias in an almost global sense. Stability properties of that observer was investigated in [10], [16], [20] assuming availability of only one vector measurement but considering that the associated reference vector is time-varying in Inertial frame. Although in this condition convergence rate of the observer depends on the time-varying properties of the reference vector (which can be small in certain situations), [11], [20] proved that asymptotic attitude tracking is still possible by proper coupling of the observer with a nonlinear controller. In fact, observability analysis of attitude dynamics on $\mathsf{SO}(3)$ manifold was carefully addressed in [21] and it was shown that when only a single vector measurement is available, the corresponding reference vector indeed requires to be time-varying in order to guarantee the observability. Interestingly, similar result was achieved in [2] for an estimator whose states evolved on $\mathbb{R}^9$ instead of $\mathsf{SO}(3)$.

The present work is devoted to carefully investigate the effect of non-ideal sensor measurements on the performance of a nonlinear attitude estimator. The attitude estimator which is formulated in the matrix Lie group representation of $\mathsf{SO}(3)$ was previously proposed in [15], [19], [23]. The effect of non-ideal angular velocity reading was investigated in [22] where an ultimate bound for the estimation error was presented. In this paper, we extend the approach presented in [22] to cover the non-ideal vector measurements as well. Specifically, we only consider boundedness assumption on the measurement noise and derive an ultimate bound of the attitude estimation error as a function of those bounds and the gain parameters of the observer. Then, we consider the problem of minimizing the effect of measurement noise on the steady state estimation error by proper tuning of observer

Alireza Khosravian is with the Research School of Engineering, Australian National University, ACT 2601, Australia. e-mail: alireza.khosravian@anu.edu.au

gains. A key contribution of the paper is to convert the optimization problem into a set of Linear Matrix Inequalities (LMIs) which can be employed as a systematic approach for near optimal gain tuning when large number of vector measurements are available. The presented method is of interest in practical cases since only the reference vectors and upper bounds of measurement non-ideality need to be known in order to calculate the near optimal observer gains. As a case study, the optimization problem is numerically solved in a simple case which leads to some interesting observations.

II. Definitions an Notations

We denote the trace, the smallest and the largest singular value of the matrix A by $\mathrm{tr}(A)$, $\underline{\sigma}(A)$ and $\overline{\sigma}(A)$, respectively. Also for any $a \in \mathbb{R}^n$, $\|a\|$ denotes the euclidian norm of the vector a. The identity matrix is denoted by I. The skew operator $S(.)$ maps any vector $a = [a_1\ a_2\ a_3]^\top \in \mathbb{R}^3$ to its corresponding skew-symmetric matrix by

$$S(a) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

The unskew operator is defined such that $S^{-1}(S(a)) = a$. For any $a, b \in \mathbb{R}^3$, $R \in \mathsf{SO(3)}$, $B \in \mathbb{R}^{3\times3}$ and any skew-symmetric matrix $K \in \mathbb{R}^{3\times3}$, the following properties hold:

Property 1: $RS(a)R^\top = S(Ra)$.

Property 2: $\mathrm{tr}(S(a)B) = -a^\top S^{-1}(B - B^\top)$.

Property 3: $RS^{-1}(K) = S^{-1}(RKR^\top)$.

Property 4: $S(a)S(b) = ba^\top - a^\top bI$.

Property 5: $S^{-1}(ab^\top - ba^\top) = b \times a$.

The set of special orthogonal matrices is defined by $\mathsf{SO(n)} = \{R \in \mathbb{R}^{3\times3} : R^\top R = I,\ \det(R) = 1\}$. Any element $\tilde{R}$ of $\mathsf{SO(3)}$ can be identified by an angle of rotation $\theta \in \mathbb{R}$, $0 \leq \theta \leq \pi$ and an axis of rotation $\lambda \in \mathbb{R}^3, \|\lambda\| = 1$. Given the pair (θ, λ), the corresponding rotation matrix $\tilde{R}$ can be obtained using Rodrigues' formula

$$\tilde{R} = \mathrm{rot}(\theta, \lambda) = I + \sin\theta S(\lambda) + (1 - \cos\theta)S^2(\lambda). \quad (1)$$

The distance of $\tilde{R} \in \mathsf{SO(3)}$ to the identity matrix is denoted by $\|\tilde{R} - I\|_F$ and defined by Frobenius norm as $\|\tilde{R} - I\|_F^2 = \mathrm{tr}((I - \tilde{R})^\top (I - \tilde{R})) = 2\mathrm{tr}(I - \tilde{R}) = 4(1 - \cos\theta)$ where θ is defined by (1).

III. Problem formulation

Let $R \in \mathsf{SO}(3)$ denotes the rotation from body-fixed frame of a rigid body to the inertial frame, also known as the attitude matrix. Let ω denotes the angular velocity of the body-fixed frame with respect to the inertial frame expressed in the body frame. The rigid-body kinematics is given by

$$\dot{R} = RS(\omega). \quad (2)$$

The rigid body angular velocity can be measured by a 3-axis rate gyro. We assume that the gyro measurement, $\omega_n \in \mathbb{R}^3$, is disturbed by a measurement noise such that

$$\omega_n = \omega + n_\omega, \quad (3)$$

where $n_\omega \in \mathbb{R}^3$ is the gyro measurement noise bounded by $\|n_\omega\| \leq \overline{n}_\omega$. A vector measurement $v_n \in \mathbb{R}^3$ can provide partial measurement of attitude by

$$v_n = v_b + n_v, \quad (4)$$

$$v_b = R^\top v_r, \quad (5)$$

where $n_v \in \mathbb{R}^3$ denotes the measurement noise and $v_r \in \mathbb{R}^3$ is a reference vector associated to v_b. For instance, in the case of using a 3-axis magnetometer to obtain a vector measurement, v_b and v_n are respectively the actual and measured earth's magnetic field vector at the position of rigid body and are expressed in body frame. In this case, v_r is the constant vector of the earth's magnetic field at the position of rigid body which is expressed in inertial frame. In this paper, we assume that the reference vectors associated to vector measurements are constant with respect to time and they are known a priori.

The attitude estimation problem is to design an observer which utilizes the vector measurements and angular velocity reading to estimate the attitude matrix of rigid body. Suppose that $m \geq 2$ vector measurements, $v_{b1}, v_{b2}, ..., v_{bm}$ and their related reference vectors, $v_{r1}, v_{r2}, ..., v_{rm}$ are available and each v_{bi} is related to its corresponding v_{ri} by (5). Under these conditions, the following attitude estimator on $\mathsf{SO}(3)$ was proposed in [15], [19], [23].

$$\dot{\hat{R}} = \hat{R}S(\hat{\omega}), \quad (6a)$$

$$\hat{\omega} = \omega - \gamma_\omega, \quad (6b)$$

$$\gamma_\omega = \hat{R}^\top S^{-1}(\hat{R}V_b\Lambda V_r^\top - V_r\Lambda V_b^\top \hat{R}^\top), \quad (6c)$$

where $\hat{R}$ is the estimated attitude matrix and $\Lambda \in \mathbb{R}^{m\times m}$ is a constant positive definite gain matrix. Also, $V_r \in \mathbb{R}^{3\times m}$ and $V_b \in \mathbb{R}^{3\times m}$ are respectively defined by

$$V_r = \begin{bmatrix} v_{r1} & v_{r2} & ... & v_{rm} \end{bmatrix}, \quad (7)$$

$$V_b = \begin{bmatrix} v_{b1} & v_{b2} & ... & v_{bm} \end{bmatrix}. \quad (8)$$

To investigate the convergence of observer, the following attitude estimation error matrix is considered.

$$\tilde{R} = \hat{R}R^\top. \quad (9)$$

It is obvious that $\hat{R} = R$ if and only if $\tilde{R} = I$. It is known that $\mathsf{SO(3)}$ is not homeomorphic to any Euclidean vector space. This implies that the region of attraction of any equilibrium point dose not cover whole $\mathsf{SO(3)}$ space [1], [3], [12]. Due to this inherent topological limitation, the notion of almost global stability [1], [3] is used to summarize the properties of observer (6).

Proposition 1: ([15], [19], [23]) Consider the observer (6) for the attitude kinematics (2). Suppose that two or more vector measurements are available and at least two reference vectors are non-collinear. The error $\tilde{R}$ is almost globally asymptotic and locally exponentially stable around the identity matrix. That is, for any $\tilde{R}(t_0) \in R_A$ we have

$$\|\tilde{R}(t) - I\|_F \leq \|\tilde{R}(t_0) - I\|_F\, e^{-\alpha_R(t-t_0)}, \quad (10)$$

where $\alpha_R = 0.5(1+\cos\theta(t_0))\underline{\sigma}(P)$, $P = \mathrm{tr}(\Lambda_r)I - \Lambda_r$, $\Lambda_r = V_r\Lambda V_r^\top$ and the region of attraction is given by

$$R_A = \mathsf{SO}(3)\backslash\{\tilde{R} \in \mathsf{SO}(3): \ \mathrm{tr}(I-\tilde{R}) = 4\} \\ = \{\tilde{R} = \mathrm{rot}(\theta,\lambda) : \lambda \in \mathsf{S}^2, \theta \neq \pi\},$$

where θ and λ are defined by (1). □

In practical situations, the output of sensors is polluted by measurement noise. Hence, in this paper, we are interested in investigating practical stability of observer (6) in presence of measurement noise. Another interesting question which is tackled here is to tune the constant observer gains according to the sensor properties such that it optimizes the steady state behavior of the attitude estimator.

Remark 1: In practical situations, the outputs of gyro or attitude sensors may be corrupted by unknown biases. It is common to assume that these biases are constant in order to adaptively compensate those biases in the attitude estimation algorithm. For instance, [15], [19], [23] couple a suitable bias estimator to the observer (6) such that it adaptively estimates a constant gyro bias. Also, [8] proposes a method for adaptive commensuration of effects of constant gyro bias as well as bias of vector measurements. Here, we only consider the steady state behavior of attitude estimator. Hence, we focus on the basic observer (6) and we do not consider the coupled adaptation algorithms proposed in the above mentioned references. □

IV. PRACTICAL STABILITY OF ATTITUDE ESTIMATOR

Stability of the attitude observer (6) in presence of bounded rate gyro noise has been investigated in [22]. We consider vector measurement noise in addition to the angular velocity noise and extend the results of [22] to obtain sufficient conditions for practical stability of the observer.

Denote $v_{ni}, \ i = 1,\dots,m$ as the measurement of $v_{bi}, \ i = 1,\dots,m$ and assume each v_{ni} is disturbed by a measurement noise $n_i \in \mathbb{R}^3$ according to (4) and the measurement noise is bounded by $\|n_i\| < \overline{n}_i$. Define $V_n = [v_{n1} \ v_{n2} \ \dots \ v_{nm}] \in \mathbb{R}^{3\times m}$ which represents the measured outputs of system. Assume that the angular velocity measurement is disturbed by measurement noise according to (3). Under the above mentioned assumptions, the estimator still takes the similar form as (6a), but we should replace V_b and ω by V_n and ω_n respectively. So, the estimator is formulated as

$$\dot{\hat{R}} = \hat{R}S(\hat{\omega}), \tag{11a}$$

$$\hat{\omega} = \omega_n - \gamma_n, \tag{11b}$$

$$\gamma_n = \hat{R}^\top S^{-1}(\hat{R}V_n\Lambda V_r^\top - V_r\Lambda V_n^\top \hat{R}^\top). \tag{11c}$$

Here we suppose a special case that Λ is diagonal, i.e. $\Lambda = \mathrm{diag}(k_1, \ k_2, \ \dots, \ k_m)$ where $k_i, \ i = 1,\dots,m$ are positive scalars. Under this condition, the following theorem generalizes practical stability of the observer discussed in [22, Theorem 10].

Theorem 1: Suppose that the upper bounds of angular velocity noise and vector measurement noises satisfy

$$\frac{1}{\underline{\sigma}(P)}(\sum_{i=1}^{m} k_i\|v_{ri}\|\overline{n}_i + \overline{n}_\omega) < 1, \tag{12}$$

where $P = \mathrm{tr}(\Lambda_r)I - \Lambda_r$ and $\Lambda_r = V_r\Lambda V_r^\top$. Define

$$\sin\theta_{\text{min}} := \frac{1}{\underline{\sigma}(P)}(\sum_{i=1}^{m} k_i\|v_{ri}\|\overline{n}_i + \overline{n}_\omega) \tag{13}$$

and $\theta_{\text{max}} := \pi - \theta_{\text{min}}$. For any initial condition satisfying

$$\|I - \tilde{R}(t_0)\|_F^2 < 4\frac{\underline{\sigma}(P)}{\overline{\sigma}(P)}(1-\cos\theta_{\text{max}}), \tag{14}$$

there exists a finite $T \in \mathbb{R}$ such that

$$\|I - \tilde{R}(t)\|_F^2 < 4\frac{\overline{\sigma}(P)}{\underline{\sigma}(P)}(1-\cos\theta_{\text{min}}) \tag{15}$$

for all $t \geq T + t_0$. □

Proof of Theorem 1 is given in the appendix.

V. NEAR OPTIMAL TUNING OF OBSERVER GAINS

By Theorem 1, we presented an upper bound for the steady state estimation error in the presence of measurement noise. This section is devoted to present a method to suitably tune the observer gain, Λ, such that minimizes the upper bound of estimation error. This suitable observer gain is called near optimal gain. First, in section V-A, we use a simple trick to convert the minimization procedure to a simpler optimization problem. Next, in section V-B, we reformulate the simplified optimization problem as a convex optimization problem which can be solved using a set of Linear Matrix Inequalities (LMIs). Then, we preset some interesting results of the optimization procedure in section VI.

A. optimization problem

Multiplying the sides of (14) by (15) and considering $\cos\theta_{\text{max}} = -\cos\theta_{\text{min}}$ yields

$$\|I - \tilde{R}(t_0)\|_F^2\|I - \tilde{R}(t)\|_F^2 < 16\sin^2\theta_{\text{min}}. \tag{16}$$

Moving $\sin\theta_{\text{min}}$ from (13) to (16) implies

$$\|I - \tilde{R}(t)\|_F < \frac{4}{\underline{\sigma}(P)\|I-\tilde{R}(t_0)\|_F}(\sum_{i=1}^{m} k_i\|v_{ri}\|\overline{n}_i + \overline{n}_\omega). \tag{17}$$

Since $v_{r1}, \ v_{r2}, \ \overline{n}_1, \ \overline{n}_2$ and $\overline{n}_\omega$ are known and also $\|I - \tilde{R}(t_0)\|_F$ is constant, our optimization problem is simplified to

$$\min_{k_i>0, \ i=1, \ \dots, \ m} \left\{\frac{1}{\underline{\sigma}(P)}(\sum_{i=1}^{m} k_i\|v_{ri}\|\overline{n}_i + \overline{n}_\omega)\right\}, \tag{18}$$

which can be solved by commercial software packages designed to resolve nonlinear optimization problems, such as `fmincon` function of MATLAB®.

B. tuning the observer gains using LMIs

Although minimization problem (18) can be solved using `fmincon` function, converting it into a set of Linear Matrix Inequalities (LMIs) is very useful since there are very efficient tools and software packages to numerically resolve LMIs as a kind of convex optimization problems. This is specifically important when a large number of vector measurements is available or when the reference vectors slowly vary with time and optimal gains need to be recalculated from time to time. It is worth mentioning that the dimension of the search space of optimization problem (18) is equal to the number of available vector measurements, m. Hence, as m increases, the complexity of (18) increases proportionally.

Assume that there exists a positive scalar, σ, such that

$$\sigma I < P. \tag{19}$$

Obviously, this inequality is held by choosing $\sigma = \underline{\sigma}(P)$, but we intentionally consider a more general case in order to be able to convert our problem into LMI. Using inequality (19) and following the proof of Theorem 1, the minimization (18) can be rewritten as

$$\min_{k_i>0,\ i=1,\ \dots,\ m} \left\{ \frac{1}{\sigma}\Big(\sum_{i=1}^{m} k_i \|v_{ri}\| \overline{n}_i + \overline{n}_\omega\Big) \right\}. \tag{20}$$

Now, suppose there exists a positive scalar ξ such that

$$\frac{1}{\sigma}\Big(\sum_{i=1}^{m} k_i \|v_{ri}\| \overline{n}_i + \overline{n}_\omega\Big) < \xi. \tag{21}$$

The problem (20) is equivalent to minimizing ξ in (21). Define the new variables $k_{\sigma i} := \frac{k_i}{\sigma}$ $(i = 1,\ \dots, m)$ and $\sigma_{\text{inv}} = \sigma^{-1}$. We have

$$\sum_{i=1}^{m} k_{\sigma i} \|v_{ri}\| \overline{n}_i + \sigma_{\text{inv}} \overline{n}_\omega < \xi \tag{22}$$

On the other hand, recalling $P = \operatorname{tr}(V_r \operatorname{diag}(k_1, k_2, \dots, k_m) V_r^\top) I - V_r \operatorname{diag}(k_1, k_2, \dots, k_m) V_r^\top$, the inequality (19) can also be rewritten in the following form with respect to the new variables $k_{\sigma i},\ (i = 1,\ \dots, m)$

$$\operatorname{tr}(V_r \operatorname{diag}(k_{\sigma 1}, \dots, k_{\sigma m}) V_r^\top) I - V_r \operatorname{diag}(k_{\sigma 1}, \dots, k_{\sigma m}) V_r^\top > I. \tag{23}$$

The inequalities (22) and (23) are standard LMIs with respect to the variables $k_{\sigma i}$ $(i = 1,\ \dots, m)$, σ_{inv}, and ξ. Consequently, the minimization problem (18) can be reformulated as the following set of LMIs.

$$\min_{k_{\sigma i}\ (i=1,\dots,m),\ \sigma_{\text{inv}},\ \xi} (\xi):$$
$$\begin{cases} k_{\sigma i} > 0\ (i = 1,\ \dots, m), \\ \sigma_{\text{inv}} > 0, \\ \xi > 0, \\ \operatorname{tr}(V_r \operatorname{diag}(k_{\sigma 1}, \dots, k_{\sigma m}) V_r^\top) I - V_r \operatorname{diag}(k_{\sigma 1}, \dots, k_{\sigma m}) V_r^\top > I, \\ \sum_{i=1}^{m} k_{\sigma i} \|v_{ri}\| \overline{n}_i + \sigma_{\text{inv}} \overline{n}_\omega < \xi. \end{cases}$$

VI. A NUMERICAL EXAMPLE AND SOME INTERESTING OBSERVATIONS

In this section, we assume no gyro noise (i.e. $\overline{n}_\omega = 0$) and we try to numerically calculate the near optimal gains for a range of vector measurement noise bounds in order to discuss some interesting results of the optimization procedure. Let us assume that two vector measurements are available (i.e. $m = 2$) and define $n_{r1} := \frac{\overline{n_1}}{\|v_{r1}\|}$, $n_{r2} := \frac{\overline{n_2}}{\|v_{r2}\|}$, $k_{r1} := k_1 \|v_{r1}\|^2$, $k_{r2} := k_2 \|v_{r2}\|^2$, $K_{\text{rat}} := \frac{k_{r1}}{k_{r2}}$ and $N_{\text{rat}} := \frac{n_{r1}}{n_{r2}}$. The physical interpretation of n_{r1} is the normalized measurement noise in the first vector measurement , i.e. n_{r1} represents the percentage of measurement noise which exists in v_{n1}. Similarly, n_{r2} is the normalized measurement noise in the second vector measurement. Also, k_{r1} and k_{r2} represent the normalized gains for the vector measurements in which the amplitude of corresponding reference vectors are also observed. Actually, in order to truly compare the gains associated with the first and second vector measurements, we should compare k_{r1} and k_{r2}. Hence, K_{rat} is used in comparison between the observer gains related to first and second vectors. Using the above definitions, one can show that $\underline{\sigma}(P) = \frac{1}{2} k_{r2}(K_{\text{rat}} + 1 - \sqrt{K_{\text{rat}}^2 - 2\eta_{12} K_{\text{rat}} + 1})$ where

$$\eta_{12} = 1 - 2\Big(\frac{v_{r1}^\top v_{r2}}{\|v_{r1}\| \|v_{r2}\|}\Big)^2.$$

Note that η_{12} depends only on the difference angle between the directions of v_{r1} and v_{r2} and it is bounded by $-1 \leq \eta_{12} \leq 1$. For example $\eta_{12} = 1$ implies that v_{r1} and v_{r2} are perpendicular, whereas $\eta_{12} = -1$ implies that v_{r1} and v_{r2} are either the same or in opposite directions. Using the mentioned notations, we can simplify the optimization problem (18) to

$$\min_{K_{\text{rat}}>0} \left\{ \frac{K_{\text{rat}} + N_{\text{rat}}^{-1}}{K_{\text{rat}} + 1 - \sqrt{K_{\text{rat}}^2 - 2\eta_{12} K_{\text{rat}} + 1}} \right\}. \tag{24}$$

Note that in this case the 2-dimensional optimization problem (18) is reduced to the 1-dimensional optimization problem (24). Hence, only the gain ratio, K_{rat}, affects the optimization procedure which means that k_1 and k_2 have no importance by themselves but their ratio determines the optimality. After computing K_{rat} from (24), we should choose $k_2 = \frac{k_1 \|v_{r1}\|^2}{K_{\text{rat}} \|v_{r2}\|^2}$ to minimize the effect of measurement noise. We can use the extra degree of freedom to chose a suitable value for k_1 in order to achieve a desired convergence rate for the observer.

Using the same ratiocinations as it was presented in section V-B, one can convert (24) into the following set of LMIs.

$$\min_{k_{\sigma 1}, k_{\sigma 2}, \xi} (\xi):$$
$$\begin{cases} k_{\sigma 1} > 0, \\ k_{\sigma 2} > 0, \\ \xi > 0, \\ \operatorname{tr}(V_r \operatorname{diag}(k_{\sigma 1}, k_{\sigma 2}) V_r^\top) I - V_r \operatorname{diag}(k_{\sigma 1}, k_{\sigma 2}) V_r^\top > I, \\ k_{\sigma 1} \|v_{r1}\| \overline{n}_1 + k_{\sigma 2} \|v_{r2}\| \overline{n}_2 < \xi. \end{cases}$$

Hence, the optimization problem can be formulated without using the variable σ_{inv}. This recalls the fact that we have also reduced the dimension of the optimization problem in (24). It is worth mentioning that although σ_{inv} is omitted in above LMIs, we can yet calculate K_{rat} using $K_{\text{rat}} := \frac{k_{\sigma 1}}{k_{\sigma 2}}$. Once again, the optimization procedure determines the ratio of k_1 to k_2 but the amount of k_1 and k_2 does not affect the optimization by themselves.

The optimization problem (24) is solved for $-1 < \eta_{12} < 1$ and $0.01 < N_{\text{rat}} < 1$ and the optimal solution K_{rat}^* is illustrated in Fig. 1. The following observations are immediately recognized from Fig. 1:

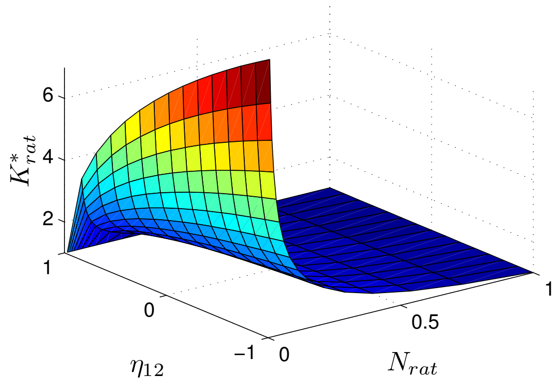

Fig. 1. Near optimal gain ratio (K^*_{rat}) versus various amount of noise ratio (N_{rat}) and direction angle between reference vectors (η_{12})

1) For a fixed η_{12}, K_{rat} decreases when N_{rat} increases. That is to say, the larger is the amount of noise in v_{n1} with respect to v_{n2}, the less gain should be chosen for the first vector measurement with respect to the second measurement.

2) Independent of η_{12}, we have $K^*_{\text{rat}} \to 1$ as $N_{\text{rat}} \to 1$. That is, in order to achieve minimum estimation error when there are equal normalized measurement noises in the vectors, we should choose the same normalized gains for both vectors independent of the direction of reference vectors.

3) Independent of N_{rat}, we have $K^*_{\text{rat}} \to 1$ as $\eta_{12} \to 1$. That is, in order to achieve minimum estimation error when reference vectors are perpendicular, we should choose the same normalized gains for both vectors independent of the amount of measurement noises. This is an interesting result, since perpendicular vectors offer measurements in completely independent directions and hence their gain should be chosen equally, independent of their measurement noises.

4) $K^*_{\text{rat}} \to +\infty$ as $\eta_{12} \to -1$ and $N_{\text{rat}} \to 0$. It means that when there is much smaller noise in the first vector measurement comparing to the second vector, and the difference between the direction of reference vectors are small, then we should choose a much higher gain for the first vector measurement than the second one.

VII. Conclusion

This paper investigates the effect of measurement noise on the performance of a nonlinear attitude estimator. Supposing both non-ideal vector measurements and non-ideal gyro output are available, an ultimate bound of the attitude estimation error is derived as a function of observer gains, reference vectors, and upper bounds of measurement noises. Moreover, the problem of tuning the observer gains to minimize the upper bound of steady state estimation error is formulated as a convex optimization problem which can be solved using a set of LMIs. Some interesting results of the optimization procedure is discussed with an illustrative numerical example. Although only the noise in the body frame is considered in this paper, the results presented here can be extended without significant modifications to support bounded noise/disturbnce in the inertial frame as well. As a future work, we will focus on obtaining a tighter upper bound for the estimation error and optimizing the steady state behavior of the observer accordingly. Illustrating the efficiency of the optimization procedure in practical scenarios is considered as another future work. Additionally, investigating the practical stability of the attitude observer when it is coupled with a bias estimators to compensate for gyro bias can be addressed in future.

Appendix

proof of Theorem 1

The proof is an extension of the proof of [22, Theorem 10] and is based on using the derivation of boundedness properties for nonlinear systems [9, Theorem 4.18]. Differentiating (9) with respect to time yields the error dynamics $\dot{\tilde{R}} = \dot{\hat{R}} R^\top + \hat{R}\dot{R}^\top = \hat{R}S(\hat{\omega})R^\top - \hat{R}S(\omega)R^\top = \hat{R}S(\hat{\omega}-\omega)R^\top$. Using Property 1, the attitude error dynamics is obtained as

$$\dot{\tilde{R}} = \tilde{R}S(R(\hat{\omega}-\omega)). \tag{25}$$

Consider the Lyapunov candidate $W = \text{tr}((I-\tilde{R})\Lambda_r)$ where $\Lambda_r = V_r\Lambda V_r^\top$. Since Λ is positive definite and rank$(V_r) \geq 2$, it can be verified that W is a positive definite function [7]. Differentiating W with respect to time, we have $\dot{W} = -\text{tr}(\dot{\tilde{R}}\Lambda_r)$. Using (25) yields $\dot{W} = -\text{tr}(\tilde{R}S(R(\hat{\omega}-\omega))\Lambda_r)$. Next we try to rewrite $\dot{W}$ in terms of γ_ω. To this end, we use Property 1 to obtain $\dot{W} = -\text{tr}(S(\hat{R}(\hat{\omega}-\omega))\tilde{R}\Lambda_r)$. Using Property 2 and noting $V_b = R^\top V_r$ we have

$$\begin{aligned}\dot{W} &= S^{-\top}(\tilde{R}\Lambda_r - \Lambda_r\tilde{R}^\top)\hat{R}(\hat{\omega}-\omega),\\ &= S^{-\top}(\hat{R}R^\top V_r\Lambda V_r^\top - V_r\Lambda V_r^\top R\hat{R}^\top)\hat{R}(\hat{\omega}-\omega),\\ &= S^{-\top}(\hat{R}V_b\Lambda V_r^\top - V_r\Lambda V_b^\top\hat{R}^\top)\hat{R}(\hat{\omega}-\omega).\end{aligned}$$

Hence, by (6c), $\dot{W}$ is simplified to $\dot{W} = \gamma_\omega^\top(\hat{\omega}-\omega)$. Replacing for $\hat{\omega}$ from (11b), we yield

$$\dot{W} = \gamma_\omega^\top(n_\omega - \gamma_n). \tag{26}$$

On the other hand, invoking the algebraic manipulations derived in Appendix A, we have $\gamma_n = \gamma_\omega + \gamma_d$ where

$$\gamma_d = -\sum_{i=1}^{m} k_i(\hat{R}^\top v_{ri}) \times n_i. \tag{27}$$

Replacing γ_n in (26), we obtain

$$\dot{W} = -\gamma_\omega^\top\gamma_\omega + \gamma_\omega^\top\gamma_d + \gamma_\omega^\top n_\omega \leq -\|\gamma_\omega\|\big(\|\gamma_\omega\| - \|\gamma_d\| - \|n_\omega\|\big). \tag{28}$$

Using (27), and noting $\|\hat{R}^\top v_{ri}\| = \|v_{ri}\|$, we have

$$\|\gamma_d\| \leq \sum_{i=1}^{m} k_i\|v_{ri}\|\overline{n}_i. \tag{29}$$

Also, considering the angle-axis representation $\tilde{R} = \text{rot}(\theta,\lambda)$ and using the same ratiocinations adopted in [11, Lemma 2], it can be shown that $\gamma_\omega = R^\top Q^\top P\lambda$ where $Q = \sin\theta I + (1-\cos\theta)S(\lambda)$. Recalling $0 \leq \theta \leq \pi$ and $\|\lambda\| = 1$, we have

$$\|\gamma_\omega\| = \|R^\top Q^\top P\lambda\| \geq \underline{\sigma}(Q)\underline{\sigma}(P) \geq \underline{\sigma}(P)\sin\theta. \tag{30}$$

Employing (29) and (30), we can simplify (28) to

$$\dot{W} \leq -\|\gamma_\omega\|(\|\gamma_d\| + \|n_\omega\|)\big(\frac{\sin\theta}{\sin\theta_{\min}} - 1\big),$$

where $\sin\theta_{\min}$ is given by (13). Finally, considering condition (12), we obtain $\dot{W} < 0$ for all $\theta \in [\theta_{\min}, \theta_{\max}]$. Similar to [22], we define $r_{\min} := \|I - \tilde{R}_{\min}\|_F^2$, $\tilde{R}_{\min} = \mathrm{rot}(\theta_{\min}, \lambda)$ and $r_{\max} := \|I - \tilde{R}_{\max}\|_F^2$, $\tilde{R}_{\max} = \mathrm{rot}(\theta_{\max}, \lambda)$. Since $\|I - \tilde{R}\|_F^2 = 4(1 - \cos\theta)$ is monotonically increasing on $\theta \in (0, \pi)$, we conclude that all $\|I - \tilde{R}\|_F^2 \in [r_{\min}, r_{\max}]$ imply $\dot{W} < 0$, which characterizes a compact set in $\mathsf{SO(3)}$ where the Lyapunov function is decreasing.

On the other hand, we can rewrite W as $W = \frac{1}{4}\|I - \tilde{R}\|_F^2 \lambda^\top P \lambda$ [7], [11], [22], [23]. Hence, the Lyapunov function satisfies $\alpha_1(\|I - \tilde{R}\|_F^2) \leq W \leq \alpha_2(\|I - \tilde{R}\|_F^2)$ where $\alpha_1(x) = \frac{1}{4}\underline{\sigma}(P)x$, $\alpha_2(x) = \frac{1}{4}\overline{\sigma}(P)x$. Now, defining the sets $\Omega_{t,r_{\min}} = \{\tilde{R} \in \mathsf{SO(3)} : W \leq \alpha_2(r_{\min})\}$, $\Omega_{t,r_{\max}} = \{\tilde{R} \in \mathsf{SO(3)} : W \leq \alpha_1(r_{\max})\}$, and $C_{t,r_{\min},r_{\max}} = \Omega_{t,r_{\max}} \backslash \Omega_{t,r_{\min}}$; any point $\tilde{R} \in C_{t,r_{\min},r_{\max}}$ satisfies $\|I - \tilde{R}\|_F^2 \in [r_{\min}, r_{\max}]$, as shown by using

$$\begin{aligned} W \leq \alpha_1(r_{\max}) &\Rightarrow \alpha_1(\|I - \tilde{R}\|_F^2) \leq \alpha_1(r_{\max}) \\ &\Rightarrow \|I - \tilde{R}\|_F^2 \leq r_{\max}. \\ W \geq \alpha_2(r_{\min}) &\Rightarrow \alpha_2(\|I - \tilde{R}\|_F^2) \geq \alpha_2(r_{\min}) \\ &\Rightarrow \|I - \tilde{R}\|_F^2 \geq r_{\min}. \end{aligned}$$

So, we conclude that for any $\tilde{R} \in C_{t,r_{\min},r_{\max}}$ we have $\dot{W} < 0$. Consequently, any solution starting in $C_{t,r_{\min},r_{\max}}$ reaches $\Omega_{t,r_{\min}}$ in a finite time, $T + t_0$, and it remains in $\Omega_{t,r_{\min}}$ for all $t > T + t_0$ since $\dot{W} < 0$ in the corresponding boundary. The condition (14) yields $\tilde{R}(t_0) \in \Omega_{t,r_{\max}}$, and any $\tilde{R}(t) \in \Omega_{t,r_{\min}}$ yields (15) which completes the proof of Theorem 1. ■

Appendix A: ([22], [23]) By virtue of $V_n \Lambda V_r^\top = \sum_{i=1}^m k_i v_{ni} v_{ri}^\top$, we can rewrite γ_n as

$$\begin{aligned} \gamma_n &= S^{-1}(V_n \Lambda V_r^\top \hat{R} - \hat{R}^\top V_r \Lambda V_n^\top) \\ &= \sum_{i=1}^m k_i S^{-1}(v_{ni} v_{ri}^\top \hat{R} - \hat{R}^\top v_{ri} v_{ni}^\top). \end{aligned}$$

In light of Property 5, we obtain $\gamma_n = \sum_{i=1}^m k_i(\hat{R}^\top v_{ri}) \times v_{ni}$. Similarly, one can rewrites γ_ω as $\gamma_\omega = \sum_{i=1}^m k_i(\hat{R}^\top v_{ri}) \times v_{bi}$. Noting that $v_{ni} = v_{bi} + n_i$, we have $\gamma_n = \gamma_\omega + \gamma_d$ as given by (27). ■

ACKNOWLEDGMENT

Some parts of this work was done when the author was with the Electrical Engineering Department of Sharif University of Technology. He is currently with the Research School of Engineering of Australian National University where he has finalized this work. The author would like to thank Dr. Mehrzad Namvar, Dr. Jochen Trumpf, and Prof. Robert Mahony for their valuable feedback during preparation of this paper.

REFERENCES

[1] D. Angeli, “An almost global notion of input-to-state stability,” *IEEE Trans. Autom. Control*, vol. 49, no. 6, pp. 866–874, 2004.

[2] P. Batista, C. Silvestre, and P. Oliveira, “A GES attitude observer with single vector observations,” *Automatica*, vol. 48, pp. 388–395, 2012.

[3] S. P. Bhat and D. S. Bernstein, “A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon,” *Systems and Control Letters*, vol. 39, no. 1, pp. 63–70, January 2000.

[4] S. Bonnabel, P. Martin, and P. Rouchon, “A non-linear symmetry-preserving observer for velocity-aided inertial navigation,” in *American Control Conference, 2006*. IEEE, 2006, pp. 5–pp.

[5] J. L. Crassidis and F. L. Markley, “Unscented filtering for spacecraft attitude estimation,” *Journal of Guidance, Control, and Dynamics*, vol. 26, 2003.

[6] J. L. Crassidis, F. L. Markley, and Y. Cheng, “Survey of nonlinear attitude estimation methods,” *Journal of Guidance, Control, and Dynamics*, vol. 30, pp. 12–28, 2007.

[7] R. Cunha, C. Silvestre, and J. Hespanha, “Output-feedback control for stabilization on SE(3),” *Systems and Control Letters*, vol. 57, no. 12, pp. 1013–1022, December 2008.

[8] H. F. Grip, T. I. Fossen, T. A. Johansen, and A. Saberi, “Attitude estimation using biased gyro and vector measurements with time-varying reference vectors,” *IEEE Trans. Autom. Control*, vol. 57, no. 5, pp. 1332–1338, 2012.

[9] H. K. Khalil, *Nonlinear Systems*. 3rd edition, Prentice Hall.

[10] A. Khosravian and M. Namvar, “Globally exponential estimation of satellite attitude using a single vector measurement and gyro,” in *Proc. 49th IEEE Conf. on Decision and Control*, USA, December 2010.

[11] ——, “Rigid body attitude control using a single vector measurement and gyro,” *IEEE Trans. Autom. Control*, vol. 57, no. 5, pp. 1273–1279, 2012.

[12] D. E. Koditschek, “The application of total energy as a lyapunov function for mechanical control systems,” *Control Theory and Multibody Systems*, vol. 97, p. 131151, 1989.

[13] E. Lefferts, F. Markley, and M. Shuster, “Kalman filtering for spacecraft attitude estimation,” *Journal of Guidance, Control, and Dynamics*, vol. 5, no. 5, pp. 417–429, 1982.

[14] R. Mahony, T. Hamel, and J. M. Pflimlin, “Complementary filter design on the special orthogonal group SO(3),” in *Proc. IEEE Conf. on Decision and Control, and the European Control Conf.*, Spain, December 2005.

[15] ——, “Nonlinear complementary filters on the special orthogonal group,” *IEEE Trans. Autom. Control*, vol. 53, no. 5, pp. 1203–1218, June 2008.

[16] R. Mahony, T. Hamel, J. Trumpf, and C. Lageman, “Nonlinear attitude observers on SO(3) for complementary and compatible measurements: A theoretical study,” in *Proc. joint 48th IEEE Conf. on Decision and Control and 28th Chinese Control Conf.*, China, December 2009.

[17] M. Malisoff, M. Krichman, and E. Sontag, “Global stabilization for systems evolving on manifolds,” *Journal of Dynamical and Control Systems*, vol. 12, no. 2, p. 161184, 2006.

[18] J. M. Pflimlin, T. Hamel, and P. Soueres, “Nonlinear attitude and gyroscope’s bias estimation for VTOL UAV,” *International Journal of Systems Science*, vol. 38, 2007.

[19] H. Rehbinder and B. Ghosh, “Pose estimation using line-based dynamic vision and inertial sensors,” *IEEE Trans. Autom. Control*, vol. 48, no. 2, pp. 186–199, 2003.

[20] D. Seo and M. R. Akella, “Separation property for the rigid-body attitude tracking control problem,” *Journal of Guidance, Control, and Dynamics*, vol. 30, no. 6, pp. 1569–1576, November-December 2007.

[21] J. Trumpf, R. Mahony, T. Hamel, and C. Lageman, “Analysis of non-linear attitude observers for time-varying reference measurements,” *IEEE Trans. Autom. Control*, vol. 57, no. 11, pp. 2789–2800, 2012.

[22] J. F. Vasconcelos, R. Cunha, C. Silvestre, and P. Oliveira, “Stability of a nonlinear attitude observer on SO(3) with nonideal angular velocity measurements,” in *Proc. European Control Conference 2009*, Hungary, August 2009.

[23] J. Vasconcelos, C. Silvestre, and P. Oliveira, “A nonlinear observer for rigid body attitude estimation using vector observations,” in *Proc. 17th IFAC World Congress*, Korea, July 2008.