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Dual Gravitons in AdS_4/CFT_3 and the Holographic Cotton Tensor

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Abstract

We argue that gravity theories in AdS_4 are holographically dual to either of two three-dimensional CFT's: the usual Dirichlet CFT_1 where the fixed graviton acts as a source for the stress-energy tensor, and a dual CFT_2 with a fixed dual graviton which acts as a source for a dual stress-energy tensor. The dual stress-energy tensor is shown to be the Cotton tensor of the Dirichlet CFT. The two CFT's are related by a Legendre transformation generated by a gravitational Chern-Simons coupling. This duality is a gravitational version of electric-magnetic duality valid at any radius r, where the renormalized stress-energy tensor is the electric field and the Cotton tensor is the magnetic field. Generic Robin boundary conditions lead to CFT's coupled to Cotton gravity or topologically massive gravity. Interaction terms with CFT_1 lead to a non-zero vev of the stress-energy tensor in CFT_2 coupled to gravity even after the source is removed. We point out that the dual graviton also exists beyond the linearized approximation, and spell out some of the details of the non-linear construction.

Contents

1 Introduction

In AdS/CFT, the boundary conditions on the two independent modes of the graviton with, respectively, slow and fast fall-off at infinity encode the (representative of the conformal class of) boundary metrics and the one-point function of the stress-energy tensor of the CFT [1, 2]. For four-dimensional pure gravity, this is summarized in the holographic relation:

$$\langle T_{ij}(x)\rangle = \frac{3\ell^2}{16\pi G_N}\, g_{(3)ij}(x)\ , \tag{1}$$

that is, the third coefficient in the boundary expansion of the graviton $g_{ij}(r, x)$ is the renormalized holographic stress-energy tensor (see notation in Appendix A). The stress-energy tensor should be seen here as a functional computed as the CFT response to the background boundary metric $g_{(0)}$.

This result is obtained from the identification of the on-shell bulk action with the generating functional of connected correlation functions in the CFT, leading to:

$$\delta S[g_{(0)}] = (\mathrm{eom}) + \frac{1}{2}\int \mathrm{d}^3x\,\sqrt{g_{(0)}}\,\langle T_{ij}(x)\rangle\,\delta g^{ij}_{(0)}(x)\ . \tag{2}$$

The bulk action, including boundary terms and counterterms, is given in Appendix A. With Dirichlet boundary conditions, this gives the definition of the renormalized stress-energy tensor after sending the regulator to zero.

In the Neumann variational problem, the stress-energy tensor is fixed, in particular equation (2) sets it to zero. Mixed boundary conditions are obtained, in the case of scalar fields, by adding boundary terms that correspond to multiple-trace deformations of the CFT. For gravity, Neumann and mixed boundary conditions were not studied in the past, except for brane-world scenarios [3], for at least two reasons:

1) The Neumann and mixed variational problems do not always lead to a consistent bulk quantization scheme. This is due to the fact that generically the mode with fast fall-off is normalizable whereas the mode with slow fall-off is non-normalizable. Therefore, the latter must be fixed at the boundary, as in a Dirichlet boundary problem. Only in special cases, such as scalar fields in the range of masses $-\frac{d^2}{4} < m^2 < \frac{d^2}{4} + 1$, or if the theory has a cutoff, are the Neumann and mixed boundary conditions admissible.

2) A Neumann variational problem would naively identify the graviton as the dual operator, which does not make sense in a CFT_3. Indeed, if the quantity that is being held fixed is the canonical momentum –the renormalized holographic stress-energy tensor– rather than the asymptotic value of the metric, the former will act as a source for the boundary graviton since they are coupled in the on-shell bulk action. However, a spin two operator of dimension zero is below the unitarity bound. If the CFT is coupled to gravity and the graviton is integrated over, the non-zero expectation value of the graviton would not make sense at all, as a gravity theory does not have local operators. At best, it might make sense in some perturbative sense, where the CFT is expanded about a suitable background.

The resolution of the first problem has been known for some time. Ishibashi and Wald showed four years ago [4] that in four dimensions, both the Dirichlet and the Neumann boundary conditions lead to a well-posed initial value problem by showing that there is a suitable self-adjoint extension of a certain differential operator which plays the role of the Hamiltonian. This self-adjoint extension determines the asymptotic boundary conditions, so the freedom in choosing self-adjoint extensions of this operator corresponds to the freedom in choosing suitable boundary conditions at infinity. Their analysis showed that in four dimensions, the self-adjoing extensions are such that Dirichlet, Neumann boundary conditions as well as Robin boundary conditions are possible. In the latter case a linear combination of the leading and the subleading modes of the graviton is held fixed.

In this paper we propose a solution to the second problem above, giving the correct holographic interpretation of the Neumann and mixed problems. The key question we will address is: what are the observables of the dual theory? Here, it is useful to first recall the scalar field case in the special range of masses mentioned above, where duality interchanges sources in CFT_1 with one-point functions in CFT_2, and viceversa. Concretely, for Dirichlet quantization the mode ϕ_+ with fast fall-off corresponds to the dual operator $\langle\mathcal{O}_{(\Delta_+)}\rangle$ of dimension Δ_+, whereas the slow mode is a fixed source $\phi_- = J$. In the second quantization scheme, these roles are interchanged and ϕ_- corresponds to the one-point function of an operator of dimension Δ_- whereas ϕ_+ is held fixed. Deforming the CFT by a multiple-trace operator, one can flow towards the Dirichlet CFT [5, 6].

For gravity, the picture is more involved. Generically, the energy-momentum tensor does not depend on the boundary graviton. However, its expectation value does depend on it. This can be seen from the bulk via the regularity condition. Now in three dimensions one may classically associate to a given stress-energy tensor a graviton, constructed through the Cotton tensor. Suppose we have CFT_1 with fixed graviton h_{ij} and stress-energy tensor $\langle T_{ij}\rangle$. In three dimensions, given h_{ij}, we can map it to a conserved, traceless object of dimension 3, via the three-dimensional Cotton tensor (see Appendix C.2):

$$\langle\tilde{T}_{ij}\rangle = C_{ij}[h]\ . \tag{3}$$

On the other hand, given $\langle T_{ij}\rangle$ we derive from it, up to zero modes, a transverse, traceless tensor $\tilde{h}_{ij}$ of dimension zero:

$$\tilde{h}_{ij} = \frac{4\varepsilon}{\Box^3}\,C_{ij}(\langle T\rangle)\ , \tag{4}$$

where from now on $\varepsilon = -1$ in Lorentzian signature, and $\varepsilon = 1$ in the Euclidean. The two theories are related by a Legendre transform which is a gravitational version of electric-magnetic duality. Thus the bulk is dual to two possible CFT's, depending on whether the on-shell action is interpreted as the generator of connected correlation functions of the stress-energy tensor or as the effective action. In the absence of matter, the operators seen by the bulk are the correlators of the respective stress-energy tensors. Duality essentially interchanges CFT_1, with fixed source h_{ij} and stress-energy tensor $\langle T_{ij}\rangle$, with CFT_2, with dual source $\tilde{h}_{ij}$ and stress-energy tensor $\langle\tilde{T}_{ij}\rangle$. This dual graviton has opposite parity from the original one.

The bulk action connects both CFT's through the coupling (2). Using (3), the variation (2) produces the Cotton tensor and the resulting term becomes of the gravitational

Chern-Simons type connecting both gravitons. This is the term that generates the Legendre transformation. This is the same result as for abelian gauge fields, where the effective action is a BF-term that defines the S-duality operation. Notice that all this is true independent of the linearization.

The two fixed gravitons are related by the S-duality operation that will be defined later,

$$S(h_{ij}) = -\tilde{h}_{ij} \ . \tag{5}$$

This operation squares to minus one. It performs a Legendre transformation on the action, and hence modifies the holographic dictionary. The usual dictionary identifies the on-shell action with the generating functional of connected correlation functions of the stress-energy tensor in CFT_1. After S-duality, the bulk on-shell action can be defined to give the boundary effective action of CFT_2 instead.

There are mixed boundary conditions corresponding to coupling the theory to dynamical gravity and integrating over the graviton. To illustrate this, let us consider a slightly intermediate problem where the boundary theory is only coupled to Cotton gravity, and one integrates over the conformal metrics plus other degrees of freedom of the CFT. The mixed boundary condition corresponding to this fixes a linear combination of the boundary metric and the stress-energy tensor of the original theory. This means adding to the bulk action a gravitational Chern-Simons term $k/4\pi S_{\text{CS}}$ (defined in Appendix C.1) and taking the variation:

$$\delta S = (\text{eom}) + \frac{1}{2}\int \mathrm{d}^3x\, \sqrt{g}\left(\frac{3\ell^2}{2\kappa^2}\, g_{(3)ij} - \frac{k}{4\pi}\, C_{ij}(g)\right)\delta g^{ij} \ , \tag{6}$$

and recall that $g_{(3)}$ is the stress-energy tensor of the unperturbed Dirichlet theory, (1). The mixed boundary condition reads

$$g_{(3)ij} = \frac{k\kappa^2}{6\pi\ell^2}\, C_{ij} \tag{7}$$

where C_{ij} is the Cotton tensor. Linearizing, the analytic continuation of the Euclidean regularity condition (which will be derived in section 2) $\bar{h}_{(3)} = \frac{1}{3}\Box^{3/2}\bar{h}_{(0)}$ gives:

$$\Box^{1/2}\bar{h}_{(0)} = -\left(\frac{\kappa}{\ell}\right)^2 \frac{k}{6\pi}\, \epsilon_{ikl}\partial_k \bar{h}_{(0)jl} \ . \tag{8}$$

By dualizing this equation, we find that the coefficient on the right-hand side must be ± 1, and hence that the coefficient of the Chern-Simons term is given in terms of Newton's constant as:

$$k = \pm 6\pi\left(\frac{\ell}{\kappa}\right)^2 \ . \tag{9}$$

This value also corresponds to the gravitational instantons recently found in [7, 8]. We will show that equation (8) has solutions with non-trivial graviton profile. They are specified by a single parameter, which determines the value of the action, and in that sense they have no dynamics. The above illustrates the fact that the boundary graviton may have non-trivial configurations.

Obviously, (8) also admits $k = 0$ and $k = \infty$. $k = 0$ is simply the unperturbed theory with zero stress-energy tensor. At $k = \infty$ the gravitational Chern-Simons term dominates and we get Cotton gravity, i.e. zero Cotton tensor. This sets the transverse, traceless part of the graviton fluctuations to zero, thus getting back the Dirichlet boundary condition for the transverse, traceless graviton. For generic k as in (9), $g_{(3)}$ acts as a stress-energy tensor and the original CFT is coupled to Cotton gravity. Although the perturbation considered here is classically marginal, we do not get a marginal line of boundary conditions, but rather marginal points (9), $k = 0$, and $k = \infty$.

Instanton solutions (8) are topological, but one can get proper boundary dynamics by considering non-marginal perturbations, if one couples the CFT to dynamical gravity. The additional points (9) of the topological situation become a line and we are left with only two "fixed points", $k = 0$, $k = \infty$.

The degrees of freedom of boundary gravitons are then best described by two scalars γ and ψ, in the following way. It turns out that the transverse, traceless part of the graviton can be written as:

$$\bar{h}_{ij}(r,p) = \gamma(r,p)\, E_{ij}(p) - i\psi(r,p)\, D_{ij}(p) \tag{10}$$

where

$$\begin{aligned} E_{ij} &= \frac{\bar{p}^*_i \bar{p}^*_j}{\bar{p}^{*2}} - \frac{1}{2}\,\Pi_{ij} \\ D_{ij} &= \frac{1}{2p^3}\left(\bar{p}^*_i \epsilon_{jkl} + \bar{p}^*_j \epsilon_{ikl}\right) p_k \bar{p}^*_l \end{aligned} \tag{11}$$

are the two unique transverse, traceless tensors of rank two that can be constructed from p, p^*. They are each other's duals and will play an important role in the following, as they determine the index structure and duality properties of the metric.

The dual graviton $\tilde{h}$ has the same expansion as h:

$$\tilde{h}_{ij}(r,p) = \tilde{\gamma}(r,p)\, E_{ij}(p) - i\tilde{\psi}(r,p)\, D_{ij}(p)\ , \tag{12}$$

and duality interchanges $\gamma \leftrightarrow \tilde{\gamma}$, $\psi \leftrightarrow \tilde{\psi}$.

For generic Robin boundary conditions, the coupling between the two gravitons leads to a theory with non-zero stress energy tensor even in the absence of a source, whereas the vev in the original theory is zero.

The content of the paper is as follows. In section 2 we give the bulk solutions that are used later, including regularity of the Euclidean solution and the value of the on-shell action. In section 3, mixed boundary conditions are discussed and it is shown that they give rise to topologically massive gravity on the boundary. We solve the instanton equations explicitly and compute the on-shell value of the action. Section 4 is the main section, where we work out gravitational electric-magnetic duality: we show that it follows from a symmetry of the equations of motion, show that it acts as a Legendre transformation, we construct the dual graviton at the non-linear level, give the bulk interpretation and we discuss holography of the mixed boundary conditions. We find that the stress-energy tensor can spontaneously acquire a non-zero vev. We also compute two-point functions and discuss cosmological topologically massive gravity. In section 5 we discuss our results and give some future directions.

As this work was being completed, a nice preprint [9] appeared dealing with the first problem mentioned before. The results by Ishibashi and Wald were found to hold for $d \leq 4$ and were conjectured to hold for higher dimensions as well, the key issue being to take into account the counterterms that render the action finite in the definition of the symplectic structure. Very recent related work also appeared in [10].

2 Bulk dynamics

We will solve Einstein's equations perturbatively about the AdS_4 background. We take the standard Fefferman-Graham form of the metric:

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\ell^2}{r^2}\left(\mathrm{d}r^2 + g_{ij}(r,x)\mathrm{d}x^i\mathrm{d}x^j\right) \\ g_{ij}(r,x) &= \eta_{ij} + h_{ij}(r,x) \ , \end{aligned} \tag{13}$$

where the metric fluctuations have the following expansion:

$$h_{ij}(r,x) = h_{(0)ij}(x) + r^2 h_{(2)ij}(x) + r^3 h_{(3)ij} + \dots \tag{14}$$

In four dimensions there are no logarithmic terms and one can show that the linear term is absent. We have fixed the gauge $h_{rr} = h_{ir} = 0$. We will raise and lower indices with η_{ij} (δ_{ij} in the Euclidean) and denote $h = h_{ii}$. Einstein's equations about the linearized background take the form:

$$\begin{aligned} h'' - \frac{1}{r}\,h' &= 0 \\ \partial_j h'_{ij} - \partial_i h' &= 0 \\ h''_{ij} - \frac{2}{r}\,h'_{ij} + \Box h_{ij} + \partial_i\partial_j - \partial_i\partial_k h_{jk} - \partial_j\partial_k h_{ik} - \frac{1}{r}\,h'\eta_{ij} &= 0 \ , \end{aligned} \tag{15}$$

where the primes denote r-derivatives. We immediately find that the trace and transverse parts of the graviton are quadratic in r,

$$\begin{aligned} h(r,x) &= h_{(0)}(x) + r^2\,h_{(2)}(x) \\ \partial_j h_{ij}(r,x) &= \partial_j h_{(0)ij}(x) + r^2\,\partial_j h_{(2)ij}(x) \ , \end{aligned} \tag{16}$$

and all higher-order coefficients are conserved and traceless, $h_{(n)ii} = \partial_j h_{(n)ij} = 0$ for $n > 2$. Using this, we can rewrite the last line in (15) as an equation for purely the transverse, traceless part of the graviton defined in (118):

$$\bar{h}'_{ij} - \frac{2}{r}\,\bar{h}'_{ij} + \Box\bar{h}_{ij} = 0 \ . \tag{17}$$

In Euclidean signature, the general solution is:

$$\bar{h}_{ij}(r,p) = f_1(r,p)\,a_{ij}(p) + f_{(3)}(r,p)\,b_{ij}(p) \ , \tag{18}$$

where f_1 and f_3 are the two independent solutions of the above equation written in momentum space:

$$\begin{aligned} f_1(r,p) &= \cosh pr - pr\sinh pr \\ f_3(r,p) &= -\sinh pr + pr\cosh pr \ . \end{aligned} \tag{19}$$

Near the boundary we recover the expansion (14) in the distance to the boundary:

$$\begin{aligned} \bar{h}_{ij}(r,p) &= \left(1 - \frac{1}{2}\, p^2 r^2 + \ldots\right) \bar{h}_{(0)ij}(p) + (r^3 + \ldots)\, \bar{h}_{(3)ij} \\ \bar{h}_{(0)ij}(p) &= a_{ij}(p) \\ \bar{h}_{(2)ij}(p) &= -\frac{1}{2}\, p^2 \bar{h}_{(0)ij} \\ \bar{h}_{(3)ij}(p) &= \frac{p^3}{3}\, b_{ij}(p) \ . \end{aligned} \tag{20}$$

The series decomposes into two independent series with even and odd powers of r. $h_{(0)}$ multiplies even powers and $h_{(3)}$ multiplies the odd powers. By (1), $h_{(3)}$ is the holographic stress-energy tensor. The remaining non-transverse, traceful parts of the metric are given in (143).

It is convenient for later use to rewrite the above in the form (10) where the two independent components of the metric are made completely explicit. Demanding that (10) satisfies the bulk equations of motion, i.e. that it has the form (18) (or its Lorentzian counterpart), we get the following expansion:

$$\begin{aligned} \gamma(r,p) &= \gamma(p) f_0(r,p) + \delta(p) f_3(r,p) \\ \psi(r,p) &= \psi(p) f_0(r,p) + \chi(p) f_3(r,p) \ . \end{aligned} \tag{21}$$

The first and third coefficients then read:

$$\begin{aligned} \bar{h}_{(0)ij}(p) &= \gamma(p)\, E_{ij}(p) - i\psi(p)\, D_{ij}(p) \\ \bar{h}_{(3)ij}(p) &= \frac{\varepsilon}{3} |p|^3 \left(\delta(p)\, E_{ij}(p) - i\chi(p)\, D_{ij}(p)\right) \ . \end{aligned} \tag{22}$$

In Appendix B.1 we give the main properties of the tensors E, D.

2.1 Regularity and on-shell action

The above Euclidean solution blows up exponentially at $r = \infty$ unless $a_{ij} = b_{ij}$, which gives the regularity condition:

$$h_{(3)ij}(x) = \frac{1}{3}\, |\Box|^{3/2} h_{(0)ij} \ , \tag{23}$$

hence relating the one-point function of the dual operator to the source, as usual.

In the Lorentzian, the solution oscillates and there is no regularity condition of the type found for Euclidean solutions:

$$h_{ij}(r,p) = a_{ij}(p)\left(\cos(|p|r) + |p|r\sin(|p|r)\right) + b_{ij}(p)(|p|r\cos(|p|r) - \sin(|p|r)) \ . \tag{24}$$

The on-shell action can be easily obtained as follows. From the definition of the stress-energy tensor (2), using $h_{(3)ij}(p) = f(p)\,h_{(0)ij}(p)$ for *arbitrary* $f(p)$ (of which the regular solutions (23) are a special case), and integrating, we get

$$S_{\text{on-shell}} = \frac{3\ell^2}{8\kappa^2}\int \mathrm{d}^3x\,\bar{h}_{(0)ij}\bar{h}_{(3)ij} =: W[h_{(0)}]\ , \tag{25}$$

where gauge-dependent terms do not contribute, as can be checked using the results in Appendix D. This expression is independent of the choice of linear boundary condition, that is the choice of $f(p)$; the fact that the relation is linear uniquely specifies the on-shell value of the action up to quadratic order [11]. The same result is obtained expanding the action as in [12] and subtracting the divergent part with the usual counterterms [2].

3 Dynamical gravity on the boundary

One important difference between fields of spin 1 and 2, and scalar fields in AdS_4, is that the natural linear boundary conditions preserving conformal invariance involve derivatives rather than being algebraic. Generalized boundary conditions involving derivatives therefore may have zero modes that propagate purely on the boundary and lead to boundary degrees of freedom. This was first pointed out for abelian gauge fields in [11]; for gravity, it was pointed out in [13] and [9]. Instanton boundary conditions for $U(1)$ gauge fields involve a parity-odd operator which is linear in spatial derivatives [11]; instanton boundary conditions for the graviton involve a parity-odd, dimension three operator, the Cotton tensor, and they correspond to coupling the boundary theory to Cotton gravity, as explained in the introduction. More general boundary conditions may be obtained by coupling the theory to Einstein gravity. This will be worked out in this section and the next.

3.1 Linear boundary conditions

The simplest linear boundary condition reads:

$$\Box^{1/2}\bar{h}_{(0)ij} = \pm\,\epsilon_{ikl}\partial_k\bar{h}_{(0)jl}\ . \tag{26}$$

As in (7), this can be obtained adding a gravitational Chern-Simons term to the action. It may also be obtained from bulk configurations as in [7, 8], by considering solutions with self-dual Weyl tensor:

$$C_{\mu\nu\alpha\beta} = \frac{1}{2}\,\epsilon_{\mu\nu\lambda\sigma}\,C_{\alpha\beta}{}^{\lambda\sigma}\ . \tag{27}$$

Asymptotically, this leads to (26).

The linear boundary condition (26) can be written in terms of the curvature tensor as

$$\bar{R}_{ij} = \mp\frac{1}{\Box^{1/2}}\,C_{ij}\ , \tag{28}$$

where $\bar{R}_{ij} = \Pi_{ij}{}^{kl} R_{kl}$, and Π_{ijkl} is the spin-two projector defined in Appendix C.2. Hence, only the transverse, traceless part of the curvature is involved. We will now solve this equation. It is useful to first rewrite it as a self-duality equation

$$h_{ij} = \pm d_{ij}[h] \tag{29}$$

for some operator d. One can formally give the following non-local expression for it:

$$d = \frac{2}{(-\varepsilon \Box)^{3/2}} C \ , \tag{30}$$

where C is the Cotton tensor. Notice that the operator in the denominator is positive definite in both signatures. One easily checks that d squares to one,

$$d^2 = 1 \ . \tag{31}$$

Local expressions are obtained by multiplying both sides with the Laplacian; the above definition, though formal, is very useful for book-keeping.

The most general self-dual solution of (26) then takes the following form:

$$\bar{h}_{ij} = E_{ij} \pm d_{ij}[E] \ , \tag{32}$$

where E_{ij} is a general transverse, traceless tensor. By construction, the solution is (anti)self-dual. E_{ij} can be constructed uniquely up to an overall factor. This is done in Appendix B. In momentum space, up to an overall factor it takes the form:

$$E_{ij} = \frac{\bar{p}^*_i \bar{p}^*_j}{\bar{p}^{*2}} - \frac{1}{2} \Pi_{ij} \ . \tag{33}$$

In general, we may expand the graviton as

$$\bar{h}_{(0)ij} = \gamma E_{ij} + \frac{1}{p} \psi' \epsilon_{ikl} p_k E_{jl} \tag{34}$$

where $\bar{p}^*_i$ is the transverse projection of $p^*_i = (-E, \vec{p})$, and Π_{ij} is the transverse projector.

3.2 Topologically massive gravity

Instanton boundary conditions give the traceless, transverse part of three-dimensional topologically massive gravity, (26). We can get the full topologically massive gravity by coupling it to Einstein gravity with Newton's constant $1/\mu$. The boundary condition for the resulting Neumann problem is:

$$R_{ij}[g_{(0)}] - \frac{1}{\mu} C_{ij}[g_{(0)}] = \frac{3\ell^2}{4\mu\kappa^2} g_{(3)ij} \ , \tag{35}$$

where we have used the fact that the curvature scalar is zero, $R = 0$. $g_{(3)ij}$ acts here as a source for Einstein's equations. It is as a function of the boundary metric $g_{(0)}$ determined by the particular CFT state. At the linearized level, and in the ground state, its functional form is given by analytic continuation from the regular Euclidean solutions (23). For general states, and at the non-linear level, the relation will differ from that.

In the purely Neumann quantization scheme the stress-tensor vanishes while still getting interesting boundary dynamics. For simplicity we now set $g_{(3)} = 0$. It is easy to check that by virtue of $R = 0$ the theory is, at the non-linear level, described by a conserved, traceless tensor –the curvature– satisfying:

$$\left(\Box + \epsilon\mu^2\right) R_{ij} - 3\left(R^2_{ij} - \frac{1}{3}\, g_{ij}\mathrm{Tr}(R^2)\right) = 0 \ , \tag{36}$$

where $R^2_{ij} := R_{ik}R^k_j$. Later on this equation will be generalized to the cosmological setting.

3.3 On-shell action for instantons

Instantons in gauge theory have two important properties. They are topological, i.e. they have vanishing stress-energy tensor and zero Hamiltonian; secondly, their topological class is characterized by an integer k giving the on-shell value of the action.

Bulk gravitational instantons of the type (27) have similar properties as we will now show. The self-duality equation (26) imposes $\psi' = \pm\gamma$ in (34), therefore

$$\bar{h}_{(0)ij} = \gamma\left(E_{ij} \pm \frac{1}{p}\,\epsilon_{ikl}p_k E_{jl}\right) \ , \tag{37}$$

where the metric is (anti-)self-dual.

The on-shell action can be easily computed from the bulk up to second order in the perturbations. Notice that for a general solution (34), we have

$$h_{(0)ij}(p)h_{(0)ij}(-p) = \frac{1}{2}\left(\gamma^2 + \varepsilon\psi'^2\right) \ , \tag{38}$$

where $\varepsilon = 1$ in the Euclidean, $\varepsilon = -1$ in Lorentzian signature. The Euclidean on-shell action is

$$S_{\text{on-shell}} = \frac{3\ell^2}{8\kappa^2}\int \mathrm{d}^3x\, \bar{h}_{(0)ij}(x)\bar{h}_{(3)ij}(x) = \frac{\pi^2\ell^2}{8G_N}\, K \ , \tag{39}$$

with $K = \int \mathrm{d}^3p\, |p|^3\, \gamma(p)\gamma(-p)$. γ should be chosen such that this integral is finite. On the other hand, the Hamiltonian is zero. This result is similar to the one for $U(1)$ gauge fields [11] in AdS_4. It would be interesting to see whether there are any topological restrictions on the possible values of K.

4 Duality

We have shown that in three dimensions, given a stress-energy tensor $\langle T_{ij}\rangle_h$, we can define a graviton via the Cotton tensor: $\langle T_{ij}\rangle_h = C_{ij}[\tilde{h}]$. A priori, this graviton does not satisfy any dynamics but is given by the response of the one-point function in a particular state to the fixed graviton source h_{ij}. Given h_{ij}, we can also construct a new conserved, traceless, dimension three tensor using the Cotton tensor. The question is whether we may interpret the latter as a stress-energy tensor $\langle\tilde{T}_{ij}\rangle$ in some dual theory.

From the bulk point of view, the interchange of h_{ij} and $\tilde{h}_{ij}$ corresponds to a change in boundary conditions from Dirichlet to Neumann (and, as we have seen, there are also massive interpolating solutions). In this section we will first show that the symmetry interchanging h_{ij} and $\tilde{h}_{ij}$ is a duality of the electric-magnetic type, for any value of the radial coordinate r and not only at the boundary. The magnetic variable is the Cotton tensor, and the electric variable turns out to be the renormalized holographic stress-energy tensor. We will explicitly show the symmetry of the bulk equations of motion that leads to this result. Then we will show that CFT_1 and CFT_2 are related by a Legendre transformation which allows us to identify the generating functional of one theory with the effective action of the other under the familiar dictionary [14, 15], thus showing $C_{ij}[h]=\langle\tilde{T}_{ij}\rangle$.

4.1 Duality symmetry of the equations of motion

We will identify the duality symmetry of the bulk equations of motion that allows us to define electric and magnetic components of the graviton. It will be useful to first recall the case of abelian gauge fields in the bulk of AdS [11]. The solution of the bulk equations of motion for a Maxwell field is:

$$\bar{A}_i(r,p)=A_i(p)\cos(|p|r)+\frac{1}{|p|}\,E_i(p)\sin(|p|r)\ . \tag{40}$$

In Coulomb gauge, the bulk electric and magnetic fields are

$$\begin{aligned}B_i(r,x)&:=\epsilon_{ijk}\partial_j\bar{A}_k(r,x)\\ E_i(r,x)&:=\partial_r\bar{A}_i(r,x)\ .\end{aligned} \tag{41}$$

Given an electric-magnetic change of boundary onditions, $B'_i(x)=-E_i(x)$, $E'_i(x)=B_i(x)$, one readily checks that the bulk fields (41) also satisfy:

$$\begin{aligned}B'_i(r,x)&=-E_i(r,x)\\ E'_i(r,x)&=B_i(r,x)\ .\end{aligned} \tag{42}$$

This is a non-trivial property of the equations of motion. $E_i(x)$ and $B_i(x)$ are two independent quantities on the boundary, namely the one-point function of a holographic one-point function of a global symmetry current, and the (curl of) the corresponding source. Their bulk extensions (41) are related to each other via the first r- and boundary spatial derivatives: $E(r,x)=\partial_r\left(\frac{1}{\Box^{1/2}}*\mathrm{d}B(r,x)\right)$. This implies $B(r,x)=\partial_r\left(\frac{1}{\Box^{1/2}}*\mathrm{d}B'(r,x)\right)$, which holds by virtue of the equation of motion, $B(r,x)=B(p)\cos(|p|r)+|p|A'(p)\sin(|p|r)$, where the r-derivative exchanges the two oscillating branches.

In the gravity case, we need to construct bulk electric and magnetic quantities which when interchanged give solutions of the equations of motion. It is a priori not entirely obvious which are the correct quantities, as the electric and magnetic boundary conditions interchange just the boundary conditions $h_{(0)}$ and $\tilde{h}_{(0)}$. And even though these quantities readily extend to bulk fields $\bar{h}_{ij}(r,p) = f_0(r,p)\,\bar{h}_{(0)ij}$, $\tilde{h}_{ij}(r,p) = f_3(r,p)\,\tilde{h}_{(0)ij}$ satisfying the equations of motion, electric-magnetic variables should belong to the *same* CFT.

Based on what he have already said, one may readily expect that electric-magnetic duality exchanges the stress-energy tensor at radius r with the Cotton tensor at radius r. Indeed, the pair $(\langle T_{ij}\rangle, C_{ij}[\tilde{h}])$, rather than $(h_{ij}, \tilde{h}_{ij})$, belong to the same CFT. How do we check that this is a symmetry of the equations of motion, as in (42)?

The Cotton tensor has dimension 3. It is therefore natural to compare it with the third time derivative of the graviton (where "time" is the r-direction). Writing (24) as

$$\bar{h}_{ij}[a,b] = a_{ij}(p)(\cos(|p|r) + |p|r\sin(|p|r)) + b_{ij}(p)(|p|r\cos(|p|r) - \sin(|p|r)) \ , \tag{43}$$

one easily finds by inspection

$$\bar{h}'''_{ij}[-b,a] = |p|^3\bar{h}_{ij}[a,b] - 3\,\frac{|p|}{r}\,\bar{h}'_{ij}[a,b] \ . \tag{44}$$

This is the basic symmetry that allows to define electric-magnetic duality in the bulk. Expressing $\bar{h}'''_{ij}$ in lower derivatives via the quations of motion, $\bar{h}'''_{ij} = \frac{2}{r^2}\,h'_{ij} - |p|^2\bar{h}'_{ij} - \frac{2|p|^2}{r}\,\bar{h}_{ij}$, we get

$$\begin{aligned} -\bar{h}_{ij}[-b,a] &= -\frac{1}{|p|^3r^2}\,\bar{h}'_{ij}[a,b] + \frac{1}{|p|r}\,\bar{h}_{ij}[a,b] - \frac{1}{|p|}\,\bar{h}'_{ij}[a,b] \\ &=: \frac{1}{|p|^3}\,P_{ij}[a,b] \end{aligned} \tag{45}$$

P_{ij} becomes, at the boundary, the stress-energy tensor: $P_{ij}(0,p) = -3h_{(3)}$. In fact, we have for any r:

$$\langle T_{ij}(x)\rangle_r = -\frac{\ell^2}{2\kappa^2}\,P_{ij}(r,x) - \frac{\ell^2}{2\kappa^2}\,|p|^2\bar{h}'_{ij}(r,x) \ , \tag{46}$$

and so they differ by a local term that vanishes at the boundary. Thus, up to this finite renormalization, $P_{ij}(r,x)$ is the renormalized stress-energy tensor at radius r.

However, (45) is still not the right symmetry, which should interchange the two gravitons, $h_{(0)}$ and $\tilde{h}_{(0)}$ instead of a_{ij} and b_{ij}. One easily finds

$$\begin{aligned} 2C_{ij}(\bar{h}[-\tilde{a},a]) &= -|p|^3P_{ij}[a,\tilde{a}] \\ 2C_{ij}(P[-\tilde{a},a]) &= +|p|^3\bar{h}_{ij}[a,\tilde{a}] \end{aligned} \tag{47}$$

This is the form electric-magnetic duality takes in the bulk. It may be obtained as a combination of the operation $d = 2C/\Box^{3/2}$ defined earlier, and the discrete operation $s(a) = -b$, $s(b) = a$. We will denote it by $S := sd$. It acts as expected:

$$\begin{aligned} S(\bar{h}_{(0)}) &= -\tilde{h}_{(0)} \\ S(\tilde{h}_{(0)}) &= +h_{(0)} \ . \end{aligned} \tag{48}$$

We may now define the electric and magnetic variables

$$\begin{aligned} \mathcal{E}_{ij}(r,x) &= -\frac{\ell^2}{2\kappa^2}\,P_{ij}(r,x) \\ \mathcal{B}_{ij}(r,x) &= \frac{\ell^2}{\kappa^2}\,C_{ij}[\bar{h}(r,x)] \ , \end{aligned} \tag{49}$$

such that

$$\begin{aligned} \mathcal{E}_{ij}(0,x) &= \langle T_{ij}(x) \rangle \\ \mathcal{B}_{ij}(0,x) &= \frac{\ell^2}{\kappa^2}\, C_{ij}[\bar{h}_{(0)}] \ . \end{aligned} \tag{50}$$

S-duality then acts as

$$\begin{aligned} S(\mathcal{B}) &= -\mathcal{E} \\ S(\mathcal{E}) &= +\mathcal{B} \ , \end{aligned} \tag{51}$$

and obviously $S^2 = -1$. Thus, *gravitational S-duality interchanges the renormalized stress-energy tensor with the Cotton tensor at radius r*.

There is a second important possibility, which is defining $\mathcal{B}(r,x) = 2C_{ij}[\bar{h}]$ without the factor of ℓ^2/κ^2. In that case, h and $\tilde{h}$ still satisfy (48), except the coupling ℓ/κ now also transforms under S-duality as $\ell'/\kappa' = \pm 2\kappa/\lambda$. This possibility of inverting Newton's constant was pointed out in the electromagnetic case [11], and in the gravitational case in [16]. Of course, such a transformation brings one out of the supergravity regime, and one may only trust the resulting solution if the original one is exact.

4.2 The gravitational Legendre transformation

The analysis of Ishibashi and Wald [4] implies that, as far as boundary conditions are concerned, the four-dimensional graviton behaves in a similar way to scalars in the range of masses $-\frac{d^2}{4} < m^2 < -\frac{d^2}{4} + 1$ ($d = 3$). The one-parameter line of mixed boundary conditions mimicks the IR flow of boundary CFT's deformed by higher-trace operators. The two fixed points are characterized by two CFT's related by a Legendre transformation.

The gravity picture is quite different because the CFT is deformed by coupling it to gravity. However, electric-magnetic duality still relates the two "fixed points", the Dirichlet and Neumann problems. We will now show that electric-magnetic duality acts on the action as a Legendre transformation. For comparison, it is useful to recall here how this works for scalar fields. In the usual CFT with an operator of dimension Δ_+, ϕ_+ is the fast decaying bulk mode whereas ϕ_- is the slowly decaying mode which is fixed at the boundary. The bulk on-shell action is holographically identified with the generating functional of the boundary CFT as a function of this mode: $S_{\text{on-shell}}[\phi_-] = -W[J]$ with $\phi_- = J$. Then, up to contact terms, we have [2]:

$$\langle \mathcal{O}_{(\Delta_+)} \rangle_J = -\frac{\delta W[J]}{\delta J} = -(\Delta_+ - \Delta_-)\, \phi_+ \ . \tag{52}$$

The dual theory is then obtained by first defining the Legendre transformation [17]:

$$\mathcal{W}[\phi_-, \phi_+] = W[\phi_-] + \int \mathrm{d}^3x \,\sqrt{g_{(0)}}\, \phi_+(x)\phi_-(x) \ . \tag{53}$$

Extremizing this functional with respect to ϕ_-, $\frac{\delta W}{\delta \phi_-} + \phi_+ = 0$, we get a solution $\phi_- = \phi_-[\phi_+]$. The dual generating functional is now defined by evaluating $\mathcal{W}$ at the extremum so it becomes purely a function of ϕ_+,

$$\tilde{W}[\phi_+] = \mathcal{W}[\phi_-[\phi_+], \phi_+] = W[\phi_-]| + \int \mathrm{d}^3x\, \sqrt{g_{(0)}}\, \phi_-\phi_+| \ , \tag{54}$$

where the right-hand side is evaluated at the extremum. Thus, the generating functional of the dual theory is identified with the effective action of the original theory, but now $\phi_+ = \tilde{J}$ is fixed. The dual operator is:

$$\langle \tilde{\mathcal{O}}_{(\Delta_-)} \rangle_{\tilde{J}} = \frac{\delta \tilde{W}[\phi_+]}{\delta \phi_+} = \phi_- \ . \tag{55}$$

In [11] it was shown that for abelian gauge fields, a Chern-Simons term generates a Legendre transformation between a theory with a global current of dimension 2, and a theory with a dual current constructed from a dual gauge field.

The gravitational electric-magnetic analog of the above involves a gravitational Chern-Simons term, as we show next. In the usual CFT, $T_{ij} = \frac{2}{\sqrt{g}}\frac{\delta W}{\delta g^{ij}}$ where $W[g]$ is the on-shell bulk action. We construct the dual CFT in the usual way:

$$\mathcal{W}[g, \tilde{g}] = W[g] + V[g, \tilde{g}] \ . \tag{56}$$

We want to calculate $V[g, \tilde{g}]$. At the extremum, $\frac{\delta \mathcal{W}}{\delta g^{ij}} = 0$, we have $\frac{1}{\sqrt{g}}\frac{\delta V}{\delta g^{ij}} = -\frac{1}{2}\langle T_{ij}\rangle$, where the last variation is simply a partial derivative. Linearizing and using the fact that $\langle T_{ij}\rangle$ can be written as:

$$\langle T_{ij}\rangle = \frac{\ell^2}{\kappa^2}\, C_{ij}[\tilde{h}] \ , \tag{57}$$

we get

$$V[h, \tilde{h}] = -\frac{\ell^2}{2\kappa^2}\int \mathrm{d}^3x\, h^{ij} C_{ij}[\tilde{h}] = -\frac{\ell^2}{2\kappa^2}\int \mathrm{d}^3x\, \tilde{h}^{ij} C_{ij}[h] \ . \tag{58}$$

Since $C_{ij} = \frac{1}{\sqrt{g}}\frac{\delta S_{\mathrm{CS}}}{\delta g^{ij}}$, V can be rewritten as the variation of the gravitational Chern-Simons term:

$$V[h, \tilde{h}] = -\frac{\ell^2}{2\kappa^2}\int \mathrm{d}^3x\, h^{ij}\frac{\delta^2 S_{\mathrm{CS}}[g]}{\delta g^{ij}\delta g^{kl}}\,\tilde{h}^{kl} =: S_{\mathrm{CS}}[h, \tilde{h}] \ . \tag{59}$$

We may now define

$$\tilde{W}[\tilde{h}] = W[h] + V[h, \tilde{h}] \ , \tag{60}$$

where h is an extremum of the action. In the case at hand, using (25):

$$W[h] = \frac{\ell^2}{8\kappa^2}\int \mathrm{d}^3x\, h_{(0)ij}\Box^{3/2} h_{(0)ij} \ , \tag{61}$$

we get back the regularity condition $h_{(3)} = \frac{1}{3}\,\Box^{3/2} h_{(0)}$, as we should. The latter is rewritten as $C_{ij}(\tilde{h}) = \frac{1}{2}\,\Box^{3/2} h_{ij}$. In this case, the dual functional is:

$$\tilde{W}[\tilde{h}] = -\frac{\ell^2}{8\kappa^2}\int \mathrm{d}^3x\, \tilde{h}_{(0)ij}\Box^{3/2}\tilde{h}_{(0)ij} \ . \tag{62}$$

We can now compute the dual stress-energy tensor:

$$\langle \tilde{T}_{ij} \rangle = -2\,\frac{\delta \tilde{W}[\tilde{h}]}{\delta \tilde{h}^{ij}} = \frac{\ell^2}{\kappa^2}\, C_{ij}[g]\ . \tag{63}$$

From (50), we find that the dual stress-energy tensor is actually the magnetic field:

$$\langle \tilde{T}_{ij}(x) \rangle_{\mathrm{CFT}_2} = \mathcal{B}_{ij}(0,x) = \mathcal{E}'_{ij}(0,x) = S\left(\langle T_{ij}(x)\rangle_{\mathrm{CFT}_1}\right)\ , \tag{64}$$

and this definition extends in the natural way to any finite r.

The same result can be obtained by applying S-duality (48) directly on the generating functional (25):

$$W'[\tilde{h}] := S(W[h]) = W[h] - \frac{\ell^2}{2\kappa^2}\int \mathrm{d}^3x\, \bar{h}_{ij} C_{ij}[\tilde{h}]\ . \tag{65}$$

Given that the relation between the generating functionals of CFT_1 and CFT_2 is a Legendre transformation, and since the graviton in one theory becomes after differentiation the stress-energy tensor of the dual theory, we may identify the generating functional of one theory with the effective action of the other:

$$\begin{aligned} \Gamma[\langle T_{ij}\rangle(\tilde{h})] &= +\tilde{W}[\tilde{h}] \\ \tilde{\Gamma}[\langle \tilde{T}_{ij}\rangle(h)] &= -W[h]\ . \end{aligned} \tag{66}$$

Thus, in the dual theory the holographic dictionary is modified. This modification is as in the case of duality for spin-zero and spin-one fields in the special range of masses [14, 11].

4.3 The non-linear dual graviton

The above considerations can be extended to the non-linear theory. Let us first consider the issue of non-linear boundary conditions. Generic non-linear Neumann boundary conditions take the form:

$$\frac{3\ell^2}{4\kappa^2}\, g_{(3)ij} = \frac{\ell^2}{2\kappa^2}\, C_{ij}[\tilde{g}] = \mu\left(R_{ij}[g] - \frac{1}{2}\, g_{ij} R[g] + \lambda\, g_{ij}\right) - C_{ij}[g]\ . \tag{67}$$

λ is a boundary cosmological constant, which we will take to be positive $\lambda > 0$. As we will now show, this equation arises as a Neumann boundary condition of the bulk action deformed by boundary terms; therefore, it should be read as determining $g_{(3)}$ in terms of $g_{(0)}$. It must be supplemented by a regularity condition (in Euclidean signature) or by a specification of the state (in Lorentzian signature). These may take the form of a relation between $g_{(3)}$ and $g_{(0)}$, as we found in the linear case (see e.g. (23)); in the non-linear case this relation will be very involved (see [18] for progress in this direction). In turn, because (67) is a differential rather than an algebraic equation, the metric $g_{(0)}$ satisfies a dynamical equation. For $g_{(3)} = 0$, this is the topologically massive gravity we encountered earlier. Thus, we modify the action with terms:

$$S = S_{\mathrm{EH}} + \frac{\mu\ell^2}{4\kappa^2}\int \mathrm{d}^3x\sqrt{g}\,(R[g] - 2\lambda) - \frac{\ell^2}{4\kappa^2}\, S_{\mathrm{cs}}\ , \tag{68}$$

and $S_{\rm EH}$ is the bulk Einstein-Hilbert action, including the Hawking-Gibbons term and counterterms, given in (97).

It is important to notice the role of the bulk Einstein-Hilbert term here. Effectively, it acts as a Chern-Simons coupling between $\tilde{g}$ and g. Indeed, using the non-linear version of (57), we find in the on-shell action:

$$\delta S_{\rm EH} = \frac{\ell^2}{2\kappa^2}\int \mathrm{d}^3x\sqrt{g}\,\delta g^{ij}C_{ij}[\tilde{g}]\ . \tag{69}$$

Thus, the bulk produces for us a conformal, diffeomorphism invariant, local Lorentz invariant and fully non-linear coupling between the two gravitons. This is analogous to the situation with (purely linear) Abelian gauge fields, see the discussion in [19, 11].

For non-abelian theories, it is well-known that there are obstructions to the classical electric-magnetic duality. As a prerequisite for the existence of a duality transformation at the non-linear level in the gravity case, it is crucial that the existence of the dual graviton $\tilde{g}_{ij}$ can be established beyond the linear approximation. The former question will not be addressed in this paper but we will now argue that the latter is the case.

Given a dual graviton $\tilde{g}_{ij}$ we can always construct a dual stress-energy tensor using the non-linear Cotton tensor:

$$\langle T_{ij}\rangle = \frac{\ell^2}{\kappa^2}\,C_{ij}[\tilde{g}]\ , \tag{70}$$

which is automatically traceless and conserved. We need to answer the opposite question: given the stress tensor $\langle T_{ij}\rangle$, can we associate to it a dual graviton $\tilde{g}_{ij}$? In other words, is (70) true for any CFT? This is the case if the Cotton tensor can be inverted, which up to zero modes is perturbatively the case around an appropriate background. Dualizing (70), we get

$$\varepsilon\,\tilde{\epsilon}_i{}^{kl}\tilde{\nabla}_k\langle T_{lj}\rangle = -\Box\tilde{R}_{ij}+\frac{1}{4}\tilde{\nabla}_i\tilde{\nabla}_j\tilde{R}+\frac{1}{4}\,\tilde{g}_{ij}\Box\tilde{R}+3\tilde{R}^2_{ij}-\tilde{g}_{ij}\,\mathrm{Tr}\tilde{R}^2-\frac{1}{2}\left(3\tilde{R}_{ij}-\tilde{g}_{ij}\tilde{R}\right)\tilde{R}\ . \tag{71}$$

This equation should be regarded as determining $\tilde{g}_{ij}$, given the left-hand side. Finding $\tilde{g}_{ij}$ is thus equivalent to solving Einstein's equations. We had already established that, at the linear order, the above equation has solutions, given by (4). The above equation now also gives the higher order terms in a systematic way. Thus, if perhaps not unique in the presence of zero modes, $\tilde{g}$ exists at the non-linear level.

4.4 Bulk interpretation

The role of the Chern-Simons term (59), like the AB-Chern Simons terms of [19], is to Legendre transform a theory with stress-energy tensor $\langle T_{ij}\rangle$ where h_{ij} is fixed, into a theory with stress-energy tensor $\langle\tilde{T}_{ij}\rangle$ where $\tilde{h}_{ij}$ is fixed. From the bulk point of view, it transforms the Dirichlet problem into a Neumann boundary problem, as we have already seen. It is instructive to rewrite the above formulas in terms of the bulk quantities:

$$Z[g] = \int_g \mathcal{D}G_{\mu\nu}\,e^{-S[G]}\ , \tag{72}$$

such that the metric at the boundary approaches g_{ij}. Expanding about a flat boundary metric, taking the Legendre transform and using (56)-(60) gives:

$$\int \mathcal{D}h_{ij}\, Z[h]\, e^{V[h,\tilde{h}]} = \int \mathcal{D}h_{ij}\, e^{\mathcal{W}[h,\tilde{h}]} = e^{\tilde{W}[\tilde{h}]} =: \tilde{Z}[\tilde{h}] \ , \tag{73}$$

where we have taken the saddle-point approximation for $\mathcal{W}[h,\tilde{h}]$, obtaining the dual functional. Thus, we find

$$\tilde{Z}[\tilde{h}] = \int \mathcal{D}h_{ij}\, e^{S_{\text{cs}}[h,\tilde{h}]}\, Z[h] \ , \tag{74}$$

and the gravitational Chern-Simons term indeed transforms from Dirichlet to Neumann. This relation is of course inverted by Legendre transforming back:

$$\tilde{\mathcal{W}}[h,\tilde{h}] := \tilde{W}[\tilde{h}] - V[h,\tilde{h}] = -\frac{\ell^2}{8\kappa^2}\int \tilde{h}^{ij}\Box^{3/2}\tilde{h}_{ij} + \frac{\ell^2}{2\kappa^2}\int \tilde{h}^{ij}C_{ij}[h] \ . \tag{75}$$

At the extremum, $\frac{\delta\tilde{\mathcal{W}}}{\delta\tilde{h}^{ij}} = 0$, we again get the regularity condition, which then gives $\tilde{\mathcal{W}}| = \frac{\ell^2}{8\kappa^2}\int h^{ij}\Box^{3/2}h_{ij} = W[h]$, so

$$\int \mathcal{D}\tilde{h}_{ij}\, e^{-S_{\text{cs}}[h,\tilde{h}]}\, \tilde{Z}[\tilde{h}] = \int \mathcal{D}\tilde{h}_{ij}\, e^{\tilde{\mathcal{W}}[h,\tilde{h}]} = e^{W[h]} = Z[h] \ , \tag{76}$$

as it should.

4.5 Mixed boundary conditions

We have shown that, depending on the boundary conditions, we can interpret the same bulk theory either as a functional $Z[h]$, where the graviton h_{ij} is fixed and is a source for the stress-energy tensor $\langle T_{ij}\rangle$, or in terms of its Legendre transformed functional $\tilde{Z}[\tilde{h}]$ where the dual graviton $\tilde{h}_{ij}$ is a source for the dual stress-energy tensor. The latter equals the Cotton tensor of the original theory, (63).

It was shown by Ishibashi and Wald [4] that a mixed boundary problem again gives rise to a well-defined quantization problem in the bulk (see also [9]). It is therefore natural to seek a generalization of the above where we replace the functional $W[h]$ by some new functional $\mathcal{W}[J]$ that depends on a source that is a linear combination of h and $\tilde{h}$:

$$J_{ij}(x) = h_{ij}(x) + \lambda\,\tilde{h}_{ij}(x) \ . \tag{77}$$

Since we only want to modify the boundary conditions and not the bulk dynamics, we have to identify $W = S_{\text{on-shell}}$ as before. However, the form of the holographic stress-energy tensor will change, as we need to vary the action with respect to J_{ij} not h_{ij}. This means that we need to modify the definition of $V[h,\tilde{h}]$ as this was obtained from the standard stress-energy tensor. We define:

$$\mathcal{W}[h,J] = S_{\text{on-shell}}[h] + V[h,J] \ . \tag{78}$$

It is easy to check that we get (77) provided $V[h,J] = \frac{\ell^2}{2\kappa^2\lambda}\int \left(\frac{1}{2}\,h^{ij} - J^{ij}\right) C_{ij}[h]$:

$$\begin{aligned}
\mathcal{W}[h,J] &= \frac{\ell^2}{2\kappa^2}\int \left(\frac{1}{2}\,h^{ij}C_{ij}[\tilde{h}] + \frac{1}{\lambda}\left(\frac{1}{2}\,h^{ij} - J^{ij}\right) C_{ij}[h]\right) \\
\delta\mathcal{W} &= -\frac{\ell^2}{2\kappa^2\lambda}\int C_{ij}[J - h - \lambda\,\tilde{h}]\,\delta h^{ij} = 0\ ,
\end{aligned} \tag{79}$$

indeed giving back (77). Since we have not changed the bulk solution but only the boundary conditions, $\tilde{h}$ is still determined by regularity or whatever other condition is imposed on a particular solution in Lorentzian signature. In the case of regular solutions, (77) gives:

$$J_{ij} = h_{ij} + \frac{\lambda}{(-\varepsilon\Box)^{3/2}}\,2C_{ij}[h]\ . \tag{80}$$

This determines h_{ij} in terms of the source, however not completely. There may be zero modes

$$h_{0ij} + \frac{\lambda}{(-\varepsilon\Box)^{3/2}}\,2C_{ij}[h_0] = h_{0ij} + \lambda\,d_{ij}[h_0] = 0\ . \tag{81}$$

We recognize here the self-duality equation (29), and we know that its only non-zero solutions are for $\lambda = \pm 1$. Thus, we will distinguish the cases $\lambda = \pm 1$ and $\lambda \neq \pm 1$.[1]

For any λ, the dual stress-energy tensor is

$$\langle \tilde{T}_{ij} \rangle_J = -2\frac{\delta S}{\delta J^{ij}} = \frac{\ell^2}{\lambda\kappa^2}\,C_{ij}[h]\ . \tag{82}$$

The case $\lambda = \pm 1$ and the non-zero stress-energy tensor vev

The case $\lambda = \pm 1$ brings us back to the instanton equation (29), which we solved in (32). In this special case, J can be easily shown to be self/anti-self dual: $J = h \pm d[h]$ implies $J = \pm d[J]$. In turn, this implies (anti-) self-dualilty of the dual stress-energy tensor. We can solve for $h(J)$, getting: $h = h_0 + \frac{1}{2}\,J$. Setting the source to zero, we get

$$\langle \tilde{T}_{ij} \rangle_{J=0} = \pm\frac{\ell^2}{\kappa^2}\,C_{ij}[h_0] = -\frac{\ell^2}{2\kappa^2}(-\varepsilon\Box)^{3/2}h_{0ij}\ . \tag{83}$$

The one-point function of the dual stress-energy tensor in CFT_2 coupled to gravity does not vanish when the source is set to zero. This result is not surprising when there are two coupled gravitons. Although by setting $J_{ij} = 0$ we have removed the graviton source, h_{0ij} (the graviton of the original Dirichlet theory) couples to J_{ij}. After setting the source to zero, the theory has non-vanishing stress and shear. However, notice that by construction the stress-energy tensor is always conserved. Also, the stress-energy tensor is zero if h_0 is a conformally flat fluctuation. In CFT_1, on the other hand, by construction we have $\langle T_{ij} \rangle_h = 0$ for any h.

The case $\lambda \neq \pm 1$

Now the zero-mode h_0 disappears and we solve for h completely in terms of the source:

$$h = \frac{1}{1-\lambda^2}\left(J - \lambda\,d[J]\right)\ . \tag{84}$$

[1] Recently, a similar phenomenon has been found in the context of AdS_3, where $\lambda = \pm 1$ was called the chiral point [20]. Here, we find extra degrees of freedom at $\lambda = \pm 1$.

The one-point function:

$$\langle \tilde{T}_{ij} \rangle_J = -\frac{\ell^2}{2\kappa^2(1-\lambda^2)} (-\varepsilon\Box)^{3/2} \left(J_{ij} - \frac{1}{\lambda}\, d_{ij}[J] \right) \tag{85}$$

indeed vanishes if $J = 0$.

A further generalization of (77) can be achieved by introducing a mass parameter μ and requiring:

$$J_{ij} = h_{(0)ij} + \frac{3}{\mu^3}\, h_{(3)ij} \ . \tag{86}$$

This is analogous to the usual one-parameter family of boundary conditions that one gets for scalar fields when the CFT is augmented by double-trace deformations. The piece we now add to the action is $V[h, J] = \frac{\mu^3}{2} \int (\frac{1}{2}\, h^2 + Jh)$. Using regularity, we rewrite the above as

$$J_{ij} = \left(1 + \frac{(-\varepsilon\Box)^{3/2}}{\mu^3} \right) h_{ij} \ , \tag{87}$$

and again it has zero modes These are massive gravitons that satisfy $(-\varepsilon\Box)^{3/2} h_0 + \mu^3 h_0 = 0$. We then solve

$$h = h_0 + \frac{\mu^3}{(-\varepsilon\Box)^{3/2} + \mu^3}\, J \ . \tag{88}$$

Again, the dual stress-energy tensor acquires a non-zero expectation value, even if $J = 0$:

$$\langle \tilde{T}_{ij} \rangle_J = -\mu^3 h_{0ij} - \frac{\mu^6}{(-\varepsilon\Box)^{3/2} + \mu^3}\, J_{ij} \ , \tag{89}$$

whereas $\langle T_{ij} \rangle_h = 0$, as it should.

4.6 Duality of two-point functions

Having computed the on-shell action (39), and taking into account the gravitational Chern-Simons term, we obtain the two-point function of the stress-energy tensor in the standard way:

$$\langle T_{ij} T_{kl} \rangle = \frac{\ell^2}{\kappa^2}\, |p|^3\, \Pi_{ijkl} + \frac{ip^2}{\mu}\, \epsilon_{imn} p_n \Pi_{jmkl} \ , \tag{90}$$

which has the standard form including the parity-odd term.

The two-point function of the dual stress-energy tensor is computed by differentiating $\tilde{W}[\tilde{h}]$ with respect to the dual graviton instead. For standard duality, where κ/ℓ is inert, $\tilde{W}[\tilde{h}]$ was explicitly computed in (62). The dual two-point function takes the same form as (90).

However, as pointed out at the end of section 4.1, there is a second possibility where the coupling transforms under S-duality as $\ell'/\kappa' = 2\kappa/\ell$. In that case, the two-point function exhibits the behavior found from field theory in [21] (without including the Chern-Simons term).

All this is analogous to duality for two-point functions of spin-1 currents studied in [11]. In that case, it was found that full duality involves the gauge coupling as well as the θ-angle. For gravity, the analogous coupling is μ. It should be straightforward to generalize the bulk analysis for spin one to the case of spin two along the lines of [11] and check that under duality Newton's constant and the mass μ mix in the way predicted from field theory in [21].

4.7 Cosmological topologically massive gravity

Finally we show that the theory found in (67) can at the non-linear level be described in terms of a massive, conserved and traceless tensor I_{ij}. Setting the second graviton to zero, we write:

$$R_{ij}[g] - \frac{1}{2}\, g_{ij}\, R[g] + \lambda\, g_{ij} = \mu\, C_{ij}[g] \ . \tag{91}$$

It automatically follows that the space has constant scalar curvature, $R = 6\lambda$. Therefore we may define a conserved, traceless tensor from the curvature as follows:

$$I_{ij} := R_{ij}[g] - \frac{1}{3}\, g_{ij}\, R[g] = R_{ij}[g] - 2\lambda\, g_{ij} \ . \tag{92}$$

Notice that $I_{ij} = 0$ gives maximally symmetric solutions. The equation of motion can be rewritten as

$$I_{ij} = \frac{1}{\mu}\, \epsilon_i{}^{kl} \nabla_k I_{jl} \ . \tag{93}$$

One can check that I_{ij} satisfies

$$\left(\Box + \varepsilon\mu^2 - 3\lambda\right) I_{ij} - 3\left(I^2_{ij} - \frac{1}{3}\, g_{ij} I^2\right) = 0 \ , \tag{94}$$

where $I^2_{ij} = I_{ik} I^k_j$. This equation describes a non-linear massive conserved and traceless tensor. At the linearized level, the above equation was also found in [22] (see also [23]). Linearizing, we get

$$\delta I_{ij} = \frac{1}{\mu}\, \epsilon_i{}^{kl} \nabla_k \delta I_{lj} \ . \tag{95}$$

In De Donder gauge:

$$\delta I_{ij} = -\frac{1}{2}\, \Box h_{ij} + \lambda\, h_{ij} - \lambda\, g_{ij}\, h \ , \tag{96}$$

where $\nabla^j \delta I_{ij} = \delta I^i_i = 0$ and $(\Box + 4\lambda)h = 0$.

All of the above are valid for any value of λ. Only for $\lambda > 0$ does the holographic analysis of the previous sections go through; for $\lambda < 0$, part of the bulk has to be excised, the new boundary having a common two-dimensional bondary with the usual AdS boundary. We leave the study of holography in this interesting case for the future.

5 Discussion

We have shown evidence for the existence of two CFT's with different parity being dual to the same bulk system. This is a generic fact about gravity, and the dual graviton is always present if the stress-energy tensor is non-zero. Therefore, we expect *any* gravity theory in AdS_4 with conformal boundary conditions to be dual to either of two CFT's. This type of duality is already known for conformal matter in a fixed AdS_4 background, therefore it is reasonable to expect it to hold for the coupled gravity-matter system. In that case, the holographic stress-energy tensor satisfies non-trivial Ward identities [2], therefore only part of it is given by the Cotton tensor. The rest of it will presumably contain all of the dual operators present in the CFT. It would be interesting to study this in detail. The bulk action gives the non-linear coupling between both gravitons in a BF-term.

The two gravitons found here have different parity. This is reminiscent of the situation in [24] where E_{11} requires the gravitational degrees of freedom to be described in terms of dual fields. For all the dualities of this type found in AdS_4 so far, there is an embedding in M-theory [11, 15]. However, there is the important difference that our dual graviton is not a bulk graviton but lives in three dimensions.

Gravity can become dynamical on the boundary of AdS_4 by appropriate choice of boundary conditions. Whereas algebraic boundary conditions generically fully determine the bulk and boundary values of the fields, Robin-type differential boundary conditions specify the boundary values only up to zero modes. These zero modes satisfy dynamical equations which are holographically identified with the dynamics of the boundary graviton. Depending on the boundary conditions, the CFT is coupled to Cotton gravity or to topologically massive gravity. The results are similar to the case of deformations for scalar fields and $U(1)$ gauge fields. The latter is dual to the topologically massive gauge theory [11]. In the scalar case, deforming the action by irrelevant operators, the theory flowed towards the IR fixed point. In the case of bulk gravity studied in this paper, boundary gravitons are fixed in the Dirichlet and Neumann problems, which are limiting cases of the mixed boundary conditions. Allowing general mixed boundary conditions amounts to coupling the CFT to Einstein gravity where the graviton becomes dynamical.

We found examples where the coupling between both gravitons spontaneously generates a non-zero vev for the stress-energy tensor of CFT_2 coupled to gravity after the source is removed. This background stress-energy tensor is nevertheless conserved. In CFT_1 coupled to gravity, on the other hand, the vev is always zero for the boundary conditions that we have considered. In three dimensions there are no conformal anomalies. However, there are gravitational anomalies in topological theories. It would be interesting to see whether this effect is related to the gravitational anomaly of the type discussed in [25]. Notice that the non-vanishing stress-energy tensor found here is parity-odd and proportional to the Cotton tensor. It would be worth studying this mechanism in the light of recent developments in the three-dimensional CFT [26, 27].

The Legendre transformation relating both theories is electric-magnetic duality, the electric field being the *renormalized* stress-energy tensor and the magnetic field being the Cotton tensor. This leads to a modification of the holographic dictionary in the dual theory, where the on-shell bulk action can now be defined to be the effective action of CFT_2. Gravitational electric-magnetic duality in flat space and in spaces with a cosmological constant has been discussed earlier in [24, 28]. Our approach differs from earlier works in that we have given the holographic interpretation of the duality. For this work it is essential that the boundary quantities are renormalized. Another crucial difference with previous work is the existence of a dual graviton on the boundary and a dual stress-energy tensor. The existence of this graviton was established at the non-linear level.

Duality in the case of $U(1)$ gauge fields in the bulk has had important applications to condensed matter systems [29], as it relates materials with completely different properties. S-duality has been used to make new predictions for the quantum Hall effect in graphene [30]. One may expect duality to play an important role in the non-abelian case as well (see [31] where the bulk configuration was conjectured to be dual to a superconductor). In this paper we have found the holographic dictionary for gravitational electric-magnetic duality, and it is natural to ask which condensed matter systems it relates to each other. In particular, it is suggested here that the two CFT's have different parity, and it would be interesting to understand the implications of this for the specific materials.

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A Holographic renormalization and the Neumann problem

A.1 Holographic renormalization: Dirichlet problem

The renormalized Einstein-Hilbert action is:

$$\begin{aligned} S_{\rm EH} &= S_{\rm bulk} + S_{\rm GH} + S_{\rm ct} \\ &= -\frac{1}{2\kappa^2}\int \mathrm{d}^{d+1}x\sqrt{G}\,(R[G]-2\Lambda) - \frac{1}{2\kappa^2}\int \mathrm{d}^dx\sqrt{\gamma}\,2K \\ &\quad +\frac{1}{2\kappa^2}\int \mathrm{d}^dx\sqrt{\gamma}\left(\frac{2(d-1)}{\ell}+\frac{\ell}{d-2}\,R[\gamma]\right) \end{aligned} \tag{97}$$

where $\kappa^2 = 8\pi G_N$, $\Lambda = -\frac{d(d-1)}{2\ell^2}$ and γ is the induced metric on the boundary and $K = \gamma^{ij}K_{ij}$. For $d \geq 4$, further counterterms appear.

We choose coordinates

$$\mathrm{d}s^2 = \mathrm{d}R^2 + e^{2R/\ell}g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{\ell^2}{r^2}\left(\mathrm{d}r^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j\right)\ , \tag{98}$$

with $r/\ell = e^{-R/\ell}$ and

$$K_{ij} = \frac{1}{2}\,\partial_R\gamma_{ij} = \frac{\ell}{r^2}\,g_{ij} - \frac{\ell}{2r}\,g'_{ij}\ . \tag{99}$$

The variation of the action is:

$$\delta(S_{\rm bulk}+S_{\rm GH}) = \frac{1}{2\kappa^2}\int \mathrm{d}^dx\sqrt{\gamma}\left(K^{ij}-\gamma^{ij}K\right)\delta\gamma_{ij}\ . \tag{100}$$

The remaining variations are

$$\delta S_{\rm ct} = -\frac{1}{2\kappa^2}\int {\rm d}^d x\sqrt{\gamma}\left(\frac{d-1}{\ell}\,\gamma_{ij} - \frac{\ell}{d-2}\left(R_{ij}[\gamma] - \frac{1}{2}\,\gamma_{ij}R[\gamma]\right)\right)\delta\gamma^{ij}\ . \tag{101}$$

We can now compute the boundary stress-energy tensor:

$$\langle T_{ij}\rangle = \frac{2}{\sqrt{g_{(0)}}}\frac{\delta S_{\rm EH}}{\delta g^{ij}_{(0)}} = \lim_{\epsilon\to 0}\left(\frac{\ell}{\epsilon}\right)^{d-2}\frac{2}{\sqrt{\gamma(\epsilon,x)}}\frac{\delta S_{\rm EH}}{\gamma^{ij}(\epsilon,x)} \tag{102}$$

evaluated at $r=\epsilon$ and taking the limit. We get:

$$\langle T_{ij}\rangle = -\frac{1}{\kappa^2}\left(\frac{\ell}{\epsilon}\right)^{d-2}\left[K_{ij} - \gamma_{ij}K + \frac{d-1}{\ell}\,\gamma_{ij} - \frac{\ell}{d-2}\left(R_{ij}[\gamma] - \frac{1}{2}\,\gamma_{ij}R[\gamma]\right)\right] \tag{103}$$

In three dimensions, this gives

$$\langle T_{ij}\rangle = \frac{3}{2}\left(\frac{\ell}{\kappa}\right)^2 g_{(3)ij}\ , \tag{104}$$

as it should.

B Boundary graviton and dual graviton

In three dimensions, we may expand the graviton in terms of p_i, p^*_i and a third independent polarization vector w_i, as follows:

$$h_{ij}(r,p) = w_iw_j + b\,(p^*_iw_j + p^*_jw_i) + c\,(p_iw_j + p_jw_i) + d\,(p^*_ip_j + p^*_jp_i) + e\,p_ip_j + \phi\,\frac{p^*_ip^*_j}{\bar p^{*2}}\ . \tag{105}$$

There is a lot of redundancy in this expression. Obviously, c, d, and e are gauge parameters. Furthermore, the b-dependent term can be removed by a shift of w_i. However, this is only useful when a single graviton is around; if there are two gravitons around, the b-term cannot be removed.

We are interested in the transverse, traceless part of h_{ij}:

$$\bar h_{ij}(r,p) = \Pi_{ijkl}\left(w_kw_l + \phi\,\frac{p^*_ip^*_j}{\bar p^{*2}} + bp^*_kw_l + bp^*_lw_k\right)\ , \tag{106}$$

where we have reabsorbed $a(r,p)$ in $w_i(r,p)$ and rescaled $b(r,p)$ by the same factor.

$w_i(r,p)$ is a two-dimensional vector with fixed direction and arbitrary scale factor. Its transverse part is $\bar w_i = \Pi_{ij}w_j$. A convenient choice for w is one where $(p_i, \bar p^*_i, \bar w_i)$ form an orthogonal basis:

$$\begin{aligned}\bar w_i(r,p) &= \sqrt{\varphi(r,p)}\,v_i(p)\\ v_i(p) &= v\,\epsilon_{ijk}p_j\bar p^*_k\ ,\end{aligned} \tag{107}$$

and $v = 1/\sqrt{\varepsilon p^2 \bar{p}^{*2}}$ such that $v_i v^i = 1$. Defining $\gamma = \phi - \varphi$ and $-i\psi = 2\sqrt{\varphi} b v p^3$ and filling it in (106), we get the form (10).

Regular bulk solutions are obtained imposing (23). Filling this in (18), we get:

$$\begin{aligned} \bar{h}_{ij}(r,p) &= e^{-|p|r}(1+|p|r)\,\bar{h}_{ij}(p) \\ \bar{w}_i(r,p) &= e^{-|p|r/2}\sqrt{1+|p|r}\,\bar{w}_i(p) \\ \phi(r,p) &= e^{-|p|r}(1+|p|r)\,\phi(p) \ . \end{aligned} \tag{108}$$

As we reviewed in (3)-(4), in three dimensions, given a conserved, traceless stress-energy tensor, it is always possible to introduce a dual graviton $\tilde{h}_{(0)}$:

$$h_{(3)ij}(p) = -ip^2 \epsilon_{ikl} p_k \tilde{h}_{(0)jl}(p) \ , \tag{109}$$

again a traceless transverse, dimension zero tensor. $\tilde{h}_{(0)ij}$ extends to a bulk graviton $\tilde{h}_{ij}(r,p)$, defined as the Cotton tensor of the holographic stress-energy tensor evaluated at cutoff r, $\langle T_{ij}\rangle_r$. $\tilde{h}_{(0)}$ has the same expansion as $h_{(0)}$:

$$\tilde{h}_{(0)ij} = \tilde{\gamma}(p) E_{ij}(p) - i\tilde{\psi}(p) D_{ij}(p) \ . \tag{110}$$

Comparing with (22), we get

$$\begin{aligned} \tilde{\psi}(p) &= -\frac{i}{3}\frac{p^2}{\bar{p}^{*2}}\,\delta(p) \\ \tilde{\gamma}(p) &= -\frac{i}{3}\frac{\bar{p}^{*2}}{p^2}\,\chi(p) \end{aligned} \tag{111}$$

in Lorentzian signature. In the Euclidean, the factors of $-i$ are absent. Again, these expressions extend to the bulk in the obvious way.

B.1 Properties of the tensors E_{ij} and D_{ij}

The transverse, traceless tensors E_{ij} and D_{ij} are constructed from the independent vectors $(p_i, \bar{p}^*_i, \epsilon_{ijk} p_j \bar{p}^*_k)$ and are defined in (11). They are each others duals:

$$\begin{aligned} D_{ij} &= \frac{\bar{p}^{*2}}{p^3}\,\epsilon_{ikl} p_k E_{jl} \\ E_{ij} &= -\frac{\varepsilon p}{\bar{p}^{*2}}\,\epsilon_{ikl} p_k D_{jl} \ . \end{aligned} \tag{112}$$

They also satisfy:

$$\begin{aligned} E_{ik} E^k_j &= \frac{1}{4}\Pi_{ij} \\ D_{ik} D^k_j &= -\frac{\varepsilon}{4}\left(\frac{\bar{p}^{*2}}{p^2}\right)\Pi_{ij} \\ E_{ik} D^k_j &= \frac{1}{4p^2}\left(\bar{p}^*_i \epsilon_{jkl} - \bar{p}^*_j \epsilon_{ikl}\right) p_k \bar{p}^*_l = \frac{1}{4}\frac{\bar{p}^{*2}}{p^2}\,\epsilon_{ijk} p_k \end{aligned}$$

$$
\begin{aligned}
E_{ij}E^{ij} &= \frac{1}{2} \\
E_{ij}D^{ij} &= 0 \\
D_{ij}D^{ij} &= \frac{\varepsilon}{2}\left(\frac{\bar{p}^{*2}}{p^2}\right)^2 .
\end{aligned} \tag{113}
$$

C Three-dimensional curvature tensors

C.1 The Cotton tensor and the gravitational Chern-Simons term

The derivation of the Cotton tensor from the variation of the gravitational Chern-Simons action is standard. Up to a total derivative, the latter can be written either in terms of the spin connection or in terms of the connection 1-form. We will use the spin connection formulation, see for example [32, 33]. Writing:

$$
S_{\rm cs} = -\frac{1}{4}\int \mathrm{Tr}\left(\omega\wedge \mathrm{d}\omega + \frac{2}{3}\,\omega\wedge\omega\wedge\omega\right) , \tag{114}
$$

we get

$$
\delta S_{\rm cs} = -\frac{1}{2}\int \mathrm{Tr}\left(\delta\omega\wedge R\right) = -\frac{1}{2}\int \epsilon^{ijk}R_{ijl}{}^{m}\delta\Gamma^{l}_{km} = -\int C^{ij}\delta g_{ij} . \tag{115}
$$

The last line defines the Cotton tensor,

$$
C_{ij} = \frac{1}{2}\,\epsilon_i{}^{kl}\nabla_k\left(R_{jl} - \frac{1}{4}\,g_{jl}R\right) . \tag{116}
$$

C.2 Linearized tensors

The projector is defined as follows. The transverse part of h_{ij} is

$$
h^{\perp}_{ij} = h_{ij} - \frac{1}{\Box}\left(\partial_i\partial_k h_{jk} + \partial_j\partial_k h_{ik}\right) + \frac{\partial_i\partial_j}{\Box^2}\partial_k\partial_l h_{kl} . \tag{117}
$$

The transverse, traceless part is now

$$
\begin{aligned}
\bar{h}_{ij} &= h^{\perp}_{ij} + \frac{1}{2}\left(\frac{\partial_i\partial_j}{\Box} - \delta_{ij}\right)h^{\perp} \\
&= h_{ij} - \frac{1}{\Box}\left(\partial_i\partial_k h_{jk} + \partial_j\partial_k h_{ik}\right) + \frac{1}{2}\,\delta_{ij}\frac{\partial_k\partial_l}{\Box}h_{kl} + \frac{1}{2}\,\frac{\partial_i\partial_j\partial_k\partial_l h_{kl}}{\Box^2} + \frac{1}{2}\left(\frac{\partial_i\partial_j}{\Box} - \delta_{ij}\right)h
\end{aligned} \tag{118}
$$

Explicitly, the spin-2 projector is given in terms of the spin-1 projectors:

$$
\begin{aligned}
\Pi_{ijkl} &= \frac{1}{2}\left(\Pi_{ik}\Pi_{jl} + \Pi_{il}\Pi_{jk} - \Pi_{ij}\Pi_{kl}\right) \\
\Pi_{ij} &= \delta_{ij} - \frac{\partial_i\partial_j}{\Box} .
\end{aligned} \tag{119}
$$

Of course, it satisfies

$$
\Pi_{ijmn}\Pi_{mnkl} = \Pi_{ijkl} . \tag{120}
$$

The three-dimensional Ricci, Schouten, and Cotton tensors are then

$$
\begin{aligned}
\delta R_{ij} &= -\frac{1}{2}\,\Box\bar{h}_{ij} + \frac{1}{4\Box}\left(\partial_i\partial_j + \delta_{ij}\Box\right)\left(\Box h - \partial_k\partial_l h_{kl}\right) \\
\delta R &= \Box h - \partial_k\partial_l h_{kl} \\
\delta P_{ij} &= -\frac{1}{2}\,\Box\bar{h}_{ij} + \frac{1}{4}\partial_i\partial_j\left(h - \frac{\partial_k\partial_k}{\Box}h_{kl}\right) \\
\delta C_{ij} &= \frac{1}{2}\,\epsilon_{ikl}\partial_k\Box\bar{h}_{jl}\ .
\end{aligned}
\tag{121}
$$

The Cotton tensor depends only on the transverse, traceless part of the graviton.

D The Lorentzian solution

We will solve the three linearized bulk equations of motion (15) including the gauge-dependent terms. We expand

$$
h_{ij}(r,x) = \bar{h}_{ij}(r,x) + \delta_{ij}\,\phi(r,x) + \partial_i\xi_j(r,x) + \partial_j\xi_i(r,x)\ . \tag{122}
$$

The physical part $\bar{h}_{ij}(r,x)$ has already been solved for. The trace ϕ and gauge parameters ξ_i satisfy:

$$
\begin{aligned}
\partial_k\left(\xi_i'' - \frac{1}{r}\,\xi_i'\right) + \frac{3}{2}\left(\phi'' - \frac{1}{r}\,\phi'\right) &= 0 \quad (123)\\
\Box\xi_i' - \partial_i\partial_k\xi_k' - 2\partial_i\phi' &= 0 \quad (124)\\
\partial_i\xi_j'' + \partial_j\xi_i'' - \frac{2}{r}\left(\partial_i\xi_j' + \partial_j\xi_i' + \delta_{ij}\partial_k\xi_k'\right) + \partial_i\partial_j\phi + \delta_{ij}\left(\phi'' - \frac{5}{r}\,\phi' + \Box\phi\right) &= 0\ . \quad (125)
\end{aligned}
$$

The derivative of (124) automatically gives:

$$
\Box\phi' = 0\ . \tag{126}
$$

Taking linear combinations of (124) and filling in (123), we get

$$
\Box P_i = \frac{1}{2}\,\partial_i\Phi \tag{127}
$$

where $P_i = \xi_i'' - \frac{1}{r}\,\xi_i'$, $\Phi = \phi'' - \frac{1}{r}\,\phi'$. Taking further derivatives, and after some manipulation, we get

$$
\begin{aligned}
\Box P_i &= 0 \\
\Phi &= \Phi(r) \\
\partial_i P_i &= -\frac{3}{2}\Phi(r)\ .
\end{aligned}
\tag{128}
$$

In these variables, (123) is rewritten as

$$
\partial_i P_i = -\frac{3}{2}\,\Phi\ . \tag{129}
$$

Using $\Phi = \Phi(r)$, taking a further derivative we get

$$\partial_i \partial_k \xi_k'' = \frac{1}{r}\,\partial_i \partial_k \xi_k' \ . \tag{130}$$

We now turn to (125). Taking its trace, we get

$$\frac{2}{3}\partial_k \left(\xi_k'' - \frac{5}{r}\,\xi_k'\right) + \phi'' - \frac{5}{r}\,\phi' + \frac{4}{3}\,\Box\phi = 0 \ . \tag{131}$$

Filling it back in, we get the traceless equation

$$\partial_i \xi_j'' + \partial_j \xi_i'' - \frac{2}{3}\,\delta_{ij}\partial_k \xi_k'' - \frac{2}{r}\left(\partial_i \xi_j' + \partial_j \xi_i' - \frac{2}{3}\,\delta_{ij}\partial_k \xi_k'\right) + \partial_i\partial_j\phi - \frac{1}{3}\,\delta_{ij}\Box\phi = 0 \ . \tag{132}$$

Hitting this equation with ∂_j we get

$$\Box\xi_i'' + \frac{1}{3}\,\partial_i\partial_k\xi_k'' - \frac{2}{r}\left(\Box\xi_i' + \frac{1}{3}\,\partial_i\partial_k\xi_k'\right) + \frac{2}{3}\,\partial_i\Box\phi = 0 \ . \tag{133}$$

Equation (131) gives

$$\begin{aligned} \partial_k\xi_k' &= -\frac{3}{2}\,\phi' + \frac{1}{2}\,r\Box\phi \\ \partial_k\xi_k'' &= -\frac{3}{2}\,\phi'' + \frac{1}{2}\,\Box\phi \end{aligned} \tag{134}$$

Equation (124) now reads

$$\begin{aligned} \Box\xi_i' &= \frac{1}{2}\,\partial_i\phi' + \frac{1}{2}\,r\partial_i\Box\phi \\ \Box\xi_i'' &= \frac{1}{2}\,\partial_i\phi'' + \frac{1}{2}\,\partial_i\Box\phi \end{aligned} \tag{135}$$

In order to solve (125), we decompose ξ_i. In order to facilitate this, we define

$$\begin{aligned} Q_i &= \xi_i'' - \frac{2}{r}\,\xi_i' \\ Q &= \phi'' - \frac{2}{r}\,\phi' \end{aligned} \tag{136}$$

and rewrite it as

$$\partial_i Q_j + \partial_j Q_i + \delta_{ij} Q + \partial_i\partial_j\phi = 0 \ . \tag{137}$$

We decompose

$$Q_i = p_i a + \bar{p}_i^* b + \bar{q}_i \tag{138}$$

where $\bar{q}\cdot p = \bar{q}\cdot p^* = 0$. We get:

$$\begin{aligned} a &= -\frac{i}{2}\,\phi \\ b &= 0 \\ p^2 &= 0 \end{aligned} \tag{139}$$

and either $\bar{q} \sim p$ or $\bar{q} \sim \bar{q}$. The choice is immaterial. We will choose $\bar{q} \sim p$, in that case $\bar{q}_i$ in the expansion of Q_i can be included in a. So we get from the above the solution

$$\begin{aligned} \phi'' - \frac{2}{r} f' &= 0 \\ \xi_i'' - \frac{2}{r} \xi_i' &= -\frac{1}{2} \partial_i \phi \\ \Box\phi = \Box\xi_i &= 0 \; . \end{aligned} \tag{140}$$

The solution is as follows:

$$\begin{aligned} \phi(r,x) &= \phi_{(0)}(x) + r^3 \phi_{(3)}(x) \\ \xi_i(r,x) &= \xi_{(0)i}(x) + r^2 \xi_{(2)i}(x) + r^3 \xi_{(3)i}(x) \; , \end{aligned} \tag{141}$$

where

$$\begin{aligned} \xi_{(2)i} &= \frac{1}{4} \partial_i \phi_{(0)}(x) \\ \partial_k \xi_{(3)k} &= -\frac{3}{2} \phi_{(3)} \\ \Box\phi_{(0)} &= 0 \\ \partial_i \phi_{(3)} &= 0 \; . \end{aligned} \tag{142}$$

Hence, the full expansions of the coefficients (14) are:

$$\begin{aligned} h_{(0)ij}(x) &= \bar{h}_{(0)ij}(x) + \delta_{ij}\, \phi_{(0)} + \partial_i \xi_{(0)j} + \partial_j \xi_{(0)i} \\ h_{(2)ij}(x) &= \bar{h}_{(2)ij}(x) + \frac{1}{2} \partial_i \partial_j \phi_{(0)} \\ h_{(3)ij}(x) &= \bar{h}_{(3)ij}(x) + \delta_{ij}\, \phi_{(3)} + \partial_i \xi_{(3)j} + \partial_j \xi_{(3)i} \; , \end{aligned} \tag{143}$$

and of course they satisfy

$$\begin{aligned} h_{(2)ii} = \partial^j h_{(2)ij} &= 0 \\ h_{(3)ii} = \partial^j h_{(3)ij} &= 0 \; . \end{aligned} \tag{144}$$

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