Chern-Simons Theory, 2d Yang-Mills, and

Lie Algebra Wanderers

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Abstract

We work out the relation between Chern-Simons, 2d Yang-Mills on the cylinder, and Brownian motion. We show that for the unitary, orthogonal and symplectic groups, various observables in Chern-Simons theory on S^3 and lens spaces are exactly given by counting the number of paths of a Brownian particle wandering in the fundamental Weyl chamber of the corresponding Lie algebra. We construct a fermionic formulation of Chern-Simons on S^3 which allows us to identify the Brownian particles as B-model branes moving on a non-commutative two-sphere, and construct 1- and 2-matrix models to compute Brownian motion ensemble averages.

Contents

1 Introduction

The idea that the large N limit of quantum field theories gives closed string theories [1, 2] has gained much in tractability in the context of topological strings, where, as Gopakumar and Vafa [3] showed –in the case of the conifold geometric transition– the associated gauge theory is a *bosonic*, topological gauge theory, namely three-dimensional Chern-Simons theory on S^3. Recently, it has become more and more clear that other low-dimensional bosonic theories may also play a central role in string theory computations, and in particular, building on the old idea of Gross [4], 2d Yang-Mills [5, 6, 7]. Furthermore, these bosonic theories can often be mapped to statistical mechanical systems [8, 6, 9], which provides one with useful computational tools [10]. Also, it is by now clear that Chern-Simons and two-dimensional Yang-Mills (in its usual and its quantum deformed versions) are closely connected [6, 7, 11], generalizing previous work which related them at zero coupling [12, 13, 14] (see also [15]).

In this paper we work out the relation between Chern-Simons, 2d Yang-Mills on the cylinder, and Brownian motion found in [8, 6]. We show that for unitary, orthogonal and symplectic groups, observables in Chern-Simons theory on S^3 are exactly given by counting the number of paths of a Brownian particle wandering in the fundamental Weyl chamber of the corresponding algebra, or equivalently by a number of non-intersecting movers on a line where certain boundary conditions are imposed. In particular, we compute the partition function, the expectation value of the unknot, and the expectation value of the Hopf link. Our results suggest that Brownian motion might give a rather general and independent way of computing knot and 3-manifold invariants.

We also comment on the relation between 2d Yang-Mills on the cylinder and Brownian motion in an affine Weyl chamber, already worked out in [6]. This gave a connection between 2d Yang-Mills on the cylinder and Chern-Simons on lens spaces that made possible the computation of the modular transformation properties of 2d Yang-Mills on the cylinder, which for particular external states reduces to the quantum-deformed 2d Yang-Mills on the sphere. Thus, these can be translated into modular transformation properties of certain A-model amplitudes.

We construct a fermionic formulation of Chern-Simons theory which allows us to identify the Brownian particles with B-model branes moving on a non-commutative sphere.

Finally, we show how hermitian 1- and 2-matrix models can be used to compute Brownian motion observbles.

This is the companion paper of [8]. Sections 2 and 3 include pedagogical introductions to Brownian motion and Chern-Simons theory, respectively. In sections 3, 4 and 5 we work out the relation between Chern-Simons and Brownian motion in various cases. In section 5 we discuss 2d Yang-Mills, in section 6 we work out the fermionic description of Chern-Simons and its relation to topological strings, and in section 7 we introduce matrix models for Brownian motion. In the appendices we give technical details, including a discussion of framing and a path integral counting of the Brownian motion paths.

2 Random walks and Brownian motion

Almost a hundred years ago, Einstein gave the first mathematical explanation of the phenomenon of 'Brownian motion', the zig-zag-like random motion characteristic of pollen grains suspended in water, observed by the botanist Robert Brown around 1827. Einstein's model was a discrete 'random walk' model, that we will briefly review.

2.1 Random walks

In its simplest guise, we can regard the random walk of a particle as follows. At every tick of the clock, the particle can jump either right or left with equal probability. We want to compute the probabilty that after time t the particle will have traveled a distance x. We set $t = n\tau$, where n denotes the total number of steps and τ the average time between one step and the next one; and $x = ml$, where m is the number of steps in the positive x-direction minus the number of negative steps. The probability that after n steps the particle has made m steps in the x-direction is then:

$$P(n|m) = \frac{n!}{[\frac{1}{2}(n+m)]![\frac{1}{2}(n-m)]!} \frac{1}{2^n} \; , \tag{1}$$

where the binomial distribution comes from the fact that the particle takes a total of $\frac{1}{2}(n+m)$ steps in the positive x-direction, and $\frac{1}{2}(n-m)$ in the negative direction. Of course we can rewrite this probability in terms of x and t and take the limit of small τ, l. If we do this while keeping

$$D = l^2/2\tau \tag{2}$$

fixed, applying Stirling's formula we get for the probability density

$$P(x)\mathrm{d}x = \frac{\mathrm{d}x}{\sqrt{4\pi Dt}} e^{-x^2/4Dt} \; . \tag{3}$$

An important result of Einstein's computation was the computation of the diffusion constant D in terms of the microscopic scales l and τ.

The probability distribution (1) satisfies an interesting property, characteristic of birth processes [16, 17]. If we make P into an infinite matrix $P_{ij}(t)$, where i and j denote positions along the x-direction, and m the number of steps between the state i and the state j, then we can write the time evolution as follows:

$$P'(t) = AP(t) \; . \tag{4}$$

A is an infinite matrix whose only non-zero entries are on the diagonal, and one position above or below it, that is, only the matrix elements A_{ii}, $A_{i,i+1}$ and $A_{i,i-1}$ are non-vanishing. This expresses the fact that the particle moves one step at the time right or left, but cannot jump. The intial condition is obviously:

$$P(0) = \mathrm{id} \; . \tag{5}$$

Notice that, if we take $\tau \to 0$, this results in the delta-function shape of the continuous case.

Equation (4) is really the infinitesimal version of the heat equation satisfied by the continuous density (3). There is a nice reformulation of this equation as a difference equation [18]. Writing $A_{ii} = -(\lambda_i + \mu_i)$, $A_{ij} = \lambda_i$ if $j = i+1$ and μ_i if $j = i-1$, the matrix determines a system of polynomials by means of the recurrence relations

$$\begin{aligned} Q_0(x) &= 1 \\ -xQ_0(x) &= -(\lambda_0 + \mu_0)Q_0(x) + \lambda_0 Q_1(x) \\ -xQ_n(x) &= \mu_n Q_{n-1}(x) - (\lambda_n + \mu_n)Q_n(x) + \lambda_n Q_{n+1}(x) \; . \end{aligned} \tag{6}$$

Karlin and McGregor further showed that for any system of this type there exists a positive measure ψ for which the following orthogonal relations hold:

$$\int_0^\infty \mathrm{d}\psi(x) \, Q_i(x)Q_j(x) = \frac{\delta_{ij}}{\pi_i} \tag{7}$$

4

where the moments are given by $\pi_0 = 1$, $\pi_n = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n}$. Such a measure is called a solution of the Stieltjes moment problem. Extending the range of integration to all reals, one can construct solutions of the analogous Hamburger moment problem.

The probability matrix is now given in terms of these polynomials by

$$P_{ij}(t) = \pi_j \int_0^\infty \mathrm{d}\psi(x)\, e^{-xt} Q_i(x) Q_j(x) \ , \tag{8}$$

and an important result of Karlin and Mc Gregor is that the determinant

$$P\left(t; \psi; \{i\}, \{j\}\right) = \det P_{ij}(t) \tag{9}$$

is positive definite. This extends to the continuous case as well, a property that we will use later on.

2.2 Vicious walkers and Brownian motion probabilities

In this paper we consider Brownian motion of a particular type: the so-called *vicious walkers* [19]. Vicious walkers are movers whose trajectories are not allowed to intersect during the whole motion. Let us however consider the case of harmless movers first. These perform Brownian motion, but their trajectories are allowed to intersect at any time. For a single mover moving on a line, we compute the probability distribution of going from poing x to point y in time t, given a diffusion coefficient D in the medium. This is given by

$$p_t(x, y) = \frac{1}{\sqrt{4\pi D t}}\, e^{-(x-y)^2/4Dt} \ . \tag{10}$$

This Brownian motion probability has some elementary but important properties [20]. We will list them here for this simple case since they generalize to higher dimensions and underlie the matrix model description of Brownian motion and Chern-Simons.

1) It is the unique kernel of the solutions of the heat equation with prescribed boundary conditions:

$$\frac{\partial}{\partial t}\, p_t(x, y) = D \Delta p_t(x, y) \ , \tag{11}$$

which is usually denoted by $K(x, y; t)$. From now on we will set $D = \frac{1}{2}$.

2) It can be obtained as the continuum limit of a discrete random walk, as we reviewed in the previous section.

3) It is a positive-definite quantity and symmetric under interchange of x and y. The former property is trivial in the one-dimensional case but it is a non-trivial result due to Karlin and McGregor [16, 17] for non-intersecting paths in higher dimensions. This connects the probabilistic interpretation with the Stieltjes-Wigert polynomials that naturally appear in Chern-Simons theory [21], as we will see.

4) Finally, $K(x, y; t)$, being a probability, has the extensivity property

$$\int \mathrm{d}\mu(z)\, K(x, z; t) K(z, y; s) = K(x, y; t + s) \ . \tag{12}$$

Here, the integral runs over all reals[1]. This is sometimes called the Chapman-Kolmogoroff equation, and is a consequence of the Markov property ('lack of memory' of the process). This condition also completely fixes the normalization of K, which in the case (10) integrates to 1. In the generalization we will consider later on, the normalization is still fixed by the extensivity property, but it is not equal to 1 since we are dealing with conditional probabilities.

It is interesting to note that these expressions can be generalized along the lines of [20] to include first order derivatives in x in the heat equation. These roughly correspond to an external gravitational field, and the solutions are given in terms of Bessel functions. It would be interesting to see whether there is a gauge theory analog of such terms.

The generalization of the above for more than one particle is obvious:

$$p_{t,N}(x,y) = \frac{1}{(2\pi t)^{N/2}} \, e^{-\frac{|x-y|^2}{2t}}, \tag{13}$$

and N is the number of particles. This is the case that the particles are non-interacting, and it can equivalently be regarded as the product of the probabilities of N single movers on a line, i.e. the probability for N movers on a line to start at positions $y_1, \dots, y_N$ and end up at $x_1, \dots, x_N$ after time t.

From the point of view of the heat equation, however, (13) is not quite the higher-dimensional generalization of the heat kernel of (11), where now the Laplacian would be an N-dimensional flat space Laplacian. The fundamental solution is only obtained after we take the determinant, as we will now explain.

From the point of view of the heat equation we are naturally led to consider *vicious* walkers, that is, walkers whose trajectories do not intersect at any time (see Fig. 1). The quantity that we are then interested in is the probability of going from a state $x_1, \dots, x_N$ to a state $y_1, \dots, y_N$ in time t during which the particles perform free Brownian motion but are otherwise non-intersecting. That is, we do not enforce the non-intersecting condition by adding a force by hand, but consider rather those paths among all the possible paths such that the particles do not intersect. Their quotient then gives the desired probability density. Further, we label the particles such that at time $t = 0$, $x_1 > x_2 > \dots > x_N$. Since the trajectories are non-intersecting and the particles are distinguishable, this condition remains true at all times.

It is this condition $x_1 > x_2 > \dots x_N$ that allows us to think of the motion in terms of a single particle moving in the fundamental Weyl chamber of $U(N)$. Indeed, if x_i now instead of labeling the position of a particle on the line, labels the ith coordinate of a particle in an N-dimensional space, then the motion is restricted to the fundamental Weyl chamber of $U(N)$. Later on we will develop this point of view further and see how it generalizes to other groups.

The probability in the non-intersecting case is easy to compute thanks to the fact that the heat equation is linear and we can superpose its solutions. Indeed, we can use the method of images to write the number of non-intersecting walks as the number of free walks, from which we appropriately subtract "bad" walks. This is most easily done by thinking of this as motion of a single particle in N dimensions, in which case the "bad" walks are walks which at some point hit one of the walls of the fundamental Weyl chamber. This is explained in section 5.

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[1]In this, our conventions are different from those in [20], where – for the non-compact range – only functions over the positive reals are considered. Further, the heat equation considered here corresponds to the case $\gamma = 0$ in [20], in which case the Bessel function is simply a cosine and the integration measure is simply $d\mu(z) = \sqrt{2/\pi} dz$. Therefore, in the case considered here and with the conventions of [20] the kernel would be $K(x,y;t) = 1/\sqrt{t} \exp(-(x^2 + y^2)/2t) \cos(ixy/t)$.
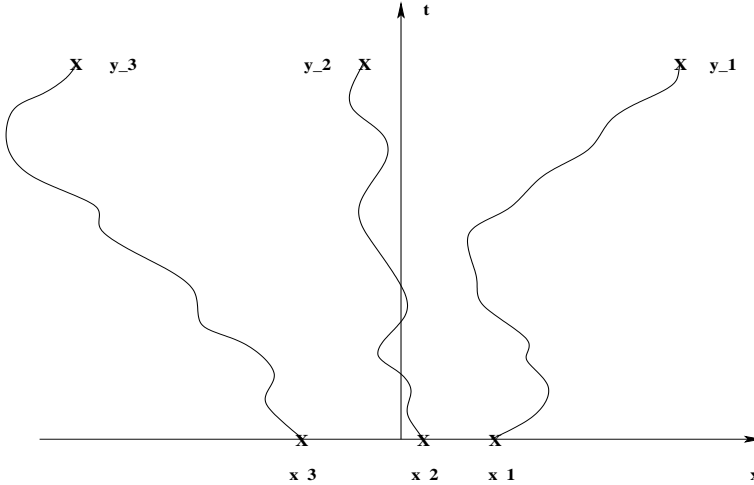
Figure 1: Three vicious walkers performing Brownian motion from x_1, x_2, x_3 to y_1, y_2, y_3.

Using the method of images (see also section 5), the probability is given by:

$$
\begin{aligned}
p_{t,N}(\lambda, \mu) &= \frac{1}{(2\pi t)^{N/2}} \, e^{-\frac{|\lambda|^2 + |\mu|^2}{2t}} \, \det(e^{\lambda_i \mu_j / t})_{1 \le i < j \le N} \\
&= \frac{1}{(2\pi t)^{N/2}} \, \det(e^{-\frac{(\lambda_i - \mu_j)^2}{2t}})_{1 \le i < j \le N} ,
\end{aligned}
\tag{14}
$$

where we now labeled the initial and final posititions as two vectors λ and μ in $\mathbb{R}^N$.

It is easy to see that this probability vanishes, as it should, when any two particles hit each other or, alternatively, when the particle moving in the fundamental Weyl chamber hits one of the walls. Indeed, in that case $\mu_i = \mu_{i+1}$ for some i, and the determinant vanishes identically.

From the representation in the last line of (14) we immediately see that the interacting probability distribution inherits from the free Brownian motion case its homogeneity, that is, it is invariant under simultaneous constant shifts $\lambda_i \to \lambda_i + c$, $\mu_i \to \mu_i + c$. In the next section we will study some more properties of $p_{t,N}(\lambda, \mu)$.

Let us briefly discuss how this interacting probability satisfies the requirements 1)-4). As for 1), consider the N-dimensional heat equation

$$
\frac{\partial}{\partial t} f(\lambda_1, \ldots, \lambda_N) = \frac{1}{2} \Delta f(\lambda_1, \ldots, \lambda_N) .
\tag{15}
$$

We are interested in antisymmetric solutions of this equation, $f(\lambda_1, \ldots, \lambda_N)$, with boundary condition $f_0(\lambda)$ at $t = 0$. The unique solution with these boundary conditions is [22]

$$
f(\lambda) = \frac{1}{N!} \prod_{i=1}^{N} \int_{\mathbb{R}^N} \mathrm{d}\mu_i \, K(\lambda, \mu; t) \, f_0(\mu)
\tag{16}
$$

and

$$
K(\lambda, \mu; t) = \frac{1}{(2\pi t)^{N/2}} \, \det(e^{-(\lambda_i - \mu_j)^2 / 2t})
\tag{17}
$$

is precisely the Brownian motion density $p_{t,N}(\lambda, \mu)$.

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Properties 2) and 3) were already discussed earlier, so we will skip the details. Positivity of the kernel $K(x, y) = p_{t,N}(x, y)$ was proven in [17], and its symmetry with respect to the vectors x, y is clear from the explicit expression.

The extensivity property deserves a few more comments. In this case it reads

$$p_{t+s,N}(x, y) = \int_C d^N z \, p_{t,N}(x, z) \, p_{s,N}(z, y) \tag{18}$$

as we prove in appendix A. This simply expresses the fact that the probability of going from x to y in time $t+s$ is the same as that of going from x to y in time t, times that of going from y to z in time s, where we integrate over all possible intermediate states y. Notice that the normalizations in (14), and in particular also the fact that we integrate over the fundamental Weyl chamber C only and not over $\mathbb{R}^N$, are crucial for the proper normalization of (18). By definition, $p_{t,N}(x, y)$ –being a conditional probability– does not integrate to one, as not all possible paths from x to y are non-intersecting.

Let us anticipate that (18) is equivalent to a matrix model computation of the partition function of Chern-Simons theory on S^3. Actually it is more general than that, providing also a computation of Wilson loops.

Notice that Fisher [19] includes an additional "framing" factor:

$$\tilde{p}_{t,N}(\lambda, \mu) = e^{-N\sigma t} \frac{1}{(2\pi t)^{N/2}} e^{-\frac{|\lambda|^2 + |\mu|^2}{2t}} \det(e^{\lambda_i \mu_j / t})_{1 \leq i < j \leq N} . \tag{19}$$

This corresponds to the total partition function, and is related to the total weights of the probabilities in the lattice of the underlying discrete random walk.

In the next few sections we show how these probabilities are related to Chern-Simons quantities.

3 Chern-Simons theory as Brownian motion: the $U(N)$ case

3.1 Chern-Simons theory and surgery

In this subsection we briefly review some aspects of Chern-Simons theory in three dimensions that we will need in what follows.

Recall that Chern-Simons theory is a quantum field theory whose action is built of a Chern-Simons term involving as gauge field a gauge connection associated to a group G on a three-manifold M [23]. The action is:

$$S(A) = \frac{k}{4\pi} \int_M \text{Tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\Big) . \tag{20}$$

k is the coupling constant and the trace is taken in the fundamental representation. This action is invariant under gauge transformations; however, in order for the theory to make sense under large gauge transformations the parameter k needs to be an integer. In the partition function one then integrates over all possible gauge field configurations:

$$Z = \int \mathcal{D}A \, e^{iS_{\text{CS}}} . \tag{21}$$

At large k, the action (20) gives the equation of motion $F = 0$; classically, therefore, the Chern-Simons path integral is dominated by flat connections.

Notice that in order to define the action (20) there is no need to choose a metric; indeed, this theory is diffeomorphic invariant at the classical level, and depends only on the global properties of the manifold on which it is defined. This remains true at the quantum level, up to a subtlety. To evaluate the partition function, one needs to fix the gauge symmetry. In doing so, a choice of metric is required. This by itself does not present a problem; although the resulting terms are not topological invariant but depend on the choice of metric, one can add a counterterm (even though there are no divergences involved) that requires a choice of metric, such that the total combination is a topological invariant. In doing so, however, one needs to fix a choice of trivialization of the tangent bundle, in other words a framing of the manifold. Thus, the resulting theory is a topological invariant of framed manifolds.

Non-trivial observables in Chern-Simons theory are Wilson loops:

$$W_\lambda(C) = \text{Tr}_\lambda \mathcal{P} \exp \int_C A_\mu \mathrm{d}x^\mu \ . \tag{22}$$

This Wilson loop is labeled by a representation λ of the gauge group, and a closed loop C in M. This knot can have non-zero crossing number, which is the minimal number of crossings needed when we draw a two-dimensional picture of the knot (for an introduction to knot theory, see [24, 25]). Knots are topological invariants, in that they depend only on the topology of the manifold and on the topological class of the embedding of the loop in the manifold. Indeed, again one can see from (22) that the definition is independent of the choice of metric. We will later see how to compute Wilson loops in practice.

It is clear that one can also consider expectation values of products of Wilson loops on knots that consist of several links. In that case, every link comes with its own representation λ_i, and one should sum over the different links in (22). For knots consisting of n links, the associated invariant is

$$W_{\lambda_1 \ldots \lambda_n} = \langle W_{\lambda_1}(C_1) \ldots W_{\lambda_n}(C_n) \rangle \ . \tag{23}$$

This expression factorizes if the links are disjoint. The classic result of [23] is that these quantum field theory expectation values can actually be computed exactly, and are in fact given by certain polynomials (actually rational functions, when properly normalized) in the exponentiated coupling and rank of the gauge group, reproducing known knot invariants in the mathematical literature.

Let us ilustrate this topological invariance in the abelian case. Consider for simplicity the case of two loops, C_1 and C_2. The equations of motion in the presence of these sources take the form:

$$\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda = -\frac{4\pi}{k} J^\mu \ , \tag{24}$$

something reminiscent of Maxwell's equations in four dimensions, where the gauge potential is playing the role of the magnetic field and the source for the Wilson loops is in this case

$$J^\mu(x) = e_1 \int_0^{2\pi} \mathrm{d}s \frac{\mathrm{d}y^\mu}{\mathrm{d}s} \delta^{(3)}(x - y(s)) + e_2 \int_0^{2\pi} \mathrm{d}t \frac{\mathrm{d}z^\mu}{\mathrm{d}t} \delta^{(3)}(x - z(t)) \tag{25}$$

where e_1 and e_2 are the charges of the particles going around the loop, and $y^\mu(s)$ and $z^\mu(t)$ are embeddings of the loops in $\mathbb{R}^3$. The above can be easily solved giving the gauge potential as an integral around the loop. In the classical limit we get:

$$\langle e^{i\int \mathrm{d}^3 x J^\mu A_\mu} \rangle = e^{\frac{ie_1}{2} \oint_{C_1} A_\mu \mathrm{d}x^\mu + \frac{ie_2}{2} \oint_{C_2} A_\mu \mathrm{d}x^\mu} \ , \tag{26}$$

and filling in the value of the gauge field this gives:

$$\langle W \rangle = e^{-\frac{2\pi i}{k} \sum_{a,b} e_a e_b \chi(C_a, C_b)} \tag{27}$$

and the sum runs over $a = 1, 2$. $\chi(C_1, C_2)$ is here the Gauss linking number:

$$\chi(C_1, C_2) = \frac{1}{4\pi} \oint_{C_1} dx^\mu \oint_{C_2} dy^\nu \, \epsilon_{\mu\nu\lambda} \frac{(x-y)^\lambda}{|x-y|^3} \; . \tag{28}$$

As long as C_1 and C_2 do not intersect at any point, the Gauss linking number depends only on topology. It is given by:

$$\chi(C_1, C_2) = \frac{1}{2} \sum_p \epsilon(p) \; , \tag{29}$$

where p are the points where (on a two-dimensional picture of the link) the links cross, and we should not count self-intersections of the knots, which appear on projection of the knot on the plane. $\epsilon(p)$ is a sign assigned to the crossing, with a $+1$ for an upper pass and a -1 for an underpass. Notice that when the orientation of one of the loops is reversed, the overall sign changes.

The expectation value (27) also contains two self-interaction terms where $a = b$ that we have not considered so far. The trouble with such a term is that, even though it is well-defined and finite, it is not a topological invariant. One obtains a topological invariant if one provides knots with a framing, that is, one chooses a normal vector field along the knot, or, in other words, one "fattens" the knots, making them to small strips, and defines the self-linking number to be the linking number of boths sides of the strip. This is really a point-splitting regularization. The final result does actually depend on the choice of framing; different choices of framing change the self-linking number by an integer, and in this way we can compare different framings. Although there are no divergences involved, this is analogous to a choice of counterterms in quantum field theory. This discussion actually extends to the non-abelian, full quantum case, and we will quote the result here for future reference. A Wilson loop with n links in representations $\lambda_1, \ldots, \lambda_n$ changes according to:

$$\langle W_{\lambda_1 \ldots \lambda_n} \rangle \to e^{2\pi i \sum_{i=1}^n s_i h(\lambda_i)} \langle W_{\lambda_1 \ldots \lambda_n} \rangle \; , \tag{30}$$

where $h(\lambda)$ is the conformal weight of the representation λ in the WZW model:

$$h(\lambda) = \frac{C(\lambda)}{2(k+g)} \; , \tag{31}$$

C being the Casimir operator and g the rank of the gauge group[2]. s_i is an integer, denoting the number of units by which we change the self-linking number of the ith knot, according to the counting of crossings described above.

Let us briefly recall canonical quantization of Chern-Simons [23]. This gives an alternative viewpoint to path integral techniques, and in particular it allows us to compute Chern-Simons quantities in more complicated manifolds. The idea is to construct a three-manifold M by *surgery*, that is, by cutting the manifold into tractable pieces and then gluing them together. Locally, the manifold looks like $\Sigma \times \mathbb{R}$, where we regard $\mathbb{R}$ as "time", and by fixing temporal gauge one easily sees that the Hilbert space associated to Σ is the Hilbert space of flat connections on Σ modulo gauge transformations. We refer to [23] for a discussion of this.

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[2]For a summary of our group theory conventions, see appendix D.

We will be interested in the cases where one can obtain M by gluing together two solid tori after performing some diffeomorphism. The Hilbert space associated with the boundary of each solid torus is then the Hilbert space of the WZW model [23], which is the space of integrable representations of highest weight at level k. In particular, the vacuum state, corresponding to the solid torus without insertions of Wilson lines, is represented by the Weyl vector $|\rho\rangle$ (see appendix D), which is the trivial representation or an empty Young tableau. Inserting a Wilson loop in representation λ around the non-contractible cycle of the solid torus gives a state in the Hilbert space, $|\lambda\rangle$, where λ is integrable.

Let us consider how to get S^3 from this. To get S^3, we need to identify the A-cycle of one of the tori with the B-cycle of the other one, and viceversa. In order to do that, we act with the $SL(2, \mathbb{Z})$ generators S and T:

$$
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} ,
$$

$$
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \tag{32}
$$

where T acts on the complex structure of the torus as $\tau \to \tau + 1$, that is, it shifts the A-cycle by a B-cycle, and S acts as $\tau \to -1/\tau$, i.e., it sends A to $-B$ and B to A. From the latter operation we clearly get an S^3 after gluing both tori. Thus, we have to compute the matrix element

$$
Z_{\mathrm{CS}}(S^3) = \langle \rho | S | \rho \rangle = S_{00} . \tag{33}
$$

That is, we need to represent S on the states $|\lambda\rangle$. This is the well-known modular matrix of the WZW model, which we quote here together with the other generator T for future reference:

$$
\begin{aligned}
S_{\lambda\mu} &= \frac{i^{|\Delta_+|}}{(k+g)^{r/2}} |P/Q^\vee|^{-\frac{1}{2}} \sum_{w \in W} \epsilon(w) e^{-\frac{2\pi i}{k+g}(\lambda, w \cdot \mu)} \\
T_{\lambda\mu} &= \delta_{\lambda\mu} e^{\frac{2\pi i C(\lambda)}{2(k+g)} - \frac{2\pi i c}{24}} .
\end{aligned} \tag{34}
$$

The central charge is $c = k \dim g/(k + g)$, $C(\lambda)$ is the Casimir of the representation λ, Δ_+ is the set of positive roots, P is the weight lattice, and $Q^\vee$ is the coroot lattice.

In the gluing we could have acted with additional external factors of T. However, this only amounts to a change of framing of the S^3, as discussed earlier. We now see that framing factors are proportional to T as in (34). In what follows, we construct S^3 with the transformation TST.

By considering more general $SL(2, \mathbb{Z})$ transformations, we can construct more generic manifolds such as lens spaces and Seifert manifolds. The most general manifold that we can obtain from gluing of two solid tori (that is, by doing surgery on an unknot inside S^3) is by applying the $SL(2, \mathbb{Z})$ matrix

$$
U = \begin{pmatrix} p & r \\ q & s \end{pmatrix} , \tag{35}
$$

where $ps - qr = 1$. The representation of this matrix on affine characters is [26, 27, 28]:

$$
U_{\lambda\mu}^{(p,q)} = \frac{(i \operatorname{sgn} q)^{|\Delta_+|}}{(l|q|)^{r/2}} e^{-\frac{id\pi}{12} \Phi(U^{(p,q)})} |P/Q|^{1/2} \sum_{n \in Q^\vee/qQ^\vee} \sum_{w \in W} \epsilon(w) e^{\frac{i\pi}{lq}[p\lambda^2 - 2\lambda(ln + w\mu) + s(ln + w\mu)^2]}
$$

$$
\tag{36}
$$

where Q is the root lattice, $\Phi(U^{(p,q)})$ is the Rademacher function

$$\Phi \begin{bmatrix} p \ r \\ q \ s \end{bmatrix} = \frac{p+s}{q} - 12s(p,q) \ , \tag{37}$$

and $s(p,q)$ is the Dedekind sum

$$s(p,q) = \frac{1}{4q} \sum_{n=1}^{q-1} \cot \frac{\pi n}{q} \cot \frac{\pi np}{q} \ . \tag{38}$$

In section 5 we will work out in detail the case of the lens space $S^3/\mathbb{Z}_p$. More general manifolds can be obtained by surgery on more general knots. See for example [26, 27, 28] for Seifert spaces.

Finally, we give the explicit expressions for the partition function and Wilson loops of Chern-Simons in S^3, which we will use later. The partition function on S^3 for gauge group $U(N)$ is given by working out (33):

$$Z_{\mathrm{CS}}(S^3) = \frac{1}{(k+N)^{N/2}} \prod_{j=1}^{N-1} (2\sin \frac{\pi j}{k+N})^{N-j}. \tag{39}$$

The unnormalized expectation value of a Wilson loop around the unknot C is:

$$W_\lambda(C) = \frac{1}{(k+N)^{N/2}} \prod_{i<j} 2\sin \frac{\pi(\lambda_i - \lambda_j)}{k+N} \ . \tag{40}$$

3.2 Chern-Simons on S^3: the probability of reunion

In this section we work out in detail the relation between Brownian motion and Chern-Simons theory for the case of the partition function, which corresponds to the probability of reunion of random walkers.

We will now evaluate the above probability density in the simplest case: the probablity of reunion, and see how this gives us the partition function of Chern-Simons on S^3 with gauge group $U(N)$. The probability of reunion is defined [19] as the probability of, given that we start with a configuration where all the movers are equally spaced,

$$\lambda_j = (c-j)a \tag{41}$$

ending up with the same configuration after time t: $\mu_j = \lambda_j$. The special boundary condition (41) we will call λ_{0j}. Notice that a is the initial spacing between two movers:

$$\lambda_{0j+1} - \lambda_{0j} = a \tag{42}$$

and at this point it can be positive or negative. Since we arrange our coordinates such that $\lambda_1 > \lambda_2 > \ldots \lambda_N$, we will take a to be positive. Let us here emphasize that, even though in the discrete case a would naturally have an interpretation as a lattice length and hence a minimal distance between two movers, that is not the case for Brownian motion, where two particles can be arbitrarily close to each other as long as they do not intersect. Notice that to get the trivial representation, labeled by the Weyl vector, we have to take (see appendix D)

$$\lambda_{0j} = \rho_j = \frac{N+1}{2} - j \tag{43}$$

12

which in particular satisfies $\sum_{i=1}^{N} \rho_i = 0$, a property that we will use later. However, as mentioned earlier, the overall constant is irrelevant as it only corresponds to a constant shift. In this and the next subsection we do not keep track of coupling-independent phase factors.

In appendix C we show that the probability of reunion is given by

$$
\begin{aligned}
p_{t,N}(\lambda_0, \lambda_0) &= \frac{1}{(2\pi t)^{N/2}} \prod_{k=1}^{N} (1 - q^k)^{N-k} \\
&= \frac{1}{(2\pi t)^{N/2}} q^{\frac{1}{6}N(N^2-1)} \prod_{k=1}^{N} (1 - q^{-k})^{N-k} \ ,
\end{aligned}
\tag{44}
$$

where $q = e^{-a^2/t}$. If we compare this with (39) (see also (107)), we see that up to normalizations both expressions are the same[3] if we set $a^2 = 1$ and

$$
-\frac{1}{t} = g_s = \frac{2\pi i}{k + N} \ .
\tag{45}
$$

Then we get

$$
Z_{\mathrm{mm}} = p_{t,N}(\lambda_0, \lambda_0)
\tag{46}
$$

and in canonical framing

$$
Z_{\mathrm{CS}}(S^3) = e^{-\frac{1}{12}g_s N(N^2-1)} p_{t,N}(\lambda_0, \lambda_0) \ .
\tag{47}
$$

This is the basic result in [8] concerning the relation between the partition function of Chern-Simons on S^3 and the probability of reunion of N vicious Brownian movers.

3.3 Wilson lines: the unknot and the Hopf link

The next non-trivial case is to consider Brownian motion where we fix the initial positions to be equally spaced, but leave the final positions arbitrary. Thus we fix

$$
\mu = \rho a
\tag{48}
$$

where ρ is the Weyl vector (see appendix D) and leave λ arbitrary. The resulting determinant (14) can now be readily computed. In a first step we use the standard Vandermonde determinant formula (158). In a second step, we use the identity (161). The result is:

$$
\det(e^{-(\lambda_i - \rho_j a)^2/2t}) = e^{-\frac{|\lambda|^2}{2t} - \frac{1}{24t}N(N^2-1)} \prod_{i<j} 2\sinh\left(\frac{\lambda_i - \lambda_j}{2t} a\right) \ .
\tag{49}
$$

Of course, we have simply proved the Weyl denominator formula for the symmetric group. As a simple check, filling in $\lambda = \rho$ gives us

$$
p_{t,N}(\rho, \rho) = \frac{1}{(2\pi t)^{N/2}} e^{-\frac{1}{12t}N(N^2-1)} \prod_{i<j} 2\sinh(\frac{j-i}{2t}) \ ,
\tag{50}
$$

---

[3]Notice that at this point the relation $-1/t = g_s$ could involve a plus sign instead of a minus sign. This would result in a different choice of framing, which is therefore immaterial for the interpretation as the partition function of Chern-Simons on S^3. We will see, however, that this arbitrariness is fixed by comparing our expressions with the WZW modular S-matrix, which will give the minus sign. Notice, however, that ultimately this is irrelevant as all expressions are holomorphic in the coupling. In particular, the relation $1/t = g_s$ should work as well if we identify the Brownian motion probability with the complex conjugate of S.

which reduces to (14) after working out the product and filling in the value of t.

What quantity could the more general equation (49) possibly be in Chern-Simons theory? In fact it is nothing but the expectation value of a Wilson loop on S^3 in a representation λ of $U(N)$, as we can see from (40). This Wilson loop winds an unknot, which in S^3 is the unique knot with one link component since there are no non-trivial cycles. The Wilson loop is however not in canonical framing. To see this, it is easiest to write

$$p_{t,N}(\lambda, \rho) = e^{\frac{2\pi i}{12}N(N^2-1)} \langle 0|TST|\lambda \rangle \tag{51}$$

where the modular matrices T and S are given as in (34). In appendix E we will analyze this framing in detail.

Having written down the representation (51), it is now straightforward to generalize this to the probability density with arbitrary initial and final states: we simply compute the operator TST within arbitrary external states λ and μ, corresponding to the two boundary contitions:

$$p_{t,N}(\lambda, \mu) = e^{\frac{2\pi i}{12}N(N^2-1)} \langle \mu|TST|\lambda \rangle \ , \tag{52}$$

as is easily checked.

4   Brownian motion with a wall: $SO(N)$ and $Sp(N)$ Chern-Simons

An interesting generalization of the Brownian movers on a line is to see what happens when we impose an additional restriction by inserting a wall at the origin, $\lambda = 0$ [29]. Thus we consider Brownian motion of N non-intersecting movers on the half-line $\mathbb{R}_+$, that is, the coordinates of the movers now satisfy $\lambda_1 > \lambda_2 > \ldots > \lambda_N > 0$. We immediately see that this is the Weyl chamber of SO and Sp, and we will return to this interpretation momentarily.

Using the method of images, we can again compute this probability from the free case (10). Namely, on top of the reflections about the forbidden walls $\lambda_i = \lambda_j$, $i \neq j$, we need to reflect around the origin. We get

$$p_{t,N}(\lambda, \mu) = \sum_{\sigma \in S_N} \epsilon(\sigma) \sum_{\varepsilon_1 \ldots \varepsilon_N = \pm 1} (-1)^{\varepsilon_1 + \ldots + \varepsilon_N} \prod_{i=1}^{N} p_t^0(\varepsilon_i \lambda_{\sigma(i)} - \mu_i) \tag{53}$$

where $p_t^0(\lambda, \mu) = \frac{1}{\sqrt{2\pi t}} e^{-(\lambda-\mu)^2/2t}$ is the free Brownian motion probability for one particle. This can be rewritten as:

$$
\begin{aligned}
p_{t,N}(\lambda, \mu) &= \frac{1}{(2\pi t)^{N/2}} e^{-\frac{|\lambda|^2+|\mu|^2}{2t}} \sum_{\sigma \in S_N} \epsilon(\sigma) \sum_{\varepsilon_1 \ldots \varepsilon_N = \pm 1} (-1)^{\varepsilon_1 + \ldots + \varepsilon_N} e^{\sum_{i=1}^{N} \varepsilon_i \lambda_{\sigma(i)} \mu_i/t} \\
&= \frac{1}{(2\pi t)^{N/2}} e^{-\frac{|\lambda|^2+|\mu|^2}{2t}} \sum_{\sigma \in S_N} \epsilon(\sigma) \prod_{i=1}^{N} (e^{\lambda_{\sigma(i)} \mu_i/t} - e^{-\lambda_{\sigma(i)} \mu_i/t}) \\
&= \frac{1}{(2\pi t)^{N/2}} e^{-\frac{|\lambda|^2+|\mu|^2}{2t}} \det(e^{\lambda_i \mu_j/t} - e^{-\lambda_i \mu_j/t})_{1 \leq i < j \leq N} \ .
\end{aligned}
\tag{54}
$$

We now come to the connection with the $SO(N)$ and $Sp(N)$ groups. (53) is really a sum over the Weyl chamber of B_N. Notice that the Weyl chambers of B_N ($SO(2N+1)$) and C_N ($Sp(2N)$) are the same, and indeed we can interpret (54) as Brownian motion probabilities of a single mover moving from λ to μ in the Weyl chamber of $SO(2N+1)$ or $Sp(2N)$.

Asymptotic expressions for (54) were given in [29] at large t for the case of initial equal spacing condition. However, we can compute (54) rather explicitly using the Weyl denominator formula. Remember that the equal spacing condition amounts to setting λ to be the Weyl vector, $\lambda = \rho$. Explicitly,

$$\rho = \sum_{i=1}^{N}(N+1-i)e_i \qquad \text{for } Sp(2N)$$

$$\rho = \sum_{i=1}^{N}(N+\frac{1}{2}-i)e_i \qquad \text{for } Sp(2N) . \tag{55}$$

Thus, $\rho_{\text{Sp}} = \rho_{\text{SO}} + \frac{1}{2}(1,1,\ldots,1)$. Obviously, although the Weyl chambers are the same, their root systems are different. In particular, some of the positive roots differ by a factor of 2. That gives the factor of $1/2$ difference in the above formulas for the Weyl vectors. Now setting the initial condition $\lambda = \rho$ equal to the Weyl vector, both for SO and Sp groups we can rewrite (54) as:

$$p_{t,N}(\rho,\mu) = \frac{1}{(2\pi t)^{N/2}} e^{-\frac{|\rho|^2+|\mu|^2}{2t}} \prod_{\alpha>0}(e^{(\alpha,\mu)/2t} - e^{-(\alpha,\mu)/2t})$$

$$= \frac{1}{(2\pi t)^{N/2}} e^{\frac{-|\rho|^2+|\mu|^2}{2t}} 2^{|\Delta|_+} \prod_{\alpha>0} \sinh\frac{(\alpha,\mu)}{2t} . \tag{56}$$

We now have to specify explicitly what the positive roots α_{ij} are. For $SO(2N+1)$ they are:

$$\{e_i - e_j\}_{i<j}$$
$$\{e_i + e_j\}_{i<j}$$
$$\{e_i\} , \tag{57}$$

with $i,j = 1,\ldots,N$. Thus we get:

$$p_{t,N}(\rho,\mu) = \frac{1}{(2\pi t)^{N/2}} e^{-\frac{|\rho|^2+|\mu|^2}{2t}} 2^{|\Delta|_+} \prod_{i<j} \sinh\frac{\mu_i-\mu_j}{2t} \sinh\frac{\mu_i+\mu_j}{2t} \prod_{k=1}^{N} \sinh\frac{\mu_i}{2t} . \tag{58}$$

By working with the determinantal expression directly rather than using the Weyl formula, the two first sinh can be combined into a single Vandermonde term [30]. Having written the expression for generic μ, we now specialize to $\mu = \rho$ also, so that we can compare with the partition function of Chern-Simons. We get:

$$p_{t,N}(\rho,\rho) = \frac{1}{(2\pi t)^{N/2}} e^{-|\rho|^2/t} 2^{|\Delta|_+} \prod_{k=1}^{2N+1} \left(\sinh\frac{k}{2t}\right)^{f(k)} \tag{59}$$

where $f(k)$ is given in (183) of appendix F. In fact, the above is also the answer for $Sp(2N)$. In that case, the positive roots are

$$\{e_i - e_j\}_{i<j}$$
$$\{e_i + e_j\}_{i<j}$$
$$2\{e_i\} . \tag{60}$$

Thus we get the expression (58) with an additional factor of 2 in the last term. Again, specializing to $\mu = \rho_{\text{Sp}}$, we get (59) with $f(k)$ as in formula (184) of appendix F.

Let us now compare with the partition function of Chern-Simons on S^3. It is equal to:

$$Z_{\text{CS}}(S^3) = S_{00} = (k+g)^{-1/2} \prod_{\alpha > 0} 2 \sin \frac{\pi(\alpha, \rho)}{k+g} \tag{61}$$

Up to the usual framing factor and roots of unity, this is precisely formula (56) with boundary condition $\mu = \rho$. In particular, equation (59) precisely agrees with the expressions in [31] for the $SO(2N+1)$ and $Sp(2N)$ partition function of Chern-Simons on S^3. The identification between the coupling and the level is given by (45), as before. The framing factor $e^{-|\rho|^2/t}$ is given by $|\rho|^2 = \frac{1}{12}N(4N^2 - 1)$ for $SO(2N+1)$, and $|\rho|^2 = \frac{1}{6}N(N+1)(2N+1)$ for $Sp(2N)$.

It should now also be clear that the more general expressions $p_{t,N}(\rho, \mu)$ and $p_{t,N}(\lambda, \mu)$ correspond to the expectation values of the unknot and the Hopf link, respectively.

Having done $SO(2N+1)$ and $Sp(2N)$, whose root systems are given by B_N and C_N, respectively, it is now natural to look at $SO(2N)$, which corresponds to D_N. The Weyl chamber is now given by $\lambda_1 > \lambda_2 > \ldots \lambda_N$, and in addition $\lambda_{N-1} > -\lambda_N$. This means that $\lambda_{N-1} > |\lambda_N|$ independently of the sign of λ_N. We can interpret this as Brownian motion of N particles on a line with a wall at $\lambda_N = 0$ such that, when the Nth particle crosses it, a mirror particle traveling along the mirror trajectory $|\lambda_N|$ is emitted, such that λ_{N-1} is not allowed to intersect the trajectory of the mirror particle (see Fig. 2). Notice that such paths are counted twice, once from the particle that goes to negative λ_N, and once from the one that stays at $\lambda_N \geq 0$.

The positive roots of $SO(2N)$ are:

$$\alpha_{ij}^{\pm} = e_i \pm e_j \tag{62}$$

with $1 \leq i < j \leq N$. The Weyl vector is:

$$\rho = \sum_{i=1}^{N}(N-i)e_i \ . \tag{63}$$

The probability is now easy to work out and goes as before:

$$p_{t,N}(\lambda, \mu) = \frac{1}{2} \frac{1}{(2\pi t)^{N/2}} e^{-\frac{|\lambda|^2 + |\mu|^2}{2t}} [\det(e^{\lambda_i \mu_j/t} + e^{-\lambda_i \mu_j/t}) + \det(e^{\lambda_i \mu_j/t} - e^{\lambda_i \mu_j/t})] \ . \tag{64}$$

Notice that the last term is precisely the probability for B_N and C_N. This term vanishes if some $\mu_i = 0$. By construction this was never the case for B_N and C_N, since the Weyl chamber precisely satisfied the condition $\lambda_1 > \lambda_2 > \ldots > \lambda_N > 0$. However, for D_N this is possible for the Nth particle, which can indeed reach $\lambda_N = 0$ as depicted in Fig. 2. In fact, from (63) we see that the Weyl vector does have a zero. Thus in computing values of $p_{t,N}(\lambda, \mu)$ with either λ or μ equal to ρ, the second term drops out. We can now expand the determinant for $\mu = \rho$ to get:

$$p_{t,N}(\lambda, \rho) = \frac{1}{(2\pi t)^{N/2}} e^{-\frac{|\lambda|^2 + |\rho|^2}{2t}} \prod_{i<j} 4 \sinh \frac{\lambda_i - \lambda_j}{2} \sinh \frac{\lambda_i + \lambda_j}{2} \ , \tag{65}$$

which is the expected result for the expectation value of a Wilson loop for the group $SO(2N)$. Filling in also $\lambda = \rho$, what we get is the expression for the partition function of $SO(2N)$ Chern-Simons on S^3 (61), as expected. The explicit expression is given in appendix F.

A further interesting case to analyze would be that of particles moving between two walls.
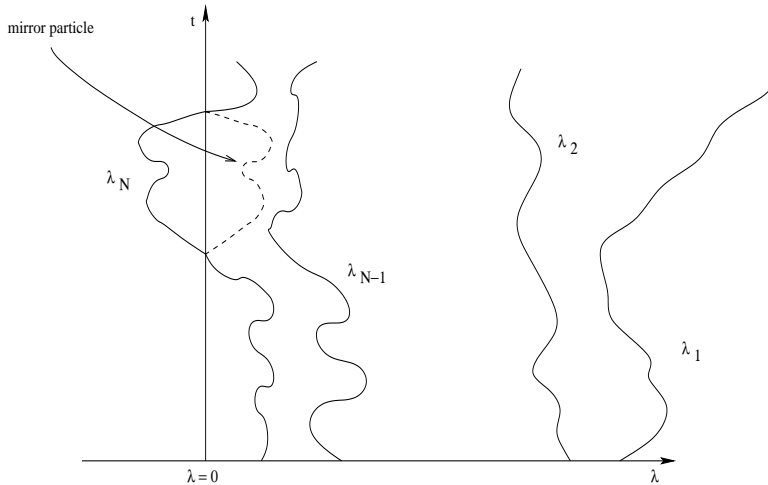
Figure 2: Two-dimensional representation of motion in the Weyl chamber of $SO(2N+1)$ and $Sp(2N)$.

5 Lie algebra wanderers: general formulation

We can generalize the above to any Weyl group associated to a root system [32]. In particular, the root system can be finite (A-G series) or affine. The affine case was worked out in detail in [6] and it was shown to be equivalent to 2dYM on the cylinder.

Thus we consider Brownian motion of a particle in the Weyl chamber C of some root system Δ. The Weyl group, W, generates $\mathbb{R}^r$ from the action on C, and r is the rank. Let λ and μ be two points in $\mathbb{R}^r$ (which, as usual, in the finite case will be chosen such as to correspond to representations of the corresponding group, and in the affine case will correspond to the integrable representations only). We can compute the conditional probability of going from λ to μ in time t without leaving the fundamental Weyl chamber by the method of images, as follows. The argument is basically a continuous version of a well-known argument by Zeilberger (see [32] and references therein). We denote the total number of walks[4] from λ to μ in time t by $p_t^0(\lambda, \mu)$. Notice that $p_t^0(\lambda, \mu)$ is the free Brownian motion probability,

$$p_{t,r}^0(\lambda, \mu) = \frac{1}{(2\pi t)^{r/2}} \, e^{|\lambda - \mu|^2/2t} \, , \tag{66}$$

where r is the rank of the gauge group. These are walks in $\mathbb{R}^r$. To count the number of walks that stay within C, we divide all the walks in $p_t^0(\lambda, \mu)$ into good and bad waks, according to whether they do or do not remain within the Weyl chamber. We will call these $p_{t,r}(\lambda, \mu)$ and $q_t(\lambda, \mu)$, respectively, and we are obviously interested in computing $p_{t,r}(\lambda, \mu)$. It is clear that we have:

$$p_{t,r}(\lambda, \mu) = p_{t,r}^0(\lambda, \mu) - q_{t,r}(\lambda, \mu) \, . \tag{67}$$

The key step is now to use the method of images to show that the bad walks satisfy

$$\sum_{w \in W} \epsilon(w) \, q_{t,r}(w(\lambda), \mu) = 0 \, . \tag{68}$$

---

[4]This is actually a probability, but in this section we will continue to talk of the number of paths; of course, the difference is just dividing out an infinite factor.

That is, for each bad walk from $w(\lambda)$ to μ we can construct a mirror walk by reflecting with respect to the wall that was crossed last. Let us call the root that this wall corresponds to $\alpha \in \Delta$. The wall is by definition perpendicular to α, $(x, \alpha) = 0$. The mirror walk in question is then another bad walk, constructed by taking the mirror part of the walk from $w_\alpha w(\lambda)$ to the crossing point, and from there continuing to μ along the original walk. This gives a unique pairing between walks in the sum (68). Since w and $w_\alpha w$ have different sign, all such pairs cancel out.

Now it is also clear that

$$p_{t,r}(w(\lambda), \mu) = q_{t,r}(w(\lambda), \mu) \ \text{ if } w \neq \text{id} , \tag{69}$$

since $w(\lambda)$ always lies outside C. Combining the above three formulas, we get

$$p_t(\lambda, \mu) = \sum_{w \in W} \epsilon(w) p_t^0(w(\lambda), \mu) = \frac{1}{(2\pi t)^{r/2}} \sum_{w \in W} \epsilon(w) e^{-|\lambda - w\mu|^2/2t} . \tag{70}$$

This is the desired generalization of the non-intersecting probability, and we should note that it holds both if W is finite or affine [6].

We would next like to compute it in some natural "vacuum" state and compare it with Chern-Simons theory. We will do this for the simple algebras. We take $\mu = \rho$ where ρ is the Weyl vector labeling the trivial representation. Using the Weyl denominator formula, we get:

$$p_{t,r}(\lambda, \rho) = \frac{1}{(2\pi t)^{r/2}} e^{-\frac{|\lambda|^2 + |\rho|^2}{2t}} \prod_{\alpha > 0} 2 \sinh \frac{(\alpha, \lambda)}{2t} , \tag{71}$$

where α are the positive roots. Under the identification $-1/t = 2\pi i/(k+g)$, this expression is the (unnormalized) expectation value of a Wilson loop around the unknot.

The partition function is obtained by setting $\lambda = \rho$:

$$Z_{\text{CS}}(S^3) = \frac{(-i)^{r/2}}{(k+g)^{r/2}} e^{-\frac{2\pi i}{12(k+g)} g \dim g} \prod_{\alpha > 0} 2 \sin \frac{\pi(\alpha, \rho)}{k+g} \tag{72}$$

where we used the Freudenthal-de Vries strange formula $|\rho|^2 = \frac{g}{12} \dim g$.

Notice that normalizing $p_{t,r}(\lambda, \rho)$ by $p_{t,r}(\rho, \rho)$ gives the quantum dimension:

$$\prod_{\alpha > 0} \frac{[(\lambda, \alpha)]}{[(\rho, \alpha)]} . \tag{73}$$

We can also rewrite the above directly in terms of the WZW modular matrices. Using the definitions (34) and the value of the central charge $c = k \dim g/(k+g)$, we get

$$p_{t,r}(\lambda, \mu) = e^{\frac{2\pi i}{12} \dim g} (TST)_{\lambda\mu} . \tag{74}$$

Obviously, S itself corresponds to the interaction term in the Brownian motion probability, and T is the Boltzmann factor, which can be seen as a propagator in the diffusive medium for translations over time t. It is now also clear that $p_{t,r}(\lambda, \mu)$ itself gives the (unnormalized) expectation value of the Hopf link invariant with representations λ and μ.

In this language it is now much more clear why we are getting the partition function of Chern-Simons on S^3, and we can give an intuitive physical picture in terms of surgery that we will explain in more detail later. The boundary conditions λ and μ correspond to states in the

Chern-Simons Hilbert space associated with two solid tori. That is, we have two solid tori where the cycle that is not filled in carries a representation λ, μ, respectively, of the gauge group G. The probability tells us to propagate a random walker from λ to μ but performing a transformation TST, that is, we glue the two solid tori with a modular transformation $\tau \to \tau/(\tau + 1)$ before gluing. This gives us an S^3 in a framing that is not the canonical one discussed in section 3, but is the matrix model framing, which is the natural framing in the mirror B-model on the resolved conifold, as we will see later [34]. We will get back to this in section 6. We discuss other framings in the appendix E.

The matrix model will be analyzed in great detail in section 7.

5.1 Lens spaces and q-deformed 2d Yang-Mills

We now discuss the case of the lens space $S^3/\mathbb{Z}_p$. This can be obtained by taking the affine Lie algebra above, with an important subtlety that was discussed in [6].

The affine Weyl chamber is

$$\tilde{W} = W \ltimes T \ , \tag{75}$$

where T denotes translations in the coroot lattice $Q^\vee$. The Brownian motion probability in this fundamental Weyl chamber is then

$$q_{t,r}(\lambda, \mu) = \frac{1}{(2\pi t)^{r/2}} \sum_{\gamma \in lQ^\vee} \sum_{w \in W} \epsilon(w) e^{-\frac{1}{2t}|\gamma + \lambda - w\mu|^2} \ . \tag{76}$$

For real positive t, this is well-known to converge; that can for example be shown by rewriting it in terms of Θ-functions [6]. When $1/t$ is $2\pi i$ times a rational number, however, as is the case if we want to compare with Chern-Simons theory, the sum is not well-defined because the exponential is periodic. As explained in [6], a finite expression is obtained if one mods out by this periodicity. Alternatively, one may regularize $-1/t = 2\pi i p/l + \epsilon$, where p and l are integers and ϵ real positive, and factor out an overall divergence, which goes like $\epsilon^{-r/2}$. This is useful if one wants to compare Chern-Simons on lens spaces and 2d Yang-Mills on the cylinder. Indeed, the appropriately normalized (76) gives the partition function of two-dimensional Yang-Mills theory on the cylinder with the corresponding gauge group, where λ and μ label the holonomies of the gauge field at the two ends of the cylinder. More precisely [6]:

$$Z_{2\mathrm{dYM}}(g, g'; t) = \frac{(-il)^r |P/Q^\vee|}{S_{0\lambda'} S_{0\mu'}} \, q_{t,r}(\lambda, \mu) \ , \tag{77}$$

where we used the fact that the partition function is a class function of g and g' to conjugate them into $e^{\lambda'}$ and $e^{-\mu'}$, respectively, and we normalized $\lambda' = 2\pi\lambda/l$, $\mu' = 2\pi i\mu/l$. The above was used to find the modular transformation properties of the partition function of 2d Yang-Mills on the cylinder [6]. It was further shown that:

$$\frac{(ST^pS)_{\lambda\mu}}{S_{\lambda\rho} S_{\rho\mu}} = \left(\frac{p^2\epsilon}{\pi}\right)^{r/2} Z_{2\mathrm{dYM}}(g, g'; t') \ , \tag{78}$$

up to a phase, and $t' = (2\pi i/l)^2 t$. This generalized previous relations between 2dYM and WZW matrices [12, 13], to the case where the coupling times the area is $2\pi i$ times a rational number. As discussed in [6], the different normalizations are just what one expects, and the overall divergence is obviously related to the fact that the Hilbert space is infinite, while we are

imposing the periodicity mentioned before. This is however only an overall divergence that can be renormalized.

One could however ask whether it makes sense to keep t real and still talk about modular transformation matrices. In fact, in [7] it was shown that this is possible, and defines a quantum deformed 2d Yang-Mills which is defined for values of g_s which are not $2\pi i$ times a rational number[5]. As shown in [7], in that case the Hilbert space is infinite and one then typically sums over all representations and not only the integrable ones. The theory in question is the BF-theory introduced in [13], where the scalar ϕ (the B of BF) is compact. The partition function was shown to be:

$$Z_{\text{qYM}}(\Sigma_g) = \sum_{\lambda} \left(\frac{S_{\lambda 0}}{S_{00}} \right)^{2-2g} q^{pC_2(\lambda)/2} e^{i\theta\lambda} \ . \tag{79}$$

In [7] this was called quantum deformed 2dYM because $S_{\lambda 0}$ are the quantum dimensions rather than the ordinary dimensions which appear in 2dYM. Let us consider the case of the sphere, $g = 0$, with $\theta = 0$. In this case, this is just a particular case of the partition function of ordinary 2dYM on the cylinder where $\lambda = \mu = \rho$, and so the results of [6] relate this to $q_{t,r}(\rho, \rho)$, (76). Such an expression was also found in [7]. In [6] it was also noticed that 2dYM on the cylinder and the BF theory of Blau and Thompson are related, and from the above we see that indeed

$$Z_{\text{2dYM}}(\rho, \rho) = Z_{\text{qYM}}(S^2) \ . \tag{80}$$

Notice that this holds for any value of p. The modular transformation properties of this theory were worked out in [6] and we can now obtain from them the modular transformation properties of the corresponding A-model topological string amplitudes. It would be worth to study this in more detail. Notice also that an explicit expression for $Z_{\text{2dYM}}(\lambda, \mu)$ with its modular transformation properties are also available [6]. It would be interesting to see if these can be interpreted as insertions on the sphere on the q-deformed 2dYM side.

Finally, let us remark that it is possible to obtain matrix model expressions for quantum deformed 2d Yang-Mills, which are useful to compute the large N expansion. In some simple cases, it is not hard to see that these are the same Chern-Simons matrix model expressions, analytically continued in the coupling. We will get back to this issue in the near future [35].

6 Brownian particles as mirror B-model branes

6.1 The open topological string side

Chern-Simons on S^3 describes A-model open topological strings on the deformed conifold T^*S^3:

$$xu + yv = \mu \ . \tag{81}$$

In this section we briefly recall some aspects of the construction that we will use later on (see, for example, [34]).

It will be useful to recall the canonical quantization of Chern-Simons on S^3 [36, 34]. The Hilbert space is as usual the Hilbert space of the torus. Denote the holonomies of the gauge field around the A- and B-cycles by u and v, respectively. On the torus one can always conjugate the gauge field into the Cartan subalgebra [36], and so

$$[u_i, v_j] = g_s \delta_{ij} \tag{82}$$

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[5]This theory computes A-model amplitudes on a Calabi-Yau which is a sum of line bundles over a Riemann surface, where the Riemann surface is interpreted as the Riemann surface where the quantum deformed 2d Yang-Mills lives.

where i, j run from 1 to N, and as usual $g_s = 2\pi i/(k + N)$, thus incorporating the shift $k \to k + N$. On the solid torus, however, as explained in [34], large gauge transformations that shift v in the coroot lattice are not a symmetry, and so u and v are non-compact variables. The partition function is then given by

$$Z_{\text{CS}}(S^3) = \langle 0_v | e^{\text{Tr } u^2/2g_s} | 0_v \rangle \ , \tag{83}$$

where $|0_v\rangle$ is the vacuum state, i.e. a solid torus with a trivial representation. Writing the above in the u-basis then gives the matrix model expression for the partition function:

$$\int \mathrm{d}u_1 \ldots \mathrm{d}u_N \, e^{\sum_{i=1}^N u_i^2/2g_s} \prod_{i<j} \sinh^2 \frac{u_i - u_j}{2} \tag{84}$$

up to normalization.

It is useful to consider the mirror of this. It is given by resolving the conifold geometry

$$xy = (e^u - 1)(e^v - 1) \tag{85}$$

on the open string B-model side by blowing up a $\mathbb{P}^1$ at $x = y = u = v = 0$ [37]. One then has N D1-branes wrapping the $\mathbb{P}^1$, whose action is given by the dimensional reduction of the holomorphic Chern-Simons action to the world-volume of the D1's [38, 39] which describes the normal directions to the branes in the Calabi-Yau[6]. As shown by Dijkgraaf and Vafa, this theory reduces to a matrix model, whose solution gives the Riemann surface appearing in the deformed closed string geometry [39]. This is well-known and we will not review it here.

More generally, if we have some gluing operator $U_{\lambda\mu}^{(p,q)} \in SL(2,\mathbb{Z})$ which is built out the S and T modular matrices, one again obtains a matrix model on the B-model side. In Chern-Simons theory, the general expression for U is known and it was given in (36).

6.2 Brownian motion fermions and the B-model

The Brownian motion problem is an intrinsically dissipative one. It is therefore natural to ask what is the quantum mechanical system associated to it, that is, the quantum mechanical system whose amplitudes are given by the Brownian motion probabilities. Since the probabilities are positive quantities and the amplitudes are complex numbers, we do expect that some analytic continuation will be involved. In this section we construct such a quantum mechanical system, and we find that it is given by a system of fermions on a non-commutative space, which can be identified with B-model branes [41, 42].

Our aim is to set up a quantum mechanical problem such that

$$p_t(\lambda, \mu) = \langle \mu | U(t) | \lambda \rangle \ . \tag{86}$$

We would like to find the states $|\lambda\rangle$, and the operator $U(t)$ which gives the time evolution. Keeping in mind the role of λ and μ as holonomies of the gauge field around the two cycles of the torus and the canonical quantization of Chern-Simons reviewed in the previous section, it is natural to require

$$[\lambda_i, \mu_j] = g_s \delta_{ij} \tag{87}$$

but to still regard λ_i and μ_i as the coordinates of a system of N particles, like in the original Brownian motion problem, since we are interested in describing motion of quantum mechanical

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[6]For a careful treatment of such non-holomorphic embeddings, see [40].

particles. Furthermore, recalling the non-intersecting condition, it is now natural to take the particles to be fermions rather than bosons. That is, if we have a multi-particle state of the form

$$\varphi_1(\lambda_1)\varphi(\lambda_2)\cdots\varphi(\lambda_N) , \tag{88}$$

we consider Slater wave-functions:

$$\psi(x) = \frac{1}{\sqrt{N!}} \det(\varphi_i(\lambda_j))_{1 \leq i < j \leq N} . \tag{89}$$

In what follows we will also use the short-hand notation

$$|\lambda\rangle = \frac{1}{\sqrt{N!}} \det(|\lambda_i\rangle_j) . \tag{90}$$

Of course, we should check that this is an orthonormal set of states of antisymmetric wave-functions. This is easy to check:

$$\langle\lambda|\lambda'\rangle = \det \delta(\lambda_i - \lambda'_j) . \tag{91}$$

We also have

$$\frac{1}{N!} \int_{\mathbb{R}^N} d^N\lambda \, |\lambda\rangle\langle\lambda|\lambda'\rangle = |\lambda'\rangle \tag{92}$$

so really

$$\frac{1}{N!} \int_{\mathbb{R}^N} d^N\lambda \, |\lambda\rangle\langle\lambda| = 1 . \tag{93}$$

The factor of $N!$ comes from the natural measure induced by the integration over the Weyl chamber:

$$\int_C d^N\lambda \, f(\lambda) = \frac{1}{N!} \int_{\mathbb{R}^N} d^N\lambda \, f(\lambda) \tag{94}$$

where C is the set $\infty > \lambda_1 > \lambda_2 > \ldots > \lambda_N > -\infty$, and f is a symmetric function.

Now we want to be able to represent (87) in this fermionic Hilbert space. This is of course just a Fourier transformation. For a single particle we have

$$\varphi_i(\mu) = \frac{1}{\sqrt{2\pi g_s}} \int_{-\infty}^{\infty} d\lambda \, e^{\lambda\mu/g_s} \varphi(\lambda) . \tag{95}$$

Obviously, we then have

$$\psi(\mu) = \frac{1}{(2\pi g_s)^{N/2}} \int_{\mathbb{R}^N} d^N\lambda \, e^{\operatorname{Tr}\lambda\mu/g_s} \psi(\lambda) = \frac{1}{N!} \frac{1}{(2\pi g_s)^{N/2}} \int_{\mathbb{R}^N} d^N\lambda \, \det(e^{\lambda\mu/g_s}) \psi(\lambda) . \tag{96}$$

We immediately compute

$$\langle\lambda|\mu\rangle = \frac{1}{(2\pi g_s)^{N/2}} \det(e^{\lambda_i\mu_j/g_s}) . \tag{97}$$

Fourier transforming back gives:

$$\psi(\lambda) = (-2\pi/g_s)^{N/2} \int_{\mathbb{R}^N} d^N\mu \, e^{-\operatorname{Tr}\lambda\mu/g_s} \psi(\mu) . \tag{98}$$

We are now in a position to compute the operator $U(t)$. By definition, it will satisfy

$$U(t+s) = U(t)U(s) \tag{99}$$

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or in other words

$$\langle\mu|U(t+s)|\lambda\rangle = \int_C \mathrm{d}^N\nu\, \langle\mu|U(t)|\nu\rangle\langle\nu|U(s)|\lambda\rangle \ . \tag{100}$$

Computing $\langle\mu|U(g_s)|\lambda\rangle$, we find

$$\frac{\langle\lambda|U(g_s)|\mu\rangle}{\langle\lambda|\mu\rangle} = e^{-\frac{\lambda^2+\mu^2}{2g_s}} \ . \tag{101}$$

Since this holds for any state, it also holds as an operator equation that

$$U = e^{-\mu^2/2g_s}e^{-\lambda^2/2g_s} \ . \tag{102}$$

Of course, we could just as well define new states

$$\phi(\lambda) = e^{-\lambda^2/2g_s}\varphi(\lambda) \ . \tag{103}$$

These transform as

$$\phi(\lambda) = (-2\pi/g_s)^{N/2} \int_{\mathbb{R}^N} \mathrm{d}^N\mu\, e^{-\lambda^2/2g_s}e^{-\mathrm{Tr}\lambda\mu/g_s}\phi(\mu) \ , \tag{104}$$

and their inner product gives the Brownian motion probability.

From (97) we see that the matrix elements $\langle\lambda|\mu\rangle$ again give the $SL(2,\mathbb{Z})$ S-matrix. This is a non-trivial fact and justifies the choice of fermionic wavefunctions. On the other hand, U is a product of T-matrices $T_{\lambda\lambda}T_{\mu\mu}$. The latter is by construction so, as we demanded $p_t(\lambda,\mu) = (TST)_{\lambda\mu}$.

It is now natural to idenfity the wave-functions $\varphi(\lambda)$ and $\varphi(\mu)$, which transform according to (96), as the B-model fermions of [41]. In fact, in view of the discussion in the previous subsection, λ and μ should really be regarded as as holomorphic coordinates rather than real coordinates. This leads us to consider Brownian motion in the complexification of $U(N)$, that is, to replace the A_{N-1} Weyl chamber by its complexified version gl_N. That is, we are considering $GL(N)$ rather than $U(N)$. In that case, u and v really describe coordinates on a non-commutative $\mathbb{P}^1$, and the Brownian particles behave like B-model fermions when going from one patch to the other. Indeed, they transform in a metaplectic representation of $SL(2,\mathbb{Z})$, and gluing one patch to the other with $SL(2,\mathbb{Z})$ transformations is done via Fourier transformation. This is true both for $\varphi(\lambda)$, which transforms with S, and for the redefined fermions $\varphi(\lambda)$, which transform with TST. In a way, this closes the chain of dualities, since we started with Chern-Simons theory and we end up with the fermions describing branes of the Kodaira-Spencer theory [41, 42]. Notice that to achieve this we chose a *different* representation of the Hilbert space from the one considered in the previous section, where we discussed the A-model.

The fact that λ and μ correspond to the holonomies of the gauge field around the cycles of the torus suggests that we should interpret the original Brownian motion description as some Brownian motion in field space. It would be interesting to see how this comes about in the path integral formulation.

7 Matrix models for Chern-Simons and Brownian motion

7.1 One-matrix model and the Stieltjes-Wigert polynomials

The matrix model for the partition function of Chern-Simons on S^3 with gauge group $U(N)$ is [28]:

$$Z_{\mathrm{mm}} = \frac{1}{N!}\prod_{k=1}^N \int_{-\infty}^\infty \frac{\mathrm{d}y_k}{2\pi} e^{-y_k^2/2g_s} \prod_{i<j} \left(2\sinh\frac{y_i-y_j}{2}\right)^2 \ . \tag{105}$$

The matrix model of Chern-Simons in S^3 was solved in [21] by the method of orthogonal polynomials. The key step was to realize that a coordinate transformation

$$y_i = \log x_i \tag{106}$$

brings it into a form for which the associated orthogonal polynomials are explicitly known. With the transformation (106), the Gaussian potential in (105) becomes a log-normal weight function $w(x) = e^{-\log^2 x_i/g_s}$ [43]. On the other hand, the sinh Vandermonde repulsion term, characteristic of unitary matrix models, becomes the usual Vandermonde interaction $x_i - x_j$. The relation between the perturbation theory of the Chern-Simons matrix model and topological invariants computed in a Hermitian matrix model has been discussed in detail in [28].

Notice that the quantum dimensions naturally appear in the expressions for the Stieltjes-Wigert orthogonal polynomials [43]. Of course, this translates into the quantum dimensions that typically appear in expectation values in Chern-Simons theory.

With these ingredients, the matrix model is easily solved and gives [21]

$$Z_{\mathrm{mm}} = \left(\frac{g_s}{2\pi}\right)^{N/2} e^{\frac{1}{6} g_s N(N^2-1)} \prod_{k=1}^{N-1} (1 - q^{-k})^{N-k} \tag{107}$$

where as usual $q = e^{g_s}$. We should note here that the relation to the partition function in canonical framing is[7] [28]:

$$
\begin{aligned}
Z_{\mathrm{CS}}(S^3) &= e^{-\frac{1}{12} g_s N(N^2-1) - \frac{i\pi}{4} N^2} Z_{\mathrm{mm}} \\
&= \frac{1}{(k+N)^{N/2}} \prod_{j=1}^{N-1} \left(2\sin \frac{\pi j}{k+N}\right)^{N-j},
\end{aligned} \tag{108}
$$

where in the last line we used the standard relation between the topological string coupling and the Chern-Simons coupling constant (45).

In what follows we will make more comments on this matrix model –which is the same as the one for Brownian motion– and the computation of more general expectation values.

Let us recall that the most characteristic property of the Chern-Simons matrix model is that it is a unitary matrix model with a Vandermonde interaction that is a sinh rather than the usual polynomial Vandermonde determinant. We will explore some interesting properties of this type of matrix model elsewhere [35]. In a sense, this matrix model is much simpler than the ordinary ones because, in fact, all the integrals involved in (105) are Gaussian after expanding the factors of sinh, and in particular contain no polynomial terms, which would result in the usual logarithmic interactions. Thus they can actually be solved trivially without recourse to matrix models. The only non-trivial part is to keep track of the combinatorics, which can for example be done using the formulas in appendix A. Physically, the interesting part of this elementary exercise is that it gives us a probabilistic interpretation of the partition function of Chern-Simons on S^3. Indeed, in the appendix we prove that the Hopf link satisfies (18). Now recalling the identifications (47) and (45), we see immediately that, if we set $x = y = \rho$ and $s = t$, the left-hand side of (18) gives the value of the partition function of Chern-Simons on S^3 (with the appropriate framing as follows from (47)) as a function of the coupling, whereas the

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[7]Note that, in contrast with the conventions in [8], we have included a factor of $e^{-\pi i N^2/4}$ in the definition of the Chern-Simons partition function in (108). Also, in [8] we dropped overall sign factors.

right-hand side gives the matrix model expression:

$$e^{\frac{1}{24}g_s N(N^2-1)} Z_{\text{CS}}(S^3, g_s/2) = \frac{1}{N!} e^{\frac{1}{12}g_s N(N^2-1)} \int_{\mathbb{R}^N} \frac{\mathrm{d}^N \lambda}{(2\pi)^N} e^{-|\lambda|^2/g_s} \prod_{i<j} \left(2 \sinh \frac{\lambda_i - \lambda_j}{2}\right)^2 .$$

(109)

Thus, if we set x and y to the vacuum, we see that the extensivity property of the Brownian motion probability gives us a derivation of the matrix model expression of the partition function of Chern-Simons. It is interesting to note that this partition function comes as a function of $g_s/2$ rather than g_s. That is due to the fact that the left-hand side of (18) (with $s = t$) depends on $2t$ rather than t. This is reminiscent of a renormalization group equation. In fact, there is no need to set the external states to the vacuum. For generic external states x and y, (18) can be rewritten in terms of expectation values of Wilson lines in the corresponding representations. Consider the more general formula (154) derived in the appendix (with $\beta = 0$):

$$\int_C \mathrm{d}^N z \, e^{-\frac{\alpha}{2}|z|^2} p_{t,N}(x, z) \, p_{s,N}(z, y) = e^{-\frac{\alpha}{2\Delta}(\frac{s}{t}|x|^2 + \frac{t}{s}|y|^2)} p_{\Delta,N}(x, y)$$

(110)

where $\Delta = \alpha st + s + t$. Rewriting this in terms of (unnormalized) Wilson loops (dropping phase factors)

$$p_{t,N}(\lambda, \mu) = e^{-\frac{1}{2t}(|\lambda|^2 + |\mu|^2)} W_{\lambda\mu}(t)$$

(111)

where we wrote the Hopf link as a function of t rather than the level, we get

$$e^{-\frac{1}{2\Delta}(\frac{\alpha s(t-1)+s}{t}|\lambda|^2 + \frac{\alpha t(s-1)+t}{s}|\mu|^2)} W_{\lambda\mu}(\Delta) = \int_C \mathrm{d}^N \nu \, e^{-\frac{\Delta}{2st}|\nu|^2} W_{\lambda\mu}(t) W_{\nu\mu}(s) .$$

(112)

It would be interesting to find a field theory interpretation of this equation.

Let us add that, although using matrix models and the Stieltjes-Wigert polynomials to compute integrals of the type (105) might seem a little overdone, the power of these techniques should come to full use when computing more complicated observables, like those involving general torus links. We hope to come back to this in the near future.

The probability of survival for N Brownian movers was defined in [19] as the probability that the N movers survive after an amount t of time, or, in other words, that given some intial condition μ, after time t the trajectories of none of the movers crossed. Given that $p_{t,N}(\lambda, \mu)$ is the probability of moving from μ to λ without intersecting, to get the probability that the movers have not intersected at t we have to integrate over λ:

$$\begin{aligned} p_{\text{survival}}(\lambda) &= \int_C \mathrm{d}^N \lambda \, p_{t,N}(\lambda, \mu) \\ &= \frac{1}{N!} \int_{\mathbb{R}^N} \mathrm{d}^N \lambda \, |p_{t,N}(\lambda, \mu)| \\ &= \frac{1}{N!} \frac{1}{(2\pi t)^{N/2}} \int_{\mathbb{R}^N} \mathrm{d}^N \lambda \, e^{-\frac{|\lambda|^2 - |\mu|^2}{2t}} |\det(e^{\lambda_i \mu_j/t})| \end{aligned}$$

(113)

where as usual the integral is over the Weyl chamber. In the second line we made use of the fact that $p_{t,N}$ is antisymmetric in λ. This can be approximated for large t as follows [29]. We choose t large and rescale λ such that $|\lambda|/t$ is small but keeping $|\lambda|^2/t$ large, and take the boundary condition so that $|\mu|^2/t$ is small (and the particles are equally spaced). Keeping only linear terms, we get

$$p_{\text{survival}}(\lambda) = \frac{1}{N!} \frac{1}{(2\pi t)^{N/2}} t^{-N(N-1)/2} \prod_{i<j} \frac{\mu_i - \mu_j}{i - j} \int_{\mathbb{R}^N} \mathrm{d}^N \lambda \, e^{-\frac{|\lambda|^2 + |\mu|^2}{2t}} \prod_{i<j} |\lambda_i - \lambda_j| .$$

(114)

It is clear that the factor of $t^{-N(N-1)/2}$ came from the Vandermonde interaction. It is amusing to note that this combines with the first term to give a factor of $t^{-N^2/2}$. The above integral can be readily done using Selberg's integral [44] and is given in terms of Γ-functions. After doing the integral, the total t-dependence is $t^{-N(N-1)/4}$ [29]. Expression (114) was also found by Fisher [19].

Notice that from the above, for large t, we have

$$p_{t,N}(\lambda, \mu) \sim \frac{1}{t^{N^2/2}} e^{-\frac{|\lambda|^2 + |\mu|^2}{2t}} \Delta(\lambda) \ . \tag{115}$$

This expansion can be used to compute more general observables:

$$\langle f \rangle = \frac{1}{N!} \int_{\mathbb{R}^N} \mathrm{d}^N \lambda \, |p_{t,N}(\lambda, \mu)| \, f(\lambda) \tag{116}$$

where we assumed that f is symmetric in μ. One can again approximate this for large t as

$$\langle f \rangle = \frac{1}{N!} \frac{1}{(2\pi t)^{N/2}} t^{-N(N-1)/2} \prod_{i<j} \frac{\mu_i - \mu_j}{i - j} \int_{\mathbb{R}^N} \mathrm{d}^N \lambda \, e^{-\frac{|\lambda|^2 + |\mu|^2}{2t}} \prod_{i<j} |\lambda_i - \lambda_j| \, f(\lambda) \ . \tag{117}$$

Again, for a large class of symmetric functions one can make use of Selberg's integral to evaluate this explicitly.

For antisymmetric functions, we write

$$\hat{f}(\lambda) = \Delta(\lambda) \, f(\lambda) \tag{118}$$

where f is symmetric. Then we simply get:

$$\langle \hat{f}(\lambda) \rangle = \frac{1}{N!} \frac{1}{(2\pi t)^{N/2}} t^{-N(N-1)/2} \prod_{i<j} \frac{\mu_i - \mu_j}{i - j} \int_{\mathbb{R}^N} \mathrm{d}^N \lambda \, e^{\frac{-|\lambda|^2 + |\mu|^2}{2t}} \Delta(\lambda)^2 f(\lambda) \tag{119}$$

that is the Hermitian matrix model with $\beta = 2$. This is the usual case, and one can use the results of for example [45] to compute the kernel. Its short-distance behavior is well-known:

$$K(\lambda, \mu) = \frac{\sin(N\pi(\lambda - \mu)\rho(\lambda))}{\pi(\lambda - \mu)} \tag{120}$$

for $\lambda - \mu \ll 1$. At this point one can make use of the usual matrix model techniques to compute Brownian motion averages in the approximation of large t, which corresponds to weak coupling on the field theory side.

7.2 Brownian motion averages and the Hermitian two-matrix model

We have seen how Brownian motion is related to one-matrix models. The probability $p_{t,N}(\lambda, \mu)$ can however be interpreted more generally as an integration kernel in the context of two-matrix models. For simplicity we will do this for the case of $U(N)$. Brownian motion two-matrix models have also appeared in [46].

We start with a simple example for the case $N = 2$. Assume we are interested in the average of the final distance between the two particles, $\langle \mu_1 - \mu_2 \rangle$ after time t, given some initial configuration $\lambda_1 - \lambda_2$. As usual, we take $\lambda_1 > \lambda_2$, $\mu_1 > \mu_2$. We can compute this as follows:

$$\langle \mu_1 - \mu_2 \rangle = \frac{1}{Z} \int_{\mu_1 > \mu_2} \mathrm{d}\mu_1 \mathrm{d}\mu_2 \, p_t(\mu, \lambda)(\mu_1 - \mu_2) \tag{121}$$

and in this case, explicitly,

$$p_t(\mu, \lambda) = \frac{1}{2\pi t} e^{-\frac{|\lambda|^2 + |\mu|^2}{2t}} (e^{\frac{\lambda_1 \mu_1 + \lambda_2 \mu_2}{t}} - e^{\frac{\lambda_1 \mu_2 + \lambda_2 \mu_1}{t}}) \ . \tag{122}$$

The normalization is given by the probability of survival

$$Z = \int_{\mu_1 > \mu_2} p(\mu, \lambda) \ . \tag{123}$$

So we get

$$\langle \lambda_1 - \lambda_2 \rangle = \frac{1}{Z} \int_{\mu_1 > \mu_2} p(\mu, \lambda)(\mu_1 - \mu_2) = \lambda_1 - \lambda_2 \ , \tag{124}$$

and Z was computed asymptotically in (114). In what follows we will drop such normalizations.

One can more generally consider averages of functions in both variables, $f(\lambda, \mu)$, corresponding to a process from λ to μ in time t, and one may also be interested in integrating over both boundary conditions, initial and final ones. Consider now for example:

$$\langle \frac{(\mu_1 - \mu_2)(\lambda_1 - \lambda_2)}{(1 + \lambda_1^2)^2(1 + \lambda_2^2)^2} \rangle = 2\pi^3 \tag{125}$$

where we set $t = 1$. From the above it is already clear that such integrals become more and more complicated as N becomes large, and also one needs to make sure that they converge. In the above, the presence of the denominators ensured convergence. Notice that in this case –but it holds more generally– after performing the μ-integrals all the Gaussian factors disappeared, and so in general one has to consider rational functions rather than polynomials.

In the following we will show how two-matrix models can be useful in this type of computations, by mapping the density $p_t(\lambda, \mu)$ to an integration kernel of a two-matrix model.

Thus we consider quantities of the following type:

$$\int_C \prod_{k=1}^N \mathrm{d}\lambda_k \mathrm{d}\mu_k \ \det(e^{-(\lambda_i - \mu_j)^2/2t})_{1 \le i < j \le N} \ \hat{f}(\lambda, \mu) \ , \tag{126}$$

and the integration region C is as usual the fundamental Weyl chamber. The hat on f is to indicate that it is an antisymmetric function under interchange of any two λ_i or μ_j. Notice that this is not necessary, as antisymmetric functions do not vanish when integrated over the fundamental Weyl chamber (in contradistinction to their vanishing when we integrate over the whole of $\mathbb{R}^N$). Nevertheless they are useful in that they can be extended to integrals over $\mathbb{R}^N$ due to the identity:

$$\int_C \prod_{i=1}^N \mathrm{d}\lambda_i \ f(\lambda) = \frac{1}{N!} \int_{\mathbb{R}^N} \prod_{i=1}^N \mathrm{d}\lambda_i \ f(\lambda) \ , \tag{127}$$

where C is the fundamental Weyl chamber, for a symmetric total integrand f, whereas for antisymmetric functions:

$$\int_C \prod_{i=1}^N \mathrm{d}\lambda_i \ \hat{f}(\lambda) = \frac{1}{N!} \sum_{w \in W} \epsilon(w) \int_{w(C)} \prod_{i=1}^N \mathrm{d}\lambda_i \ \hat{f}(\lambda) \tag{128}$$

where $w(C)$ is the Weyl chamber reached by acting with an element of the Weyl group (in this case a permutation) on the fundamental Weyl chamber. Of course, we can always turn

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an antisymmetric function into a symmetric one by taking its absolute value, and in that case one can extend the integral to $\mathbb{R}^N$, as we saw in the case of the one-matrix model. For the time being we will limit ourselves to integrals over symmetric functions, that is, antisymmetric functions $\hat{f}$ in (126). For these ovserbables, the ordering of the particles is crucial. Without loss of generality we can write

$$\hat{f}(\lambda, \mu) = \Delta(\lambda)\Delta(\mu) f(\lambda, \mu) \tag{129}$$

where $\Delta(\lambda)$ is the usual Vandermonde determinant

$$\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) . \tag{130}$$

Thus we have

$$
\begin{aligned}
\langle\hat{f}\rangle &= N! \int_C \prod_{i=1}^{N} \mathrm{d}\lambda_i \mathrm{d}\mu_i \, \Delta(\lambda)\Delta(\mu) \det(e^{-(\lambda_i-\mu_j)^2/2t}) \, f(\lambda, \mu) \\
&= \frac{1}{N!} \int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}\lambda_i \mathrm{d}\mu_i \, \Delta(\lambda)\Delta(\mu) \det(e^{-(\lambda_i-\mu_j)^2/2t}) \, f(\lambda, \mu) .
\end{aligned} \tag{131}
$$

Remember that $f(\lambda, \mu)$ is a symmetric function. In the last expression, any antisymmetric piece would not contribute. We will further assume that $\hat{f}$ is rational in λ, μ and chosen such that the integral converges. We included the factor of $N!$ and dropped a factor of $\frac{1}{(2\pi t)^{N/2}}$ (which we will nevertheless take into account later on) to simplify normalizations.

Before moving to the two-matrix model, it is convenient to rewrite the above in an alternative way. We note the following formula for arbitrary symmetric function:

$$\int_{\mathbb{R}^N} \mathrm{d}^N y \, \Delta(y) \, \det(e^{x_i y_j})_{1\leq i<j\leq N} \, f(y) = N! \int_{\mathbb{R}^N} \mathrm{d}^N y \, \Delta(y) \, e^{\sum_{i=1}^{N} x_i y_i} \, f(y) . \tag{132}$$

This is easily proven by writing out the definition of the determinant as a summation over the symmetric group, interchanging the order of summation and integration, and relabeling the y-coordinates. Similar manipulations are illustrated in appendix A. This results in a summation over S_N that gives the factor of $N!$. Of course, the right-hand side is still antisymmetric in the x_i's, a property that will be crucial in what follows.

We can now rewrite expectation values of antisymmetric functions $\hat{f}$ as

$$\langle\hat{f}\rangle = \int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}\lambda_i \mathrm{d}\mu_i \, \Delta(\lambda)\Delta(\mu) e^{-\sum_{i=1}^{N}(\lambda_i-\mu_i)^2/2t} \, f(\lambda, \mu) . \tag{133}$$

Again, any antisymmetric piece in f drops out after symmetrization.

It is not hard to recognize in (131) a two-matrix model. The form (131) is the one used in [22] which results from the Itzykson-Zuber integral, the simpler one (133) is the one advocated in [47]. It is a two-matrix model interacting via a potential $\sum_{i=1}^{N} x_i y_i$, as we will now see.

Consider then the Hermitian two-matrix ensemble [22]:

$$Z = \int [dL][dM] \, e^{-V(L)-V(M)+\beta \mathrm{Tr}(LM)} . \tag{134}$$

Averages in this ensemble are integrals of the form

$$\langle f(L, M)\rangle = \int [dL][dM] \, e^{-V(L)-V(M)+\beta \mathrm{Tr}(LM)} f(L, M) . \tag{135}$$

Typically, $f(L, M)$ is a rational function of the moments $\operatorname{Tr} L^m$, $\operatorname{Tr} L^n$.

We will clearly be interested in quadratic potentials[8] $V(M) = M^2/2t$ and $\beta = 1/t$, in which case the above takes the form:

$$\langle f \rangle = \int [dL][dM]\, e^{-\operatorname{Tr}(M-L)^2/2t} f(L, M) \ , \tag{136}$$

for some function $f(L, M)$. For now, however, we will leave the parameters generic. The measure $[dL]$ has been worked out in [22, 47] and it contains two parts: an integral over the diagonal entries of L, and an integral over the angular part. Indeed, we can diagonalize L by a unitary matrix U as

$$L = U\, \Lambda U^\dagger \ . \tag{137}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$. This, however, cannot be done simultaneously for L and M. Thus there remains an integral over unitary matrices that we have to perform explicitly. We have [47]

$$\mathrm{d}L = \mathrm{d}\Lambda \mathrm{d}U\, \Delta^2(\lambda) \ , \quad \mathrm{d}\Lambda = \prod_{i=1}^{N} \mathrm{d}\lambda_i \ . \tag{138}$$

References [22, 47] then prove that the above reduces to

$$\langle f \rangle = \frac{(2\pi)^{N(N-1)}}{(\prod_1^N p!)^2} \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \mathrm{d}\mu_i\, \Delta(\lambda) \Delta(\mu) e^{-V(\lambda)-V(\mu)} I(\lambda, \mu) f(\lambda, \mu) \ , \tag{139}$$

and we are assuming that $f(L, M)$ is a function of the moments of L, M and no mixing terms. $I(\lambda, \mu)$ is the Itzykson-Zuber integral, given by

$$I(\lambda, \mu; t) = t^{N(N-1)/2} \prod_1^N p! \frac{\det(e^{\lambda_i \mu_j/t})}{\Delta(\lambda)\Delta(\mu)} \tag{140}$$

and it is the result of the integration over the angular variables. We see that up to normalization this is precisely the integral[9] (131):

$$\int [dU]\, e^{-\frac{1}{2t}\operatorname{Tr}(\Lambda - UMU^\dagger)^2} = t^{N(N-1)/2} \prod_1^{N-1} p! \frac{\det(e^{-(\lambda_i-\mu_j)^2/2t})}{\Delta(\lambda_i)\Delta(\mu_i)} \ . \tag{141}$$

Putting everything together, this gives the Brownian motion ensemble (131).

One can use uniqueness of the solution of the heat equation with respect to the unitary invariant Laplacian operator on Hermitian matrices and on the Cartan subalgebra, respectively, to compute the coefficient appearing in the transformation from $\mathrm{d}M$ to an integral over the diagonal. What is of interest for us is the fact that the solution kernel of the former equation is

$$f(L, M; t) = \frac{1}{(2\pi t)^{N^2/2}} e^{-\operatorname{Tr}(L-M)^2/2t} \ , \tag{142}$$

---

[8]Notice, however, that in this special case Z is not well defined: it diverges like $\sim \Lambda^{N^2}$ where Λ is a cutoff on the L, M-integrals, or, alternately, as $\sim 1/(\beta - 1/t)^{N/2}$ [48] as $\beta \to 1/t$. This is however not the Brownian motion normalization.

[9]In this formula M denotes the diagonal entries of M and not M itself.

and this explains the t-dependence in (140). Let us now turn to the heat equation on the Cartan subalgebra. This is of course the heat equation we considered before:

$$\frac{\partial}{\partial t} f = \frac{1}{2} \Delta f \ , \tag{143}$$

and we look for antisymmetric solutions of this equation, $f(\lambda_1, \ldots, \lambda_N)$ with boundary condition $f_0(\lambda)$ at $t = 0$. These are given by [22]

$$f(\lambda) = \prod_{i=1}^{N} \int \mathrm{d}\mu_i \, K(\lambda, \mu; t) \, f_0(\mu) \tag{144}$$

and

$$K(\lambda, \mu; t) = \frac{1}{(2\pi t)^{N/2}} \frac{1}{N!} \, \det(e^{-(\lambda_i - \mu_j)^2/2t}) \tag{145}$$

is precisely the Brownian motion density $p_{t,N}(\lambda, \mu)$. Of course, any *symmetric* solution of the heat equation can be obtained by rescaling the corresponding symmetric solution with the Vandermonde determinant.

This is of course analogous to the case of Brownian motion in the affine Weyl chamber of $U(N)$/two-dimensional Yang-Mills [6], where the partition function is the heat kernel of solutions of the heat equation for symmetric functions.

Having found a correspondence between Brownian motion averages and one- and two-matrix models, we can now use the techniques available for these models, in particular large-N techniques. In particular, we have the following correlation function:

$$\rho_N(\lambda_1, \ldots, \lambda_N; \mu) = \frac{1}{N!} \frac{1}{Z} \, \Delta(\lambda) \, \det(e^{-(\lambda_i - \mu_j)^2/2t}) \tag{146}$$

in terms of which we can define our Brownian motion averages for antisymmetric functions, as before. The normalization factor is

$$Z = \int \mathrm{d}^N \lambda \, \Delta(\lambda) \det(e^{-(\lambda_i - \mu_j)^2/2t}) \ . \tag{147}$$

This is the analog of the density of eigenvalues in the presence of a coupling to an external source μ. It is related to the two-point kernel as follows:

$$\rho_N(\lambda_1, \ldots, \lambda_N) = \frac{1}{N!} \det \underline{K}(\lambda_i, \lambda_j) \ . \tag{148}$$

We supressed explicit μ-dependence because here we are considering it as a source, but these expressions depend on μ. An explicit expression for $\underline{K}$ has been computed in [45], and the large-N asymptotics and short distance behavior have been analyzed in [45, 49]. It is well-known that K can be expressed in terms of orthogonal polynomials. The generalization of this to the present case is in [45]. One can also integrate out some of the coordinates λ_i to get a correlation function depending on a smaller number n, $n < N$, of eigenvalues. In that case the determinantal identity (148) continues to hold [45].

Notice that this kernel is not the kernel computed before; symmetric solutions of the heat equation can be obtained from it, and it satisfies (12). In fact, it is related to the Brownian motion kernel as follows:

$$\det \underline{K}(\lambda_i, \lambda_j; \mu) = \frac{1}{\int \mathrm{d}^N \lambda \, \Delta(\lambda) K(\lambda, \mu)} \, \Delta(\lambda) K(\lambda, \mu) \ . \tag{149}$$

Two-matrix models are solved in [22, 47]. [50] has found exact formulas for mixed correlation functions of the type $\operatorname{Tr} L^n M^m$ using biorthogonal polynomials.

Notice that the same two-matrix model was conjectured to play a role in the computation of correlators of branes in Calabi-Yau crystals [42]. Since in the previous section we already saw that Brownian motion particles can be interpreted as B-model fermions, it would be interesting to see if the above formulas can be used in the context of [42], and whether they have a Chern-Simons interpretation.

8   Discussion and outlook

In this paper we have shown that for a number of observables the mathematical structures of Chern-Simons theory and Brownian motion for non-intersecting movers are almost identical, for the $U(N)$, $SO(N)$ and $Sp(N)$ groups and for the manifolds S^3 and $S^3/\mathbb{Z}_p$. The reformulation of the classical Brownian problem as counting of quantum mechanical paths in the Weyl chamber of the gauge group points to a deeper connection between Chern-Simons and Brownian motion, namely at the level of the path integral where we count paths on field space. Indeed, from the canonical quantization approach it became clear that the coordinates of the particles describe the holonomies of the gauge field around the cycles of the torus along which one does the gluing.

An intriguing connection with the B-topological string was found at the level of fermions. The natural fermionic formulation that we find for the Brownian particles is identical to the one in [41], thus identifying our particles with the branes of the B-model. In the case of S^3 that we analyzed, we found that these fermions move on a non-commutative space with two patches. For this we needed to complexify the Weyl chamber, thus effectively replacing $U(N)$ by $GL(N)$. Brownian propagation then translates into a coordinate transformation from one patch to the other which is a Fourier transformation.

The case of lens spaces corresponds to motion in the affine Lie algebra, and is related to 2d Yang-Mills theory. [6] related Chern-Simons on lens spaces to 2d Yang-Mills on the cylinder at a value of the coupling that is $2\pi i$ times a rational number. However, the results in [7] suggest to consider also the analytically continued Chern-Simons. Since integer values of the coupling are required in Chern-Simons in order to preserve invariance under large gauge transformations, the deformed theory is defined by the quantum deformed 2d Yang-Mills, which for the case of the sphere is the same as usual 2d Yang-Mills on the cylinder, as we showed. Thus, the modular transformation properties worked out in [6] should correspond to those of the A-model on the corresponding bundle over the two-sphere. It would be interesting to investigate this further. In fact, Brownian motion would seem to be more naturally associated with the quantum-deformed 2d Yang-Mills than with ordinary Yang-Mills, due to the exponential nature of the probabilities. This quantum deformed 2d YM [7] is an interesting theory in its own right. The partition function was evaluated in [6, 7] for the sphere and the torus, and by using the manipulations in appendix A it should be readily extendable to the higher genus case. In particular, one can obtain matrix model expressions, which are useful to work out the large N limit. We will come back to this in the future [51].

It is likely that the generalization of the general modular matrix $U_{\lambda\mu}^{(p,q)}$ is simply given by the same expression (36), where one analitically continues in the coupling and sums over the whole coroot lattice. It would be interesting to check this. In that case one can use the relation to the Θ-functions to study its modular transformation properties, as in [6], and one could study the reformulation of Chern-Simons theory in terms of 2d qYM for more general manifolds.

One important issue is to extend the Brownian motion description of Chern-Simons to more

general manifolds and knot invariants. Particularly interesting is the case of knots consisting of three links or more, since so far we encountered only up to two-linked knots. This is of course due to the fact that we only have two boundary conditions to our Brownian motion problem. This suggests that for extra links – which will carry additional representations – one will have to impose more boundary conditions on the problem. One could for example consider the conditional probability of starting with a configuration λ, ending up with a configuration μ, *given* that the intermediate situation at some time t was an intermediate state ν. It would be interesting to check whether this associates an invariant with representations λ, μ, ν to a three-link knot. It would be very interesting to work along these lines further, as this might give an independent and general way of computing knot invariants. It might be useful here to really view the Brownian motion picture as a braid diagram, and consider the associated algebra.

An interesting open problem is to develop matrix model technology further so one is able to compute for example general torus links, and in particular their large N expansions. The same is true for three-manifolds M where M is a Seifert homology sphere, so one can work out the corresponding geometric transitions on T^*M. As far as Brownian motion is concerned, we saw that in order to replace S^3 by $S^3/\mathbb{Z}_p$ one had to replace the finite fundamental Weyl chamber by the affine Weyl chamber, which amounts to modding out by translations in the coroot lattice. It would be interesting to generalize this procedure to other manifolds, and in particular see if there is a general way of modifying the manifold by modifying the shape of the region where the particles are allowed to walk.

Recently, the melting crystal picture was applied to Chern-Simons on S^3 directly [52]. It should be possible to see directly what melting crystals and Brownian motion have to do with each other.

Finally, it would be interesting to understand the role of the heat equation within Chern-Simons theory (perhaps in terms of stochastic quantization [53], as a Fokker-Planck equation) and in the topological string itself.

Acknowledgements

A   Extensivity properties of the kernel

In this appendix we give the proof of the extensivity formula (18):

$$p_{t+s,N}(x,y) = \int_C \mathrm{d}^N z\, p_{t,N}(x,z)\, p_{s,N}(z,y) \tag{150}$$

and work out several other identities of the same type. We first write $p_{t,N}(x,y)$ as:

$$p_{t,N}(x,y) = \frac{1}{(2\pi t)^{N/2}} \sum_{\sigma \in S_N} \epsilon(\sigma)\, e^{-\sum_{i=1}^N (x_i^2 + y_i^2 - 2x_i y_{\sigma(i)})/2t} \tag{151}$$

so that the right-hand side of (150) is

$$\frac{1}{N!} \frac{1}{(4\pi^2 st)^{N/2}} \sum_{\sigma,\sigma'} \epsilon(\sigma\sigma') \int_{\mathbb{R}^N} \mathrm{d}^N z\, e^{-\sum_{i=1}^N [(z_{\sigma(i)} - x_i)^2/2s + (z_{\sigma'(i)} - y_i)^2/2t]} \ . \tag{152}$$

We now relabel the z_i's and perform a transformation $\sigma \to \sigma'\sigma$. The σ' sum then results in a factor of $N!$. Bringing the remaining terms together and performing the Gaussian integrals, we are left with:

$$\frac{1}{(2\pi(s+t))^{N/2}} \sum_{\sigma \in S_N} \epsilon(\sigma) e^{-\sum_{i=1}^N (x_i - y_{\sigma(i)})^2 / 2(s+t)} = p_{s+t,N}(x,y) , \tag{153}$$

which is what we wanted to prove.

In fact we can easily extend this computation to the following:

$$\int_C d^N z \, e^{-\frac{\alpha}{2}|z|^2 + \beta \sum_{i=1}^N z_i} \, p_{t,N}(x,z) \, p_{s,N}(z,y)$$
$$= e^{-\frac{1}{2\Delta}\left[\alpha(\frac{s}{t}|x|^2 + \frac{t}{s}|y|^2) - 2\beta \sum_{i=1}^N (sx_i + tz_i) - N\beta^2 ts\right]} p_{\Delta,N}(x,y) \tag{154}$$

where $\Delta = \alpha st + s + t$. We can also replace $\beta \sum_{i=1}^N x_i$ by $\sum_{i=1}^N \beta_i x_i$ in the integrand, and then the final expression contains an additional symmetrization with respect to β_i.

Notice that we can construct $q_{t,r}(\lambda,\mu)$ – or, equivalently, the partition function of Chern-Simons on lens spaces – from two S^3 partition functions. Indeed, up to normalizations, we have

$$q_{t',r}(\lambda,\mu) = \left(\frac{\alpha}{2\pi}\right)^{r/2} \sum_{n \in Q^\vee/pQ^\vee} \int_{\mathbb{R}^N} d^r z \, e^{-\frac{\alpha}{2}z_i^2 + n_i z_i / t} p_{t,r}(\lambda,z) p_{t,r}(z,\mu) \tag{155}$$

where the coupling is $t' = \alpha^2 t$. The left-hand side is the operator $ST^q S$. One can also show extensivity of $q_{t,r}(\lambda,\mu)$ itself. Analogous expressions can be derived for q-deformed 2dYM.

B Path integral formulation of the path counting

Computing non-intersecting Brownian motion probabilities involves picking an infinite set of non-intersecting paths out of the infinite set of all free Brownian motion paths. It is therefore natural to look for a path integral formulation, where particles are traveling from an initial state to a final state along Brownian paths. It is well-known that free Brownian paths correspond to quantum mechanical paths, in other words, the free Brownian motion probability is the quantum mechanical path integral of a free particle. Thus, all we have to do is take into account the fact that the particles are non-intersecting. By elementary manipulations, we get:

$$p_{t,N}(\lambda,\mu) = \det \int_{\lambda_i}^{\mu_j} \mathcal{D}x_1(t) \ldots \mathcal{D}x_N(t) e^{iS[x(t)]} \tag{156}$$

where the determinant is taken with respect to the boundary conditions λ_i and μ_j, and $S[x(t)]$ is the action for a free particle. It is interesting to discretize this as follows:

$$p_{t,N}(\lambda,\mu) =$$
$$= \lim_{\epsilon \to 0} \left(\frac{m}{2\pi i \epsilon}\right)^{MN/2} \int_C dx_1^2 \ldots dx_N^2 \ldots \int_C dx_1^{M-1} \ldots dx_N^{M-1} \prod_{i=1}^{M-1} \det(e^{\frac{im}{2\epsilon}(x_a^i - x_b^{i+1})^2})_{ab} \tag{157}$$

where we simultaneously take the limit $\epsilon \to 0$ and $M \to 0$. The coordinates x_i^a label the discretized coordinates, where $i = 1, \ldots, N$ and $a = 1, \ldots, M$. Notice that we were able to restrict all integration regions to the Weyl chamber, as expected. Thus, this can be seen as a path integral for motion inside the Weyl chamber.

It is likely that the above expression presents an additional hidden supersymmetry when expressing the determinant as a fermionic integral, in the limit $M \to \infty$, as in [54]. It would be interesting to work this out further.

C   Some determinantal identities

In this appendix we collect the steps leading to (44). We use the following identities:

$$\det(e^{\lambda_i(j-1)}) = \prod_{1 \leq i < j \leq N} (e^{\lambda_j} - e^{\lambda_i})$$

$$\prod_{1 \leq i < j \leq N} q^{\lambda_j} = q^{\sum_{i=2}^{N}(i-1)\lambda_i} . \tag{158}$$

The first is the standard Vandermonde identity, and the latter is also easily Checked.

We start from the last equality in (14) with $\lambda_i = \mu_i = (c-i)a$ and extract the Gaussian factors:

$$\det(e^{-\frac{(\lambda_i-\mu_j)^2}{2t}})_{1 \leq i < j \leq N} = \frac{1}{(2\pi t)^{N/2}} e^{-\sum_{i=1}^{N}(i-1)^2 a^2/t} \det(e^{(j-1)(i-1)a^2/t}) , \tag{159}$$

where we used the freedom to shift the initial and final positions to recast the determinant in a form in which we can apply the standard Vandermonde determinant formula (158). Then using both of (158), we see that the prefactors cancel and we finally get

$$\det(e^{-(j-i)^2 a^2/2t}) = \prod_{k=1}^{N}(1 - q^k)^{N-k} \tag{160}$$

with $q = e^{-a^2/t}$.

For the computation of the expectation value of a Wilson we need in addition the identity

$$\prod_{i<j}(e^{x_j} - e^{x_i}) = \prod_k e^{\frac{N-1}{2} x_k} \prod_{i<j} 2 \sinh \frac{x_j - x_i}{2} . \tag{161}$$

This is easyly checked for low values of N, and proved by induction.

D   Representations and partitions

We first introduce some Lie algebra conventions. In this appendix we deal with the case of $U(N)$ and $SU(N)$. We start with $U(N)$. We use standard notations (see for example [55]), where ω_i, $i = 1, \ldots, N$, denotes the set of fundamental weights, and ϵ_i are unit vectors in $\mathbb{R}^N$. They are related by:

$$\omega_i = \sum_{j=1}^{i} \epsilon_j - \frac{i}{N} \sum_{j=1}^{N} \epsilon_j. \tag{162}$$

The simple roots are given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \ldots, N-1$. From here it follows that $|\alpha|^2 = 2$ and so $\alpha = \alpha^\vee$. Now we can expand a highest weight of an irrep

$$\lambda = \sum_{i=1}^{N} \lambda_i \omega_i = \sum_{i=1}^{N} (\ell_i - \kappa)\epsilon_i \tag{163}$$

where

$$\kappa = \frac{1}{N} \sum_{j=1}^{N} j\lambda_j \tag{164}$$

is the number of boxes in the Young tablaux divided by N, and

$$\ell_i = \lambda_i + \lambda_{i+1} + \ldots \lambda_N . \tag{165}$$

Using this definition, it is not hard to check that (163) holds.

A Weyl chamber is a connected set in $\mathbb{R}^N$ left out after we delete all hyperplanes orthogonal to the roots. It is defined by

$$C_w = \{\lambda | (w\lambda, \alpha_i) \geq 0, i = 1, \ldots, r\}, \; w \in W . \tag{166}$$

The fundamental Weyl chamber corresponds to the identity element and is denoted C_0. Obviously, $\lambda_i \geq 0$ for all $i = 1, \ldots, N$ if λ is in the fundamental Weyl chamber. This implies that the ℓ's satisfy $\ell_i \geq \ell_{i+1}$. Defining coordinates

$$h_i = \ell_i + \frac{N+1}{2} - i , \tag{167}$$

we see that the fundamental Weyl chamber is the domain $h_1 > h_2 > \ldots h_N$. These are the variables that are usually used to label representations, and which we have labeled by λ and μ (and we will continue to do so). They can be written as

$$h = \ell + \rho , \tag{168}$$

where ρ is the Weyl vector

$$\rho = \sum_{i=1}^{N} \omega_i = \sum_{i=1}^{N} \left(\frac{N+1}{2} - i \right) \epsilon_i \tag{169}$$

and has norm

$$|\rho|^2 = \frac{1}{12} N(N^2 - 1) . \tag{170}$$

The fundamental weights satisfy:

$$(\omega_i, \omega_j) = F_{ij} = (C^{-1})_{ij} \frac{|\alpha|^2}{2} = (C^{-1})_{ij} \tag{171}$$

where in the last line we specialized to $U(N)$. C is the Cartan matrix. The inner product for weights is

$$(\lambda, \mu) = \sum_{ij} \lambda_i \mu_j F_{ij} . \tag{172}$$

For $U(N)$, the Casimir becomes especially simple in the new basis:

$$\begin{aligned} C_2(\lambda) &= (\lambda, \lambda + 2\rho) \\ &= |\ell + \rho|^2 - |\rho|^2 = |h|^2 - |\rho|^2 = \sum_{i=1}^{N} h_i^2 - \frac{1}{12} N(N^2 - 1) . \end{aligned} \tag{173}$$

The dimension of a representation is given by

$$\dim R = \prod_{1 \leq i < j \leq N} \frac{h_i - h_j}{j - i} \tag{174}$$

E A note on framing

In previous sections we related Brownian motion and Chern-Simons quantities. Framing in Chern-Simons theory was already explained in section 3. Both in the case of the partition function and for the unknot we saw that the expressions resulting from Brownian motion were not quite in canonical framing, but in matrix model framing. That is, every state came multiplied with a Boltzmann factor. We will now explain how such factors can be modified by changing the boundary conditions of the Brownian motion. We will consider translations and dilatations of the boundary conditions. Besides the possible interpretation as framing factors in Chern-Simons, these properties of the probability densities are interesting in their own right.

We start with an example that will however cover all of the physical effects we want to analyze. We take initial and final states that are equally spaced, but are not identical:

$$
\begin{aligned}
\lambda_i &= (c - i)a \\
\mu_i &= (c' - i)a' \ .
\end{aligned}
\tag{175}
$$

Let us further introduce the notation

$$
\begin{aligned}
\alpha &= ca - c'a' \\
\beta &= a - a' \ .
\end{aligned}
\tag{176}
$$

We are thus set to compute the determinant (14) for the above values of λ, μ. This is not hard to do and the procedure is similar to the way we found the probability of reunion. Basically, we expand the exponential in such a way that we can use formula (160). The result is

$$
\det\left(e^{-(\lambda_i - \mu_j)^2/2t}\right) = e^{[-\frac{1}{6}\beta^2 N(N+1)(2N+1) - N\alpha(\alpha + \beta(N+1))]\frac{1}{2t}} \prod_{k=1}^{N-1} \left(1 - e^{-kaa'/t}\right)^{N-k} \ .
\tag{177}
$$

We see that α and β can be chosen such that the exponential equals a framing factor $\frac{\pi i k(N^2-1)}{12(k+N)}n$ for some integer n. So in principle we can reach any framing we want by translating the initial or the final positions of all the particles by a constant factor, or by rescaling them. Let us however stress that these are not *symmetries*: the probability densities change by exponential factors. However, once we have computed the probability for a specific configuration, we can compute that of any other configuration connected to it by translations or dilatations with the above transformation rules.

Not all of the above transformations have a natural interpretation in terms of framing though; the translations do not. To see this, it is best to look not at the "probability of reunion"[10], but more generally to the transformation of $p_{t,N}(\lambda, \mu)$ under dilatations and translations. Let us first consider the latter. We already know that the probability density is invariant under translations by equal factors, $\lambda \to \lambda + c$, $\mu \to +c$. We now consider the following:

$$
p_{t,N}(\lambda + ca, \mu + c'a) = e^{-\sum_{i=1}^{N}(\lambda_i - \mu_i)(c-c')\frac{a}{t} - N(c-c')^2\frac{a^2}{2t}} \, p_{t,N}(\lambda, \mu) \ .
\tag{178}
$$

Clearly, the first factor does not have much chance of being interpreted as a framing factor except in an $SU(N)$ theory where it vanishes. For $U(N)$, the framing factors go with the second Casimir of the representation, not with the first (see section 3). Of course, this factor vanishes

---

[10]We will continue to use this name, even though with the boundary conditions (176) the particles do not go back to their initial positions, now even up to an overall translation.

when $|\lambda| = |\mu|$, and in particular for the Weyl vector, which satisfies $|\rho| = \sum_{i=1}^{N} \rho_i = 0$ (see appendix D). But there is no reason to restrict oneself to this type of boundary conditions, so this does not generally reproduce the effect of framing.

The effect of dilatations on the external states however is much more similar to framing. This corresponds in (176) to $\alpha = \beta$. Let us now again consider the most general case of arbitrary initial and final states. To this end it is easiest to work directly with (14). If we rescale

$$
\begin{aligned}
\lambda &\rightarrow \lambda a \\
\mu &\rightarrow \mu a' \,,
\end{aligned}
\tag{179}
$$

we get

$$
p_{t,N}(\lambda a, \mu a') = \frac{1}{(2\pi t)^{N/2}} \, e^{-\frac{|\lambda|^2}{2t}a^2 - \frac{|\mu|^2}{2t}a'^2} \, \det(e^{\lambda_i \mu_j aa'/t}) \,.
\tag{180}
$$

Of course, this reduces to (177) for $\alpha = \beta$ and the probability of reunion. However, we do not want to change the last interaction term, which corresponds to the modular matrix S and does not change under framing. At this point we can either generalize the relation between t and the Chern-Simons coupling to include the extra factors of a, a', or we can consider rescalings of time as well:

$$
p_{t',N}(\lambda', \mu') = \frac{1}{(2\pi aa't)^{N/2}} \, e^{-\frac{|\lambda|^2}{2t}a^2 - \frac{|\mu|^2}{2t}a'^2} \, \det(e^{\lambda_i \mu_j/t}) \,.
\tag{181}
$$

This now corresponds, up to an overall constant, to a matrix element

$$
\langle \mu | T^n S T^m | \lambda \rangle
\tag{182}
$$

which is what we wanted. For this to be a framing, $a^2 = n$ and $a'^2 = m$ are integers. With a and a' real, notice that this can only *augment* the framing and not diminish it.

F   Partition function of SO and Sp Chern-Simons

F.1 $SO(2N+1)$ and $Sp(2N)$

Let us now give $f(k)$. For $SO(2N+1)$:

$$
f(k) = \begin{cases}
1 & k = j - \frac{1}{2} \ (j = 1, \ldots, N) \\
N - k/2 & k \text{ even} \\
N - k/2 - \frac{1}{2} & k \text{ odd} \,.
\end{cases}
\tag{183}
$$

For $Sp(2N)$:

$$
f(k) = \begin{cases}
N - k/2 - \frac{1}{2} & k \text{ odd} \leq N \\
N - k/2 & k \text{ even} < N \\
N - k/2 + \frac{1}{2} & k \text{ odd} > N \\
N - k/2 + 1 & k \text{ even} > N \,.
\end{cases}
\tag{184}
$$

F.2 $SO(2N)$

For $SO(2N)$:

Again, using Weyl's denominator formula this can be brought to the form

$$
\begin{aligned}
Z_{\text{CS}}(S^3) &= S_{00} = (k+g)^{-N/2} \prod_{i<j} \sinh \frac{j-i}{2t} \sinh \frac{2N-j-i}{2t} \\
&= (k+2N-2)^{-N/2} \prod_{k=1}^{N-1} (\sinh k/2t)^{f(k)} \prod_{k=N}^{2N-1} (\sinh k/2t)^{f(k)-1} \\
&= (k+2N-2)^{-N/2} \prod_{k=1}^{2N-1} (\sinh k/2t)^{f(k)} / \prod_{k=N}^{2N-1} \sinh k/2t
\end{aligned}
\tag{185}
$$

and $f(k) = N - 1/2k$ if k is even, and $N - 1/2k + 1/2$ if it is odd, in other words,

$$
f(k) = [N - \frac{k+1}{2}] \ .
\tag{186}
$$

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