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Planckian Scattering and Black Holes

Sebastian de Haro[1]

Spinoza Institute, Utrecht University
Leuvenlaan 4, 3584 CE Utrecht
The Netherlands

and

Institute for Theoretical Physics
Utrecht University
Princetonplein 5, 3584 CC Utrecht
The Netherlands

Recently, 't Hooft's S-matrix for black hole evaporation, obtained from the gravitational interactions between the in-falling particles and Hawking radiation, has been generalised to include transverse effects. The action describing the collision turned out to be a string theory action with an antisymmetric tensor background. In this article we show that the model reproduces both the correct longitudinal and transverse dynamics, even when one goes beyond the eikonal approximation or particles collide at nonvanishing incidence angles. It also gives the correct momentum tranfer that takes place in the process. Including a curvature on the horizon provides the action with an extra term, which can be interpreted as a dilaton contribution. The amplitude of the scattering is seen to reproduce the Veneziano amplitude in a certain limit, as in earlier work by 't Hooft. The theory resembles a "holographic" field theory, in the sense that it only depends on the horizon degrees of freedom, and the in- and out-Hilbert spaces are the same. The operators representing the coordinates of in- and out-going particles are non-commuting, and Heisenberg's uncertainty principle must be corrected by a term proportional to the

[1]e-mail: haro@phys.uu.nl

ratio of the ingoing momentum $p_{\rm in}$ to the impact parameter b, $\Delta x \Delta p \geq \frac{\hbar}{2} + \mathcal{O}(\ell_{\rm Pl}^2 p_{\rm in}/b)$. Reducing to 2+1 dimensions, we find that the coordinates satisfy an SO(2,1) algebra.

Introduction

Over the last years, the relation between black holes and strings has been stressed both from the string theory side [1] and from gravity [2]. This has led to the hypothesis [3] that two-dimensional closed surfaces (in four dimensions), and in particular black hole horizons, may carry all the information that is hidden inside them. One of the concrete attempts to realise this idea (see also [4] for more recent developments) is 't Hooft's S-matrix description of black holes [5]. He calculated the microscopical S-matrix for scattering between in-falling particles and out-coming radiation in a black hole background. The essential ingredient was the description of the dynamics of the horizon, that is deformed by the presence of the particles, thus modifying the trajectories of out-coming particles. It turned out that the action which appears in this S-matrix contains the Nambu-Goto action of the membrane (without time) or Euclidean string. In reference [6], the S-matrix was covariantly generalised. Nevertheless, as stressed in that paper, since the starting point was the S-matrix as calculated in Minkowski space, an eventual curvature in the transverse direction was not taken into account. Furthermore, the original S-matrix had been calculated in the limit where the transverse momenta can be neglected, which corresponds to a large impact parameter, so it was not obvious that the new expression would include the effect of these momenta. In this paper we study both problems in more detail, describing highly energetic particles by means of 2-dimensional field theory.

In the first section we review the classical analysis of geodesics in the gravitational field of a massless particle, and the Dray-'t Hooft treatment of the transverse curvature. More details can be found in the appendices. We use these results in section 2 to calculate the momentum transfer during the collision. Our approach is perturbative in the ratio of the in-going momentum to the impact parameter times Planck's length squared. Then in section 3 we check that the covariant equations of motion reproduce the trajectories of this scattering process, both the longitudinal and the transverse ones. In section 4 we show how to include a curvature in the S-matrix. We find that, in string theory language, this amounts to a dilaton term in the action, with a specific expression for the dilaton in terms of the positions of the particles. There is thus a strong similarity between this model for Planckian scattering and a $3+1$-dimensional string theory with a graviton, an antisymmetric tensor and a dilaton. Working out the scattering amplitude, in the limit of vanishing transverse momentum one gets the Veneziano formula.

In section 5 we study the quantum mechanics of the model. We find that Hilbert space is severely reduced due to the identification between momentum and position operators. Actually, the Hilbert space of the in-going particles is identified with that of the out-coming ones. Their trajectories must be described by non-commuting coordinates. We also get a noncommutative algebra for the in- and out- momentum operators, in particular for the transverse ones, which satisfies the Jacobi identity. We find evidence that Heisenberg's uncertainty principle has to be corrected with a term proportional to Newton's constant. This is a consequence of a certain "merging" of the different coordinates due to the gravitational interaction. Indeed, during the collision there is a momentum transfer which is reproduced by this new term. Although our results are not totally conclusive, since the nonlocality of certain expressions poses constraints on the validity of the model and leads to several technical problems, the proposal does give correct results within the used approximation. We verify that the canonical momentum operator contains the information about the recoil of the particles due to their mutual interaction. Another check of the model is performed in the last section, where we compactify one dimension and compare with the $2+1$-dimensional case, where one can overcome the mentioned problems. We find exact agreement. In the lower dimensional case, the position operators obey an SO(2,1) algebra. Also in this case, the Heisenberg uncertainty principle has to be modified.

1 Classical scattering at Planckian energies

The gravitational back-reaction of highly energetic particles on the metric is the starting point of the S-matrix Ansatz. The underlying philosophy is that the back-reaction of the in-falling particles modifies the out-coming trajectories. This modification would allow one to make statements about the quantum state of the black hole. The same holds for the back-reaction of out-coming Hawking radiation, which modifies the location of the horizon. It was shown by Dray and 't Hooft [7] that the effect of a particle on the metric is to shift the geodesics with a translation factor that is proportional to the momentum of the incoming particle, as we can see in Fig. 1. This phenomenon is known as shockwave. In this section we review some known facts about shockwaves, which we use in later sections.

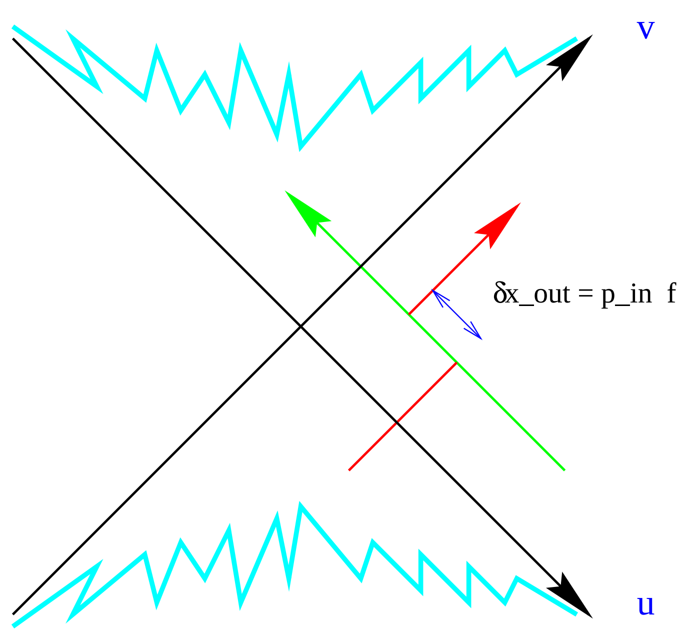

Figure 1: Effect of the shockwave in the lightcone directions

The strength of the gravitational shift is given by

$$\delta x_{\text{out}} = 8\pi G\, p_{\text{in}}\, f(\tilde{x}), \tag{1}$$

where $p_{\rm out}$ and $x_{\rm in}$ are, respectively, the momentum and positions of the out- and in-going particles in the longitudinal direction, and $\tilde{x} \equiv (x, y)$ are the directions transverse to the motion of the particles, that are assumed to collide almost frontally. f, the shift function, is the propagator of the two-dimensional surface transverse to the trajectory of the incoming particle. In flat Minkowski space,

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2, \tag{2}$$

it is given by

$$f = \frac{1}{4\pi} \log \tilde{x}^2. \tag{3}$$

For more details, we refer the reader to [7].

Furthermore, there is a deflection of the trajectories in the transverse plane (see Fig. 2), described by

$$\cot\alpha + \cot\beta = 8\pi G\, p_v \partial_y f(y), \tag{4}$$

where we have chosen our axes so that the deflection is on the $y-v$-plane. So we see that the function f fully describes the geometry, since it gives us both effects. If we define light-cone coordinates $u = -t + z$ and $v = t + z$, then before the collision the in-going particle travels along the trajectory $t = -z$, and the out-coming particle along $t = z$. The whole effect can then be captured in the metric

$$\mathrm{d}s^2 = 2\mathrm{d}v\,(\mathrm{d}u - \mathsf{f}_v(\tilde{x})\,\delta(v)\mathrm{d}v) + \mathrm{d}x^2 + \mathrm{d}y^2, \tag{5}$$

where $\mathsf{f}_v(\tilde{x}) \equiv -\frac{1}{T}\int \mathrm{d}^2\tilde{x}' P_v(\tilde{x}')\, f(\tilde{x} - \tilde{x}')$, with $T \equiv \frac{1}{8\pi G}$. f is now the shift function. This is a straightforward generalisation for the case that the total momentum is not concentrated at one point, but is a distribution over the shockwave. This allows us to describe an arbitrary amount of ingoing particles all sitting on the plane $v = 0$[2], travelling with total momentum $P^u = P_v$. The ith out-coming particle has an initial momentum $p_i^v = p_u^i$, and the in-going momentum distribution $P_v(\tilde{x})$ is typically equal to

$$P_v(\tilde{x}) = \sum_{i=1}^{N} p_v^i \delta(\tilde{x} - \tilde{x}^i), \tag{6}$$

if there are N particles with transverse positions $\tilde{x}^i$ on the plane of the shockwave. All particles satisfy the mass-shell relation $p_u p_v = 0$.

One can reproduce these two effects by solving the Euler-Lagrange equations for this metric. The mathematical subtleties of dealing with a metric including distributions have been analysed in [8]. Here, however, we will suffice with a more heuristic treatment, which is good enough in the simple cases we will be considering. The first of the equations of motion in this metric gives

$$\ddot{v} = 0, \tag{7}$$

where the dot denotes the derivative with respect to the affine parameter λ along the geodesic. This allows us to use v as a time coordinate. The other equations then amount

[2] When going to quantum mechanics, this condition will be relaxed.

to the following solutions:

$$
\begin{aligned}
u(v) &= u(0) - \frac{1}{2T}\,\mathrm{sgn}\,(v) \int \mathrm{d}^2\tilde{x}'\, P_v(\tilde{x}') \left(f(\tilde{x}_0 - \tilde{x}') + v\frac{\partial x^i}{\partial v}(0)\,\partial_i f(\tilde{x}_0 - \tilde{x}') \right) \\
x^i(v) &= x^i(0) + p_0^i v + \frac{1}{2T}\, v\,\mathrm{sgn}\,(v) \int \mathrm{d}^2\tilde{x}'\, P_v(\tilde{x}')\,\partial_i f(\tilde{x}_0 - \tilde{x}'),
\end{aligned}
\tag{8}
$$

where we defined $\tilde{x}_0 \equiv \tilde{x}(0)$ and used the identity $f(v)\,\delta(v) = f(0)\,\delta(v)$. As a boundary condition, we have chosen that the initial momentum in the u-direction is zero, and in the transverse i-direction[3] it is p^i.

If we now concentrate on the $y-v$ plane, and filling in equation (6), differentiating (8) yields

$$
\frac{\partial y}{\partial v} = \frac{1}{2T}\,\mathrm{sgn}\,(v) \int \mathrm{d}^2\tilde{x}' P_v(\tilde{x}')\,\partial_y f(\tilde{x}_0 - \tilde{x}'), \tag{9}
$$

which reduces to the result outlined in appendix A if there is just one particle going in. The latter is based on the direct calculation of geodesics in the linearised Schwarzschild field of a light particle, after which the massless limit is taken, together with the limit that the particle travels at the speed of light.

In the following we will linearise our expressions in the expansion parameter ε, defined as $\varepsilon \equiv G\,p_{\mathrm{in}} b$, where p_{in} is the in-going momentum and b the impact parameter, given by the transverse separation between the colliding particles. Notice that, since f is logarithmic in the transverse distance, $\partial_i f \sim \frac{1}{b}$.

The first of (8) gives us the shift (1) in the longitudinal coordinate u as a consequence of the ingoing particle plus a correction that —if the ingoing particles have no momentum along the transverse directions—, as it is $\mathcal{O}(\varepsilon^2)$, can be neglected . The second of (8) represents the kink in the trajectory of the out-coming particle, equation (4).

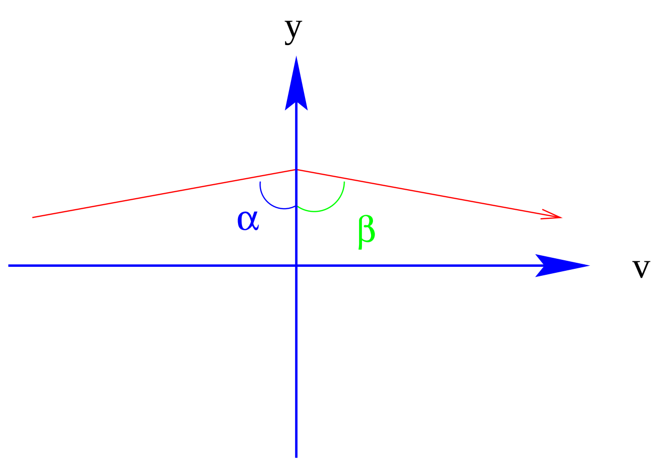

Figure 2: Effect of the shockwave in the longitudinal direction

[3]The latter will often be taken equal to zero.

This method for calculating geodesics by a shift in the metric, which is Penrose's scissors-and-paste method [9], applies to more general space-times. Indeed, starting with a metric of the form

$$\mathrm{d}s^2 = 2A(u,v)\,\mathrm{d}u\mathrm{d}v + g(u,v)\,h_{ij}(\tilde{x})\mathrm{d}x^i\mathrm{d}x^j, \tag{10}$$

which is a vacuum solution of Einstein's equations, Dray and 't Hooft showed that one obtains a solution describing a photon sitting at $v = 0$, $(x, y) = 0$, by shifting u by a factor of $p_v f$, if at $v = 0$ the following conditions are fulfilled[4]:

$$\begin{aligned} A_{,u} &= 0 = g_{,u}, \\ \frac{A}{g}\triangle f - \frac{g_{,uv}}{g} f &= \frac{A^2}{\sqrt{\det\, g_{\mu\nu}}}\,\delta^{(2)}(\tilde{x}-\tilde{x}_0), \end{aligned} \tag{11}$$

with the Laplacian $\triangle = \frac{1}{\sqrt{h}}\,\partial_i\sqrt{h}h^{ij}\partial_j$. In the Minkowski case, the solution of the second of (11) is given by (3).

2 Beyond the eikonal approximation: momentum transfer

Until now we have discussed how the trajectories of out-coming particles are modified by the shockwaves of the ingoing particles. Next we will consider the momentum transfer, which is related to the ingoing momentum, and we will need later to compare with the results coming from the S-matrix.

From Fig. 2, we learn that

$$\tan\gamma = \frac{p_y}{p_u}, \tag{12}$$

where the angle γ is defined by $\gamma = \pi - \alpha - \beta$, and α and β are defined as in the figure. p_y and p_u are the momentum of the out-coming particle in the y and v-directions, respectively, *after* it passes the shock wave. These quantities are different from the momenta before the interaction, which we denote by p^0_μ. We do not explicitly write the superscripts in or out, since it should be clear from the context whether the momentum refers to the in-going or out-coming particle[5].

We now take the initial transverse momentum to be zero, $p^0_y = 0$. This means that $\alpha = \pi/2$ and hence

$$\cot\alpha + \cot\beta = \tan\gamma = \frac{1}{T}\,p_v\,\partial_y f(y_0). \tag{13}$$

One can easily calculate that the exchange of momentum in the v-direction, to first order in ε, is equal to zero, so that $p_u \simeq p^0_u$. Therefore, we have

$$p_y = \frac{1}{T}\,p_u p_v \partial_y f. \tag{14}$$

[4] We have interchanged the u and v coordinates with respect to the Dray-'t Hooft's notation, so that the photon goes *into* the black hole if we consider Kruskal coordinates. We also have a slightly different definition for f: $\sqrt{\det\, g_{\mu\nu}}\, f = -T\,G$, where G is the Green's function used by Dray and 't Hooft.

[5] In this section we assume there is only one particle coming in.

Since $p_y \sim \frac{\partial y}{\partial v}$ and the particle is travelling at the speed of light, this can also be directly deduced from the second of (9) (see section 3).

If the initial transverse momentum is nonzero, differentiating (8) once yields

$$\frac{\partial u}{\partial v} = -\frac{1}{2T}\,\mathrm{sgn}\,(v)\frac{\partial x^i}{\partial v}\,p_v \partial_i f(\tilde{x}_0), \tag{15}$$

so for the out-coming particle we have

$$p_v = -\frac{1}{T}\,p_0^i p_v\,\partial_i f(\tilde{x}_0). \tag{16}$$

This can also be obtained from the mass-shell relation $p_\mu p^\mu = 0$. Hence, although in the transverse plane there is a momentum transfer, as we observe from (14), in the forward direction this transfer is of $\mathcal{O}(\varepsilon^2)$, and thus negligible to first order in ε. This is a specific feature of the lightcone coordinates, where the energy is proportional to the square of the transverse momentum, instead of being linear in it. Observe also that, since we have $\partial_i f \sim \frac{1}{b}$, for large transverse separations (compared to the Planck length) also the transverse momentum transfer is negligible. We are then in the regime where the eikonal approximation is valid.

3 An action principle for Planckian scattering

In reference [6] an action was proposed in terms of which an S-matrix can be constructed for the scattering process described in section 1. It was obtained by a covariant generalisation of 't Hooft's action. The latter had been found by considering the effect of the shockwave on the wavefunction of out-coming particles. For details about this method we refer to [2]. Here we show that, with a further refinement we introduce in this section, one can find an equation of motion which reproduces both the longitudinal and the transverse effects discussed previously.

The candidate action is[6]

$$S \stackrel{?}{=} -\frac{T}{2}\int \mathrm{d}^2\tilde{\sigma}\,\left(\sqrt{h}h^{ij}g_{\mu\nu}\partial_i X^\mu \partial_j X^\nu + \frac{1}{T}\,\epsilon_{\mu\nu\alpha\beta}X^\mu P^\nu\,\epsilon^{ij}\partial_i X^\alpha \partial_j X^\beta\right). \tag{17}$$

Here, X^μ is the field that describes the trajectories of particles, modified by the presence of the momentum P^ν of the other particles. (17) has the equation of motion

$$\triangle X^\mu \stackrel{?}{=} \frac{1}{2T\sqrt{h}}\,\epsilon^{\mu\nu\alpha\beta}\,\epsilon^{ij}\left(\partial_i X_\alpha \partial_j X_\beta P_\nu + 2\partial_j (P_\nu X_\beta)\partial_i X_\alpha\right), \tag{18}$$

which has a nonlocal solution. The first term under of (18) can indeed be checked to reproduce the known effects, as we will do later. However, the second factor is rather troublesome, for various reasons. One of them is that it gives extra contributions which do not appear in the gravity calculation of section 1. This term comes from the variation

[6] We have adopted the usual definition, in which a factor of $\sqrt{-\det g_{\mu\nu}}$ or $\sqrt{h}$ is included in the symbols $\epsilon_{\mu\nu\alpha\beta}$ and ϵ_{ij}, respectively, so that they transform as tensors. Although the ϵ-tensors appear here under the integral sign, we actually have neglected a term with a derivative acting on them. We also leave out of (22) a term involving a certain Christoffel symbol. These terms, however, only show up when we depart from flat geometries.

of X^α and X^β in the action. Indeed, (17) cubic in the field X^μ, whereas the original 't Hooft action was only linear. We therefore replace (17) by the following action:

$$S = -\frac{T}{2}\int \mathrm{d}^2\tilde{\sigma}\,\left(\sqrt{h}h^{ij}g_{\mu\nu}\partial_i X^\mu\partial_j X^\nu + \frac{1}{T}\,\epsilon_{\mu\nu\alpha\beta}X^\mu P^\nu\,\epsilon^{ij}\partial_i X_0^\alpha\partial_j X_0^\beta\right). \tag{19}$$

Here, X_0^μ is the position of the test particle *before* the interaction, *i.e.*, unmodified by the shockwave. It is a fixed background, which in 't Hooft's calculation equals $X_0^\mu = (0,0,\sigma,\tau)$. Therefore, there is now just one X^μ in the second term of (19) to be varied, coupled to an external source term:

$$S = -T\int \mathrm{d}^2\tilde{\sigma}\sqrt{h}\,\left(\frac{1}{2}\,h^{ij}g_{\mu\nu}\partial_i X^\mu\partial_j X^\nu + J_\mu X^\mu\right), \tag{20}$$

with

$$J_\mu = \frac{1}{2T\sqrt{h}}\,\epsilon_{\mu\nu\alpha\beta}\,W_0^{\alpha\beta}\,P^\nu, \tag{21}$$

and the orientation tensor $W^{\mu\nu}$ is defined in the following way: $W_0^{\mu\nu} \equiv \epsilon^{ij}\partial_i X_0^\mu\partial_j X_0^\nu$. The source is thus a given function. The equation of motion is now

$$\triangle X^\mu(\tilde{\sigma}) = J^\mu(\tilde{\sigma}), \tag{22}$$

and is solved by

$$X^\mu(\tilde{\sigma}) = X_0^\mu(\tilde{\sigma}) + \int \mathrm{d}^2\tilde{\sigma}'\sqrt{h}\,J^\mu(\tilde{\sigma}')f(\tilde{\sigma}-\tilde{\sigma}'), \tag{23}$$

where f is the Green's function defined by

$$\triangle f(\tilde{\sigma}-\tilde{\sigma}') = \frac{1}{\sqrt{h}}\,\delta(\tilde{\sigma}-\tilde{\sigma}'). \tag{24}$$

X_0^μ is here a harmonic function of $\tilde{\sigma}$, depending on the boundary conditions on X^μ.

Now the action (20) reproduces the longitudinal shifts of the shockwave only. To include the transverse effects as well, we have to do better. The validity of (20) is limited to the case of negligible transverse momenta, which corresponds to the limit $\varepsilon \approx 0$. Now, following [6] and [17], our assumption is that a more general expression is provided by replacing X_0^μ by X^μ in (22). We then get the equation of motion

$$\triangle X^\mu(\tilde{\sigma}) = \frac{1}{2T\sqrt{h}}\,\epsilon^{\mu\nu\alpha\beta}\,\epsilon^{ij}\partial_i X_\alpha\partial_j X_\beta P_\nu. \tag{25}$$

As we will next show, this indeed gives the transvese effects.

Let us make one remark about (19). One would be inclined to replace it by (17), which does not depend on the background field X_0^μ. However, as argued at the beginning of this section, when we calculate the equation of motion this action gives an extra term which does not yield the correct shift equations. Therefore, it cannot be assumed to be the total action, but probably it has to receive new contributions that cancel the extra term in (18). Because of the ϵ-tensor, we have not been able to find such a term. So we take (25) as our starting point for an improved theory, (19) corresponding to the eikonal limit.

Now it is easy to check that, for the longitudinal variables u and v, (25) gives the correct result. We choose the gauge $X^\mu = (u, v, x, y)$, and $\tilde{\sigma} \equiv \tilde{x}$. As in section 1, the in-going particles now only have momentum P_v, and the out-coming particles momentum P_u. Then (25) gives, for the u-component,

$$u = u_0 - \frac{1}{T} \int \mathrm{d}^2\tilde{x}' \, P_v(\tilde{x}') \, f(\tilde{x} - \tilde{x}'). \tag{26}$$

There is however a subtlety when we want to compare the solution of the equations of motion of the action (26) to the trajectories directly calculated from gravity. Our action was built up by S-matrix considerations of in-going and out-coming wavepackets. In particular, the second term of the action, (20), which involves the momentum of the in-going particle, came up when considering out-coming trajectories *after* the shift, in other words, trajectories for which $v \geq 0$. All earlier trajectories remained unchanged. So, for the second term of (26) is only valid for trajectories $v \geq 0$, we have to include a step function $\theta(v)$ in order to compare it to the trajectories coming from the gravity calculation. Since we have the relation $\theta(v) = \frac{1}{2} + \frac{1}{2} \operatorname{sgn}(v)$, we can just replace the Heaviside function by $\frac{1}{2} \operatorname{sgn}(v)$, including the constant piece in u_0. This is allowed, since physically[7] what is of interest is only the relation (4), which is not modified by such redefinitions of u_0. Taking this into account (and doing the same for the in-going particles), we get for the u- and v-directions

$$\begin{aligned} u &= u_0 - \frac{1}{2T} \operatorname{sgn}(v) \int \mathrm{d}^2\tilde{x}' \, P_v(\tilde{x}') \, f(\tilde{x} - \tilde{x}'), \\ v &= v_0 + \frac{1}{2T} \operatorname{sgn}(u) \int \mathrm{d}^2\tilde{x}' \, P_u(\tilde{x}') \, f(\tilde{x} - \tilde{x}'), \end{aligned} \tag{27}$$

which agrees with (8).

Next we check that also the transverse modes, which were not included in the original 't Hooft action [5], automatically come out of the covariant equation of motion (25). We concentrate on the motion in the $y - v$, $x - v$ planes. (25) yields

$$x^i = x^i_0 + \frac{1}{T} \int \mathrm{d}^2\tilde{x}' \, P_v(\tilde{x}') \, \partial_i v(\tilde{x}') f(\tilde{x} - \tilde{x}'), \tag{28}$$

the index i running over the values 2 and 3 in the transverse plane, $(x^2, x^3) = (x, y)$, and so, upon taking the derive with respect to v (notice that now v is regarded as the affine parameter, just as in in section 1, so that the transverse variables become dependent on it) and changing variables, (28) becomes

$$\frac{\partial x^i}{\partial v} = \frac{1}{T} \int P_v \partial_i v \, \partial_v f = \frac{1}{T} \int P_v \partial_i f(\tilde{x} - \tilde{x}'). \tag{29}$$

The dependence on the index i thus corroborates the symmetry in the choice between x and y. If f, for example, only depends on y, x will be constant with respect to v, and viceversa.

To compare (29) to the trajectories (8) we still have to include the sgn-function. However, when taking the derivative with respect to v, we would get a δ-function contribution. Probably, the sgn-function, which must be included already on the right-hand side

[7] The same can be said of the solution (8) to the equations of motion, for which a Heaviside-function would have done as well.

of $\triangle y = J^y$, goes under the integral sign of y and therefore is not affected by the derivative; maybe at the end, taking v as the affine parameter, independent of the transverse coordinates, it could be taken out of the integral sign, giving

$$\frac{\partial x^i}{\partial v} = \frac{1}{2T}\,\mathrm{sgn}\,(v) \int P_v \partial_i f(\tilde{x} - \tilde{x}'), \tag{30}$$

which is (8). This point, however, is still unclear and has to be studied in more detail. In the end we will make some remarks in this respect.

Equation (29) obviously also contains information about the transverse momentum transfer that takes place during the collision. Writing $p^\mu = p\,\frac{\partial x^\mu}{\partial v}$ (as v is the affine parameter for the out-coming particles), (30) implies

$$P^i(\tilde{x}) = \frac{1}{T}\,P^v(\tilde{x}) \int \mathrm{d}^2\tilde{x}'\,P_v(\tilde{x}')\,\partial_i f(\tilde{x} - \tilde{x}'). \tag{31}$$

The momentum P^v on the right-hand side of (31) is needed to cancel the factor of p that appears on the left-hand side of (31) when we replace $\frac{\partial x^i}{\partial v}$ by p^i, since for the trajectory of the out-coming particles *before* the collision holds $x^\mu = (0, v, \tilde{x}_0)$. This is the expression (15) found from kinematics of the process.

We thus see that the covariant equation of motion (25) reproduces both the longitudinal and the transverse effects of the shockwave, even though the original calculation only used the longitudinal effect. But as remarked, its solution is nonlocal, which means X^μ does not transform as a scalar under general reparametrisations of $\tilde{\sigma}$. It, however, does transform properly under the Lorentz group. In sections 5 and 6 we will see how to obtain local expressions.

The second term of (20) being linear in X^μ simplifies the S-matrix enormously, since the path integral can be evaluated exactly. We are interested in an expression of the kind

$$\langle \mathrm{out}|\mathrm{in}\rangle = \mathcal{N}\,\langle \epsilon^{-iT\int J_\mu X^\mu}\rangle, \tag{32}$$

where $J^\mu = J^\mu_{\mathrm{in}} + J^\mu_{\mathrm{out}}$ and the expectation value on the right-hand side is taken with respect to the reference S-matrix element that is defined by $\langle 1\rangle = \mathcal{N}^{-1}$ (see [5]). We integrate over all fields X^μ and possibly (although this is not totally clear yet) over the metric h_{ij}. For simplicity, we temporally gauge-fix the transverse metric to be flat, ignoring ghosts and the anomaly in the path-integral measure. Evaluation of the path integral then gives

$$\langle \mathrm{out}|\mathrm{in}\rangle = \exp\left(-\frac{iT}{2}\int \mathrm{d}^2\tilde{\sigma}\,\mathrm{d}^2\tilde{\sigma}'\,J_\mu(\tilde{\sigma})J^\mu(\tilde{\sigma}')f(\tilde{\sigma} - \tilde{\sigma}')\right). \tag{33}$$

Notice that

$$J_\mu J^\mu = -\frac{1}{2T^2}\,P_\mu P^\mu W_{\alpha\beta}W^{\alpha\beta} - \frac{1}{T^2}\,P_\mu W^{\mu\alpha}W_{\alpha\beta}P^\beta. \tag{34}$$

In the limit that the transverse momentum of one of the particles can be neglected, for which this action is valid, the second term of (34) vanishes, and we get

$$\langle \mathrm{out}|\mathrm{in}\rangle = \exp\left(\frac{i}{2T}\int \mathrm{d}^2\tilde{\sigma}\,\mathrm{d}^2\tilde{\sigma}'\,g_{\mu\nu}P^\mu_{\mathrm{in}}(\tilde{\sigma})P^\nu_{\mathrm{out}}(\tilde{\sigma}')f(\tilde{\sigma} - \tilde{\sigma}')\right). \tag{35}$$

This is the nonlocal form of the 't Hooft S-matrix. Substituting the value of the momentum distribution, (6), the amplitude gives an expression which depends on the momentum of each particle but also on the transverse location at which it enters or leaves the horizon. But one is interested in an amplitude which only depends on the momenta, and hence should integrate over all the possible points at which a particle can impinge the horizon. The amplitude (35) thus becomes

$$\begin{aligned}\mathcal{A} &= \langle p_1^{\mathrm{out}} \cdots p_M^{\mathrm{out}} | p_1^{\mathrm{in}} \cdots p_N^{\mathrm{in}} \rangle \\ &= \int \prod_{i,j=1}^{NM} \mathrm{d}^2\tilde{x}^i \, \mathrm{d}^2\tilde{x}^j \, \exp\left(\frac{i}{2T} \sum_{i,j=1}^{NM} \int \mathrm{d}^2\tilde{x} \, \mathrm{d}^2\tilde{x}' \, g_{\mu\nu} p_i^\mu p_j^\nu f(\tilde{x}^i - \tilde{x}^j) \right) \\ &= \int \prod_{i,j=1}^{NM} \mathrm{d}^2\tilde{x}^i \, \mathrm{d}^2\tilde{x}^j \, |\tilde{x}^i - \tilde{x}^j|^{iG\, p_i \cdot p_j}, \end{aligned} \tag{36}$$

using (3). Regularising this integral by dividing out the volume of the symmetry group [10], one gets the Koba-Nielsen generalisation of the Veneziano amplitude, as remarked in [11]. The string constant is imaginary.

4 Including the curvature of the horizon: relation to strings

In this section we study the case that the horizon of the black hole where the particles interact is not flat, but has a nonvanishing (two-dimensional) Ricci scalar, and the effect of this on the S-matrix. In reference [6] this possibility was not worked out. It was nevertheless argued that a dilaton term is expected to appear in the string theory action that effectively describes the dynamics at the horizon. But because one was unable to find it —for it couples to the Ricci scalar on the world-sheet, which was zero in the Rindler approximation— it was introduced by hand. We will see that this term naturally appears if one considers curved metrics from the beginning.

In order to do this we return to the equations for the shift, equation (11), that the function f must satisfy at the plane $v = 0$. Although with a particular choice of coordinates, these conditions include the effect of the transverse curvature. We now look for a more general form of this expression, which can be incorporated into the S-matrix. This is readily found if one considers that the second term on the left-hand side of (11) is a relic of the two-dimensional Ricci-tensor. Indeed, according to the calculation outlined in Appendix B, the Ricci tensor of the vacuum metric (10) equals

$$R_{ij} = R_{ij}^{(2)} - h_{ij} \frac{g_{,uv}}{A}, \tag{37}$$

where R_{ij} is the transverse part of Ricci tensor obtained from the full four-dimensional Riemann tensor, see (101), and $R_{ij}^{(2)}$ is the Ricci tensor corresponding to the two-dimensional metric h_{ij}. Now we notice that R_{ij} satisfies the vacuum Einstein equations[8], and hence we have that solutions of this equation must satisfy

$$R^{(2)} = \frac{2}{A} \, g_{,uv}. \tag{38}$$

Therefore we can write the second of equations (11) as

$$\left(\triangle - \frac{1}{2} R^{(2)} \right) f = \frac{1}{\sqrt{h}} \, \delta^{(2)}(\tilde{x} - \tilde{x}_0), \tag{39}$$

[8]It corresponds to the metric before we apply the shift, see the Appendix B.

$\triangle$ and R being calculated in the same metric h_{ij}.

It is easy to include this extra contribution in our action. To this purpose we apply the philosophy of [6]. When the curvature term was zero, it was argued that the action reproducing (39) is given by (19). The first term gives the free propagator, which is the first summand on the right-hand side of (39). The second term of the action is an interaction term, which provides the source on the right-hand side of (39). Now it is clear that the term involving the curvature should be a mass term, and therefore the action becomes

$$S = -\frac{T}{2}\int \mathrm{d}^2\tilde{\sigma}\,\left(\sqrt{h}h^{ij}g_{\mu\nu}\partial_i X^\mu \partial_j X^\nu + \frac{1}{2}\sqrt{h}R^{(2)}X_\mu X^\mu + \frac{1}{T}\,\epsilon_{\mu\nu\alpha\beta}X^\mu P^\nu\,\epsilon^{ij}\partial_i X_0^\alpha \partial_j X_0^\beta\right). \tag{40}$$

The equation of motion is

$$(\triangle - \frac{1}{2}\,R^{(2)})\,X^\mu = \frac{1}{2T\sqrt{h}}\,\epsilon^{\mu\nu\alpha\beta}\,W^0_{\alpha\beta}\,P_\nu, \tag{41}$$

which generalises (39). This is of course (25) with the extra curvature term, and reminds of a focusing theorem. The general solution to this equation is again (23),

$$X^\mu(\tilde{\sigma}) = X_0^\mu(\tilde{\sigma}) + \frac{1}{2T}\int \mathrm{d}^2\tilde{\sigma}'\,\epsilon^{\mu\nu\alpha\beta}\,W^0_{\alpha\beta}P_\nu\,f(\tilde{\sigma}-\tilde{\sigma}'), \tag{42}$$

but with f satisfying (39) instead of (24).

We now go to the string interpretation of (40). We temporarily bypass the problem of the extra contribution to the equation of motion in (18), assuming that if one is able to construct an action that allows for transverse momenta and gives (25) back, the X_0^μ in (40) will be replaced by X^μ, with additional terms that cancel the extra contribution to (18).

The first term in (40) is the Polyakov term which describes the free propagating string, and, as remarked in [6], the third looks formally like an antisymmetric tensor field $B_{\mu\nu}$ coupled to a Wess-Zumino term. But in this model the antisymmetric tensor field is determined by the position and momenta of the particles, and reads

$$B_{\mu\nu}(X) = \epsilon_{\mu\nu\alpha\beta}\,X^\alpha P^\beta, \tag{43}$$

from which a field-strength is derived which is just the dual of the momentum,

$$H_{\mu\nu\alpha} = 3\,\epsilon_{\mu\nu\alpha\beta}\,P^\beta. \tag{44}$$

The second term of the left hand side of equation (41) had, up to a factor of $\frac{1}{2}$, already been found in reference [12]. However, Verlinde and Verlinde found the first and second terms of the action (40) from rather different considerations, namely by a separation of the Einstein action in a strongly coupled and a weakly coupled piece, corresponding to the two physical length scales involved in the problem (a longitudinal and a transverse one)(see section 5 for a connection with this idea). But here it arises altogether with a term that can possibly be interpreted as an antisymmetric tensor background. Due to this formal similarity with the world-sheet action of string theory, it is tempting to regard the curvature term as a dilaton contribution:

$$S_{\rm dil} = -T\int \mathrm{d}^2\tilde{\sigma}\,\sqrt{h}R^{(2)}\Phi(X), \tag{45}$$

where the dilaton is given by

$$\Phi(X) = \frac{1}{2}\, g_{\mu\nu} X^\mu X^\nu. \tag{46}$$

If this is correct, then the content of the effective theory induced at the horizon is exactly the same as that of string theory, including the antisymmetric tensor and the dilaton, with explicit expressions for these fields. However, as one can see from (45), the dilaton comes in with the same power in the string coupling as the Polyakov term, so that its contribution in the path integral is of the same order as that of the free term. Therefore, in general, conformal invariance will be destroyed, and it will arise only for the massless case, which corresponds to flat shockwaves.

5 A generalised Heisenberg principle

One would like to know what the Hilbert space structure of this model is, that is, how operators act on Hilbert space and what their eigenvalues are. The quantisation of this model for the cases where the transverse momenta are exactly zero and the fluctuations of the shockwave are small was already discussed in [6]. Now we try to deal with the more general case.

We first consider the action of the operators $\hat{P}^\mu$ and $\hat{X}^\mu$ on the state vectors $|P\rangle$ and $|X\rangle$. We obviously have

$$\begin{aligned} \hat{P}^\mu|P\rangle &= P^\mu|P\rangle; \\ \hat{X}^\mu|X\rangle &= X^\mu|X\rangle. \end{aligned} \tag{47}$$

Furthermore, the states $|P\rangle$ and $|X\rangle$ as introduced in the path integral were related by a Fourier transformation [6], so that we have

$$[\hat{X}^\mu(\tilde{\sigma}), \hat{P}^\nu(\tilde{\sigma}')] = i g^{\mu\nu}\, \delta(\tilde{\sigma} - \tilde{\sigma}'). \tag{48}$$

Indeed, when constructing the S-matrix, one used the operator $\hat{P}^\mu$ to generate the shifts of the wavefunction that reproduce the shockwave effect when two particles cross each other. (48) therefore agrees with the interpretation of X^μ as the eigenvalue of a position operator, and of P^μ as that of the momentum operator (more properly, the distribution of momentum along the shockwave).

Furthermore, we have to require that the results of measurements of the positions and momenta of the particles obey the equation of motion (41). This holds both for the in-going and for the out-coming particles. Therefore, we have to require that the eigenvalues of these operators obey this equation as well. The only way we see to do this is by identifying the corresponding operators in Hilbert space (notice that this connects the Hilbert spaces of *different* particles, since (41) relates the trajectory of one particle to the momentum of the other). So, following [5], we propose (41) to be also a relation between the *operators* X^μ and P^μ acting on the Hilbert spaces of the different particles[9]. This imposes a strong constraint on the space of states, for it identifies position states

[9] By an abuse of notation, we use the same symbol for the operators $\hat{X}^\mu$, $\hat{P}^\nu$ as for the corresponding eigenvalues X^μ and P^ν.

with momentum states in a certain way. Later on we will see that also the in- and the out-Hilbert spaces are the same.

Equation (48), combined with the operator analog of (41), now leads (as shown in [6]) to the following commutation relation:

$$[X^\mu(\tilde\sigma), X^\nu(\tilde\sigma')] = -\tfrac{i}{2T}\,\epsilon^{\mu\nu\alpha\beta}\,W_{\alpha\beta} f(\tilde\sigma-\tilde\sigma'). \tag{49}$$

One has to keep in mind that the two operators X^μ here correspond to different (bundles of) particles.

However, it was already remarked that in general, and in particular when we consider departures from a flat shock-wave, since the right-hand side of (49) becomes itself an operator, equation (48) will receive $\mathcal{O}(\varepsilon)$ corrections, which, nevertheless, do not modify (49) to first order in ε. We now investigate these corrections. We anticipate that the fact that we are dealing with fields that depend on the world-sheet variables $\tilde\sigma$ in a nontrivial fashion makes things very complicated, and we were not able to find an analog of (49) that holds for arbitrary shock-waves. Yet, considering the almost-flat case still gives us a lot of information.

Our Ansatz (see appendix C) for the extra term that appears in (48) is the following

$$[X^\mu(\tilde\sigma), P^\nu(\tilde\sigma')] = i\,(g^{\mu\nu}+A^{\mu\nu})\,\delta(\tilde\sigma-\tilde\sigma'), \tag{50}$$

where

$$A^{\mu\nu} = -\frac{1}{T\sqrt{h}}\,\epsilon^{\mu\nu\alpha\beta}\epsilon^{ij}\partial_i X_\alpha \int \mathrm{d}^2\tilde\sigma''\,P_\beta(\tilde\sigma'')\,\partial_j f(\tilde\sigma-\tilde\sigma''). \tag{51}$$

The general form of this expression can be calculated from the Jacobi identity, although — due to the nonlocality— one does not get the correct $\tilde\sigma$-dependence from this (in appendix C we show where the problem comes from). However, one can justify the guess (50). At the end of this section we will see that, combined with the equations of motion, (50) represents the momentum transfer and makes the momentum operator be the generator of translations, as one would expect in a field theory, and in section 6 we will perform an independent check of this expression by dimensionally reducing it and comparing it with the 2+1 dimensional case, where the nonlocality problems can be avoided.

Notice that for large impact parameter, $\frac{1}{b}\ll 1$ in Planck units, the extra term of (50) can be neglected.

The modification of the canonical commutation relation (48) in the presence of gravitational interactions has also been predicted (although in different contexts) by several authors [13][14]:

$$\Delta x\Delta p \geq \frac{\hbar}{2} + \mathcal{O}\left(\frac{\ell_{\mathrm{Pl}}^2\, p_{\mathrm{in}}}{b}\right). \tag{52}$$

To understand the physical meaning of this, we first draw an analogy with string theory. The action of a string interacting with an antisymmetric tensor field background $B_{\mu\nu}$ can, in complex coordinates, be written as

$$S = -\frac{T}{2}\int \mathrm{d}z\mathrm{d}\overline{z}\,(g_{\mu\nu}+B_{\mu\nu})\,\overline{\partial}X^\mu\partial X^\nu. \tag{53}$$

There are two conserved quantities, let us call them Π_μ and $\overline{\Pi}_\mu$. Choosing $\overline{z}$ to be the time variable in these complex "light-cone" coordinates, the canonical momentum is

$$\Pi_\mu = \frac{\delta S}{\delta \overline{\partial} X^\mu} = -(g_{\mu\nu} + B_{\mu\nu})\partial X^\nu \equiv (g_{\mu\nu} + B_{\mu\nu})P^\nu. \tag{54}$$

It obeys the equal-time commutation relation

$$[X^\mu(z, \overline{z}), \Pi^\nu(z', \overline{z})] = ig^{\mu\nu}\,\delta(z - z'). \tag{55}$$

In analogy with our model (where ε was the expansion parameter), let us assume that the field $B_{\mu\nu}$ is proportional to some very small parameter η. Then an Ansatz consistent with (54) to first order in η is

$$P_\mu = (g_{\mu\nu} - B_{\mu\nu})\Pi^\nu + \mathcal{O}(\eta^2). \tag{56}$$

This variable is the "flat-space" canonical momentum, that is, the generator of translations in the absence of the B-field. It satisfies

$$[X^\mu(z, \overline{z}), P^\nu(z', \overline{z})] = i(g^{\mu\nu} + B^{\mu\nu})\,\delta(z - z'). \tag{57}$$

So the presence of the B-field modifies the canonical momentum and also the commutation relation between the would-be canonical momentum P^μ, and X^μ.

A similar thing is happening in our model. The commutator (50) is not the canonical one, which suggests that we have to look for another variable which is the canonical momentum. This brings us to the following definition:

$$P^\mu_{\text{can}}(\tilde{\sigma}) \equiv (g^{\mu\nu} + A^{\mu\nu})\,P_\nu(\tilde{\sigma}). \tag{58}$$

One can easily check that taking the commutator of this operator with X^μ gives

$$[X^\mu(\tilde{\sigma}), P^\nu_{\text{can}}(\tilde{\sigma}')] = ig^{\mu\nu}\delta(\tilde{\sigma} - \tilde{\sigma}') + \mathcal{O}(\varepsilon^2). \tag{59}$$

Notice that, although $A^{\mu\nu}$ is itself also an operator, to first order in ε this fact does not affect the canonical commutation relation (59), since, for the in-going particles, $A^{\mu\nu}$ is built up of out-coming operators, which, to zeroth order in ε (since $A^{\mu\nu}$ is itself already $\mathcal{O}(\varepsilon)$), commute with the position operator of the in-going particles.

The generalised commutator (50) has a simple interpretation if we go back to the shock-wave picture underlying the action (40). Before the interaction takes place, the different coordinates are not yet coupled through the equations of motion (41) or (8), *i.e.*, the out-coming fields X^μ do not yet depend on $\tilde{\sigma}$. However, *after* the interaction, the coordinates have got mixed and P_u can, for example, generate sideways displacements as well. So it is natural to identify the canonical momentum P^μ_{can} with the momentum before the interaction[10], which we denote by P^μ_0, and P^μ with the momentum after the shockwave. The latter describes the momentum transfer, and can be seen to be a measure for the recoil of the particles. This aspect had not been taken into account in earlier works. This holds both for the in-going and the out-coming particles. Since to zeroth order in ε both momenta agree, we will drop the subscript 0 in expressions which are already $\mathcal{O}(\varepsilon)$.

[10] However, the existence of such an operator again violates the Jacobi identity. Probably there is here some subtlety with the time variable involved, since P^μ_{can} is defined for times before the collision, when the coordinates X^μ still commute among themselves.

Although P^μ is a non-canonical operator, when writing (50) out in components we will conclude that it generates translations in the sense of field theory.

This definition turns out to be very powerful since it allows us to calculate expressions for the momentum transfer after the shock-wave interaction. An attempt to derive an algebra that would hold even if there is this recoil was pioneered in [15].

So we now go to Minkowski space to study the momentum transfer between the in-going and the outcoming particles. We allow the momentum distributions to be in any direction, $P_\mu^{0,\mathrm{in}} = (P_u, P_v, P_x, P_y)_{\mathrm{in}}$ and $P_\mu^{0,\mathrm{out}} = (P_u, P_v, P_x, P_y)_{\mathrm{out}}$. The only constraint is that the particles have to be massless. So, in particular, we do not require the transverse momenta to be zero, differently from the standard approach. For convenience, we split the indices in a longitudinal and a transverse part, $\mu = (\alpha, i)$, denoted by (the first few) Greek and Latin letters, respectively. For the transverse parts, we again have the gauge $x_0^i = \sigma^i$. With the above discussed definition of momentum before and after crossing the shockwave, we can invert (58) to

$$P^\mu = (g^{\mu\nu} - A^{\mu\nu})P_\nu^0 + \mathcal{O}(\varepsilon^2), \tag{60}$$

both for the in- and for the out-coming particles.

We now can calculate the transverse momentum transfer that takes place during the interaction[11]:

$$\begin{aligned} P_i(\tilde\sigma) &= \frac{1}{T}\, P_v(\tilde x) \int \mathrm{d}^2\tilde\sigma'\, P_u(\tilde\sigma')\, \partial_i f(\tilde\sigma - \tilde\sigma') - \frac{1}{T}\, P_u(\tilde\sigma) \int \mathrm{d}^2\tilde\sigma'\, P_v(\tilde\sigma')\, \partial_i f(\tilde\sigma - \tilde\sigma') \\ &\equiv \frac{1}{T}\, \epsilon^{\alpha\beta} P_\alpha \int \mathrm{d}^2\tilde\sigma'\, P_\beta\, \partial_i f(\tilde\sigma - \tilde\sigma'). \end{aligned} \tag{61}$$

In the last line we have used covariant notation. The labels "in" or "out" are left out, since it should be clear that the operator on the right-hand side of (63) which is evaluated at the same point $\tilde\sigma$ as $P_i(\tilde\sigma)$, corresponds to the same particle, whereas the operators which are integrated over give the contributions of the other particles. Notice that even if the initial transverse momentum is zero, so that the two particles have a head-on collision, after the interaction it will not vanish anymore. The two particles will 'spin' around each other for a very short time. This is in accordance with the picture of section 2. Equation (61) is (up to a minus sign) indeed in complete agreement with (14) and (31), which were calculated from kinematical considerations and by considering the transverse trajectories, respectively. If the "hard particle" has only longitudinal initial momentum, so that the equation of motion (26) (written in the 2-2 splitting of space-time) for the longitudinal part can be used,

$$X^\alpha = \frac{1}{T} \int \epsilon^{\alpha\beta} P_\beta\, f, \tag{62}$$

we get

$$P_i(\tilde\sigma) = P_\alpha \partial_i X^\alpha. \tag{63}$$

Then, (61) can also be written as

$$P_i(\tilde\sigma) = \frac{\epsilon}{T}\, P^\alpha(\tilde\sigma) \int \mathrm{d}^2\tilde\sigma'\, P_\alpha(\tilde\sigma')\, \partial_i f(\tilde\sigma - \tilde\sigma'), \tag{64}$$

[11] Remember that we dropped the 0 on the right-hand side.

where $\epsilon = 1$ if P_i is an operator corresponding to the in-going particles and $\epsilon = -1$ for the out-operators. This is 't Hooft's sign convention for momenta [5], where all in-going momenta are defined to be positive, and out-coming momenta to be negative. Indeed, if initially the in-going particles only have momentum P_v, and the out-coming ones only momentum P_u, (64) gives us

$$\begin{aligned} P_i^{\rm in}(\tilde\sigma) &= \frac{1}{T}\, P_v^{\rm in}(\tilde\sigma) \int {\rm d}^2\tilde\sigma'\, P_u^{\rm o\ ut}(\tilde\sigma')\, \partial_i f(\tilde\sigma - \tilde\sigma') \\ P_i^{\rm out}(\tilde\sigma) &= -\frac{1}{T}\, P_u^{\rm out}(\tilde\sigma) \int {\rm d}^2\tilde\sigma'\, P_v^{\rm in}(\tilde\sigma')\, \partial_i f(\tilde\sigma - \tilde\sigma'), \end{aligned} \tag{65}$$

Let us now consider the longitudinal momentum transfer. Using (60), this amounts to

$$\begin{aligned} P_u(\tilde\sigma) &= -\frac{1}{T}\, P^i \int {\rm d}^2\tilde\sigma'\, P_u\, \partial_i f + \frac{1}{T}\, P_u \int {\rm d}^2\tilde\sigma'\, P^i\, \partial_i f, \\ P_v(\tilde\sigma) &= +\frac{1}{T}\, P^i \int {\rm d}^2\tilde\sigma'\, P_v\, \partial_i f - \frac{1}{T}\, P_v \int {\rm d}^2\tilde\sigma'\, P^i\, \partial_i f. \end{aligned} \tag{66}$$

In covariant notation,

$$P^\alpha = \frac{1}{T}\, \epsilon^{\alpha\beta} \left(P_\beta \int P^i\, \partial_i f - P^i \int P_\beta\, \partial_i f \right). \tag{67}$$

In the case that the "hard particle" moves along one of the lightcone directions without transverse momentum, the first term vanishes, and (67) can be written as

$$P^\alpha(\tilde\sigma) = -P^i(\tilde\sigma)\, \partial_i X^\alpha(\tilde\sigma). \tag{68}$$

Notice that this expression does vanish if the initial transverse momentum of the test particle is zero. So, although for initial zero transverse momentum there is still a transverse momentum transfer, in the longitudinal plane this transfer is zero to first order in ε. This agrees with the Verlinde idea [12] of separating the gravitational action into two sectors, one corresponding to the longitudinal and the other corresponding to the transverse modes, with two different coupling constants. They argued that, for low momentum transfer (*i.e.*, for large transverse separations), the transverse modes become classical, and only the longitudinal ones have to be treated quantum mechanically. This corresponds to 't Hooft's analysis. From (68) we learn indeed that the longitudinal momentum transfer is negligible to first order in ε if the transverse momentum is negligible, so that the eikonal approximation can be used. However, we also find that even if the initial transverse momenta are zero, when the impact parameter becomes small, there is a transverse momentum transfer, and the transverse modes have to be treated quantum mechanically as well. The eikonal approximation can then not be trusted anymore. Put differently, the transverse modes can be treated classically if the momentum transfer is small, but that (transverse) transfer is not small for Planckian longitudinal momenta, and as soon as the distance between the particles is comparable to the Planck length. The formalism developed here, although not satisfactory for the reasons explained, seems to hold even when the transverse momenta are not negligible, and beyond the eikonal approximation.

Let us now take a closer look at the commutation relation (50),

$$[X^\mu(\tilde\sigma), P^\nu(\tilde\sigma')] = iG^{\mu\nu}\, \delta(\tilde\sigma - \tilde\sigma'), \tag{69}$$

where the "generalised metric" is defined as

$$G^{\mu\nu} \equiv g^{\mu\nu} + A^{\mu\nu}. \tag{70}$$

Writing (69) out in components, the relevant equations are

$$\begin{aligned}
[u(\tilde{\sigma}), p_i(\tilde{\sigma}')] &= i\partial_i u\, \delta(\tilde{\sigma} - \tilde{\sigma}'); \\
[v(\tilde{\sigma}), p_i(\tilde{\sigma}')] &= i\partial_i v\, \delta(\tilde{\sigma} - \tilde{\sigma}'); \\
[x^i(\tilde{\sigma}), p_u(\tilde{\sigma}')] &= i\partial_u x^i\, \delta(\tilde{\sigma} - \tilde{\sigma}'); \\
[x^i(\tilde{\sigma}), p_v(\tilde{\sigma}')] &= i\partial_v x^i\, \delta(\tilde{\sigma} - \tilde{\sigma}'),
\end{aligned} \tag{71}$$

where we remind that the equations of motion give $x^i = x^i(u(\tilde{\sigma}), v(\tilde{\sigma}))$. In covariant notation, the commutators (71) become

$$[x^\mu(\tilde{\sigma}), p_\nu(\tilde{\sigma}')] = i\partial_\nu x^\mu(\tilde{\sigma})\delta(\tilde{\sigma} - \tilde{\sigma}'). \tag{72}$$

As expected, the operator p^μ generates translations in the direction x^μ. In quantum mechanics, the different coordinates are independent of each other, and therefore the right-hand side of (72) reduces to $i\delta^\mu_\nu\, \delta(\tilde{\sigma} - \tilde{\sigma}')$, the canonical commutator. But in this case, since we have a two-dimensional field theory, the longitudinal and the transverse coordinates become mutually dependent, so that (72) is nonvanishing even for different indices μ and ν (notice that, for $\mu \neq \nu$, it is nonzero only if one index is a world-sheet index i, and the other a target-space index α). So p^μ generates translations as in (a two-dimensional) field theory.

One can also get an algebra for the commutator of the p^μ's among themselves. This cannot be done covariantly from the equation of motion (41) because of the ϵ-tensor. However, one can do it in the coordinate system we are using. With some algebra, one comes to (the operators referring all to the in- or all to the out-states)

$$\begin{aligned}
[p_\alpha(\tilde{\sigma}), p_i(\tilde{\sigma}')] &= ip_\alpha(\tilde{\sigma}')\, \partial_i \delta(\tilde{\sigma} - \tilde{\sigma}'); \\
[p_i(\tilde{\sigma}), p_j(\tilde{\sigma}')] &= ip_i(\tilde{\sigma}')\partial_j \delta(\tilde{x} - \tilde{x}') + ip_j(\tilde{\sigma})\partial_i \delta(\tilde{\sigma} - \tilde{\sigma}'),
\end{aligned} \tag{73}$$

Now we can also obtain an algebra that relates the in- and the out-operators. Using (73), we get:

$$\begin{aligned}
[p_v^{\rm in}(\tilde{\sigma}), p_i^{\rm out}(\tilde{\sigma}')] &= -iT\, \partial_i u(\tilde{\sigma}')\, f^{-1}(\tilde{\sigma} - \tilde{\sigma}'); \\
[p_u^{\rm out}(\tilde{\sigma}), p_i^{\rm in}(\tilde{\sigma}')] &= iT\, \partial_i v(\tilde{\sigma}')\, f^{-1}(\tilde{\sigma} - \tilde{\sigma}'); \\
[p_i^{\rm in}(\tilde{\sigma}), p_j^{\rm out}(\tilde{\sigma}')] &= -iT\, \partial_i v(\tilde{\sigma})\, \partial_j u(\tilde{\sigma}')\, f^{-1}(\tilde{\sigma} - \tilde{\sigma}') \\
&+ \frac{i}{T}\, p_v^{\rm in}(\tilde{\sigma})\, p_u^{\rm out}(\tilde{\sigma}')\, \partial_j f(\tilde{\sigma} - \tilde{\sigma}').
\end{aligned} \tag{74}$$

In reference [5] it was not possible to find correct expressions for (74) because, in particular, the expected expression for the last of (74) did not satisfy the Jacobi identity when combined with (73). One can check that the above expression satisfies it exactly.

The algebra (74) is very nonlocal and, furthermore, nonlinear, and so not very useful. It, however, can be significantly simplified by defining the total momentum

$$P_\mu = \int \mathrm{d}^2\tilde{\sigma}\, p_\mu(\tilde{\sigma}). \tag{75}$$

Now one can check that for these variables (74) leads to correct expressions:

$$\begin{aligned}
[p_i^{\rm in}(\tilde\sigma), P_j^{\rm out}] &= i\partial_j p_i^{\rm in}(\tilde\sigma);\\
[p_i^{\rm out}(\tilde\sigma), P_j^{\rm in}] &= i\partial_j p_i^{\rm out}(\tilde\sigma);\\
[p_\alpha^{\rm in}(\tilde\sigma), P_i^{\rm out}] &= i\partial_i p_\alpha^{\rm in}(\tilde\sigma);\\
[p_\alpha^{\rm out}(\tilde\sigma), P_i^{\rm in}] &= i\partial_i p_\alpha^{\rm out}(\tilde\sigma),
\end{aligned} \tag{76}$$

the total transverse momentum generating translations, which is the usual situation in field theory. However, a local expression is lacking.

One can check that the algebra between the transverse in-operators themselves or the out-operators themselves is similar to (76). However, we do not expect the theory to have two different generators of transverse translations. So we must have

$$P^i_{\rm in} = P^i_{\rm out}. \tag{77}$$

Integrating (61) we indeed see that this is the case. The same holds for the lightcone directions, as one sees from equation (66)). Therefore, for the integrated momentum operators we get the constraint

$$P^\mu_{\rm in} = P^\mu_{\rm out}. \tag{78}$$

Recalling that these operators correspond to the momentum transfer, this clearly implies that there is momentum conservation during the collision. Since this holds for the operators acting on any state, as a condition on Hilbert space this constraint is very strong: it means that the Hilbert space of the in-going particles is the same as that of the outcoming particles! This suggests that it is enough to look at Hawking radiation to know what the structure of the inner of the black hole is.

Some of these results had already been predicted in [15], but here they come naturally out of the formalism. Furthermore, we get (74) and (76) for free. Of course, it remains to be clarified why in (118) we get an extra term. As said before, the reason must be the nonlocal dependence of $X^\mu(\tilde\sigma)$ on $\tilde\sigma$. Therefore, we do not pretend to have proven (51). But the fact that it means that the momenta generate translations in the different directions and gives the correct momentum transfer (also obtained in section 2 from kinematics), suggests that this term will stay there when one has a theory with better defined variables and a local algebra, instead of having to "guess" the result, as we did in deriving its presence from consistency of the Jacobi identity. Therefore, we have taken it as our starting point. Actually, in the next section we perform another check that this expression gives the right result. So, although the picture is not complete and the formalism is non-local, there is strong evidence that in the presence of gravitational interactions, Heisenberg's principle has to be modified similarly to (69). Notice furthermore that (69) *does* give a local algebra, (72), after using the equations of motion.

6 Reduction to 2+1 dimensions

The S-matrix Ansatz becomes much simpler if we consider a three-dimensional world, or, better said, a world with one of the space-time (and world-sheet) directions compactified on a circle of radius R_3, say $0 \leq \sigma_2 = y \leq R_3$. To derive the S-matrix in this $2+1$

dimensional world[12], we depart from the $3+1$-dimensional action (40) and then dimensionally reduce it, like in Kaluza-Klein theories. The fields and momenta are assumed to be independent of this fourth coordinate. Correspondingly, at the path-integral level the integration over y will drop out. The action (40) then reduces to

$$S = -\frac{T_3}{2}\int \mathrm{d}\sigma \left(g_{\mu\nu}\,\partial x^\mu \partial x^\nu + \frac{2}{T_4}\,\epsilon_{\mu\nu\alpha}\, x^\mu p^\nu \partial x_0^\alpha \right). \tag{79}$$

We have defined $\sigma \equiv \sigma_1$, $\partial \equiv \frac{\partial}{\partial\sigma}$, and $\epsilon_{\mu\nu\alpha} \equiv \epsilon_{\mu\nu\alpha\sigma_2}$. Notice that we have a renormalized overall coupling, $T_3 \equiv T_4 R_3$, which gives us an effective $2+1$-dimensional gravitational constant:

$$G_3 = \frac{G_4}{R_3}. \tag{80}$$

However, the relative coupling in the second term, $\frac{1}{T_4}$, does not change. This is the coupling that will appear in the equations of motion, and these therefore involve the radius of the periodic dimension[13]. From (80) we see that, for fixed G_4, if we send R_3 to zero the three-dimensional gravitational constant diverges, whereas for $R_3 \to \infty$ (the compactified dimension being "lifted"), G_3 becomes very small.

If the action (40) appeared as a string theory action, (79) looks like the action of a relativistic particle, whose motion would be parametrized by a (spacelike) variable σ. The equation of motion of (79), with x_0^μ replaced by x^μ, is

$$\partial^2 x^\mu = \frac{R}{T}\,\epsilon^{\mu\nu\alpha}\, p_\nu\, \partial x_\alpha. \tag{81}$$

This, again, is our starting point to include transverse effects. Now, as in the preceding, we postulate the canonical commutation relation between x^μ and p^μ,

$$[x^\mu(\sigma), p^\nu(\sigma')] = i g^{\mu\nu}\delta(\sigma-\sigma'). \tag{82}$$

Furthermore, when quantizing this system, we have to take into account the equation of motion (81), which gives the leading contribution to the path integral. So, as in the $3+1$-dimensional case, commuting (81) with x^ν yields

$$[x^\mu(\sigma), x^\nu(\sigma')] = -\frac{iR}{T}\,\epsilon^{\mu\nu\alpha}\,\partial x_\alpha(\sigma')\, f(\sigma-\sigma'), \tag{83}$$

which implies

$$[\partial x^\mu(\sigma), \partial x^\nu(\sigma')] = \frac{iR}{T}\epsilon^{\mu\nu\alpha}\,\partial x_\alpha\, \delta(\sigma-\sigma'). \tag{84}$$

Notice that this is an SO(2,1) invariant algebra. It is local, and therefore seems to be a good starting point for a covariant theory in $2+1$ dimensions. The same algebra has been obtained by 't Hooft using another method [17].

It is useful to define variables that anyhow have a better dependence on the location at the horizon. 't Hooft has defined the following variables:

$$a_A^\mu \equiv \int_A \mathrm{d}^2\sigma\, \partial x^\mu = x^\mu(A_1) - x^\mu(A_0). \tag{85}$$

[12] For another approach, see [16].

[13] From now on we will take G_3 as the "true" gravitational coupling in the lower dimension, and will therefore denote T_3 by T.

These have the nice property:

$$[a^\mu_A, a^\nu_A] = \frac{iR}{T}\,\epsilon^{\mu\nu\alpha}\,a^A_\alpha, \tag{86}$$

where for simplicity we restrict ourselves to the same region A in σ-space. 't Hooft has remarked that this gives a time variable that is quantized in units of $t_{\rm Pl}/R$. Another useful quantity is the total momentum flowing through A,

$$p^\mu_A \equiv \int_A \mathrm{d}\sigma\, p^\mu(\sigma). \tag{87}$$

The canonical commutator then becomes

$$[a^\mu_A, p^\nu_A] = ig^{\mu\nu}. \tag{88}$$

But, as in the four-dimensional case, equation (88) together with (86) does not satisfy the Jacobi identity, since the operators $x\mu$ and p^μ are no longer independent of each other. One can easily check that, in order to satisfy the Jacobi identity, one has to add an extra term to (88). Analogously to the $3+1$-dimensional case, (88) is modified in the following way:

$$[a^\mu_A, p^\nu_A] = iG^{\mu\nu}, \tag{89}$$

with the "generalized metric"

$$G^{\mu\nu} = g^{\mu\nu} - \frac{R}{T}\,\epsilon^{\mu\nu\alpha}\,p^A_\alpha. \tag{90}$$

Because of the equation of motion (81), the algebra (89) can be expressed in terms of the a^μ's alone. We first note the following relations (we use the notation $u_A \equiv a^u_A$, $v_A \equiv a^v_A$, etc.):

$$\begin{aligned} p^A_v &= \frac{T}{R}\left(\frac{\partial x}{\partial v}\right)_A = -\frac{T}{R}\,(\partial u)_A \\ p^A_u &= -\frac{T}{R}\left(\frac{\partial x}{\partial u}\right)_A = \frac{T}{R}\,(\partial v)_A. \end{aligned} \tag{91}$$

Now we can write (89) out in components:

$$\begin{aligned} [u_A, p^A_x] &= i(\partial u)_A \\ [v_A, p^A_x] &= i(\partial v)_A \\ [x_A, p^A_u] &= i\left(\frac{\partial x}{\partial u}\right)_A \\ [x_A, p^A_v] &= i\left(\frac{\partial x}{\partial v}\right)_A, \end{aligned} \tag{92}$$

while all other commutators remain unchanged. Hence, p^μ is the generator of translations in the a^μ-direction, as expected.

The algebras (86) and (89) do not have the $\tilde{\sigma}$-dependence anymore. Therefore they do not suffer the problems mentioned before. One can therefore think of using them as a test for our proposal (51). Upon dimensional reduction of (50), one gets

$$[x^\mu(\sigma), p^\nu(\sigma')] = i\left(g^{\mu\nu} - \frac{R}{T}\,\epsilon^{\mu\nu\alpha}\int \mathrm{d}\sigma''\, p_\alpha(\sigma'')\; pf(\sigma-\sigma'')\right)\delta(\sigma-\sigma'). \tag{93}$$

This equation still has a bad σ-dependence. But now if we go over to the variables a^μ_A and p^ν_A, using the fact that in one dimension $f(\sigma-\sigma') = \frac{1}{2}\,|\sigma-\sigma'|$, one can check that (93) exactly gives (89). In the same way, one can check that direct dimensional reduction of the four-dimensional commutator (49) gives (84), with no need to integrate. One could think that the agreement is due to the fact that the $2+1$-dimensional action has been obtained by dimensional reduction of the $3+1$-dimensional one. This, however, is not exactly the case. The $2+1$-dimensional algebra (86) does not have the nonlocality problems of its higher dimensional analogous, and is closed. (89) was derived directly from this local algebra. What is nontrivial is that when reducing the algebras (49) and (50), the problems of the four-dimensional case disappear. It could well have happened that we got different expressions from (86) or (90), as is probably the case with (118) (although we did not check this). This further advocates for (50) as being correct in our gauge.

7 Discussion and conclusions

Although at a very rudimentary stage, some evidence has been found in favour of the holographic hypothesis, according to which the relevant degrees of freedom of a black hole sit at the horizon. In particular, we find that the in- and out-Hilbert spaces are the same. Via the shockwave model for the black hole, we indeed get a field theory living on the horizon. This field theory has (or is defined by) in- and out-operators which add momentum or generate translations, and satisfy an algebra that can be brought to a local form. The action describing the dynamics of the surface has an interpretation as a string theory action, with background fields that are given by the momenta and positions of the particles. In particular, we have found an antisymmetric tensor background and a dilaton term, which breaks conformal invariance at the classical level. The scattering amplitude one obtains is the well-known Veneziano formula already found in [11], with an imaginary Regge slope that is related to Newton's constant.

The collisions under study are completely characterized by the changes in the trajectories of the particles and the momentum transfer. We checked explicitly that the equation of motion gives the longitudinal *and* the transverse trajectories. Both effects are generated by the single function f. We were able to calculate the recoil of the particle during the process as well, an aspect which had not been well understood in earlier works.

At the quantum level, the new action gives four non-commuting coordinates, as already shown in [6], but the commutator is nonlocal. This breaks covariance under general world-sheet reparametrizations. Furthermore, we find that due to the dependence of the four coordinates on the world-sheet coordinates (that is, the position at which the particles impinge the shockwave), the Heisenberg algebra has to be corrected with a term proportional to G, whose form however we could not demonstrate due to nonlocality, and is therefore still at the speculative stage. At this point the need for new variables became evident. Nevertheless, the proposed correction to the Heisenberg commutator was shown to exactly give the correct momentum transfer that takes place during the collision, and for the fact that the momentum operator P^μ generates translations in the direction X^μ. The latter is a consequence of the reduction of the number states of the Hilbert space, since momentum and position operators are identified in a specific way, and the coordinates undergo a certain "merging" due to the gravitational interaction. We also obtained an algebra for the in and the out momentum states. This algebra, however, is nonlocal and very nonlinear. Only for the total momentum operator we get a simple algebra, which is the one expected. The arguments seem to go beyond the eikonal approximation. In general, all the coordinates have to be treated quantum mechanically, and not just the longitudinal ones. Our results corroborate the arguments by Verlinde [12], according to which for small initial transverse momenta and for large impact parameter, the eikonal approximation can be used. Indeed, in that case the momentum transfer is zero. But for small impact parameter, the transverse momentum transfer cannot be neglected. The longitudinal momentum transfer is only relevant if the particles collide at nonzero incidence angles or if we go to higher orders in ε.

Dimensionally reducing to $2+1$ dimensions things became much easier. The results obtained in this case confirmed those of the $3+1$-dimensional case, and in particular the correction of the Heisenberg algebra by a term proportional to the momentum. Both results agree exactly! A discrete structure of Hilbert space is found in the lower dimensions.

Another point to study is the Hilbert space structure of the $3+1$-dimensional S-matrix. In the presence of interactions, the wavefunctions seem to be modified not exactly by a plane wave factor, but rather by a Chern-Simons factor. The reason is that the displacements are generated in the directions perpendicular to the motion. Also, little can be said about the finiteness of the number of microstates of the black hole at this stage, especially because, although we do obtain a closed algebra for the different operators, there is still a very nontrivial $\tilde{\sigma}$-dependence. We see two possibilities to say more on this issue. One is to try and find new variables, like in the 2+1 dimensional case, to see if one can get an algebra that generates a discrete spectrum. The other is to take the background fields $B_{\mu\nu}(X)$ and $\Phi(X)$ seriously, and do a calculation in 2+1 or 3+1-dimensional gravity on this background to see what one gets.

Time is still a problem in this S-matrix picture. The world-sheet of the string, corresponding to the horizon, is Euclidean, and the interactions take place instantaneously. The issue of time has arisen several times, in particular when discussing quantisation, suggesting that we should include it in the S-matrix from the beginning, by means of step functions which keep track of which particles passed before, something similar to

$$S \stackrel{?}{=} -T\int \mathrm{d}^2\tilde{\sigma}\sqrt{h}\,\left[\frac{1}{2}\,g_{\mu\nu}\,\partial_i X^\mu\,\partial^i X^\mu + \Big(J^{\mathrm{out}}_\mu\,\theta(\lambda_{\mathrm{out}}-\lambda_{\mathrm{in}}) + J^{\mathrm{in}}_\mu\,\theta(\lambda_{\mathrm{in}}-\lambda_{\mathrm{out}})\Big)\,X^\mu\right]. \quad (94)$$

At the end, one would choose a gauge like $\lambda_{\mathrm{in}} = u$ or $\lambda_{\mathrm{out}} = v$, depending on which variable is a good affine parameter along the geodesic. Maybe one has to integrate the action (94) over the parameter λ, thus obtaining a three-dimensional world sheet. But all these aspects escape our present knowledge.

One would prefer to first have a local theory which one can then covariantly generalise. The approach advocated here, however, although not satisfactory since it contains nonlocal expressions in the intermediate steps, does give results which in the end can be expressed locally and taken as a starting point for an improved theory.

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Appendix A: Geodesics in the field of a massless particle

It is interesting to see how one gets the result (9) from a direct calculation of the geodesics in the gravitational field of a particle that travels at the speed of light. This is done in reference [7]. The result is the shift in the metric mentioned above (see equation (5)), which can be used as a starting point for calculating the effect in more general spacetimes, as explained in section 1.

Here we briefly review the calculation of the transverse trajectory $y(v)$ of a test particle in this field.

One starts with the linearised Schwarzschild metric of a light particle with mass m. One then calculates the geodesics up to $\mathcal{O}(m)$ by the Euler-Lagrange equations. The result is, for the transverse coordinate y,

$$y(y_0) = y_-(y_0) - 2m\sqrt{1+\frac{\lambda^2 E^2}{y_0^2}}, \tag{95}$$

where λ is the affine parameter along the geodesic, E is the energy of the in-going particle and y_0 is the transverse distance between both particles as they cross each other.

This is the solution obtained when the in-going particle is at rest. However, since we are interested in the effect of a massless particle that travels at the speed of light, we have to boost the solution by an infinite γ-factor and, at the same time, take the massless limit. Since the particle is boosted in the z-direction, the boost does not have any effect on y. However, taking the massless limit does have an effect. First of all, one has to choose a good affine parameter in the boosted frame. It turns out that, since v is proportional to the product $m\lambda$, outside the plane of the ingoing particle, i.e., for $v \neq 0$, λ diverges everywhere in the massless limit. Therefore, v is to be taken as the affine parameter rather than λ outside the shockwave. So we write y in (95) as a function of $v = m\lambda\,\frac{E}{p}$:

$$y = y_- - 2m\sqrt{1+\frac{v^2p^2}{m^2y_0^2}} = y_- - \frac{2}{y_0}\,|v|p\,\sqrt{1+m^2}. \tag{96}$$

In the massless limit we get

$$y = y_- - \frac{2}{y_0} v\,\mathrm{sgn}\,(v)\,p \tag{97}$$

and

$$\frac{\partial y}{\partial v} = \frac{\partial y_-}{\partial v} - \frac{2}{y_0}\,\mathrm{sgn}\,(v)\,p \tag{98}$$

which is (in units where $G = 1$) the expression (9), since in Minkowski space the function f is logarithmic in the transverse distance y_0.

Appendix B: The induced two-dimensional Ricci tensor

In this Appendix we calculate the transverse part of the Ricci tensor of the metric

$$\mathrm{d}s^2 = 2A(u,v)\,\mathrm{d}u\mathrm{d}v + g(u,v)\,h_{ij}(x^i)\mathrm{d}x^i\mathrm{d}x^j, \tag{99}$$

defined as

$$R_{ij} \equiv R^{\mu}{}_{i\mu j}. \tag{100}$$

We use the formula

$$R^{\mu}{}_{i\mu j} = \frac{1}{\sqrt{-\det g}}\,\partial_\mu\left(\sqrt{-\det g}\,\Gamma^{\mu}_{ij}\right) - \partial_i\partial_j\left(\log\sqrt{-\det g}\right) - \Gamma^{\mu}_{\nu i}\Gamma^{\nu}_{j\mu}, \tag{101}$$

which will slightly simplify the calculation. We notice that one can write

$$\begin{aligned}
\sqrt{-\det g} &= Ag\sqrt{h} \\
\Gamma^{\alpha}_{ij} &= -\frac{1}{2}\,g^{\alpha\beta}h_{ij}\partial_\beta g \\
\Gamma^{i}_{j\alpha} &= \frac{1}{2g}\,\delta^i_j\partial_\alpha g \\
\Gamma^{i}_{\alpha j} &= \frac{1}{2g}\,\delta^i_j\partial_\alpha g \\
\Gamma^{\alpha}_{\beta i} &= \Gamma_{i\alpha\beta} = 0,
\end{aligned} \tag{102}$$

where the indices μ,ν run from 1 to 4, α and β take the values 1, 2, and i,j take the values $3,4$. Plugging all this in equation (101), we get:

$$\begin{aligned}
R_{ij} &= R^{(2)}_{ij} - \frac{1}{2Ag}\,h_{ij}\partial_\alpha\left(Agg^{\alpha\beta}\partial_\beta g\right) - \Gamma^{\alpha}_{ki}\Gamma^{k}_{j\alpha} - \Gamma^{k}_{\alpha i}\Gamma^{\alpha}_{jk} \\
&= R^{(2)}_{ij} - \frac{1}{A}\,h_{ij}\,g_{,uv}.
\end{aligned} \tag{103}$$

Here $R^{(2)}_{ij}$ is the two-dimensional Ricci tensor calculated in the metric h_{ij}. Now we require R_{ij} to be a solution of the Einstein vacuum equation

$$R_{ij} = 0, \tag{104}$$

which amounts to (38).

Next we briefly show how (104) comes about in the calculation of Dray and 't Hooft. Their method was to start with the metric (99),

$$\mathrm{d}\tilde{s}^2 = 2A(u,v)\,\mathrm{d}u\mathrm{d}v + g(u,v)\,h_{ij}(x^i)\mathrm{d}x^i\mathrm{d}x^j. \tag{105}$$

This is taken to be a vacuum solution of the Einstein equation (*e.g.*, flat Minkowski space). The tilde is used to indicate that this is a *different* solution from the one we are interested in, since we look for solutions with a massless particle as a source. One then applies the shift to the u-coordinate, that is, one keeps (105) for $v<0$ but replaces u by $u+\theta u$ for $v>0$ (after the particle has fell in). In this way one obtains a different metric,

$$\mathrm{d}s^2 = 2A(\hat{u},\hat{v})\,\mathrm{d}\hat{v}(\mathrm{d}\hat{u} - \delta(v)\mathrm{d}\hat{v}) + g(\hat{u},\hat{v})\,h_{ij}(\hat{x}^i)\mathrm{d}\hat{x}^i\mathrm{d}\hat{x}^j. \tag{106}$$

On top applying the shift, we have made the coordinate transformation

$$\begin{aligned}\hat{u} &= u+\theta f\\ \hat{v} &= v\\ \hat{x}^i &= x^i.\end{aligned} \tag{107}$$

In terms of these coordinates, all the curvature is concentrated at $\hat{v}=0$. Outside the plane of the shockwave, we have flat space.

This metric should be a solution containing a photon located at $\hat{v}=0$, $\hat{x}^i=0$. Therefore we insert (107) into Einstein's equation, with a source

$$T^{\hat{u}\hat{u}} = 4p\,\delta(\hat{v})\,\delta(\tilde{x}). \tag{108}$$

One then gets

$$R_{\hat{i}\hat{j}} = R^{(2)}_{ij} - \frac{1}{A}\,h_{ij}\left[\hat{g}_{,\hat{u}\hat{v}} + \hat{g}_{,\hat{u}\hat{u}}\,f(\hat{x}^i)\,\delta(\hat{v})\right], \tag{109}$$

and this must be zero, since there is no momentum in the transverse direction. Furthermore, since the coordinates (u,v,x,y) satisfied $R_{\mu\nu}=0$, in particular they must satisfy $R_{ij}=0$, and using (104), we see that the first to terms of (109) amount to this tensor. Therefore they can be dropped. We are left with

$$R_{\hat{i}\hat{j}} = -h_{ij}\frac{\hat{g}_{,\hat{u}\hat{u}}}{\hat{A}}f(x^i)\,\delta(\hat{v}) = 0. \tag{110}$$

This, together with the contributions from the other components of the Ricci-tensor, gives us the conditions (28)[14].

Appendix C: Commutator algebra

In this appendix we describe the problem mentioned in section 5. Consider the operator identification (41), needed for consistent quantisation. Commuting this equation with $X^\nu(\tilde{\sigma}')$, and applying (48), gives

$$[\triangle X^\mu(\tilde{\sigma}), X^\nu(\tilde{\sigma}')] = -\tfrac{i}{2T}\,\epsilon^{\mu\nu\alpha\beta}\,W_{\alpha\beta}\,\delta(\tilde{\sigma}-\tilde{\sigma}'). \tag{111}$$

This algebra is local, but one can easily see that it cannot be integrated to give us the commutator $[X^\mu, X^\nu]$ if $W^{\mu\nu}$ explicitly depends on $\tilde{\sigma}$. Naively, one would integrate (111) to give

$$[X^\mu(\tilde{\sigma}), X^\nu(\tilde{\sigma}')] = -\tfrac{i}{2T}\,F^{\mu\nu}(\tilde{\sigma}')\,f(\tilde{\sigma}-\tilde{\sigma}'), \tag{112}$$

where $F^{\mu\nu}(\tilde{\sigma}) = \epsilon^{\mu\nu\alpha\beta}\,W_{\alpha\beta}(\tilde{\sigma})$, but evidently, apart from being again non-local, when $F^{\mu\nu}$ is really $\tilde{\sigma}'$-dependent, this cannot be right, since the right-hand side has to be symmetric with respect to $\tilde{\sigma},\tilde{\sigma}'$. Indeed, starting from the equation of motion for $\triangle X^\nu(\tilde{\sigma}')$ and commuting this with $X^\mu(\tilde{\sigma})$ would give us $F^{\mu\nu}(\tilde{\sigma})$ instead of $F^{\mu\nu}(\tilde{\sigma}')$ on the right-hand side of (112). One can think of summing both expressions, or to look for more a general algebra like

$$[X^\mu(\tilde{\sigma}), X^\nu(\tilde{\sigma}')] = -\tfrac{i}{2T}\int \mathrm{d}^2\tilde{\sigma}''\,F^{\mu\nu}(\tilde{\sigma}'')\,g(\tilde{\sigma},\tilde{\sigma}';\tilde{\sigma}''), \tag{113}$$

[14] In that equation we have dropped the carets, since we work only in the shifted coordinate system.

where g is a Green function whose properties can be derived by applying the Laplacians $\triangle$ and $\triangle'$ to (113). However, from Fourier analysis one can easily see that such a function does not exist.

't Hooft has proposed another expression for the commutator [2], but one can easily check that —again— this only works for $F^{\mu\nu}$ not depending on $\tilde{\sigma}$.

So we were not able to get a consistent algebra for the fields X^μ in the general case, although there may be more general Ansatze than (113). Therefore, for the moment we will work with the non-local candidate (112), which *is* valid if X^μ is the operator for the "hard" particles, and X^ν that for the soft ones. In this case, the commutator does not need to be symmetric with respect to $\tilde{\sigma}$ and $\tilde{\sigma}'$, since the $W^{\mu\nu}$-background then refers to the the "soft" particle, and the whole picture is consistent. Here we will assume that we can treat both effects (the back-reaction of the in- and of the out-going particles) independently.

One can easily see that the effect of the nonlinearities on the right-hand side of (112) is negligible to first order in ε. Notice that the right-hand side of equation (112) is already of order ε. Furthermore, from (41) we learn that, to zeroth order in ε, $\partial_i X^\mu$ only depends on $\tilde{\sigma}$ through the Virasoro expansion. So we are allowed to use (112) as long as we put the free oscillations of X^μ equal to zero. We are then left with

$$X^\mu(\tilde{\sigma}) = X_0^\mu + p_i^\mu \sigma^i + \delta X^\mu, \tag{114}$$

where p_i is some integration constant. In that case, $\partial_i X^\mu = p_i^\mu + \mathcal{O}(\varepsilon)$ and, to first order in ε, (112) is indeed valid. The $\mathcal{O}(\varepsilon)$ terms of (114) will only have an effect when we investigate the Jacobi identity.

As remarked in [6], when taking into account the operator nature of $F^{\mu\nu}$, the commutator (112) together with (48) does not satisfy the Jacobi identity:

$$[[\triangle X^\mu(\tilde{\sigma}), X^\nu(\tilde{\sigma}')], P^\alpha(\tilde{\sigma}'')] + \text{cyclic} = 0, \tag{115}$$

but it is violated by a term proportional to G. This suggests that a linear term in the gravitational constant should be added to (48). The origin of this term is the fact that, due to the interactions, the momentum of the out-coming particles changes after they cross the shockwave, as the coordinate X^μ changes as well. This is explained in the section 5.

After some algebra, one can calculate that, in order to satisfy the Jacobi identity, (48) has to be modified in the following way:

$$[X^\mu(\tilde{\sigma}), P^\nu(\tilde{\sigma}')] = i g^{\mu\nu}\, \delta(\tilde{\sigma} - \tilde{\sigma}') + i A^{\mu\nu}(\tilde{\sigma}, \tilde{\sigma}'), \tag{116}$$

with

$$A^{\mu\nu}(\tilde{\sigma}, \tilde{\sigma}') \stackrel{?}{=} \frac{1}{T\sqrt{h}}\, \epsilon^{\mu\nu\alpha\beta} \epsilon^{ij} \partial_i X_\alpha \partial_j \delta(\tilde{\sigma} - \tilde{\sigma}') \int \mathrm{d}^2\tilde{\sigma}''\, P_\beta(\tilde{\sigma}'')\, f(\tilde{\sigma} - \tilde{\sigma}''). \tag{117}$$

Although, for notational simplicity, we have not written down the labels in or out, it is important to keep track of which are the operators corresponding to the in or the out-coming particles. $A^{\mu\nu}$ can be written as

$$\begin{aligned} A^{\mu\nu}(\tilde{\sigma}, \tilde{\sigma}') \stackrel{?}{=} &- \frac{1}{T\sqrt{h}}\, \epsilon^{\mu\nu\alpha\beta} \epsilon^{ij} \partial_i X_\alpha \Big(\delta(\tilde{\sigma} - \tilde{\sigma}') \imath nt \mathrm{d}^2\tilde{\sigma}''\, P_\beta(\tilde{\sigma}'')\, \partial_j f(\tilde{\sigma} - \tilde{\sigma}'') \\ &- \partial_j \delta(\tilde{\sigma} - \tilde{\sigma}') \int \mathrm{d}^2\tilde{\sigma}''\, P_\beta(\tilde{\sigma}'')\, f(\tilde{\sigma}' - \tilde{\sigma}'') \Big). \end{aligned} \tag{118}$$

This expression cannot however be correct. As we argue at the end of section 5 and are able to check in section 6, only the first term in this expression can contribute. Indeed, this term, combined with the equations of motion, does give a local algebra; furthermore, it provides the correct momentum transfer and agrees with the 2+1-dimensional result when we compactify one dimension. The reason that we also get the second term is the awkward $\tilde{\sigma}$-dependence in a nonlocal way, which should be removed by defining different variables. Indeed, equation (115) is very badly defined, since it depends on how we let the Laplacian work on the commutators. So from this one can only get the general shape of the expression (117), but not the precise $\tilde{\sigma}, \tilde{\sigma}'$-dependence. The latter we fix by the two arguments already mentioned (correctness of the momentum transfer and comparison with the 2+1-dimensional case, where the problem is overcome by defining new variables). Therefore, we take the first term as an Ansatz, see (51). In spite of the fact that it contains a δ-function, the algebra (50) is still nonlocal because of the $\tilde{\sigma}$-dependence of f. Nevertheless, after substitution of the equations of motion it will give us a local expression.

We now see two possibilities to solve the problem about how to get the commutator (112). The most promising one is to look for new variables which do not depend on $\tilde{\sigma}$ but only on the boundary of some region of the horizon. On the other hand, the fact that the covariant approach advocated here does not seem to work in general may be a pointer that one has to go back to the original philosophy where one keeps track of what in-going and out-coming particles are, and of which are "hard" particles and which are "test" ones, by writing $X^\mu(\tilde{\sigma}) = X^\mu_{\rm in}(\tilde{\sigma}) + X^\mu_{\rm out}(\tilde{\sigma})$, as we have implicitly assumed here, and include theta-functions depending on the parametrisation of the geodesic, $\theta(\lambda_{\rm in} - \lambda_{\rm out})$ and $\theta(\lambda_{\rm out} - \lambda_{\rm in})$. This would break the symmetry of (113), so that the conditions on the algebra would be less stringent.

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