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Article (Accepted version) (Refereed)

Valuation Compressions in VCG-Based Combinatorial Auctions*

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April 18, 2018

Abstract

The focus of classic mechanism design has been on truthful direct-revelation mechanisms. In the context of combinatorial auctions the truthful direct-revelation mechanism that maximizes social welfare is the Vickrey-Clarke-Groves mechanism. For many valuation spaces computing the allocation and payments of the VCG mechanism, however, is a computationally hard problem. We thus study the performance of the VCG mechanism when bidders are forced to choose bids from a subspace of the valuation space for which the VCG outcome can be computed efficiently. We prove improved upper bounds on the welfare loss for restrictions to additive bids and upper and lower bounds for restrictions to non-additive bids. These bounds show that increased expressiveness can give rise to additional equilibria of poorer efficiency.

1 Introduction

The goal of mechanism design is to devise mechanisms consisting of an allocation rule and a payment rule that implement desirable outcomes in strategic equilibrium. A fundamental result in mechanism design theory, the so-called *revelation principle*, asserts that any equilibrium outcome of any mechanism can be obtained as a truthful equilibrium of a direct-revelation mechanism. However, as has been pointed out in prior work [9], the revelation principle says nothing about the computational complexity of such a truthful direct-revelation mechanism.

In the context of combinatorial auctions the truthful direct-revelation mechanism that maximizes social welfare is the *Vickrey-Clarke-Groves (VCG) mechanism* [37, 8, 20]. Unfortunately, for many valuation spaces computing the VCG allocation and payments is a computationally hard problem. This is, for example, the case for subadditive, fractionally subadditive, and submodular valuations [24]. We thus study the performance of the VCG mechanism in settings in which the bidders are forced to use bids from a subspace of the valuation space for which the allocation and payments can be computed efficiently. This is obviously the case for additive bids, where the VCG-based mechanism can be interpreted as a separate second-price auction for each item. Another class of bids for which this is the case is the class of OXS bids, which stands for ORs of XORs of singletons and includes additive bids, and the even more general bidding space GS, which stands for gross substitutes. For OXS bids polynomial-time algorithms for finding a maximum weight

*A preliminary version of these results appeared in the *Proceedings of the 9th International Conference on Web and Internet Economics*, Cambridge, MA, USA, 2013.

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matching in a bipartite graph such as the algorithms of Tarjan [36] and Fredman and Tarjan [19] can be used. For GS bids there is a fully polynomial-time approximation scheme due to Kelso and Crawford [23] and polynomial-time algorithms based on linear programming [12] and convolutions of $M^{\#}$-concave functions [29, 28, 30].

One consequence of restrictions of this kind, that we refer to as *valuation compressions*, is that there is typically no longer a truthful dominant-strategy equilibrium that maximizes social welfare. We therefore analyze the *Price of Anarchy*, i.e., the ratio between the optimal social welfare and the worst possible social welfare at equilibrium. Note that even with fully expressive bids the Price of Anarchy of the VCG mechanism is not necessarily equal to one. This is because the VCG mechanism may admit other, inefficient equilibria and the Price of Anarchy metric does not restrict to dominant-strategy equilibria if they exist. We focus on equilibrium concepts such as correlated equilibria and coarse correlated equilibria, which naturally emerge from learning processes in which the bidders minimize internal or external regret [18, 21, 25, 5].

Our Contribution We study valuation compressions in the VCG mechanism and the resulting Price of Anarchy for a broad range of (complement-free) valuations and restrictions to bids with different degrees of expressiveness. We consider pairs of valuations and bids from all levels of the hierarchy that ranges from subadditive over fractionally subadditive, submodular, GS, and OXS to additive valuations.

We start our analysis by considering fractionally subadditive (or less general) valuations. For this class of valuations it is known that the restriction to additive bids leads to a Price of Anarchy with respect to coarse correlated and Bayes-Nash equilibria of exactly 2 [6, 3]. We show an upper bound of 2 for these equilibrium concepts that applies to non-additive bids. Our proof, just as the previous results for additive bids, goes through a proof technique called weak smoothness [35]. Establishing weak smoothness, however, requires novel techniques as non-additive bids lead to non-additive payments for which most of the techniques developed in prior work do not apply.

We also provide a matching lower bound of 2 by showing that for fractionally subadditive valuations the VCG mechanism satisfies a property known as *outcome closure* [27]. This property guarantees that the set of pure Nash equilibria for a less general bid space is contained in the set of pure Nash equilibria for a more general bid space. As a consequence the same construction that leads to a pure Nash equilibrium that obtains only one half of the achievable social welfare for additive bids due to Christodoulou et al. [6] also constitutes a pure Nash equilibrium with non-additive bids.

We then turn to subadditive valuations. Prior work has shown that for additive bids the Price of Anarchy is 2 for pure Nash equilibria and at most $O(\log(m))$, where m is the number of items, for coarse correlated equilibria [3]. The best known bound for additive bids and Bayes-Nash equilibria is 4 [17]. The $O(\log(m))$-bound of Bhawalkar and Roughgarden [3] goes through weak smoothness and therefore applies to coarse correlated and Bayes-Nash equilibria. It can be extended to non-additive bids using our techniques developed for this setting, implying an upper bound of $O(\log(m))$ for these equilibrium concepts. The improved bound of 4 of Feldman et al. [17] does not go through any of the existing smoothness techniques. To capture this type of argument we introduce a novel smoothness notion, that we refer to as *relaxed smoothness*, and show that it also implies bounds for coarse correlated equilibria. We thus obtain an upper bound of 4 on the Price of Anarchy with respect to coarse correlated equilibria and restrictions from subadditive valuations to additive bids.

We complement our upper bounds on the Price of Anarchy with respect to coarse correlated and Bayes-Nash equilibria with a lower bound of 2.4 that applies to pure Nash equilibria and restrictions to OXS bids. Together with the upper bound of 2 on the Price of Anarchy with respect to pure

		valuations			
		gross substitutes	submodular	fractionally sub.	subadditive
bids	additive		[2,2]		[2,**4**]
	OXS				
	gross substitutes	X		[**2**,**2**]	
	submodular	X	X		[**2.4**,**O(log(m))**]
	fractionally sub.	X	X	X	

Table 1: Summary of our results (bold) and the related work (regular) for coarse correlated equilibria and minimization of external regret through repeated play. The range indicates upper and lower bounds on the Price of Anarchy.

Nash equilibria and restrictions to additive bids of Bhawalkar and Roughgarden [3] this shows a strict increase in the Price of Anarchy as we transition from additive bids to the next larger class OXS.

Finally, as a step towards understanding the complexity of finding an equilibrium of the VCG mechanism with restricted bids, we show that for subadditive valuations and additive bids deciding whether there exists a pure Nash equilibrium is $\mathcal{NP}$-hard.

Our analysis leaves a number of interesting open questions, both regarding the computation of equilibria and regarding improved upper and lower bounds. Some of these questions, in particular those regarding the computational complexity of equilibria have been addressed in follow-up work.

Related Work The Price of Anarchy in combinatorial auctions in which agents are forced to use additive bids has been studied in [6, 3, 17] for simultaneous second-price auctions and in [22, 17, 7] for simultaneous first-price auctions. The case where all items are identical, but additional items contribute less to the valuation and agents are required to place additive bids has been analyzed in [26, 11].

Our work differs from these works in that it also considers non-additive bids, which allows to quantify the impact of expressiveness. A similar question was in parallel and independently of us studied by Babaioff et al. [1]. They show that some of our findings actually apply to the broader class of mechanisms that maximize declared welfare, i.e., the sum of the winning agents' bids. This includes the VCG-based mechanisms studied here but also other mechanisms such as the Walrasian mechanism.

On a technical level what ties most of these results together is that they use the smoothness framework to prove Price of Anarchy guarantees. Smooth games were originally defined and analyzed in [32, 33]. This work has been extended to mechanisms in [35]. Our work adds to this line of work by providing an alternative smoothness notion, which gives more flexibility in choosing the deviating bids and thus more power to prove stronger bounds. Also related is more recent work by Dütting and Kesselheim [15], which provides a purely combinatorial characterization of algorithms that lead to mechanisms with a small Price of Anarchy. Our key technical lemmas are closely related to the notion of permeability defined in this paper.

The equilibrium computation question has previously been addressed in [6], who gave a polynomial-time algorithm for computing a pure Nash equilibrium for restrictions from submodular valuations to additive bids and an exponential-time procedure for finding a pure Nash equilibrium for restrictions from fractionally subadditive valuations to additive bids. Follow-up work showed hardness results for pure Nash equilibria [13], for Bayes-Nash equilibria [4], and for no-regret learning [10]. Motivated by this, Daskalakis and Syrgkanis [10] considered an alternative equilibrium concept for which polynomial-time learning algorithms exist, while Dütting and Kesselheim [16] proved welfare guarantees for game-playing dynamics that apply out of equilibrium.

An alternate (non-constructive) way of proving lower bounds on the Price of Anarchy was recently presented in [34]. This work, amongst others, shows that simultaneous first-price auctions achieve the best possible Price of Anarchy guarantee for subadditive valuations among all "simple" mechanisms.

The impact of more or less expressiveness in mechanisms was also studied in [2] and [27, 14]. The former points out that in a combinatorial auction setting the *best* welfare that can be achieved strictly increases with a suitably defined notion of expressiveness. The latter shows how in sponsored search auctions restrictions of the bid space to a subspace of the valuation space can improve the set of equilibria as a whole by eliminating low revenue equilibria. Our work complements this line of work by showing that in combinatorial auctions more expressiveness can give rise to additional equilibria of poorer efficiency.

2 Preliminaries

Combinatorial Auctions In a *combinatorial auction* there is a set N of n *agents* and a set M of m *items*. Each agent $i \in N$ employs preferences over bundles of items, represented by a *valuation function* $v_i : 2^M \to \mathbb{R}_{\geq 0}$. We use V_i for the *class of valuation functions* of agent i, and $V = \prod_{i \in N} V_i$ for the class of valuation profiles of all agents $i \in N$. We write $v = (v_i, v_{-i}) \in V$, where v_i denotes agent i's valuation and v_{-i} denotes the valuations of all agents other than i. We assume that the valuation functions are *normalized* and *monotone*, i.e., $v_i(\emptyset) = 0$ and $v_i(S) \leq v_i(T)$ for all $S \subseteq T$.

A mechanism $\mathcal{M} = (f, p)$ collects bids and computes an allocation and payments. A bid b_i by agent i is a bidding function $b_i : 2^M \to \mathbb{R}_{\geq 0}$. We use B_i to denote the *class of bidding functions* of agent i and $B = \prod_{i \in N} B_i$ for the class of possible bid profiles. We write $b = (b_i, b_{-i}) \in B$, where b_i denotes agent i's bid and b_{-i} denotes the bids of all agents but i. The mechanism chooses the allocation according to *allocation rule* $f : B \to \mathcal{P}(M)$, where $\mathcal{P}(M)$ denotes all possible partitions X of the set of items M into n sets $X_1, \ldots, X_n$, and payments using *payment rule* $p : B \to \mathbb{R}^n_{\geq 0}$.

We define the *social welfare* of an *allocation* X as the sum $\mathrm{SW}(X) = \sum_{i \in N} v_i(X_i)$ of the agents' valuations and use $\mathrm{OPT}(v)$ to denote the maximal achievable social welfare. An allocation rule f *maximizes declared welfare* if for all bids b it chooses the allocation $f(b)$ that maximizes the sum of the agents' bids, i.e., $\sum_{i \in N} b_i(f_i(b)) = DW_b(M) := \max_{X \in \mathcal{P}(M)} \sum_{i \in N} b_i(X_i)$. We assume *quasi-linear preferences*, i.e., agent i's *utility* under mechanism $\mathcal{M}$ given valuations v and bids b is $u_i(b, v_i) = v_i(f_i(b)) - p_i(b)$.

We focus on the *Vickrey-Clarke-Groves (VCG)* mechanism of [37, 8, 20]. Define $DW_{b_{-i}}(S) = \max_{X \in \mathcal{P}(S)} \sum_{j \neq i} b_j(X_j)$ for all $S \subseteq M$. The VCG mechanisms starts from an allocation rule f that maximizes declared welfare and computes the payment of each agent i as $p_i(b) = DW_{b_{-i}}(M) - DW_{b_{-i}}(M \setminus f_i(b))$. As the payment $p_i(b)$ only depends on the bundle $f_i(b)$ allocated to agent i and the bids b_{-i} of the agents other than i, we also use $p_i(f_i(b), b_{-i})$ to denote agent i's payment.

If the bids are additive then the VCG prices are additive, i.e., for every agent i and every bundle $S \subseteq M$ we have $p_i(S, b_{-i}) = \sum_{j \in S} \max_{k \neq i} b_k(j)$. Furthermore, the set of items that an agent wins in the VCG mechanism are the items for which he has the highest bid, i.e., agent i wins item j against bids b_{-i} if $b_i(j) \geq \max_{k \neq i} b_k(j) = p_i(j)$ (ignoring ties). Many of the complications in this paper come from the fact that these two observations do *not* apply to non-additive bids.

Valuation Compressions Our main object of study in this paper are *valuation compressions*, i.e., restrictions of the bidding space B to a subspace of the valuation space V.[1] More specifically, we consider valuation spaces V and bid spaces B from the following hierarchy due to Lehmann et al. [24]:

$$\text{OS} \subset \text{OXS} \subset \text{GS} \subset \text{SM} \subset \text{XOS} \subset \text{CF}.$$

The class OS is the class of *additive* functions. The name comes from the fact that it can be expressed syntactically as ORs-of-XSs (see below). The acronym GS stands for *gross substitutes* and the acronym SM for *submodular*. The most general class in this hierarchy, CF, for *complement free*, is the class of *subadditive* functions.

The classes OXS and XOS are defined syntactically. To define them we need the class of singleton functions, which we denote by XS. Functions from this class assign the same value to all bundles that contain a specific item and assign a value of zero to all other bundles. We also need two operators OR and XOR. The OR ($\vee$) operator is defined as $(u \vee w)(S) = \max_{T \subseteq S}(u(T) + w(S \setminus T))$ and the XOR ($\otimes$) operator is defined as $(u \otimes w)(S) = \max(u(S), w(S))$. Note that the class of functions that can be expressed as OR of XS valuations is precisely the class OS of additive functions.

The class OXS is the class of functions that can be described as ORs of XORs of XS functions and the class XOS is the class of functions that can be described by XORs of ORs of XS functions. Note that the latter class is simply the maximum over a set of additive functions. So a valuation v_i from this class can be expressed by a set of additive functions $a_i^1, \dots, a_i^k$ such that $v_i(S) = \max_{\ell=1,\dots,k} \sum_{j \in S} a_i^\ell(j)$ for all $S \subseteq M$.

Another important class is the class β-XOS, where $\beta \geq 1$, of β-fractionally subadditive functions. A valuation v_i is β-*fractionally subadditive* if for every subset of items T there exists an additive valuation a_i such that (a) $\sum_{j \in T} a_i(j) \geq v_i(T)/\beta$ and (b) $\sum_{j \in S} a_i(j) \leq v_i(S)$ for all $S \subseteq T$. The special case $\beta = 1$ corresponds to the class XOS of *fractionally subadditive* valuations. It can be shown that the class CF is contained in $O(\log(m))$-XOS (see, e.g., Theorem 5.2 in [3]).

Solution Concepts We use game-theoretic reasoning to analyze how agents interact with the mechanism, a desirable criterion being stability according to some solution concept. In the *complete information* model the agents are assumed to know each others' valuations, and in the *incomplete information* model the agents' only know from which (possibly distinct) distributions the valuations of the other agents are drawn. In the remainder we focus on the complete information case. The definitions and our results for incomplete information are given in Appendix A.

The static game-theoretic solution concepts that we consider in the complete information setting are:

$$\text{DSE} \subset \text{PNE} \subset \text{MNE} \subset \text{CE} \subset \text{CCE},$$

where DSE stands for *dominant strategy equilibrium*, PNE stands for *pure Nash equilibrium*, MNE stands for *mixed Nash equilibrium*, CE stands for *correlated equilibrium*, and CCE stands for *coarse correlated equilibrium*.

In our analysis we only need the definitions of pure Nash and coarse correlated equilibria. Bids $b \in B$ constitute a *pure Nash equilibrium (PNE)* for valuations $v \in V$ if for every agent $i \in N$ and every bid $b_i' \in B_i$, $u_i(b_i, b_{-i}, v_i) \geq u_i(b_i', b_{-i}, v_i)$. A distribution $\mathcal{B}$ over bids $b \in B$ is a *coarse correlated equilibrium (CCE)* for valuations $v \in V$ if for every agent $i \in N$ and every pure deviation $b_i' \in B_i$, $\mathrm{E}_{b \sim \mathcal{B}}[u_i(b_i, b_{-i}, v_i)] \geq \mathrm{E}_{b \sim \mathcal{B}}[u_i(b_i', b_{-i}, v_i)]$.

[1] This definition is consistent with the notion of *simplification* in [27, 14]. It differs from the notion of mechanism expressiveness in [2], which is based on the shattering dimension of the underlying mechanism.

The dynamic solution concept that we consider in this setting is regret minimization. A sequence of bids $b^1, \ldots, b^T$ incurs *vanishing average external regret* if for all agents i, $\sum_{t=1}^{T} u_i(b_i^t, b_{-i}^t, v_i) \geq \max_{b_i'} \sum_{t=1}^{T} u_i(b_i', b_{-i}^t, v_i) - o(T)$ holds, where $o(\cdot)$ denotes the little-oh notation. The empirical distribution of bids in a sequence of bids that incurs vanishing external regret converges to a coarse correlated equilibrium (see, e.g., Chapter 4 of [31]).

Price of Anarchy We quantify the welfare loss from valuation compressions by means of the *Price of Anarchy (PoA)*.

The PoA with respect to PNE for valuations $v \in V$ is defined as the worst ratio between the optimal social welfare $\mathrm{OPT}(v)$ and the welfare $\mathrm{SW}(b)$ of a PNE $b \in B$,

$$\mathrm{PoA}(v) = \sup_{b\text{: PNE}} \frac{\mathrm{OPT}(v)}{\mathrm{SW}(b)} \ .$$

Similarly, the PoA with respect to MNE, CE, and CCE for valuations $v \in V$ is the worst ratio between the optimal social welfare $\mathrm{OPT}(v)$ and the expected welfare $\mathrm{E}_{b\sim\mathcal{B}}[\mathrm{SW}(b)]$ of a MNE, CE, or CCE $\mathcal{B}$,

$$\mathrm{PoA}(v) = \sup_{\mathcal{B}\text{: MNE, CE or CCE}} \frac{\mathrm{OPT}(v)}{\mathrm{E}_{b\sim\mathcal{B}}[\mathrm{SW}(b)]} \ .$$

It is not difficult to see that the Price of Anarchy can be arbitrarily bad even if there is only a single item for sale. As argued in the related literature, however, this requires agents to grossly overstate their values for certain bundles of items, which seems unnatural (see, e.g., [17]). We therefore impose the assumption that agents avoid such overbidding strategies by restricting the action space (and the set of possible deviations from an equilibrium bid profile) B_i of each agent i to bids b_i such that $b_i(S) \leq v_i(S)$ for all $S \subseteq M$.

3 Fractionally Subadditive Valuations

We begin our analysis with valuation compressions from β-fractionally subadditive valuations to less general bids. We show an upper bound on the Price of Anarchy with respect to coarse correlated equilibria and Bayes-Nash equilibria of 2β. We also show a lower bound of 2 on the Price of Anarchy with respect to pure Nash equilibria for valuation compressions from fractionally subadditive valuations to less general bids. We thus show that for fractionally subadditive valuations increased expressiveness neither improves nor deteriorates the Price of Anarchy.

3.1 Upper Bounds

We establish our upper bounds on the Price of Anarchy by showing that the VCG mechanism with restricted bids is weakly smooth. Weak smoothness is a parametrized property of mechanisms that requires that for every valuation profile and every bid profile there exists a "good" unilateral deviation for each agent. The deviations are only allowed to depend on the valuation profile, and they are considered to be good if in sum over all agents they ensure high enough utilities.

Definition 1 (Syrgkanis and Tardos [35]). A mechanism $\mathcal{M} = (f, p)$ is *weakly* (λ, μ_1, μ_2)*-smooth* for $\lambda, \mu_1, \mu_2 \geq 0$ if for every valuation profile $v \in V$ and bid profile $b \in B$ there exists a bid $a_i(v)$ for every agent $i \in N$ that does not require agent i to overbid such that

$$\sum_{i\in N} u_i((a_i, b_{-i}), v_i) \quad \geq \quad \lambda \cdot \mathrm{OPT}(v) \ - \ \mu_1 \cdot \sum_{i\in N} p_i(f_i(b), b_{-i}) \ - \ \mu_2 \cdot \sum_{i\in N} b_i(f_i(b)) \ .$$

Syrgkanis and Tardos [35] show that a weakly (λ, μ_1, μ_2)-smooth mechanism achieves a Price of Anarchy of at most $\frac{\max(\mu_1,1)+\mu_2}{\lambda}$ with respect to both coarse correlated and Bayes-Nash equilibria.

Theorem 1. *Consider running the VCG mechanism for β-fractionally subadditive valuations and fractionally subadditive bids. Then the VCG mechanism is weakly $(1/\beta, 1, 1)$-smooth.*

We will prove this result with the help of two lemmas. We consider these lemmas, which show how to deal with non-additive bids, as our main technical contribution.

Lemma 1. *Consider running the VCG mechanism for β-fractionally subadditive valuations and fractionally subadditive bids. Then for all valuations $v \in V$, every agent $i \in N$, and every bundle of items $Q_i \subseteq M$ there exists an additive bid $a_i \in B_i$ that only depends on Q_i and v_i and does not require agent i to overbid such that for all bids $b_{-i} \in B_{-i}$,*

$$u_i(a_i, b_{-i}, v_i) \geq \frac{v_i(Q_i)}{\beta} - p_i(Q_i, b_{-i}) \ .$$

Proof. Fix valuations v, agent i, and bundle Q_i. As $v_i \in \beta$-XOS there exists an additive bid $a_i \in \mathrm{OS}$ for which $\sum_{j \in X_i} a_i(j) \leq v_i(X_i)$ for all $X_i \subseteq Q_i$, and $\sum_{j \in Q_i} a_i(j) \geq \frac{v_i(Q_i)}{\beta}$. Consider bids b_{-i}. Recall our notation for the maximum declared welfare that is achievable by distributing items $S \subseteq M$ among the agents $j \neq i$, which we defined to be $DW_{b_{-i}}(S) = \max_{X \in \mathcal{P}(S)} \sum_{j \neq i} b_j(X_j)$. As the VCG mechanism selects the outcome that maximizes the sum of the bids,

$$a_i(f_i(a_i, b_{-i})) + DW_{b_{-i}}(M \setminus f_i(a_i, b_{-i})) \geq a_i(Q_i) + DW_{b_{-i}}(M \setminus Q_i) \ .$$

We have chosen a_i such that $a_i(f_i(a_i, b_{-i})) \leq v_i(f_i(a_i, b_{-i}))$ and $a_i(Q_i) \geq v_i(Q_i)/\beta$. Thus,

$$\begin{aligned} v_i(f_i(a_i, b_{-i})) + DW_{b_{-i}}(M \setminus f_i(a_i, b_{-i})) &\geq a_i(f_i(a_i, b_{-i})) + DW_{b_{-i}}(M \setminus f_i(a_i, b_{-i})) \\ &\geq a_i(Q_i) + DW_{b_{-i}}(M \setminus Q_i) \\ &\geq \frac{v_i(Q_i)}{\beta} + DW_{b_{-i}}(M \setminus Q_i) \ . \end{aligned}$$

Subtracting $DW_{b_{-i}}(M)$ from both sides gives

$$v_i(f_i(a_i, b_{-i})) - p_i(f_i(a_i, b_{-i}), b_{-i}) \geq \frac{v_i(Q_i)}{\beta} - p_i(Q_i, b_{-i}) \ .$$

As $u_i((a_i, b_{-i}), v_i) = v_i(f_i(a_i, b_{-i})) - p_i(f_i(a_i, b_{-i}), b_{-i})$ this shows that $u_i((a_i, b_{-i}), v_i) \geq v_i(Q_i)/\beta - p_i(Q_i, b_{-i})$ as claimed. □

Lemma 2. *Consider running the VCG mechanism for β-fractionally subadditive valuations and fractionally subadditive bids. For every allocation $Q_1, \dots, Q_n$ and all bids $b \in B$,*

$$\sum_{i=1}^{n} \Big[p_i(Q_i, b_{-i}) - p_i(f_i(b), b_{-i}) \Big] \leq \sum_{i=1}^{n} b_i(f_i(b)) \ .$$

Proof. We have $p_i(Q_i, b_{-i}) = DW_{b_{-i}}(M) - DW_{b_{-i}}(M \setminus Q_i)$ and $p_i(f_i(b), b_{-i}) = DW_{b_{-i}}(M) - DW_{b_{-i}}(M \setminus f_i(b))$ because the VCG mechanism is used. Thus,

$$\sum_{i=1}^{n} \Big[p_i(Q_i, b_{-i}) - p_i(f_i(b), b_{-i}) \Big] = \sum_{i=1}^{n} \Big[DW_{b_{-i}}(M \setminus f_i(b)) - DW_{b_{-i}}(M \setminus Q_i) \Big] \ . \tag{1}$$

We have $DW_{b_{-i}}(M \setminus f_i(b)) = \sum_{k\neq i} b_k(f_k(b))$ and $DW_{b_{-i}}(M \setminus Q_i) \geq \sum_{k\neq i} b_k(f_k(b) \cap (M \setminus Q_i))$ because $(f_k(b) \cap (M \setminus Q_i))_{i\neq k}$ is a feasible allocation of the items $M \setminus Q_i$ among the agents $-i$. Thus,

$$\begin{aligned}
&\sum_{i=1}^{n} \Big[DW_{b_{-i}}(M \setminus f_i(b)) - DW_{b_{-i}}(M \setminus Q_i)\Big] \\
&\qquad \leq \sum_{i=1}^{n} \Big[\sum_{k\neq i} b_k(f_k(b)) - \sum_{k\neq i} b_k(f_k(b) \cap (M \setminus Q_i))\Big] \\
&\qquad \leq \sum_{i=1}^{n} \Big[\sum_{k=1}^{n} b_k(f_k(b)) - \sum_{k=1}^{n} b_k(f_k(b) \cap (M \setminus Q_i))\Big] \\
&\qquad = \sum_{i=1}^{n}\sum_{k=1}^{n} b_k(f_k(b)) - \sum_{i=1}^{n}\sum_{k=1}^{n} b_k(f_k(b) \cap (M \setminus Q_i)) \ . \qquad (2)
\end{aligned}$$

The second inequality holds due to the monotonicity of the bids. Since XOS = 1-XOS for every agent k, bid $b_k \in$ XOS, and set $f_k(b)$ there exists a bid $a_{k,f_k(b)} \in$ OS such that

$$b_k(f_k(b)) = a_{k,f_k(b)}(f_k(b)) = \sum_{j\in f_k(b)} a_{k,f_k(b)}(j) \ , \qquad \text{and}$$

$$b_k(f_k(b) \cap (M \setminus Q_i)) \geq a_{k,f_k(b)}(f_k(b) \cap (M \setminus Q_i)) = \sum_{j\in f_k(b)\cap(M\setminus Q_i)} a_{k,f_k(b)}(j) \ .$$

As $Q_1, \ldots, Q_n$ is a partition of M every item is contained in exactly one of the sets $Q_1, \ldots, Q_n$ and hence in $n-1$ of the sets $M \setminus Q_1, \ldots, M \setminus Q_n$. By the same argument for every agent k and set $f_k(b)$ every item $j \in f_k(b)$ is contained in exactly $n-1$ of the sets $f_k(b) \cap (M \setminus Q_1), \ldots, f_k(b) \cap (M \setminus Q_n)$. Thus, for every fixed k we have that $\sum_{i=1}^{n} b_k(f_k(b) \cap (M \setminus Q_i)) \geq (n-1) \cdot \sum_{j\in f_k(b)} a_{k,f_k(b)}(j) = (n-1) \cdot a_{k,f_k(b)}(f_k(b)) = (n-1) \cdot b_k(f_k(b))$. It follows that

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{k=1}^{n} b_k(f_k(b)) - \sum_{i=1}^{n}\sum_{k=1}^{n} b_k(f_k(b) \cap (M \setminus Q_i)) \\
&\qquad \leq n \cdot \sum_{k=1}^{n} b_k(f_k(b)) - (n-1) \cdot \sum_{k=1}^{n} b_k(f_k(b)) = \sum_{i=1}^{n} b_k(f_k(b)) \ . \qquad (3)
\end{aligned}$$

The claim follows by combining inequalities (1), (2), and (3). □

Proof of Theorem 1. Applying Lemma 1 to the optimal bundles $O_1, \ldots, O_n$ and summing over all agents i,

$$\sum_{i\in N} u_i(a_i, b_{-i}, v) \geq \frac{1}{\beta} \cdot \mathrm{OPT}(v) - \sum_{i\in N} p_i(O_i, b_{-i}) \ .$$

Applying Lemma 2 we obtain

$$\sum_{i\in N} u_i(a_i, b_{-i}, v) \geq \frac{1}{\beta} \cdot \mathrm{OPT}(v) - \sum_{i\in N} p_i(f_i(b), b_{-i}) - \sum_{i\in N} b_i(X_i(b)) \ .$$ □

Remark Lemma 2 and hence Theorem 1 apply whenever bids are at least additive and at most fractionally subadditive. So, for instance, these results apply with OXS bids when agents are restricted to use at most $k \geq 1$ XORs.

3.2 Lower Bounds

In order to show our lower bound on the Price of Anarchy with respect to pure Nash equilibria, we will show that the exact same construction that is used in [6] to show a lower bound of 2 for restrictions to additive bids also constitutes a pure Nash equilibrium when agents are allowed to use more general bids. We do so by showing that for fractionally subadditive valuations the VCG mechanism satisfies a property of mechanisms known as outcome closure.

Definition 2 (Milgrom [27])**.** A mechanism satisfies *outcome closure* for a given class V of valuation functions and a restriction of the class B of bidding functions to a subclass B' of bidding functions if for every $v \in V$, every agent $i \in N$, all bids $b'_{-i} \in B'_{-i}$, and every bid $b_i \in B_i$ there exists a bid $b'_i \in B'_i$ for which $u_i(b'_i, b'_{-i}, v_i) \geq u_i(b_i, b'_{-i}, v_i)$.

Milgrom [27] shows that if a mechanism satisfies outcome closure, then every pure Nash equilibrium under B' is also a pure Nash equilibrium under B.

Theorem 2. *Consider running the VCG mechanism for fractionally subadditive valuations and a set of allowable bids that is contained in the class of fractionally subadditive functions and includes all additive functions. Then the Price of Anarchy with respect to pure Nash equilibria is at least* 2*.*

Proof. It suffices to show that the VCG mechanism satisfies outcome closure for V and the restriction of B to B', where B and B' can be any of the mentioned classes of bidding functions. To prove this fix valuations $v \in V$, bids $b'_{-i} \in B'_{-i}$, and consider an arbitrary bid $b_i \in B_i$ by agent i. Denote the bundle that agent i gets under (b_i, b'_{-i}) by Q_i and denote his payment by $p_i = p_i(Q_i, b'_{-i})$. By Lemma 1 there exists a bid $b'_i \in B'_i$ that does not overbid and that satisfies $u_i(b'_i, b'_{-i}, v_i) \geq v_i(Q_i) - p_i(Q_i, b'_{-i}) = u_i(b_i, b'_{-i}, v_i)$. This proves that outcome closure is satisfied. □

4 Subadditive Valuations

We now turn to subadditive valuations. Our analysis from the previous section already shows that the Price of Anarchy with respect to both coarse correlated and Bayes-Nash equilibria is upper bounded by $O(\log(m))$. In this section we introduce a new smoothness notion that allows to improve the guarantee for coarse correlated equilibria and additive bids to 4. We also present a lower bound for pure Nash equilibria and restrictions to OXS bids that shows that the Price of Anarchy with respect to pure Nash equilibria strictly increases as we go from additive to OXS bids.

4.1 Relaxed Smoothness

Recall that weak smoothness is a parametrized property of mechanisms that requires that for every valuation profile and every bid profile there exists a good deviation for each agent, where the deviation may only depend on the valuation profile. Our smoothness notion, that we refer to as relaxed smoothness, also allows the deviation of an agent to depend on the distribution of the other agents' bids.

Definition 3. A mechanism $\mathcal{M} = (f, p)$ is *relaxed* (λ, μ_1, μ_2)*-smooth* for $\lambda, \mu_1, \mu_2 \geq 0$ if for every valuation profile $v \in V$, every distribution over bids $\mathcal{B}$, and every agent i there exists a bid $a_i(v, \mathcal{B}_{-i})$ that does not require agent i to overbid such that

$$\sum_{i \in N} \operatorname*{E}_{b_{-i}} \left[u_i((a_i, b_{-i}), v_i)\right] \;\geq\; \lambda \cdot \mathrm{OPT}(v) \;-\; \mu_1 \cdot \sum_{i \in N} \operatorname*{E}_{b} \left[p_i(f_i(b), b_{-i})\right] \;-\; \mu_2 \cdot \sum_{i \in N} \operatorname*{E}_{b}[b_i(f_i(b))] \;.$$

Relaxed smoothness, just like weak smoothness, implies a bound on the Price of Anarchy for both coarse correlated and Bayes-Nash equilibria. We present the result for coarse correlated equilibria. The result for Bayes-Nash equilibria can be found in Appendix A.

Theorem 3. *If a mechanism $\mathcal{M} = (f, p)$ is relaxed (λ, μ_1, μ_2)-smooth, then the Price of Anarchy for coarse correlated equilibria is at most*

$$\frac{\max\{\mu_1, 1\} + \mu_2}{\lambda} \enspace .$$

Proof. Fix valuations v. Consider a coarse correlated equilibrium $\mathcal{B}$. For each b from the support of $\mathcal{B}$ denote the allocation for b by $f(b) = (f_1(b), \ldots, f_n(b))$. Let $a = (a_1, \ldots, a_n)$ be defined as in Definition 3. Then,

$$\begin{aligned}
\mathop{\mathrm{E}}_{b \sim \mathcal{B}}\left[\mathrm{SW}(f(b))\right] &= \sum_{i \in N} \mathop{\mathrm{E}}_{b \sim \mathcal{B}}\left[u_i(b, v_i)\right] + \sum_{i \in N} \mathop{\mathrm{E}}_{b \sim \mathcal{B}}\left[p_i(f_i(b), b_{-i})\right] \\
&\geq \sum_{i \in N} \mathop{\mathrm{E}}_{b_{-i} \sim \mathcal{B}_{-i}}\left[u_i((a_i, b_{-i}), v_i)\right] + \sum_{i \in N} \mathop{\mathrm{E}}_{b \sim \mathcal{B}}\left[p_i(f_i(b), b_{-i})\right] \\
&\geq \lambda \cdot \mathrm{OPT}(v) - (\mu_1 - 1) \cdot \sum_{i \in N} \mathop{\mathrm{E}}_{b \sim \mathcal{B}}\left[p_i(f_i(b), b_{-i})\right] - \mu_2 \cdot \sum_{i \in N} \mathop{\mathrm{E}}_{b \sim \mathcal{B}}\left[b_i(f_i(b))\right] \enspace ,
\end{aligned}$$

where the first equality uses the definition of $u_i(b, v_i)$ as the difference between $v_i(f_i(b))$ and $p_i(f_i(b), b_{-i})$, the first inequality uses the fact that $\mathcal{B}$ is a coarse correlated equilibrium, and the second inequality holds because $a = (a_1, \ldots, a_n)$ is defined as in Definition 3.

Since agents do not overbid this can be rearranged to give

$$(1 + \mu_2) \cdot \mathop{\mathrm{E}}_{b \sim \mathcal{B}}[\mathrm{SW}(f(b))] \geq \lambda \cdot \mathrm{OPT}(v) - (\mu_1 - 1) \cdot \sum_{i \in N} \mathop{\mathrm{E}}_{b \sim \mathcal{B}}[p_i(f_i(b), b_{-i})] \enspace .$$

For $\mu_1 \leq 1$ the second term on the right hand side is lower bounded by zero and the result follows by rearranging terms. For $\mu_1 > 1$ we use that $\mathrm{E}_{b \sim \mathcal{B}}[p_i(f_i(b), b_{-i})] \leq \mathrm{E}_{b \sim \mathcal{B}}[v_i(f_i(b))]$ to lower bound the second term on the right hand side and the result follows by rearranging terms. □

4.2 Upper Bound for Additive Bids

The advantage of relaxed smoothness over weak smoothness is that it gives us additional freedom in choosing deviations for each agent. We next show how the proof technique of Feldman et al. [17] can be used to show relaxed smoothness of the VCG mechanism for restrictions to additive bids.

Proposition 1. *Consider running the VCG mechanism for subadditive valuations and additive bids. Then the VCG mechanism is relaxed $(1/2, 0, 1)$-smooth.*

To prove this result we need two auxiliary lemmas.

Lemma 3. *Consider running the VCG mechanism for subadditive valuations and additive bids. Then for every agent $i \in N$, every bundle of items $Q_i \subseteq M$, and every distribution $\mathcal{B}_{-i}$ over the bids $b_{-i} \in B_{-i}$ of the agents other than i there exists an additive bid $a_i \in B_i$ that only depends on Q_i and $\mathcal{B}_{-i}$ and does not require agent i to overbid such that*

$$\mathop{\mathrm{E}}_{b_{-i} \sim \mathcal{B}_{-i}}[u_i((a_i, b_{-i}), v_i)] \geq \frac{1}{2} \cdot v_i(Q_i) - \mathop{\mathrm{E}}_{b_{-i} \sim \mathcal{B}_{-i}}[p_i(Q_i, b_{-i})] \enspace .$$

Proof. Let $\epsilon > 0$. Consider bids b_{-i} of the agents $-i$. The bids b_{-i} induce a price $p_i(j) = \max_{k \neq i} b_k(j)$ for each item j. Let T be a maximal subset of items from Q_i such that $v_i(T) < p_i(T) + |T| \cdot \epsilon$. Define the *truncated* prices q_i as follows:

$$q_i(j) = \begin{cases} p_i(j) & \text{for } j \in Q_i \setminus T, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The distribution $\mathcal{B}_{-i}$ on the bids b_{-i} induces a distribution $\mathcal{C}_i$ on the prices p_i as well as a distribution $\mathcal{D}_i$ on the truncated prices q_i.

We would like to consider a bid b_i by agent i that is drawn from the same distribution $\mathcal{D}_i$ as the truncated prices. To explicitly deal with ties, we will increase the bid on each item in $Q_i \setminus T$ by ϵ. So, to determine the bid b_i we first draw b'_i from $\mathcal{D}_i$ and then let $b_i(S) = b'_i(S) + |S \cap Q_i \setminus T| \cdot \epsilon$ for all S. We need to argue that the resulting bids are additive and that they do not entail overbidding. The first condition is satisfied because additive bids lead to additive prices, and so the truncated prices are additive. To see that the second condition is satisfied assume by contradiction that for some non-empty set $S \subseteq Q_i \setminus T$, $b_i(S) = q_i(S) + |S| \cdot \epsilon > v_i(S)$. As $p_i(S) = q_i(S)$ it follows that

$$v_i(S \cup T) \leq v_i(S) + v_i(T) < p_i(S) + |S| \cdot \epsilon + p_i(T) + |T| \cdot \epsilon = p_i(S \cup T) + |S \cup T| \cdot \epsilon \ ,$$

which contradicts our definition of the set T as a maximal subset of Q_i for which the inequality $v_i(T) \leq p_i(T) + |T| \cdot \epsilon$ holds.

Consider bid b_i. Let $f_i(b_i, p_i)$ be the set of items won with bid b_i against prices p_i. Let $g_i(b_i, q_i)$ be the subset of items from Q_i won with bid b_i against the truncated prices q_i. As $p_i(j) = q_i(j)$ for $j \in Q_i \setminus T$ and $p_i(j) \geq q_i(j)$ for $j \in T$ we have $g_i(b_i, q_i) \subseteq f_i(b_i, p_i) \cup T$. Thus, using the fact that v_i is subadditive, $v_i(g_i(b_i, q_i)) \leq v_i(f_i(b_i, p_i)) + v_i(T)$. By the definition of the prices p_i and the truncated prices q_i we have $p_i(Q_i) - q_i(Q_i) = p_i(T) > v_i(T) - |T| \cdot \epsilon \geq v_i(T) - |Q_i| \cdot \epsilon$. By combining these inequalities we obtain

$$v_i(f_i(b_i, p_i)) + p_i(Q_i) \geq v_i(g_i(b_i, q_i)) + q_i(Q_i) - |Q_i| \cdot \epsilon \ .$$

Taking expectations over the prices $p_i \sim \mathcal{C}_i$ and the truncated prices $q_i \sim \mathcal{D}_i$ gives

$$\mathop{\mathrm{E}}_{p_i \sim \mathcal{C}_i}\left[v_i(f_i(b_i, p_i)) + p_i(Q_i) \right] \geq \mathop{\mathrm{E}}_{q_i \sim \mathcal{D}_i}\left[v_i(g_i(b_i, q_i)) + q_i(Q_i) \right] - |Q_i| \cdot \epsilon \ .$$

Now recall the process by which we generate b_i. Let us make the dependence on b'_i visible by writing $b_i(b'_i)$. Next we take expectations over $b'_i \sim \mathcal{D}_i$ on both sides of the previous inequality. Then we bring the $p_i(Q_i)$ term to the right and the $q_i(Q_i)$ term to the left. Finally, we exploit that the expectation over $q_i \sim \mathcal{D}_i$ of $q_i(Q_i)$ is the same as the expectation over $b'_i \sim \mathcal{D}_i$ of $b'_i(Q_i)$ to obtain

$$\begin{aligned} &\mathop{\mathrm{E}}_{b'_i \sim \mathcal{D}_i}\left[\mathop{\mathrm{E}}_{p_i \sim \mathcal{C}_i}\left[v_i(f_i(b_i(b'_i), p_i)) \right] \right] - \mathop{\mathrm{E}}_{b'_i \sim \mathcal{D}_i}\left[b'_i(Q_i) \right] \\ &\qquad \geq \mathop{\mathrm{E}}_{b'_i \sim \mathcal{D}_i}\left[\mathop{\mathrm{E}}_{q_i \sim \mathcal{D}_i}\left[v_i(g_i(b_i(b'_i), q_i)) \right] \right] - \mathop{\mathrm{E}}_{p_i \sim \mathcal{C}_i}\left[p_i(Q_i) \right] - |Q_i| \cdot \epsilon \ . \end{aligned} \tag{4}$$

Now, using the fact that b'_i and q_i are drawn from the same distribution $\mathcal{D}_i$, we can lower bound the first term on the right-hand side of the preceding inequality by

$$\begin{aligned} &\mathop{\mathrm{E}}_{b'_i \sim \mathcal{D}_i}\left[\mathop{\mathrm{E}}_{q_i \sim \mathcal{D}_i}\left[v_i(g_i(b_i(b'_i), q_i)) \right] \right] \\ &\qquad = \frac{1}{2} \cdot \mathop{\mathrm{E}}_{b'_i \sim \mathcal{D}_i}\left[\mathop{\mathrm{E}}_{q_i \sim \mathcal{D}_i}\left[v_i(g_i(b_i(b'_i), q_i)) + v_i(g_i(b_i(q_i), b'_i)) \right] \right] \geq \frac{1}{2} \cdot v_i(Q_i) \ , \end{aligned} \tag{5}$$

where the inequality in the last step comes from the fact that the subset $g_i(b_i(b'_i), q_i)$ of Q_i won with bid $b_i(b'_i)$ against prices q_i and the subset $g_i(b_i(q_i), b'_i)$ of Q_i won with bid $b_i(q_i)$ against prices b'_i cover all of Q_i and, thus, because v_i is subadditive, it must be that $v_i(g_i(b_i(b'_i), q_i)) + v_i(g_i(b_i(q_i), b'_i)) \geq v_i(Q_i)$.

Note that agent i's utility for bid b_i against bids b_{-i} is given by his valuation for the set of items $f_i(b_i, p_i)$ minus the price $p_i(f_i(b_i, p_i))$. Note further that the price $p_i(f_i(b_i, p_i))$ that he faces is at most his bid $b_i(f_i(b_i, p_i))$. Finally note that his bid $b_i(f_i(b_i, p_i))$ is at most $b'_i(Q_i \setminus T) + |Q_i \setminus T| \cdot \epsilon \leq b'_i(Q_i) + |Q_i| \cdot \epsilon$ because of the way we generate b_i from b'_i and because b'_i is drawn from $\mathcal{D}_i$. Together with inequality (4) and inequality (5) this shows that

$$\begin{aligned}
&\mathop{\mathrm{E}}_{b'_i \sim \mathcal{D}_i}\left[\mathop{\mathrm{E}}_{b_{-i} \sim \mathcal{B}_{-i}}\left[u_i((b_i(b'_i), b_{-i}), v_i)\right]\right] \\
&\qquad\qquad \geq \mathop{\mathrm{E}}_{b'_i \sim \mathcal{D}_i}\left[\mathop{\mathrm{E}}_{p_i \sim \mathcal{C}_i}\left[v_i(f_i(b_i(b'_i), p_i))\right]\right] - \mathrm{E}\left[b'_i(Q_i)\right] - |Q_i| \cdot \epsilon \\
&\qquad\qquad \geq \frac{1}{2} \cdot v_i(Q_i) - \mathop{\mathrm{E}}_{p_i \sim \mathcal{C}_i}\left[p_i(Q_i)\right] - 2 \cdot |Q_i| \cdot \epsilon \ .
\end{aligned}$$

Since this inequality is satisfied in expectation if bid b_i is generated by drawing b'_i from distribution $\mathcal{D}_i$ there must be at least one a'_i from the support of $\mathcal{D}_i$ and hence a deterministic bid a_i that satisfies it. The claim follows by taking $\epsilon \to 0$. □

Lemma 4. *Consider running the VCG mechanism for subadditive valuations and additive bids. Then for every partition $Q_1, \dots, Q_n$ of the items and all bids b,*

$$\sum_{i \in N} p_i(Q_i, b_{-i}) \leq \sum_{i \in N} b_i(f_i(b)) \ .$$

Proof. For every agent i and each item $j \in Q_i$ we have $p_i(j, b_{-i}) = \max_{k \neq i} b_k(j) \leq \max_k b_k(j)$. Hence an upper bound on the sum $\sum_{i \in N} p_i(Q_i, b_{-i})$ is given by $\sum_{i \in N} \max_k b_k(j)$. The VCG mechanisms selects allocation $f_1(b), \dots, f_n(b)$ such that $\sum_{i \in N} b_i(f_i(b))$ is maximized. □

Proof of Proposition 1. The claim follows by applying Lemma 3 to every agent i and the corresponding optimal bundle O_i, summing over all agents i, and using Lemma 4 to bound $\mathrm{E}_{b_{-i} \sim \mathcal{B}_{-i}}[\sum_{i \in N} p_i(O_i, b_{-i})]$ by $\mathrm{E}_{b \sim \mathcal{B}}[\sum_{i \in N} b_i(f_i(b))]$. □

Remark In the proof of Lemma 3 it is important that *additive* bids b_{-i} by the agents other than i induce *additive* VCG payments $p_i(S, b_{-i}) = \sum_{j \in S} p_i(j) = \sum_{j \in S} \max_{k \neq i} b_k(j)$. The VCG payments induced by more general bids can typically *not* be expressed within the same bidding language.

4.3 Lower Bound for OXS Bids

We conclude this section by proving a lower bound on the Price of Anarchy with respect to pure Nash equilibria for restrictions to OXS bids, which is strictly larger than the corresponding upper bound for additive bids.

Theorem 4. *Consider running the VCG mechanism for subadditive valuations and a set of allowable bids that is contained in the class of fractionally subadditive functions and includes all OXS functions. Assume further that there are at least $n \geq 2$ agents and $m \geq 6$ items. Then for every $\delta > 0$ there exist valuations v such that the Price of Anarchy for pure Nash equilibria is at least $2.4 - \delta$.*

The proof of this theorem makes use of the following auxiliary lemma, which relates the maximum bid on any of the subsets of a set that are by one element smaller to the bid on the set itself.

Lemma 5. *For every fractionally subadditive bid $b_i \in XOS$ and every set of items $X \subseteq M$ it holds that*

$$\max_{S \subseteq X, |S|=|X|-1} b_i(S) \geq \frac{|X|-1}{|X|} \cdot b_i(X) \ .$$

Proof. As $b_i \in$ XOS there exists an additive bid a_i such that $\sum_{j \in X} a_i(j) = b_i(X)$ and for every $S \subseteq X$ we have $b_i(S) \geq \sum_{j \in S} a_i(j)$. There are $|X|$ many ways to choose $S \subseteq X$ such that $|S| = |X| - 1$ and these $|X|$ many sets will contain each of the items $j \in X$ exactly $|X| - 1$ times. Thus, $\sum_{S \subseteq X, |S|=|X|-1} b_i(S) \geq (|X| - 1) \cdot b_i(X)$. For any set $T \in \arg\max_{S \subseteq X, |S|=|X|-1} b_i(S)$, using the fact that the maximum is at least as large as the average, we therefore have $b_i(T) \geq (|X| - 1)/|X| \cdot b_i(X)$. □

Proof of Theorem 4. There are 2 agents and 6 items. The items are divided into two sets X_1 and X_2, each with 3 items. The valuations of agent $i \in \{1, 2\}$ are given by (all indices are modulo two)

$$v_i(S) = \begin{cases} 12 & \text{for } S \subseteq X_i,\ |S| = 3 \\ 6 & \text{for } S \subseteq X_i,\ 1 \leq |S| \leq 2 \\ 5 + 1\epsilon & \text{for } S \subseteq X_{i+1},\ |S| = 3 \\ 4 + 2\epsilon & \text{for } S \subseteq X_{i+1},\ |S| = 2 \\ 3 + 3\epsilon & \text{for } S \subseteq X_{i+1},\ |S| = 1 \\ \max_{j \in \{1,2\}}\{v_i(S \cap X_j)\} & \text{otherwise.} \end{cases}$$

The variable ϵ is a sufficiently small positive number. The valuation v_i of agent i is subadditive, but not fractionally subadditive. (The problem for agent i is that the valuation for X_i is too high given the valuations for $S \subset X_i$.)

The welfare maximizing allocation awards set X_1 to agent 1 and set X_2 to agent 2. The resulting welfare is $v_1(X_1) + v_2(X_2) = 12 + 12 = 24$.

We claim that the following profile of bids $b = (b_1, b_2)$ can be expressed within OXS and constitutes a pure Nash equilibrium:

$$b_i(S) = \begin{cases} 0 & \text{for } S \subseteq X_i \\ 5 + 1\epsilon & \text{for } S \subseteq X_{i+1},\ |S| = 3 \\ 4 + 2\epsilon & \text{for } S \subseteq X_{i+1},\ |S| = 2 \\ 3 + 3\epsilon & \text{for } S \subseteq X_{i+1},\ |S| = 1 \\ \max_{j \in \{1,2\}}\{b_i(S \cap X_j)\} & \text{otherwise.} \end{cases}$$

Given b VCG awards set X_2 to agent 1 and set X_1 to agent 2 for a welfare of $v_1(X_2)+v_2(X_1) = 2 \cdot (5 + \epsilon) = 10 + 2\epsilon$, which is by a factor $2.4 - 12\epsilon/(25 + 5\epsilon)$ smaller than the optimum welfare.

We can express b_i as ORs of XORs of XS bids as follows: Let $X_i = \{a, b, c\}$ and $X_{i+1} = \{d, e, f\}$. Let h_d, h_e, h_f and ℓ_d, ℓ_e, ℓ_f be XS bids that value d, e, f at $3 + 3\epsilon$ and $1 - \epsilon$, respectively. Then $b_i(T) = (h_d(T) \otimes h_e(T) \otimes h_f(T)) \vee \ell_d(T) \vee \ell_e(T) \vee \ell_f(T)$.

To show that b is a Nash equilibrium we can focus on agent i (by symmetry) and on deviating bids a_i that win agent i a subset S of X_i (because agent i currently wins X_{i+1} and $v_i(S) = \max\{v_i(S \cap X_1), v_i(S \cap X_2)\}$ for sets S that intersect both X_1 and X_2).

Note that the price that agent i faces on the subsets S of X_i are superadditive: For $|S| = 1$ the price is $(5 + \epsilon) - (4 + 2\epsilon) = 1 - \epsilon$, for $|S| = 2$ the price is $(5 + \epsilon) - (3 + 3\epsilon) = 2 - 2\epsilon$, and for $|S| = 3$ the price is $5 + \epsilon$.

Case 1: $S = X_i$. We claim that this case cannot occur. To see this observe that because $a_i \in$ XOS, Lemma 5 shows that there must be a 2-element subset T of S for which $a_i(T) \geq 2/3 \cdot a_i(S)$. On the one hand this shows that $a_i(S) \leq 9$ because otherwise $a_i(T) \geq 2/3 \cdot a_i(S) > 6$ in contradiction to our assumption that a_i does not overbid. On the other hand to ensure that VCG assigns S to agent i we must have $a_i(S) \geq a_i(T) + (3 + 3\epsilon)$. Thus $a_i(S) \geq 2/3 \cdot a_i(S) + (3 + 3\epsilon)$ and, hence, $a_i(S) \geq 9(1 + \epsilon)$. We conclude that $9 \geq a_i(S) \geq 9(1 + \epsilon)$, which gives a contradiction.

Case 2: $S \subset X_i$. In this case agent i's valuation for S is 6 and his payment is at least $1 - \epsilon$ as we have shown above. Thus, $u_i(a_i, b_{-i}) \leq 5 + \epsilon = u_i(b_i, b_{-i})$, i.e., the utility does not increase with the deviation. □

5 Computational Complexity of Equilibria

Our final result concerns the computational complexity of finding a pure Nash equilibrium. It states that for restrictions from subadditive valuations to additive bids it is $\mathcal{NP}$-hard to decide whether a pure Nash equilibrium exists. The same decision problem is simple for fractionally subadditive valuations because pure Nash equilibria are guaranteed to exist [6].

Theorem 5. *Consider running the VCG mechanism for agents with subadditive valuations and additive bids. Then it is $\mathcal{NP}$-hard to decide whether there exists a pure Nash equilibrium.*

Proof. We reduce from 3-Partition. Given an instance of 3-Partition consisting of a multiset of $3m$ positive integers $w_1, \ldots, w_{3m} \in (B/4, B/2)$ that sum up to mB, we construct an instance of a combinatorial auction in which the agents have subadditive valuations in polynomial time as follows:

The set of agents is $B_1, \ldots, B_m$ and $C_1, \ldots, C_m$. The set of items is $\mathcal{I} \cup \mathcal{J}$, where $\mathcal{I} = \{I_1, \ldots, I_{3m}\}$ and $\mathcal{J} = \{J_1, \ldots, J_{3m}\}$. Let $\mathcal{J}_i = \{J_i, J_{m+i}, J_{2m+i}\}$. Every agent B_i has valuations

$$v_{B_i}(S) = \max\{v_{\mathcal{I},B_i}(S), v_{\mathcal{J},B_i}(S)\},$$

where

$$v_{\mathcal{I},B_i}(S) = \sum_{I_e \in \mathcal{I} \cap S} w_e \quad \text{and} \quad v_{\mathcal{J},B_i}(S) = \begin{cases} 10B & \text{if } |\mathcal{J}_i \cap S| = 3 \ , \\ 5B & \text{if } |\mathcal{J}_i \cap S| \in \{1, 2\} \ , \\ 0 & \text{otherwise.} \end{cases}$$

Every agent C_i has valuations

$$v_{C_i}(S) = \begin{cases} 16B & \text{if } |\mathcal{J}_i \cap S| = 3 \ , \\ 8B & \text{if } |\mathcal{J}_i \cap S| \in \{1, 2\} \ , \\ 0 & \text{otherwise.} \end{cases}$$

The valuations for the items in $\mathcal{J}$ are motivated by an example for valuations without a pure Nash equilibrium in [3]. Note that our valuations are subadditive.

We show first that if there is a solution of our 3-Partition instance then the corresponding auction has a pure Nash equilibrium. Let us assume that $P_1, \ldots, P_m$ is a solution of 3-Partition. We obtain a pure Nash equilibrium when every agent B_i bids w_j for each I_j with $j \in P_i$ and zero for the other items; and every agent C_i bids $4B$ for each item in $\mathcal{J}_i$. The first step is to show that no agent B_i would change his strategy. The utility of B_i is B, because B_i's payment is zero. As the valuation function of B_i is the maximum of his valuation for the items in $\mathcal{I}$ and the items in $\mathcal{J}$ we can study the strategies for $\mathcal{I}$ and $\mathcal{J}$ separately. If B_i changed his bid and won another item in $\mathcal{I}$, B_i would have to pay his valuation for this item because there is an agent B_j bidding on it, and, thus, his utility would not increase. As B_i does not overbid, B_i could win at most one item of the items in $\mathcal{J}_i$. His value for the item would be $5B$, but the payment would be C_i's bid of $4B$. Thus, his utility would not be larger than B if B_i won an item of $\mathcal{J}$. Hence, B_i does not want to change his bid. The second step is to show that no agent C_i would change his strategy. This follows since the utility of every agent C_i is $16B$, and this is the highest utility that C_i can obtain.

We will now show two facts that follow if the auction is in a pure Nash equilibrium: (1) We first show that in every pure Nash equilibrium every agent B_i must have a utility of at least B. To see this denote the bids of agent C_i for the items J_i, J_{m+i}, J_{2m+i} in $\mathcal{J}_i$ by c_1, c_2, and c_3 and assume w.l.o.g. that $c_1 \leq c_2 \leq c_3$. As agent C_i does not overbid, $c_2 + c_3 \leq 8B$, and, thus, $c_1 \leq 4B$. If agent B_i bade $5B$ for J_i, B_i would win J_i and his utility would be at least B, because B_i has to pay C_i's bid for J_i. As B_i's utility in the pure Nash equilibrium cannot be worse, his utility in the pure Nash equilibrium has to be at least B. (2) Next we show that in a pure Nash equilibrium agent B_i cannot win any of the items in $\mathcal{J}_i$. For a contradiction suppose that agent B_i wins at least one of the items in $\mathcal{J}_i$ by bidding b_1, b_2, and b_3 for the items J_i, J_{m+i}, J_{2m+i} in $\mathcal{J}_i$. Then agent C_i does not win the whole set $\mathcal{J}_i$ and his utility is at most $8B$. As agent B_i does not overbid, $b_i + b_j \leq 5B$ for $i \neq j \in \{1, 2, 3\}$. Then, $b_1 + b_2 + b_3 \leq 7.5B$. Agent C_i can however bid $c_1 = b_1 + \epsilon$, $c_2 = b_2 + \epsilon$, $c_3 = b_3 + \epsilon$ for some $\epsilon > 0$ without violating no-overbidding to win all items in $\mathcal{J}_i$ for a utility of at least $16B - 7.5B > 8B$. Thus, C_i's utility increases when C_i changes his bid, i.e., the auction is not in a pure Nash equilibrium.

Now we use facts (1) and (2) to show that our instance of 3-Partition has a solution if the auction has a pure Nash equilibrium. Let us assume that the auction is in a pure Nash equilibrium. By (1) we know that every agent B_i gets at least utility B. Combined with (2) we know that every agent B_i wins only items in $\mathcal{I}$, pays zero, and has exactly utility B. Recall that all w_e with $e \in \{1, \ldots, 3m\}$ satisfy $B/4 < w_e < B/2$. Thus the valuation of an agent B_i is larger than $4 \cdot B/4 = B$ for a subset of $\mathcal{I}$ with more than three items and is smaller than $2 \cdot B/2 = B$ for a subset of $\mathcal{I}$ with less than three items. Hence, every bidder B_i gets exactly three items in $\mathcal{I}$ and the assignment of the items in $\mathcal{I}$ corresponds to a solution of 3-Partition. □

A Relaxed Smoothness and Bayes-Nash Equilibria

In this appendix we show that relaxed smoothness also implies an upper bound on the Price of Anarchy with respect to Bayes-Nash equilibria. For this result it is important that valuations are distributed independently. In fact, when valuations are correlated the theorem that we show does not apply.

Definitions In the incomplete information setting valuations are drawn independently from not necessarily identical distributions. Denote the distribution from which agent i's valuation is drawn by $\mathcal{D}_i$. Let $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$. Agent i knows his value v_i and the distributions $\mathcal{D}_{-i}$ from which the other agents' valuations are drawn, but he does not observe the realizations of these random draws.

A collection of possibly randomized bidding functions $\mathsf{b}_i : V_i \to B_i$, for $1 \leq i \leq n$, is a *mixed Bayes-Nash equilibrium* if for every agent i, every valuation v_i in the support of $\mathcal{D}_i$, and every pure deviation $b_i' \in B_i$,

$$\mathop{\mathrm{E}}_{v_{-i}\sim\mathcal{D}_{-i}}\big[u_i(\mathsf{b}_i(v_i),\mathsf{b}_{-i}(v_{-i}),v_i)\big] \geq \mathop{\mathrm{E}}_{v_{-i}\sim\mathcal{D}_{-i}}\big[u_i(b_i',\mathsf{b}_{-i}(v_{-i}),v_i)\big].$$

The Price of Anarchy with respect to mixed Bayes-Nash equilibria is the ratio between the expected optimal social welfare and the expected welfare of the worst mixed Bayes-Nash equilibrium

$$\mathrm{PoA}_{\mathrm{MBNE}} = \sup_{\mathsf{b}\text{: MBNE}} \frac{\mathrm{E}_{v\sim\mathcal{D}}[\mathrm{OPT}(v)]}{\mathrm{E}_{v\sim\mathcal{D}}[\mathrm{SW}(\mathsf{b}(v))]}.$$

Theorem We are now ready to prove that relaxed smoothness also implies a bound on the Price of Anarchy for Bayes-Nash equilibria.

Theorem 6. *If a mechanism $\mathcal{M} = (f, p)$ is relaxed (λ, μ_1, μ_2)-smooth then the Price of Anarchy for mixed Bayes-Nash equilibria is at most*

$$\frac{\max\{\mu_1, 1\} + \mu_2}{\lambda}.$$

Proof. Consider a mixed Bayes-Nash equilibrium b. For each agent i let $\mathcal{B}_{-i}$ denote the distribution over bids $b_{-i} \in B_{-i}$ induced by b_{-i} and $\mathcal{D}_{-i}$. Let $\mathcal{B}$ denote the distribution over bids $b \in B$ induced by b and $\mathcal{D}$.

In the Bayesian setting Definition 3 is not directly applicable because the deviating bid a_i in this definition may depend on v_{-i}, which is not known to agent i. What we will do instead is the following: We let each agent i "hallucinate" valuations $\hat{v}_{-i}$ for the agents other than i, where the "hallucinated" valuations $\hat{v}_{-i}$ are distributed according to $\mathcal{D}_{-i}$, and let him use the corresponding deviations.

Since no agent wants to deviate to the resulting randomized bid we obtain the following lower bound on the social welfare:

$$\begin{aligned}
\mathop{\mathrm{E}}_{v\sim\mathcal{D}}\big[\mathrm{SW}(\mathsf{b}(v))\big] &= \sum_{i=1}^{n} \mathop{\mathrm{E}}_{v_i\sim\mathcal{D}_i}\left[\mathop{\mathrm{E}}_{b_{-i}\sim\mathcal{B}_{-i}}\big[u_i(\mathsf{b}_i(v_i), b_{-i}, v_i) + p_i(f_i(\mathsf{b}_i(v), b_{-i}), b_{-i})\big]\right] \\
&\geq \sum_{i=1}^{n} \mathop{\mathrm{E}}_{v_i\sim\mathcal{D}_i}\left[\mathop{\mathrm{E}}_{b_{-i}\sim\mathcal{B}_{-i}}\Big[\mathop{\mathrm{E}}_{\hat{v}_{-i}^{(i)}\sim D_{-i}} u_i\big(a_i(v_i, \hat{v}_{-i}^{(i)}, \mathcal{B}_{-i}), b_{-i}, v_i\big)\Big]\right] \\
&\quad + \mathop{\mathrm{E}}_{v_i\sim\mathcal{D}_i}\left[\mathop{\mathrm{E}}_{b_{-i}\sim\mathcal{B}_{-i}}\big[p_i(f_i(\mathsf{b}_i(v), b_{-i}), b_{-i})\big]\right] \\
&= \mathop{\mathrm{E}}_{v\sim\mathcal{D}}\left[\sum_{i=1}^{n} \mathop{\mathrm{E}}_{b_{-i}\sim\mathcal{B}_{-i}}\big[u_i\big(a_i(v_i, v_{-i}, \mathcal{B}_{-i}), b_{-i}, v_i\big)\big]\right] \\
&\quad + \mathop{\mathrm{E}}_{v_i\sim\mathcal{D}_i}\left[\mathop{\mathrm{E}}_{b_{-i}\sim\mathcal{B}_{-i}}\big[p_i(f_i(\mathsf{b}_i(v), b_{-i}), b_{-i})\big]\right] \\
&\geq \mathop{\mathrm{E}}_{v\sim\mathcal{D}}\left[\lambda \cdot OPT(v) - \mu_1 \cdot \sum_{i=1}^{n} \mathop{\mathrm{E}}_{b\sim\mathcal{B}}[p_i(f_i(b), b_{-i})] - \mu_2 \cdot \sum_{i=1}^{n} \mathop{\mathrm{E}}_{b\sim\mathcal{B}}[b_i(f_i(b))]\right] \\
&\quad + \mathop{\mathrm{E}}_{v_i\sim\mathcal{D}_i}\left[\mathop{\mathrm{E}}_{b_{-i}\sim\mathcal{B}_{-i}}\big[p_i(f_i(\mathsf{b}_i(v), b_{-i}), b_{-i})\big]\right] \\
&= \lambda \mathop{\mathrm{E}}_{v\sim\mathcal{D}}[OPT(v)] - (\mu_1 - 1)\sum_{i=1}^{n} \mathop{\mathrm{E}}_{b\sim\mathcal{B}}[p_i(f_i(b), b_{-i})] - \mu_2 \sum_{i=1}^{n} \mathop{\mathrm{E}}_{b\sim\mathcal{B}}[b_i(f_i(b))],
\end{aligned}$$

where the first equality uses quasi-linearity of the utilities, the following inequality uses the equilibrium condition, the second equality uses stochastic independence and linearity of expectation, the next inequality uses the smoothness guarantee, and the third and final equality follows by rearranging terms.

Since agents do not overbid this can be rearranged to give

$$(1+\mu_2) \mathop{\mathrm{E}}_{v\sim\mathcal{D}}[\mathrm{SW}(\mathsf{b}(v))] \geq \lambda \mathop{\mathrm{E}}_{v\sim\mathcal{D}}[OPT(v)] - (\mu_1 - 1) \mathop{\mathrm{E}}_{v\sim\mathcal{D}}\left[\sum_{i=1}^{n} p_i(f_i(\mathsf{b}(v)), \mathsf{b}_{-i}(v))\right].$$

For $\mu_1 \leq 1$ the second term on the right hand side is lower bounded by zero and the result follows by rearranging terms. For $\mu_1 > 1$ we use that $\mathrm{E}_{v\sim\mathcal{D}}[p_i(f_i(\mathsf{b}(v)), \mathsf{b}_{-i}(v))] \leq \mathrm{E}_{v\sim\mathcal{D}}[v_i(f_i(\mathsf{b}(v)))] = \mathrm{E}_{v\sim\mathcal{D}}[SW(\mathsf{b}(v))]$ to lower bound the second term on the right hand side and the result follows by rearranging terms. □

Acknowledgements

The work of Paul Dütting was supported by an SNF Postdoctoral Fellowship. The work of Monika Henzinger and Martin Starnberger was funded by the Vienna Science and Technology Fund (WWTF) through project ICT10-002, and by the University of Vienna through IK I049-N.

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