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Systems of variational inclusion problems and differential inclusion problems with applications

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Abstract In this paper, we study the existence theorems of systems of variational inclusion problems. From these existence results, we study the existence theorems of systems of variational differential inclusion problems, mathematical program with systems of variational inclusion constraints, and mathematical program with systems of equilibrium constraints.

Keywords Systems of variational inclusion problem · Systems of variational differential inclusion problems · Systems of equilibrium problems · Mathematical program with systems of variational differential inclusion constraints · Mathematical program with systems of equilibrium constraints

1 Introduction

Let U and V be two Banach spaces and let $T := [a, b]$ be a time interval of the real line. Let $H : T \times U \times V \multimap U$, $F : T \times U \multimap U$, and $S : U \times T \multimap V$ be multivalued maps. The control problem is the differential equations with control parameters

$$\dot{x} = f(t, x, u), \quad u \in S(x, t) \tag{1}$$

where $\dot{x}$ denote the time derivative of $x(t)$. The differential inclusion problem is the problem of finding x such that

$$\dot{x} \in F(t, x), \quad t \in T. \tag{2}$$

It has been well recognized that differential inclusions, which are certainly of their own interest provide a useful generalization control systems generated by (1) via $F(t, x) = f(t, x, S(x, t))$. In some cases, especially when the set $F(t, x)$ is a convex set for each $(t, x) \in T \times X$, the differential inclusions (2) admits parametric representations of type (1),

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but in general they can not be reduced to parametric control system and should be studied for their own sake. There are many results in the literatures studied differential inclusion problems. One can refer to [19,21] and the references therein for the differential inclusion problems.

Let E be a topological vector space (in short t.v.s.), X be a nonempty subset of E, and $f : X \times X \multimap E$ be a function with $f(x, x) \geq 0$ for all $x \in X$, then the scalar equilibrium problem is to find $\bar{x} \in X$ such that $f(\bar{x}, y) \geq 0$ for all $y \in X$. The equilibrium problem contains optimization problems, variational inequalities problem, the Nash equilibrium problem, fixed point problems, complementary problems, and Ekeland's variational principle as special case (see [2,8,12]). This problem was extensively investigated and generated to the vector equilibrium for single valued or multivalued maps [8–15] and references therein.

Let I be an index set. For each $i \in I$, let Z_i be a real t.v.s., X_i and Y_i be nonempty closed convex subsets of locally convex spaces E_i and W_i, respectively. Let $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$. For each $i \in I$, let $S_i : X \times Y \multimap X_i$, $T_i : X \multimap Y_i$, $G_i : X \times Y \times Y_i \multimap Z_i$ be multivalued maps. Recently, Lin [10] studied the following type of systems of variational inclusion problem:

Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, and $0 \in G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in T_i(\bar{x})$.

Lin [10] used Himmelberg's fixed point theorem to study the existence theorem of this problem. By this result, he gave some applications. For detail, one can refer to Lin [10].

In this paper, let $A_i : X \multimap X_i$, $F_i : X \multimap X_i$, $H_i : X \times X \multimap Z_i$, $C_i : X \multimap Z_i$, and $G_i : X \times X \times X_i \multimap Z_i$ be multivalued maps. Let $\bar{A}_i : X \multimap X_i$ and $\bar{F}_i : X \multimap X_i$ be defined by $\bar{A}_i(x) = \{y_i \in X_i : (x, y_i) \in \mathrm{cl}_{X \times X_i}\mathrm{Gr}(A_i)\}$ and $\bar{F}_i(x) = \{y_i \in X_i : (x, y_i) \in \mathrm{cl}_{X \times X_i}\mathrm{Gr}(F_i)\}$, where $\mathrm{cl}_{X \times X_i}\mathrm{Gr}(A_i)$ denote the closure of $\mathrm{Gr}(A_i)$ in $X \times X_i$. Throughout this paper, we use these notations unless otherwise specified. We study the following variational inclusion problems:

(VIP-1) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $\bar{x}_i \in \bar{A}_i(\bar{x})$, $\bar{y}_i \in \bar{F}_i(\bar{x})$, and $0 \in G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in A_i(\bar{x})$;

(VIP-2) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $\bar{x}_i \in \bar{A}_i(\bar{x})$, $\bar{y}_i \in \bar{F}_i(\bar{x})$, and $H_i(\bar{x}, \bar{y}) \cap G_i(\bar{x}, \bar{y}, v_i) = \emptyset$ for all $v_i \in A_i(\bar{x})$.

Note that our problem (VIP-1) is different from the problem studied in Lin [8–10]. From the existence theorems of (VIP-1), we study the following problems:

(a) Systems of variational differential inclusion problems:
Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $\bar{x}_i \in \bar{A}_i(\bar{x})$, $\bar{y}_i \in \bar{F}_i(\bar{x})$, and $\frac{d\bar{x}}{dt} \in G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in A_i(\bar{x})$. That is, $\bar{x}_i \in \bar{A}_i(\bar{x})$, $\bar{y}_i \in \bar{F}_i(\bar{x})$, and $\frac{d\bar{x}}{dt} \in \bigcap_{v_i \in A_i(\bar{x})} G_i(\bar{x}, \bar{y}, v_i)$;

(b) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $\bar{x}_i \in \bar{A}_i(\bar{x})$, $\bar{y}_i \in \bar{F}_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, v_i) \cap C_i(\bar{x}) \neq \emptyset$ for all $v_i \in A_i(\bar{x})$;

(c) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $\bar{x}_i \in \bar{A}_i(\bar{x})$, $\bar{y}_i \in \bar{F}_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, v_i) \nsubseteq -\mathrm{int}C_i(\bar{x})$ for all $v_i \in A_i(\bar{x})$;

(d) Variational fixed point problem:
Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $\bar{x}_i \in \bar{A}_i(\bar{x})$, $\bar{y}_i \in \bar{F}_i(\bar{x})$, and $\bar{x} \in G_i(\bar{x}, \bar{y}, v_i)$ for all $v_i \in A_i(\bar{x})$.
From problem (VIP-2), we study the following problems:

(e) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $\bar{x}_i \in \bar{A}_i(\bar{x})$, $\bar{y}_i \in \bar{F}_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, v_i) = \frac{d\bar{x}}{dt}$ for all $v_i \in A_i(\bar{x})$, where $G_i : X \times X \times X_i \to Z$ is a function and Z is a t.v.s.;

(f) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $\bar{x}_i \in \bar{A}_i(\bar{x})$, $\bar{y}_i \in \bar{F}_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, v_i) \subset C_i(\bar{x})$ for all $v_i \in A_i(\bar{x})$;
(g) Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $\bar{x}_i \in \bar{A}_i(\bar{x})$, $\bar{y}_i \in \bar{F}_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, v_i) \cap (-\text{int}C_i(\bar{x})) = \emptyset$ for all $v_i \in A_i(\bar{x})$;
(h) Variational stationary point problem:
Find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i \in I}, (\bar{y}_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $\bar{x}_i \in \bar{A}_i(\bar{x})$, $\bar{y}_i \in \bar{F}_i(\bar{x})$, and $G_i(\bar{x}, \bar{y}, v_i) = \{\bar{x}\}$ for all $v_i \in A_i(\bar{x})$.

Let Z be a real t.v.s., D a proper closed convex cone in Z. A point $\bar{y} \in A$ is called a vector minimal point of A if for any $y \in A$, $y - \bar{y} \notin -D \setminus \{0\}$. The set of vector minimal points of A is denoted by $\text{Min}_D A$.

Problems (a) and (e) contain control parameter, but problem (2) does not have control parameter. Problems (a) and (e) are different from any differential inclusion problems studied in the literatures. Problem (1) is not a special case of (a).

Let $h : X \times Y \multimap W$ and W be a real t.v.s. ordered by a closed convex cone D. As applications of our results, we study the following mathematical programs with systems of variational differential inclusion constraints and mathematical programs with systems of equilibrium constraints:

(i) $\text{Min}_D h(x, y)$ subject to $(x, y) = ((x_i)_{i \in I}, (y_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $x_i \in \bar{A}_i(x)$, $y_i \in \bar{F}_i(x)$, and $\frac{dx}{dt} \in G_i(x, y, v_i)$ for all $v_i \in A_i(x)$;
(j) $\text{Min}_D h(x, y)$ subject to $(x, y) = ((x_i)_{i \in I}, (y_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $x_i \in \bar{A}_i(x)$, $y_i \in \bar{F}_i(x)$, and $G_i(x, y, v_i) = \{\frac{dx}{dt}\}$ for all $v_i \in A_i(x)$;
(k) $\text{Min}_D h(x, y)$ subject to $(x, y) = ((x_i)_{i \in I}, (y_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $x_i \in \bar{A}_i(x)$, $y_i \in \bar{F}_i(x)$, and $G_i(x, y, v_i) \subset C_i(x)$ for all $v_i \in A_i(x)$;
(l) $\text{Min}_D h(x, y)$ subject to $(x, y) = ((x_i)_{i \in I}, (y_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $x_i \in \bar{A}_i(x)$, $y_i \in \bar{F}_i(x)$, and $G_i(x, y, v_i) \cap C_i(x) \neq \emptyset$ for all $v_i \in A_i(x)$;
(m) $\text{Min}_D h(x, y)$ subject to $(x, y) = ((x_i)_{i \in I}, (y_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $x_i \in \bar{A}_i(x)$, $y_i \in \bar{F}_i(x)$, and $G_i(x, y, v_i) \not\subseteq -\text{int}C_i(x)$ for all $v_i \in A_i(x)$;
(n) $\text{Min}_D h(x, y)$ subject to $(x, y) = ((x_i)_{i \in I}, (y_i)_{i \in I}) \in X \times X$ such that for each $i \in I$, $x_i \in \bar{A}_i(x)$, $y_i \in \bar{F}_i(x)$, and $G_i(x, y, v_i) \cap (-\text{int}C_i(x)) = \emptyset$ for all $v_i \in A_i(x)$.

In this paper, we study the existence theorems of systems of variational inclusion problem. From these results, we give simple proofs of existence theorems of solution for systems of equilibrium problems which are recently studied by Lin et al. [11]. From these existence theorems of variational inclusion problem, we study the existence theorems of systems of differential inclusion problems, systems of variational fixed point, systems of variational stationary points, mathematical problems with systems of equilibrium constraints, mathematical problem with systems of differential inclusion constraints, and systems of generalized vector quasi-equilibrium problem with upper and lower bounds.

Our results are different from any existence results of these types of problems. For detail, one can refer to [1,3–7,9,10,13–15,17,18,20,21] and references therein.

2 Preliminaries

Throughout this paper, all topological spaces (in short t.s.) are assumed to be Hausdorff. Let X and Y be t.s., $T : X \multimap Y$ be a multivalued map, T is said to be upper semicontinuous (in short u.s.c.) (respectively lower semicontinuous (in short l.s.c.) at $x \in X$ if for every open set U in Y with $T(x) \subseteq U$ (resp. $T(x) \cap U \neq \emptyset$), there exists an open neighborhood $V(x)$ of

x such that $T(x') \subseteq U$ (resp. $T(x') \cap U \neq \emptyset$) for all $x' \in V(x)$; T is said to be u.s.c. (resp. l.s.c.) on X if T is u.s.c. (resp. l.s.c.) at every point of X; T is continuous at x if T is both u.s.c. and l.s.c. at x; T is closed if $\mathrm{Gr}(T) = \{(x, y) \in X \times Y : y \in T(x), x \in X\}$ is a closed set in $X \times Y$. We also define $\bar{T} : X \multimap Y$ by $\bar{T}(x) = \{y \in Y : (x, y) \in \mathrm{cl}_{X\times Y}\mathrm{Gr}(T)\}$, where $\mathrm{cl}_{X\times Y}\mathrm{Gr}(T)$ denote the closure of $\mathrm{Gr}(T)$ in $X \times Y$.

The following lemmas and theorems are needed in this paper.

Lemma 2.1 [20] *Let X and Y be t.s., and $T : X \multimap Y$ be a multivalued map. Then T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$ and any net $\{x_\alpha\}_{\alpha\in\Lambda}$ in X converges to x, there exists a net $\{y_\alpha\}_{\alpha\in\Lambda}$ such that $y_\alpha \in T(x_\alpha)$ for all $\alpha \in A$ and $y_\alpha \to y$.*

Lemma 2.2 [19] *Let Z be a t.v.s. and C be a closed convex cone in Z. If A is a nonempty compact subset of Z, then $Min_C A \neq \emptyset$.*

Lemma 2.3 [16] *Let X and Y be Hausdorff t.v.s., $F, G : X \multimap Y$ be multivalued maps. Let $F + G : X \multimap Y$ be defined by $(F + G)(x) := F(x) + G(x)$ for each $x \in X$.*

(a) If F is an u.s.c. multivalued map with nonempty compact values and G is closed, then $F + G$ is closed;
(b) If F is l.s.c. and G is open, then $F + G$ is open.

Theorem 2.1 [1] *Let X and Y be t.s., and $T : X \multimap Y$ be a multivalued map.*

(i) *If T is an u.s.c. multivalued map with nonempty closed values, then T is closed;*
(ii) *If Y is a compact space and T is closed, then T is u.s.c.;*
(iii) *If X is compact and T is an u.s.c. multivalued map with nonempty compact values, then $T(X)$ is compact.*

Definition 2.1 Let X be a nonempty convex subset of a vector space E, Y be a nonempty convex subset of a vector space H and Z be a real t.v.s.. Let $F : X \times Y \multimap Z$ and $C : X \multimap Z$ be multivalued maps such that for each $x \in X$, $C(x)$ is a closed convex cone. For each $x \in X$,

(i) F is $C(x)$—quasiconvex if for any $y_1, y_2 \in Y$ and $\lambda \in [0, 1]$, either

$$F(x, y_1) \subseteq F(x, \lambda y_1 + (1-\lambda)y_2) + C(x),$$

or

$$F(x, y_2) \subseteq F(x, \lambda y_1 + (1-\lambda)y_2) + C(x).$$

(ii) F is $\{0\}$—quasiconvex-like if for any $y_1, y_2 \in Y$ and $\lambda \in [0, 1]$, either

$$F(x, \lambda y_1 + (1-\lambda)y_2) \subseteq F(x, y_1),$$

or

$$F(x, \lambda y_1 + (1-\lambda)y_2) \subseteq F(x, y_2).$$

(iii) F is 0—quasiconvex if for any $y_1, y_2 \in Y$ and $\lambda \in [0, 1]$, either

$$F(x, y_1) \subseteq F(x, \lambda y_1 + (1-\lambda)y_2),$$

or

$$F(x, y_2) \subseteq F(x, \lambda y_1 + (1-\lambda)y_2).$$

Definition 2.2 [3] Let E be a t.v.s and X be a subset of E such that $X = \bigcup_{n=1}^{\infty} G_n$, where $\{G_n\}_{n=1}^{\infty}$ is an increasing (in the sense that $G_n \subseteq G_{n+1}$) sequence of nonempty compact sets. A sequence $\{y_n\}_{n=1}^{\infty}$ in X is said to be *escaping* from X(relative to $\{G_n\}_{n=1}^{\infty}$) if for each $n \in \mathbb{N}$, there exists $M > 0$ such that $y_k \notin G_n$ for all $k \geq M$.

Theorem 2.2 [11] *Let I be any index set. For each $i \in I$, let X_i be a nonempty subset of a locally convex t.v.s. E_i and let $X = \Pi_{i\in I} X_i$. For each $i \in I$, let $X_i \times X_i = \bigcup_{j=1}^{\infty} G_{i,j}$, where $\{G_{i,j}\}_{j=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of a locally convex t.v.s. $E_i \times E_i$. For each $i \in I$, assume that:*

(i) *$A_i : X \multimap X_i$ and $F_i : X \multimap X_i$ are l.s.c. multivalued maps with nonempty convex values;*
(ii) *$P_i : X \times X \multimap X_i$ has an open graph and $x_i \notin coP_i(x, y)$ for all $(x, y) \in X \times X$;*
(iii) *for each sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in $X \times X$ with $(x_n, y_n) \in G_n = \Pi_{i\in I} G_{i,n}$ for each $n \in \mathbb{N}$, which is escaping from $X \times X$ relative to $\{G_n\}_{n=1}^{\infty}$, there exists $m \in \mathbb{N}$ and $(\tilde{x}_m, \tilde{y}_m) \in G_m$ such that $\pi_i(\tilde{x}_m) \in A_i(x_m) \cap P_i(x_m, y_m)$ and $\pi_i(\tilde{y}_m) \in F_i(x_m)$ for all $i \in I$, where $\pi_i(x)$ is the projection of $x \in X$ onto X_i.*

Then there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in X \times X$ such that for each $i \in I, \hat{x}_i \in \bar{A}_i(\hat{x}), \hat{y}_i \in \bar{F}_i(\hat{x})$, and $A_i(\hat{x}) \cap P_i(\hat{x}, \hat{y}) = \emptyset$.

Theorem 2.3 [11] *Let I be an index set. For each $i \in I$, let X_i be a nonempty convex subset of a locally convex t.v.s. E_i and let $X = \Pi_{i\in I} X_i$. For each $i \in I$, assume that:*

(i) *$A_i : X \multimap X_i$ and $F_i : X \multimap X_i$ are l.s.c. multivalued maps with nonempty convex values;*
(ii) *$P_i : X \times X \multimap X_i$ has an open graph and $x_i \notin coP_i(x, y)$ for all $(x, y) \in X \times X$;*
(iii) *there exist nonempty compact subsets K and M of X and nonempty compact convex subsets $\tilde{D}_i$ and D_i of X_i for each $i \in I$ with the property that for each $(x, y) \in (X \times X)\setminus(K \times M)$, there exists $j \in I$ such that $A_j(x) \cap P_j(x, y) \cap \tilde{D}_j \neq \emptyset$ and $F_j(x) \cap D_j \neq \emptyset$.*

Then there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ such that for each $i \in I, \hat{x}_i \in \bar{A}_i(\hat{x}), \hat{y}_i \in \bar{F}_i(\hat{x})$, and $A_i(\hat{x}) \cap P_i(\hat{x}, \hat{y}) = \emptyset$.

3 Existence theorems of systems of generalized quasi-variational inclusions problems

From now onward unless otherwise specified, for each $i \in I$, let X_i be a nonempty closed convex subset of a locally convex Hausdorff t.v.s. E_i and let $X = \Pi_{i\in I} X_i$. For each $i \in I$, let Z_i be a t.v.s. and $C_i : X \multimap Z_i$ be a multivalued map such that for each $x \in X$, $C_i(x)$ is a proper closed convex cone with $\text{int}C_i(x) \neq \emptyset$.

Theorem 3.1 *For each $i \in I$, let $G_i : X \times X \times X_i \multimap Z_i$ be a closed multivalued map with nonempty values. For each $i \in I$, assume that:*

(i) *$F_i, A_i : X \multimap X_i$ are l.s.c. multivalued maps with nonempty convex values;*
(ii) *for each $x = (x_i)_{i\in I} \in X$ and $y \in X, 0 \in G_i(x, y, x_i)$;*
(iii) *for each $(x, y) \in X \times X, u_i \multimap G_i(x, y, u_i)$ is $\{0\}$-quasiconvex-like;*
(iv) *there exist nonempty compact subsets K and M of X and nonempty compact convex subsets $\hat{D}_i$ and D_i of X_i for each $i \in I$ such that for each $(x, y) \in (X \times X)\setminus(K \times M)$, there exist $j \in I, \hat{u}_j \in \hat{D}_j$, and $\hat{v}_j \in D_j$ such that $\hat{u}_j \in A_j(x), \hat{v}_j \in F_j(x)$, and $0 \notin G_j(x, y, \hat{u}_j)$.*

Then there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ *such that for each* $i \in I, \hat{x}_i \in \bar{A}_i(\hat{x}), \hat{y}_i \in \bar{F}_i(\hat{x})$*, and* $0 \in G_i(\hat{x}, \hat{y}, u_i)$ *for all* $u_i \in A_i(\hat{x})$.

Proof For each $i \in I$, let $P_i : X \times X \multimap X_i$ be defined by $P_i(x, y) = \{u_i \in X_i : 0 \notin G_i(x, y, u_i)\}$ for $(x, y) \in X \times X$. By (iii), $P_i(x, y)$ is a convex set for each $(x, y) \in X \times X$. By (ii), $x_i \notin P_i(x, y) = \text{co}P_i(x, y)$ for each $(x, y) \in X \times X$. For each $i \in I$, P_i has an open graph in $X \times X \times X_i$. Indeed, let $(x, y, u_i) \in \text{cl}[\text{Gr}(P_i)]^c$, then there exists a net $(x^\alpha, y^\alpha, u_i^\alpha)_{\alpha\in\Lambda}$ in $(\text{Gr}(P_i))^c$ such that $(x^\alpha, y^\alpha, u_i^\alpha) \to (x, y, u_i)$, where Λ is an index set. One has $(x^\alpha, y^\alpha, u_i^\alpha) \in X \times X \times X_i$ and $0 \in G_i(x^\alpha, y^\alpha, u_i^\alpha)$. Since X_i and X are closed sets and G_i is closed, $(x, y, u_i) \in X \times X \times X_i$ and $0 \in G_i(x, y, u_i)$. Hence, $(x, y, u_i) \in (\text{Gr}(P_i))^c$ and $(\text{Gr}(P_i))^c$ is a closed set. This shows that $\text{Gr}(P_i)$ is open and P_i has an open graph. By (iv), for each $(x, y) \in X \times X \backslash (K \times M)$, there exists $j \in I$ such that $A_j(x) \cap P_j(x, y) \cap \tilde{D}_j \neq \emptyset$ and $F_j(x) \cap D_j \neq \emptyset$. Then by Theorem 2.3 there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ such that for each $i \in I, \hat{x}_i \in \bar{A}_i(\hat{x}), \hat{y}_i \in \bar{F}_i(\hat{x})$, and $P_i(\hat{x}, \hat{y}) \cap A_i(\hat{x}) = \emptyset$. That is, $0 \in G_i(\hat{x}, \hat{y}, u_i)$ for all $u_i \in A_i(\hat{x})$. □

Remark 3.1 Theorem 3.1 is different from Theorem 3.1 in [10]. The continuity assumptions on F_i and A_i in Theorem 3.1 [10] and Theorem 3.1 are different. The convexity assumptions on G_i are different. In Theorem 3.1, we assume a coercive condition, but in Theorem 3.1 in [10], we don't assume the coercive assumptions. The proofs and conclusions of these two theorems are also different.

Theorem 3.2 *For each* $i \in I$*, let* $G_i : X \times X \times X_i \multimap Z_i$ *be a l.s.c. multivalued map with nonempty values. For each* $i \in I$*, assume that:*

- (i) $F_i, A_i : X \multimap X_i$ *are l.s.c. multivalued maps with nonempty convex values;*
- (ii) $H_i : X \times X \multimap Z_i$ *is a multivalued map with nonempty values and open graph;*
- (iii) *for each* $x = (x_i)_{i\in I} \in X$ *and* $y \in X$, $H_i(x, y) \cap G_i(x, y, x_i) = \emptyset$;
- (iv) *for each* $(x, y) \in X \times X, u_i \multimap G_i(x, y, u_i)$ *is* $\{0\}$*-quasiconvex;*
- (v) *there exist nonempty compact subsets* K *and* M *of* X *and nonempty compact convex subsets* $\hat{D}_i$ *and* D_i *of* X_i *for each* $i \in I$*, such that for each* $(x, y) \in (X \times X) \setminus (K \times M)$*, there exist* $j \in I, \hat{u}_j \in \hat{D}_j$*, and* $\hat{v}_j \in D_j$ *such that* $\hat{u}_j \in A_j(x), \hat{v}_j \in F_j(x)$*, and* $H_j(x, y) \cap G_j(x, y, \hat{u}_j) \neq \emptyset$.

Then there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ *such that for each* $i \in I, \hat{x}_i \in \bar{A}_i(\hat{x}), \hat{y}_i \in \bar{F}_i(\hat{x})$*, and* $H_i(\hat{x}, \hat{y}) \cap G_i(\hat{x}, \hat{y}, v_i) = \emptyset$ *for all* $v_i \in A_i(\hat{x})$.

Proof For each $i \in I$, let $F_i : X \times X \times X_i \multimap Z_i$ be defined by $Q_i(x, y, v_i) = Z_i \setminus [-H_i(x, y) + G_i(x, y, v_i)]$ for each $(x, y, v_i) \in X \times X \times X_i$. Then by Lemma 2.3 , Q_i is closed. By (iv), for each $x = (x_i)_{i\in I} \in X$ and $y \in X$, $v_i \multimap Q_i(x, y, v_i)$ is $\{0\}$-quasiconvex-like. Then Theorem 3.2 follows from Theorem 3.1. □

Remark 3.2 Theorem 3.2 is also true if "G_i is l.s.c. and H_i is open" is replaced by "G_i is open and H_i is l.s.c.".

Theorem 3.3 *For each* $i \in I$*, let* $G_i : X \times X \times X_i \multimap Z_i$ *be an u.s.c. multivalued map with nonempty compact values. For each* $i \in I$*, assume that:*

- (i) $F_i, A_i : X \multimap X_i$ *are l.s.c. multivalued maps with nonempty convex values;*
- (ii) $H_i : X \times X \multimap Z_i$ *is a multivalued map with nonempty values and closed graph;*
- (iii) *for each* $x = (x_i)_{i\in I} \in X$ *and* $y \in X$, $H_i(x, y) \cap G_i(x, y, x_i) \neq \emptyset$;

(iv) *for each* $(x, y) \in X \times X$, $u_i \multimap G_i(x, y, u_i)$ *is* $\{0\}$*-quasiconvex-like;*

(v) *there exist nonempty compact subsets* K *and* M *of* X *and nonempty compact convex subsets* $\hat{D}_i$ *and* D_i *of* X_i *for each* $i \in I$*, such that for each* $(x, y) \in (X \times X) \setminus (K \times M)$*, there exist* $j \in I, \hat{u}_j \in \hat{D}_j$*, and* $\hat{v}_j \in D_j$ *such that* $\hat{u}_j \in A_j(x), \hat{v}_j \in F_j(x)$*, and* $H_j(x, y) \cap G_j(x, y, \hat{u}_j) = \emptyset$.

Then there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ *such that for each* $i \in I, \hat{x}_i \in \bar{A}_i(\hat{x}), \hat{y}_i \in \bar{F}_i(\hat{x})$*, and* $H_i(\hat{x}, \hat{y}) \cap G_i(\hat{x}, \hat{y}, v_i) \neq \emptyset$ *for all* $v_i \in A_i(\hat{x})$.

Proof Let $Q_i : X \times X \times X_i \multimap Z_i$ be defined by $F_i(x, y, v_i) = -H_i(x, y) + G_i(x, y, v_i)$ for each $(x, y, v_i) \in X \times X \times X_i$. By Lemma 2.3, Q_i is closed. By (iv), for each $(x, y) \in X \times X$, $v_i \multimap Q_i(x, y, v_i)$ is $\{0\}$-quasiconvex-like. Then Theorem 3.3 follows from Theorem 3.1. □

Theorem 3.4 *For each* $i \in I$*, let* $X_i = \bigcup_{j=1}^{\infty} Q_{i,j}$*, where* $\{Q_{i,j}\}_{j=1}^{\infty}$ *is an increasing sequence of nonempty compact convex subset of* E_i*, and* $G_i : X \times X \times X_i \multimap Z_i$ *is a closed multivalued map with nonempty values. Assume that conditions (i)–(iii) of Theorem 3.1 and the following condition hold:*

(iv′) *for each sequence* $\{(x^n, y^n)\}_{n=1}^{\infty}$ *in* $X \times X$ *with* $(x^n, y^n) \in Q_n = \Pi_{i\in I} Q_{i,n}$ *for each* $n \in \mathbb{N}$*, which is increasing from* $X \times X$ *relative to* $\{Q_n\}_{n=1}^{\infty}$*, there exist* $m \in \mathbb{N}$ *and* $(\tilde{x}^m, \tilde{y}^m) \in Q_m$ *such that* $\pi_i(\tilde{x}^m) \in A_i(x^m), \pi_i(\tilde{y}^m) \in F_i(x^m)$ *and* $0 \notin G_i(x^m, y^m, \pi_i(\tilde{x}^m))$ *for all* $i \in I$*, where* $\pi_i(x)$ *is the projection of* $x \in X$ *onto* X_i.

Then there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in X \times X$ *such that for each* $i \in I, \hat{x}_i \in \hat{A}_i(\hat{x}), \hat{y}_i \in \hat{F}_i(\hat{x})$*, and* $0 \in G_i(\hat{x}, \hat{y}, u_i)$ *for all* $u_i \in A_i(\hat{x})$.

Proof For each $i \in I$, let P_i be defined as in Theorem 3.1. Applying Theorem 2.2 and following the same argument as in Theorem 3.1, we can prove Theorem 3.4. □

Remark 3.3

(a) In Theorem 3.3, if $H_i(x, y) = C_i(x)$ is a closed multivalued map with nonempty values, then Theorem 3.3 will be reduced to Theorem 3.2.2 in [11]. Besides, if $H_i(x, y) = Z_i \setminus (-\text{int}C_i(x))$, then Theorem 3.3 will be reduced to Theorem 3.2.4 in [11].

(b) In Theorem 3.2, if $H_i(x, y) = Z_i \setminus C_i(x)$ is an open multivalued map, then Theorem 3.2 will be reduced to Theorem 3.2.1 in [11]. If $H_i(x, y) = -\text{int}C_i(x)$ is an open multivalued map, then Theorem 3.2 will be reduced to Theorem 3.2.3 in [11].

(c) Applying Theorem 3.4, we can obtain similar results as Theorems 3.2 and 3.3. From the similar results of Theorems 3.2 and 3.4, we can prove Theorems 3.2.5–3.2.8 in [11].

4 Applications

The following notations are needed in the following theorem. Let D be an open set in $\mathbb{R}$, we denote $BC^1(D) = \{x|\frac{dx}{dt} : D \to \mathbb{R}$ is a bounded continuous function and $x : D \to \mathbb{R}$ is a bounded function$\}$ and $BC(D) = \{x|x : D \to \mathbb{R}$ is a bounded function$\}$. If $f \in BC^1(D)$, we define $\|f\| = \sup_{t\in D} |\frac{df(t)}{dt}| + \sup_{t\in D} |f(t)|$. If $g \in BC(D)$, we define $\|g\| = \sup_{t\in D} |g(t)|$. For $f, g \in BC^1(D)$, we define $(f + g)(t) = f(t) + g(t)$. It is easy to see that $BC^1(D)$ is a normed linear space.

Let D be an open set in $\mathbb{R}$, $F : BC^1(D) \to BC(D)$ be defined by $F(x) = \frac{dx}{dt}$ for each $x \in BC^1(D)$. Then $F : BC^1(D) \to BC(D)$ is a closed function. Indeed,

$||F(x)||_C = ||\frac{dx}{dt}||_C \leq ||\frac{dx}{dt}||_C + ||x||_C = ||x||_{C^1}$ and this shows that F is continuous. Therefore, F is a closed function.

The following theorems study the existence of the systems of variational differential inclusion problems.

Theorem 4.1 *Let I be any index set. For each $i \in I$, let D_i be an open set in $\mathbb{R}$, X_i a nonempty closed convex subset of $BC^1(D_i)$, $X = \prod_{i \in I} X_i$, $Z_i = BC(D_i)$, and $G_i : X \times X \times X_i \multimap Z_i$ be an u.s.c. multivalued map with nonempty compact values. For each $i \in I$, assume that:*

- (i) *$F_i, A_i : X \multimap X_i$ are l.s.c. multivalued maps with nonempty convex values;*
- (ii) *for each $x = (x_i)_{i \in I} \in X$ and $y \in X$, $\frac{dx_i}{dt} \in G_i(x, y, x_i)$;*
- (iii) *for each $(x, y) \in X \times X$, $u_i \multimap G_i(x, y, u_i)$ is $\{0\}$-quasiconvex-like;*
- (iv) *there exist nonempty compact subsets K and M of X and nonempty compact convex subsets $\hat{D}_i$ and D_i of X_i for each $i \in I$ such that for each $(x, y) \in (X \times Y) \setminus (K \times M)$, there exist $j \in I, \hat{u}_j \in \hat{D}_j$, and $\hat{v}_j \in D_j$ such that $\hat{u_j} \in A_j(x), \hat{v}_j \in F_j(x)$, and $\frac{dx_i}{dt} \notin G_j(x, y, \hat{u}_j)$.*

Then there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i \in I}, (\hat{y}_i)_{i \in I}) \in K \times M$ such that for each $i \in I, \hat{x}_i \in \bar{A}_i(\hat{x}), \hat{y}_i \in \bar{F}_i(\hat{x})$, and $\frac{d\hat{x}_i}{dt} \in G_i(\hat{x}, \hat{y}, v_i)$ for all $v_i \in A_i(\hat{x})$.

Proof Let $H_i : X \times X \multimap Z_i$ be defined by $H_i(x, y) = \frac{dx_i}{dt}$. Then H_i is open. Then Theorem 4.1 follows from Theorem 3.3. □

Theorem 4.2 *For each $i \in I$, let $G_i : X \times X \times X_i \multimap X$ be an u.s.c. multivalued map with nonempty compact values. For each $i \in I$, assume that:*

- (i) *$F_i, A_i : X \multimap X_i$ are l.s.c. multivalued maps with nonempty convex values;*
- (ii) *for each $x = (x_i)_{i \in I} \in X, y \in X, x \in G_i(x, y, x_i)$;*
- (iii) *for each $(x, y) \in X \times X$, $u_i \multimap G_i(x, y, u_i)$ is $\{0\}$-quasiconvex-like;*
- (iv) *there exist a nonempty compact subsets K and M of X and a nonempty compact convex subsets $\tilde{D}_i$ and D_i of X_i for each $i \in I$ such that for each $(x, y) \in (X \times X) \setminus (K \times M)$, there exist $j \in I, \hat{u}_j \in \hat{D}_j$, and $\hat{v}_j \in D_j$ such that $\hat{u}_j \in A_j(x), \hat{v}_j \in F_j(x)$, and $\hat{x} \notin G_j(x, y, \hat{u}_j)$.*

Then there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i \in I}, (\hat{y}_i)_{i \in I}) \in K \times M$ such that for each $i \in I, \hat{x}_i \in \bar{A}_i(\hat{x}), \hat{y}_i \in \bar{F}_i(\hat{x})$, and $\hat{x} \in G_i(\hat{x}, \hat{y}, v_i)$ for all $v_i \in A_i(\hat{x})$.

Proof Let $H : X \times X \multimap X$ be defined by $H(x, y) = -\{x\}$. Then Theorem 4.2 follows from Theorem 4.1. □

Lemma 4.1 *Let X be a nonempty subset of a topological space E, Z be a real t.v.s., and D be a nonempty closed convex cone in Z. Let $F : X \multimap Z$ be a l.s.c. multivalued map. Suppose that for each $x \in X$, $IMin_D F(x) \neq \emptyset$. Let $H : X \multimap Z$ be defined by $H(x) := IMin_D F(x)$ for each $x \in X$. Then H is a closed function.*

Proof If $(x, y) \in \mathrm{cl}(\mathrm{Gr}(H))$, then there exists a net $\{(x_\alpha, z_\alpha)\}_{\alpha \in \Lambda}$ in $\mathrm{Gr}(H)$ such that $(x_\alpha, z_\alpha) \to (x, z)$. One has $z_\alpha \in H(x_\alpha) = \mathrm{IMin}_D F(x_\alpha)$. Hence, $F(x_\alpha) - z_\alpha \in D$. Let $u \in F(x)$. Since F is l.s.c., there exists a net $\{u_\alpha\}_{\alpha \in \Lambda}$ such that $u_\alpha \in F(x_\alpha)$ for all $\alpha \in \Lambda$ and $u_\alpha \to u$. We have $u_\alpha - z_\alpha \in D$. Since D is a closed set, $u - z \in D$. Therefore, $F(x) - z \subseteq D$ and $z \in \mathrm{IMin}_D F(x) = H(x)$. This shows that $(x, z) \in \mathrm{Gr}(H)$ and H is closed. Since D is proper, it is easy to see that H is a function. □

Lemma 4.2 Let X be a nonempty subset of a t.v.s. E, Z be a real t.v.s., and D be a proper closed convex cone in Z with nonempty interior. Let $F : X \multimap Z$ be a continuous multivalued map with nonempty compact values. Let $m : X \multimap Z$ be defined by $m(x) := \mathrm{WMin}_D F(x)$ for each $x \in X$. Then m is a closed multivalued map with nonempty closed values.

Proof Since $F(x)$ is compact for each $x \in X$, it follows from Lemma 2.2 that $\emptyset \neq \mathrm{Min}_D F(x) \subseteq \mathrm{WMin}_D F(x) = m(x)$ for each $x \in X$. If $(x, z) \in \mathrm{cl}(\mathrm{Gr}(m))$, then there exists a net $\{(x_\alpha, z_\alpha)\}_{\alpha \in \Lambda}$ in $\mathrm{Gr}(m)$ such that $(x_\alpha, z_\alpha) \to (x, z)$. One has $z_\alpha \in m(x_\alpha) \subseteq F(x_\alpha)$. Since F is an u.s.c. multivalued map with nonempty closed values, it follows from Theorem 2.1 that F is closed and $z \in F(x)$. We want to show that $z \in m(x)$. Suppose that $z \notin m(x)$. Then there exists $\omega \in F(x)$ such that $\omega - z \in -\mathrm{int}D$. But F is l.s.c., there exists a net $\{\omega_\alpha\}_{\alpha \in \Lambda}$ with $\omega_\alpha \in F(x_\alpha)$ for all $\alpha \in \Lambda$ such that $\omega_\alpha \to \omega$. Therefore, $\omega_\alpha - z_\alpha \in -\mathrm{int}D$ for some α. This is impossible since $z_\alpha \in m(x_\alpha)$. Therefore, $z \in m(x)$ and m is closed. □

Theorem 4.3 *For each $i \in I$, let Z_i be a real t.v.s. and C_i be a proper closed convex cone in Z_i, and let $G_i : X \times X \times X_i \multimap Z_i$ be a continuous multivalued map with nonempty compact values. For each $i \in I$, suppose that the conditions (i) and (iii) of Theorem 4.1 and the following condition are satisfied:*

(iv) *there exist nonempty compact subsets K and M of X and nonempty compact convex subsets $\hat{D}_i$ and D_i of X_i for each $i \in I$ such that for each $(x, y) \in (X \times X) \backslash (K \times M)$, there exist $j \in I, \hat{u}_j \in \hat{D}_j$, and $\hat{v}_j \in D_j$ such that $\hat{u_j} \in A_j(x), \hat{v}_j \in F_j(x)$, and $G_j(x, y, \hat{u}_j) \cap WMin_{C_i} G_i(x, y, x_i) = \emptyset$.*

Then there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i \in I}, (\hat{y}_i)_{i \in I}) \in K \times M$ such that for each $i \in I, \hat{x}_i \in \bar{A}_i(\hat{x}), \hat{y}_i \in \bar{F}_i(\hat{x})$, and $G_i(\hat{x}, \hat{y}, v_i) \cap WMin_{C_i} G_i(\hat{x}, \hat{y}, \bar{x}_i) \neq \emptyset$ for all $v_i \in A_i(\hat{x})$.

Proof Let $H_i : X \times X \multimap Z_i$ be defined by $H_i(x, y) := \mathrm{WMin}_{C_i} G_i(x, y, x_i)$ for each $(x, y) \in X \times X$. By Lemma 4.2, H_i is closed. For each $(x, y) \in X \times X$, $\mathrm{WMin}_{C_i} G_i(x, y, x_i) = H_i(x, y) \subseteq G_i(x, y, x_i)$. Then Theorem 4.3 follows from Theorem 3.3. □

For the special case of Theorem 4.1, we have the following results.

Corollary 4.1 *For each $i \in I$, let $G_i : X \times X \times X_i \multimap \mathbb{R}$ be a continuous multivalued map with nonempty compact values. For each $i \in I$, suppose that the conditions (i) and (ii) of Theorem 3.1 and the following are satisfied:*

(iv) *there exist nonempty compact subsets K and M of X and nonempty compact convex subsets $\hat{D}_i$ and D_i of X_i for each $i \in I$ such that for each $(x, y) \in (X \times X) \setminus (K \times M)$, there exist $j \in I, \hat{u}_j \in \hat{D}_j$, and $\hat{v}_j \in D_j$ such that $\hat{u_j} \in A_j(x), \hat{v}_j \in F_j(x)$, and $MinG_j(x, y, x_j) \notin G_j(x, y, \hat{u}_j)$.*

Then there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i \in I}, (\hat{y}_i)_{i \in I}) \in K \times M$ such that for each $i \in I, \hat{x}_i \in \bar{A}_i(\hat{x}), \hat{y}_i \in \bar{F}_i(\hat{x})$, and $Min_{C_i} G_i(\hat{x}, \hat{y}, \hat{x}_i) \in G_i(\hat{x}, \hat{y}, v_i)$ for all $v_i \in A_i(\hat{x})$.

Corollary 4.2 *Let X be a nonempty closed convex subset of a t.v.s. and $G : X \multimap R$ be a continuous multivalued map with nonempty compact values. Assume that:*

(i) *G is $\{0\}$-quasiconvex-like;*
(ii) *there exists a nonempty compact subset K of X and a nonempty convex subset $\hat{D}$ of X such that for each $x \in X \setminus K$, there exists $\hat{u} \in \hat{D}$ such that $MinG(x) \notin G(\hat{u})$.*

Then there exists $\hat{x} \in X$ such that $MinG(\hat{x}) \in \cap_{y \in X} G(y)$.

Proof Let $A, F : X \multimap X$ be defined by $A(x) := X$ and $F(x) := X$ for each $x \in X$. Then Corollary 4.2 follows from Corollary 4.1. □

Applying Lemma 4.1 and following the same argument as in Theorem 4.3, we can prove Theorem 4.4.

Theorem 4.4 *In Theorem 4.3, suppose that* $IMinG_i(x, y, x_i) \neq \emptyset$ *for each* $x = (x_i)_{i\in I} \in X$ *and* $y \in X$. *If the condition (iv) of Theorem 4.3 is replaced by (iv)′, where*

> there exist nonempty compact subsets K and M of X and nonempty compact convex subsets $\hat{D}_i$ and D_i of X_i for each $i \in I$ such that for each $(x, y) \in (X \times X) \setminus (K \times M)$, there exist $j \in I$, and $\hat{u}_j \in \hat{D}_j$, $\hat{v}_j \in D_j$ such that $\hat{u}_j \in A_j(x)$, $\hat{v}_j \in F_j(x)$, and $\mathrm{IMin}_{C_i} G_j(x, y, x_j) \notin G_j(x, y, \hat{u}_j)$.

Then there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ *such that for each* $i \in I$, $\hat{x}_i \in \bar{A}_i(\hat{x})$, $\hat{y}_i \in \bar{F}_i(\hat{x})$, *and* $IMin_{C_i} G_i(\hat{x}, \hat{y}, \bar{x}_i) \in G_i(\hat{x}, \hat{y}, v_i)$ *for all* $v_i \in A_i(\hat{x})$.

We can also apply Theorem 3.2 to study systems of generalized equilibrium problem with upper and lower bounded.

Theorem 4.5 *For each* $i \in I$, *let* $a_i, b_i \in \mathbb{R}$ *with* $a_i < b_i$ *and* $G_i : X \times X \times X_i \multimap \mathbb{R}$ *be a l.s.c multivalued map with nonempty values. For each* $i \in I$, *suppose that the conditions (i) and (iii) of Theorem 3.1 and the following conditions are satisfied:*

(ii) *for each* $x = (x_i)_{i\in I}$ *and* $y \in X$, $G_i(x, y, x_i) \subseteq [a_i, b_i]$;
(iv) *there exist nonempty compact subsets* K *and* M *of* X *and nonempty compact convex subsets* $\hat{D}_i$ *and* D_i *of* X_i *for each* $i \in I$ *such that for each* $(x, y) \in (X \times X) \setminus (K \times M)$, *there exist* $j \in I$, $\hat{u}_j \in \hat{D}_j$, *and* $\hat{v}_j \in D_j$ *such that* $\hat{u}_j \in A_j(x)$, $\hat{v}_j \in F_j(x)$, *and* $G_j(x, y, \hat{u}_j) \nsubseteq [a_j, b_j]$.

Then there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ *such that for each* $i \in I$, $\hat{x}_i \in \bar{A}_i(\hat{x})$, $\hat{y}_i \in \bar{F}_i(\hat{x})$, *and* $G_i(\hat{x}, \hat{y}, v_i) \subseteq [a_i, b_i]$ *for all* $v_i \in A_i(\hat{x})$.

Proof Let $H_i : X \times X \multimap \mathbb{R}$ be defined by $H_i(x, y) := R\backslash[a_i, b_i]$ for each $(x, y) \in X \times X$. Then Theorem 4.5 follows from Theorem 3.2. □

Theorem 4.6 *For each* $i \in I$, *let* $G_i : X \times X \times X_i \multimap X$ *be an l.s.c. multivalued map with nonempty values. For each* $i \in I$, *assume that:*

(i) $F_i, A_i : X \multimap X_i$ *are l.s.c. multivalued maps with nonempty convex values;*
(ii) *for each* $x = (x_i)_{i\in I} \in X$ *and* $y \in X$, $G_i(x, y, x_i) = \{x\}$;
(iii) *for each* $(x, y) \in X \times Y$, $u_i \multimap G_i(x, y, u_i)$ *is* $\{0\}$*-quasiconvex;*
(iv) *there exist nonempty compact subsets* K *and* M *of* X *and nonempty compact convex subsets* $\hat{D}_i$ *and* D_i *of* X_i *for each* $i \in I$ *such that for each* $(x, y) \in (X \times X) \setminus (K \times M)$, *there exist* $j \in I$, $\hat{u}_j \in \hat{D}_j$, *and* $\hat{v}_j \in D_j$ *such that* $\hat{u}_j \in A_j(x)$, $\hat{v}_j \in F_j(x)$, *and* $x \in G_j(x, y, \hat{u}_j)$.

Then there exists $(\hat{x}, \hat{y}) = ((\hat{x}_i)_{i\in I}, (\hat{y}_i)_{i\in I}) \in K \times M$ *such that for each* $i \in I$, $\hat{x}_i \in \bar{A}_i(\hat{x})$, $\hat{y}_i \in \bar{F}_i(\hat{x})$, *and* $G_i(\hat{x}, \hat{y}, v_i) = \{\hat{x}\}$ *for all* $v_i \in A_i(\hat{x})$.

Proof Let $H(x, y) = -(X \setminus \{x\})$. It is clear that $H_i : X \times Y \multimap X$ has open graph. Then Theorem 4.6 follows from Theorem 3.2. □

5 Mathematical program with system of variational differential inclusion constraints or systems of equilibrium constraints

As an application of Theorem 4.1, we have the following existence theorem of mathematical program with system of variational differential inclusions constraints.

Theorem 5.1 *In Theorem 4.1, if we assume further that $h : X \times X \multimap W$ is an u.s.c. multivalued map with nonempty compact values, where W is a real t.v.s. ordered by a closed convex cone D. Then there exists a solution to the problem:*

(MPIC) $Min_D h(x, y)$ subject to $(x, y) = ((x_i)_{i\in I}, (y_i)_{i\in I}) \in X \times X$ such that for each $i \in I$, $x_i \in \bar{A}_i(x)$, $y_i \in \bar{F}_i(x)$, and $\frac{dx_i}{dt} \in F_i(x, y, v_i)$ for all $v_i \in A_i(x)$.

Proof For each $i \in I$, let $L_i = \{(x, y) \in X \times X : (x, y) = ((x_i)_{i\in I}, (y_i)_{i\in I}), x_i \in \bar{A}_i(x), y_i \in \bar{F}_i(x)$, and $\frac{dx_i}{dt} \in F_i(x, y, v_i)$ for all $v_i \in A_i(x)\}$. Let $L = \cap_{i\in I} L_i$. By Theorem 4.1, $L_i \neq \emptyset$. As we see in Theorem 4.1, the map $(x, y, v_i) \multimap -\frac{dx_i}{dt} + F_i(x, y, v_i)$ is closed. L_i is closed. Indeed, if $(x, y) \in \text{cl}(L_i)$, then there exists a sequence $((x^n, y^n))_{n\in N}$ in L_i such that $(x^n, y^n) \to (x, y)$. Let $(x^n, y^n) = ((x_i^n)_{i\in I}, (y_i^n)_{i\in I})$ and $(x, y) = ((x_i)_{i\in I}, (y_i)_{i\in I})$. One has $(x^n, y^n) \in X \times X, x_i^n \in \bar{A}_i(x^n), y_i^n \in \bar{F}_i(x^n)$, and $0 \in -\frac{dx_i^n}{dt} + G_i(x^n, y^n, v_i)$ for all $v_i \in A_i(x^n)$. Let $v_i \in A_i(x)$. Since A_i is l.s.c., there exists a sequence $\{v_i^n\}_{n\in N}$ such that $v_i^n \to v_i$. Hence $0 \in -\frac{dx_i^n}{dt} + G_i(x^n, y^n, v_i^n)$. But $\bar{A}_i : X \multimap X_i$, $\bar{F}_i : X \multimap X_i$ are closed, and X is a closed set, we see $(x, y) \in X \times X, x_i \in \bar{A}_i(x), y_i \in \bar{F}_i(x)$, and $0 \in -\frac{dx_i}{dt} + G_i(x, y, v_i)$ for all $v_i \in A_i(x)$. This shows that L_i is closed and $L = \cap_{i\in I} L_i$ is closed. By (iv) of Theorem 4.1, $L \subseteq K \times M$. This shows that L is compact. Since $h : X \times X \multimap U$ is an u.s.c. multivalued map with nonempty compact values, $h(L)$ is a compact set. Then Theorem 5.1 follows from Lemma 2.2 □

Theorem 5.2 *Let G_i be the same as in Theorem 3.2. For each $i \in I$, suppose that conditions (i) and (iv) of Theorem 3.2 and and the following conditions are satisfied:*

- (ii) *$C_i : X \multimap Z_i$ is a closed multivalued map with nonempty values;*
- (iii) *for each $x = (x_i)_{i\in I} \in X$ and $y \in X$, $G_i(x, y, x_i) \subset C_i(x)$;*
- (v) *there exist nonempty compact subsets K and M of X and nonempty compact convex subsets $\hat{D}_i$ and D_i of X_i for each $i \in I$ such that for each $(x, y) \in (X \times X)\backslash(K \times M)$, there exist $j \in I, \hat{u}_j \in \hat{D}_j$, and $\hat{v}_j \in D_j$ such that $\hat{u}_j \in A_j(x), \hat{v}_j \in F_j(x)$, and $G_j(x, y, \hat{u_j}) \nsubseteq C_j(x)$;*
- (vi) *$h : X \times X \multimap W$ is an u.s.c. multivalued map with nonempty compact values, where W and D are the same as in Theorem 5.1.*

Then there exists a solution to the problem:

$Min_D h(x, y)$ subject to $(x, y) = ((x_i)_{i\in I}, (y_i)_{i\in I}) \in X \times X$ such that for each $i \in I$, $x_i \in \bar{A}_i(x)$, $y_i \in \bar{F}_i(x)$, and $G_i(x, y, v_i) \subset C_i(x)$ for all $v_i \in A_i(x)$.

Proof For each $i \in I$, let $L_i = \{(x, y) \in X \times X : (x, y) = ((x_i)_{i\in I}, (y_i)_{i\in I}), x_i \in \bar{A}_i(x), y_i \in \bar{F}_i(x)$, and $G_i(x, y, v_i) \subset C_i(x)$ for all $v_i \in A_i(x)\}$. Let $L = \cap_{i\in I} L_i$. Let $H_i : X \times Y \multimap Z_i$ be defined by $H_i(x, y) = (Z_i \setminus (C_i(x)))$ for each $(x, y) \in X \times X$. By Theorem 3.2, we can get $L \neq \emptyset$.

Since $C_i : X \multimap Z_i$ is closed. L_i is a closed set for each $i \in I$. Indeed, let $(x, y) \in \text{cl}(L_i)$, then there exists a net $(x^\alpha, y^\alpha) \in L_i$ such that $(x^\alpha, y^\alpha) \to (x, y)$. Let $(x^\alpha, y^\alpha) = ((x_i^\alpha), (y_i^\alpha))$ and $(x, y) = ((x_i)_{i\in I}, (y_i)_{i\in I})$. Then $x_i^\alpha \to x_i, y_i^\alpha \to y_i, x_i^\alpha \in \bar{A}_i(x^\alpha), y_i^\alpha \in \bar{F}_i(x^\alpha)$, and $G_i(x^\alpha, y^\alpha, v_i) \subset C_i(x^\alpha)$ for all $v_i \in A_i(x^\alpha)$. Let $v_i \in A_i(x)$. Since A_i is l.s.c., there exists a net $v_i^\alpha \in A_i(x^\alpha)$ for all α such that $v_i^\alpha \to v_i$. Hence $G_i(x^\alpha, y^\alpha, v_i^\alpha) \subset C_i(x^\alpha)$. Let $s_i \in G_i(x, y, v_i)$. Since G_i is l.s.c., there exists a net $\{s_i^\alpha\}_{\alpha\in\Lambda}$ such that $s_i^\alpha \in G_i(x^\alpha, y^\alpha, v_i^\alpha)$ for all α and $s_i^\alpha \to s_i$. Since $\bar{A}_i$, $\bar{F}_i$ and C_i are closed, we have $(x, y) \in X \times Y$, $x_i \in \bar{A}_i(x)$, $y_i \in \bar{F}_i(x)$ and $s_i \in C_i(x)$. This shows that $G_i(x, y, v_i) \subset C_i(x)$ for all $v_i \in A_i(x)$. Hence $(x, y) \in L_i$ and L_i is closed. Hence, L is closed. But by (v), $L \subseteq K \times M$. Therefore, L is compact. Then we follow the same argument as in Theorem 5.1, we can prove Theorem 5.2. □

Remark 5.1 It is easy to see that Theorems 5.1 and 5.2 are true, if $h : X \times Y \to \mathbb{R}$ is a l.s.c. function.

Theorem 5.3 *In Theorem 5.2, if conditions (ii), (iii), and (v) are replaced by* $(ii)_1$, $(iii)_1$ *and,* $(v)_1$, *where*

- $(ii)_1$ $W_i : X \multimap Z_i$ *defined by* $W_i(x) = Z\backslash(-intC_i(x))$ *is a closed multivalued map;*
- $(iii)_1$ *for each* $x = (x_i)_{i\in I} \in X$ *and* $y \in X$, $G_i(x, y, x_i) \cap (-intC_i(x)) = \emptyset$;
- $(v)_1$ *there exist nonempty compact subsets* K *and* M *of* X *and nonempty compact convex subsets* $\hat{D}_i$ *and* D_i *of* X_i, *for each* $i \in I$ *such that for each* $(x, y) \in (X \times X)\backslash(K \times M)$, *there exist* $j \in I$, $\hat{u}_j \in \hat{D}_j$, *and* $\hat{v}_j \in D_j$ *such that* $\hat{u}_j \in A_j(x)$, $\hat{v}_j \in F_j(x)$, *and* $G_j(x, y, \hat{u}_j) \cap (-intC_j(x)) \neq \emptyset$;

Then there exists a solution to the problem:
$Min_D h(x, y)$ *subject to* $(x, y) = ((x_i)_{i\in I}, (y_i)_{i\in I}) \in X \times X$ *such that for each* $i \in I$, $x_i \in \bar{A}_i(x)$, $y_i \in \bar{F}_i(x)$, *and* $G_i(x, y, u_i) \cap (-intC_i(x)) = \emptyset$ *for all* $u_i \in A_i(x)$.

Proof For each $i \in I$, let $L_i = \{(x, y) \in X \times X, (x, y) = ((x_i)_{i\in I}, (y_i)_{i\in I}), x_i \in \bar{A}_i(x), y_i \in \bar{F}_i(x)$, and $G_i(x, y, u_i) \subset Z_i \setminus (-\text{int}C_i(x))\}$, and $L = \cap_{i\in I} L_i$. Following the similar argument as in Theorem 5.2, we can prove Theorem 5.3. □

Theorem 5.4 *For each* $i \in I$, *let* $G_i : X \times X \times X_i \multimap Z_i$ *be a u.s.c multivalued map with nonempty compact values. For each* $i \in I$, *suppose that conditions (i), (iii) of Theorem 3.3, conditions (ii), (iv) are replaced by* $(ii)_2$, $(iv)_2$, *and assume that (v), where*

- $(ii)_2$ *for each* $x = (x_i)_{i\in I} \in X$ *and* $y \in X$, $G_i(x, y, x_i) \cap C_i(x) \neq \emptyset$;
- $(iv)_2$ *there exist nonempty compact subsets* K *and* M *of* X *and nonempty compact convex subsets* $\hat{D}_i$ *and* D_i *of* X_i *for each* $i \in I$ *such that for each* $(x, y) \in (X \times X)\backslash(K \times M)$, *there exist* $j \in I$, $\hat{u}_j \in \hat{D}_j$, *and* $\hat{v}_j \in D_j$ *such that* $\hat{u}_j \in A_j(x)$, $\hat{v}_j \in F_j(x)$, *and* $G_j(x, y, \hat{u}_j) \cap C_j(x) = \emptyset$;
- (v) $C_i : X \multimap Z_i$ *is a closed map with nonempty values;*
- (vi) $h : X \times X \multimap W$ *is an u.s.c. multivalued map with nonempty compact values, where* W *and* D *be the same as in Theorem 5.1.*

Then there exists a solution to the problem:
$Min_D h(x, y)$ *subject to* $(x, y) = ((x_i)_{i\in I}, (y_i)_{i\in I}) \in X \times X$ *such that for each* $i \in I$, $x_i \in \bar{A}_i(x)$, $y_i \in \bar{F}_i(x)$, *and* $G_i(x, y, u_i) \cap C_i(x) \neq \emptyset$ *for all* $u_i \in A_i(x)$.

Proof For each $i \in I$, let $L_i = \{(x, y) \in X \times X \mid (x, y) = ((x_i)_{i\in I}, (y_i)_{i\in I}), x_i \in \bar{A}_i(x), y_i \in \bar{F}_i(x)$, and $G_i(x, y, u_i) \cap C_i(x) \neq \emptyset$ for all $u_i \in A_i(x)\}$ and $L = \cap_{i\in I} L_i$. By Theorem 3.1, we can prove that $L \neq \emptyset$. Furthermore, by assumptions on C_i and G_i and Lemma 2.3, we can also show that L is a compact set. Following the same argument as in Theorem 5.1, we can prove Theorem 5.4. □

Remark 5.2

(a) Theorems 5.2–5.4 are true if $h : X \times Y \to \mathbb{R}$ is a l.s.c. function.
(b) Theorems 5.2–5.4 are different from any results in [11,14,15,18].

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