The boundedness of the Riesz transform on a metric cone

Peijie Lin

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Declaration

The work in this thesis is my own except where otherwise stated.

Peijie Lin

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Abstract

In this thesis we study the boundedness, on $L^p(M)$, of the Riesz transform T associated to a Schrödinger operator with an inverse square potential $V = \frac{V_0}{r^2}$ on a metric cone M defined by

$$T = \nabla\left(\Delta + \frac{V_0(y)}{r^2}\right)^{-\frac{1}{2}}.$$

Here $M = Y \times [0, \infty)_r$ has dimension $d \geq 3$, and the smooth function V_0 on Y is restricted to satisfy the condition $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2 > 0$, where Δ_Y is the Laplacian on the compact Riemannian manifold Y.

The definition of T involves the Laplacian Δ on the cone M. However, the cone is not a manifold at the cone tip, so we initially define the Laplacian away from the cone tip, and then consider its self-adjoint extensions. The Friedrichs extension is adopted as the definition of the Laplacian.

Using functional calculus, T can be written as an integral involving the expression $(\Delta + \frac{V_0(y)}{r^2} + \lambda^2)^{-1}$. Therefore if we understand the resolvent kernel of the Schrödinger operator $\Delta + \frac{V_0(y)}{r^2}$, we have information about T. We construct and at the same time collect information about this resolvent kernel, and then use the information to study the boundedness of T.

The two most interesting parts in the construction of the resolvent kernel are the behaviours of the kernel as $r, r' \to 0$ and $r, r' \to \infty$. To study them, a process called the blow-up is performed on the domain of the kernel. We use the b-calculus to study the kernel as $r, r' \to 0$, while the scattering calculus is used as $r, r' \to \infty$.

The main result of this thesis provides a necessary and sufficient condition on p for the boundedness of T on $L^p(M)$. The interval of boundedness depends on V_0 through the first and second eigenvalues of $\Delta_Y + V_0(y) + (\frac{d-2}{2})^2$.

- When the potential function V is positive, we have shown that the lower

threshold is 1, and the upper threshold is strictly greater than the dimension d.

- When the potential function V is negative, we have shown that the lower threshold is strictly greater than 1, and the upper threshold is strictly between 2 and d.

- Our results for $p \leq 2$ are contained in the work of J. Assaad, but we use different methods in this thesis. Our boundedness results for $p \geq \frac{d}{2}$ for positive inverse square potentials, and for $p > 2$ for negative inverse square potentials, are new.

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