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# Flexoelectric effect on vibration responses of piezoelectric nanobeams embedded in viscoelastic medium based on nonlocal elasticity theory

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**Abstract:** In this study, vibration characteristics of a piezoelectric nanobeam embedded in viscoelastic medium are investigated based on nonlocal Euler-Bernoulli beam theory. In doing this, the governing equations of motion and boundary conditions for vibration analysis are first derived by using Hamilton's principle, where nonlocal effect, piezoelectric effect, flexoelectric effect and viscoelastic medium are considered simultaneously. Subsequently, the transfer function method is employed to obtain the natural frequencies and corresponding mode shapes in closed form for the embedded piezoelectric nanobeam with arbitrary boundary conditions. The proposed mechanics model is validated by comparing the obtained results with those available in the literature, where good agreement is achieved. The effects of nonlocal parameter, boundary conditions, slenderness ratio, flexoelectric coefficient and viscoelastic medium on vibration responses are also examined carefully for the embedded nanobeam. The results demonstrate the efficiency and robustness of the developed model for vibration analysis of a complicated multi-physics system comprising piezoelectric nanobeam with flexoelectric effect, viscoelastic medium and electrical loadings.

**Keywords:** piezoelectric nanobeam; flexoelectric effect; nonlocal Euler-Bernoulli beam theory; viscoelastic medium

## 1. Introduction

Piezoelectric nanobeams have various practical applications in smart devices and

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systems due to their exceptional electromechanical coupling effect [1,2], such as nanotransducers, nanoresonators, nanosensors and nanogenerators [3-6]. It is thus essential to quantify and understand the vibration behaviors of piezoelectric nanobeams. As a universal electromechanical mechanism in all piezoelectric materials, flexoelectricity has been reported to have strong influence on the vibration responses of piezoelectric nanobeams [7-9]. As a result, the study of flexoelectric effect on vibration responses of embedded piezoelectric nanobeams may provide valuable information for the above mentioned potential applications of piezoelectric nanobeams.

Numerous studies have been performed so far by researchers to examine the mechanical properties of piezoelectric nanobeams with flexoelectric effect. Based on the flexoelectricity theory and strain gradient theory, a size-dependent bending model was developed by Qi et al. [10] to investigate the static bending of an electro-elastic bilayer nanobeam. The influence of flexoelectric effect on the static bending and free vibration was examined by Yan and Jiang [11] for a simply supported piezoelectric nanobeam. In this study, the governing equations of motion were derived by using Hamilton's principle and the explicit expressions of deflection and natural frequencies were also obtained. A modified couple stress theory and Euler-Bernoulli beam theory were proposed by Li and Luo [6] to study the effects of couple stress, flexoelectricity and piezoelectricity on vibration characteristics of piezoelectric microbeams. In this study, the results showed that the effective bending rigidity of the piezoelectric microbeam was hardened due to enhanced flexoelectric effect. Liang et al. [9] examined the effects of surface and flexoelectricity on static bending of piezoelectric nanobeams based on Euler-Bernoulli beam theory and variational principle. The study pointed out that the flexoelectric effect had a momentous influence on the bending rigidity of the piezoelectric nanobeam. A micro scale Timoshenko beam model was developed by Yue et al. [12] to investigate the static bending and free vibration problems for a piezoelectric nanobeam with simply supported boundary conditions. Here, the governing equations of motion and related boundary conditions were derived by using the variational principle and Hamilton's principle. It is noted that in the above studies the size effect was taken into account by considering the flexoelectric effect of piezoelectric nanobeams based on strain gradient theory.

However, some researchers point out that nonlocal elasticity theory should be incorporated to the strain gradient theory for more accurate prediction of mechanical behavior of nanostructures [13-15]. On this basis, to include the nonlocal effect in vibration analysis of piezoelectric nanobeams, Ebrahimi and Barati [8] investigated the vibration characteristics of a flexoelectric nanobeam resting on Winkler-Pasternak elastic foundation based on nonlocal elasticity theory. In this study, Hamilton's principle was used to derive the governing equations of motion and a Galerkin-based method was applied to obtain the natural frequencies. Based on nonlocal Timoshenko beam theory, Ke and Wang [16] derived the governing equations and boundary conditions for vibration analysis of piezoelectric nanobeams by utilizing Hamilton's principle and computed the natural frequencies for various boundary conditions by using differential quadrature method. Nonlocal Timoshenko beam theory was also used by Ke et al. [17] to investigate the nonlinear vibration of piezoelectric nanobeams, which were subjected to an applied voltage and a uniform temperature change. Thermo-electro-mechanical vibration was examined by Ansari et al. [18] for a postbuckled piezoelectric Timoshenko nanobeam based on nonlocal elasticity theory. In the study, the governing equations of motion were derived using Hamilton's principle and then solved via generalized differential quadrature (GDQ) method.

It is noted that piezoelectric nanobeams are often embedded in a medium in many of their nanotechnology applications, such as nanosensors and nanogenerators [5,8,19]. In particular, some media employed in the nanotechnology applications normally exhibit viscoelastic behavior. To the best of the present authors' knowledge, the vibration responses of piezoelectric nanobeams with both flexoelectric effect and nonlocal effect have not been reported in the literature when the surrounding viscoelastic medium is taken into account. The information, however, is essential for the engineering applications of piezoelectric nanobeams in nanotechnology. Hence, the objective of the present work is to study the vibration responses of a piezoelectric nanobeam with flexoelectric and nonlocal effects, which is embedded in viscoelastic medium and subjected to electrical loadings. Here, the nonlocal Euler-Bernoulli beam model is employed to derive the governing equations and a numerical approach named the transfer function method (TFM) is proposed to calculate the natural frequencies of nanobeams with arbitrary boundary conditions. Subsequently, a detailed

parametric study is conducted to investigate the effects of nonlocal parameter, boundary conditions, slenderness ratio, flexoelectric coefficient and viscoelastic medium on the vibration responses of piezoelectric nanobeams.

## 2. Mathematical modeling

Here let us consider a piezoelectric nanobeam with flexoelectric and nonlocal effects, which is embedded in viscoelastic medium and subjected to electrical loadings, as shown in Fig. 1. The piezoelectric nanobeam is modeled as nonlocal Euler-Bernoulli beam with length  $L$ , thickness  $h$  and width  $b$ . The viscoelastic medium is described by a visco-Pasternak foundation model, whose Winkler's modulus parameter is  $k_w$ , Pasternak's modulus parameter is  $k_G$  and damping parameter is  $c_t$ . A Cartesian coordinate system  $oxyz$  is also defined, in which the  $x$ -,  $y$ - and  $z$ -axes are taken along the length, width and thickness directions of the nanobeam, respectively. To account for the flexoelectric effect, the electric Gibbs free energy density  $G_b$  can be written as [6]

$$G_b = -\frac{1}{2}\kappa_{ij}E_iE_j - \frac{1}{2}b_{ijkl}E_{i,j}E_{k,l} + \frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - e_{ijk}E_i\varepsilon_{jk} - \mu_{ijkl}(E_k\varepsilon_{ij,l} - \varepsilon_{ij}E_{k,l}) \quad (1)$$

where  $\kappa_{ij}$ ,  $b_{ijkl}$ ,  $c_{ijkl}$ ,  $e_{ijk}$  and  $\mu_{ijkl}$  are dielectric constant tensor, nonlocal electrical coupling coefficient tensor, elastic stiffness tensor, piezoelectric coefficient tensor and flexoelectric coefficient tensor, respectively. In addition,  $E_i$  and  $E_j$  are electric field vectors,  $\varepsilon_{ij}$  and  $\varepsilon_{kl}$  are strain tensors,  $\varepsilon_{ij,l}$  and  $E_{k,l}$  are the gradients of strain and electric field.

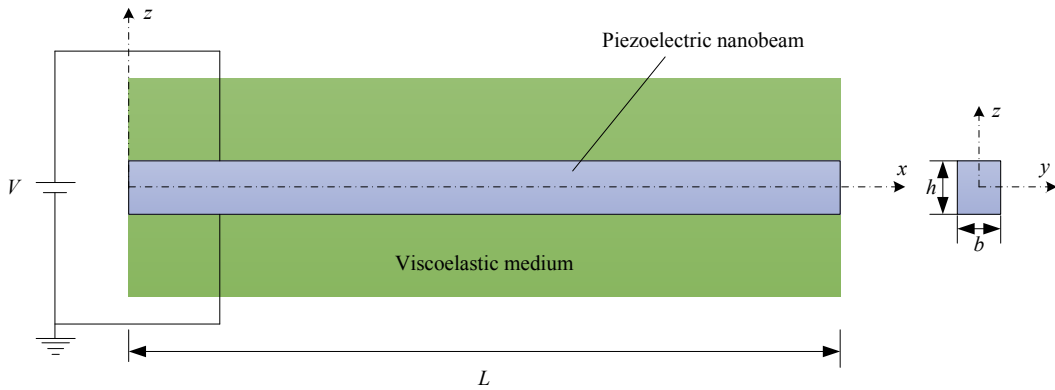


Fig. 1 Schematic of a piezoelectric nanobeam embedded in viscoelastic medium under

electrical field.

For an Euler-Bernoulli beam, the displacement field can be expressed as

$$u = -z \frac{\partial w}{\partial x}, v = 0, w = w(x, t). \quad (2)$$

where  $u$ ,  $v$  and  $w$  are the displacement components along the  $x$ -,  $y$ - and  $z$ -directions on the cross section. Accordingly, the components of strains and strain gradients are given as

$$\varepsilon_{xx} = -z \frac{\partial^2 w}{\partial x^2}, \varepsilon_{xx,z} = -\frac{\partial^2 w}{\partial x^2}. \quad (3)$$

Based on nonlocal Euler-Bernoulli beam theory [20,21], the constitutive equations for piezoelectric nanobeams with flexoelectric effect can be derived in terms of the electric Gibbs free energy density as

$$\left(1 - (e_0 a)^2 \nabla^2\right) \sigma_{xx} = c_{11} \varepsilon_{xx} - e_{31} E_z + \mu_{31} E_{z,z} \quad (4)$$

$$\left(1 - (e_0 a)^2 \nabla^2\right) \tau_{xxz} = -\mu_{31} E_z \quad (5)$$

$$\left(1 - (e_0 a)^2 \nabla^2\right) D_z = \kappa_{33} E_z + e_{31} \varepsilon_{xx} + \mu_{31} \varepsilon_{xx,z} \quad (6)$$

$$\left(1 - (e_0 a)^2 \nabla^2\right) Q_{zz} = b_{33} E_{z,z} - \mu_{31} \varepsilon_{xx} \quad (7)$$

where  $\sigma_{xx}$ ,  $\tau_{xxz}$ ,  $D_z$  and  $Q_{zz}$  denote the nonlocal stress, higher-order stress, electric displacement and electric quadrupole, respectively. Moreover,  $e_0 a$  is the nonlocal parameter and  $\nabla$  denotes the Hamilton arithmetic operator. In addition, the poling direction of the piezoelectric material is assumed to coincide with the  $z$ -direction and only the electric field in the  $z$ -direction is considered [12], i.e.,

$$E_z = -\frac{\partial \Phi}{\partial z} \quad (8)$$

where  $\Phi$  is the electric potential. In the absence of free electric charges, Gauss's law requires

$$-\frac{\partial^2 Q_{zz}}{\partial z^2} + \frac{\partial D_z}{\partial z} = 0 \quad (9)$$

Substituting Eqs. (6) and (7) into Eq. (9) yields

$$\kappa_{33}E_{z,z} + e_{31}\varepsilon_{xx,z} + 2\mu_{31}\varepsilon_{xx,zz} - b_{33}E_{z,zzz} = 0 \quad (10)$$

In addition, the boundary conditions of electric potential  $\Phi$  are given by

$$\Phi\left(-\frac{h}{2}\right) = 0, \Phi\left(\frac{h}{2}\right) = V. \quad (11)$$

From Eqs. (3), (10) and (11), the expression of electric potential  $\Phi$  can be determined as

$$\begin{aligned} \Phi = & -\frac{e_{31}}{2\kappa_{33}}\frac{\partial^2 w}{\partial x^2}z^2 - \frac{\mu_{31}}{2\kappa_{33}}\frac{\partial^2 w}{\partial x^2}z + \frac{V}{h}z + \frac{e_{31}h^2}{8\kappa_{33}}\frac{\partial^2 w}{\partial x^2} + \frac{V}{2} \\ & + \frac{e_{31}}{\eta^2\kappa_{33}}\frac{e^{\eta z} + e^{-\eta z}}{e^{\eta z/2} + e^{-\eta z/2}}\frac{\partial^2 w}{\partial x^2} + \frac{\mu_{31}h}{4\eta^2\kappa_{33}}\frac{e^{\eta z} - e^{-\eta z}}{e^{\eta z/2} - e^{-\eta z/2}}\frac{e^{\eta z} + e^{-\eta z}}{e^{\eta z/2} + e^{-\eta z/2}} \end{aligned} \quad (12)$$

where  $\eta = \sqrt{\kappa_{33}/b_{33}}$ . For simplification of analysis, the high-order nonlocal electrical coupling effect in Eq. (12) is neglected by taking  $b_{33} = 0$ , and we have [6]

$$\Phi = -\frac{e_{31}}{2\kappa_{33}}\frac{\partial^2 w}{\partial x^2}z^2 - \frac{\mu_{31}}{2\kappa_{33}}\frac{\partial^2 w}{\partial x^2}z + \frac{V}{h}z + \frac{e_{31}h^2}{8\kappa_{33}}\frac{\partial^2 w}{\partial x^2} + \frac{V}{2} \quad (13)$$

Substituting Eq. (13) into Eqs. (4)-(8) leads to

$$E_z = \frac{e_{31}}{\kappa_{33}}\frac{\partial^2 w}{\partial x^2}z + \frac{\mu_{31}}{2\kappa_{33}}\frac{\partial^2 w}{\partial x^2} - \frac{V}{h} \quad (14)$$

$$E_{z,z} = \frac{e_{31}}{\kappa_{33}}\frac{\partial^2 w}{\partial x^2} \quad (15)$$

$$\left(1 - (e_0 a)^2 \nabla^2\right) \sigma_{xx} = -c_{11} \frac{\partial^2 w}{\partial x^2} z - \frac{e_{31}^2}{\kappa_{33}} \frac{\partial^2 w}{\partial x^2} z + \frac{e_{31}\mu_{31}}{2\kappa_{33}} \frac{\partial^2 w}{\partial x^2} + e_{31} \frac{V}{h} \quad (16)$$

$$\left(1 - (e_0 a)^2 \nabla^2\right) \tau_{xz} = -\frac{e_{31}\mu_{31}}{\kappa_{33}} \frac{\partial^2 w}{\partial x^2} z - \frac{\mu_{31}^2}{2\kappa_{33}} \frac{\partial^2 w}{\partial x^2} + \mu_{31} \frac{V}{h} \quad (17)$$

$$\left(1 - (e_0 a)^2 \nabla^2\right) D_z = -\frac{\mu_{31}}{2} \frac{\partial^2 w}{\partial x^2} - \kappa_{33} \frac{V}{h} \quad (18)$$

The governing equations and boundary conditions for vibration analysis of piezoelectric nanobeams can be obtained by using Hamilton's principle, i.e.,

$$\int_0^t (\delta \Pi_k + \delta \Pi_F - \delta \Pi_s) dt = 0 \quad (19)$$

where the strain energy  $\Pi_s$ , kinetic energy  $\Pi_k$  and external work  $\Pi_F$  can be calculated

by the following equations

$$\Pi_s = \frac{1}{2} \int_0^L \int_A (\sigma_{xx} \varepsilon_{xx} + \tau_{xxz} \varepsilon_{xx,z}) dA dx = -\frac{1}{2} \int_0^L \left( M_{xx} \frac{\partial^2 w}{\partial x^2} + P_{xxz} \frac{\partial^2 w}{\partial x^2} \right) dx \quad (20)$$

$$\Pi_k = \frac{1}{2} \int_0^L \rho A \left( \frac{\partial w}{\partial t} \right)^2 dx \quad (21)$$

$$\Pi_F = -\frac{1}{2} \int_0^L \left[ b N_Q w + N_{xx} \left( \frac{\partial w}{\partial x} \right)^2 \right] dx \quad (22)$$

here  $M_{xx}$  is the internal bending moment,  $P_{xxz}$  is the higher-order axial couple,  $N_{xx}$  is the internal stress resultant and  $N_Q$  is the reaction of the viscoelastic medium, which can be expressed as

$$M_{xx} = \int_A \sigma_{xx} z dA, P_{xxz} = \int_A \tau_{xxz} dA, N_{xx} = \int_A \sigma_{xx} dA. \quad (23)$$

$$N_Q = k_w w - k_G \nabla^2 w + c_t \frac{\partial w}{\partial t} \quad (24)$$

Substituting Eqs. (20)-(22) into Hamilton's principle (19), the governing equation of embedded piezoelectric nanobeams with flexoelectric effect can be derived by integrating it by parts and setting the coefficients in front of  $\delta w$  to zero

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 P_{xxz}}{\partial x^2} + N_{xx} \frac{\partial^2 w}{\partial x^2} - b \left( k_w w - k_G \nabla^2 w + c_t \frac{\partial w}{\partial t} \right) = \rho A \frac{\partial^2 w}{\partial t^2} \quad (25)$$

The corresponding boundary conditions also can be obtained as follows

$$w = 0 \text{ or } \frac{\partial M_{xx}}{\partial x} + \frac{\partial P_{xxz}}{\partial x} + N_{xx} \frac{\partial w}{\partial x} = 0 \quad (26)$$

$$\frac{\partial w}{\partial x} = 0 \text{ or } M_{xx} + P_{xxz} = 0 \quad (27)$$

From Eqs. (3), (16) and (17), we have

$$\left( 1 - (e_0 a)^2 \nabla^2 \right) M_{xx} = - \left( c_{11} + \frac{e_{31}^2}{\kappa_{33}} \right) \frac{b h^3}{12} \frac{\partial^2 w}{\partial x^2} \quad (28)$$

$$\left( 1 - (e_0 a)^2 \nabla^2 \right) P_{xxz} = - \frac{\mu_{31}^2 b h}{2 \kappa_{33}} \frac{\partial^2 w}{\partial x^2} + \mu_{31} b V \quad (29)$$

$$\left( 1 - (e_0 a)^2 \nabla^2 \right) N_{xx} = e_{31} b V \quad (30)$$

Inserting Eqs. (28)-(30) in the governing Eq. (25) leads to



$$\begin{aligned}
& -\left(\bar{M}_{xx} + \bar{P}_{xxz}\right) \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + \bar{N}_{xx} \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \left(1 - \alpha^2 \frac{\partial^2}{\partial \bar{x}^2}\right) \left(\bar{k}_w \bar{w} - \bar{k}_G \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \bar{c}_t \frac{\partial \bar{w}}{\partial t}\right) \\
& = \left(1 - \alpha^2 \frac{\partial^2}{\partial \bar{x}^2}\right) \eta \frac{\partial^2 \bar{w}}{\partial t^2}
\end{aligned} \tag{31}$$

in which the dimensionless terms are defined for the sake of convenience and generality as follows

$$\begin{aligned}
\bar{x} &= \frac{x}{L}, \bar{w} = \frac{w}{L}, \alpha = \frac{e_0 a}{L}, \bar{c}_{11} = c_{11} L, \eta = \frac{h}{L}, \bar{k}_w = \frac{k_w L^2}{\bar{c}_{11}}, \bar{k}_G = \frac{k_G}{\bar{c}_{11}}, \\
\bar{c}_t &= \frac{c_t L}{\sqrt{\rho L \bar{c}_{11}}}, t = \frac{t}{L} \sqrt{\frac{\bar{c}_{11}}{\rho L}}, \bar{P}_{xxz} = \frac{\mu_{31}^2 \eta}{2 \kappa_{33} \bar{c}_{11} L}, \bar{N}_{xx} = \frac{e_{31} V}{\bar{c}_{11}}, \bar{M}_{xx} = \left(1 + \frac{e_{31}^2 L}{\kappa_{33} \bar{c}_{11}}\right) \frac{\eta^3}{12}.
\end{aligned} \tag{32}$$

Using these dimensionless terms, the boundary conditions can be rewritten as

$$\bar{w} = 0 \text{ or } \left(\bar{M}_{xx} + \bar{P}_{xxz}\right) \frac{\partial^3 \bar{w}}{\partial \bar{x}^3} - \bar{N}_{xx} \frac{\partial \bar{w}}{\partial \bar{x}} = 0 \tag{33}$$

$$\frac{\partial \bar{w}}{\partial \bar{x}} = 0 \text{ or } \left(\bar{M}_{xx} + \bar{P}_{xxz}\right) \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \frac{\mu_{31} V}{\bar{c}_{11} L} = 0 \tag{34}$$

The governing Eq. (31) satisfying the boundary conditions in Eqs. (33) and (34) can be solved by assuming the solutions in the form

$$\bar{w}(\bar{x}) = \bar{W}(\bar{x}) e^{i\Omega t} \tag{35}$$

where  $\Omega$  is the dimensionless angular frequency, and  $\bar{W}$  is the corresponding mode shape.

Also, the dimensionless angular frequency  $\Omega$  can be given by

$$\Omega = \omega L \sqrt{\frac{\rho L}{\bar{c}_{11}}} \tag{36}$$

Substituting Eq. (35) into Eqs. (31), (33) and (34), one has

$$\begin{aligned}
& -\left(\bar{M}_{xx} + \bar{P}_{xxz}\right) \frac{\partial^4 \bar{W}}{\partial \bar{x}^4} + \bar{N}_{xx} \frac{\partial^2 \bar{W}}{\partial \bar{x}^2} - \left(1 - \alpha^2 \frac{\partial^2}{\partial \bar{x}^2}\right) \left(\bar{k}_w \bar{W} - \bar{k}_G \frac{\partial^2 \bar{W}}{\partial \bar{x}^2} + i\Omega \bar{c}_t \bar{W}\right) \\
& = -\Omega^2 \left(1 - \alpha^2 \frac{\partial^2}{\partial \bar{x}^2}\right) \eta \bar{W}
\end{aligned} \tag{37}$$

$$\bar{W} = 0 \text{ or } \left(\bar{M}_{xx} + \bar{P}_{xxz}\right) \frac{\partial^3 \bar{W}}{\partial \bar{x}^3} - \bar{N}_{xx} \frac{\partial \bar{W}}{\partial \bar{x}} = 0 \tag{38}$$

$$\frac{\partial \bar{W}}{\partial \bar{x}} = 0 \text{ or } \left(\bar{M}_{xx} + \bar{P}_{xxz}\right) \frac{\partial^2 \bar{W}}{\partial \bar{x}^2} - \frac{\mu_{31} V}{\bar{c}_{11} L} = 0 \tag{39}$$

It should be noted the proposed mechanics model above is a simplified model, which is only available for vibration analysis of piezoelectric nanobeams as the electric field and strain gradient are homogeneous in the direction of width  $b$ . In addition, the governing Eq. (37) is a fourth-order ordinary differential equation for  $\bar{W}$ , and the coefficients in front of  $\bar{W}$  and its derivatives are the functions of  $\Omega$ . In the next section, the natural frequencies and corresponding mode shapes in closed form are calculated for the embedded piezoelectric nanobeam with arbitrary boundary conditions by using the TFM.

### 3. Transfer function method

To achieve the eigenvalues and frequency response functions, the state vector  $\boldsymbol{\eta}(\bar{x}, \Omega)$  is defined as

$$\boldsymbol{\eta}(\bar{x}, \Omega) = \left[ \bar{W}, \frac{d\bar{W}}{d\bar{x}}, \frac{d^2\bar{W}}{d\bar{x}^2}, \frac{d^3\bar{W}}{d\bar{x}^3} \right]^T \quad (40)$$

where the superscript "T" denotes matrix transpose. Then the governing Eq. (37) can be rewritten in a matrix form as

$$\frac{d\boldsymbol{\eta}(\bar{x}, \Omega)}{d\bar{x}} = \mathbf{F}(\Omega)\boldsymbol{\eta}(\bar{x}, \Omega) \quad (41)$$

where

$$\mathbf{F}(\Omega) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ f_1 & 0 & f_2 & 0 \end{bmatrix} \quad (42)$$

and

$$f_1 = \frac{\Omega^2 \eta - \bar{k}_w - i\Omega \bar{c}_t}{\bar{M}_{xx} + \bar{P}_{xxz} + \alpha^2 \bar{k}_w}, f_2 = \frac{\bar{N}_{xx} + \bar{k}_G + \alpha^2 \bar{k}_w + \alpha^2 i\Omega \bar{c}_t - \alpha^2 \Omega^2 \eta}{\bar{M}_{xx} + \bar{P}_{xxz} + \alpha^2 \bar{k}_w}. \quad (43)$$

Also, the boundary conditions can be expressed in a matrix form as

$$\mathbf{M}(\Omega)\boldsymbol{\eta}(0, \Omega) + \mathbf{N}(\Omega)\boldsymbol{\eta}(1, \Omega) = 0 \quad (44)$$

where  $\mathbf{M}(\Omega)$  and  $\mathbf{N}(\Omega)$  are the boundary condition set matrices at the left and the right ends of the nanobeam, respectively. To demonstrate the approach, several typical boundary condition set matrices are given as examples in the following.

(1) For clamped-clamped (C-C) boundary conditions, we have

$$\mathbf{M}(\Omega) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{N}(\Omega) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (45)$$

(2) For simply supported-simply supported (S-S) boundary conditions, one has

$$\mathbf{M}(\Omega) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & M_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{N}(\Omega) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & M_1 & 0 \end{bmatrix}. \quad (46)$$

where

$$M_1 = \frac{\mu_{31}V}{\bar{c}_{11}L(\bar{M}_{xx} + \bar{P}_{xxz})} \quad (47)$$

(3) For clamped-free (C-F) boundary conditions,  $\mathbf{M}(\Omega)$  is given by Eq. (45), and

$\mathbf{N}(\Omega)$  can be expressed as

$$\mathbf{N}(\Omega) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & M_1 & 0 \\ 0 & -\bar{N}_{xx} & 0 & \bar{M}_{xx} + \bar{P}_{xxz} \end{bmatrix} \quad (48)$$

The solution of Eq. (41) can be calculated as

$$\boldsymbol{\eta}(\bar{x}, \Omega) = e^{F(\Omega)\bar{x}} \boldsymbol{\eta}(0, \Omega) \quad (49)$$

Substituting Eq. (49) into Eq. (44), one has

$$\left[ \mathbf{M}(\Omega) + \mathbf{N}(\Omega) e^{F(\Omega)} \right] \boldsymbol{\eta}(0, \Omega) = 0 \quad (50)$$

Thus the dimensionless frequencies  $\Omega$  of the nanobeam can be obtained by solving the following transcendental characteristic equation

$$\det \left[ \mathbf{M}(\Omega) + \mathbf{N}(\Omega) e^{F(\Omega)} \right] = 0 \quad (51)$$

In addition, the mode shape corresponding to the dimensionless frequency  $\Omega_j$  can be expressed as

$$\boldsymbol{\eta}(\bar{x}, \Omega_j) = e^{F(\Omega_j)\bar{x}} \boldsymbol{\eta}(0, \Omega_j) \quad (52)$$

According to Eq. (36), the natural frequencies  $\omega$  of the embedded piezoelectric nanobeam with flexoelectric effect can be calculated from

$$\omega = \frac{\Omega}{L} \sqrt{\frac{\bar{c}_{11}}{\rho L}} \quad (53)$$

#### 4. Numerical results and discussion

In this section, the formulation achieved above is first validated by comparing the obtained results with those available in the literature. This is followed by a detailed parametric study of the effects of nonlocal parameter, boundary conditions, slenderness ratio, flexoelectric coefficient and viscoelastic medium on the vibration responses of nanobeams. In doing this, the material of the piezoelectric nanobeam with flexoelectric effect is assumed to be BaTiO<sub>3</sub>. Unless otherwise stated, the values of some parameters used for numerical calculations are given as follows: the length of the nanobeam  $L=40\text{nm}$ , thickness  $h=2\text{nm}$ , width  $b=h$ , mass density  $\rho=7500\text{kg/m}^3$ , elastic stiffness  $c_{11}=131\text{GPa}$ , flexoelectric coefficient  $\mu_{31}=1\times 10^{-6}\text{C/m}$ , dielectric constant  $\kappa_{33}=12.56\times 10^{-9}\text{C/V}\cdot\text{m}$  and piezoelectric coefficient  $e_{31}=-4.35\text{C/m}^2$ . These geometrical and material properties of the piezoelectric nanobeam are adopted from the papers [6,11,12].

The dimensionless fundamental frequencies of piezoelectric nanobeams with various boundary conditions and slenderness ratios  $L/h$  in comparison with those of Ref. [17] are listed in Table 1. In this calculation, the values of basic parameters are the same as those in the paper [17], and the effects of flexoelectricity and viscoelastic medium are omitted by taking  $\mu_{31}=0$ ,  $k_w=0$ ,  $k_G=0$  and  $c_f=0$ . It can be seen from Table 1 that the results of this study are in good agreement with those in the literature, which verifies the accuracy and efficiency of the proposed method for vibration analysis of piezoelectric nanobeams.

For future comparisons with other researchers, Table 2 presents the first three dimensionless frequencies of piezoelectric nanobeams in the absence of medium (i.e.,  $k_w=0$ ,  $k_G=0$  and  $c_f=0$ ) and in the presence of viscoelastic medium ( $k_w=1\text{GPa/nm}$ ,  $k_G=0.25\text{GPa}\cdot\text{nm}$  and  $c_f=10^{-4}\text{GPa}\cdot\text{ns/nm}$ ) under typical boundary conditions, including C-F, S-S and C-C. From Table 2 we can observe that the imaginary parts related to damping ratios appear in the natural frequencies of nanobeams when the viscoelastic medium is considered. The reason for

this is that the damping effect of the viscoelastic medium is introduced into the system. It also can be seen that boundary conditions have strong influence on the real parts of natural frequencies but have no effect on the imaginary parts. Furthermore, nonlocal parameter  $\alpha$  has no effect on the imaginary parts of natural frequencies. This is because that the imaginary parts of natural frequencies are only related to the damping of the viscoelastic medium, in which nonlocal effect is not taken into account.

Table 1 Dimensionless fundamental frequencies of piezoelectric nanobeams with various boundary conditions and slenderness ratios  $L/h$  in comparison with those of Ref. [17].

BCs		$L/h$					
		6	8	10	16	20	30
S-S	Present	0.4571	0.3428	0.2743	0.1714	0.1371	0.0914
	Ke et al. [17]	0.4570	0.3428	0.2742	0.1714	0.1371	0.0914
C-S	Present	0.7079	0.5309	0.4248	0.2655	0.2124	0.1416
	Ke et al. [17]	0.7077	0.5310	0.4250	0.2658	0.2127	0.1420

Table 2 First three dimensionless frequencies of piezoelectric nanobeams with different boundary conditions and nonlocal parameter  $\alpha$ .

BCs	In the absence of medium			In the presence of viscoelastic medium		
	$\alpha=0.0$	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.0$	$\alpha=0.1$	$\alpha=0.2$
	1.5331	1.5399	1.5607	2.9086+0.0319i	2.9121+0.0319i	2.9232+0.0319i
C-F	9.6082	8.6976	7.0049	9.9219+0.0319i	9.0429+0.0319i	7.4294+0.0319i
	26.9033	21.1570	14.4468	27.0176+0.0319i	21.3022+0.0319i	14.6586+0.0319i
	4.3036	4.1058	3.6440	4.9835+0.0319i	4.7930+0.0319i	4.4039+0.0319i
S-S	17.2145	14.5761	10.7191	17.3921+0.0319i	14.7853+0.0319i	11.0019+0.0319i
	38.7327	28.1868	18.1521	38.8125+0.0319i	28.2964+0.0319i	18.3218+0.0319i
	9.7557	8.8189	7.0870	10.0648+0.0319i	9.1597+0.0319i	7.5069+0.0319i
C-C	26.8912	21.1150	14.4430	27.0067+0.0319i	21.2952+0.0319i	14.6549+0.0319i
	52.7197	35.4709	21.8228	52.7787+0.0319i	35.5585+0.0319i	21.9649+0.0319i

Since nonlocal parameter  $\alpha$  has no effect on the imaginary parts of natural frequencies for embedded piezoelectric nanobeams, only the effect of nonlocal parameter  $\alpha$  on the real parts of the first three natural frequencies are presented in Fig. 2 to Fig. 4. Here the frequency ratios  $\text{Re}(\omega_{\text{NL}}/\omega_{\text{L}})$  in the vertical axis are used to denote the real part of the ratios  $\omega_{\text{NL}}/\omega_{\text{L}}$  between the two frequencies  $\omega_{\text{NL}}$  and  $\omega_{\text{L}}$ , which are the damped frequencies obtained based on the nonlocal and local (classical) mechanics theories, respectively. It can be seen from Fig. 2 that the real parts of the first frequency ratios  $\omega_{\text{NL}}/\omega_{\text{L}}$  decrease significantly with rising nonlocal parameter  $\alpha$  for both S-S and C-C nanobeams but increase slightly for C-F nanobeams. This suggests that the rigidity of the embedded nanobeam is reduced due to enhanced nonlocal effect for S-S and C-C boundary conditions but hardened for C-F boundary conditions. A similar phenomenon was also described by Lei et al. [20] and Lu et al. [22]. Furthermore, such an effect of nonlocal parameter  $\alpha$  turns out to be less pronounced when the viscoelastic medium is considered. For example, as  $\alpha$  increases from 0 to 0.2 the real parts of the first frequency ratios  $\omega_{\text{NL}}/\omega_{\text{L}}$  for S-S nanobeams decrease about 15.33% in the absence of viscoelastic medium but about 10.99% in the presence of viscoelastic medium. The strong influence of boundary conditions on vibration responses of nanobeams also can be observed in Fig. 2. The values of the first frequency ratios  $\omega_{\text{NL}}/\omega_{\text{L}}$  follow the order: C-C < S-S < C-F, which implies that the nonlocal effect on the natural frequencies is more substantial when the stronger constraints are imposed on the boundaries. The effect of nonlocal parameter  $\alpha$  on the higher frequency ratios  $\omega_{\text{NL}}/\omega_{\text{L}}$  is also investigated in Fig. 3 and Fig. 4. It can be seen from the figures that the real parts of the higher frequency ratios  $\omega_{\text{NL}}/\omega_{\text{L}}$  decrease significantly with  $\alpha$  no matter what boundary conditions are imposed on the boundaries. In addition, the nonlocal effect on the higher frequency ratios turns out to be much larger than its effect on the fundamental ones. For instance, as  $\alpha$  changes from 0 to 0.2 the real parts of the second frequency ratios  $\omega_{\text{NL}}/\omega_{\text{L}}$  for S-S nanobeams with and without viscoelastic medium decrease about 52.79% and 53.14%, respectively, in Fig. 3 much larger than the values shown above for the first frequency ratios. As expected, the effect of viscoelastic medium on the frequency ratios becomes less pronounced as the mode number increases.

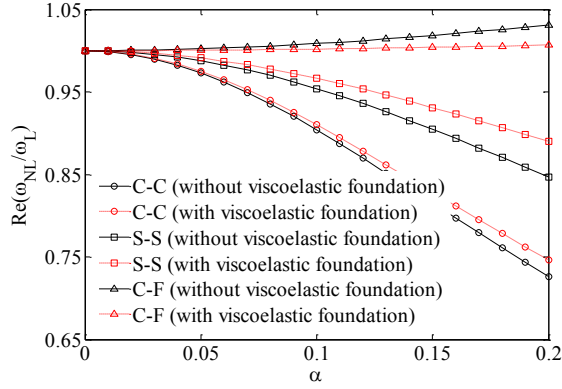


Fig. 2 The effect of nonlocal parameter  $\alpha$  on the real parts of the first frequency ratios  $\omega_{NL}/\omega_L$  for piezoelectric nanobeams with various boundary conditions.

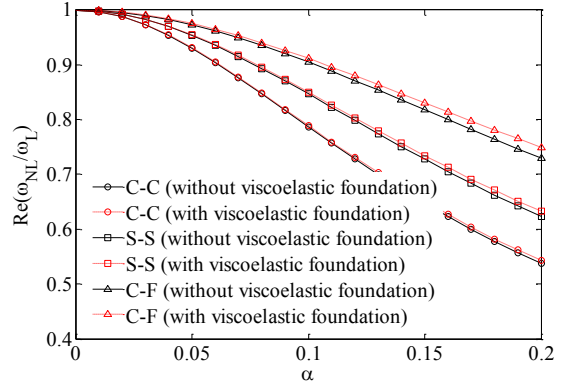


Fig. 3 The effect of nonlocal parameter  $\alpha$  on the real parts of the second frequency ratios  $\omega_{NL}/\omega_L$  for piezoelectric nanobeams with various boundary conditions.

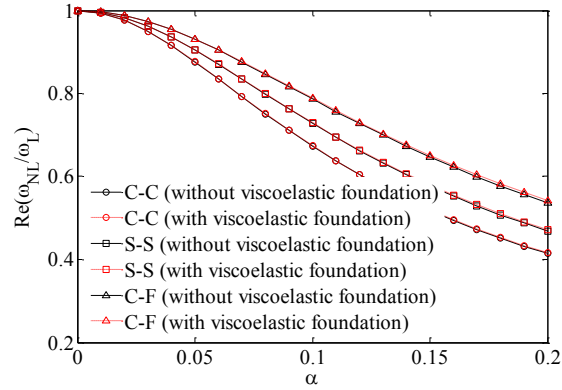


Fig. 4 The effect of nonlocal parameter  $\alpha$  on the real parts of the third frequency ratios  $\omega_{NL}/\omega_L$  for piezoelectric nanobeams with various boundary conditions.

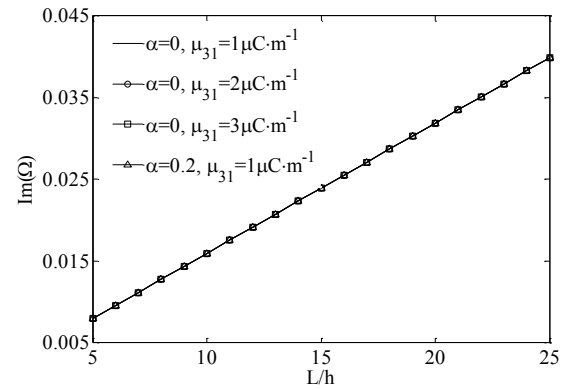


Fig. 5 The effect of slenderness ratio  $L/h$  on the imaginary parts of the first dimensionless frequencies for S-S piezoelectric nanobeams with various flexoelectric coefficient  $\mu_{31}$ .

Fig. 5 depicts the effect of slenderness ratio  $L/h$  on the imaginary parts of the first dimensionless frequencies for embedded S-S piezoelectric nanobeams with various flexoelectric coefficient  $\mu_{31}$ . In this case, the thickness of the nanobeam is taken as  $h=2\text{nm}$  and the length  $L$  changes to satisfy different slenderness ratio  $L/h$ . It can be observed from Fig. 5 that the imaginary parts of the first dimensionless frequencies increase almost linearly with an

increase in slenderness ratio  $L/h$ . The possible reason for this is that the damping effect of viscoelastic medium introduced into the system becomes more pronounced as the length  $L$  increases. Both flexoelectric coefficient  $\mu_{31}$  and nonlocal parameter  $\alpha$  have no effect on the imaginary parts of natural frequencies. Hence, the effect of slenderness ratio  $L/h$  on the real parts of the first three natural frequencies is mainly examined in Fig. 6 to Fig. 8. From the figures we can find that the real parts of the first three natural frequencies decrease significantly as slenderness ratio  $L/h$  increases for various flexoelectric coefficient  $\mu_{31}$ . This is because that the rigidity of the nanobeam is reduced with rising slenderness ratio  $L/h$ . In addition, the effect of slenderness ratio  $L/h$  turns out to be more substantial as flexoelectric coefficient  $\mu_{31}$  and mode number increase. For example, as  $L/h$  increases from 5 to 25 the first and the second natural frequencies decrease about 12.73 and 54.93 at  $\mu_{31}=1\mu\text{C}/\text{m}$  but about 40.91 and 165.17 at  $\mu_{31}=3\mu\text{C}/\text{m}$ . The significant effect of flexoelectric coefficient  $\mu_{31}$  also can be observed in Fig. 6 to Fig. 8, where the real parts of the first three natural frequencies increase significantly with rising  $\mu_{31}$ . This effect of  $\mu_{31}$  is found to be less pronounced as slenderness ratio  $L/h$  increases. For instance, as  $\mu_{31}$  changes from  $1\mu\text{C}/\text{m}$  to  $3\mu\text{C}/\text{m}$  the real parts of the first natural frequencies increase about 34.43 at  $L/h=5$  but only about 6.25 at  $L/h=25$ .

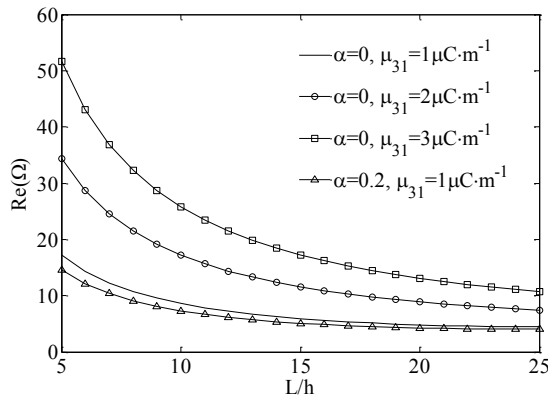


Fig. 6 The effect of slenderness ratio  $L/h$  on the real parts of the first dimensionless frequencies for S-S piezoelectric nanobeams with various flexoelectric coefficient  $\mu_{31}$ .

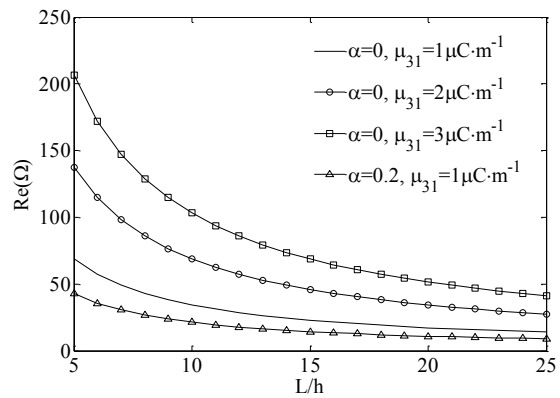


Fig. 7 The effect of slenderness ratio  $L/h$  on the real parts of the second dimensionless frequencies for S-S piezoelectric nanobeams with various flexoelectric coefficient  $\mu_{31}$ .



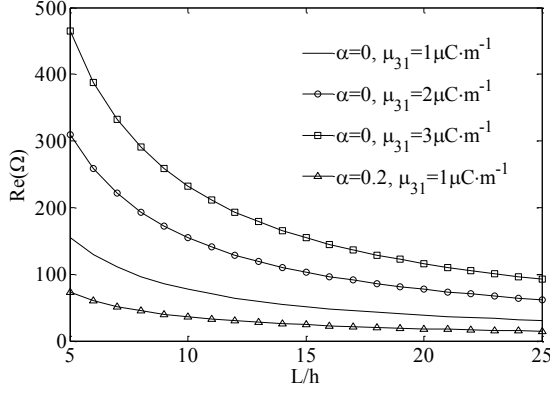


Fig. 8 The effect of slenderness ratio  $L/h$  on the real parts of the third dimensionless frequencies for S-S piezoelectric nanobeams with various flexoelectric coefficient  $\mu_{31}$ .

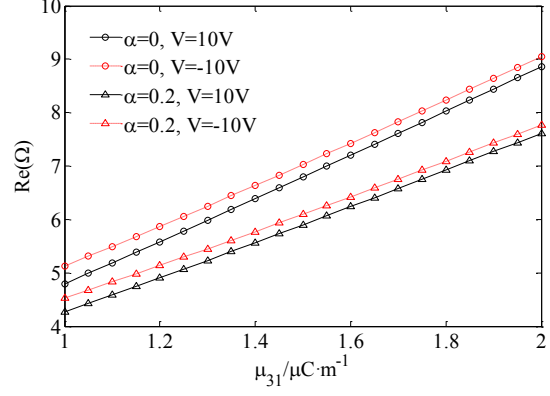


Fig. 9 The effect of flexoelectric coefficient  $\mu_{31}$  on the real parts of the first dimensionless frequencies for S-S piezoelectric nanobeams with various nonlocal parameter  $\alpha$  and electric voltage  $V$ .

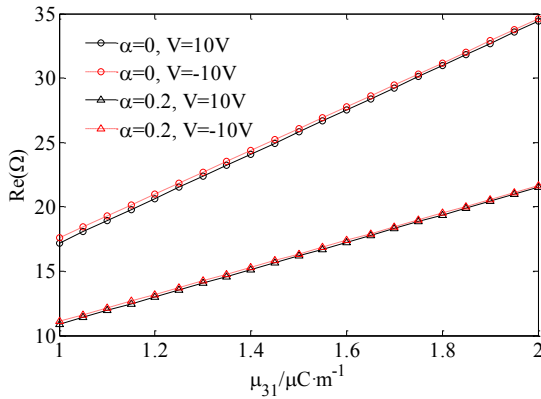


Fig. 10 The effect of flexoelectric coefficient  $\mu_{31}$  on the real parts of the second dimensionless frequencies for S-S piezoelectric nanobeams with various nonlocal parameter  $\alpha$  and electric voltage  $V$ .

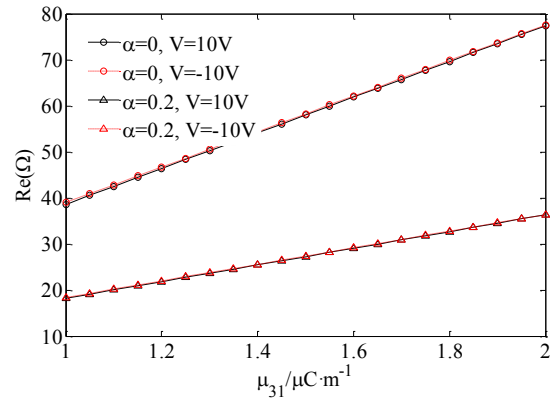


Fig. 11 The effect of flexoelectric coefficient  $\mu_{31}$  on the real parts of the third dimensionless frequencies for S-S piezoelectric nanobeams with various nonlocal parameter  $\alpha$  and electric voltage  $V$ .

Fig. 9 to Fig. 11 present the effect of flexoelectric coefficient  $\mu_{31}$  on the real parts of the first three dimensionless frequencies for embedded S-S piezoelectric nanobeams with various nonlocal parameter  $\alpha$  and electric voltage  $V$ . From the figures it can be observed that flexoelectric coefficient  $\mu_{31}$  has a strong influence on the natural frequencies of piezoelectric

nanobeams. The real parts of the first three dimensionless frequencies increase almost linearly as flexoelectric coefficient  $\mu_{31}$  increases. This implies that the rigidity of the embedded nanobeam is hardened due to enhanced flexoelectric effect. In addition, the effect of flexoelectric coefficient  $\mu_{31}$  turns out to be more substantial as nonlocal parameter  $\alpha$  decreases or mode number increases. For example, as  $\mu_{31}$  increases from  $1\mu\text{C}/\text{m}$  to  $2\mu\text{C}/\text{m}$  the first and the third natural frequencies with  $V=10\text{V}$  increase, respectively, about 3.35 and 18.01 at  $\alpha=0.2$  but about 4.06 and 38.57 at  $\alpha=0$ . Compared with nonlocal parameter  $\alpha$ , the effect of electric voltage  $V$  on the sensibility of natural frequencies to flexoelectric coefficient  $\mu_{31}$  is small.

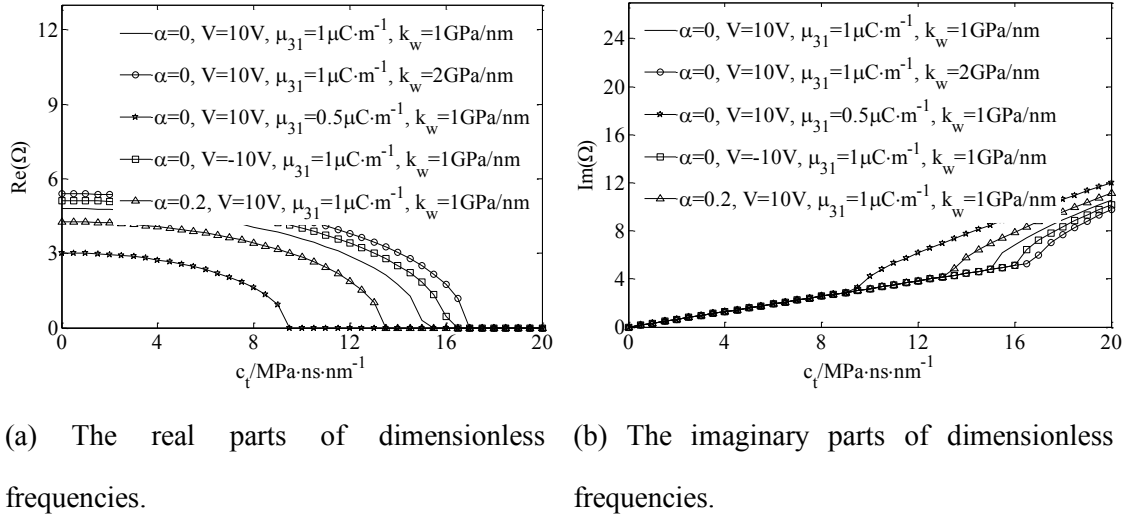


Fig. 12 The variation of the first dimensionless frequencies for S-S piezoelectric nanobeams with damping parameter  $c_t$ .

As the final numerical example, the effect of viscoelastic medium on the vibration responses of embedded piezoelectric nanobeams is examined. To this end, Fig. 12 and Fig. 13 show the variations of the first two complex natural frequencies versus damping parameter  $c_t$  for S-S piezoelectric nanobeams. It can be seen from Fig. 12 and Fig. 13 that the real parts of the first two natural frequencies remain zero as damping parameter  $c_t$  is larger than a certain value, which is denoted by  $(c_t)_{\text{crit}}$  to represent the critical value of  $c_t$  for nonoscillatory eigenfrequencies. Accordingly, a sharp change also can be observed in the imaginary parts of the first two natural frequencies. In addition, the value of  $(c_t)_{\text{crit}}$  increases significantly with an increase in Winkler's modulus parameter  $k_w$ , flexoelectric coefficient  $\mu_{31}$  and mode number or

a decrease in nonlocal parameter  $\alpha$  and electric voltage  $V$ . It also can be seen from the figures that as damping parameter  $c_t$  is smaller than  $(c_t)_{crit}$ , the real parts of the first two natural frequencies decrease nonlinearly with rising damping parameter  $c_t$ , but the imaginary parts increase almost linearly.

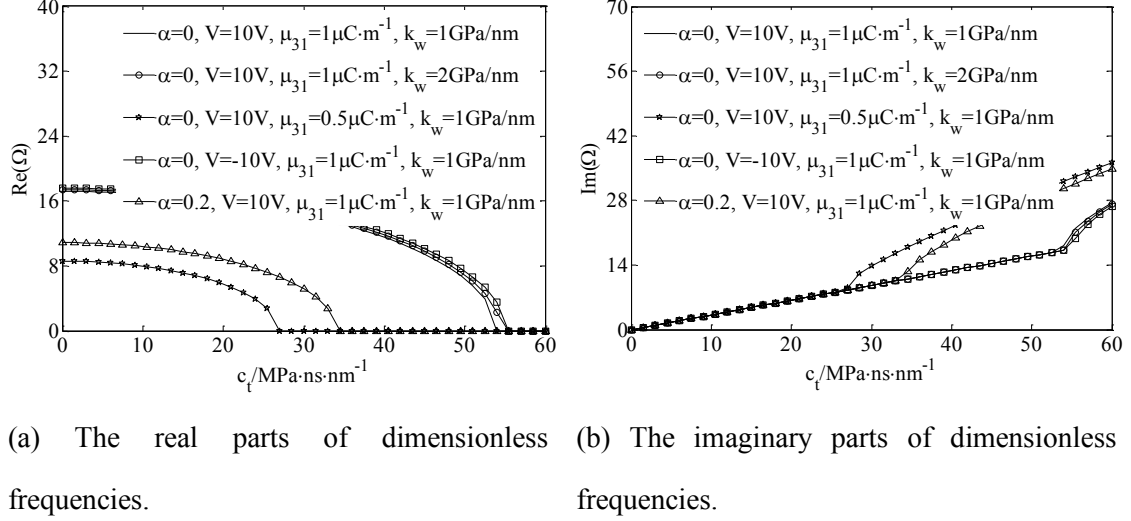


Fig. 13 The variation of the second dimensionless frequencies for S-S piezoelectric nanobeams with damping parameter  $c_t$ .

## 5. Conclusions

Based on nonlocal Euler-Bernoulli beam theory, the transverse vibration was investigated for a piezoelectric nanobeam embedded in viscoelastic medium with various boundary conditions. Considering nonlocal effect, piezoelectric effect and flexoelectric effect simultaneously, the governing equations of motion and boundary conditions for vibration analysis are first derived by using Hamilton's principle. Subsequently, the natural frequencies and corresponding mode shapes in closed form for embedded piezoelectric nanobeams with arbitrary boundary conditions were obtained by utilizing the transfer function method (TFM). In benchmark cases, the proposed model was validated by comparing the obtained results with those available in the literature, where good agreement has been achieved. In addition, a detailed parametric study was also conducted to investigate the effects of nonlocal parameter, boundary conditions, slenderness ratio, flexoelectric coefficient and viscoelastic medium on the vibration responses of nanobeams. Some of the key contributions made in this study

include:

- Novelty of this work includes simultaneous consideration of nonlocal effect, piezoelectric effect, flexoelectric effect, viscoelastic surrounding medium and electrical loadings for nanobeam dynamics.
- The increase in nonlocal parameter  $\alpha$  leads to a significant decrease in the fundamental frequencies of both C-C and S-S nanobeams but an increase in those of C-F nanobeams. Such effect of  $\alpha$  becomes more substantial with rising frequency modes and boundary condition stiffness.
- Flexoelectric coefficient  $\mu_{31}$  has a strong influence on vibration responses of nanobeams. The natural frequencies of the embedded nanobeams increase significantly as  $\mu_{31}$  increases. This effect of  $\mu_{31}$  turns out to be less pronounced as nonlocal parameter  $\alpha$  and slenderness ratio  $L/h$  increase or mode number decreases.
- Nonlocal parameter  $\alpha$ , boundary conditions, flexoelectric coefficient  $\mu_{31}$  and mode numbers have no effect on the imaginary parts of natural frequencies, which are only related to the damping of the viscoelastic surrounding medium.
- The critical values of damping parameter  $c_t$  to obtain nonoscillatory eigenfrequencies also can be determined by using the proposed method. Moreover, the value of  $(c_t)_{crit}$  increases significantly as  $k_w$ ,  $\mu_{31}$  and mode number increase or  $\alpha$  and  $V$  decrease.

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