

# The boundedness of the Riesz transform on a metric cone

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# Declaration

The work in this thesis is my own except where otherwise stated.

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# Abstract

In this thesis we study the boundedness, on  $L^p(M)$ , of the Riesz transform  $T$  associated to a Schrödinger operator with an inverse square potential  $V = \frac{V_0}{r^2}$  on a metric cone  $M$  defined by

$$T = \nabla \left( \Delta + \frac{V_0(y)}{r^2} \right)^{-\frac{1}{2}}.$$

Here  $M = Y \times [0, \infty)_r$  has dimension  $d \geq 3$ , and the smooth function  $V_0$  on  $Y$  is restricted to satisfy the condition  $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2 > 0$ , where  $\Delta_Y$  is the Laplacian on the compact Riemannian manifold  $Y$ .

The definition of  $T$  involves the Laplacian  $\Delta$  on the cone  $M$ . However, the cone is not a manifold at the cone tip, so we initially define the Laplacian away from the cone tip, and then consider its self-adjoint extensions. The Friedrichs extension is adopted as the definition of the Laplacian.

Using functional calculus,  $T$  can be written as an integral involving the expression  $(\Delta + \frac{V_0(y)}{r^2} + \lambda^2)^{-1}$ . Therefore if we understand the resolvent kernel of the Schrödinger operator  $\Delta + \frac{V_0(y)}{r^2}$ , we have information about  $T$ . We construct and at the same time collect information about this resolvent kernel, and then use the information to study the boundedness of  $T$ .

The two most interesting parts in the construction of the resolvent kernel are the behaviours of the kernel as  $r, r' \rightarrow 0$  and  $r, r' \rightarrow \infty$ . To study them, a process called the blow-up is performed on the domain of the kernel. We use the  $b$ -calculus to study the kernel as  $r, r' \rightarrow 0$ , while the scattering calculus is used as  $r, r' \rightarrow \infty$ .

The main result of this thesis provides a necessary and sufficient condition on  $p$  for the boundedness of  $T$  on  $L^p(M)$ . The interval of boundedness depends on  $V_0$  through the first and second eigenvalues of  $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2$ .

- When the potential function  $V$  is positive, we have shown that the lower

threshold is 1, and the upper threshold is strictly greater than the dimension  $d$ .

- When the potential function  $V$  is negative, we have shown that the lower threshold is strictly greater than 1, and the upper threshold is strictly between 2 and  $d$ .
- Our results for  $p \leq 2$  are contained in the work of J. Assaad, but we use different methods in this thesis. Our boundedness results for  $p \geq \frac{d}{2}$  for positive inverse square potentials, and for  $p > 2$  for negative inverse square potentials, are new.



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