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Tag der mündlichen Prüfung: ....

Multiple Permitting
and
Bounded Turing Reducibilities

Betreuer: Professor Dr. Klaus Ambos-Spies

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Abstract

We look at various properties of the computably enumerable (c.e.) not totally ω-c.e. Turing degrees. In particular, we are interested in the variant of multiple permitting given by those degrees. We define a property of left-c.e. sets called *universal similarity property* which can be viewed as a universal or uniform version of the property of array noncomputable c.e. sets of agreeing with any c.e. set on some component of a very strong array. Using a multiple permitting argument, we prove that the Turing degrees of the left-c.e. sets with the universal similarity property coincide with the c.e. not totally ω-c.e. degrees. We further introduce and look at various notions of so-called *universal array noncomputability* and show that c.e. sets with those properties can be found exactly in the c.e. not totally ω-c.e. Turing degrees and that they guarantee a special type of multiple permitting called *uniform multiple permitting.* We apply these properties of the c.e. not totally ω-c.e. degrees to give alternative proofs of well-known results on those degrees as well as to prove new results. E.g., we show that a c.e. Turing degree contains a left-c.e. set which is not cl-reducible to any complex left-c.e. set if and only if it is not totally ω-c.e. Furthermore, we prove that the nondistributive finite lattice $\mathcal{S}_7$ can be embedded into the c.e. Turing degrees precisely below any c.e. not totally ω-c.e. degree.

We further look at the question of join preservation for bounded Turing reducibilities r and r' such that r is stronger than r'. We say that join preservation holds for two reducibilities r and r' if every join in the c.e. r-degrees is also a join in the c.e. r'-degrees. We consider the class of monotone admissible (uniformly) bounded Turing reducibilities, i.e., the reflexive and transitive Turing reducibilities with use bounded by a function that is contained in a (uniformly computable) family of strictly increasing computable functions. This class contains for example identity bounded Turing (ibT-) and computable Lipschitz (cl-) reducibility. Our main result of Chapter 3 is that join preservation fails for cl and any strictly weaker monotone admissible uniformly bounded Turing reducibility. We also look at the dual question of meet preservation and show that for all monotone admissible bounded Turing reducibilities r and r' such that r is stronger than r', meet preservation holds. Finally, we completely solve the question of join and meet preservation in the classical reducibilities 1, m, tt, wtt and T.

Zusammenfassung

Wir betrachten verschiedene Eigenschaften der aufzählbaren nicht vollständig ω-aufzählbaren (not totally ω-c.e.) Turing-Grade. Wir interessieren uns insbesondere für die Varianten des multiplen Permittings, die diese Grade ermöglichen. Wir definieren eine Eigenschaft links-aufzählbarer Mengen, die wir *universelle Ähnlichkeitseigenschaft* (*universal similarity property*) nennen und die man als universelle oder uniforme Version jener Eigenschaft Array-nichtberechenbarer aufzählbarer Mengen, mit jeder aufzählbaren Menge auf einer Komponente eines Very-Strong-Arrays übereinzustimmen, auffassen kann. Mithilfe eines multiplen Permitting-Arguments beweisen wir, dass die Turing-Grade der links-aufzählbaren Mengen mit der universellen Ähnlichkeitseigenschaft mit den aufzählbaren nicht vollständig ω-aufzählbaren Turing-Graden übereinstimmen. Weiterhin definieren und betrachten wir verschiedene Begriffe der sogenannten *universellen Array-Nicht-berechenbarkeit* (*universal array noncomputability*) und zeigen, dass aufzählbare Mengen mit diesen Eigenschaften genau in den aufzählbaren nicht vollständig ω-aufzählbaren Turing-Graden liegen und dass sie eine spezielle Art des multiplen Permittings ermöglichen, die wir *uniformes multiples Permitting* (*uniform multiple permitting*) nennen. Wir wenden diese Eigenschaften der aufzählbaren nicht vollständig ω-aufzählbaren Turing-Grade an, um alternative Beweise bekannter Ergebnisse, die diese Grade betreffen, zu führen und um neue Ergebnisse zu beweisen. Beispielsweise zeigen wir, dass ein aufzählbarer Turing-Grad genau dann eine links-aufzählbare Menge enthält, die nicht cl-reduzierbar auf eine komplexe links-aufzählbare Menge ist, wenn er nicht vollständig ω-aufzähbar ist. Außerdem beweisen wir, dass der nichtdistributive endliche Verband $\mathcal{S}_7$ genau unterhalb jedes aufzählbaren nicht vollständig ω-aufzählbaren Grades in die aufzählbaren Turing-Grade eingebettet werden kann.

Wir betrachten außerdem die Frage nach der Join-Erhaltung für beschränkte Turing-Reduzierbarkeiten r und r' sodass r stärker als r' ist. Join-Erhaltung gilt für zwei Reduzierbarkeiten r und r', wenn jeder Join in den aufzählbaren r-Graden auch ein Join in den aufzählbaren r'-Graden ist. Wir betrachten die Klasse der monotonen zulässigen (uniform) beschränkten Turing-Reduzierbarkeiten, das heißt, der reflexiven und transitiven Turing-Reduzierbarkeiten, deren Use-Funktion durch eine Funktion beschränkt ist, die in einer (uniform berechenbaren) Familie streng monoton steigender berechenbarer Funktionen liegt. Diese Klasse enthält zum Beispiel die ibT-Reduzierbarkeit und die cl-Reduzierbarkeit. Das Hauptergebnis des dritten Kapitels besagt, dass Join-Erhaltung für cl und eine echt stärkere monotone zulässige uniform beschränkte Turing-Reduzierbarkeit nicht gelten kann. Wir gehen auch auf die duale Frage nach der Meet-Erhaltung ein und zeigen, dass diese für alle monotonen zulässigen beschränkten Turing-Reduzierbarkeiten r und r' gilt, sodass r stärker als r' ist. Abschließend beantworten wir die Frage nach der Join- und Meet-Erhaltung in den klassischen Reduzierbarkeiten 1, m, tt, wtt und T vollständig.

Contents

Chapter 1

Preliminaries

In this chapter, we give the most general definitions, notation and conventions used and assumed everywhere (unless mentioned otherwise). Further definitions will be given when needed. Familiarity with basic concepts of computability (see e.g. Soare [Soa87] or Downey and Hirschfeldt [DH10]) is assumed.

Natural numbers are denoted by lower case letters like x, y, z, x_0, x_1, x_2 or a, b, c. Letters s, t, u mostly refer to stages of constructions or approximations while letters f, g, h mostly denote total functions and lower case Greek letters like φ, ψ stand for partial functions. Finite strings are denoted by lower case Greek letters like σ, τ. Sets of natural numbers are denoted by upper case letters like A, B, C or A_0, A_1, A_2. We denote Turing functionals with upper case Greek letters (e.g., Φ, Ψ) and their use functions with the corresponding lower case Greek letters (e.g., φ, ψ).

We identify a set A of natural numbers with its characteristic function $A : \omega \to \{0, 1\}$ as well as with its characteristic sequence $A = A(0)A(1)A(2)\ldots$. $A \upharpoonright n$ denotes the finite string $A(0)A(1)\ldots A(n-1)$ or, depending of the context, the finite set $A \cap \{0, \ldots, n-1\}$. Unless stated otherwise, we assume that $e, n, s, t, x, y, z \geq 0$ and that $i \leq 1$.

We begin with reviewing the definition of computable approximations and enumerations.

Definition 1. *(a) A* computable approximation *of a set A is a sequence $\{A_s\}_{s\geq 0}$ of finite sets A_s such that $\lim_{s\to\infty} A_s(x) = A(x)$ for all $x \geq 0$ and such that there is a computable function f with $A_s = D_{f(s)}$ for all $s \geq 0$, i.e., $f(s)$ is the canonical index of A_s.*

(b) A function $\hat{f} : \omega \times \omega \to \omega$ is a computable approximation *of a unary function f if $\hat{f}$ is computable and $\lim_{s\to\infty} \hat{f}(x, s) = f(x)$ holds for all $x \geq 0$. If the choice of $\hat{f}$ is clear from the context, we let $f_s(x) = \hat{f}(x, s)$ for all x and s.*

(c) A computable enumeration *of a set A is a computable approximation of A such that for all x and s, $A_{s+1}(x) \geq A_s(x)$ holds.*

Unless mentioned otherwise, in the following, without loss of generality (w.l.o.g.) we assume that for a given computable approximation $\{A_s\}_{s\geq 0}$ of a set A, $A_s \subseteq \{0, \ldots, s-1\}$ holds for all s.

We turn to looking at universal functions and sets.

Definition 2. *(a) An $n+1$-ary partial computable function φ is* universal for the n-ary partial computable functions *(an n-*universal partial computable function*) if, for every n-ary partial* *computable function ψ, there is an index e such that for all $x_0, \ldots x_{n-1}$, $\varphi_e(x_0, \ldots, x_{n-1}) = \varphi(e, x_0, \ldots, x_{n-1}) = \psi(x_0, \ldots, x_{n-1})$ holds.*

(b) *A c.e. set W is* universal for the (unary) computably enumerable sets *(a* universal computably enumerable set*) if for every c.e. set A, there is an index e such that $W_e = \{x : \langle e, x\rangle \in W\} = A$.*

In the following, we sometimes use the term *universal function* to refer to a unary universal partial computable function. Unless mentioned otherwise, φ denotes the standard universal function obtained by goedelization of the unary Turing machines (for more details, refer to [Soa87]). We let φ_e denote the eth branch of φ, i.e., $\varphi_e(x) = \varphi(e, x)$ for all e and x. Note that each φ_e is a unary partial computable function. We may approximate φ in stages, i.e., we let $\varphi_{e,s}(x) = \varphi_e(x)$ if $\varphi_e(x)$ is defined within the first s steps and $\varphi_{e,s}(x)\uparrow$ otherwise for all e, s and x. W.l.o.g., we assume that for all e, x and s, if $\varphi_{e,s}(x)\downarrow$, then $e, x, \varphi_e(x) < s$ holds. We may now use the universal function φ to define a universal c.e. set by letting $W = \{\langle e, x\rangle : \varphi_e(x)\downarrow\}$. An approximation to W is given by $W_{e,s} = \{x : \varphi_{e,s}(x)\downarrow\}$.

We now give the definition of Turing functionals.

Definition 3. *(a) An* oracle Turing machine *is a Turing machine (for the formal definition of a Turing machine see [Soa87, pp. 11-13]) with an additional read-only tape which we call the oracle tape.*

(b) *A* Turing functional *is a partial function Φ_M : POWER$(\omega) \times \omega \to \omega$ computed by an oracle Turing machine M depending on the oracle set.*

(c) *The partial function computed by M with the oracle A is denoted by $\Phi_M^A : \omega \to \omega$.*

(d) *We define the* use function *φ of a Turing functional Φ by*

$$\varphi^X(x) = \begin{cases} y+1 & \textit{if } \Phi^X(x)\downarrow \textit{ and } y \textit{ is the greatest oracle query in the computation} \\ 0 & \textit{if } \Phi^X(x)\downarrow \textit{ and there are no oracle queries in the computation} \\ \uparrow & \textit{otherwise.} \end{cases}$$

(e) *We let $\{\Phi_e\}_{e\geq 0}$ be a standard enumeration of the Turing functionals obtained by goedelization of the oracle Turing machines and we let φ_e be the use function of Φ_e.*

We use computable enumerations of the functionals and of the corresponding use functions which are denoted by an additional index s given by the stage of the enumeration. So, for instance, $\Phi_{e,s}^X$ is the result of computing Φ_e^X for s steps and $\varphi_{e,s}^X$ is the corresponding use. Formally, we define:

$$\Phi_{e,s}^X(x) = \begin{cases} \Phi_e^X(x) & \text{if the computation of } \Phi_e^X(x) \text{ converges in } \leq s \text{ steps} \\ \uparrow & \text{otherwise.} \end{cases}$$

$$\varphi_{e,s}^X(x) = \begin{cases} \varphi_e^X(x) & \text{if the computation of } \Phi_e^X(x) \text{ converges in } \leq s \text{ steps} \\ \uparrow & \text{otherwise.} \end{cases}$$

We assume that if $\Phi_{e,s}^X(x)$ is defined then $e, x, \Phi_{e,s}^X(x), \varphi_{e,s}^X(x) < s$. Recall the well-known use principle that we will explicitly and implicitly use in many constructions.

Lemma 4. *For all sets A and B and for all Turing functionals Φ, the following holds.*

$$\Phi^A(x)\downarrow \;\&\; B\restriction\varphi^A(x) = A\restriction\varphi^A(x)$$
$$\Downarrow$$
$$\Phi^B(x) = \Phi^A(x) \;\&\; \varphi^B(x) = \varphi^A(x).$$

In the course of the following chapters, we look at the following reducibilities.

Definition 5. *(a) A set A is* one-one (1-) reducible *to a set B ($A \leq_1 B$) (*via f*) if there is a computable one-to-one function f such that $x \in A$ if and only if $f(x) \in B$.*

(b) A set A is many-one (m-) reducible *to a set B ($A \leq_{\mathrm{m}} B$) (*via f*) if there is a computable function f such that $x \in A$ if and only if $f(x) \in B$.*

(c) A set A is truth-table (tt-) reducible *to a set B ($A \leq_{\mathrm{tt}} B$) (*via g *and* h*) if there are computable functions $g : \omega \to \omega^*$ and $h : \omega \times \{0,1\}^* \to \{0,1\}$ such that for all x, if $g(x) = y_0, \ldots, y_n$, then $A(x) = h(x, B(y_0), \ldots, B(y_n))$.*

(d) A set A is Turing (T-) reducible *to a set B ($A \leq_{\mathrm{T}} B$) if there is a Turing functional Φ such that $\Phi^B(x) = A(x)$ for all $x \geq 0$.*

(e) A set A is f-bounded Turing (f-T) reducible *to a set B ($A \leq_{f-\mathrm{T}} B$) if there is a Turing functional Φ such that $\Phi^B(x) = A(x)$ and $\varphi^B(x) \leq f(x) + 1$ for all x.*

(f) A set A is weak truth-table (wtt-) reducible *to a set B ($A \leq_{\mathrm{wtt}} B$) if A is f-bounded Turing reducible to B for some computable function f.*

(g) A set A is linearly bounded Turing (lbT-) reducible *to a set B ($A \leq_{\mathrm{lbT}} B$) if A is f-bounded Turing reducible to B for a linearly bounded function f, i.e., for a function f satisfying $f(x) \leq k_0 x + k_1$ for some numbers $k_0, k_1 > 0$ and all numbers $x \geq 0$.*

(h) A set A is (i+k)-bT reducible *to a set B ($A \leq_{(\mathrm{i}+k)\mathrm{bT}} B$) if A is f-bounded Turing reducible to B for $f(x) = x + k$ ($k \geq 0$).*

(i) A set A is computably Lipschitz (cl-) reducible *to a set B ($A \leq_{\mathrm{cl}} B$) if A is* (i+k)*-bT-reducible to B for some $k \geq 0$.*

(j) A set A is identity bounded Turing (ibT-) reducible *to a set B ($A \leq_{\mathrm{ibT}} B$) if A is f-bounded Turing reducible to B for the identity function $f(x) = x$.*

(k) We call the r-reducibilities for $r = \mathrm{ibT}, \mathrm{cl}$ strongly bounded Turing reducibilities.

Convention: In the following, r will, if not stated otherwise, refer to any of the reducibilities $r = 1$, m, tt, T, wtt, lbT, cl, ibT. For reducibilities $\leq_r$ which are preorderings (i.e., reflexive and transitive), we define r-equivalence by

$$A =_r B \Leftrightarrow (A \leq_r B \;\&\; B \leq_r A).$$

Then $=_r$ is an equivalence relation. We call the equivalence class of a set A under $=_r$ the r-*degree* of A:

$$deg_r(A) = \{B : B =_r A\}.$$

An r-degree is c.e. if it contains a c.e. set. The partial ordering of the c.e. r-degrees is denoted by $(\mathbf{R}_r, \leq)$ and c.e. r-degrees are denoted by bold face lowercase letters ($\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\ldots$). For $r \in$ {tt, T, wtt, lbT, cl, ibT}, we let $\mathbf{0}_r$ denote the r-degree of the computable sets. If the reducibility r is clear from the context, we sometimes write $\mathbf{0}$ for $\mathbf{0}_r$. We slightly abuse notation by writing $f \leq_r \mathbf{a}$ and $A \leq_r \mathbf{a}$ for a function f, a set A and an r-degree $\mathbf{a}$ when we mean $deg_r(f) \leq \mathbf{a}$ and $deg_r(A) \leq \mathbf{a}$, respectively. We will sometimes only use *degree* to refer to a Turing degree.

A property P of (c.e.) sets is called *r-invariant* if for all (c.e.) sets A and B such that $P(A)$ and $A =_r B$ hold, $P(B)$ holds, as well. An r-degree $\mathbf{a}$ *bounds* a set A if $A \leq_r \mathbf{a}$ holds. For a class $\mathcal{C}$ of sets of natural numbers and a reducibility r, we call a set A *r-hard for $\mathcal{C}$* if $B \leq_r A$ holds for all $B \in \mathcal{C}$ and we call A *r-complete for $\mathcal{C}$* if A is r-hard for $\mathcal{C}$ and $A \in \mathcal{C}$. An r-degree is *r-hard* (*r-complete*) *for $\mathcal{C}$* if it contains a set which is r-hard (r-complete) for $\mathcal{C}$. In the following, we simply say *r-hard* and *r-complete* instead of r-hard and r-complete for the class of the c.e. sets, respectively.

We further recall the definitions of meets, joins and (semi-) lattices, lattice embeddings as well as maximal and minimal pairs.

Definition 6. *Let $(\mathcal{P}, \leq_{\mathcal{P}})$ be a partial ordering.*

(a) Two elements $a_0, a_1 \in \mathcal{P}$ have a greatest lower bound *or* meet *if there is an element $b \in \mathcal{P}$ such that $b \leq_{\mathcal{P}} a_0, a_1$ and such that for all $c \in \mathcal{P}$ with $c \leq_{\mathcal{P}} a_0, a_1$, $c \leq_{\mathcal{P}} b$ holds. We then write $a_0 \wedge a_1 = b$.*

(b) Two elements $a_0, a_1 \in \mathcal{P}$ have a least upper bound *or* join *if there is an element $b \in \mathcal{P}$ such that $a_0, a_1 \leq_{\mathcal{P}} b$ and such that for all $c \in \mathcal{P}$ with $a_0, a_1 \leq_{\mathcal{P}} c$, $b \leq_{\mathcal{P}} c$ holds. We then write $a_0 \vee a_1 = b$.*

(c) A partial ordering is a lower semilattice *if every pair of elements has a meet and it is an* upper semilattice *if every pair of elements has a join.*

(d) A partial ordering is a lattice *if it is a lower semilattice and an upper semilattice.*

(e) An upper semilattice $(\mathcal{U}, \leq_{\mathcal{U}})$ is distributive *if for all elements a_0, a_1 and b of $\mathcal{U}$, if $b \leq_{\mathcal{U}} a_0 \vee a_1$, then there are elements $b_0, b_1 \in \mathcal{U}$ such that $b_0 \leq_{\mathcal{U}} a_0$, $b_1 \leq_{\mathcal{U}} a_1$ and $b = b_0 \vee b_1$ hold and it is* nondistributive *otherwise.*

(f) A lattice $(\mathcal{L}, \leq_{\mathcal{L}})$ is modular *if for all elements a, b and c of $\mathcal{L}$, if $a \leq_{\mathcal{L}} c$, then $a \vee (b \wedge c) = (a \vee b) \wedge c$.*

Note that every distributive lattice is modular.

Definition 7. *Let $(\mathcal{L}, \leq_{\mathcal{L}})$ be a lattice and let $(\mathcal{U}, \leq_{\mathcal{U}})$ be an upper semilattice. A function $p : \mathcal{L} \to \mathcal{U}$ is called a* lattice embedding *(*of $\mathcal{L}$ into $\mathcal{U}$*) if p is one-to-one and the following hold for all $a, a_0, a_1, b \in \mathcal{L}$.*

(i) $a \leq_{\mathcal{L}} b$ if and only if $p(a) \leq_{\mathcal{U}} p(b)$.

(ii) If $a_0 \wedge a_1 = b$, then $p(a_0) \wedge p(a_1) = p(b)$ and if $a_0 \vee a_1 = b$, then $p(a_0) \vee p(a_1) = p(b)$.

A lattice embedding $p : \mathcal{L} \to \mathcal{U}$ *is* zero-preserving *if* $\mathcal{L}$ *has a least element* a *and* $p(a)$ *is the least element of* $\mathcal{U}$*.*

For a lattice $(\mathcal{L}, \leq_{\mathcal{L}})$ and an upper semilattice $(\mathcal{U}, \leq_{\mathcal{U}})$, we say that $\mathcal{L}$ can be *embedded into* $\mathcal{U}$ if there is a lattice embedding of $\mathcal{L}$ into $\mathcal{U}$ and, for $c \in C$, we say that $\mathcal{L}$ can be *embedded into* $\mathcal{U}$ *below* c or that c *bounds a (lattice) embedding of* $\mathcal{L}$ if there is a lattice embedding p of $\mathcal{L}$ into $\mathcal{U}$ such that $p(a) \leq_{\mathcal{U}} c$ for all $a \in \mathcal{L}$.

Definition 8. *For a reducibility* r*, two c.e. (left-c.e.)* r*-degrees* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *are called a* maximal pair in the c.e. (left-c.e.) r-degrees *or an* r-maximal pair in the c.e. (left-c.e.) degrees *if they do not have a common upper bound, i.e., if there is no c.e. (left-c.e.)* r*-degree* $\mathbf{b}$ *such that* $\mathbf{a}_0, \mathbf{a}_1 \leq \mathbf{b}$ *holds. If, for two c.e. (left-c.e.) sets* A_0 *and* A_1*,* $deg_r(A_0)$ *and* $deg_r(A_1)$ *form a maximal pair in the c.e. (left-c.e.)* r*-degrees, we also call* A_0 *and* A_1 *an* r-maximal pair in the c.e. (left-c.e.) sets*.*

Definition 9. *For a reducibility* r*, two nonzero c.e.* r*-degrees* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *are called a* minimal pair in the (c.e.) r-degrees *or an* r-minimal pair *if* $\mathbf{a_0} \wedge \mathbf{a_1} = \mathbf{0}_r$*. If, for two noncomputable c.e. sets* A_0 *and* A_1*,* $deg_r(A_0)$ *and* $deg_r(A_1)$ *form a minimal pair in the (c.e.)* r*-degrees, we also call* A_0 *and* A_1 *an* r-minimal pair*.*

Finally, for some constructions, we use the full binary tree $T = \{0, 1\}^{<\omega}$ to model guesses about requirements. Here, the elements of T are called *nodes* and $|\alpha|$ denotes the length of node α. For nodes α and β, α is *below* β and β is *above* α ($\alpha \sqsubset \beta$) if α is a proper initial segment of β. We write $\alpha \sqsubseteq \beta$ if $\alpha \sqsubset \beta$ or $\alpha = \beta$. We say that α is to the *left* of β and β is to the *right* of α ($\alpha <_{\mathrm{L}} \beta$) if there is a node γ such that $\gamma 0 \sqsubseteq \alpha$ and $\gamma 1 \sqsubseteq \beta$. Finally, we say that α is *less* than β ($\alpha < \beta$ or $\beta > \alpha$) if $\alpha <_{\mathrm{L}} \beta$ or $\alpha \sqsubset \beta$.

Chapter 2

Not Totally ω-C.E. Degrees and Multiple Permitting

2.1 Introduction

Permitting arguments are widely used in all kinds of constructions in computability theory. If we wish to construct a set A below a given – or equally constructed – set B, we make sure that the approximation of A can only change below some $x+1$ if the approximation of B changes below $f(x)+1$ for some appropriate computable function f. The most straightforward version of permitting is guaranteed by any approximation of any noncomputable set B. It is argued that, given a strictly increasing computable sequence of numbers $x_0, x_1, \ldots$ together with a strictly increasing computable sequence of stages $s_0, s_1, \ldots$, then for some x_n, B has to change below x_n+1 after stage s_n, i.e., permitting has to be given by B for some x_n at some point, otherwise the set would be computable. However, this version of permitting is only sufficient if the construction requires *one* change of the constructed set to make sure a single requirement is met. In more involved constructions, however, we need to successively change A on several numbers or intervals to meet one requirement. Here, we need what is called *multiple permitting*.

When examining constructions involving multiple permitting, Downey, Jockusch and Stob [DJS90] introduced the class of *array noncomputable* (*a.n.c.*) c.e. sets and showed that those sets allow certain multiple permitting constructions. They exploited this fact to investigate properties of the c.e. sets which are a.n.c. Building on this approach, many properties where discovered that can be captured with constructions using the multiple permitting given by a.n.c. c.e. sets. E.g., it has been shown by Barmpalias, Downey and Greenberg [BDG10] that a c.e. Turing degree **a** contains a left-c.e. set which is not cl-reducible to any random left-c.e. set if and only if **a** is a.n.c., i.e., if **a** contains an a.n.c. c.e. set. This extends the result by Barmpalias and Lewis [BL06a] that there exists a left-c.e. set that is not cl-reducible to any random left-c.e. set. The notion of cl-reducibility has been introduced by Downey, Hirschfeldt and LaForte ([DHL01], [DHL04]) and is closely related to randomness notions. For more background on cl-reducibility, refer to the introduction to Chapter 3.

Another interesting class of degrees related to multiple permitting is the class of the c.e. not totally ω-c.e. degrees. They were introduced by Downey, Greenberg and Weber [DGW07] and are a subclass of the a.n.c. c.e. degrees. They have shown that the c.e. not totally ω-c.e. degrees are exactly the c.e. degrees below which there is a critical triple in the c.e. Turing degrees. Thus, these degrees are definable in the c.e. Turing degrees. The question whether the same holds for the a.n.c. c.e. degrees is open. The c.e. not totally ω-c.e. degrees allow a stronger version of multiple permitting than the one given by the a.n.c. c.e. degrees. This is often referred to as *not-totally-ω-c.e. permitting* and widely used in the literature. E.g., the proof that any c.e. not totally ω-c.e. degree bounds a critical triple in the c.e. Turing degrees uses it. Later, Barmpalias, Downey and Greenberg [BDG10] applied it to show that in any c.e. not totally ω-c.e. degree, there is a set which is not wtt-reducible to any hypersimple set. Another application can be found in [BDN12], where Brodhead, Downey and Ng show that there is a computably bounded random set in any c.e. not totally ω-c.e. degree.

When analyzing various constructions using not-totally-ω-c.e. permitting mentioned above, we find that they all follow a similar pattern. Usually, a known construction of a set with a certain property is changed in a way that it can be combined with not-totally-ω-c.e. permitting and possibly with coding. Our goal is to isolate the permitting and coding, which hitherto had to be explicitly taken care of, from the actual constructions. We define a property of left-c.e. sets based on an extension of array noncomputability from c.e. sets to the left-c.e. sets called the *universal similarity property* and prove that any c.e. not totally ω-c.e. degree contains a left-c.e. set with this property. To see this, we use exactly the well-know approach of combining the construction of such a set with not-totally-ω-c.e. permitting and with coding. Then, it can be shown that all left-c.e. sets with the universal similarity property have most of the special properties mentioned above. For that matter, it is enough to analyze the basic constructions and to show that any requirement of the construction can be met within an interval of computably bounded length. This is a modular approach to dealing with c.e. not totally ω-c.e. degrees. The left-c.e. sets with the universal similarity property can thus be viewed as generic sets for the c.e. not totally ω-c.e. degrees.

We exploit this observation to obtain a new result on the c.e. not totally ω-c.e. degrees, namely that they are exactly the c.e. degrees which contain a left-c.e. set which is not cl-reducible to any left-c.e. complex set. This result has been conjectured by Greenberg and it parallels the theorem by Barmpalias, Downey and Greenberg on a.n.c. c.e. degrees and cl-reducibility to random left-c.e. sets. To prove our result, we transfer it to an equivalent theorem on wtt-hard sets and maximal pairs in the left-c.e. ibT-degrees where ibT-reducibility has been introduced by Soare [Soa04]. For more details, again, refer to the introduction to Chapter 3. Maximal pairs in the c.e. as well as in the left-c.e. ibT-degrees have been extensively studied. The existence of ibT-maximal pairs in the c.e. sets has been shown by Barmpalias and, independently, by Fan and Lu [FL05] and the existence of ibT-maximal pairs in the left-c.e. sets has been shown by Yu and Ding [YD04]. Ambos-Spies, Ding, Fan and Merkle [ASDFM13] have shown that the Turing degrees containing halves of ibT-maximal pairs in the c.e. sets are just the array noncomputable c.e. degrees (and, similarly, for the wtt-degrees). In contrast to this, Fan and Yu [FY11] have shown that any left-c.e. set is half of an ibT-maximal pair in the left-c.e. sets. The Fan-Yu result implies that there is an ibT-maximal pair in the left-c.e. sets where one of the halves is c.e. This fact had previously been shown by Fan [Fan09] already, by using a more direct argument. By an observation of Downey and

Hirschfeldt [DH10], however, no pair of c.e. sets is ibT-maximal in the left-c.e. sets. In [ASLM], Ambos-Spies, Losert and Monath have extended Fan's result. We combine this extension with the existence of left-c.e. sets with the universal similarity property in c.e. not totally ω-c.e. degrees to prove the already mentioned result that each c.e. not totally ω-c.e. degree contains a left-c.e. set which is not cl-reducible to any complex left-c.e. set.

We will also introduce various notions of universal array noncomputability that demonstrate the fact that the property of a c.e. degree to be not totally ω-c.e. can be viewed as a uniform or universal version of a c.e. degree being a.n.c. We show that the notions coincide with each other up to wtt-equivalence and that the T-degrees of c.e. sets with these notions are exactly the c.e. not totally ω-c.e. T-degrees. We also use the notions of universal array noncomputability to show that the c.e. not totally ω-c.e. degrees capture exactly the notion of *uniform multiple permitting* which we define based on the formalization of multiple permitting notions by Ambos-Spies in [ASa]. Finally, we give an application of uniform multiple permitting by showing that the nondistributive finite lattice $\mathcal{S}_7$, which contains a critical triple, can be embedded into the c.e. Turing degrees exactly below every c.e. not totally ω-c.e. degree.

The outline of this chapter is as follows. In Section 2.2, we give basic definitions and facts and review important results from the literature on a.n.c. c.e. degrees as well as on c.e. not totally ω-c.e. degrees. In Section 2.3, we show that a c.e. Turing degree contains a left-c.e. set with the universal similarity property if and only if it is not totally ω-c.e. We then apply this result to give an alternative proof of the Brodhead-Downey-Ng result on computably bounded random sets and to prove Greenberg's conjecture using a result on maximal pairs in the left-c.e. ibT-degrees. In Section 2.4, we define several notions of universal array noncomputablity for c.e. sets. We make some basic observations on those notions and see that they capture the notion of uniform multiple permitting. Furthermore, we show that the different notions coincide with each other up to wtt-equivalence. We proceed to prove that c.e. sets with the various universal array noncomputability properties can be found exactly in the c.e. not totally ω-c.e. T-degrees. Finally, we apply uniform multiple permitting to show that the lattice $\mathcal{S}_7$ can be embedded below any c.e. not totally ω-c.e. Turing degree.

2.2 Preliminaries

In this section, we give more background and details on the notions we will have a closer look at in this chapter. Many notions and results in this and in the next section have been developed together with the Heidelberg Logic Group (Klaus Ambos-Spies, Nan Fang, Wolfgang Merkle and Martin Monath) and some of them will appear in the forthcoming paper [ASLM] by Ambos-Spies, Losert and Monath. We begin with a few basic definitions.

Definition 10. *(a) A computable approximation* $\{A_s\}_{s\geq 0}$ *of a set* A *is a* computable almost-enumeration *of* A *if the following holds.*

$$\forall x \forall s (x \in A_s \setminus A_{s+1} \Rightarrow \exists y < x (y \in A_{s+1} \setminus A_s)).$$

(b) A set A *is* almost-computably enumerable *(*almost-c.e. *or* a.c.e. *for short) if there is a computable almost-enumeration of* A.

As one can easily see, a real α is left-c.e. if and only if the set A such that $\alpha = 0.A$ is almost-c.e. So we may identify left-c.e. reals, left-c.e. sets and almost-c.e. sets in the following. Recall the following definition of a very strong array by Downey, Jockusch and Stob.

Definition 11 ([DJS90])**.** *A sequence $\mathcal{F} = \{F_n\}_{n\geq 0}$ of finite sets is a* very strong array *(or* v.s.a. *for short) if the following hold.*

- *There is a computable function f such that $F_n = D_{f(n)}$, i.e., $f(n)$ is the canonical index of F_n.*
- *$F_n \cap F_m = \emptyset$ if $n \neq m$.*
- *$F_0 \neq \emptyset$ and for all $n \geq 0$, $|F_{n+1}| > |F_n|$.*

For all n, we call F_n a component *of $\mathcal{F}$.*

Note that in the original definition in [DJS90] the components F_n of a very strong array are required to form a partition of the natural numbers. Here, we follow Downey and Hirschfeldt [DH10] and drop this requirement. We will later see that this does not make a difference when it comes to defining array noncomputablity. However, we need the following definition of special types of very strong arrays.

Definition 12. *Let $\mathcal{F} = \{F_n\}_{n\geq 0}$ be a v.s.a.*

(a) $\mathcal{F}$ is complete *(a* c.v.s.a.*) if $\bigcup_{n\geq 0} F_n = \omega$ holds.*

(b) $\mathcal{F}$ is a very strong array of intervals *(*v.s.a.i.*) if for all n, F_n is an interval and $\max F_n < \min F_{n+1}$ holds.*

(c) A complete very strong array of intervals *(*c.v.s.a.i.*) is a v.s.a.i. which is complete.*

By definition, any v.s.a. is given by a single computable function. This implies that we may uniformly computably enumerate all very strong arrays (together with the initial segments of very strong arrays) as well as the complete very strong arrays of intervals (again, including initial segments) in the following way.

For a given partial function ψ, let $F_0^\psi \downarrow = D_{\psi(0)}$ if $\psi(0)\downarrow$ and $D_{\psi(0)} \neq \emptyset$ and for given n, let $F_{n+1}^\psi \downarrow = D_{\psi(n+1)}$ if F_n^ψ and $\psi(n+1)$ are defined, $|D_{\psi(n+1)}| > |F_n^\psi|$ and for all $n' \leq n$, $D_{\psi(n+1)} \cap F_{n'}^\psi = \emptyset$. Moreover, for the case of complete very strong arrays of intervals, we let $I_0^\psi \downarrow = F_0^\psi$ if $F_0^\psi\downarrow$ and F_0^ψ is an interval with $\min F_0^\psi = 0$ and, for $n \geq 0$, we let $I_{n+1}^\psi \downarrow = F_{n+1}^\psi$ if $I_n^\psi\downarrow$, $F_{n+1}^\psi\downarrow$ and F_{n+1}^ψ is an interval such that $\min F_{n+1}^\psi = \max I_n^\psi + 1$. A uniform approximation of the F_n^ψ is obtained by letting $F_{n,s}^\psi = F_n^\psi$ if $F_n^\psi\downarrow$ and $\psi_s(m)\downarrow$ for all $m \leq n$ and letting $F_{n,s}^\psi\uparrow$ otherwise for all n. Similarly for I_n^ψ.

Now, let $\mathcal{F}^\psi = \{F_n^\psi\}_{n\geq 0}$ if $F_n^\psi\downarrow$ for all n (then, $\mathcal{F}^\psi$ is a v.s.a.), otherwise let $\mathcal{F}^\psi = \{F_n^\psi\}_{n<m}$ for the least m such that $F_m^\psi\uparrow$ (then, $\mathcal{F}^\psi$ is an initial segment of a v.s.a.) and define $\mathcal{I}^\psi$ analogously. Given a universal function φ, we obtain enumerations $\{\mathcal{F}^{\varphi_e}\}_{e\geq 0}$ and $\{\mathcal{I}^{\varphi_e}\}_{e\geq 0}$ of all very strong arrays and their initial segments and of all complete very strong arrays of intervals and their initial segments, respectively. In the following, if the choice of the universal function φ is clear from the context, we write F_n^e, I_n^e, $\mathcal{F}^e$ and $\mathcal{I}^e$ in place of $F_n^{\varphi_e}$, $I_n^{\varphi_e}$, $\mathcal{F}^{\varphi_e}$ and $\mathcal{I}^{\varphi_e}$, respectively.

Definition 13. *For a very strong array* $\mathcal{F} = \{F_n\}_{n\geq 0}$*, two sets* A *and* B *are* $\mathcal{F}$-similar *(denoted by* $A \sim_{\mathcal{F}} B$*) if the following holds.*

$$\exists^\infty n : A \cap F_n = B \cap F_n.$$

We are now ready to define array noncomputability following Downey, Jockusch and Stob.

Definition 14 ([DJS90])**.** *(a) Given a very strong array* $\mathcal{F}$*, a c.e. set* A *is* $\mathcal{F}$-array noncomputable *(*$\mathcal{F}$-a.n.c.*) if, for every c.e. set* B*,* A *is* $\mathcal{F}$*-similar to* B*.*

(b) A c.e. set A *is* array noncomputable *(*a.n.c.*) if there is a v.s.a.* $\mathcal{F}$ *such that* A *is* $\mathcal{F}$*-a.n.c.*

(c) A c.e. degree $\mathbf{a}$ *is* array noncomputable *(*a.n.c.*) if there is an a.n.c. c.e. set* $A \in \mathbf{a}$.

Note that a c.e. set which is array noncomputable is not computable. The requirement to make a c.e. set a.n.c. may be weakened as follows.

Proposition 15 ([DJS90])**.** *Let* $\mathcal{F} = \{F_n\}_{n\geq 0}$ *be a v.s.a. and let* A *be a c.e. set such that, for any c.e. set* B*, the following holds.*

$$\exists n \; (A \cap F_n = B \cap F_n).$$

Then A *is* $\mathcal{F}$*-a.n.c.*

The property of array noncomputable c.e. sets of being $\mathcal{F}$-similar to any given c.e. set for some v.s.a. $\mathcal{F}$ has been exploited to prove that a.n.c. c.e. sets have various properties. E.g., they are not maximal (as shown by Downey, Jockusch and Stob in [DJS90]; for more details, refer there). Moreover, properties of *almost*-c.e. sets of a.n.c. c.e. degree have been studied in the past, e.g., the following connection between array noncomputability and cl-reducibility to random almost-c.e. sets established by Barmpalias, Downey and Greenberg. We first recall the definition of (Martin-Löf) randomness.

Definition 16. *(a) A* Martin-Löf test *(or* ML-test *for short) is a uniformly c.e. sequence* $\{U_n\}_{n\geq 0}$ *of c.e. sets* $U_n \subseteq \{0,1\}^*$ *such that, for* $n \geq 0$*,* $\mu([U_n]) < 2^{-n}$*.*

(b) A real $\alpha \in \{0,1\}^\omega$ passes *an ML-test* $\{U_n\}_{n\geq 0}$ *if* $\alpha \notin \bigcap_{n\geq 0}[U_n]$*; and* $\{U_n\}_{n\geq 0}$ covers α *otherwise.*

(c) A set A passes *the ML-test* $\{U_n\}_{n\geq 0}$ *(is* covered *by* $\{U_n\}_{n\geq 0}$*) if the characteristic sequence* α *of* A *passes the ML-test* $\{U_n\}_{n\geq 0}$ *(is covered by* $\{U_n\}_{n\geq 0}$*).*

(d) A real α *(set* A*) is* (Martin-Löf) random *(or (*ML-*)* random *for short) if* α *(*A*) passes all ML-tests.*

Theorem 17 ([BDG10])**.** *For a c.e. degree* $\mathbf{a}$*, the following are equivalent.*

(i) $\mathbf{a}$ *is a.n.c.*

(ii) There is an almost-c.e. set in $\mathbf{a}$ *that is not cl-reducible to any random almost-c.e. set.*

As, by Definition 14, array noncomputablity is only defined for c.e. sets (though the definition can be extended; see [DJS96]) and a.n.c. c.e. sets are only $\mathcal{F}$-similar to all c.e. sets for some v.s.a. $\mathcal{F}$, this cannot be exploited to prove any results on *almost*-c.e. sets of a.n.c. c.e. degree. We will later review a characterization of the a.n.c. c.e. degrees that has been used to prove results like Theorem 17. However, it seems promising to transfer the similarity property from Definition 14 to the case of almost-c.e. set. This cannot be done in a straightforward way, as it can be shown that for any v.s.a. $\mathcal{F}$, there is no almost-c.e. set which is $\mathcal{F}$-similar to *all* almost-c.e. sets. Therefore, we consider sets that are locally almost-c.e. with respect to a fixed v.s.a. Formally, we have the following definition.

Definition 18. *Let $\mathcal{F} = \{F_n\}_{n\geq 0}$ be a very strong array.*

(a) A computable approximation $\{A_s\}_{s\geq 0}$ is $\mathcal{F}$-compatible *if the following hold.*

$$\forall n \forall x \in F_n \forall s (x \in A_s \setminus A_{s+1} \Rightarrow \exists y \in F_n (y < x \ \& \ y \in A_{s+1} \setminus A_s)),$$
$$\forall x \forall s (x \notin \bigcup_{n\geq 0} F_n \Rightarrow A_s(x) \leq A_{s+1}(x)).$$

(b) A set A is $\mathcal{F}$-compatibly almost-c.e. *(*$\mathcal{F}$-almost-c.e. *or* $\mathcal{F}$-a.c.e. *for short) if there is an $\mathcal{F}$-compatible computable approximation of A.*

(c) A set A is purely $\mathcal{F}$-compatibly almost-c.e. *(*purely $\mathcal{F}$-almost-c.e. *or* purely $\mathcal{F}$-a.c.e. *for short) if A is $\mathcal{F}$-a.c.e. and $A \subseteq \bigcup_{n\geq 0} F_n$ holds.*

Note that for any v.s.a. $\mathcal{F}$, any $\mathcal{F}$-compatible computable approximation is a computable almost-enumeration, hence any $\mathcal{F}$-almost-c.e. set is almost-c.e. Note that the converse is not true, i.e., there is an almost-c.e. set which is not $\mathcal{F}$-almost-c.e. for any v.s.a. $\mathcal{F}$. We may now define the following notion of array noncomputability for almost-c.e. sets, paralleling Definition 14.

Definition 19. *(a) An almost-c.e. set is* array noncomputable for the $\mathcal{F}$-almost-c.e. sets *(*$\mathcal{F}$-a.c.e.-a.n.c.*) for a v.s.a. $\mathcal{F}$ if it is $\mathcal{F}$-similar to any $\mathcal{F}$-a.c.e set.*

(b) An almost-c.e. set is array noncomputable for the almost-c.e. *sets (*a.c.e.-a.n.c.*) if it is $\mathcal{F}$-a.c.e.-a.n.c. for some v.s.a. $\mathcal{F}$.*

(c) A c.e. degree is array noncomputable for the almost-c.e. *sets (*a.c.e.-a.n.c.*) if it contains an a.c.e.-a.n.c. almost-c.e. set.*

Very similarly to Proposition 15, it does not matter whether we require an almost-c.e. set to agree with a given $\mathcal{F}$-a.c.e. set on infinitely many components of a v.s.a. $\mathcal{F}$ or on just one component to make it $\mathcal{F}$-a.c.e.-a.n.c.

Proposition 20 ([ASLM])**.** *Let $\mathcal{F} = \{F_n\}_{n\geq 0}$ be a v.s.a. and let A be an almost-c.e. set such that, for any $\mathcal{F}$-a.c.e. set B, the following holds.*

$$\exists n \ (A \cap F_n = B \cap F_n).$$

Then A is $\mathcal{F}$-a.c.e.-a.n.c.

It has been shown that the a.c.e.-a.n.c. c.e. degrees coincide with the a.n.c. c.e. degrees.

Theorem 21 ([ASLM])**.** *A c.e. (wtt- or T-) degree is a.c.e.-a.n.c. if and only if it is a.n.c.*

We will later see that this fact can be exploited to give an alternative proof of direction (i)⇒(ii) of Theorem 17. When studying the Turing degrees bounding critical triples, Downey, Greenberg and Weber introduced the notion of c.e. totally ω-c.e. degrees.

Definition 22 ([DGW07])**.** *(a) A function g is* h-c.e. *for a function h if there is a computable approximation of g such that following holds.*

$$\forall x \geq 0 \; (|\{s : g_{s+1}(x) \neq g_s(x)\}| \leq h(x)).$$

(b) A function g is ω-c.e. *if there is a computable function h such that g is h-c.e.*

(c) A c.e. Turing degree $\mathbf{a}$ *is* totally ω-c.e. *if every function $g \leq_{\mathrm{T}} \mathbf{a}$ is ω-c.e.*

Note that, in the above definition, w.l.o.g. we may assume that $g_0(x) = 0$ for $x \geq 0$ and that h is strictly increasing. In the following we tacitly make these assumptions. The relation between a.n.c. c.e. degrees and c.e. not totally ω-c.e. degrees can be deduced from the following characterization of the array noncomputable c.e. degrees given by Downey, Jockusch and Stob.

Lemma 23 ([DJS90], [DJS96])**.** *The following are equivalent for a c.e. degree* $\mathbf{a}$.

(i) $\mathbf{a}$ *is a.n.c.*

(ii) For every computable function h, there is a function $g \leq_{\mathrm{T}} \mathbf{a}$ that is not h-c.e.

Note that this implies that every c.e. degree which is not totally ω-c.e is a.n.c. This fact leads to the conjecture that there is a (uniform or universal) characterization of the c.e. not totally ω-c.e. degrees in terms of the similarity properties discussed above. This yields the following definition.

Definition 24. *An almost-c.e. set has the* universal similarity property *(*u.s.p.*) if it is $\mathcal{F}$-a.c.e.-a.n.c. for every v.s.a. $\mathcal{F}$.*

We will see in the next section that almost-c.e. sets with the universal similarity property can be found exactly in the c.e. not totally ω-c.e. Turing degrees. We will then exploit this fact to prove several theorems on c.e. not totally ω-c.e. degrees using the universal similarity property. For some constructions, it is more convenient to only consider complete very strong arrays of intervals. The following proposition, for which we say that a v.s.a. $\hat{\mathcal{F}} = \{\hat{F}_n\}_{n \geq 0}$ *dominates* a v.s.a. $\mathcal{F} = \{F_n\}_{n \geq 0}$ if, for any number n, there is a number m such that $F_m \subseteq \hat{F}_n$, shows that for the case of array noncomputability for the almost-c.e. sets, this does not make a difference.

Proposition 25 ([ASLM])**.** *(a) For any v.s.a. $\mathcal{F} = \{F_n\}_{n \geq 0}$ there is a complete very strong array of intervals $\mathcal{I} = \{I_n\}_{n \geq 0}$ which dominates $\mathcal{F}$.*

(b) Let $\hat{\mathcal{F}} = \{\hat{F}_n\}_{n \geq 0}$ and $\mathcal{F} = \{F_n\}_{n \geq 0}$ be very strong arrays such that $\hat{\mathcal{F}}$ dominates $\mathcal{F}$. Then any $\hat{\mathcal{F}}$-a.c.e.-a.n.c. almost-c.e. set ($\hat{\mathcal{F}}$-a.n.c. c.e. set) is $\mathcal{F}$-a.c.e.-a.n.c. ($\mathcal{F}$-a.n.c.).

2.3 On Sets with the Universal Similarity Property

In this section, we investigate the characteristics of almost-c.e. sets with the universal similarity property. We show that such sets can be found precisely in the c.e. not totally ω-c.e. Turing degrees and use this fact to (re-)prove several results on those degrees.

We start with establishing the existence of almost-c.e. sets with the universal similarity property. For the construction of such sets, it is useful to note that, by Proposition 25, it suffices to consider complete very strong arrays of intervals.

Proposition 26. *Let A be an a.c.e. set such that, for any complete v.s.a.i. $\mathcal{I}$, A is $\mathcal{I}$-a.c.e.-a.n.c. Then A has the universal similarity property.*

In the following we construct an almost-c.e. set with the universal similarity property. This construction will be refined in the next subsection where we show that almost-c.e. sets with the universal similarity property can be found in any c.e. not totally ω-c.e. degree.

Theorem 27. *There is an almost-c.e. set with the universal similarity property.*

For the proof we need a computable enumeration of the (locally) almost-c.e. sets. Let $\{V_{e,s}\}_{e,s\geq 0}$ be a computable enumeration of computable almost-enumerations $\{V_{e,s}\}_{s\geq 0}$ ($e \geq 0$) such that for every c.v.s.a.i. $\mathcal{I}$ and for every $\mathcal{I}$-a.c.e. set B, there is an index $e \geq 0$ such that, for the a.c.e. set $V_e = \lim_{s\to\infty} V_{e,s}$, $V_e = B$ holds and the almost-enumeration $\{V_{e,s}\}_{s\geq 0}$ is $\mathcal{I}$-compatible. (Note that such an enumeration exists. By standard techniques we can define a computable enumeration $\{V_{e,s}\}_{e,s\geq 0}$ of computable almost-enumerations $\{V_{e,s}\}_{s\geq 0}$ ($e \geq 0$) such that, for any computable almost-enumeration $\{B_s\}_{s\geq 0}$, there is an index e such that the almost-enumeration $\{V_{e,s}\}_{s\geq 0}$ is a *delayed version* of $\{B_s\}_{s\geq 0}$, i.e., such that there is a strictly increasing computable sequence $\{s_n\}_{n\geq 0}$ of stages such that $s_0 = 0$ and such that $V_{e,s} = B_{s_n}$ for any s and n such that $s_n \leq s < s_{n+1}$. So it suffices to note that if a computable almost-enumeration is $\mathcal{I}$-compatible then so is any delayed version of it.)

Proof of Theorem 27. Note that, for any c.v.s.a.i. $\mathcal{I}$ and for any $\mathcal{I}$-a.c.e. set B, there are numbers e_0 and e_1 such that $\mathcal{I}^{e_0} = \mathcal{I}$, $V_{e_1} = B$ and the computable almost-enumeration $\{V_{e_1,s}\}_{s\geq 0}$ of B is compatible with $\mathcal{I}$. So, by Proposition 26, in order to construct an almost-c.e. set A with the universal similarity property, it suffices to meet the following requirements for all $e \geq 0$ where $e = \langle e_0, e_1\rangle$.

$$R_e : \text{If } \mathcal{I}^{e_0} \text{ is a c.v.s.a.i. and } V_{e_1} \text{ is } \mathcal{I}^{e_0}\text{-almost-c.e. via } \{V_{e_1,s}\}_{s\geq 0}$$
$$\text{then there is a number } n \text{ such that } A \cap I_n^{e_0} = V_{e_1} \cap I_n^{e_0}.$$

An a.c.e. set A meeting these requirements can be constructed by a standard finite injury argument using the following basic strategy for meeting a single requirement R_e (where the computable almost-enumeration $\{A_s\}_{s\geq 0}$ is defined such that $A_s \subseteq \omega \restriction s$).

At the stage $s+1$ at which the attack on R_e is started, appoint $x_e = s+1$ as a *follower* (note that x_e is not in A_s). Then wait for a stage $s' > s$ such that, for some $n < s'$, $I^{e_0}_{n,s'} \downarrow$ and $x_e < \min I^{e_0}_n$. (Note that such a stage s' must exist if the hypothesis of R_e holds.) At the least such stage $s'+1$ (if any) and for the least corresponding n, put x_e into A_{s+1} and let

$A_{s+1} \cap I_n^{e_0} = V_{e_1,s'+1} \cap I_n^{e_0}$. (By $x_e < \min I_n^{e_0}$, this is compatible with making $\{A_s\}_{s\geq 0}$ a computable almost-enumeration.) Moreover, initialize all lower priority requirements (thereby ensuring that these requirements do not interfere with the definition of A on $I_n^{e_0}$). Finally, at any stage $s'' > s'$ such that $V_{e_1,s''+1} \cap I_n^{e_0} \neq V_{e_1,s''} \cap I_n^{e_0}$ correct A on $I_n^{e_0}$ by letting $A_{s''+1} \cap I_n^{e_0} = V_{e_1,s''+1} \cap I_n^{e_0}$, provided that, up to this stage, the almost-enumeration $\{V_{e_1,s}\}_{s\geq 0}$ of V_{e_1} is consistent with the assumption that V_{e_1} is $\mathcal{I}^{e_0}$-a.c.e. via $\{V_{e_1,s}\}_{s\geq 0}$. (Note that, by the latter restriction, the definition of $A_{s''+1}$ is compatible with making $\{A_s\}_{s\geq 0}$ a computable almost-enumeration.)

Obviously, this finitary strategy ensures that, assuming that the hypothesis of R_e holds and that R_e is not injured after stage s, $A \cap I_n^{e_0} = V_{e_1} \cap I_n^{e_0}$ for n as above whence requirement R_e is met. The actual construction of A, a standard finite injury argument coordinating the above finitary strategies for meeting the individual requirements, is straightforward. □

2.3.1 Sets with the Universal Similarity Property and not Totally ω-C.E. Degrees

We now combine the basic construction of an almost-c.e. set with the universal similarity property with the not-totally-ω-c.e. permitting technique (see Downey, Greenberg and Weber [DGW07] as well as Barmpalias, Downey and Greenberg [BDG10]) in order to show that any c.e. not totally ω-c.e. Turing degree contains an a.c.e. set with this property. Note that not-totally-ω-c.e. permitting is a stronger variant of multiple permitting than the one given by the a.n.c. c.e. degrees.

Theorem 28. *Let* **a** *be a c.e. Turing degree which is not totally ω-c.e. There is an almost-c.e. set $A \in$* **a** *which has the universal similarity property.*

Proof. Fix a c.e. set $C \in \mathbf{a}$ and let $\{C_s\}_{s\geq 0}$ be a computable enumeration of C (where w.l.o.g. $C_{s+1} \setminus C_s \neq \emptyset$ for all s). It suffices to give a computable almost-enumeration $\{A_s\}_{s\geq 0}$ of an almost-c.e. set A such that A has the universal similarity property and $A =_{\mathrm{T}} C$.

Just as in the proof of Theorem 27, in order to ensure that A has the universal similarity property, it suffices to meet the requirements

$$R_e : \text{If } \mathcal{I}^{e_0} \text{ is a c.v.s.a.i. and } V_{e_1} \text{ is } \mathcal{I}^{e_0}\text{-almost-c.e. via } \{V_{e_1,s}\}_{s\geq 0}$$
$$\text{then there is a number } n \text{ such that } A \cap I_n^{e_0} = V_{e_1} \cap I_n^{e_0}.$$

for all $e \geq 0$ where, here and in the following, $e = \langle e_0, e_1 \rangle$. The strategy to meet requirement R_e is the one given in the proof of Theorem 27 above. The strategy has to be adjusted, however, since we have to ensure that $A \leq_{\mathrm{T}} C$ and $C \leq_{\mathrm{T}} A$.

In order to guarantee $A \leq_{\mathrm{T}} C$, the changes of A required by the strategy for meeting R_e have to be permitted by C. Since, for a single follower, this permission might not be given, now a finite sequence of followers x and corresponding intervals I_n^e – in the following denoted by $x_{e,k}$ and $J_{e,k}$, respectively – is used, where a new follower is appointed if, for the existing followers $x_{e,k}$ and the associated intervals $J_{e,k}$, the attacks are blocked, i.e., $x_{e,k}$ is not allowed to enter A or A is not allowed to change on the associated interval $J_{e,k}$ – since the required permission by C is not given. In order to formalize the permitting-constraint and in order to argue that – despite of this constraint – the strategy for meeting R_e remains finitary and succeeds, we have to exploit that

$\mathbf{a} = deg_{\mathrm{T}}(C)$ is not totally ω-c.e. By the latter fix a total function $g \leq_{\mathrm{T}} C$ which is not ω-c.e. Let Γ be a Turing functional such that $g = \Gamma^C$ and fix the computable approximation of g such that $g_s(x) = \Gamma_s^{C_s}(x)$ if the right hand side is defined and $g_s(x) = 0$ otherwise (for all $x, s \geq 0$).

Then $x_{e,k}$ is allowed to enter A at stage $s+1$ or A is allowed to change on the associated interval $J_{e,k}$ at stage $s+1$ only if $g_{s+1}(k) \neq g_s(k)$ where $x_{e,k}$ is the $(k+1)$st follower of R_e (in order of magnitude) at the end of stage s. As we will argue below, this constraint suffices to ensure $A \leq_{\mathrm{T}} C$ while, on the other hand, the fact that (by choice of g) the number of stages s with $g_{s+1}(k) \neq g_s(k)$ is not computably bounded in k guarantees that one of the attacks on R_e will succeed.

In order to guarantee $C \leq_{\mathrm{T}} A$ we use a movable marker γ. The markers $\gamma(e)$ are put down in order and if $\gamma(e)$ is put down at stage $s+1$ then $\gamma(e)$ is put down on $s+1$. Once put down, $\gamma(e)$ may be lifted later. Eventually, however, $\gamma(e)$ reaches a final position, i.e., is put down and not lifted later. Lifting a marker is subject to the following constraints. First (for technical convenience), if $\gamma(e)$ is lifted at stage $s+1$ then all markers $\gamma(e')$ with $e' > e$ (defined at stage s) are simultaneously lifted in order to guarantee that the numbers for which the markers are defined form an initial segment of ω. Second (and crucially), $\gamma(e)$ may be lifted at stage $s+1$ only if A is changing on $\gamma(e)$ or a smaller number. The latter ensures that A controls the moves of γ (assuming that A is almost-c.e. via $\{A_s\}_{s\geq 0}$). In particular, A can tell whether the position of a marker attained at some stage is permanent or not whence A can compute the final position of any marker and the stage at which it is attained first. So, in order to compute C from A it suffices to ensure that if e enters C at stage $s+1$ and the marker $\gamma(e)$ is defined at stage s then $\gamma(e)$ is lifted at stage $s+1$ (whence $C(e) = C_{s'}(e)$ for the stage s' at which $\gamma(e)$ is put down on its final position). Finally, we ensure that if $\gamma(e)$ is defined at stage s then $\gamma(e)$ is not in A_s whence we may lift $\gamma(e)$ at stage $s+1$ by enumerating $\gamma(e)$ into A at stage $s+1$.

More formally, by letting $\gamma(e,s)$ denote the position of $\gamma(e)$ at stage s, we define a partial computable function $\gamma : \omega^2 \to \omega$ with computable domain (where $\gamma(e,s)$ is specified at stage s of the construction) having the following properties (for $e, e', s \geq 0$).

(γ_1) $\gamma(e,0)\uparrow$.

(γ_2) If $\gamma(e,s+1) \neq \gamma(e,s)$ then *either* $\gamma(e,s)\uparrow$ and $\gamma(e,s+1) = s+1$ *or* $\gamma(e,s)\downarrow$, $\gamma(e,s+1)\uparrow$ and $A_{s+1} \upharpoonright \gamma(e,s)+1 \neq A_s \upharpoonright \gamma(e,s)+1$.

(γ_3) If $\gamma(e,s)\downarrow$ and $e' < e$ then $\gamma(e',s)\downarrow$ and $\gamma(e',s) < \gamma(e,s)$.

(γ_4) If $\gamma(e,s)\downarrow$ then $\gamma(e,s) \notin A_s$.

(γ_5) $\gamma^*(e) = \lim_{s\to\infty} \gamma(e,s) \in \omega$ exists.

(γ_6) If $e \in C_{s+1} \setminus C_s$ then $\gamma(e,s+1)\uparrow$.

Claim 1. Assume that $\{A_s\}_{s\geq 0}$ is a computable almost-enumeration of A and that γ has the above properties. Then $C \leq_{\mathrm{T}} A$.

Proof. Given e, $C(e)$ is computed from A (uniformly in e) as follows. Using A as an oracle find the least stage s such that $\gamma(e,s)\downarrow$ and $A_s \upharpoonright \gamma(e,s)+1 = A \upharpoonright \gamma(e,s)+1$. By ($\gamma_5$), such a stage exists and, by (γ_2) and (γ_6), $C_s(e) = C(e)$. □

In order to avoid conflicts between the strategy for meeting the requirements R_e and the coding of C into A using the movable marker γ, we put down a marker at stage $s+1$ only if no follower is appointed at stage $s+1$ and vice versa. So no follower is a marker position and vice versa. Moreover, by giving marker $\gamma(e)$ higher priority than requirement $R_{e'}$ for $e' \geq e$, by initialization we can ensure that no marker sits in an interval currently associated with a lower priority follower. On the other hand, an interval $J_{e',k}$ associated with a higher priority follower $x_{e',k}$ ($e' < e$) cannot be completely cleared from all lower priority markers $\gamma(e)$. At the stage where $J_{e',k}$ becomes associated with $x_{e',k}$ no number enters A whence (by (γ_2)) we must not lift any marker. Once $R_{e'}$ starts to let A mimic $V_{e'_1}$ on $J_{e',k}$, however, the enumeration of $x_{e',k}$ into A allows to lift the lower priority markers $\gamma(e)$ in $J_{e',k}$. Hence – as soon as it becomes relevant – the interval $J_{e',k}$ will be cleared of all coding markers.

So all in all (up to finite injuries in terms of initializing a requirement respectively lifting a marker) the coding of C into A does not interfere with the strategy for meeting the requirements R_e and vice versa.

Having explained the basic features of the proof and having introduced some of the required notions, we now turn the formal construction. An index e is *eligible* at stage $s+1$ if the enumeration of $\mathcal{I}^{e_0}$ and the almost-enumeration of V_{e_1} up to stage $s+1$ do not contradict the assumption that V_{e_1} is $\mathcal{I}^{e_0}$-a.c.e., i.e., if there are no $n, t \leq s$ such that $I^{e_0}_{n,s+1} \downarrow$, $V_{e_1,t+1} \cap I^{e_0}_n \neq V_{e_1,t} \cap I^{e_0}_n$ and $x \notin V_{e_1,t+1}$ for the least $x \in I^{e_0}_n$ such that $V_{e_1,t+1}(x) \neq V_{e_1,t}(x)$. Note that e is eligible at all stages $s+1 \geq 1$ if the hypothesis of R_e holds. Moreover, if e is not eligible at stage $s+1$ then e is not eligible at all stages $t > s$. We now turn to the construction of A where we let A_s denote the finite part of A constructed by the end of stage s.

Construction.

Stage 0. Stage 0 is vacuous. I.e., $A_0 = \emptyset$, $\gamma(x, 0) \uparrow$ for all x and no requirement has a follower at the end of stage s.

Stage $s+1$. A requirement R_e *requires attention* at stage $s+1$ if $e < s$, e is eligible at stage $s+1$ and one of the following holds.

(i) No follower is assigned to R_e at the end of stage s.

(ii) (i) does not hold, $x_{e,0} < \cdots < x_{e,n}$ ($n \geq 0$) are the followers assigned to R_e at the end of stage s and there is a number $k < n$ such that, for the interval $J_{e,k}$ associated with $x_{e,k}$, $A_s \cap J_{e,k} \neq V_{e,s+1} \cap J_{e,k}$ and $g_s(k) \neq g_{s+1}(k)$.

(iii) (i) and (ii) do not hold, $x_{e,0} < \cdots < x_{e,n}$ ($n \geq 0$) are the followers assigned to R_e at the end of stage s, for all $k < n$, $A_s \cap J_{e,k} \neq V_{e,s+1} \cap J_{e,k}$ holds where $J_{e,k}$ is the interval associated with $x_{e,k}$, and there is a number $m \leq s$ such that $I^{e_0}_{m,s+1} \downarrow$ and $x_{e,n} < \min I^{e_0}_m$.

Let c_s be the least element of $C_{s+1} \setminus C_s$. Fix $e < s$ minimal such that at least one of the following holds

(I) $e = c_s$.

(II) $\gamma(e, s)$ is undefined.

(III) R_e requires attention.

and perform the following corresponding action.

1. If (I) holds then put $\gamma(e, s)$ into A (if defined). Furthermore, for all $e' \geq e$, let $\gamma(e', s+1) \uparrow$ and *initialize* requirement $R_{e'}$, i.e., cancel all followers (and the intervals associated with them) assigned to requirement $R_{e'}$ at the end of stage s.
2. If (I) does not hold, but (II) holds, let $\gamma(e, s+1) = s+1$. Furthermore, for all $e' \geq e$, initialize requirement $R_{e'}$.
3. If (I) and (II) do not hold, but (III) holds, perform the following action according to the clause via which R_e requires attention.
   (i) Appoint $x_{e,0} = s+1$ as a follower to R_e.
   (ii) For all k that make Clause (ii) in the definition of requiring attention true, let $A_{s+1} \cap J_{e,k} = V_{e_1,s+1} \cap J_{e,k}$ and, in case $x_{e,k} \notin A_s$, put $x_{e,k}$ into A. Furthermore, for the least number y such that $A_{s+1}(y) \neq A_s(y)$ by this action and for any $e' > e$ such that $y \leq \gamma(e', s) \downarrow$, let $\gamma(e', s+1) \uparrow$.
   (iii) For the least m that makes Clause (iii) in the definition of requiring attention true, associate the interval $J_{e,n} = I^{e_0}_m$ with the follower $x_{e,n}$ of R_e. Furthermore, appoint $x_{e,n+1} = s+1$ as a further follower to R_e.

   In any of the subcases (i) – (iii) say that R_e *receives attention* or *becomes active*. Furthermore, for all $e' > e$, initialize $R_{e'}$. Finally, unless $\gamma(c_s, s) \uparrow$ or case (ii) applies and $\gamma(c_s, s+1)$ is made undefined by the action there, put $\gamma(c_s, s)$ into A and, for all $e'' \geq c_s$, let $\gamma(e'', s+1) \uparrow$.

(If not explicitly stated otherwise, any parameter depending on the stage is unchanged at stage $s+1$.)

This completes the construction. Note that the construction ensures that the followers of R_e are appointed in order of magnitude and that the greatest follower is the unique follower which has not yet an interval associated with it. In the remainder of the proof we show that the construction is correct.

Verification.

Note that the construction is effective. In order to show that A has the required properties, we prove a series of claims. Before we turn to these claims, however, we first make some observations on the construction to be used in the proofs of the claims. If not stated otherwise these observations follow from the construction and the fact that $\max I^e_n < s$ if $I^e_{n,s} \downarrow$ (recall that $I^e_{n,s} = D_{\varphi_{e,s}(n)}$ and $\varphi_{e,s}(n) < s$ by convention) by straightforward inductions on s.

If a follower x is appointed at stage $s+1$ then $x = s+1$ and no marker is put down at stage $s+1$ and if a marker $\gamma(e)$ is put down at stage $s+1$ then $\gamma(e, s+1) = s+1$ and no follower is appointed at stage $s+1$. Moreover, at any stage $s+1$ at most one follower is appointed and at most one marker is put down. So a number which becomes a follower in the course of the construction does not become a marker position and vice versa; and followers for different requirements and marker positions for different arguments differ. Moreover, by effectivity of the construction, we can tell whether a number will ever become a follower or a marker position. Another consequence of the above is that any follower x existing at stage s and any marker $\gamma(e, s)$ defined at stage s is $\leq s$. Similarly, for any interval J associated with some follower x at stage s, $\max J < s$. Since a follower is not put into A at the stage where it is appointed and since a marker is not put into A at the stage where it is put down, it follows that $A_s \subseteq \omega \upharpoonright s$.

If requirement R_e has at least one follower at the end of stage s then we let $x_{e,0}[s] < x_{e,1}[s] < \cdots < x_{e,n}[s]$ $(n \geq 0)$ be the followers of R_e at the end of stage s in order of magnitude and, for $k < n$, we let $J_{e,k}[s]$ be the interval associated with $x_{e,k}[s]$. (In the following we omit $[s]$ if the stage s is clear from context.) Note that, for $s \leq s'$ and $k \leq n$, $x_{e,k}[s] = x_{e,k}[s']$ unless R_e is initialized at a stage t with $s < t \leq s'$ (since followers are appointed in increasing order). Now given s, for $k \leq n$, $x_{e,k}$ is appointed at stage $x_{e,k} \leq s$ and $x_{e,k} \notin A_{x_{e,k}}$. Moreover, $x_{e,n}$ is the only follower of R_e at the end of stage s which is not yet associated with an interval. For $k < n$, $J_{e,k}$ becomes associated with $x_{e,k}$ at stage $x_{e,k+1}$ and

$$x_{e,k} < \min J_{e,k} \leq \max J_{e,k} < x_{e,k+1} \tag{2.1}$$

holds. Moreover, by initialization, for any $e' \neq e$ and any $R_{e'}$-follower x existing at stage s, $x < x_{e,0}$ if $e' < e$ and $x_{e,n} < x$ if $e < e'$. It follows that, for any follower x existing at stage s and any interval J which is associated with any follower at stage s, it holds that $x \notin J$. By disjointness of the sets of followers and marker positions, this implies that a follower x of R_e can be enumerated into A at stage $s+1$ only if R_e requires attention via Clause (ii) at stage $s+1$ and $x = x_{e,k}$ for some $k < n$ that makes Clause (ii) in the definition of requiring attention true. It follows that if $s+1$ is the first stage after appointment of $x_{e,k}$ at which this happens then $x_{e,k} \in A_{s+1} \setminus A_s$ (note that $x_{e,k}[t] = x_{e,k}$ for all t with $x_{e,k} \leq t \leq s$).

Finally observe that, by effectivity of the construction, the partial marker function γ is computable and has computable domain. Moreover, conditions (γ_1) – (γ_4) and (γ_6) hold. (Condition (γ_5) will be established in Claim 3 below.) Correctness of (γ_1), (γ_3) and (γ_6) is immediate by construction and so is the first part of (γ_2), while the second part of (γ_2) is immediate by construction assuming (γ_4). So it suffices to show (γ_4). For a proof of (γ_4) fix $\gamma(e, s)$ such that $\gamma(e, s)\downarrow$ and $\gamma(e, s)$ is enumerated into A at stage $s+1$. We have to show that $\gamma(e, s+1)\uparrow$. If $\gamma(e, s)$ is enumerated into A for the sake of coding then this is immediate. Since $\gamma(e, s)$ cannot be a follower, this only leaves the case that a requirement $R_{e'}$ receives attention via Clause (ii) at stage $s+1$, there is a number k that makes Clause (ii) in the definition of requiring attention true and $\gamma(e, s) \in J_{e',k}$. Now, if $e' < e$ then, by construction, $\gamma(e, s+1)\uparrow$ as required. The case that $e \leq e'$, however, cannot occur. By construction, $\gamma(e, s)$ becomes defined at stage $\gamma(e, s)$ and at this stage $R_{e'}$ is initialized. So $x_{e',k}$ is appointed after this stage whence $\gamma(e, s) < x_{e',k} < \min J_{e',k}$ contrary to assumption.

Claim 2. A is a.c.e. via $\{A_s\}_{s \geq 0}$.

Proof. By effectivity of the construction, it suffices to show that, for any number which is extracted from A at some stage a lesser number is enumerated into A at the same stage. Note that a number y can be extracted from A at stage $s+1$ only if a requirement R_e receives attention via Clause (ii) and y is in the interval $J_{e,k}$ for some number k that makes Clause (ii) in the definition of requiring attention true. So, given such s, e and k such that

$$A_{s+1} \cap J_{e,k} \neq A_s \cap J_{e,k} \tag{2.2}$$

holds, it suffices to show that there is a number $x < \min J_{e,k}$ such that $x \in A_{s+1} \setminus A_s$ or that, for $x = \mu z \in J_{e,k}(A_s(z) \neq A_{s+1}(z))$, $x \in A_{s+1}$.

Now, if $s+1$ is the least stage after the appointment of the follower $x_{e,k}$ at which R_e receives attention via Clause (ii) and k makes Clause (ii) in the definition of requiring attention true then, as observed above, $x_{e,k} \notin A_s$ whence $x_{e,k}$ is put into A_{s+1} by construction. (Note that $x_{e,k}[s'] = x_{e,k}$ for $x_{e,k} \leq s' \leq s$.) Hence, by $x_{e,k} < \min J_{e,k}$, $x = x_{e,k}$ will do.

So w.l.o.g. we may assume that there is at least one stage t such that $x_{e,k} < t+1 < s+1$ and such that R_e receives attention via Clause (ii) at stage $t+1$ and k makes Clause (ii) in the definition of requiring attention true at this stage. Let t' be the least such t and let t'' be the greatest such t. (Note that, by choice of t', $J_{e,k}$ becomes associated with $x_{e,k}$ at a stage $\leq t'$ whence $\max J_{e,k} \leq t'$.) Then, at stage $t'+1$, the interval $J_{e,k}$ is cleared of all markers. Namely, by minimality of t', $x_{e,k}$ is enumerated into A at stage $t'+1$ and all markers $\gamma(e', t')$ with $e' > e$ which are defined at stage t' and $\geq x_{e,k}$ are lifted. On the other hand, for no $e'' \leq e$, $\gamma(e'')$ is put down at any stage s' with $x_{e,k} < s' \leq s$ since otherwise the follower $x_{e,k}$ would be canceled at stage s'. Hence any marker $\gamma(e', s')$ defined at stage $s' \geq t'$ is either less than $x_{e,k}$ or greater than t'. So, by $x_{e,k} < \min J_{e,k} \leq \max J_{e,k} \leq t'$,

$$\forall e' \, \forall s' \geq t' (\gamma(e', s') \downarrow \Rightarrow \ \gamma(e', s') \notin J_{e,k}). \tag{2.3}$$

holds. It follows that

$$A_{t''+1} \cap J_{e,k} = A_s \cap J_{e,k}$$

holds. Namely, no coding marker put into A after stage t'' is in $J_{e,k}$ (by (2.3) and $t' \leq t''$); no requirement of higher priority than R_e acts at any stage s' with $t''+1 \leq s' \leq s$ (since R_e is not initialized at such a stage); any lower priority requirement changes A only on numbers $> \max J_{e,k}$ after stage t'' (since the requirement is initialized at stage $t''+1 > \max J_{e,k}$); and finally, by maximality of t'' and by (2.2), any action of R_e at a stage s' with $t''+1 < s' \leq s$ will not change A on $J_{e,k}$. Since, by choice of t'' and s and by construction, A and V_{e_1} agree on $J_{e,k}$ at stage $t''+1$ and $s+1$, it follows that

$$A_s \cap J_{e,k} = V_{e_1,t''+1} \cap J_{e,k} \ \& \ A_{s+1} \cap J_{e,k} = V_{e_1,s+1} \cap J_{e,k}.$$

So, in order to show that the least change of A on $J_{e,k}$ at stage $s+1$ is positive, it suffices to show, that $x_0 \in V_{e_1,s+1}$ where x_0 is the least number $x \in J_{e,k}$ such that $V_{e_1,s+1}(x) \neq V_{e_1,t''+1}(x)$. But, since R_e requires attention at stage $s+1$, e is eligible at stage $s+1$. Since $J_{e,k} = I_m^{e_0}$ for some m where the right hand side is defined by stage $t' < s$, it follows that that the finite sequence $\{V_{e_1,s'} \cap J_{e,k}\}_{s' \leq s+1}$ must be the initial segment of a computable almost-enumeration. So $x_0 \in V_{e_1,s+1}$ must hold, which completes the proof of Claim 2. $\square$

Claim 3. For any $e \geq 0$, $\gamma(e) = \lim_{s \to \infty} \gamma(e,s) \in \omega$ *exists and requirement* R_e *requires attention only finitely often.*

Proof. The proof is by induction on e. Fix e and, by inductive hypothesis, assume the claim to be correct for $e' < e$. Let s_0 be the greatest stage s such that $s = 0$ or $C_{s+1} \upharpoonright e+1 \neq C_s \upharpoonright e+1$ or $\gamma(e', s+1) \neq \gamma(e', s)$ for some $e' < e$ or $R_{e'}$ requires attention at stage $s+1$ for some $e' < e$. Note that such a stage s_0 exists by inductive hypothesis.

First we show that $\gamma^*(e) = \lim_{s\to\infty} \gamma(e, s) \in \omega$ exists. By choice of s_0, at any stage $s+1$ such that $s > s_0$, condition (I) fails for $e' \leq e$ and conditions (II) and (III) fail for $e' < e$. It follows that $\gamma(e)$ is not lifted after stage $s_0 + 1$. Moreover, if $\gamma(e, s_0 + 1) \uparrow$ then $\gamma(e)$ is put down at stage $s_0 + 2$. So, in any case, $\gamma(e, s) = \gamma(e, s_0 + 2) \downarrow$ for all $s \geq s_0 + 2$ whence $\gamma^*(e) = \gamma(e, s_0 + 2)$.

It remains to show that R_e requires attention only finitely often. For a contradiction assume that this is not the case. We show that there is a computable approximation $\{\tilde{g}_n\}_{n\geq 0}$ of g for which the number of mind changes is computably bounded whence g is ω-c.e. contrary to choice of g.

Let $s_1 = s_0 + 1$ if $\gamma(e, s_0 + 1) \downarrow$ and let $s_1 = s_0 + 2$ otherwise. Then, by choice of s_0, either R_e is initialized at stage s_1 or $s_1 = 1$. So, in either case, R_e does not have any followers at the end of stage s_1. Moreover, R_e receives attention whenever it requires attention after stage s_1 and R_e is not initialized after stage s_1. By the latter, any R_e-follower existing at a stage $> s_1$ is permanent (hence, if $x_{e,k}[s]$ is defined for some $s > s_1$ then $x_{e,k}[s'] = x_{e,k}[s]$ for all $s' \geq s$).

Note that, by choice of s_1, a follower of R_e is appointed at stage $s_1 + 1$ since R_e requires attention via (i) at this stage. In fact there are infinitely many R_e-followers appointed after stage s_1. Namely, assume that there are only finitely many followers appointed and fix $s_2 > s_1$ minimal such that no follower is appointed after stage s_2. Then, for any $s \geq s_2$, the followers defined at stage s are just the followers $x_{e,0} < \cdots < x_{e,n}$ $(n \geq 0)$ defined at stage s_2. So R_e does not require attention via (i) or (iii) after stage s_2. Moreover, R_e requires attention via (ii) at a stage $s+1 > s_2$ only if $g_s(k) \neq g_{s+1}(k)$ for some $k \leq n$. But, since $\{g_s\}_{s\geq 0}$ is a computable approximation of g, the latter can happen only finitely often. So R_e requires attention only finitely often contrary to assumption.

Now let

$$x_{e,0} < x_{e,1} < x_{e,2} < \ldots$$

be the permanent R_e-followers. Then, for any $n \geq 0$, $x_{e,n}$ is appointed at stage $x_{e,n}$ and, for any s such that $x_{e,n} \leq s < x_{e,n+1}$, $x_{e,0}, \ldots, x_{e,n}$ are the R_e-followers defined at stage s. Moreover, by construction, an interval $J_{e,n}$ becomes associated with $x_{e,n}$ at the stage $x_{e,n+1}$ where $x_{e,n+1}$ is appointed and

$$\forall\, k < n\ (A_{x_{e,n+1}-1} \cap J_{e,k} \neq V_{e_1, x_{e,n+1}-1} \cap J_{e,k}) \tag{2.4}$$

holds. The latter implies that, for any k, n with $k < n$

$$\begin{gathered}\text{If } g_{x_{e,n+2}-1}(k) \neq g_{x_{e,n+1}-1}(k) \text{ then there is a stage } s \text{ with } x_{e,n+1} - 1 \leq s < x_{e,n+2} - 1 \\ \text{and such that } A_{s+1} \cap J_{e,k} \neq A_s \cap J_{e,k} \text{ or } V_{e_1,s+1} \cap J_{e,k} \neq V_{e_1,s} \cap J_{e,k} \text{ holds.}\end{gathered} \tag{2.5}$$

holds. For a proof of (2.5), fix $k < n$ such that $g_{x_{e,n+2}-1}(k) \neq g_{x_{e,n+1}-1}(k)$ holds. By (2.4) it suffices to show that there is a stage t such that $x_{e,n+1} - 1 \leq t \leq x_{e,n+2} - 1$ and $A_t \cap J_{e,k} = V_{e_1,t} \cap J_{e,k}$ holds. By $g_{x_{e,n+2}-1}(k) \neq g_{x_{e,n+1}-1}(k)$ fix s such that $x_{e,n+1} - 1 \leq s < x_{e,n+2} - 1$ and $g_{s+1}(k) \neq g_s(k)$. If

$A_s \cap J_{e,k} = V_{e_1,s} \cap J_{e,k}$ then $t = s$ will do. Otherwise, R_e requires and receives attention via (ii) at stage $s+1$ and $A_{s+1} \cap J_{e,k} = V_{e_1,s+1} \cap J_{e,k}$. (Note that, since R_e requires attention infinitely often, there are infinitely many stages at which e is eligible. But this implies that e is eligible at all stages ≥ 1.) So $t = s+1$ will do.

Since $\{V_{e_1,s}\}_{s\geq 0}$ is a computable almost-enumeration and since, by Claim 2, $\{A_s\}_{s\geq 0}$ is a computable almost-enumeration, too, there are at most $2^{1+\max J_{e,k}}$ stages such that V_{e_1} changes below $1 + \max J_{e,k}$ and similarly for A. Hence (2.5) implies that

$$|\{n > k : g_{x_{e,n+2}-1}(k) \neq g_{x_{e,n+1}-1}(k)\}| \leq 2 \cdot 2^{1+\max J_{e,k}}.$$

So, if we define the computable approximation $\{\tilde{g}_n\}_{n\geq 0}$ of g by letting

$$\tilde{g}_n(k) = g_{x_{e,n+1}-1}(k),$$

then the number of mind changes of $\tilde{g}_n$ on k is bounded by $2 \cdot 2^{1+\max J_{e,k}} + k + 1$. So g is ω-c.e. contrary to choice of g. □

Claim 4. For $e \geq 0$, requirement R_e is met.

Proof. Fix e and, for a contradiction, assume that R_e is not met. Then the hypothesis of R_e is true – hence, in particular, $\mathcal{I}^{e_0}$ is a c.v.s.a.i. – and, for all n, $A \cap I_n^{e_0} \neq V_{e_1} \cap I_n^{e_0}$. Moreover, e is eligible at all stages ≥ 1. By Claim 3, fix a stage $s_0 > e$ such that no requirement $R_{e'}$ with $e' \leq e$ requires attention after stage s_0, such that $\gamma(e', s) = \gamma^*(e')$ for all $e' \leq e$ and all $s \geq s_0$ and such that $C_{s_0} \upharpoonright e + 1 = C \upharpoonright e + 1$.

Since R_e does not require attention at stage $s_0 + 1$, there is a follower assigned to R_e at the end of stage s_0. So we may fix $n \geq 0$ such that $x_{e,0}, x_{e,1}, \ldots, x_{e,n}$ and $J_{e,0}, J_{e,1}, \ldots, J_{e,n-1}$ are the finitely many followers and associated intervals assigned to R_e at the end of stage s_0. By choice of s_0, these assignments are permanent. Moreover, since $J_{e,k} \in \mathcal{I}^{e_0}$ it follows that $A \cap J_{e,k} \neq V_{e_1} \cap J_{e,k}$ (for all $k < n$). So, for any sufficiently large stage $s_1 > s_0$, $A_{s_1} \cap J_{e,k} \neq V_{e_1,s_1} \cap J_{e,k}$ for all $k < n$. Since $\mathcal{I}^{e_0}$ is a c.v.s.a.i. we may pick such a stage s_1 such that there is a number m such that $I_{m,s_1}^{e_0} \downarrow$ and $x_{e,n} < \min I_m^{e_0}$. Then Clause (ii) or Clause (iii) in the definition of requiring attention is true at stage $s_1 + 1$. So (since $e < s_1$ and e is eligible at stage $s_1 + 1$) R_e requires attention at stage $s_1 + 1$, which is a contradiction. □

Claim 5. $A \leq_{\mathrm{T}} C$.

Proof. Given x let s_x be the least stage s such that $C_s \upharpoonright x+1 = C \upharpoonright x+1$ and $g_t \upharpoonright x+1 = g \upharpoonright x+1$ for all $t \geq s$. Then s_x can be computed from C (uniformly in x). So it suffices to show that $A_s(x) = A_{s+1}(x)$ for all $s \geq s_x$ whence $A(x) = A_{s_x}(x)$.

Fix s such that $A_s(x) \neq A_{s+1}(x)$. Then, by construction, one of the following two cases must apply.

Case 1. $x = \gamma(c_s, s)$ where $c_s = \min(C_{s+1} \setminus C_s)$ and $\gamma(c_s, s)$ is enumerated into A at stage $s+1$.

Then, by (γ_2) and by construction, $c_s \leq \gamma(c_s, s)$ whence $C_{s+1} \upharpoonright x+1 \neq C_s \upharpoonright x+1$. So $s < s_x$.

Case 2. There are numbers e and k such that requirement R_e receives attention via (ii) at stage $s+1$, $g_{s+1}(k) \neq g_s(k)$ and $x = x_{e,k}$ or $x \in J_{e,k}$ where $x_{e,k}$ is the $(k+1)$st follower (in order of magnitude) of R_e at the end of stage s and $J_{e,k}$ is the interval associated with $x_{e,k}$.

Since $x_{e,k} < \min J_{e,k}$ it follows that $k \leq x_{e,k} \leq x$ whence $g_{s+1} \upharpoonright x+1 \neq g_s \upharpoonright x+1$. So $s < s_x$ is this case, too. ☐

Claim 6. $C \leq_{\mathrm{T}} A$.

Proof. As observed above already, the marker function γ is computable, has computable domain and satisfies the conditions (γ_1) – (γ_4) and (γ_6). Moreover, (γ_5) holds by Claim 3. So $C \leq_{\mathrm{T}} A$ by Claims 1 and 2. ☐

Claims 2, 4, 5 and 6 show that A has the required properties. This completes the proof of Theorem 28. ☐

It can be shown that the converse of Theorem 28 holds, too.

Theorem 29 ([ASLM])**.** *Let A be an almost-c.e. set with the universal similarity property. Then $deg_{\mathrm{T}}(A)$ is not totally ω-c.e.*

To conclude, we get the following characterization of the c.e. not totally ω-c.e. degrees in terms of the universal similarity property.

Theorem 30. *For a c.e. Turing degree* **a** *the following are equivalent.*

(i) There is an almost-c.e. set A with the universal similarity property in **a**.

(ii) **a** *is not totally ω-c.e.*

Proof. This is immediate by Theorems 28 and 29. ☐

In the following subsections, we give examples of applications of Theorem 30 proving properties of the c.e. not totally ω-c.e. degrees.

2.3.2 Sets with the Universal Similarity Property and CB-Randomness

Brodhead, Downey and Ng [BDN12] have shown that any c.e. not totally ω-c.e. Turing degree contains a (not necessarily almost-c.e.) CB-random set and that any such degree bounds a CB-random almost-c.e. set. We unify these results by showing that any almost-c.e. set with the universal similarity property is CB-random.

Theorem 31. *Any almost-c.e. set with the universal similarity property is CB-random.*

Corollary 32. *Let* **a** *be a c.e. Turing degree which is not totally ω-c.e. There is an a.c.e. set $A \in$* **a** *such that A is CB-random.*

Proof. This is immediate by Theorems 30 and 31. ☐

Before we prove Theorem 31 we review the relevant notions.

Definition 33. *(a) For any function $f : \omega \to \omega$, an ML-test $\{U_n\}_{n \geq 0}$ is f-*bounded *(or an f-*test *for short) if, for $n \geq 0$, $|U_n| \leq f(n)$.*

(b) *A Martin-Löf test* $\{U_n\}_{n\geq 0}$ *is* computably-bounded *(or a* CB-test *for short) if* $\{U_n\}_{n\geq 0}$ *is* f*-bounded for some computable function* f*.*

(c) *A real* α *(set* A*) is* f-Martin-Löf random *(or* f-ML-random *for short) if* α *(*A*) passes all* f*-tests.*

(d) *A real* α *(set* A*) is* computably-bounded random *(or* CB-random *for short) if* α *(*A*) passes all CB-tests (i.e., if* α *(*A*) is* f*-ML-random for all computable functions* f*).*

Now, since a set A is CB-random if and only if A is f-ML-random for all computable functions f, for a proof of Theorem 31, it suffices to establish the following lemma. The somewhat technical proof of the lemma can be found in the forthcoming paper [ASLM] by Ambos-Spies, Losert and Monath.

Lemma 34 ([ASLM])**.** *Let* f *be a computable function. There is a c.v.s.a.i.* $\mathcal{F} = \{F_n\}_{n\geq 0}$ *such that any* $\mathcal{F}$*-a.c.e.-a.n.c. almost-c.e. set* A *is* f*-ML-random.*

2.3.3 Sets with the Universal Similarity Property and Maximal Pairs

As a further example, we now turn to investigate maximal pairs in the almost-c.e. ibT- and cl-degrees. They have been and are still widely studied. Yu and Ding [YD04] have shown that there exists a maximal pair in the almost-c.e. ibT-degrees. This result has been extended in various directions. E.g., Fan [Fan09] has shown that there is a maximal pair in the almost-c.e. ibT-degrees such that one half is c.e. In fact, by a result of Fan and Yu [FY11], *every* noncomputable almost-c.e. set is half of a maximal pair in the almost-c.e. ibT-degrees. However, as shown by Downey and Hirschfeldt [DH10], we cannot make both halves c.e.

In [ASLM], Fan's theorem that there is an ibT-maximal pair (A, B) in the almost-c.e. sets where B is c.e. is strengthened in two directions. Namely, it is shown that the c.e. set B can be chosen to be arbitrarily sparse, i.e., to be a subset of any given infinite computable set D and that it suffices to let A be any $\mathcal{F}$-a.c.e.-a.n.c. almost-c.e. set (where the choice of $\mathcal{F}$ depends on D).

Lemma 35 (First Maximal Pair Lemma)**.** *Let* D *be an infinite computable set. There are a c.v.s.a.i.* $\mathcal{F} = \{F_n\}_{n\geq 0}$ *and a c.e. set* $B \subseteq D$ *such that, for any* $\mathcal{F}$*-a.c.e.-a.n.c. almost-c.e. set* A*,* (A, B) *is an* ibT*-maximal pair in the almost-c.e. sets.*

The quite technical proof of this lemma is based on the Yu-Ding method for constructing maximal pairs and its refinement by Barmpalias, Downey and Greenberg and can be found in the forthcoming paper [ASLM] by Ambos-Spies, Losert and Monath. The lemma immediately implies the following.

Lemma 36 (Second Maximal Pair Lemma)**.** *Let* A *be an almost-c.e. set with the universal similarity property and let* D *be any infinite computable set. There is a c.e. set* $B \subseteq D$ *such that* (A, B) *is an* ibT*-maximal pair in the almost-c.e. sets.*

As mentioned in [ASLM], Lemma 35 implies the following Theorem which, by Theorem 21, implies direction (i)$\Rightarrow$(ii) of Theorem 17.

Theorem 37. *There is a v.s.a. $\mathcal{F}$ such that no $\mathcal{F}$-a.c.e.-a.n.c. almost-c.e. set is cl-reducible to any ML-random almost-c.e. set.*

In the remainder of this subsection, we prove that a c.e. T-degree contains an a.c.e. set which is not ibT-reducible to any wtt-hard a.c.e. set if and only if it is not totally ω-c.e. We begin with applying the Second Maximal Pair Lemma to prove the following.

Theorem 38. *Let A be an almost-c.e. set with the universal similarity property and let C be a wtt-hard almost-c.e. set. Then $A \not\leq_{\mathrm{ibT}} C$.*

The theorem is immediate by the Second Maximal Pair Lemma and the following equivalence.

Lemma 39 ([ASLM])**.** *Let A be an almost-c.e. set. The following are equivalent.*

(i) A is not ibT-reducible to any wtt-hard almost-c.e. set.

(ii) For any infinite computable set D there is a computably enumerable subset B of D such that (A, B) is an ibT-maximal pair in the almost-c.e. sets.

By Theorem 30, Theorem 38 implies the following.

Theorem 40. *Let $\mathbf{a}$ be a c.e. Turing degree which is not totally ω-c.e. Then, there is an almost-c.e. set $A \in \mathbf{a}$ that is not ibT-reducible to any wtt-hard almost-c.e. set.*

We now proceed to prove the converse of Theorem 40.

Theorem 41. *Let $\mathbf{a}$ be a c.e. Turing degree which is totally ω-c.e. and let A be any almost-c.e. set in $\mathbf{a}$. There is a wtt-hard almost-c.e. set B such that $A \leq_{\mathrm{ibT}} B$.*

For the proof of Theorem 41, we need the following technical lemma.

Lemma 42 ([ASLM])**.** *Let A be a noncomputable almost-c.e. set such that $deg_{\mathrm{T}}(A)$ is totally ω-c.e. Then, the following hold.*

(a) There is a strictly increasing computable function h such that the following holds.

For any computable function f there are a computable almost-enumeration $\{A_s\}_{s\geq 0}$ of A and an infinite computable set D such that

$$\forall\, x \in D\ (|\{s : A_{s+1} \upharpoonright f(x) \neq A_s \upharpoonright f(x)\}| < h(x))$$

holds.

(b) There are a computable almost-enumeration $\{A_s\}_{s\geq 0}$ of A and a c.v.s.a.i. $\mathcal{I} = \{I_n\}_{n\geq 0}$ such that

$$\forall n \geq 0\ (|\{s : A_{s+1} \cap \bigcup_{n'\leq n} I_{n'} \neq A_s \cap \bigcup_{n'\leq n} I_{n'}\}| < |I_n| - 1) \tag{2.6}$$

Now we are ready to prove Theorem 41.

Proof of Theorem 41. If $\mathbf{a} = \mathbf{0}$ then the claim is obvious. So w.l.o.g. assume that $\mathbf{a} > \mathbf{0}$ and let A be any almost-c.e. set in $\mathbf{a}$. By Lemma 42 (b) fix a computable almost-enumeration $\{A_s\}_{s\geq 0}$ of A and a c.v.s.a.i. $\mathcal{I} = \{I_n\}_{n\geq 0}$ such that (2.6) holds. Moreover, fix a_n and b_n such that $I_n = [a_n, b_n]$.

It suffices to show that there are an almost-c.e. set B and a computable almost-enumeration $\{B_s\}_{s\geq 0}$ of B such that

$$A \leq_{\mathrm{ibT}} B \tag{2.7}$$

and such that

$$\forall\, n, s \geq 0\ (B_s(b_n) = 0) \tag{2.8}$$

holds. Then, for $C = B \cup \{b_n : n \in K\}$ where K is the halting set, C is almost-c.e., $A \leq_{\mathrm{ibT}} C$ and $K \leq_{\mathrm{wtt}} C$.

The desired computable almost-enumeration $\{B_s\}_{s\geq 0}$ is defined as follows. Let $B_0(x) = 0$ for all x. For the definition of B_{s+1} distinguish the following two cases. If $A_{s+1} = A_s$ then let $B_{s+1} = B_s$. Otherwise, let x_{s+1} be the least x such that $A_{s+1}(x) \neq A_s(x)$ and fix the unique number n_{s+1} such that $x_{s+1} \in I_{n_{s+1}}$. (Note that $A_{s+1}(x_{s+1}) = 1$ and $A_s(x_{s+1}) = 0$ since $\{A_s\}_{s\geq 0}$ is an almost-enumeration.) If $x_{s+1} \neq b_{n_{s+1}}$ then let y_{s+1} be the greatest number $y \leq x_{s+1}$ in $I_{n_{s+1}}$ such that $B_s(y) = 0$ and if $x_{s+1} = b_{n_{s+1}}$ then let y_{s+1} be the greatest number $y < x_{s+1}$ in $I_{n,s+1}$ such that $B_s(y) = 0$. In either case let

$$B_{s+1}(x) = \begin{cases} B_s(x) & \text{if } x < y_{s+1} \\ 1 & \text{if } x = y_{s+1} \\ 0 & \text{if } x > y_{s+1}. \end{cases}$$

Assuming that the number y_{s+1} exists for all s such that x_{s+1} is defined, it is obvious that $\{B_s\}_{s\geq 0}$ is a computable almost-enumeration satisfying (2.8). Moreover, for the set B almost-enumerated by $\{B_s\}_{s\geq 0}$, (2.7) holds since $\{B_s\}_{s\geq 0}$ permits $\{A_s\}_{s\geq 0}$ (namely, for any z and s such that $A_{s+1} \upharpoonright z+1 \neq A_s \upharpoonright z+1$, it holds that $B_{s+1} \upharpoonright z+1 \neq B_s \upharpoonright z+1$).

So it only remains to show that y_{s+1} exists whenever x_{s+1} exists. For a contradiction, let s_0 be the first stage such that x_{s_0+1} exists and y_{s_0+1} does not exist and let $n = n_{s_0+1}$. As y_{s_0+1} does not exist, $[a_n, x_{s_0+1}) \subseteq B_{s_0}$. We claim that there are a number $k \geq 0$ and stages $s_k < s_{k-1} < \ldots < s_0$ such that the following hold.

$$x_{s_0+1} < x_{s_1+1} < \ldots < x_{s_k+1} = b_n \tag{2.9}$$

and for all $i < k$

$$x_{s_i+1} \in B_{s_i}, \tag{2.10}$$

$$(x_{s_i+1}, x_{s_{i+1}+1}) \subseteq B_{s_{i+1}}. \tag{2.11}$$

As there is at most one x such that $B(x)$ changes from 0 to 1 at any given stage, this implies that there are at least $|[a_n, x_{s_0+1})| + |\bigcup_{i<k}(x_{s_i+1}, x_{s_{i+1}+1})| + |\{x_{s_i+1} : i < k\}| = |I_n| - 1$ many stages where B changes in I_n. By construction, this only happens if A changes in I_n, hence A changes at least $|I_n| - 1$ many times in I_n, which contradicts the choice of $\mathcal{I}$ and of the computable almost-enumeration of A.

It remains to show that (2.9) holds for some k and $s_k < s_{k-1} < \ldots < s_0$ and that (2.10) and (2.11) hold for all $i < k$. The proof is by induction. For that matter, we prove that, if a stage

$s_i \leq s_0$ such that $x_{s_i+1} \in B_{s_i}$ exists, then there is a stage $s_{i+1} < s_i$ such that the following hold

$$x_{s_{i+1}+1} > x_{s_i+1} = y_{s_{i+1}+1}, \tag{2.12}$$

$$x_{s_{i+1}+1} = b_n \text{ or } x_{s_{i+1}+1} \in B_{s_{i+1}} \tag{2.13}$$

and such that (2.11) holds. Fix $s_i \leq s_0$ such that $x_{s_i+1} \in B_{s_i}$. Then, by construction, there must be a stage $s_{i+1} < s_i$ such that $x_{s_i+1} = y_{s_{i+1}+1}$ (hence $x_{s_i+1} \in B_{s_{i+1}+1} \setminus B_{s_{i+1}}$) and such that $x_{s_i+1} \in B_{s+1}$ for all s with $s_{i+1} \leq s < s_i$. By construction, $x_{s_{i+1}+1} \geq x_{s_i+1}$, so in order to prove (2.12) assume for a contradiction that $x_{s_{i+1}+1} = x_{s_i+1}$. Then, $x_{s_i+1} \in A_{s_{i+1}+1} \setminus A_{s_i}$, i.e., x_{s_i+1} is taken out of A at some stage $s+1$ (i.e., $x_{s_i+1} \in A_s \setminus A_{s+1}$) with $s_{i+1} < s < s_i$. By construction, this implies that $x_{s+1} < x_{s_i+1}$ exists, whence, by $s < s_i \leq s_0$, $y_{s+1} \leq x_{s+1} < x_{s_i+1}$ exist, so, by construction, $x_{s_i+1} \notin B_{s+1}$ contradicting the choice of s_{i+1}. It follows that $x_{s_{i+1}+1} > x_{s_i+1}$, hence (2.12) holds. It follows by definition of $y_{s_{i+1}+1}$ that (2.11) and (2.13) hold.

Now, since y_{s_0+1} does not exist, either $x_{s_0+1} = b_n$ (then, (2.9) holds with $k = 0$ and (2.10) and (2.11) are vacuous) or $x_{s_0+1} \in B_{s_0}$. In the latter case, from the above together with the fact that there are only finitely many stages $s < s_0$, it follows by induction that for some $k > 0$, there are stages $s_k < s_{k-1} < \ldots < s_0$ such that (2.9) holds and such that (2.10) and (2.11) hold for all $i < k$. This completes the proof. □

To conclude, we have the following theorem.

Theorem 43. *For a c.e. degree* **a***, the following are equivalent.*

(i) There is an almost-c.e. set in **a** *that is not ibT-reducible to any wtt-hard almost-c.e. set.*

(ii) **a** *is not totally ω-c.e.*

Proof. Immediate by Theorems 40 and 41. □

2.3.4 Sets with the Universal Similarity Property and cl-Reducibility to Complex Sets

In this subsection, we apply Theorem 43 to characterize the Turing degrees containing almost-c.e. sets which cannot be reduced to any complex almost-c.e. set. Namely, we prove the following theorem which has been conjectured by Noam Greenberg and parallels Theorem 17.

Theorem 44. *Let* **a** *be a c.e. Turing degree. The following are equivalent.*

(i) There is an almost-c.e. set in **a** *which is not cl-reducible to any complex almost-c.e. set.*

(ii) **a** *is not totally ω-c.e.*

Here, complex sets are defined in terms of plain Kolmogorov complexity and computable orders where a computable order is a computable, nondecreasing, unbounded function.

Definition 45 (Kanovich, see e.g. [DH10])**.** *Let h be a computable order. A set A is h-*complex *if $C(A \upharpoonright n) \geq h(n)$ for all n. A set A is* complex *if A is h-complex for some computable order.*

Note that Downey and Hirschfeldt proved that there is a set that is not cl-reducible to any complex set (Theorem 9.13.2 in [DH10]). Moreover, Fan and Yu [FY11] showed that there is an *almost-c.e.* set which is not cl-reducible to any complex *almost-c.e.* set. In order to prove Theorem 44, we need a further equivalence. Kanovich (see Theorem 8.16.7 in [DH10]) has shown that a c.e. set A is complex if and only if A is wtt-complete. Since any almost-c.e. set is wtt-equivalent to a c.e. set and since Kjos-Hanssen, Merkle and Stephan [KHMS11] have observed that the class of complex sets is closed upwards under wtt-reducibility, the complex almost-c.e. sets coincide with the wtt-hard almost-c.e. sets.

Lemma 46. *An almost-c.e. set A is complex if and only if A is wtt-hard (for the class of the c.e. sets).*

We observe further that a set is cl-reducible to a wtt-hard almost-c.e. set if and only if it is ibT-reducible to such a set.

Proposition 47 ([ASLM])**.** *A set A is cl-reducible to some wtt-hard almost-c.e. set if and only if A is ibT-reducible to such a set.*

So, by the coincidence of the complex almost-c.e. sets and the wtt-hard almost-c.e. sets, direction (i)⇒(ii) of Theorem 44 follows from Theorem 41 while direction (ii)⇒(i) follows from Theorem 40 and Proposition 47.

Proof of Theorem 44. (i)⇒(ii). The proof is by contraposition. Assume that $\mathbf{a}$ is totally ω-c.e. and let A be any almost-c.e. set in $\mathbf{a}$. It suffices to show that there is a complex almost-c.e. set C such that $A \leq_{\text{cl}} C$. By Theorem 41 there is a wtt-hard almost-c.e. set C such that $A \leq_{\text{ibT}} C$ hence $A \leq_{\text{cl}} C$. So the claim follows by Lemma 46.

(ii)⇒(i). Assume that $\mathbf{a}$ is not totally ω-c.e. By Theorem 40, there is a set $A \in \mathbf{a}$ which is not ibT-reducible to any wtt-hard almost-c.e. set. We show that A is not cl-reducible to any complex almost-c.e. set. For a contradiction assume that A is cl-reducible to the complex almost-c.e. set C. Then, by Lemma 46, C is wtt-hard and, by Proposition 47, A is ibT-reducible to some wtt-hard almost-c.e. set $\hat{C}$. But this contradicts the choice of A. □

2.4 Notions of Universal Array Noncomputability

In the last section, we have seen that the c.e. not totally ω-c.e. degrees contain almost-c.e. sets that possess a property that can be viewed as a universal version of the key property of the a.n.c. c.e. sets; namely, the universal similarity property. This observation strengthens the impression based on Lemma 23 that being not totally ω-c.e. is a universal or uniform version of being a.n.c. Note that it has been shown by Downey, Jockusch and Stob [DJS90] that no c.e. set is $\mathcal{F}$-a.n.c. for *all* very strong arrays $\mathcal{F}$, so there is no equivalent of the universal similarity property for c.e. sets. In this section, however, we define typical properties of c.e. sets of not totally ω-c.e. Turing degree, namely, we introduce various notions of *universal array noncomputability*. We explore the properties of and the relationships between c.e. sets with the different universal array noncomputability properties, their wtt- as well as T-degrees. In particular, we see that those sets capture a further notion of multiple permitting, namely *uniform multiple permitting* and that their

Turing degrees are precisely the c.e. not totally ω-c.e. Turing degrees, implying that the latter are uniformly multiply permitting. Later we exploit this fact to look at more properties of the c.e. not totally ω-c.e. degrees. Many of the definitions and results in this section will appear in the forthcoming paper [ASL] by Ambos-Spies and Losert.

2.4.1 Basic Definitions and Facts

We start with the basic definitions.

Definition 48. *For any set A, we let $A^{\langle e\rangle} = \{x : \langle e, x\rangle \in A\}$ be the eth row of A.*

Recall the definition of the enumeration $\{\mathcal{F}^e\}_{e\geq 0} = \{\mathcal{F}^{\varphi_e}\}_{e\geq 0}$ of all very strong arrays (and initial segments) for a given universal function φ from the discussion following Definition 12.

Definition 49. *(a) Given a universal function φ, we call a c.e. set A* φ-universally array noncomputable *(*φ-universally a.n.c. *or* φ-u.a.n.c.*) if $A^{\langle e\rangle}$ is $\mathcal{F}^e$-a.n.c. whenever $\mathcal{F}^e$ is a very strong array.*

(b) A c.e. set is uniformly universally array noncomputable *(*uniformly u.a.n.c. *or* u.u.a.n.c.*) if it is φ-u.a.n.c. for some universal function φ.*

(c) We call a c.e. set A universally a.n.c. *(*u.a.n.c.*) if for every very strong array $\mathcal{F}$, there is an index e such that $A^{\langle e\rangle}$ is $\mathcal{F}$-a.n.c.*

We call a c.e. (Turing- or wtt-) degree φ-*u.a.n.c.* (*u.u.a.n.c*, *u.a.n.c.*) whenever it contains a φ-u.a.n.c. (u.u.a.n.c., u.a.n.c.) c.e. set.

Note that for every universal function φ, there exists a φ-universally a.n.c. c.e. set. We establish this fact in the following lemma.

Lemma 50. *Let φ be a universal function. Then, there is a c.e. set A which is φ-universally a.n.c.*

Proof. Let A be defined by

$$\langle e, x\rangle \in A \ \Leftrightarrow\ \exists n(F^e_n\downarrow \& \ x \in W_n \cap F^e_n)\ (\Leftrightarrow\ \exists n, s(F^e_{n,s}\downarrow \& \ x \in W_{n,s} \cap F^e_n))$$

for all e and x. Then, obviously, A is c.e. and if $\mathcal{F}^e$ is a v.s.a. then $A^{\langle e\rangle} \cap F^e_n = W_n \cap F^e_n$ for all n, so $A^{\langle e\rangle}$ is $\mathcal{F}^e$-a.n.c. Hence, by definition, A is φ-u.a.n.c. □

The following relations among the notions are immediate by definition.

Proposition 51. *Let φ be a universal function. For any c.e. set A, the following holds.*

$$\begin{gathered} A \text{ is } \varphi\text{-u.a.n.c.} \\ \Downarrow \\ A \text{ is u.u.a.n.c.} \\ \Downarrow \\ A \text{ is u.a.n.c.} \end{gathered}$$

In the following, we show that the implications in Proposition 51 are strict.

Lemma 52. *Let φ be a universal function. There is a c.e. set A which is uniformly u.a.n.c. but not φ-u.a.n.c.*

Proof. Fix an index e_0 such that $\mathcal{F}^{\varphi_{e_0}}$ is a very strong array and an index e_1 such that $\mathcal{F}^{\varphi_{e_1}}$ is not a very strong array. Note that e_0 and e_1 exist by choice of φ. Let ψ be defined as follows. For $e \neq e_0, e_1$, let $\psi(e,x) = \varphi(e,x)$ for all x and let $\psi(e_0,x) = \varphi(e_1,x)$ and $\psi(e_1,x) = \varphi(e_0,x)$ for all x. It is easy to verify that ψ is a universal function, whence, by Lemma 50, there exists a ψ-u.a.n.c. c.e. set $\hat{A}$. Now, define A by letting $A^{\langle e_0 \rangle} = \emptyset$ and $A^{\langle e \rangle} = \hat{A}^{\langle e \rangle}$ for all $e \neq e_0$. Then, A is c.e. and, by choice of $\hat{A}$ and as $\mathcal{F}^{\psi_{e_0}} = \mathcal{F}^{\varphi_{e_1}}$ is not a v.s.a., A is ψ-u.a.n.c. and hence u.u.a.n.c. However, by choice of e_0, $\mathcal{F}^{\varphi_{e_0}}$ is a v.s.a. As $A^{\langle e_0 \rangle} = \emptyset$, $A^{\langle e_0 \rangle}$ is not $\mathcal{F}^{e_0}$-a.n.c., hence A is not φ-u.a.n.c. This completes the proof of the lemma. □

Lemma 53. *There is a c.e. set A that is u.a.n.c. but not uniformly u.a.n.c.*

Proof. We computably enumerate such a set A.

Let φ together with a computable approximation $\{\varphi_s\}_{s \geq 0}$ be a 2-universal partial computable function and let ψ together with a computable approximation $\{\psi_s\}_{s \geq 0}$ be some fixed 1-universal partial computable function. For fixed e, let φ^e denote the eth branch of φ. Note that for any e, φ^e is a two-ary partial computable function. Furthermore, for every e, as discussed following Definition 12, let $\mathcal{I}^e = \mathcal{I}^{\psi_e}$ be the complete very strong array of intervals (or initial segment) defined by ψ_e and for every e and k, let $\mathcal{F}^{e,k} = \mathcal{F}^{\varphi^e_k}$ be the v.s.a. (or initial segment) defined by φ^e_k. It suffices to construct a c.e. set A such that the following requirements are met.

$\hat{R}_{2e}$: If $\mathcal{I}^e$ is a c.v.s.a.i. then $A^{\langle k \rangle}$ is $\mathcal{I}^e$-a.n.c. for some k.

$\hat{R}_{2e+1}$: If φ^e is universal for the (unary) partial computable functions,
then there is some $k \geq 0$ such that $\mathcal{F}^{e,k}$ is a v.s.a. and $A^{\langle k \rangle}$ is not $\mathcal{F}^{e,k}$-a.n.c.

Here, the requirements $\hat{R}_{2e}$ guarantee that A is u.a.n.c. (notice that, by Proposition 25, it is enough to meet $\hat{R}_{2e}$ for all complete very strong arrays of intervals) whereas the requirements $\hat{R}_{2e+1}$ make sure that A is not uniformly u.a.n.c.

Our strategy is as follows. For a splitting $\{J_n\}_{n \geq 0}$ of ω into intervals, we replace our requirements by the following stronger requirements.

R_{2e} : If $\mathcal{I}^e$ is a c.v.s.a.i. then, for some $k \in J_e$, $A^{\langle k \rangle}$ is $\mathcal{I}^e$-a.n.c.

R_{2e+1} : For any $e' < e$, if there is $k' \in J_e$ such that $\mathcal{F}^{e',k'}$ is a v.s.a.
then, for some $k \in J_e$, $\mathcal{F}^{e',k}$ is a v.s.a. and $A^{\langle k \rangle}$ is not $\mathcal{F}^{e',k}$-a.n.c.

Note that R_{2e} immediately implies $\hat{R}_{2e}$. Moreover, for any e' such that $\varphi^{e'}$ is universal, there are infinitely many k such that $\mathcal{F}^{e',k}$ is a v.s.a. In particular, for any splitting $\{J_n\}_{n \geq 0}$ of ω into intervals, it follows that there are $e > e'$ and $k' \in J_e$ such that $\mathcal{F}^{e',k'}$ is a v.s.a. So, for any e', there is $e > e'$ such that R_{2e+1} implies $\hat{R}_{2e'+1}$.

Our strategy is as follows. To meet a single requirement R_{2e}, we aim to guarantee that $A^{\langle k\rangle}$ is $\mathcal{I}^e$-a.n.c. for some $k \in J_e$. For that matter, for fixed k and n, we pick an unused component $I^e_{f(k,n)}$ of $\mathcal{I}^e$ and make sure that $A^{\langle k\rangle} \cap I^e_{f(k,n)} = W_n \cap I^e_{f(k,n)}$. We ensure this by defining f in stages and, whenever $f(k,n,s)$ is defined at some stage s and $A^{\langle k\rangle}_s \cap I^e_{f(k,n,s)} \neq W_{n,s+1} \cap I^e_{f(k,n,s)}$ holds, we arrange that $A^{\langle k\rangle}_{s+1} \cap I^e_{f(k,n,s)} = W_{n,s+1} \cap I^e_{f(k,n,s)}$ holds.

To meet a single requirement R_{2e+1}, for each $e' < e$ and $k \in J_e$ such that $\mathcal{F}^{e',k}$ is a v.s.a. – if any – we try to make sure that $A^{\langle k\rangle}$ is computable. As we cannot decide whether $\mathcal{F}^{e',k}$ is a v.s.a. or not, we choose to act at so-called $\mathcal{F}^{e',k}$-expansionary stages – i.e., stages where new components of $\mathcal{F}^{e',k}$ are defined – for the sake of meeting requirement R_{2e+1}. Note that, whenever $\mathcal{F}^{e',k}$ is a v.s.a., there are infinitely many $\mathcal{F}^{e',k}$-expansionary stages.

To avoid conflicts between the different types of requirements, we only act on the odd stages to meet the requirements of the form R_{2e} and only on the even stages to meet the requirements of the form R_{2e+1}. Moreover, we choose to try to make $A^{\langle k\rangle}$ $\mathcal{I}^e$-a.n.c. for *every* $k \in J_e$ for the sake of R_{2e}. We might need to destroy action taken for that matter in order to meet a requirement R_{2e+1} which possibly aims to make $A^{\langle k\rangle}$ computable. At a given stage $s'+1$, we then choose to enumerate *all* numbers from $I^e_{f(k,n,s')}$ (if defined) into $A^{\langle k\rangle}$ for $n > n_e(e',k,s')$ where $n_e(e',k,s)$ is appropriately chosen depending on e and $e' < e$ and bounded in s in case that k is minimal in J_e such that $\mathcal{F}^{e',k}$ is indeed a v.s.a. We can then argue that, for every $e' < e$, for the least $k \in J_e$ such that $\mathcal{F}^{e',k}$ is a v.s.a. (if any), this strategy guarantees that $A^{\langle k\rangle}$ is either cofinite or finite, depending on whether $\mathcal{I}^e$ is a c.v.s.a.i. or not. In any case, $A^{\langle k\rangle}$ will then be computable, hence not $\mathcal{F}^{e',k}$-a.n.c. By choosing the splitting $\{J_n\}_{n\geq 0}$ in a way that $|J_n| = n+1$, we make sure that there is a number $k \in J_e$ that is not the least $k' \in J_e$ such that $\mathcal{F}^{e',k'}$ is a v.s.a. for any $e' < e$. For the least such k, we argue that the strategy for meeting R_{2e} succeeds.

Before giving the actual construction, we define the following notions. Let $\{J_n\}_{n\geq 0}$ be a splitting of ω into intervals such that $|J_n| = n+1$ for all n, i.e., let $x_0 = 0$ and for every $n \geq 0$, let $x_{n+1} = x_n + n + 1$ and let $J_n = [x_n, x_{n+1})$. As the fact whether some $\mathcal{F}^{e,k}$ is a v.s.a. or not is not computable, we have to use approximations. We define the following *length of agreement function.* For all e, k and s, we let

$$l(e,k,s) = \mu n(F^{e,k}_{n,s}\uparrow).$$

A stage $2s$ is $\mathcal{F}^{e,k}$*-expansionary* if $s = 0$ or $l(e,k,2s) > l(e,k,2s')$ holds for all $s' < s$. Moreover, if e is fixed, then for $e' < e$ and $k \in J_e$, let

$$n_e(e',k,2s+1) = \max_{k'\in J_e, k'<k} l(e',k',2s+1).$$

Now we are ready to give the formal construction of A where we let A_s denote the finite part of A constructed at the end of stage s.

Construction.

Stage 0 is vacuous, i.e., $A_0 = \emptyset$ and $f(k,n,0)\uparrow$ for all k and n.

Stage $2s+1$. For all $e < s$ and all $k \in J_e$, perform the following actions.

1. For all n such that $f(k,n,2s)\downarrow$ and such that

$$W_{n,2s+1} \cap I^e_{f(k,n,2s)} \neq A^{\langle k\rangle}_{2s} \cap I^e_{f(k,n,2s)} \tag{2.14}$$

holds, let $A^{\langle k\rangle}_{2s+1} \cap I^e_{f(k,n,2s)} = W_{n,2s+1} \cap I^e_{f(k,n,2s)}$. For all such n, say that R_{2e} *becomes active* (*via* Clause 1 and k and n).

2. For the least n such that $f(k,n,2s)\uparrow$, if there is a number m such that

   (i) $I^e_{m,2s+1}\downarrow$,

   (ii) $I^e_m \cap A^{\langle k\rangle}_{2s} = \emptyset$ and

   (iii) if $n > 0$, then $m > f(k,n-1,2s)$

   hold, let $f(k,n,2s+1) = m$ for the least such m, hence $I^e_{f(k,n,2s+1)} = I^e_m$, and say that R_{2e} *becomes active* (*via* Clause 2 and k and n).

Unless mentioned otherwise, let $A^{\langle k\rangle}_{2s+1} = A^{\langle k\rangle}_{2s}$ and $f(k,n,2s+1) = f(k,n,2s)$ for all k and n.

Stage $2s+2$. For any $e < s$ and for each $k \in J_e$, perform the following action. For any $e' < e$ such that $2s+2$ is $\mathcal{F}^{e',k}$-expansionary and

$$n_e(e',k,2s+1) < l(e',k,2s+1) \tag{2.15}$$

holds, for all $n > n_e(e',k,2s+1)$ such that $f(k,n,2s+1)\downarrow$, let $f(k,n,2s+2)\uparrow$ and put all $x \in I^e_{f(k,n,2s+1)}$ into $A^{\langle k\rangle}$. For all such e, k, e' and n, we say that R_{2e+1} *becomes active* (*via* k, e' and n).

Unless mentioned otherwise, let $A^{\langle k\rangle}_{2s+2} = A^{\langle k\rangle}_{2s+1}$ and $f(k,n,2s+2) = f(k,n,2s+1)$ for all k and n.

This completes the construction.

Verification.

In the following we prove a series of claims showing that A has the required properties, i.e., that A is c.e. and that all requirements are met. Before we turn to these claims, we begin with the following observations for fixed e, $k \in J_e$, $e' < e$, n, m and s. These are, unless mentioned otherwise, immediate by construction.

If $f(k,n,s) \neq f(k,n,s+1)$ holds , then either $f(k,n,s)\uparrow$ and $f(k,n,s+1)\downarrow$ hold and R_{2e} acts via Clause 2 and k and n at stage $s+1$ or $f(k,n,s)\downarrow$ and $f(k,n,s+1)\uparrow$ hold and R_{2e+1} acts via k, e' and n for some $e' < e$. Moreover, if $f(k,n,s)\downarrow$, then $f(k,n',s)\downarrow$ and $f(k,n',s) < f(k,n,s)$ for all $n' < n$. To see this, it suffices to note that if $f(k,n,s+1)\downarrow$ and $f(k,n,s)\uparrow$ hold, then n is minimal with $f(k,n,s)\uparrow$ and, in case $n > 0$, $f(k,n,s+1) > f(k,n-1,s) = f(k,n-1,s+1)$ holds and if $f(k,n,s+1)\uparrow$ and $f(k,n,s)\downarrow$ hold, then, by construction, $f(k,n',s+1)\uparrow$ for all $n' > n$.

Note that $A^{\langle k\rangle} \subseteq \bigcup_{m\geq 0 \& I^e_m\downarrow} I^e_m$. Moreover, $A^{\langle k\rangle}_{s+1} \cap I^e_m \neq A^{\langle k\rangle}_s \cap I^e_m$ implies that $f(k,n',s) = m$ for some n'. Furthermore, if $f(k,n,s) = m$ and $f(k,n,s+1)\uparrow$, then $f(k,n',s'+1) \neq m$ for all n' and for all $s' \geq s$. Namely, for a contradiction fix the least $s' \geq s$ with $f(k,n',s'+1) = m$ for some n' and fix the least corresponding n'. By minimality of s' and by the above, we know that $f(k,n',s')\uparrow$, hence R_{2e} acts via Clause 2 and k and n' at stage $s'+1$, hence, by construction, $A^{\langle k\rangle}_{s'} \cap I^e_m = \emptyset$. On the other hand, by choice of s, R_{2e+1} becomes active via k, e' and n for some $e' < e$ at stage $s+1$, yielding $A^{\langle k\rangle}_{s+1} \cap I^e_m = I^e_m$ by construction. By minimality of s', there is no n''

such that $f(k, n'', s''+1) = m$ for any s'' with $s \leq s'' < s'$, so $A^{\langle k \rangle}_{s'} \cap I^e_m = A^{\langle k \rangle}_{s+1} \cap I^e_m = I^e_m \neq \emptyset$, a contradiction.

Claim 1. A is c.e. via $\{A_s\}_{s \geq 0}$.

Proof. Fix $k \geq 0$ and, for a contradiction, fix x and $s' \geq 0$ such that $A^{\langle k \rangle}_{s'+1}(x) < A^{\langle k \rangle}_{s'}(x)$. Then, by construction, there are e and m such that $k \in J_e$ and $x \in I^e_m$ and such that either R_{2e} or R_{2e+1} becomes active at stage $s'+1$. As requirements of the form R_{2e+1} only put numbers into A, $s' = 2s$ for some s, R_{2e} becomes active via Clause 1 and k and some n such that $f(k, n, 2s) = m$ at stage $2s+1$ and $A^{\langle k \rangle}_{2s+1} \cap I^e_m = W_{n,2s+1} \cap I^e_m$. So by assumption, $x \in A^{\langle k \rangle}_{2s} \setminus W_{n,2s+1}$. Fix the stage $s_x + 1 \leq 2s$ such that x is enumerated into $A^{\langle k \rangle}$ at stage $s_x + 1$. By construction, this can only happen in the following two cases.

> *Case 1. For some n' with $f(k, n', s_x) = m$, R_{2e} becomes active via Clause 1 and k and n' at stage $s_x + 1$.*
>
> We claim that $n' = n$. It then follows by construction that $A^{\langle k \rangle}_{s_x+1} \cap I^e_m = W_{n,s_x+1} \cap I^e_m$. Hence $x \in W_{n,s_x+1}$. But, as $\{W_{n,s}\}_{s \geq 0}$ is a computable enumeration, by $s_x + 1 \leq 2s$, this contradicts the fact that $x \notin W_{n,2s+1}$.
>
> It remains to show that $n' = n$. Assume for a contradiction that $n' \neq n$. Then, by the above observations and since $f(k, n, 2s) = m$, it follows that $f(k, n', 2s) \neq m$, hence there is a stage s'' with $s_x \leq s'' < 2s$ such that $f(k, n', s'') = m$ and $f(k, n', s''+1) \uparrow$. But, again by the above observations, this contradicts the fact that $f(k, n, 2s) = m$.
>
> *Case 2. For some $e' < e$ and n' with $f(k, n', s_x) = m$, R_{2e+1} becomes active via k, e' and n' at stage $s_x + 1$*
>
> Then, by construction, $f(k, n', s_x + 1) \uparrow$, hence, by the above observations, $f(k, n'', s) \neq m$ for all n'' and for all $s > s_x$, contradicting $f(k, n, 2s) = m$ by $2s > s_x$.

This completes the proof of Claim 1. □

For proving that all requirements are met, we need the following definitions. For fixed e, e' with $e' < e$, let

$$k^e_{e'} = \begin{cases} \mu k \in J_e(\mathcal{F}^{e',k} \text{ is a v.s.a.}) & \text{if such a } k \text{ exists} \\ -1 & \text{otherwise.} \end{cases}$$

Furthermore, for all e, let

$$k_e = \mu k \in J_e(k \notin \{k^e_{e'} : e' < e\}).$$

Note that, as $|J_e| = e+1$, k_e exists for all e. For fixed e and for all $e' < e$ and $k \in J_e$, let

$$n_e(e', k) = \lim_{s \to \infty} n_e(e', k, 2s+1).$$

Note that $n_e(e', k) \in \omega \cup \{\infty\}$. Before we turn to proving that the requirements are met, we show an auxiliary claim.

Claim 2. If $\mathcal{I}^e$ is a c.v.s.a.i., then for all e and n, $\lim_{s \to \infty} f(k_e, n, s)$ exists.

Proof. Fix e and n. The proof is by induction on n, so assume the claim holds for each $n' < n$. For $n > 0$, fix the least s_{n-1} with $f(k_e, n-1, 2s_{n-1}+1) = \lim_{s\to\infty} f(k_e, n-1, s)$ and let $s_{-1} = 0$. Moreover, fix the least $s_0 > e$ such that, for all $e' < e$ such that $\mathcal{F}^{e',k_e}$ is not a v.s.a., there is no $\mathcal{F}^{e',k_e}$-expansionary stage $2s$ with $s \geq s_0$ and such that for all $e' < e$ such that $\mathcal{F}^{e',k_e}$ is a v.s.a., $n_e(e', k_e, 2s+1) \geq n$ holds for all $s \geq s_0$.

Note that s_0 exists as, if $\mathcal{F}^{e',k_e}$ is not a v.s.a., there are only finitely many $\mathcal{F}^{e',k_e}$-expansionary stages and, if $\mathcal{F}^{e',k_e}$ is a v.s.a., then $k^e_{e'} < k_e$ by definition of $k^e_{e'}$ and of k_e, hence, by definition of $n_e(e', k_e, 2s+1)$, as $l(e', k^e_{e'}, 2s+1)$ is unbounded in s, so is $n_e(e', k_e, 2s+1)$ (moreover, the latter in nondecreasing in s).

Now, let $s_1 = \max\{s_{n-1}, s_0\}$ and, if $f(k, n, 2s_1+1)\downarrow$, let $s = s_1$. Otherwise, let $s > s_1$ be minimal such that R_{2e} becomes active at stage $2s+1$ via k and n. Note that such a stage exists by choice of s_{n-1} and as $\mathcal{I}^e$ is a c.v.s.a.i. We claim that $f(k, n, 2s+1)\downarrow$ and that $f(k, n, s'+1) = f(k, n, s')$ for all $s' \geq 2s+1$, hence $\lim_{s\to\infty} f(k, n, s)$ exists. The former is immediate by choice of s. It remains to show the latter. For a contradiction fix the least $s' \geq 2s+1$ such that $f(k, n, s'+1) \neq f(k, n, s')$. By minimality of s', $f(k, n, s')\uparrow$. This implies that R_{2e+1} acts via k, e' and n for some $e' < e$ at stage $s'+1$, hence $s' = 2s''+1$ for some s'', $2s''+2$ is $\mathcal{F}^{e',k_e}$-expansionary and $n_e(e', k_e, 2s''+1) < n$. But, by $s'' \geq s_0$, this contradicts the choice of s_0. This completes the proof of Claim 2. □

Claim 3. For all e, R_{2e} is met.

Proof. Fix e and w.l.o.g., assume that $\mathcal{I}^e$ is a c.v.s.a.i. We claim that then, $A^{\langle k_e\rangle}$ is $\mathcal{I}^e$-a.n.c. To prove this, we fix n and show that there is some m such that $A^{\langle k_e\rangle} \cap I^e_m = W_n \cap I^e_m$.

Namely, $m = \lim_{s\to\infty} f(k_e, n, s)$ will do. The existence of m is immediate by Claim 2. So it remains to show that $A^{\langle k_e\rangle} \cap I^e_m = W_n \cap I^e_m$. Fix s_n minimal such that $f(k_e, n, s) = m$ for all $s \geq s_n$. Note that by construction, $A^{\langle k_e\rangle}_{s_n} \cap I^e_m = \emptyset$ and $A^{\langle k_e\rangle} \cap I^e_m$ only changes after stage s_n if either R_{2e+1} acts via k_e, e' and n for some $e' < e$ at some stage $2s+2 > s_n$ or if R_{2e} acts via k_e, n and Clause 1. The former would by construction imply that $f(k_e, n, 2s+2)\uparrow$ which, by $2s+2 > s_n$ contradicts the choice of s_n. The latter happens at any stage $2s+1 > s_n$ such that (2.14) holds for $k = k_e$. Then, it is ensured that $A^{\langle k_e\rangle}_{2s+1} \cap I^e_m = W_{n,2s+1} \cap I^e_m$. Hence, as $W_n \cap I^e_m$ changes at most finitely often after stage s_n, $A^{\langle k_e\rangle} \cap I^e_m = W_n \cap I^e_m$ holds which completes the proof of Claim 3. □

For proving that the requirements R_{2e+1} are met as well, we need one more auxiliary claim.

Claim 4. If $\mathcal{I}^e$ is a c.v.s.a.i., then for all $k \in J_e$ and all $m \geq 0$, there is a stage $2s+1$ such that there is n with $f(k, n, 2s+1) = m$.

Proof. For a contradiction, fix some $k \in J_e$ and the least m such that $f(k, n, 2s+1) \neq m$ for all s and n.

As $\mathcal{I}^e$ is a c.v.s.a.i., we may fix $s_0 > e$ minimal with $I^e_{m,2s_0+1}\downarrow$. For each s, let n_s be the least n such that $f(k, n, 2s)\uparrow$ holds. We claim that for every $s \geq s_0$, for $n = n_s$, Clauses (i) – (iii) in Clause 2 of requirement R_{2e} becoming active hold.

By construction and by assumption on m, it then follows that R_{2e} becomes active via Clause 2 and k and n_s at every stage $2s+1$ with $s \geq s_0$ and $f(k, n_s, 2s+1) = m'$ for some $m' < m$. By the pigeon hole principle, this implies that there are $m_0 < m$, s' and s'' with $s_0 \leq s' < s''$ such that R_{2e} becomes active via Clause 2 and k and $n_{s'}$ at stage $2s'+1$ and $f(k, n_{s'}, 2s'+1) = m_0$ and such that R_{2e} becomes active via Clause 2 and k and $n_{s''}$ at stage $2s''+1$ and $f(k, n_{s''}, 2s''+1) = m_0$. It follows that $f(k, n_{s'}, 2s'') \neq m_0$, hence there is a stage s with $s' \leq s < s''$ such that $f(k, n_{s'}, 2s+1) = m_0$ and $f(k, n_{s'}, 2s+2) \uparrow$. But this contradicts the observations preceding Claim 1.

It remains to show that, for each $s \geq s_0$, (i) – (iii) hold for $n = n_s$. (i) holds by choice of s_0 and of the approximation of $\mathcal{I}^e$. (ii) follows from the assumption on m and from the observations preceding Claim 1. The proof of (iii) for $s \geq s_0$ and $n = n_s$ is by induction on $s \geq s_0$. Assume that for some $s \geq s_0$, for all s' with $s_0 \leq s' < s$, (iii) holds for $s = s'$ and $n = n_{s'}$ and, for a contradiction, assume that (iii) does not hold for $n = n_s$, i.e., that $n_s > 0$ and $f(k, n_s - 1, 2s) = m' \geq m$ hold. By assumption on m, it follows that $m' > m$. Moreover, there is $s' < s$ such that $n_{s'} = n_s - 1$ and R_{2e} becomes active via Clause 2 and k and $n_{s'}$ at stage $2s'+1$ and $f(k, n_{n'}, 2s'+1) = m'$ holds. It follows that $s' > e$ and that m' is minimal such that (i) – (iii) hold for $s = s'$, $n = n_{s'}$ and $m = m'$. So, $I^e_{m',2s'} \downarrow$, hence $s' \geq s_0$. As $s' < s$, it follows by inductive hypothesis that (i) – (iii) hold for $s = s'$ and $n = n_{s'}$ which, by $m < m'$, contradicts the minimality of m'.

This completes the proof of Claim 4. □

Claim 5. For all e, R_{2e+1} is met.

Proof. Fix e, $e' < e$ and $k' \in J_e$ such that $\mathcal{F}^{e',k'}$ is a v.s.a. We claim that then, $\mathcal{F}^{e',k^e_{e'}}$ is a v.s.a. and $A^{\langle k^e_{e'} \rangle}$ is computable and hence not $\mathcal{F}^{e',k^e_{e'}}$-a.n.c. The first part is immediate by definition of $k^e_{e'}$. For a proof of the second part, note that if $\mathcal{I}^e$ is not a c.v.s.a.i., then, by construction, $A^{\langle k \rangle}$ is finite for all $k \in J_e$, hence, in particular, $A^{\langle k^e_{e'} \rangle}$ is finite and hence computable. So w.l.o.g., we assume that $\mathcal{I}^e$ is a c.v.s.a.i. We claim that then, $A^{\langle k^e_{e'} \rangle}$ is cofinite. Note that, by definition of $k^e_{e'}$, the following hold.

$$\max_{k' \in J_e, k' < k^e_{e'}} \lim_{s \to \infty} (l(e', k', s)) < \infty \tag{2.16}$$

$$\lim_{s \to \infty} (l(e', k^e_{e'}, s)) = \infty \tag{2.17}$$

Now, let $M = \{m : \exists n(m = \lim_{s\to\infty} f(k^e_{e'}, n, s)\}$. We prove that for each $m \notin M$, $A^{\langle k^e_{e'} \rangle} \cap I^e_m = I^e_m$ holds. Namely, for fixed $m \notin M$, by Claim 4, fix the least s_m such that $f(k^e_{e'}, n, 2s_m + 1) = m$ for some n together with the corresponding n. By choice of m, it follows that there is s'_m minimal with $2s'_m + 2 > 2s_m + 1$ such that R_{2e+1} acts via $k^e_{e'}$, n and e'' for some $e'' < e$ at stage $2s'_m + 2$, which – as, by minimality of s'_m, $f(k^e_{e'}, n, 2s'_m + 1) = m$ holds – yields $A^{\langle k^e_{e'} \rangle}_{2s'_m+2} \cap I^e_m = I^e_m$, hence, by Claim 1, $A^{\langle k^e_{e'} \rangle} \cap I^e_m = I^e_m$.

To prove that $A^{\langle k^e_{e'} \rangle}$ is cofinite, as $\mathcal{I}^e$ is a c.v.s.a.i., it now suffices to show that M is finite. For that matter, it is enough to show that $N = \{n : \lim_{s\to\infty} f(k^e_{e'}, n, s) \text{ exists}\}$ is finite. Namely, for each $n > n_e(e', k^e_{e'})$, we claim that $n \notin N$. Moreover, by (2.16), $n_e(e', k^e_{e'}) < \infty$. For a proof, assume for a contradiction that there is $n > n_e(e', k^e_{e'})$ such that $\lim_{s\to\infty} f(k^e_{e'}, n, s) = m_0$ exists and fix the least $s' > e$ such that $f(k^e_{e'}, n, s) = m_0$ for all $s \geq s'$. By (2.16) and (2.17), there is an $\mathcal{F}^{e',k^e_{e'}}$-expansionary stage $2s''+2 > s'$ such that (2.15) holds for $k = k^e_{e'}$ and $s = s''$ and such that $n_e(e', k^e_{e'}, 2s''+1) = n_e(e', k^e_{e'})$. Fix the least such s''. Then, by choice of s', $f(k^e_{e'}, n, 2s''+1) \downarrow = m_0$, hence, by $n > n_e(e', k^e_{e'})$ and by choice of s'', R_{2e+1} becomes active at stage $2s''+2$ via $k^e_{e'}$, e' and n yielding $f(k^e_{e'}, n, 2s''+2)\uparrow$, contradicting $2s''+2 > s'$. It follows that N and hence M is finite, so $A^{\langle k^e_{e'}\rangle}$ is cofinite which completes the proof of Claim 5. $\square$

Claims 1, 3 and 4 show that A has the required properties. This completes the proof of Lemma 53. $\square$

Moreover, none of the universality notions of c.e. sets is wtt-invariant. Indeed, in every c.e. wtt-degree, there is a c.e. set which is not universally a.n.c. Namely, for a given c.e. set $\hat{A}$, let $A = \{\langle e, 0\rangle : e \in \hat{A}\}$. Then A is c.e., $A =_{\mathrm{wtt}} \hat{A}$ and $A^{\langle e\rangle}$ is computable for every e, hence A is not universally a.n.c.

The following bounding notion of c.e. sets, however, which is implied by the universality notions is wtt-invariant and, as we will show below, the universality notions and this bounding notion coincide up to wtt-equivalence.

Definition 54. *A c.e. set A (a c.e. degree* **a***) has the* uniform bounding property (u.b.p.) via f *if f is a strictly increasing computable function and, for any v.s.a. $\mathcal{F}$, there is an $\mathcal{F}$-a.n.c. c.e. set B such that B is f-bounded Turing reducible to A (***a***); and A (***a***) has the* uniform bounding property (u.b.p.) *if A has the u.b.p. via some f.*

Note that Downey, Jockusch and Stob [DJS90] have shown that, for any array noncomputable set c.e. A and for any v.s.a. $\mathcal{F}$, there is an $\mathcal{F}$-a.n.c. c.e. set $A_{\mathcal{F}}$ which is wtt-equivalent to A. But a computable bound $f = f_{\mathcal{F}}$ such that $A_{\mathcal{F}}$ is f-bounded Turing reducible to A in general depends on the v.s.a. $\mathcal{F}$. In contrast, for a set A with the uniform bounding property such a bound f exists which does not depend on the very strong array $\mathcal{F}$.

Proposition 55. *Let A be a universally a.n.c. c.e. set. Then A has the uniform bounding property via $f(x) = \langle x, x\rangle$.*

Note that Proposition 55 follows from that fact that, for every set A and for each e, $A^{\langle e\rangle} \leq_{f-\mathrm{T}} A$ for $f(x) = \langle x, x\rangle$.

2.4.2 U.a.n.c. Sets and Multiple Permitting

In this subsection, we look at the variant of multiple permitting guaranteed by c.e. sets with the uniform bounding property. In [DJS90], the a.n.c. c.e. degrees have been introduced in order to capture a certain type of multiple permitting. In [ASa], Ambos-Spies has formalized this notion as follows.

Definition 56 ([ASa])**.** *Let $\mathcal{F} = \{F_n\}_{n\geq 0}$ be a v.s.a., let f be a computable function, let A be a c.e. set and let $\{A_s\}_{s\geq 0}$ be a computable enumeration of A. Then A is $\mathcal{F}$-*permitting via f *and $\{A_s\}_{s\geq 0}$ if, for any partial computable function ψ,*

$$\exists^\infty n\, \forall\, x \in F_n(\psi(x)\downarrow \Rightarrow\ A \upharpoonright f(x)+1 \neq A_{\psi(x)} \upharpoonright f(x)+1)$$

holds.

*A is $\mathcal{F}$-*permitting via f *if there is a computable enumeration $\{A_s\}_{s\geq 0}$ of A such that A is $\mathcal{F}$-permitting via f and $\{A_s\}_{s\geq 0}$; A is $\mathcal{F}$-*permitting *if A is $\mathcal{F}$-permitting via some computable f; and A is* multiply permitting *if A is $\mathcal{F}$-permitting for some v.s.a. $\mathcal{F}$.*

Definition 57 ([ASa])**.** *A c.e. r-degree* **a** *is* multiply permitting *(*$\mathcal{F}$-permitting*,* $\mathcal{F}$-permitting via *f) if there is a c.e. set A in* **a** *such that A is multiply permitting ($\mathcal{F}$-permitting, $\mathcal{F}$-permitting via f).*

In [ASa], Ambos-Spies has shown that, for a v.s.a. $\mathcal{F}$ and a computable function f, a c.e. set A which is $\mathcal{F}$-permitting via f and some computable enumeration $\{A_s\}_{s\geq 0}$ is $\mathcal{F}$-permitting via f and *all* computable enumerations $\{\hat{A}_s\}_{s\geq 0}$ of A. So when dealing with the multiple permitting notions we may drop the reference to the underlying computable enumeration.

Moreover, as shown by Ambos-Spies in [ASa], too, multiple permitting captures the permitting properties of the a.n.c. c.e. sets and multiply permitting c.e. sets are just the c.e. sets which are wtt-equivalent to some a.n.c. c.e. set.

Lemma 58 (Permitting Lemma for A.N.C. C.E. Sets [ASa])**.** *Let $\mathcal{F} = \{F_n\}_{n\geq 0}$ be a v.s.a., let A be an $\mathcal{F}$-a.n.c. c.e. set, let $\{A_s\}_{s\geq 0}$ be a computable enumeration of A and let ψ be a partial computable function. Then*

$$\exists^\infty n \, \forall \, x \in F_n \; (\psi(x) \downarrow \Rightarrow \; x \in A \setminus A_{\psi(x)})$$

holds. Hence, in particular, A is $\mathcal{F}$-permitting via $f(x) = x$.

Theorem 59 ([ASa])**.** *Let* **a** *be a c.e. wtt-degree, let $\mathcal{F}$ be a very strong array and let f be a strictly increasing computable function. The following are equivalent.*

(i) **a** *is array noncomputable.*

(ii) **a** *is multiply permitting.*

(iii) There is an $\mathcal{F}$-a.n.c. c.e. set $A \in$ **a***.*

(iv) There is a c.e. set $A \in$ **a** *such that A is $\mathcal{F}$-permitting via f.*

(v) Any c.e. set $A \in$ **a** *is $\mathcal{F}$-permitting hence multiply permitting.*

By Theorem 59, any multiply permitting set c.e. A is $\mathcal{F}$-permitting for all very strong arrays $\mathcal{F}$. The bound $f_{\mathcal{F}}$ such that A is $\mathcal{F}$-permitting via $f_{\mathcal{F}}$ depends on the v.s.a. $\mathcal{F}$. So the following gives a stronger multiple permitting property defined in the spirit of the uniform bounding property.

Definition 60. *A c.e. set A is* uniformly multiply permitting via f *if f is a strictly increasing computable function and, for any v.s.a. $\mathcal{F}$, A is $\mathcal{F}$-permitting via f; and A is* uniformly multiply permitting *if A is uniformly multiply permitting via some f.*

As one can easily check, the uniform multiple permitting property is wtt-invariant and the wtt-degrees of the uniformly multiply permitting c.e. sets are closed upwards in the c.e. wtt-degrees.

Lemma 61. *Let f and g be strictly increasing computable functions and let A and B be c.e. sets such that A is uniformly multiply permitting via f and $A \leq_{g-\mathrm{T}} B$. Then B is uniformly multiply permitting via $g(f)$.*

Lemma 62. *Let A be c.e. and have the uniform bounding property via f. Then A is uniformly multiply permitting via f.*

Proof. Given a v.s.a. $\mathcal{F}$, it suffices to show that A is $\mathcal{F}$-permitting via f. By choice of A there is an $\mathcal{F}$-a.n.c. c.e. set $\hat{A}$ such that $\hat{A} \leq_{f-\mathrm{T}} A$. By Lemma 58, $\hat{A}$ is $\mathcal{F}$-permitting via $\hat{f}(x) = x$. So, by Lemma 61, A is $\mathcal{F}$-permitting via $f(\hat{f}) = f$. □

In the following subsections, we exploit this fact to examine the wtt- and T-degrees of universally a.n.c. c.e. sets.

2.4.3 On the wtt-Degrees of U.a.n.c. Sets

In this subsection, we show that, up to wtt-equivalence, the various universal array noncomputability notions coincide with each other, with the uniform bounding property and with the uniform multiple permitting property.

Theorem 63. *Let* $\mathbf{a}$ *be a c.e. wtt-degree and let* φ *be a universal function. Then, the following are equivalent.*

(i) $\mathbf{a}$ *is* φ*-universally a.n.c.*

(ii) $\mathbf{a}$ *is uniformly universally a.n.c.*

(iii) $\mathbf{a}$ *is universally a.n.c.*

(iv) $\mathbf{a}$ *has the uniform bounding property.*

(v) Any c.e. set $A \in \mathbf{a}$ *has the uniform bounding property.*

(vi) There is a c.e. set $A \in \mathbf{a}$ *which is uniformly multiply permitting.*

(vii) All c.e. sets $A \in \mathbf{a}$ *are uniformly multiply permitting.*

Proof. The equivalence (iv)⇔(v) is immediate by definition of the uniform bounding property and the equivalence (vi)⇔(vii) is immediate by Lemma 61. Moreover, the implications (i)⇒(ii) and (ii)⇒(iii) hold by Proposition 51 and the implications (iii)⇒(iv) and (iv)⇒(vi) hold by Proposition 55 and Lemma 62, respectively. So it suffices to show the implication (vi)⇒(i). This is done by proving the following two lemmas. □

Lemma 64. *Let* C *be a c.e. set, let* φ *be a universal function and let* $\hat{A} \leq_{\mathrm{wtt}} C$ *(*$\hat{A} \leq_{\mathrm{T}} C$*) be a* φ*-universally a.n.c. c.e. set. Then, there is a* φ*-universally a.n.c. c.e. set* A *with* $A =_{\mathrm{wtt}} C$ *(*$A =_{\mathrm{T}} C$*).*

Proof. We give the proof for the case of wtt-reducibility. For T-reducibility, the proof is the same, we just replace *wtt* by *T* everywhere in the proof.

Fix $e \geq 0$ such that $\mathcal{F}^e$ is not a very strong array and let

$$A(x) = \begin{cases} C(y) & \text{if } x = \langle e, y \rangle \text{ for some } y \geq 0 \\ \hat{A}(x) & \text{otherwise.} \end{cases}$$

Since $\hat{A}$ and C are c.e., it is easy to see that A is c.e., too. As we only change $\hat{A}^{\langle e \rangle}$ and $\mathcal{F}^e$ is not a very strong array, A is φ-universally a.n.c. Since $A = \hat{A} \setminus \{\langle e, x \rangle : x \geq 0\} \,\dot{\cup}\, \{\langle e, x \rangle : x \in C\}$, where the first component of the disjoint union is ibT-reducible to $\hat{A}$ (which is wtt-reducible to C) and the second component is ibT-reducible to C, it is easy to see that A is wtt-reducible to C. Furthermore, $C \leq_{\text{wtt}} A$ as for all x, $x \in C$ if and only if $\langle e, x\rangle \in A$. $\square$

Lemma 65. *Let A be a c.e. set such that A is uniformly multiply permitting and let φ be any universal function. There is a φ-u.a.n.c. c.e. set $\hat{A}$ such that $\hat{A} \leq_{\text{wtt}} A$.*

Proof. Fix a strictly increasing computable function f and a computable enumeration $\{A_s\}_{s\geq 0}$ of A such that A is uniformly multiply permitting via f and $\{A_s\}_{s\geq 0}$. It suffices to enumerate a c.e. set $\hat{A} \leq_{\text{wtt}} A$ such that, for any $e \geq 0$ such that $\mathcal{F}^e$ is a v.s.a. and for any $m \geq 0$,

$$\exists^\infty\, n\; (\hat{A}^{\langle e\rangle} \cap F^e_{\langle m,n\rangle} = W_m \cap F^e_{\langle m,n\rangle}) \tag{2.18}$$

holds.

A computable enumeration $\{\hat{A}_s\}_{s\geq 0}$ of a set $\hat{A}$ with the required properties is obtained by letting $\hat{A}_0 = \emptyset$ and by putting a number $\langle e, x\rangle$ (which is not yet in $\hat{A}_s$) into $\hat{A}_{s+1}$ if and only if there are numbers m, n such that $e, m, n, x < s$ and

- $F^e_{\langle m,n\rangle,s}\downarrow$,
- $x \in (W_{m,s} \cap F^e_{\langle m,n\rangle}) \setminus \hat{A}^{\langle e\rangle}_s$ and
- x is f-permitted by A at stage $s+1$, i.e., $A_{s+1} \upharpoonright f(x)+1 \neq A_s \upharpoonright f(x)+1$

hold.

Obviously, $\hat{A}$ is c.e. and $\hat{A} \leq_{\text{wtt}} A$. So, given e such that $\mathcal{F}^e$ is a v.s.a. and a c.e. set W_m it suffices to show that (2.18) holds. For this sake define the partial computable function ψ on $\bigcup_{n\geq 0} F^e_{\langle m,n\rangle}$ by letting

$$\psi(x) = \mu s > e, m, n, x\; [x \in W_{m,s}]$$

for $x \in F^e_{\langle m,n\rangle}$. Since, by assumption, A is $\{F^e_{\langle m,n\rangle}\}_{n\geq 0}$-permitting via f, there are infinitely many numbers n such that

$$\forall\, x \in F^e_{\langle m,n\rangle}\; (\psi(x)\downarrow \Rightarrow\; A \upharpoonright f(x)+1 \neq A_{\psi(x)} \upharpoonright f(x)+1)$$

holds. But, by definition of $\hat{A}$ and ψ, this implies that $\hat{A}^{\langle e\rangle} \cap F^e_{\langle m,n\rangle} = W_m \cap F^e_{\langle m,n\rangle}$ for any such n. $\square$

We conclude this subsection by observing that the partial ordering $\overline{\text{UANC}_{\text{wtt}}}$ of the c.e. wtt-degrees which are not universally a.n.c. forms an ideal in the c.e. wtt-degrees (i.e., it is closed downwards and under join in the c.e. wtt-degrees). Downward closure of $\overline{\text{UANC}_{\text{wtt}}}$ follows by Lemma 64. In order to obtain the closure under join, by distributivity of the c.e. wtt-degrees (shown by Lachlan; see e.g. Stob [Sto83]) and by Theorem 63, it suffices to show that, for any splitting of a c.e. set A with the uniform multiple permitting property into two disjoint c.e. sets A_0 and A_1, one of these sets is uniformly multiply permitting, too. For a proof we refer to the forthcoming paper [ASL] by Ambos-Spies and Losert.

2.4.4 U.a.n.c. Sets and Sets with the Universal Similarity Property

In the preceding subsection we have given various characterizations of the c.e. wtt-degrees which contain universally a.n.c. c.e. sets. We now look at the relation between universally a.n.c. c.e. sets and almost-c.e. sets with the universal similarity property. We show that any almost-c.e. set with the u.s.p. is wtt-equivalent to a u.a.n.c. c.e. set (Theorem 66). The converse holds only in the following weaker form: for any u.a.n.c. c.e. wtt-degree **a** there is an almost-c.e. set A with the u.s.p. such that $A \leq_{\text{wtt}} \mathbf{a}$ (Theorem 67). Namely while, by Lemma 64, the u.a.n.c. c.e. wtt-degrees are closed upwards in the c.e. wtt-degrees, it follows from Theorem 38 that no wtt-hard almost-c.e. set has the u.s.p.

Theorem 66. *Let* **a** *be a c.e. wtt-degree that contains an almost-c.e. set with the universal similarity property and let φ be a universal function. There is a φ-universally a.n.c. c.e. set $A \in \mathbf{a}$.*

Proof. Fix an almost-c.e. set $\hat{A} \in \mathbf{a}$ with the u.s.p. and let $\{\hat{A}_s\}_{s\geq 0}$ be a computable almost-enumeration of $\hat{A}$. By Lemma 64, it suffices to construct a c.e. set $A \leq_{\text{wtt}} \hat{A}$ which is φ-universally a.n.c. in stages meeting the following requirements for every $e \geq 0$ (where, here and in the following, $e = \langle e_0, e_1\rangle$).

$$\hat{R}_e : \text{If } \mathcal{F}^{e_0} \text{ is a very strong array then there is } n \text{ such that } W_{e_1} \cap F^{e_0}_{\langle e_1,n\rangle} = A^{\langle e_0\rangle} \cap F^{e_0}_{\langle e_1,n\rangle}.$$

It is easy to verify that the requirements guarantee that A is φ-u.a.n.c. Our strategy to meet $\hat{R}_e$ is to wait for a stage t_e such that $F^{e_0}_{\langle e_1,n\rangle,t_e}$ is defined for some n (note that, if $\mathcal{F}^{e_0}$ is a v.s.a., such a stage exists) and to let $A^{\langle e_0\rangle}$ copy W_{e_1} on $F^{e_0}_{\langle e_1,n\rangle}$ from stage t_e on.

To make sure that $A \leq_{\text{wtt}} \hat{A}$, we combine this strategy with permitting. I.e., whenever we need to enumerate a number x into A, we wait for permitting by $\hat{A}$, i.e. for a stage s with $\hat{A}_{s+1} \upharpoonright x+1 \neq \hat{A}_s \upharpoonright x+1$. As this permission may not be given on the fixed finite set $F^{e_0}_{\langle e_1,n\rangle}$, we assign infinitely many sets $B_{e,k} \in \mathcal{F}^{e_0}$, $k \geq 0$, (where each $B_{e,k}$ is of the form $F^{e_0}_{\langle e_1,n\rangle}$ for some n and, for $k < k'$, $n < n'$ holds) to every requirement R_e. Permission to enumerate a number x from $B_{e,k}$ into $A^{\langle e_0\rangle}$ at some stage $s+1$ is given whenever $\hat{A}_{s+1} \upharpoonright x+1 \neq \hat{A}_s \upharpoonright x+1$. For every requirement, exploiting the fact that $\hat{A}$ has the universal similarity property and is thus $\mathcal{F}$-similar to every c.e. set for every v.s.a. $\mathcal{F}$, we enumerate an auxiliary c.e. set V_e to force $\hat{A}$ to change below $x+1$. We show that, by this strategy, for some k, we receive permitting by $\hat{A}$ whenever we wait for it.

In fact, we replace the requirements $\hat{R}_e$ by the following requirements $R_{\langle e,k\rangle}$ for all $e, k \geq 0$.

$$R_{\langle e,k\rangle} : \text{If } \mathcal{F}^{e_0} \text{ is a very strong array then } W_{e_1} \cap B_{e,k} = A^{\langle e_0\rangle} \cap B_{e,k} \text{ or } \hat{A} \cap B_{e,k} \neq V_e \cap B_{e,k}.$$

Assume that A and V_e ($e \geq 0$) are c.e. sets such that $R_{\langle e,k\rangle}$ is met for all $e, k \geq 0$. We show that then $\hat{R}_e$ is met for all $e \geq 0$. Fix e and w.l.o.g. assume that the hypothesis of $\hat{R}_e$ holds, i.e., that $\mathcal{F}^{e_0}$ is a very strong array. Then, for all $k \geq 0$,

$$W_{e_1} \cap B_{e,k} = A^{\langle e_0\rangle} \cap B_{e,k} \text{ or } \hat{A} \cap B_{e,k} \neq V_e \cap B_{e,k} \tag{2.19}$$

holds. Furthermore, $\mathcal{B} = \{B_{e,k}\}_{k\geq 0}$ is a very strong array. As $\hat{A}$ has the universal similarity property, $\hat{A}$ is $\mathcal{B}$-a.c.e-a.n.c., so, since V_e is c.e. and hence $\mathcal{B}$-a.c.e., it follows that there is some $k \geq 0$ such that $\hat{A} \cap B_{e,k} = V_e \cap B_{e,k}$. By (2.19), for this k, $W_{e_1} \cap B_{e,k} = A^{\langle e_0\rangle} \cap B_{e,k}$ holds. As $B_{e,k} = F^{e_0}_{\langle e_1,n\rangle}$ for some $n \geq 0$, it follows that $W_{e_1} \cap F^{e_0}_{\langle e_1,n\rangle} = A^{\langle e_0\rangle} \cap F^{e_0}_{\langle e_1,n\rangle}$, hence $\hat{R}_e$ is met.

It remains to construct c.e. sets A and V_e ($e \geq 0$) such that $R_{\langle e,k\rangle}$ is met for all $e, k \geq 0$. For that matter, for each e and k, we simultaneously aim to let $A^{\langle e_0\rangle}$ copy W_{e_1} on $B_{e,k}$ and to make V_e different from $\hat{A}$ on $B_{e,k}$. We can then argue that, depending on how often $\hat{A}$ changes on $B_{e,k}$, we either receive permitting from $\hat{A}$ on $B_{e,k}$ often enough to copy W_{e_1} on $B_{e,k}$ or $B_{e,k}$ has enough elements to make V_e different from $\hat{A}$ on $B_{e,k}$. We now turn to the construction of A and V_e.

Construction.

At every stage t, if there is $\langle e,k\rangle \leq t$ such that there is a number $m \geq 0$ such that all of the following hold,

- $F^{e_0}_{\langle e_1,m\rangle,t}\downarrow$,
- for any $\langle e',k'\rangle < \langle e,k\rangle$, if $t_{e',k'} < t$ exists, then $\max B_{e',k'} < \min F^{e_0}_{\langle e_1,m\rangle}$ and
- if $k > 0$, then $t_{e,k-1} < t$ exists,

then fix the least such $\langle e,k\rangle$ and the least corresponding m, let $t_{e,k} = t$ and let $B_{e,k} = F^{e_0}_{\langle e_1,m\rangle}$.

If $t_{e,k}$ exists for some $k \geq 0$, we say that $B_{e,k}$ is *assigned* to requirement $R_{e,k}$ at stage $t_{e,k}$ and we say that $B_{e,k}$ is *defined* ($B_{e,k}\downarrow$) at a given stage s if $t_{e,k} \leq s$ exists. Note that whenever $B_{e,k}$ is defined at a stage s, $B_{e,k'}$ is defined at stage s' for all $k' \leq k$ and for all $s' \geq s$. Note further that, by definition of the sets $B_{e,k}$, for $\langle e,k\rangle \neq \langle e',k'\rangle$, $B_{e,k} \cap B_{e',k'} = \emptyset$ holds. Finally, note that, if $\mathcal{F}^{e_0}$ is a v.s.a., then $t_{e,k}$ exists for all k and $\{B_{e,k}\}_{k\geq 0}$ is a v.s.a. We now turn to the definition of A and V_e ($e \geq 0$) in stages where we let A_s and $V_{e,s}$ denote the finite parts of A and V_e, respectively, enumerated by the end of stage s.

Stage 0 is vacuous, i.e., we let $A_0 = \emptyset$ and $V_{e,0} = \emptyset$ for all e.

Stage $s+1$. At stage $s+1$, fix all $e \leq s$ and all k such that $B_{e,k}$ is defined at stage s, such that

$$A^{\langle e_0\rangle}_s \cap B_{e,k} \neq W_{e_1,s+1} \cap B_{e,k} \tag{2.20}$$

and such that, for $x = \min(W_{e_1,s+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_s$, at least one of the following holds.

(i) $\hat{A}_{s+1} \upharpoonright x+1 \neq \hat{A}_s \upharpoonright x+1$.

(ii) $\hat{A}_{s+1} \cap B_{e,k} = V_{e,s} \cap B_{e,k}$.

For all such e and k, perform the following action depending on which of the clauses above holds (if both clauses hold, perform both actions).

(i) Enumerate all numbers $\langle e_0, y\rangle$ such that $y \in (W_{e_1,s+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_s$ into A.

(ii) Enumerate x into V_e.

We say that $R_{\langle e,k\rangle}$ *becomes active* at stage $s+1$ for all such e and k.

This completes the construction.

Verification.

Note that the construction is effective and that the actions taken for the sake of different requirements do not interfere with each other as they are performed on disjoint sets. Moreover, A and all V_e are c.e. by construction. We prove the following claims to show that A has the required properties.

Claim 1. For $e, k \geq 0$, requirement $R_{\langle e,k\rangle}$ is met.

Proof. Fix $e, k \geq 0$. We first make a crucial observation, namely that for all stages s_0, s_1, s_2 with $0 \leq s_0 \leq s_1 \leq s_2$, the following holds.

$$\left.\begin{array}{l} x = \min(W_{e_1,s_0+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_{s_0} \\ x = \min(W_{e_1,s_2+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_{s_2} \end{array}\right\} \Rightarrow x = \min(W_{e_1,s_1+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_{s_1} \tag{2.21}$$

For a proof of (2.21) fix s_0, s_1, s_2 with $0 \leq s_0 \leq s_1 \leq s_2$ such that the hypotheses of (2.21) hold. Then it is easy to see that, since A and W_{e_1} are c.e. and by choice of s_0, s_1 and s_2, $x \in (W_{e_1,s_1+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_{s_1}$ holds, hence, for $y = \min(W_{e_1,s_1+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_{s_1}$, $y \leq x$ holds. Assume for a contradiction that $y < x$. Then, as $y \in W_{e_1,s_1+1}$, $y \in W_{e_1,s_2+1}$ holds, hence $y \in A^{\langle e_0\rangle}_{s_2}$ must hold (because $x = \min(W_{e_1,s_2+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_{s_2}$ and $y < x$). So there is a stage s_y with $s_1 \leq s_y < s_2$ such that y is enumerated into $A^{\langle e_0\rangle}$ at stage s_y+1. By construction, this only happens if $R_{\langle e,k\rangle}$ becomes active via Clause (i) at stage s_y+1. But then, all numbers in $(W_{e_1,s_y+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_{s_y}$ are enumerated into $A^{\langle e_0\rangle}$ at stage s_y+1. From the hypotheses of (2.21), it follows that $x \in (W_{e_1,s_y+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_{s_y}$, hence x is enumerated into $A^{\langle e_0\rangle}$ at stage $s_y + 1$, which, by $s_y < s_2$ contradicts the fact that $x = \min(W_{e_1,s_2+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_{s_2}$. So (2.21) must hold.

We now argue that whenever $R_{\langle e,k\rangle}$ becomes active via Clause (ii) at some stage $s + 1$, one of the following holds.

$$\hat{A}_{s+1} \cap B_{e,k} \neq \hat{A}_s \cap B_{e,k}. \tag{2.22}$$

$$V_{e,s+1} \cap B_{e,k} \neq V_{e,s} \cap B_{e,k}. \tag{2.23}$$

Fix s such that $R_{\langle e,k\rangle}$ becomes active via Clause (ii) at stage $s + 1$. Then, by construction, the following holds.

$$\hat{A}_{s+1} \cap B_{e,k} = V_{e,s} \cap B_{e,k}. \tag{2.24}$$

Furthermore, $x \in V_{e,s+1}$ for $x = \min(W_{e_1,s+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_s$. If $x \notin V_{e,s}$, then, by $x \in B_{e,k}$, (2.23) holds. So assume that $x \in V_{e,s}$. Then, there is a stage $t < s$ such that x is enumerated into V_e at stage $t+1$. As $x \in B_{e,k}$, this only happens if $R_{\langle e,k\rangle}$ becomes active via Clause (ii) at stage $t+1$ and $x = \min(W_{e_1,t+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_t$. This implies that $\hat{A}_{t+1} \cap B_{e,k} = V_{e,t} \cap B_{e,k}$ holds. Moreover, as x is enumerated into V_e at stage $t + 1$, it follows that $\hat{A}_{t+1}(x) = 0 \neq 1 = V_{e,t+1}(x)$, hence, by (2.24) and by $x \in V_{e,s}$, there must be a stage u where $t + 1 \leq u \leq s$ such that $\hat{A}_{u+1}(x) = 1 \neq 0 = \hat{A}_u(x)$. We claim that $x \in A^{\langle e_0\rangle}_{u+1}$, hence, as $x \notin A^{\langle e_0\rangle}_s$ by choice of x, $u = s$ and (2.22) holds. By (2.21) for $s_0 = t$, $s_1 = u$ and $s_2 = s$, we know that $x = \min(W_{e_1,u+1} \cap B_{e,k}) \setminus A^{\langle e_0\rangle}_u$, hence (2.20) holds for $s = u$ and, by $t < u$, $e < u$ and $B_{e,k}$ is defined at stage u. By choice of u, this implies that $R_{\langle e,k\rangle}$ becomes active via Clause (i) at stage $u + 1$, hence x is enumerated into $A^{\langle e_0\rangle}$ at stage $u + 1$.

Now assume for a contradiction that $R_{\langle e,k\rangle}$ is not met. By this assumption, the hypothesis of $R_{\langle e,k\rangle}$ is true, so $\mathcal{F}^{e_0}$ is a very strong array. This implies that $t_{e,k}$ exists and $B_{e,k}$ is defined at all stages $s \geq t_{e,k}$. Furthermore, $W_{e_1} \cap B_{e,k} \neq A^{\langle e_0\rangle} \cap B_{e,k}$ and $\hat{A} \cap B_{e,k} = V_e \cap B_{e,k}$ hold. Fix a stage

$s_0 \geq t_{e,k}$ such that

$$\begin{aligned} W_{e_1,s} \cap B_{e,k} &= W_{e_1} \cap B_{e,k}, \\ A_s^{\langle e_0 \rangle} \cap B_{e,k} &= A^{\langle e_0 \rangle} \cap B_{e,k}, \\ \hat{A}_s \cap B_{e,k} &= \hat{A} \cap B_{e,k}, && (2.25) \\ V_{e,s} \cap B_{e,k} &= V_e \cap B_{e,k} && (2.26) \end{aligned}$$

hold for all $s \geq s_0$. But then, by assumption, (2.20) holds for $s = s_0$ and $\hat{A}_{s_0+1} \cap B_{e,k} = V_{e,s_0} \cap B_{e,k}$, hence $R_{\langle e,k \rangle}$ becomes active via Clause (ii) at stage $s_0 + 1$. But then, either (2.22) or (2.23) holds for $s = s_0$, contradicting either (2.25) or (2.26). It follows that $R_{\langle e,k \rangle}$ is met. □

Claim 2. $A \leq_{\text{wtt}} \hat{A}$.

Proof. For given $x \geq 0$, fix $e_0, y \geq 0$ such that $x = \langle e_0, y \rangle$ and find a stage s_x such that $\hat{A}_s \upharpoonright y + 1 = \hat{A} \upharpoonright y + 1$ for all $s \geq s_x$. We claim that $A = A_{s_x}$. For a contradiction assume that there is a stage $s \geq s_x$ such that $A_{s+1}(x) \neq A_s(x)$. By construction, this only happens if $y \in B_{e,k}$ for some $e, k \geq 0$ (where $e = \langle e_0, e_1 \rangle$ for some $e_1 \geq 0$) and $R_{e,k}$ becomes active at stage $s+1$ via Clause (i). But this implies that $y \in (W_{e_1,s+1} \cap B_{e,k}) \setminus A_s^{\langle e_0 \rangle}$ and that $\hat{A}_{s+1} \upharpoonright \min(W_{e_1,s+1} \cap B_{e,k}) \setminus A_s^{\langle e_0 \rangle} + 1 \neq \hat{A}_s \upharpoonright \min(W_{e_1,s+1} \cap B_{e,k}) \setminus A_s^{\langle e_0 \rangle} + 1$, hence $\hat{A}_s \upharpoonright y + 1 \neq \hat{A} \upharpoonright y + 1$, a contradiction. □

This completes the proof of Theorem 66. □

Theorem 67. *Let* $\mathbf{a}$ *be a c.e. wtt-degree which is u.a.n.c. Then, there is an almost-c.e. set* $A \leq_{\text{wtt}} \mathbf{a}$ *with the universal similarity property.*

Proof. By Theorem 63, fix a c.e. set $\hat{A} \in \mathbf{a}$ and a strictly increasing computable function f such that $\hat{A}$ is uniformly multiply permitting via f and let $\{\hat{A}_s\}_{s \geq 0}$ be a computable enumeration of $\hat{A}$ such that $\hat{A}_s \subseteq \omega \upharpoonright s$. We give a computable almost-enumeration $\{A_s\}_{s \geq 0}$ of a set A such that A has the universal similarity property and $A \leq_{\text{wtt}} \hat{A}$. The latter is ensured by permitting, i.e., by guaranteeing that

$$\forall\, x, s\ (A_{s+1}(x) \neq A_s(x) \ \Rightarrow\ \hat{A}_{s+1} \upharpoonright f(\langle x, 2^{x+1} \rangle) + 1 \neq \hat{A}_s \upharpoonright f(\langle x, 2^{x+1} \rangle) + 1). \tag{2.27}$$

holds. (The term 2^{x+1} in this bound reflects the fact that a computable almost enumeration may change on the initial segment $\omega \upharpoonright x + 1$ up to 2^{x+1} many times.)

Similarly to the proof of Theorem 28, we combine the basic strategy from the proof of Theorem 27 with permitting according to (2.27). As explained there, in order to ensure that A has the universal similarity property, it suffices to meet the following requirements for all $e \geq 0$ where (here and in the following) $e = \langle e_0, e_1 \rangle$.

$$\begin{aligned} R_e :\ & \text{If } \mathcal{I}^{e_0} \text{ is a c.v.s.a.i. and } V_{e_1} \text{ is } \mathcal{I}^{e_0}\text{-almost c.e. via } \{V_{e_1,s}\}_{s \geq 0} \\ & \text{then there is a number } n \text{ such that } A \cap I_n^{e_0} = V_{e_1} \cap I_n^{e_0}. \end{aligned}$$

Recall that the computable enumeration $\{V_{e,s}\}_{e,s \geq 0}$ of computable almost-enumerations $\{V_{e,s}\}_{s \geq 0}$ ($e \geq 0$) has been defined following Theorem 27. The strategy to meet the requirements R_e while guaranteeing (2.27) is essentially the same as in the proof of Theorem 28. Note that here, we do not need coding, so the construction is somewhat simpler. Unless mentioned otherwise, all notions (e.g., eligibility) are defined as in the proof of Theorem 28. We now turn to the formal construction where, as usual, we let A_s denote the finite part of A constructed by the end of stage s.

Construction.

Stage 0 is vacuous. I.e., $A_0 = \emptyset$ and no requirement has a follower at the end of stage 0 (i.e., all requirements are initialized at stage 0).

Stage $s+1$. A requirement R_e *requires attention* at stage $s+1$ if $e < s$, R_e is eligible at stage $s+1$ and one of the following holds.

(i) No follower is assigned to R_e at the end of stage s.

(ii) (i) does not hold, $x_{e,0} < \cdots < x_{e,n}$ $(n \geq 0)$ are the followers assigned to R_e at the end of stage s and there is a number $k < n$ such that either

   (a) $A_s(x_{e,k}) = 0$ and $\hat{A}_{s+1} \upharpoonright f(\langle x_{e,k}, 2^{x_{e,k}+1}\rangle) + 1 \neq \hat{A}_s \upharpoonright f(\langle x_{e,k}, 2^{x_{e,k}+1}\rangle) + 1$
   or

   (b) $A_s(x_{e,k}) = 1$, $A_s \cap J_{e,k} \neq V_{e_1,s+1} \cap J_{e,k}$ for the interval $J_{e,k}$ associated with $x_{e,k}$, and $\hat{A}_{s+1} \upharpoonright f(\langle x, 2^{x+1}\rangle) + 1 \neq \hat{A}_s \upharpoonright f(\langle x, 2^{x+1}\rangle) + 1$ for the least $x \in J_{e,k}$ such that $A_s(x) \neq V_{e_1,s}(x)$,

   holds.

(iii) (i) and (ii) do not hold, $x_{e,0} < \cdots < x_{e,n}$ $(n \geq 0)$ are the followers assigned to R_e at the end of stage s, for all $k < n$, $A_s(x_{e,k}) = 0$ or $A_s \cap J_{e,k} \neq V_{e_1,s} \cap J_{e,k}$ holds where $J_{e,k}$ is the interval associated with $x_{e,k}$, and there is a number $m \leq s$ such that $I^{e_0}_{m,s+1} \downarrow$ and $x_{e,n} < \min I^{e_0}_m$ (note that $\max I^{e_0}_m < s+1$).

Fix $e < s$ minimal such that R_e requires attention and perform the following action corresponding to the clause via which R_e requires attention.

(i) Appoint $x_{e,0} = s+1$ as a follower to R_e.

(ii) For all k that make Clause (ii) in the definition of requiring attention true, perform the following action depending on which subclause holds.

   (a) Put $x_{e,k}$ into A_{s+1} and let $A_{s+1} \cap J_{e,k} = V_{e_1,s+1} \cap J_{e,k}$.

   (b) Let $A_{s+1} \cap J_{e,k} = V_{e_1,s+1} \cap J_{e,k}$.

(iii) For the least m that makes Clause (iii) in the definition of requiring attention true, associate the interval $J_{e,n} = I^{e_0}_m$ with the follower $x_{e,n}$ of R_e. Furthermore, appoint $x_{e,n+1} = s+1$ as a further follower to R_e.

In any of the subcases (i) – (iii) say that R_e *receives attention* or *becomes active*. Furthermore, for all $e' > e$, initialize $R_{e'}$, i.e., cancel all corresponding assignments of followers and intervals.

(If not explicitly stated otherwise, any parameter depending on the stage is unchanged at stage $s+1$.)

This completes the construction. Note that the construction ensures that the followers of R_e are appointed in order of magnitude and that the greatest follower is the only follower which has not yet an interval associated with it.

Verification.

Note that the construction is effective and that the observations made at the beginning of the verification in the proof of Theorem 28 hold here, too (if applicable). In order to show that A has the required properties, we prove a series of claims, similarly as in the proof of Theorem 28.

Claim 1. A is a.c.e. via $\{A_s\}_{s\geq 0}$.

Proof. The proof is the same as the proof of Claim 2 in the proof of Theorem 28. □

Claim 2. $A \leq_{\rm wtt} \hat{A}$.

Proof. By Claim 1, it suffices to show that (2.27) holds. But this is immediate by Clause (ii) in the definition of requiring attention since any possible change of A at stage $s+1$ is determined by the action of a requirement becoming active according to this clause at stage $s+1$. □

Claim 3. For any $e \geq 0$, requirement R_e requires attention only finitely often.

Proof. The proof is by induction on e. Fix e and, by inductive hypothesis, assume the claim to be correct for $e' < e$. Let s_0 be the greatest stage s such that $s = 0$ or $R_{e'}$ requires attention at stage s for some $e' < e$. Note that such a stage s_0 exists by inductive hypothesis. Moreover, R_e is initialized at stage s_0 and, after stage s_0, R_e becomes active whenever it requires attention and any follower or interval assigned to R_e after stage s_0 is permanent since R_e is not initialized anymore. So if $x_{e,k} = x_{e,k}[s]$ for some $s > s_0$ then $x_{e,k} = x_{e,k}[s']$ for all $s' \geq s$ and similarly for $J_{e,k}$.

Now, for a contradiction, assume that requirement R_e requires attention infinitely often. Note that R_e requires attention via (i) at most once after stage s_0. Moreover, for fixed k, R_e requires attention via (ii) and k at most finitely often since, by (2.1) and by definition of requiring attention, this only happens at a stage $s+1$ such that $\hat{A}_{s+1} \upharpoonright f(\langle x, 2^{x+1}\rangle) + 1 \neq \hat{A}_s \upharpoonright f(\langle x, 2^{x+1}\rangle) + 1$ for $x = \max J_{e,k}$. So R_e requires attention via (iii) infinitely often.

It follows that there are infinitely many followers and corresponding intervals assigned to R_e. So we can effectively list the (permanent) followers $x_{e,0} < x_{e,1} < x_{e,2} < \dots$ existing after stage s_0 and the corresponding intervals $J_{e,0}, J_{e,1}, J_{e,2}, \dots$ where $x_{e,n}$ is appointed at stage $x_{e,n} > s_0$ and $J_{e,k}$ is assigned at stage $x_{e,n+1}$. Note that, by permanence of the followers and intervals, $x_{e,k} \in A_{s+1}$ if and only if there is a stage t such that $x_{e,k+1} \leq t \leq s$ and R_e acts via (ii)(a) and k at stage $t+1$ and, for $s \geq x_{e,k+1}$,

$$A_{s+1} \cap J_{e,k} \neq A_s \cap J_{e,k} \text{ if and only if } R_e \text{ acts via (ii)(a) or (ii)(b) and } k \text{ at stage } s+1. \quad (2.28)$$

Now, by noncomputability of $\hat{A}$, there are infinitely many k such that $\hat{A}$ changes below $x_{e,k}$ (hence below $f(\langle x_{e,k}, 2^{x_{e,k}+1}\rangle) + 1$) after stage $x_{e,k+1}$. So, for infinitely many k, R_e requires attention (hence becomes active) via Clause (ii)(a) and $x_{e,k}$. So we may fix strictly increasing computable sequences of indices $\{k_n\}_{n\geq 0}$ and stages $\{t_n\}_{n\geq 0}$ such that $t_n \geq x_{e,k_n+1}$ and R_e acts via (ii)(a) and x_{e,k_n} at stage t_n+1 whence

$$\forall\, n\, \forall\, s \geq t_n\ (A_s(x_{e,k_n}) = 1). \tag{2.29}$$

Since R_e requires attention via Clause (iii) infinitely often, it follows that

$$\forall\, n\ (A \cap J_{e,k_n} \neq V_{e_1} \cap J_{e,k_n}).$$

So, for any n we may let x_n be the least x in J_{e,k_n} such that $A(x) \neq V_{e_1}(x)$. In order to get the desired contradiction, we show that, for any n, there is a stage t''_n such that

$$V_{e_1,t''_n+1}(x_n) \neq V_{e_1,t''_n}(x_n)\ \&\ \hat{A} \restriction f(\langle x_n, 2^{x_n+1}\rangle) + 1 = \hat{A}_{t''_n} \restriction f(\langle x_n, 2^{x_n+1}\rangle) + 1. \tag{2.30}$$

Then we use the fact that $\hat{A}$ is uniformly multiply permitting via f in order to show that (2.30) has to fail for some n.

We first turn to the proof of (2.30). By construction, there is a strictly increasing computable sequence $m_0 < m_1 < m_2 < \ldots$ such that $J_{e,k_n} = I^{e_0}_{m_n}$. Hence $\mathcal{I}^{e_0}$ is a c.v.s.a.i. and the subarray $\mathcal{J} = \{J_{e,k_n}\}_{n\geq 0} = \{I^{e_0}_{m_n}\}_{n\geq 0}$ of $\mathcal{I}^{e_0}$ is a v.s.a. Since R_e requires attention infinitely often – hence e is eligible infinitely often – this implies that the computable almost-enumeration $\{V_{e_1,s}\}_{s\geq 0}$ of V_{e_1} is compatible with $\mathcal{I}^{e_0}$ hence compatible with $\mathcal{J}$. So, for any n,

$$\forall\, x \in J_{e,k_n}\ \forall\, s\ (x \in V_{e_1,s} \setminus V_{e_1,s+1} \ \Rightarrow\ \exists\, y < x\ (y \in J_{e,k_n}\ \&\ y \in V_{e_1,s+1} \setminus V_{e_1,s})) \tag{2.31}$$

holds.

Now, for any n, let t'_n be the greatest stage t' such that R_e requires attention via Clause (ii) and x_{e,k_n} at stage $t'+1$; and let t''_n be the least stage t'' such that $V_{e_1,s}(x_n) = V_{e_1}(x_n)$ for all $s > t''$.

Note that, by choice of t_n, t'_n exists and $t_n \leq t'_n$. Moreover, by R_e being active at stage t'_n+1 via Clause (ii) and x_{e,k_n}, the approximations of A and V_{e_1} agree on J_{e,k_n} at stage t'_n+1 and, by maximality of t'_n and by (2.28), the approximation of A on J_{e,k_n} does not change after stage t'_n+1. Hence

$$\forall\, s > t'_n\ (A \cap J_{e,k_n} = A_s \cap J_{e,k_n} = A_{t'_n+1} \cap J_{e,k_n} = V_{e_1,t'_n+1} \cap J_{e,k_n}). \tag{2.32}$$

Moreover, since, for $y < x_n$ such that $y \in J_{e,k_n}$, $A(y) = V_{e_1}(y)$ it follows by (2.32) and (2.31) that the approximation of $V_{e_1}(y)$ does not change after stage t'_n+1 whence (by (2.32))

$$\forall\, s > t'_n\ \forall\, y < x_n\ (y \in J_{e,k_n}\ \Rightarrow\ V_{e_1}(y) = V_{e_1,s}(y) = A_s(y) = A(y)). \tag{2.33}$$

Finally, since, by choice of t''_n, $V_{e_1,s}(x_n) = V_{e_1}(x_n) \neq A(x_n)$ for all $s > t''_n$, it follows by (2.32) that $t'_n < t''_n$ and $V_{e_1,s}(x_n) \neq A_s(x_n)$ for all $s > t''_n$ whence, by (2.33),

$$\forall\, s > t''_n\ (x_n = \mu x \in J_{e,k_n}[A_s(x) \neq V_{e_1,s}(x)]). \tag{2.34}$$

Furthermore, by $t''_n > 0$ and by minimality of t''_n,

$$V_{e_1,t''_n+1}(x_n) \neq V_{e_1,t''_n}(x_n). \tag{2.35}$$

Now, by $t'_n < t''_n$, R_e does not require attention via Clause (ii) and x_{e,k_n} at any stage $s \geq t''_n + 1$. Since $A_s(x_{e,k_n}) = 1$ for such s by (2.29) and by $t_n \leq t''_n$, it follows by (2.34) and by definition of requiring attention that

$$\hat{A} \upharpoonright f(\langle x_n, 2^{x_n+1} \rangle) + 1 = \hat{A}_{t''_n} \upharpoonright f(\langle x_n, 2^{x_n+1} \rangle) + 1. \tag{2.36}$$

Obviously, (2.35) and (2.36) imply (2.30).

Having established (2.30), in the remainder of the proof we use the fact that $\hat{A}$ is uniformly multiply permitting via f to refute (2.30) for some n.

Since $\mathcal{J}$ is a v.s.a., $\mathcal{F} = \{F_n\}_{n \geq 0}$ where

$$F_n = \{\langle x, y \rangle : x \in J_{e,k_n} \ \&\ y \leq 2^{x+1}\}$$

is a very strong array, too. Define the partial computable function ψ on $\bigcup_{n \geq 0} F_n$ as follows. For $x \in J_{e,k_n}$ and $y \leq 2^{x+1}$, let $\psi(\langle x, y \rangle)$ be the least stage s such that $V_{e_1,t+1}(x) \neq V_{e_1,t}(x)$ for y many stages $t < s$ (if such an s exists). Then, as $\hat{A}$ is uniformly multiply permitting via f, there are infinitely many numbers n such that

$$\forall \langle x, y \rangle \in F_n \ (\psi(\langle x, y \rangle) \downarrow \Rightarrow \ \hat{A} \upharpoonright f(\langle x, y \rangle) + 1 \neq \hat{A}_{\psi(\langle x, y \rangle)} \upharpoonright f(\langle x, y \rangle) + 1). \tag{2.37}$$

Fix such an n. Let $p(x_n)$ be the number of stages t such that $V_{e_1,t+1}(x_n) \neq V_{e_1,t}(x_n)$. Since $\{V_{e_1,s}\}_{s \geq 0}$ is a computable almost enumeration, $p(x_n) \leq 2^{x_n+1}$. So $\psi(\langle x_n, p(x_n) \rangle)$ is defined. Moreover, by definition of ψ and by the first part of (2.30), $\psi(\langle x_n, p(x_n) \rangle) > t''_n$. It follows, by the second part of (2.30), that $\hat{A} \upharpoonright f(\langle x_n, 2^{x_n+1} \rangle) + 1 = \hat{A}_{\psi(\langle x_n, p(x_n) \rangle)} \upharpoonright f(\langle x_n, 2^{x_n+1} \rangle) + 1$. But this contradicts (2.37).

This completes the proof of Claim 3. □

Claim 4. For $e \geq 0$, requirement R_e is met.

Proof. This follows from Claim 3 exactly as Claim 4 in the proof of Theorem 28 follows from Claim 3 there. □

Claims 1, 2 and 4 imply that A has the required properties. This completes the proof of the theorem. □

2.4.5 On the T-Degrees of U.a.n.c. Sets

We now turn to the Turing degrees of universally a.n.c. c.e. sets. First note that the construction from the proof of Theorem 67 can be combined with coding in the same way as in the proof of Theorem 28. This fact, together with Theorem 63, yields the following corollary.

Corollary 68. *Let* $\mathbf{a}$ *be a c.e. Turing degree and let* φ *be a universal function. Then, the following are equivalent.*

(i) $\mathbf{a}$ *is* φ*-universally a.n.c.*

(ii) $\mathbf{a}$ *is uniformly universally a.n.c.*

(iii) $\mathbf{a}$ *is universally a.n.c.*

(iv) $\mathbf{a}$ *has the uniform bounding property.*

(v) $\mathbf{a}$ *is uniformly multiply permitting.*

(vi) There is an almost-c.e. set $A \in \mathbf{a}$ *which has the universal similarity property.*

Together with Theorem 30, this directly implies the following Theorem.

Theorem 69. *For a c.e. Turing degree* $\mathbf{a}$ *and a universal function* φ*, the following are equivalent.*

(i) $\mathbf{a}$ *is not totally* ω*-c.e.*

(ii) $\mathbf{a}$ *is* φ*-universally a.n.c.*

In the following, we prove Theorem 69 directly. We begin with the direction (i)⇒(ii).

Theorem 70. *Let* $\mathbf{a}$ *be a c.e. Turing degree that is not totally* ω*-c.e. and let* φ *be a universal function. Then, there is a c.e. set* $A \in \mathbf{a}$ *which is* φ*-universally a.n.c.*

Proof. By Lemma 64, it suffices to construct a c.e. set $A \leq_{\mathrm{T}} \mathbf{a}$ with the required property in stages meeting the following requirements for every $e \geq 0$ where, here and in the following, $e = \langle e_0, e_1 \rangle$.

$$R_e : \text{If } \mathcal{F}^{e_0} \text{ is a very strong array then there is } n \text{ such that } W_{e_1} \cap F^{e_0}_{\langle e_1, n\rangle} = A^{\langle e_0 \rangle} \cap F^{e_0}_{\langle e_1, n\rangle}. \quad (2.38)$$

As in the proof of Theorem 66, the requirements guarantee that A is φ-universally a.n.c. The strategy to meet R_e is the same as the strategy to meet $\hat{R}_e$ in the proof of Theorem 66 explained there. To make sure that $A \leq_{\mathrm{T}} \mathbf{a}$, again, we combine this strategy with permitting. I.e., whenever we want to enumerate a number x into A, we wait for permitting by some function $g \leq_{\mathrm{T}} \mathbf{a}$. As, for some e, this permission may not be given on a fixed finite set $F^{e_0}_{\langle e_1, n\rangle}$, we again assign infinitely many sets $B_{e,k} \in \mathcal{F}^{e_0}$, $k \geq 0$, to every requirement R_e, just as in the proof of Theorem 66. Here, permission to enumerate numbers from $B_{e,k}$ into A at some stage $s+1$ is given whenever $g_{s+1}(k) \neq g_s(k)$. We use the fact that we may choose g to be not ω-c.e., i.e., the mind changes of any computable approximation of g cannot be computably bounded. Exploiting this fact, by a typical not-totally-ω-c.e. permitting argument similar to the one from the proof of Theorem 28, we show that for some k, we receive permitting by g whenever we wait for it.

Fix a c.e. set $C \in \mathbf{a}$ and a function $g \leq_{\mathrm{T}} C$ together with a Turing functional Γ such that g is not ω-c.e. and such that $\Gamma^C = g$. Let $\{g_s\}_{s \geq 0}$ be the computable approximation of g where $g_s = \Gamma_s^{C_s}$ for some fixed computable enumeration of C. We now turn to the formal construction of A where we let A_s be the finite part of A enumerated by the end of stage s.

Construction.

For $e, k \geq 0$, we define the stages $t_{e,k}$ and the sets $B_{e,k}$ just as in the proof of Theorem 66, we say that $B_{e,k}$ is *assigned* to requirement R_e at stage $t_{e,k}$ if the latter exists and we say that $B_{e,k}$ is *defined* at a stage s if $t_{e,k} \leq s$ exists. Note that the observations on the stages $t_{e,k}$ and the sets $B_{e,k}$ from the proof of Theorem 66 hold here, as well. We now turn to the definition of A in stages.

Stage 0. At stage 0, let $A_0 = \emptyset$.

Stage $s+1$. At stage $s+1$, for all $e \leq s$ and all k such that $B_{e,k}$ is defined at stage s, such that

$$B_{e,k} \cap A_s^{\langle e_0 \rangle} \neq B_{e,k} \cap W_{e_1,s+1}$$

and such that $g_{s+1}(k) \neq g_s(k)$, enumerate all numbers $\langle e_0, x\rangle$ such that $x \in (B_{e,k} \cap W_{e_1,s+1}) \setminus A_s^{\langle e_0 \rangle}$ into A. We say that R_e *becomes active* via k at stage $s+1$ for all such e and k.

This completes the construction.

Verification.

Note that the construction is effective and that the actions taken for the sake of different requirements do not interfere with each other. Note that, by construction, A is c.e. Moreover, for e, k such that $B_{e,k}$ is defined at some point, numbers of the form $\langle e_0, x\rangle$ with $x \in B_{e,k}$ only enter A at some stage $s+1$ if $x \in W_{e_1,s+1}$. It follows that for all s, the following holds.

$$A_s^{\langle e_0 \rangle} \cap B_{e,k} \subseteq W_{e_1,s} \cap B_{e,k} \subseteq W_{e_1,s+1} \cap B_{e,k}. \tag{2.39}$$

We prove the following claims to show that A has the required properties.

Claim 1. For $e \geq 0$*, requirement* R_e *is met.*

Proof. Fix $e \geq 0$ and for a contradiction assume that R_e is not met. We show that there is a computable approximation $\{\tilde{g}_n\}_{n\geq 0}$ of g for which the number of mind changes is computably bounded whence g is ω-c.e. contrary to choice of g.

By assumption, the hypothesis of R_e is true, so $\mathcal{F}^{e_0}$ is a very strong array. This implies that for all $k \geq 0$, $t_{e,k}$ exists and $B_{e,k}$ is defined at all stages $s \geq t_{e,k}$. Note that, by construction, $\max B_{e,k} < \min B_{e,k+1}$ for all $k \geq 0$. As $B_{e,k} \in \{F^{e_0}_{\langle e_0,n\rangle}\}_{n\geq 0}$ for all k, by assumption $A \cap B_{e,k} \neq W_{e_1} \cap B_{e,k}$ holds for all k. This implies that there is an infinite computable set of stages $\{s_0, s_1, \ldots\}$ with $s_0 > e$ and such that $s_n \geq t_{e,n}$ for all n and such that

$$\forall\, k < n\ (A_{s_n} \cap B_{e,k} \neq W_{e_1,s_n+1} \cap B_{e,k}) \tag{2.40}$$

holds. The latter implies that, for any k, n with $k < n$,

$$\begin{aligned} &\text{if } g_{s_{n+1}}(k) \neq g_{s_n}(k) \text{ then there is a stage } s \text{ such that } s_n < s \leq s_{n+1} \text{ and} \\ &\qquad\qquad\qquad\qquad \text{such that } W_{e_1,s+1} \cap B_{e,k} \neq W_{e_1,s} \cap B_{e,k} \text{ holds.} \end{aligned} \tag{2.41}$$

holds. For a proof of (2.41), fix $k < n$ such that $g_{s_{n+1}}(k) \neq g_{s_n}(k)$ holds. Fix t minimal with $s_n \leq t < s_{n+1}$ such that $g_{t+1}(k) \neq g_t(k)$. Assume for a contradiction that $W_{e_1,s+1} \cap B_{e,k} = W_{e_1,s} \cap B_{e,k}$ for all s with $s_n < s \leq s_{n+1}$. Then, in particular, $W_{e_1,t+1} \cap B_{e,k} = W_{e_1,s_n+1} \cap B_{e,k}$. Furthermore, by construction and by minimality of t, $A_t^{\langle e_0 \rangle} \cap B_{e,k} = A_{s_n}^{\langle e_0 \rangle} \cap B_{e,k}$, so, by (2.40), $A_t^{\langle e_0 \rangle} \cap B_{e,k} \neq W_{e_1,t+1} \cap B_{e,k}$. This implies that R_e becomes active via k at stage $t+1$, so by (2.39) for $s = t$, $A_{t+1}^{\langle e_0 \rangle} \cap B_{e,k} = W_{e_1,t+1} \cap B_{e,k}$ holds. Since, by assumption, $W_{e_1,s_{n+1}+1} \cap B_{e,k} = W_{e_1,t+1} \cap B_{e,k}$ and, by construction, $A_{s_{n+1}}^{\langle e_0 \rangle} \cap B_{e,k} = A_{t+1}^{\langle e_0 \rangle} \cap B_{e,k}$, it follows that $W_{e_1,s_{n+1}+1} \cap B_{e,k} = A_{s_{n+1}}^{\langle e_0 \rangle} \cap B_{e,k}$ holds which contradicts (2.40) for $n+1$ in place of n.

Since $\{W_{e_1,s}\}_{s\geq 0}$ is a computable enumeration there are at most $|B_{e,k}|$ many stages such that W_{e_1} changes in $B_{e,k}$. Hence (2.41) implies that

$$|\{n > k : g_{s_{n+1}}(k) \neq g_{s_n}(k)\}| \leq |B_{e,k}|.$$

So, if we define the computable approximation $\{\tilde{g}_n\}_{n\geq 0}$ of g by letting

$$\tilde{g}_n(k) = g_{s_n}(k),$$

then the number of mind changes of $\tilde{g}_n$ on k is bounded by $|B_{e,k}| + k + 1$. So g is ω-c.e. contrary to choice of g. □

Claim 2. $A \leq_{\mathrm{T}} C$.

Proof. For a given $x \geq 0$, by $g = \Gamma^C$, fix a stage s_x such that $g_t \restriction x+1 = g \restriction x+1$ holds for all $t \geq s_x$. It suffices to show that $A_{s+1}(x) = A_s(x)$ for all $s \geq s_x$ whence $A(x) = A_{s_x}(x)$.

Fix s such that $A_{s+1}(x) \neq A_s(x)$. Fix $e_0, y \geq 0$ such that $x = \langle e_0, y\rangle$. Then, by construction, there are numbers e_1 and k such that, for $e = \langle e_0, e_1\rangle$, $B_{e,k}$ is defined at stage s, $y \in B_{e,k}$ and such that $g_{s+1}(k) \neq g_s(k)$. Since, by construction, $B_{e,k'}$ is defined at stage s for all $k' < k$ and $\max B_{e,k'} < \min B_{e,k} \leq y$ for all such k', it follows that $k \leq y \leq x$ whence $g_{s+1} \restriction x+1 \neq g_s \restriction x+1$. So $s < s_x$. □

This completes the proof of Theorem 70. □

We now turn to the proof of the direction (ii)⇒(i) in Theorem 69. Note that in [ASL], there is an alternative proof of this direction using a result on (non-) wtt-reducibility to hypersimple sets by Barmpalias, Downey and Greenberg [BDG10]. Here, however, we give a direct proof using not-totally-ω-c.e. permitting.

Theorem 71. *Let A be a c.e. set such that $deg_{\mathrm{T}}(A)$ is totally ω-c.e. Then, A is not universally a.n.c.*

Before giving the proof of Theorem 71, we state a technical lemma needed.

Lemma 72. *Let A be a noncomputable c.e. set such that $deg_{\mathrm{T}}(A)$ is totally ω-c.e. Then, there are a computable enumeration $\{A_s\}_{s\geq 0}$ of A with $A_0 = \emptyset$ and a complete very strong array of intervals $\mathcal{I} = \{I_n\}_{n\geq 0}$ such that the following holds.*

$$\forall n \forall e < n |\{s : A^{\langle e\rangle}_{s+1} \cap \bigcup_{n'\leq n} I_{n'} \neq A^{\langle e\rangle}_s \cap \bigcup_{n'\leq n} I_{n'}\}| < |I_n| - 1. \tag{2.42}$$

We first show how Theorem 71 follows from Lemma 72.

Proof of Theorem 71 assuming Lemma 72. If A is computable, then the claim is straightforward. So assume that A is noncomputable. By Lemma 72, fix a computable enumeration $\{A_s\}_{s\geq 0}$ of A with $A_0 = \emptyset$ and a complete very strong array of intervals $\mathcal{I} = \{I_n\}_{n\geq 0}$ such that (2.42) holds. For given $e \geq 0$, we show that $A^{\langle e\rangle}$ is not $\mathcal{I}$-a.n.c. By Definition 49, it is enough construct a c.e. set V_e such that the following holds.

$$\forall n > e(A^{\langle e\rangle} \cap I_n \neq V_e \cap I_n). \tag{2.43}$$

We give the construction of V_e in stages. We aim to make V_e different from $A^{\langle e\rangle}$ on each I_n for each $n > e$. By Lemma 72, we can argue that our strategy is successful.

At stage 0, let

$$V_{e,0}(x) = \begin{cases} 1 & \text{if } x = \max I_n \text{ for some } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

At stage $s+1$, fix all $n > e$ such that there is at least one number x with $x \in I_n$ such that $A^{\langle e \rangle}_{s+1}(x) \neq A^{\langle e \rangle}_s(x)$. For all such n, if $A^{\langle e \rangle}_{s+1} \cap I_n = V_{e,s} \cap I_n$, enumerate the greatest number y with $y \in I_n \setminus V_{e,s}$ (if any) into V_e.

Note that $V_e = \lim_{s\to\infty} V_{e,s}$ is c.e. We now show that (2.43) holds. Fix $n > e$. We show that for all $s \geq 0$,

$$A^{\langle e \rangle}_s \cap I_n \neq V_{e,s} \cap I_n \tag{2.44}$$

holds. The proof is by induction on s. For $s = 0$, $A^{\langle e \rangle}_0 \cap I_n = \emptyset \neq V_{e,0} \cap I_n$. Now assume that (2.44) holds for some $s \geq 0$. If there is no $x \in I_n$ such that $A^{\langle e \rangle}_{s+1}(x) \neq A^{\langle e \rangle}_s(x)$, by construction, $V_{e,s+1} \cap I_n = V_{e,s} \cap I_n \neq A^{\langle e \rangle}_s \cap I_n = A^{\langle e \rangle}_{s+1} \cap I_n$, hence (2.44) holds for $s+1$ in place of s. So assume that there is such a number $x \in I_n$. In case $A^{\langle e \rangle}_{s+1} \cap I_n \neq V_{e,s} \cap I_n$, by construction, $V_{e,s+1} \cap I_n = V_{e,s} \cap I_n$, hence (2.44) holds for $s+1$ in place of s in this case, too. So assume furthermore that $A^{\langle e \rangle}_{s+1} \cap I_n = V_{e,s} \cap I_n$. We claim that in this case there is a number y with $y \in I_n \setminus V_{e,s}$. Then, by construction, the greatest such y is enumerated into V_e at stage $s+1$, hence $V_{e,s+1} \cap I_n \neq V_{e,s} \cap I_n = A^{\langle e \rangle}_{s+1} \cap I_n$, so (2.44) holds for $s+1$ in place of s.

It remains to show that y exists, i.e., that $|V_{e,s} \cap I_n| < |I_n|$. Note that for all $z \in V_{e,s} \cap I_n$, $z = \max I_n$ or there is a stage $t \geq 0$ such that z is enumerated into V_e at stage $t+1$. By construction, the latter only happens if there is a number $w \in I_n$ such that $A^{\langle e \rangle}_{t+1}(w) \neq A^{\langle e \rangle}_t(w)$. By (2.42), there are at most $|I_n| - 2$ such stages t, hence there are at most $|I_n| - 1$ numbers in $V_{e,s} \cap I_n$. It follows that $|V_{e,s} \cap I_n| < |I_n|$. □

We now turn to the proof of Lemma 72.

Proof of Lemma 72. Fix h as in Lemma 42 (a), let f be the function $f(x) = \langle 2h(x), 2h(x) \rangle$ and fix the corresponding computable enumeration $\{A_s\}_{s\geq 0}$ of A with $A_0 = \emptyset$ and the infinite computable set D such that

$$\forall\, x \in D\ (|\{s : A_{s+1} \upharpoonright f(x) \neq A_s \upharpoonright f(x)\}| < h(x)) \tag{2.45}$$

holds. (Lemma 42 is stated for an almost-c.e. set A. It is easy to see that the computable almost-enumeration $\{A_s\}_{s\geq 0}$ of A given in the proof of this lemma in [ASLM] is actually a computable enumeration with $A_0 = \emptyset$ if we start with a c.e. set A together with a computable enumeration $\{\hat{A}_s\}_{s\geq 0}$ of A with $\hat{A}_0 = \emptyset$.)

Then the desired c.v.s.a.i. $\mathcal{I} = \{I_n\}_{n\geq 0}$ is defined by letting $I_n = [x_n, x_{n+1})$ where the numbers x_n are inductively defined by $x_0 = 0$ and

$$x_{n+1} = 2h(y_n) \text{ where } y_n = \mu y (y \in D \ \&\ y > x_n).$$

Note that, by choice of h, y_n and x_{n+1}, $x_{n+1} > 2x_n$. So $\mathcal{I}$ is a c.v.s.a.i. Finally, for a proof of

(2.42), it suffices to note that, for $e < n$,

$$\begin{aligned}
&|\{s : A^{\langle e\rangle}_{s+1} \cap \bigcup_{n'\leq n} I_{n'} \neq A^{\langle e\rangle}_{s} \cap \bigcup_{n'\leq n} I_{n'}\}| \\
&\leq |\{s : A_{s+1} \restriction \langle e, x_{n+1}\rangle \neq A_s \restriction \langle e, x_{n+1}\rangle\}| \\
&\leq |\{s : A_{s+1} \restriction \langle n, x_{n+1}\rangle \neq A_s \restriction \langle n, x_{n+1}\rangle\}| \\
&= |\{s : A_{s+1} \restriction \langle n, 2h(y_n)\rangle \neq A_s \restriction \langle n, 2h(y_n)\rangle\}| \\
&\leq |\{s : A_{s+1} \restriction \langle 2h(y_n), 2h(y_n)\rangle \neq A_s \restriction \langle 2h(y_n), 2h(y_n)\rangle\}| \\
&= |\{s : A_{s+1} \restriction f(y_n) \neq A_s \restriction f(y_n)\}| \\
&< h(y_n)
\end{aligned}$$

where the last inequality holds by (2.45), while, for fixed $n > e$,

$$|I_n| = 2h(y_n) - x_n > 2h(y_n) - y_n \geq h(y_n)$$

holds. $\square$

2.4.6 Embedding the $\mathcal{S}_7$, an Application of Uniform Multiple Permitting

In this subsection, we give a further example of applying Corollary 68. Namely, we show that a c.e. Turing degree bounds an embedding of the nondistributive finite lattice $\mathcal{S}_7$, which contains a critical triple, if and only if it is not totally ω-c.e. In [DGW07], Downey, Greenberg and Weber show the following.

Theorem 73 ([DGW07])**.** *A c.e. Turing degree* $\mathbf{a}$ *bounds a critical triple if and only if* $\mathbf{a}$ *is not totally* ω*-c.e.*

Here, the definition of a critical triple is given by Downey and Weinstein and we say that a c.e. Turing degree $\mathbf{a}$ *bounds* a critical triple if there is a critical triple $\mathbf{a}_0$, $\mathbf{a}_1$ and $\mathbf{b}$ in the c.e. Turing degree such that $\mathbf{a}_0, \mathbf{a}_1, \mathbf{b} \leq \mathbf{a}$ holds.

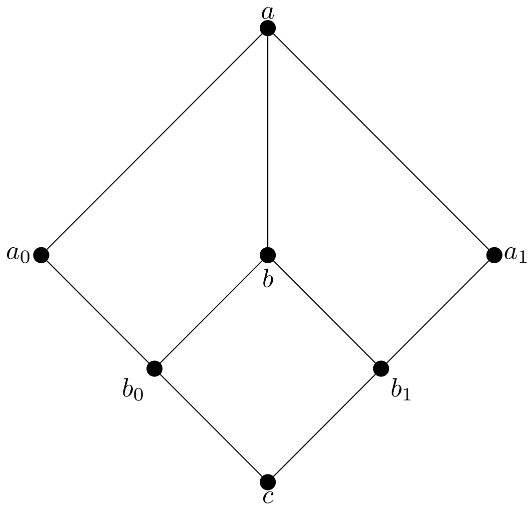
Definition 74 ([Dow90], [Wei88])**.** *Three elements* a_0*,* a_1 *and* b *of an upper semilattice* $(\mathcal{U}, \leq_{\mathcal{U}})$ *form a* critical triple *if the following hold.*

- a_0*,* a_1 *and* b *are pairwise* $\leq_{\mathcal{U}}$*-incomparable.*
- $a_0 \vee b = a_1 \vee b$*.*
- *For every* $c \in \mathcal{U}$*, if* $c \leq_{\mathcal{U}} a_0, a_1$*, then* $c \leq_{\mathcal{U}} b$*.*

Recall that the c.e. Turing degrees form an upper semilattice but not a lower semilattice. As many wellknown finite lattices contain a critical triple, this brings us to the question of embeddings of finite lattices into the c.e. Turing degrees. These have been intensively studied. It has been shown by Thomason [Tho71] and, independently, by Lerman (see Ambos-Spies and Lerman [ASL86]) that every finite distributive lattice can be embedded into the c.e. Turing degrees. Moreover, one has looked at the well-know nondistributive finite lattices $\mathcal{M}_5$ and $\mathcal{N}_5$ (where the former is modular and latter is not). Lachlan [Lac72] has given embeddings of these lattices into the c.e. Turing degrees. Moreover, it has been shown that the $\mathcal{N}_5$ can be embedded into the c.e. Turing degree below every noncomputable c.e. degree (see Ambos-Spies and Fejer [ASF88]). In contrast to this, Downey and Greenberg (see [DG06]) have shown that the $\mathcal{M}_5$ can be embedded into the c.e. Turing degrees below a c.e. degree **a** if and only if **a** is not totally $< \omega^\omega$-c.e. (where being totally $(<)$ α-c.e. for an ordinal α is a generalization of being totally ω-c.e. introduced by Downey and Greenberg; see [DG06]). Here we use Corollary 68 to show that the nondistributive finite lattice $\mathcal{S}_7$ can be embedded into the c.e. Turing degrees exactly below the c.e. not totally ω-c.e. Turing degrees. This is the first example of a finite lattice with this property.

Definition 75. *The $\mathcal{S}_7$ is the finite partial ordering $(\{a, a_0, a_1, b, b_0, b_1, c\}, \leq_{\mathcal{S}_7})$ where for $x, y \in \{a, a_0, a_1, b, b_0, b_1, c\}$, $x \leq_{\mathcal{S}_7} y$ holds if and only if $x = y$ or $x = c$ or $y = a$ or $x \in \{b_0, b_1\}$ and $y = b$ or, for $i \leq 1$, $x = b_i$ and $y = a_i$.*

The $\mathcal{S}_7$ can be illustrated as below. Note that the lattice is nondistributive and contains a critical triple, namely a_0, a_1 and b.



For better readability, we begin with the basic construction to embed the above lattice into the c.e. Turing degrees.

Theorem 76. *There is a zero-preserving lattice embedding $p : \mathcal{S}_7 \to (\mathbf{R}_{\mathrm{T}}, \leq)$.*

To prove Theorem 76, we argue that it suffices to prove the following theorem.

Theorem 77. *There are pairwise disjoint c.e. sets A_0, A_1, B_0, B_1 such that, for $i \leq 1$,*

(i) $deg_{\mathrm{T}}(A_iB_i) \wedge deg_{\mathrm{T}}(B_0B_1) = deg_{\mathrm{T}}(B_i)$,

(ii) $deg_{\mathrm{T}}(A_0B_0) \wedge deg_{\mathrm{T}}(A_1B_1) = \mathbf{0}$,

(iii) $A_i \leq_{\mathrm{T}} A_{1-i}B_0B_1$,

(iv) $A_i \not\leq_{\mathrm{T}} B_0B_1$,

(v) $B_i \not\leq_{\mathrm{T}} A_{1-i}B_{1-i}$.

(Here and in the following we write XY in place of $X \cup Y$. Note that, by disjointness of the sets X and Y we construct, $deg_{\mathrm{T}}(XY) = deg_{\mathrm{T}}(X \oplus Y)$.)

Proof of Theorem 76 assuming Theorem 77. Let A_0, A_1, B_0 and B_1 be as in Theorem 77. Then, a zero-preserving lattice embedding $p : \mathcal{S}_7 \to (\mathbf{R}_\mathrm{T}, \leq)$ is given by letting $p(a) = deg_\mathrm{T}(A_0B_0B_1)$, $p(a_i) = deg_\mathrm{T}(A_iB_i)$, $p(b) = deg_\mathrm{T}(B_0B_1)$, $p(b_i) = deg_\mathrm{T}(B_i)$ for $i \leq 1$ and $p(c) = \mathbf{0}$. It remains to show that p is indeed a zero-preserving lattice embedding. In the following, let $x, y, z \in \{a, a_0, a_1, b, b_0, b_1, c\}$.

We first show that $x \leq_{\mathcal{S}_7} y$ if and only if $p(x) \leq p(y)$. By pairwise disjointness of the sets A_i and B_i, it is immediate that $x \leq_{\mathcal{S}_7} y$ implies that $p(x) \leq p(y)$ for all $x, y \in \{a, a_0, a_1, b, b_0, b_1, c\}$. We have to show that the converse holds, too. By the former, by definition of the $\mathcal{S}_7$ and of p and by transitivity of Turing reducibility, for $i \leq 1$, it suffices to show that the following hold.

(a) $A_0B_0B_1 \not\leq_\mathrm{T} A_iB_i, B_0B_1$,

(b) $A_iB_i \not\leq_\mathrm{T} B_i, B_{1-i}$,

(c) $B_0B_1 \not\leq_\mathrm{T} B_i$,

(d) B_i is not computable.

For a proof, it is enough to note that (a) follows from (v) and (iv) by transitivity, (b) and (c) similarly follow from (iv) and (v), respectively and (d) directly follows from (v).

We now prove that $x \wedge y = z$ implies $p(x) \wedge p(y) = p(z)$. By definition of the $\mathcal{S}_7$, it suffices to show that $p(a_i) \wedge p(b) = p(b_i)$ for $i \leq 1$ and that $p(b_0) \wedge p(b_1) = p(a_0) \wedge p(a_1) = p(c)$. But this is immediate by definition of p and by (i) and (ii), respectively.

Finally we need to show that $x \vee y = z$ implies $p(x) \vee p(y) = p(z)$. Again by definition of the $\mathcal{S}_7$, it is enough to show that $p(a_0) \vee p(b) = p(a_1) \vee p(b) = p(a_0) \vee p(a_1) = p(a)$. By disjointness of the sets and by definition of p, $p(a_0) \vee p(b) = p(a)$ is immediate. By (iii), it follows that $p(a_1) \vee p(b) = p(a)$ and $p(a_0) \vee p(a_1) = p(a)$ hold, too.

Altogether, it follows that p is a lattice embedding. As c is the least element of the $\mathcal{S}_7$ and $p(c) = \mathbf{0}$ is the least element of $(\mathbf{R}_\mathrm{T}, \leq)$, p is zero-preserving, hence the proof of Theorem 76 assuming Theorem 77 is complete. □

We now give the proof of Theorem 77.

Proof of Theorem 77. In the following, let $e, s \geq 0$ and $i \leq 1$. We computably enumerate pairwise disjoint sets A_0, A_1, B_0 and B_1 with the required properties. For that matter, it suffices to meet the following requirements.

Meet requirements.

M_{3e+i} : If $\Phi_e^{A_iB_i}$ is total and $\Phi_e^{A_iB_i} = \Phi_e^{B_0B_1}$ then $\Phi_e^{A_iB_i} \leq_\mathrm{T} B_i$.

M_{3e+2} : If $\Phi_e^{A_0B_0}$ is total and $\Phi_e^{A_0B_0} = \Phi_e^{A_1B_1}$ then $\Phi_e^{A_0B_0}$ is computable.

Join requirements.

$J_i : A_i \leq_\mathrm{T} A_{1-i}B_0B_1$.

Nonordering requirements.

$N_{4e+i} : A_i \neq \Phi_e^{B_0 B_1}$.

$N_{4e+2+i} : B_i \neq \Phi_e^{A_{1-i} B_{1-i}}$.

Note that by "Posner's trick" (see Soare [Soa87], Remark IX.1.4), the meet requirements suffice to guarantee the required meets in (i) above, while the (global) join requirements and the nonordering requirements ensure (ii) and (iii), respectively.

We start by describing the strategies for meeting the different types of requirements where we introduce some of the notation required for the formal construction.

For the meet requirements we use a variant of Lachlan's branching degree construction (for the branching degree requirements M_n where $n = 3e + i$; see [Lac66]) respectively the minimal pair technique (for the minimal pair requirements M_n where $n = 3e+2$). We call a meet requirement M_n *infinitary* if its hypothesis is true and we call M_n *finitary* otherwise. We define (α-)expansionary stages as usual. For an infinitary meet requirement M_n we make sure that at a stage $s + 1$ only big numbers enter the oracles of the computations in the hypothesis of M_n unless the stage s is expansionary. Furthermore, if M_n is a branching degree requirement, i.e., $n = 3e + i$, then, whenever at some stage $s + 1$ where s is expansionary, both oracles of the computations in the hypothesis of M_n (i.e., $A_i B_i$ and $B_0 B_1$) are changed, we make a sufficiently small change in the set B_i representing the meet, too. For a minimal pair requirement M_n, $n = 3e + 2$, at a stage $s + 1$ where s is expansionary numbers may enter at most one of the oracles of the computations in the hypothesis, i.e., either $A_0 B_0$ or $A_1 B_1$, but not both.

Formally, we define the lengths of agreement as follows.

$$l(3e + i, s) = \max\{x : \forall y < x(\Phi_{e,s}^{A_{i,s} B_{i,s}}(y) \downarrow = \Phi_{e,s}^{B_{0,s} B_{1,s}}(y))\},$$
$$l(3e + 2, s) = \max\{x : \forall y < x(\Phi_{e,s}^{A_{0,s} B_{0,s}}(y) \downarrow = \Phi_{e,s}^{A_{1,s} B_{1,s}}(y))\}.$$

Note that $l(n, s)$ is computable. Moreover, for the oracle sets X and Y in the hypothesis of any meet requirement M_n ($n = 3e + j$, $j \in \{0, 1, 2\}$), the following holds.

$$\Phi_e^X \text{ total and } \Phi_e^X = \Phi_e^Y \Rightarrow \lim_{s\to\infty} l(n, s) \downarrow = \infty. \tag{2.46}$$

Since we cannot decide whether a meet requirement M_n is infinitary or finitary, we use the full binary tree $T = \{0, 1\}^{<\omega}$ to model our guesses on whether M_e is infinitary or not.

A node α of length k codes a guess about the hypotheses of the first k meet requirements $M_0, \ldots, M_{k-1}$ where, for $n < k$, $\alpha(n) = 0$ codes the guess that M_n is infinitary and $\alpha(n) = 1$ codes the guess that M_n is finitary. At any stage s of the construction we have an approximation δ_s, i.e., a guess on which of the first s meet requirements are infinitary. For the definition of δ_s, we inductively define *α-stages* for each node α as follows. Each stage $s \geq 0$ is a λ-stage. If s is an α-stage, then we call s *α-expansionary* if $l(|\alpha|, s) > l(|\alpha|, t)$ for all α-stages $t < s$ and we call s an $\alpha 0$-stage if s is α-expansionary and an $\alpha 1$-stage if s is an α-stage but not α-expansionary. Then $\delta_s \in T$ is the unique string α of length s such that s is an α-stage. Moreover, we say that α is *accessible* at stage $s + 1$ if s is an α-stage.

The *true path* $f : \omega \to \{0, 1\}$ of the construction is defined by

$$f(n) = \begin{cases} 0 & \text{if there are infinitely many } (f \restriction n)\text{-expansionary stages} \\ 1 & \text{otherwise.} \end{cases}$$

Note that f is the leftmost path through T visited infinitely often, i.e., satisfying $\delta_s \sqsubset f$ for infinitely many s. Moreover, (2.46) implies that $f(n) = 0$ for infinitary M_n. To each node α of length n, we assign a strategy $\mathcal{N}_\alpha$ for meeting requirement N_n which is based on the guess α. We show that the strategy $\mathcal{N}_{f \restriction n}$ on the true path succeeds in meeting N_n.

For the join requirements, we guarantee that every number x which may enter any set $X \in \{A_0, A_1, B_0, B_1\}$ under construction is *targeted* for this set at a stage $s \leq x$ where each number may be targeted for at most one set. (This also guarantees that every number is enumerated into at most one of the sets under construction.) Now, for the sake of J_i, whenever a number x becomes targeted for A_i at a stage s, at stage $s + 1$ an unused number $x' > s$ is appointed as a *trace* of x and targeted for A_{1-i}, B_0 or B_1. Then, if x is enumerated into A_i, simultaneously its (current) trace is enumerated into its target set. The trace x' of x may be replaced by another trace x'' at some stage $s' + 1 > s + 1$. This action, called *retargeting*, requires that the trace x' is enumerated into its target at stage $s' + 1$ and the new unused trace x'' is appointed at the same stage $s' + 1$ where the target of x'' may be freely chosen from A_{1-i}, B_0 or B_1 (but the target has to be determined by stage $\min\{x'', s' + 1\}$). Provided that, for fixed x, retargeting happens only finitely often, this guarantees that $A_i \leq_{\mathrm{T}} A_{1-i}B_0B_1$. In our construction, for any x there will be at most one retargeting. Namely if x is targeted for A_i and its first trace x' is targeted for A_{1-i} then x' may be replaced later by a trace x'' targeted for B_i. (The correctness of the reduction $A_i \leq_{\mathrm{T}} A_{1-i}B_0B_1$ in this special case is shown in Claim 5 below.)

To meet a single nonordering requirement N_n of the form $X \neq \Phi_e^Y$ for some X, Y and e, we use the well-known Friedberg-Muchnik strategy. We appoint a follower x to N_n and wait for a stage s such that $\Phi_{e,s}^{Y_s}(x) \downarrow= 0$. If there is no such stage s then we never put x into X and the requirement is met. If such a stage s exists, we make sure that we preserve the computation $\Phi_{e,s}^{Y_s}(x)$ by preserving Y up to the use of the computation (actually we preserve Y up to s which, by our convention on uses, bounds the use of the computation) and put x into X.

We have to ensure that the strategies for meeting the different requirements can be combined with each other. In order to achieve this we use some ideas introduced by Lachlan in [Lac72] in the embeddings of the two nondistributive five-element lattices $\mathcal{N}_5$ and $\mathcal{M}_5$ and by Downey, Greenberg and Weber in the construction of a degenerate critical triple of Turing degrees (Theorem 2.1 in [DGW07]) where we follow the latter quite closely. Consider a strategy $\mathcal{N}_\alpha$ for some nonordering requirement N_n. This strategy might wish to enumerate some number x into some set X under construction. If $n = 4e + 2 + i$ then $X = B_i$, hence putting x into X is uncritical for the join requirements. On the other hand, if $n = 4e + i$ then $X = A_i$. So for the sake of J_i, we have to define a trace $x_1 > x$ of x which has to be targeted for one of the sets A_{1-i}, B_0 or B_1. This trace x_1 has to enter its target when x enters A_i unless we put x_1 into its target previously and replace it by a new trace y_0 (with a possibly new target among A_{1-i}, B_0 or B_1). In general, we have to assign the trace x_1 before there is a stage s such that the computation $\Phi_{e,s}^{B_{0,s}B_{1,s}}(x)$ converges. So we cannot target x_1 for B_0 or B_1, because the use of the computation $\Phi_{e,s}^{B_{0,s}B_{1,s}}(x)$ may exceed x_1

whence the enumeration of x_1 may destroy this computation. So when x is appointed we define a trace x_1 targeted for A_{1-i}. But, for the sake of J_{1-i}, x_1 needs a trace x_2 targeted for either A_i, B_0 or B_1. Again, we cannot target x_2 for B_0 or B_1 because otherwise enumerating x_2 into B_0 or B_1 might destroy the computation $\Phi_{e,s}^{B_{0,s}B_{1,s}}(x)$ showing up later. Therefore, we target x_2 for A_i. Now, this process has to be iterated: we need a trace x_3 for x_2 targeted for A_{1-i} and so on. If $\Phi_{e,s}^{B_{0,s}B_{1,s}}(x)$ never becomes defined, this process goes on forever. This is uncritical, however, since neither x nor any of the corresponding traces has to be enumerated into its target. On the other hand, as soon as we see a converging computation $\Phi_{e,s}^{B_{0,s}B_{1,s}}(x)$ at some stage s, we may now define a trace $y_m > x_m$ of x_m (where $x_0 = x$ and x_m, $m \geq 0$, is the largest trace of x defined up to stage s and x_m is targeted for A_{j_m}, $j_m \in \{0,1\}$) such that y_m is bigger than the use of the computation $\Phi_{e,s}^{B_{0,s}B_{1,s}}(x)$ and target it for for B_{j_m}. Then enumerating y_m into B_{j_m} later will not destroy this computation. In fact, in the actual construction, we simultaneously define traces $y_m < y_{m-1} < \ldots < y_0$ where each y_l is a (potential) trace of the corresponding number x_l and y_l is targeted for the set B_{j_l} where x_l is targeted for A_{j_l}. Once the trace y_m is defined, we may simultaneously enumerate y_m into B_{j_m} and x_m into A_{j_m} and at the same time replace the trace x_m of x_{m-1} by (activating) the $B_{j_{m-1}}$-trace y_{m-1}. So, inductively, we may enumerate the pairs (x_l, y_l) into A_{j_l} and B_{j_l} for $l = m, m-1, \ldots 0$. So eventually, the follower x is enumerated into A_i thereby meeting requirement N_{4e+i}. Obviously, this procedure is compatible with the strategy for the global join requirements. Moreover, if we limit the appointment of the B_i-traces and the enumeration of the traces (x_l, y_l) into A_{j_l} and B_{j_l} to stages where α is accessible, we may argue that these actions are compatible with the constraints imposed by the meet requirements.

Note that numbers enter any set under construction only for the sake of some nonordering strategy. We call a trace targeted for A_i an A_i- or A-trace and a trace targeted for B_i a B_i- or B-trace. Similarly a number targeted for A_i is an A_i- or A-number and a number targeted for B_i is a B_i- or B-number. We call nonordering requirements N_n with $n = 4e + i$ and their corresponding strategies *critical* (as they need traces for their followers) and nonordering requirements N_n with $n = 4e + 2 + i$ and their strategies *uncritical* (as they do not). Moreover, we call a follower or (potential) trace of any nonordering strategy *active* at stage $s + 1$ if it is defined (i.e., assigned to the strategy) but not yet enumerated into its target by the end of stage s. If $\mathcal{N}_\alpha$ is critical and has an active follower x at stage $s + 1$ then we call the sequence $x_0, x_1, \ldots, x_m$ or $x_0, x_1, \ldots, x_m, y_m, y_{m-1}, \ldots, y_0$ associated with $\mathcal{N}_\alpha$ at the end of stage s where $x = x_0$, the numbers $x_1, \ldots, x_m$ are the active A-traces and the numbers $y_m, y_{m-1}, \ldots, y_0$ are the active (potential) B-traces the *entourage* of the follower x.

Construction.

We say that x is a *new large* number at stage $s + 1$ if $x > s + 1$ and x is greater than any number used in the construction so far. If we say that a (nonordering) strategy $\mathcal{N}_\alpha$ is *initialized* at stage s then any follower or trace associated with $\mathcal{N}_\alpha$ is canceled at stage s and $\mathcal{N}_\alpha$ is declared not to be satisfied at stage s. For any set X under construction we let X_s be the finite part of X enumerated by the end of stage s.

Stage 0. $X_0 = \emptyset$ for any set X under construction and any strategy is initialized.

Stage $s + 1$. A strategy $\mathcal{N}_\alpha$ with $|\alpha| = n$ *requires attention* at stage $s + 1$ if $\alpha \sqsubseteq \delta_s$, $\mathcal{N}_\alpha$ is not satisfied at the end of stage s and one of the following holds.

(i) No follower is assigned to $\mathcal{N}_\alpha$ at the end of stage s.

(ii) $\mathcal{N}_\alpha$ is critical, say $n = 4e+i$, $\mathcal{N}_\alpha$ has a follower x^α at the end of stage s, $\Phi_{e,s}^{B_{0,s}B_{1,s}}(x^\alpha) = 0$ and x^α is not realized at the end of stage s.

(iii) $\mathcal{N}_\alpha$ is critical and has a realized follower x^α at the end of stage s.

(iv) $\mathcal{N}_\alpha$ is uncritical, say $n = 4e + 2 + i$, $\mathcal{N}_\alpha$ has a follower x^α at the end of stage s, $\Phi_{e,s}^{A_{1-i,s}B_{1-i,s}}(x^\alpha) = 0$ and x^α is not realized at the end of stage s.

Fix the least α such that $\mathcal{N}_\alpha$ requires attention at stage $s + 1$ and perform the following action according to the clause above via which $\mathcal{N}_\alpha$ requires attention.

(i) Assign the least new large number x^α as a follower to $\mathcal{N}_\alpha$. If N_n is critical of the form $A_i \neq \Phi_e^{B_0B_1}$, declare the entourage of x^α to be $x_0^\alpha = x^\alpha$ and x_0^α to be targeted for A_i. If N_n is uncritical of the form $B_i \neq \Phi_e^{A_{1-i}B_{1-i}}$, declare x^α to be targeted for B_i.

(ii) Declare x^α to be *realized*. Append the least $m + 1$ new large numbers $y_m^\alpha < y_{m-1}^\alpha < \ldots < y_0^\alpha$ to the entourage $x^\alpha = x_0^\alpha, x_1^\alpha, \ldots, x_m^\alpha$ of x^α. For $0 \leq l \leq m$, declare y_l^α to be targeted for B_{j_l} where j_l is such that x_l^α is targeted for A_{j_l}. Moreover, declare y_m^α to be the B_{j_m}-trace of x_m^α. (For $l < m$, y_l^α will become the B_{j_l}-trace of x_l^α once the current A_{1-j_l}-trace x_{l+1}^α of x_l^α will be enumerated into A_{1-j_l}; so we refer to y_l^α as the potential B_{j_l}-trace of x_l^α.)

(iii) Let $x^\alpha = x_0^\alpha, x_1^\alpha \ldots, x_m^\alpha, y_m^\alpha, y_{m-1}^\alpha, \ldots, y_0^\alpha$ be the entourage of x^α at the end of stage s and fix $i \leq 1$ such that x_m^α and y_m^α are targeted for A_i and B_i, respectively. Enumerate x_m^α into A_i and y_m^α into B_i and delete x_m^α and y_m^α from the entourage of x^α. If $m > 0$ appoint y_{m-1}^α as the B_{1-i}-trace of x_{m-1}^α replacing the old A_i-trace x_m^α of x_{m-1}^α (note that x_{m-1}^α and y_{m-1}^α have been previously targeted for A_{1-i} and B_{1-i}, respectively). If $m = 0$, declare $\mathcal{N}_\alpha$ to be *satisfied*.

(iv) Enumerate x^α into its target B_i at stage $s + 1$, declare x^α to be *realized* and declare $\mathcal{N}_\alpha$ to be *satisfied*.

In any of the cases, declare that $\mathcal{N}_\alpha$ *receives attention* or *becomes active* at stage $s + 1$ (*via* follower x^α). Initialize all strategies $\mathcal{N}_\beta$ with $\alpha < \beta$. Furthermore, for any strategy $\mathcal{N}_\beta$ with $\beta < \alpha$ such that the last element x_m^β of the current entourage is targeted for some A_i, add a new large number x_{m+1}^β to the entourage as a trace of x_m^β and target it for A_{1-i}. Finally, let $X_{s+1} = X_s$ for all sets X under construction unless mentioned otherwise above and let status and parameters of strategies be unchanged unless mentioned otherwise above.

Verification.

We start with a few observations. Note that at any stage $s + 1$, exactly one strategy $\mathcal{N}_\alpha$ is active. Furthermore, if $\mathcal{N}_\alpha$ is uncritical and enumerates some number at stage $s+1$, it enumerates exactly one number into exactly one B_i. We then call $s + 1$ a *B_i-stage*. If $\mathcal{N}_\alpha$ is critical and performs some enumeration at stage $s+1$, it enumerates exactly two numbers, one into A_i and one into B_i for exactly one i. We call such a stage $s + 1$ an *(A_iB_i)-stage*. It follows that every stage at which some enumeration takes place is either a B_0-, a B_1-, an (A_0B_0)- or an (A_1B_1)-stage.

We now prove a series of claims that show that the constructed sets have the required properties.

Claim 1. All constructed sets are c.e.

Proof. Immediate by construction. □

Claim 2 (True Path Lemma). $f = \liminf_{s\to\infty} \delta_s$, i.e., for any α, $\alpha \sqsubset f$ if and only if $\alpha \sqsubset \delta_s$ for infinitely many s and there are only finitely many s such that $\delta_s <_L \alpha$. Moreover, if M_n is infinitary then $f(n) = 0$.

Proof. The first part is immediate by definition of δ_s and f. The second part follows from (2.46). □

Claim 3. Every strategy $\mathcal{N}_\alpha$ on the true path (i.e., $\alpha \sqsubset f$) is initialized only finitely often and requires attention only finitely often. Moreover, $N_{|\alpha|}$ is met.

Proof. The proof is by induction on $|\alpha|$. Given $\alpha \sqsubset f$, by Claim 2, fix s_0 minimal such that $\alpha \leq \delta_s$ for all $s \geq s_0$ and, by inductive hypothesis, fix $s_1 \geq s_0$ minimal such that no strategy $\mathcal{N}_\beta$ with $\beta \sqsubset \alpha$ requires attention after stage s_1. Then $\mathcal{N}_\alpha$ is never initialized after stage s_1 and receives attention whenever it requires attention after stage s_1. By minimality of s_1, a follower x^α is permanently assigned to $\mathcal{N}_\alpha$ at stage $s_2 + 1$ where s_2 is the least α-stage $\geq s_1$. (Note that s_2 exists because α is on the true path whence there are infinitely many α-stages.) Now, if there is no α-stage $s_3 > s_2$ such that x^α becomes realized at stage $s_3 + 1$, then $\mathcal{N}_\alpha$ does not require attention after stage $s_2 + 1$ and x^α witnesses that $N_{|\alpha|}$ met. (For the latter note that, for $|\alpha| = 4e + i$, $\Phi_{e,s}^{B_{0,s}B_{1,s}}(x^\alpha) \neq 0$ for all α-stages $s > s_2$ hence $\Phi_e^{B_0B_1}(x^\alpha) \neq 0$ and x^α is not enumerated into A_i. Similarly, for $|\alpha| = 4e + 2 + i$, $\Phi_e^{A_{1-i}B_{1-i}}(x^\alpha) \neq 0$ and $x^\alpha \notin B_i$.) So w.l.o.g. let s_3 be the least α-stage $> s_2$ such that x^α becomes realized at stage s_3+1, and distinguish the following two cases.

> *Case 1: $N_{|\alpha|}$ is critical, say $|\alpha| = 4e + i$.*
>
> Let $x^\alpha = x_0^\alpha, x_1^\alpha, \ldots, x_m^\alpha$ be the entourage of the follower x^α at stage s_3. By construction, $\Phi_{e,s_3}^{B_{0,s_3}B_{1,s_3}}(x^\alpha) = 0$, x^α is declared to be realized at stage s_3+1, and B-traces y_l^α $(0 \leq l \leq m)$ are appointed at stage $s_3 + 1$ (and there are no further traces associated with x^α appointed later). Moreover, for any α-stage $s > s_3$ such that $\mathcal{N}_\alpha$ is not satisfied at stage s, $\mathcal{N}_\alpha$ becomes active at stage $s + 1$ via Clause (iii) and two elements from x^α's entourage are enumerated into their corresponding sets. Hence, the entourage becomes smaller. As there are infinitely many α-stages, it follows that at some stage $s_4 + 1 > s_3 + 1$, x^α itself is enumerated into its target set A_i and hence $\mathcal{N}_\alpha$ is declared to be satisfied. By choice of s_1, $\mathcal{N}_\alpha$ is never initialized after stage s_4+1, thus remains satisfied forever. So $\mathcal{N}_\alpha$ does not require attention after stage $s_4 + 1$.
>
> It remains to show that $N_{|\alpha|}$ is met. Since x^α is enumerated into A_i at stage s_4+1 and since $\Phi_{e,s_3}^{B_{0,s_3}B_{1,s_3}}(x^\alpha) = 0$, it suffices to show that no number $< s_3$ enters B_0 or B_1 after stage s_3. Since no strategy $\mathcal{N}_\beta$ with $\beta < \alpha$ acts after stage s_1 and since all strategies $\mathcal{N}_\beta$ with $\beta > \alpha$ are initialized at stage $s_3 + 1$ (hence enumerate only numbers $> s_3 + 1$ into any set under construction after stage s_3), this follows from the fact that the only numbers enumerated into B_0 or B_1 by $\mathcal{N}_\alpha$ after stage s_3 are the B-traces $y_m^\alpha, \ldots, y_0^\alpha$ which are new large numbers at stage $s_3 + 1$ hence greater than s_3.

Case 2: $\mathcal{N}_{|\alpha|}$ is uncritical, say $|\alpha| = 4e+2+i$.

By construction, $\Phi_{e,s_3}^{A_{1-i,s_3}B_{1-i,s_3}}(x^\alpha) = 0$, $\mathcal{N}_\alpha$ becomes active at stage s_3+1, x^α is enumerated into B_i and $\mathcal{N}_\alpha$ is declared to be satisfied. Again, $\mathcal{N}_\alpha$ is never initialized after stage s_3+1, hence remains satisfied and does not require attention after stage s_3+1. Moreover, as in the first case, we may argue that x^α witnesses that $N_{|\alpha|}$ is met since no number $\leq s_3$ is enumerated into A_{1-i} or B_{1-i} after stage s_3. Namely, since no strategy $\mathcal{N}_\beta$ with $\beta < \alpha$ becomes active after stage s_3 and all strategies $\mathcal{N}_\beta$ with $\beta > \alpha$ are initialized at stage s_3+1 it suffices to note that $\mathcal{N}_\alpha$ does not become active after stage s_3+1 and that it does not enumerate any number into A_{1-i} or B_{1-i} at stage s_3+1. □

Claim 4. For every $n \geq 0$, M_n is met.

Proof. Fix $n \geq 0$ and w.l.o.g. assume that the hypothesis of M_n is true. Let $\alpha = f \upharpoonright n$. By the True Path Lemma, $\alpha 0 \sqsubset f$. So there are infinitely many $\alpha 0$-stages and, by Claims 2 and 3, we may fix an $\alpha 0$-stage $s_0 > n$ such that no strategy $\mathcal{N}_\beta$ with $\beta \leq \alpha 0$ becomes active after stage s_0. Let $S = \{s_l : l \geq 0\}$ where $s_0 < s_1 < s_2 < \dots$ are the $\alpha 0$-stages $\geq s_0$. Then S is computable and

$$l(n, s_0) < l(n, s_1) < l(n, s_2) < \dots.$$

Observe that, for any $l \geq 0$ and any stage t with $s_l+1 < t \leq s_{l+1}$, only strategies $\mathcal{N}_\beta$ with $\alpha 0 <_{\mathrm{L}} \beta$ may act. As those strategies are initialized at stage s_l+1, at such stages t only numbers $> s_l+1$ can enter any set under construction. So, in particular,

$$\forall\, i \leq 1\ \forall\, l \geq 0 \Big(A_{i,s_l+1} \upharpoonright s_l+1 = A_{i,s_{l+1}} \upharpoonright s_l+1 \ \&\ B_{i,s_l+1} \upharpoonright s_l+1 = B_{i,s_{l+1}} \upharpoonright s_l+1\Big). \quad (2.47)$$

Note that, by our convention on uses, this implies that any oracle computation existing at the end of stage s_l which is not injured at stage s_l+1 will not be injured by the end of stage s_{l+1}.

Now distinguish the following cases depending on the type of the meet requirement. We start with the more straightforward case of the minimal pair requirements.

Case 1: $n = 3e+2$.

Here the claim follows by the standard minimal pair argument. It suffices to show that, for given x,

$$\Phi_e^{A_0 B_0}(x) = \Phi_{e,s_k}^{A_{0,s_k} B_{0,s_k}}(x) \quad (2.48)$$

for the least k such that $l(n, s_k) > x$. For a proof of (2.48), it suffices to show that, for all $l \geq k$,

$$\exists\, i \leq 1\ (\Phi_{e,s_{l+1}}^{A_{i,s_{l+1}} B_{i,s_{l+1}}}(x) = \Phi_{e,s_l}^{A_{i,s_l} B_{i,s_l}}(x)) \quad (2.49)$$

holds. Namely, since $l(n, s_l) > x$ for all $l \geq k$, it follows by induction on l that, for both $i = 0$ and $i = 1$, $\Phi_{e,s_l}^{A_{i,s_l} B_{i,s_l}}(x) = \Phi_{e,s_k}^{A_{i,s_k} B_{i,s_k}}(x)$ for all $l \geq k$. For a proof of (2.49), by (2.47) it suffices to show that no number less than s_l enters $A_0 B_0$ at stage s_l+1 or no number less than s_l enters $A_1 B_1$ at stage s_l+1 whence $\Phi_{e,s_l}^{A_{0,s_l} B_{0,s_l}}(x)$ or $\Phi_{e,s_l}^{A_{1,s_l} B_{1,s_l}}(x)$ will be preserved by the end of stage s_{l+1}. But this is immediate by construction since any stage at which any of the sets under construction is changed is either a B_0- or (A_0, B_0)-stage or a B_1- or (A_1, B_1)-stage.

Case 2: $n = 3e + i$.

Here the claim follows by refining Lachlan's branching degree argument. It suffices to show that, for given x,

$$\Phi_e^{A_i B_i}(x) = \Phi_{e,s_k}^{A_{i,s_k} B_{i,s_k}}(x) \tag{2.50}$$

for the least k such that

(a) $l(n, s_k) > x$ and

(b) none of the numbers targeted for B_i which exist at the end of stage s_k will enter B_i after stage s_k.

Note that such a number k exists (namely, it suffices to fix the least k' such that $l(n, s_{k'}) > x$, fix the highest priority strategy $\mathcal{N}_\beta$ with $\alpha 0 \sqsubseteq \beta$ which acts after stage $s_{k'}$, let $s_{k''} + 1$ be the last stage at which $\mathcal{N}_\beta$ acts and let $k = k'' + 1$; then no $\mathcal{N}_\gamma$ with $\gamma \leq \beta$ acts after stage s_k and, for any γ such that $\beta < \gamma$ and $\mathcal{N}_\gamma$ has a follower at stage s_k, this follower has been appointed after stage $s_{k''} + 1$ whence $\alpha 0 <_{\mathrm{L}} \gamma$ and $\mathcal{N}_\gamma$ is initialized at stage $s_k + 1$) and that the least such k can be found computably in B_i.

Now, for a proof of (2.50), it suffices to define a strictly increasing function $g(r)$ ($r \geq 0$) satisfying

$$\Phi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}}}(x) = \Phi_{e,s_k}^{A_{i,s_k} B_{i,s_k}}(x), \tag{2.51}$$

$$B_{i,s_{g(r)}} \restriction \varphi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}}}(x) = B_i \restriction \varphi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}}}(x), \tag{2.52}$$

$$B_{i,s_{g(r)}} \restriction \varphi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}}}(x) = B_i \restriction \varphi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}}}(x) \tag{2.53}$$

for all $r \geq 0$. (Obviously, (2.51) implies (2.50). The other conditions are used in the inductive proof of (2.51). Also note that the function g does not have to be computable.)

The function g is inductively defined by letting $g(0) = k$ and by letting $g(r + 1) = g(r) + 1$ unless the nonordering strategy $\mathcal{N}_\beta$ which acts at stage $s_{g(r)} + 1$ is critical and acts via Clause (ii), i.e., if the follower x^β of $\mathcal{N}_\beta$ becomes realized and the corresponding B-traces are appointed at stage $s_{g(r)} + 1$. In this case let $g(r + 1) = q$ where $q > g(r)$ is minimal such that $\mathcal{N}_\beta$ does not act via x^β at any stage $\geq s_q + 1$ (i.e., $\mathcal{N}_\beta$ acts via x^β at stage $s_{g(r+1)-1} + 1$ for the last time).

Now the proof of (2.51), (2.52) and (2.53) is by (simultaneous) induction on r. For $r = 0$, $g(r) = k$. So (2.51) is immediate. Moreover, by Clause (b) in the definition of s_k, no number less than s_k enters B_i after stage s_k. So, by our convention on uses, the computations $\Phi_{e,s_k}^{A_{i,s_k} B_{i,s_k}}(x)$ and $\Phi_{e,s_k}^{B_{0,s_k} B_{1,s_k}}(x)$ are B_i-correct, hence (2.52) and (2.53) hold, too.

For the inductive step fix r such that (2.51), (2.52) and (2.53) hold. Since $l(n, s_{g(r+1)}) > l(n, s_{g(r)}) \geq l(n, s_k) > x$ it suffices to establish

$$\Phi_{e,s_{g(r+1)}}^{A_{i,s_{g(r+1)}} B_{i,s_{g(r+1)}}}(x) = \Phi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}}}(x) \text{ or } \Phi_{e,s_{g(r+1)}}^{B_{0,s_{g(r+1)}} B_{1,s_{g(r+1)}}}(x) = \Phi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}}}(x) \tag{2.54}$$

and (2.52) and (2.53) for $r+1$ in place of r. In the following we refer to the latter as $(2.52)_{r+1}$ and $(2.53)_{r+1}$.

Fix the nonordering strategy $\mathcal{N}_\beta$ which becomes active at stage $s_{g(r)}+1$ and let x^β be the follower of $\mathcal{N}_\beta$ at the end of stage $s_{g(r)}+1$. Note that $\alpha 0 \sqsubset \beta$ (by choice of s_0 since $s_{g(r)}$ is an $\alpha 0$-stage). Moreover, by definition of g, $\mathcal{N}_\beta$ acts via x^β at stage $s_{g(r+1)-1}+1$ (note that, for $g(r+1) = g(r)+1$, $s_{g(r+1)-1}+1 = s_{g(r)}+1$) whence $\mathcal{N}_\beta$ is not initialized at any stage $s+1$ with $s_{g(r)}+1 \leq s+1 \leq s_{g(r+1)-1}+1$. So, since $s_{g(r+1)}$ is the least $\alpha 0$-stage $\geq s_{g(r+1)-1}+1$,

$$\text{no strategy } \mathcal{N}_\gamma \text{ with } \gamma < \beta \text{ acts at any stage } s+1 \text{ with } s_{g(r)}+1 \leq s+1 \leq s_{g(r+1)}. \tag{2.55}$$

Next we show that

$$A_{i,s_{g(r+1)}} \restriction \varphi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}}}(x) = A_{i,s_{g(r)}} \restriction \varphi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}}}(x) \tag{2.56}$$

or

$$B_{1-i,s_{g(r+1)}} \restriction \varphi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}}}(x) = B_{1-i,s_{g(r)}} \restriction \varphi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}}}(x) \tag{2.57}$$

holds. For a contradiction assume that (2.56) and (2.57) fail. Then, by our convention on uses, $A_{i,s_{g(r+1)}} \restriction s_{g(r)} \neq A_{i,s_{g(r)}} \restriction s_{g(r)}$ and $B_{1-i,s_{g(r+1)}} \restriction s_{g(r)} \neq B_{1-i,s_{g(r)}} \restriction s_{g(r)}$. Since no strategy simultaneously enumerates numbers into A_i and into B_{i-1}, since only strategies $\mathcal{N}_\delta$ with $\alpha 0 < \delta$ may act after stage $s_{g(r)} \geq s_0$ and since all strategies $\mathcal{N}_\delta$ with $\alpha 0 <_{\mathrm{L}} \delta$ are initialized at stage $s_{g(r)}+1$, it follows that there is an $\alpha 0$-stage strictly between $s_{g(r)}$ and $s_{g(r+1)}$. By definition of g, this implies that $\mathcal{N}_\beta$ is a critical nonordering strategy and that $\mathcal{N}_\beta$ receives attention via Clause (ii) at stage $s_{g(r)}+1$. It follows that all B-numbers which are associated with $\mathcal{N}_\beta$ or with a lower priority strategy after stage $s_{g(r)}$ are appointed after this stage hence are greater than $s_{g(r)}$. By (2.55), this implies

$$\forall\, j \leq 1\ (B_{j,s_{g(r+1)}} \restriction s_{g(r)} = B_{j,s_{g(r)}} \restriction s_{g(r)}).$$

Obviously, this implies (2.57) contrary to assumption.

Now, since (2.56) *or* (2.57) holds, it follows by (2.52) and (2.53) that (at least) one of the computations $\Phi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}}}(x)$ or $\Phi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}}}(x)$ is preserved by the end of stage $s_{g(r+1)}$. So (2.54) holds.

It remains to show that $(2.52)_{r+1}$ and $(2.53)_{r+1}$ hold. If (2.56) *and* (2.57) hold then this is immediate by the inductive hypotheses (2.52) and (2.53). So, for the rest of the proof we may assume that (2.56) or (2.57) fails and, for a contradiction, we assume that $(2.52)_{r+1}$ or $(2.53)_{r+1}$ fails, too.

By failure of (2.56) or (2.57) there are a stage t_0 and a number z_0 such that $s_{g(r)} \leq t_0 < s_{g(r+1)}$ and either z_0 enters A_i at stage t_0+1 and $z_0 < \varphi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}}}(x)$ or z_0 enters B_{1-i} at stage t_0+1 and $z_0 < \varphi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}}}(x)$ (hence, in either case, $z_0 < s_{g(r)}$). Fix such t_0 and z_0. Since all strategies $\mathcal{N}_\gamma$ with $\beta < \gamma$ are initialized at stage $s_{g(r)}+1$ hence enumerate only numbers $> s_{g(r)}$ after this stage and since, by (2.55), no strategy $\mathcal{N}_\gamma$ with $\gamma < \beta$ may act at stage t_0+1, it follows that z_0 is enumerated into its target by $\mathcal{N}_\beta$ at stage t_0+1. Hence t_0

is a β-stage and we may fix the stage $t'_0 + 1 \leq z_0$ at which z_0 becomes associated with $\mathcal{N}_\beta$. So, summarizing,

$$t'_0 + 1 \leq z_0 < s_{g(r)} < t_0 + 1 \leq s_{g(r+1)-1} + 1 \leq s_{g(r+1)} \tag{2.58}$$

and (by (2.55))

$$\forall\, s\ (t'_0 + 1 \leq s + 1 \leq s_{g(r+1)} \ \Rightarrow\ \mathcal{N}_\beta \text{ is not initialized at stage } s+1). \tag{2.59}$$

On the other hand, by failure of $(2.52)_{r+1}$ or $(2.53)_{r+1}$, there are a stage $t_1 \geq s_{g(r+1)}$ and a number z_1 such that z_1 enters B_i at stage $t_1 + 1$ and either $z_1 < \varphi_{e,s_{g(r+1)}}^{A_{i,s_{g(r+1)}} B_{i,s_{g(r+1)}}}(x)$ or $z_1 < \varphi_{e,s_{g(r+1)}}^{B_{0,s_{g(r+1)}} B_{1,s_{g(r+1)}}}(x)$ (hence, in either case, $z_1 < s_{g(r+1)}$). Fix such t_1 and z_1, fix the strategy $\mathcal{N}_{\beta'}$ which enumerates z_1 into B_i at stage $t_1 + 1$ and fix the stage $t'_1 + 1$ at which z_1 is appointed as follower or trace to $\mathcal{N}_{\beta'}$.

Then t'_1 is a β'-stage, $t'_1 + 1 < z_1 < s_{g(r+1)} < t_1 + 1$ and $\mathcal{N}_{\beta'}$ is not initialized at any stage $s + 1$ such that $t'_1 + 1 \leq s + 1 \leq t_1 + 1$. In particular, $\mathcal{N}_{\beta'}$ is not initialized at stage $s_{g(r+1)} + 1$ whence $\alpha 0 \sqsubset \beta'$. So t'_1 is an $\alpha 0$-stage, hence $t'_1 \leq s_{g(r+1)-1}$. Since $\mathcal{N}_\beta$ acts at stage $s_{g(r+1)-1} + 1$ but $\mathcal{N}_{\beta'}$ is not initialized at this stage, it follows that $\beta' \leq \beta$.

Now, if $\beta' < \beta$ then $\mathcal{N}_\beta$ is initialized at stage $t'_1 + 1$. By $t'_1 + 1 < s_{g(r+1)}$ and (2.59), this implies that $t'_1 + 1 < t'_0 + 1$ hence $z_1 < z_0$. By choice of z_0 this implies

$$z_1 < \max\{\varphi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}}}(x), \varphi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}}}(x)\}.$$

Since z_1 enters B_i after stage $s_{g(r+1)}$ this contradicts (2.52) or (2.53).

This leaves the case that $\beta = \beta'$. Then z_0 and z_1 are associated with $\mathcal{N}_\beta$ but targeted for different sets. So $\mathcal{N}_\beta$ is critical. Since $\mathcal{N}_\beta$ acts at stage $s_{g(r)} + 1$ and since (by (2.58) and (2.59)) z_0 is associated with $\mathcal{N}_\beta$ at the end of stage $s_{g(r)}$, it follows that $\mathcal{N}_\beta$ acts via Clause (ii) or Clause (iii) at stage $s_{g(r)} + 1$.

First assume that $\mathcal{N}_\beta$ acts at stage $s_{g(r)} + 1$ via Clause (ii). Then any of the B-traces $y_m^\beta, \ldots, y_0^\beta$ appointed at stage $s_{g(r)} + 1$ which enters its target later does so by the end of stage $s_{g(r+1)-1} + 1$ and no additional numbers become associated with $\mathcal{N}_\beta$ by the end of stage $s_{g(r+1)}$. So any number z enumerated into B_i by $\mathcal{N}_\beta$ after stage $s_{g(r+1)}$ has to be appointed after stage $s_{g(r+1)}$ hence has to be greater than $s_{g(r+1)}$. So, in particular, $z_1 > s_{g(r+1)}$ contrary to choice of z_1.

Finally, assume that $\mathcal{N}_\beta$ acts via Clause (iii) at stage $s_{g(r)} + 1$. Then $g(r+1) = g(r) + 1$. Since t'_1 is an $\alpha 0$-stage and $t'_1 + 1 < s_{g(r+1)}$, the latter implies that $t'_1 \leq s_{g(r)}$. In fact, since z_1 becomes appointed as a B_i-trace of $\mathcal{N}_\beta$ at stage $t'_1 + 1$ and since no new trace is appointed at stage $s_{g(r)} + 1$, $t'_1 < s_{g(r)}$ and $\mathcal{N}_\beta$ becomes active via (ii) at stage $t'_1 + 1$.

Now, in order to get the desired contradiction, we look at the size of t'_1. If $t'_1 < s_{g(0)}$ then z_1 is associated with $\mathcal{N}_\beta$ at the end of stage $s_{g(0)}$ and enters B_i after stage $s_{g(0)}$. Since $g(0) = k$ this contradicts the choice of k.

So w.l.o.g. we may assume that $t_1' \geq s_{g(0)}$. Fix the unique number $r' < r$ such that $s_{g(r')} \leq t_1' < s_{g(r'+1)}$ and fix the strategy $\mathcal{N}_{\beta''}$ which acts at stage $s_{g(r')}+1$. Now, if $t_1' = s_{g(r')}$ then $\beta'' = \beta$, $\mathcal{N}_\beta$ becomes active via (ii) at stage $s_{g(r')}+1$ and z_1 is appointed as a B_i-trace at stage $s_{g(r')}+1$. So, by definition of g, z_1 cannot enter B_i after stage $s_{g(r'+1)-1}+1$. So, by $r' < r$, $t_1 + 1 \leq s_{g(r'+1)-1} + 1 < s_{g(r+1)}$ contrary to choice of t_1.

This leaves the case that $s_{g(r')} < t_1' < s_{g(r'+1)}$. Since t_1' is an $\alpha 0$-stage, it follows that $s_{g(r')+1} < s_{g(r'+1)}$. So $\mathcal{N}_{\beta''}$ becomes active via Clause (ii) at stage $s_{g(r')}+1$. Since $\mathcal{N}_\beta$ acts at stage $t_1'+1$, it follows by (2.55) (applied to β'' and r' in place of β and r) that $\beta'' \leq \beta$ and since a new trace for $\mathcal{N}_\beta$ is appointed at stage $t_1'+1$ where $s_{g(r')} < t_1' \leq s_{g(r'+1)-1}$ it follows that $\beta \neq \beta''$. So $\beta'' < \beta$. It follows that $\mathcal{N}_\beta$ is initialized at stage $s_{g(r'+1)-1}+1$ and z_1 is canceled. By the latter, $t_1 + 1 < s_{g(r'+1)-1} + 1 < s_{g(r+1)}$. But, just as in the preceding case, this contradicts the choice of t_1.

So, in any case, the assumption that $(2.52)_{r+1}$ or $(2.53)_{r+1}$ fails leads to a contradiction. This completes the proof of Claim 4. □

Claim 5. For $i \in \{0,1\}$, J_i is met.

Proof. For fixed $i \in \{0,1\}$ and $x \geq 0$, we sketch how to (uniformly) compute $A_i(x)$ using $A_{1-i}B_0B_1$ as an oracle. First, by running the construction up to stage x, find out whether there is a stage $s < x$ such that x is appointed as a follower or a trace and targeted for A_i at stage $s+1$. If there is no such stage s then $x \notin A_i$. So w.l.o.g. fix such s. By construction, either x is canceled or a trace x' of x targeted for either A_{1-i} or B_i is appointed at stage $s+2$. If x is canceled then, obviously, $x \notin A_i$. If x' is targeted for B_i, then $x \in A_i$ if and only if $x' \in B_i$. If x' is targeted for A_{1-i} and $x' \notin A_{1-i}$, then $x \notin A_i$. Finally, if x' is targeted for A_{1-i} and $x' \in A_{1-i}$ then, at the stage where x' enters A_{1-i}, a new trace y of x targeted for B_i is appointed and $x \in A_i$ if and only if $y \in B_i$. □

This completes the proof of Theorem 77. □

We now show that the $\mathcal{S}_7$ can be embedded into the c.e. Turing degrees exactly below any c.e. not totally ω-c.e. Turing degree. In contrast to Theorem 76, however, here the embedding may not preserve the least element.

Theorem 78. *A c.e. Turing degree* **a** *bounds a lattice embedding of the $\mathcal{S}_7$ into the c.e. Turing degrees if and only if* **a** *is not totally ω-c.e.*

The only if direction of Theorem 78 follows immediately from Theorem 73 as we have seen that the S_7 contains a critical triple. For a proof of the if direction, by Corollary 68 and Theorem 69, it suffices to show that any c.e. set D which is uniformly multiply permitting bounds an embedding of the S_7 into the c.e. degrees.

Theorem 79. *Let D be a c.e. set that is uniformly multiply permitting. There are pairwise disjoint c.e. sets A_0, A_1, B_0, B_1 and C such that*

$$A_0, A_1, B_0, B_1, C \leq_{\mathrm{T}} D \tag{2.60}$$

and, for $i \leq 1$,

(i) $deg_{\mathrm{T}}(A_iB_iC) \wedge deg_{\mathrm{T}}(B_0B_1C) = deg_{\mathrm{T}}(B_iC)$ *and* $deg_{\mathrm{T}}(A_0B_0C) \wedge deg_{\mathrm{T}}(A_1B_1C) = deg_{\mathrm{T}}(C)$,

(ii) $A_i \leq_{\mathrm{T}} A_{1-i}B_0B_1$ *and*

(iii) $A_i \not\leq_{\mathrm{T}} B_0B_1C$ *and* $B_i \not\leq_{\mathrm{T}} A_{1-i}B_{1-i}C$.

Again, we first prove Theorem 78 assuming Theorem 79.

Proof of Theorem 78 assuming Theorem 79. The only if direction immediately follows from the Downey-Greenberg-Weber result that no c.e. totally ω-c.e. degree bounds a critical triple as the $\mathcal{S}_7$ contains a critical triple (Theorem 73). By Corollary 68 and Theorem 69, the if direction can be deduced from Theorem 79 in a very similar way as Theorem 76 follows from Theorem 77. $\square$

It now suffices to prove Theorem 79.

Proof of Theorem 79. We combine the construction from the proof of Theorem 77 with marker permitting in order to enumerate c.e. sets A_0, A_1, B_0, B_1 and C with the required properties. Unless mentioned otherwise, all notions introduced in the proof of Theorem 77 are defined here correspondingly. Furthermore, as there we may argue that, in addition to (2.60), it is enough to meet the following requirements (where here and in the following $e \geq 0$ and $i \leq 1$).

Meet requirements.

$M_{3e+i}:$ If $\Phi_e^{A_iB_iC}$ is total and $\Phi_e^{A_iB_iC} = \Phi_e^{B_0B_1C}$ then $\Phi_e^{A_iB_iC} \leq_{\mathrm{T}} B_iC$.

$M_{3e+2}:$ If $\Phi_e^{A_0B_0C}$ is total and $\Phi_e^{A_0B_0C} = \Phi_e^{A_1B_1C}$ then $\Phi_e^{A_0B_0} \leq_{\mathrm{T}} C$.

Join requirements.

$J_i : A_i \leq_{\mathrm{T}} A_{1-i}B_0B_1$.

Nonordering requirements.

$N_{4e+i} : A_i \neq \Phi_e^{B_0B_1C}$.

$N_{4e+2+i} : B_i \neq \Phi_e^{A_{1-i}B_{1-i}C}$.

In order to satisfy (2.60), given a computable enumeration $\{D_s\}_{s\geq 0}$ of D, we define a computable marker $\gamma(x,s)$ such that γ is nondecreasing in the second argument, $\gamma(x,s) < \gamma(x,s+1)$ only if a number $\leq \gamma(x,s)$ is enumerated into D at stage $s+1$ and $\gamma^*(x) = \lim_{s\geq 0}\gamma(x,s) < \omega$ exists. (Note that this implies that γ^* is computable in D.) Then we guarantee $X \leq_{\mathrm{T}} D$ for $X = A_i, B_i, C$ by enumerating a number x into X at stage $s+1$ only if a number $\leq \max_{x'\leq x}\gamma(x',s)$ enters D at the same stage. (See the proof of Claim 6 for details.)

In order to meet the requirements we adjust the strategies introduced in the proof of Theorem 77. Recall that the only strategies which enumerate numbers into the sets under construction are the nonordering strategies. If such a strategy $\mathcal{N}_\alpha$ wants to enumerate a follower or a trace, now this has to be γ-permitted by D. This may force $\mathcal{N}_\alpha$ to enumerate numbers also at stages where α is not accessible. This is not compatible with the minimal pair strategy which allows the enumeration of "small" numbers only at successor stages of expansionary stages. This problem is overcome by adding the set C to the meets which allows us to use the branching degree strategy in place of the minimal pair strategy. For this sake, any number x associated with a nonordering strategy $\mathcal{N}_\alpha$ will be assigned a C-trace z first, before it is allowed to be enumerated into its target and if x is enumerated then the trace z is simultaneously enumerated into C. It will be crucial that these C-traces are appointed at stages at which α is accessible (and that they are greater than the stage at which they are appointed). Moreover, if for critical $\mathcal{N}_\alpha$ an A_j-trace x^α_{l+1} and the corresponding B_j-trace y^α_{l+1} and C-trace z^α_{l+1} are enumerated into their targets at a stage $s+1$ then the C-trace z^α_l for the traces x^α_l and y^α_l which might be enumerated next is appointed at a stage $> s+1$ at which α is accessible. So both sides of a computation (related to a meet requirement) we may be concerned with have recovered and z^α_l will be "big" hence not interfere with the current computations. Note that, in contrast to the A- and B-traces which are assigned in order to meet the join requirements, the C-traces are assigned in order to meet the meet requirements.

An additional effect of the permitting constraint is that we cannot argue that a single follower will eventually be permitted. So if a follower x or, in case of a critical nonordering strategy, a member of the entourage of x waits to be permitted, we have to assign a new follower. In case of the noncritical nonordering requirements, by the standard permitting argument (using that D is noncomputable) we may argue that eventually one of the followers is permitted, hence the strategy remains finitary.

For a critical nonordering strategy $\mathcal{N}_\alpha$ (say $|\alpha| = 4e+i$), the situation is more delicate. Here we need that not only one of the followers but also all of its traces are permitted. To achieve this we have to use that D is uniformly multiply permitting. So fix a strictly increasing computable function f such that D is uniformly multiply permitting via f (and $\{D_s\}_{s\geq 0}$) and for any number x let $\gamma(x,0) = f(x)+1$ be the initial position of the marker $\gamma(x)$.

Then, in order to exploit the uniform multiple permitting property of D, given a follower $x^{\alpha,p}$, at the first stage $s+1$ (if any) such that the entourage of $x^{\alpha,p}$ is complete at stage $s+1$ – i.e., such that $x^{\alpha,p}$ becomes realized at stage $s+1$ and the A-part of the entourage, $x^{\alpha,p} = x^{\alpha,p}_0, x^{\alpha,p}_1, \dots, x^{\alpha,p}_{m_p}$ is extended at stage $s+1$ by adding the B-traces $y^{\alpha,p}_{m_p}, \dots, y^{\alpha,p}_0$ (as in the basic construction) and the C-trace $z^{\alpha,p}_{m_p}$ (for the sake of the meet strategies) – we assign an interval $F = F^\alpha_p$ to $x^{\alpha,p}$ where $x^{\alpha,p} = \min F$ and $|F| \geq m_p + 2$. The latter allows us to define a (uniformly) partial computable function ψ on F, such that (assuming the strategy is on the true path) the attack via $x^{\alpha,p}$ can be completed since D γ-permits the enumeration of all of the members in the entourage of $x^{\alpha,p}$ – provided that, for any $x \in F$, D changes below $f(x)+1$ after stage $\psi(x)$ (if the latter is defined).

This leads to the following strategy: at any α-stage s such that, for all existing $\mathcal{N}_\alpha$-followers $x^{\alpha,p'}$ ($p' \leq p$), the corresponding attacks on $N_{|\alpha|}$ are stuck waiting for a required permission by D (namely either the permission to raise the γ-marker position of $x^{\alpha,p'}$ above $f(\max F^\alpha_{p'})$ or to enumerate some numbers in the entourage into their targets), we appoint a new greater follower $x^{\alpha,p+1}$ at stage $s+1$. Then we can argue that if $\mathcal{N}_\alpha$ acts infinitely often without being initialized then an infinite ascending sequence $x^{\alpha,0}, x^{\alpha,1}, x^{\alpha,2}, \dots$ of *permanent* $\mathcal{N}_\alpha$-followers (i.e., followers that are never canceled) will be defined such that the associated intervals $F^\alpha_0, F^\alpha_1, F^\alpha_2, \dots$ form a very strong array of intervals and the partial computable function ψ defined on these intervals as indicated above, contradicts the fact that D is $\{F^\alpha_n\}_{n\geq 0}$-permitting via f. So, we may argue that, for some follower $x^{\alpha,p}$, we can complete the attack and enumerate $x^{\alpha,p}$ into its target A_i.

The latter, however, is not quite sufficient in order to argue that $\mathcal{N}_\alpha$ is satisfied. Namely, it might happen that, for the follower $x^{\alpha,p}$ for which the attack is completed, the current approximation of the computation $\Phi_e^{B_0B_1C}(x^{\alpha,p})$ is injured by the action of a lesser follower $x^{\alpha,p'}$ with $p' < p$ (namely a number in the entourage of $x^{\alpha,p'}$ which is less than the use of the current approximation of $\Phi_e^{B_0B_1C}(x^{\alpha,p})$ may be enumerated into one of the oracle sets of this computation). So we cannot argue that enumerating $x^{\alpha,p}$ into A_i guarantees that $A_i(x^{\alpha,p}) \neq \Phi_e^{B_0B_1C}(x^{\alpha,p})$. This problem can be overcome, however, by the following observation. If a number of the entourage of $x^{\alpha,p'}$ is enumerated into its target (after $x^{\alpha,p}$ has been realized hence F_p^α has been defined) then this requires permission by D. But this permission by D allows to raise the γ-position of $x^{\alpha,p'}$ above $f(\max F_p^\alpha)$. So by making F_p^α sufficiently large by adding numbers for all members of the higher priority entourages of $\mathcal{N}_\alpha$, we may argue that D will permit $x^{\alpha,p'}$ to enter A_i, too. This way we may conclude that the least follower $x^{\alpha,p'}$ entering A_i will not be injured and will witness that N_{4e+i} is met.

In order to deal with the just described interactions between the followers of a critical nonordering strategy $\mathcal{N}_\alpha$, we say an $\mathcal{N}_\alpha$-follower x is *stronger* than an $\mathcal{N}_\alpha$-follower x' if $x < x'$, i.e., if x is appointed earlier than x'. Then if a number in the entourage of x is enumerated into its target (thereby possibly injuring x') we cancel x' and its entourage if the interval F' corresponding to x' has not yet been defined and we declare x' to be *injured* if F' is defined already. In the latter case x' is not canceled but no further action for x' is taken. (Note that we cannot cancel an interval F once it is assigned (unless the strategy is initialized). Otherwise, assuming that $\mathcal{N}_\alpha$ acts infinitely often (without being initialized), we cannot argue that the permanent intervals define a very strong array.)

For the formal construction we need some further notions and notation. A *new large number* x at stage $s+1$ is a number $x > s+1$ such that x is greater than all numbers y used in the construction by the end of stage s where a number y is *used* by the end of stage s if either y has been appointed as a follower or (potential) trace by the end of this stage or y is in one of the intervals F_p^α defined by the end of this stage.

For a follower $x^{\alpha,p}$ of a critical strategy $\mathcal{N}_\alpha$, an interval F_p^α becomes associated with $x^{\alpha,p}$ at the stage at which the follower becomes realized. A realized follower $x^{\alpha,p}$ becomes *admissible* at a stage $s+1$ if permission is given to let $\gamma(x^{\alpha,p}, s+1) > f(\max F_p^\alpha)$. As long as $x^{\alpha,p}$ is not realized, the entourage of $x^{\alpha,p}$ consists of A-numbers only and its members (in increasing order, i.e., in order of appointment) are denoted by $x^{\alpha,p} = x_0^{\alpha,p}, x_1^{\alpha,p}, \dots, x_m^{\alpha,p}$ ($m \geq 0$). Once realized, the entourage of $x^{\alpha,p}$ has the form $x^{\alpha,p} = x_0^{\alpha,p}, \dots, x_m^{\alpha,p}, y_m^{\alpha,p}, \dots, y_0^{\alpha,p}, z_m^{\alpha,p}$ or $x^{\alpha,p} = x_0^{\alpha,p}, \dots, x_m^{\alpha,p}, y_m^{\alpha,p}, \dots, y_0^{\alpha,p}$ where $x_0^{\alpha,p}, \dots, x_m^{\alpha,p}$ are A-numbers, $y_m^{\alpha,p}, \dots, y_0^{\alpha,p}$ are B-numbers and $z_m^{\alpha,p}$ is a C-number. In the former case the follower $x^{\alpha,p}$ is called *C-certified*. At the stage at which $x^{\alpha,p}$ becomes realized, the entourage has its maximum length. The entourage of $x^{\alpha,p}$ becomes reduced only if the follower is admissible and C-certified. If such a reduction takes place, the greatest A-, B- and C-numbers in the entourage are enumerated into their targets. Hence, just after such a reduction, the follower is not C-certified. So, following any reduction step, the follower has to get a new C-certificate before the next reduction can take place. This certification happens at a stage where α is accessible.

The follower $x^{\alpha,p}$ of a noncritical strategy $\mathcal{N}_\alpha$ has a C-trace whenever it is realized.

We say a strategy $\mathcal{N}_\alpha$ *gets permitting* via a number x at stage $s+1$ if

$$D_{s+1} \upharpoonright \gamma(x,s)+1 \neq D_s \upharpoonright \gamma(x,s)+1 \tag{2.61}$$

and, at the end of stage s, $x = x^{\alpha,p}$ is a follower of $\mathcal{N}_\alpha$ and one of the following holds.

(I) $\mathcal{N}_\alpha$ is critical and $x^{\alpha,p}$ is admissible, C-certified and not injured.

(II) $\mathcal{N}_\alpha$ is uncritical and $x^{\alpha,p}$ is realized.

We say that $\mathcal{N}_\alpha$ *gets permitting* at stage $s+1$ if $\mathcal{N}_\alpha$ gets permitting via some number x at stage $s+1$.

Now the construction is as follows.

Construction.

Stage 0. All sets under construction are empty at stage 0 and all strategies are initialized (i.e., no parameters are associated with any strategy and no number is used at stage 0). Moreover, $\gamma(x, 0) = f(x) + 1$ for all $x \geq 0$.

Stage $s+1$. A strategy $\mathcal{N}_\alpha$ with $|\alpha| = n$ *requires attention* at stage $s+1$ if $\mathcal{N}_\alpha$ is not satisfied at the end of stage s and either $\mathcal{N}_\alpha$ gets permitting at stage $s+1$ (in that case, say $\mathcal{N}_\alpha$ requires attention *via permitting*) or $\alpha \sqsubseteq \delta_s$ and one of the following holds.

(i) No follower is assigned to $\mathcal{N}_\alpha$ at the end of stage s.

(ii) $\mathcal{N}_\alpha$ is critical and has the followers $x^{\alpha,0} < \ldots < x^{\alpha,p-1}$ at the end of stage s and, for all $r \leq p-1$, $x^{\alpha,r}$ is realized and $x^{\alpha,r}$ is injured or C-certified.

(iii) $\mathcal{N}_\alpha$ is critical and has a follower $x^{\alpha,p}$ at the end of stage s such that $x^{\alpha,p}$ is not yet realized and $\Phi_{e,s}^{B_{0,s}B_{1,s}C_s}(x^{\alpha,p}) = 0$ where $n = 4e + i$.

(iv) $\mathcal{N}_\alpha$ is critical and has an admissible follower $x^{\alpha,p}$ at the end of stage s such that $x^{\alpha,p}$ is not injured and not C-certified.

(v) $\mathcal{N}_\alpha$ is uncritical and has the followers $x^{\alpha,0} < \ldots < x^{\alpha,p-1}$ at the end of stage s and all of them are realized at the end of stage s.

(vi) $\mathcal{N}_\alpha$ is uncritical and has a follower $x^{\alpha,p}$ at the end of stage s such that $x^{\alpha,p}$ is not yet realized and $\Phi_{e,s}^{A_{1-i,s}B_{1-i,s}C_s}(x^{\alpha,p}) = 0$ where $n = 4e + 2 + i$.

Fix the least α such that $\mathcal{N}_\alpha$ requires attention at stage $s+1$. If $\mathcal{N}_\alpha$ requires attention via permitting, fix the least follower $x^{\alpha,p}$ such that $\mathcal{N}_\alpha$ gets permitting via $x^{\alpha,p}$ and perform the following action according to the case via which $\mathcal{N}_\alpha$ gets permitting.

(I) Let $x_0^{\alpha,p}, x_1^{\alpha,p}, \ldots, x_m^{\alpha,p}, y_m^{\alpha,p}, \ldots y_0^{\alpha,p}, z_m^{\alpha,p}$ be the entourage of $x^{\alpha,p} = x_0^{\alpha,p}$ at the end of stage s and fix $j \leq 1$ such that $x_m^{\alpha,p}$ is targeted for A_j (hence $y_m^{\alpha,p}$ is targeted for B_j). Enumerate $x_m^{\alpha,p}$ into A_j, $y_m^{\alpha,p}$ into B_j and $z_m^{\alpha,p}$ into C. Declare that $\mathcal{N}_\alpha$ is *not C-certified.* Delete $x_m^{\alpha,p}$, $y_m^{\alpha,p}$ and $z_m^{\alpha,p}$ from the entourage of $x_0^{\alpha,p}$. If $m > 0$ then appoint $y_{m-1}^{\alpha,p}$ as the B_{1-j}-trace of $x_{m-1}^{\alpha,p}$ (replacing the old A_j-trace $x_m^{\alpha,p}$ of $x_{m-1}^{\alpha,p}$; note that $x_{m-1}^{\alpha,p}$ and $y_{m-1}^{\alpha,p}$ have been previously targeted for A_{1-j} and B_{1-j}, respectively). If $m = 0$ (i.e., the follower itself has just been enumerated into its target set), declare $\mathcal{N}_\alpha$ to be *satisfied.*

In either case fix $q \geq p$ maximal such that F_q^α is defined at the end of stage s. Let

$$\gamma(x^{\alpha,p}, s+1) = \max\{\gamma(x^{\alpha,p}, s), f(\max F_q^\alpha)\} + 1,$$

cancel all followers $x_0^{\alpha,r}$ with $r > q$ and their entourages (if any) and declare all $x_0^{\alpha,r}$ with $p < r \leq q$ to be *injured* (if any).

(II) Fix i such that $x^{\alpha,p}$ is targeted for B_i. Enumerate $x^{\alpha,p}$ into B_i, enumerate the C-trace $y^{\alpha,p}$ of $x^{\alpha,p}$ into C and declare $\mathcal{N}_\alpha$ to be *satisfied.*

Otherwise, perform the following action according to the first clause above via which $\mathcal{N}_\alpha$ requires attention where in case of (iii), (iv) and (vi) the corresponding follower $x^{\alpha,p}$ is chosen to be minimal. (Actually, it will follow by construction that at most one of these clauses may apply and if (iii), (iv) or (vi) applies then the corresponding follower $x^{\alpha,p}$ is uniquely determined.)

(i) Assign the least new large number $x^{\alpha,0}$ as a first follower to $\mathcal{N}_\alpha$. If N_n is critical of the form $A_i \neq \Phi_e^{B_0 B_1 C}$, declare the entourage of $x^{\alpha,0}$ to be $x_0^{\alpha,0} = x^{\alpha,0}$ and $x^{\alpha,0}$ to be targeted for A_i. If N_n is uncritical of the form $B_i \neq \Phi_e^{A_{1-i} B_{1-i} C}$, declare $x^{\alpha,0}$ to be targeted for B_i.

(ii) Assign the least new large number $x^{\alpha,p}$ as an additional follower to $\mathcal{N}_\alpha$, declare the entourage of $x^{\alpha,p}$ to be $x_0^{\alpha,p} = x^{\alpha,p}$ and $x^{\alpha,p}$ to be targeted for A_i where N_n is of the form $A_i \neq \Phi_e^{B_0 B_1 C}$.

(iii) Declare $x^{\alpha,p}$ to be *realized.* Let $x^{\alpha,p} = x_0^{\alpha,p}, x_1^{\alpha,p}, \dots, x_m^{\alpha,p}$ be the entourage of $x^{\alpha,p}$ at the end of stage s (note that the entourage consists of A-numbers only). Add the least new large numbers $y_m^{\alpha,p} < y_{m-1}^{\alpha,p} < \dots < y_0^{\alpha,p} < z_m^{\alpha,p}$ to the entourage of $x^{\alpha,p}$ as new (potential) traces. For $0 \leq l \leq m$, declare $y_l^{\alpha,p}$ to be targeted for B_{j_l} where j_l is such that $x_l^{\alpha,p}$ is targeted for A_{j_l} and declare $z_m^{\alpha,p}$ to be targeted for C. Moreover, declare $y_m^{\alpha,p}$ to be the B_{j_m}-trace of $x_m^{\alpha,p}$, declare $z_m^{\alpha,p}$ to be the C-trace of $x_m^{\alpha,p}$ and $y_m^{\alpha,p}$ and declare $x^{\alpha,p}$ to be *C-certified.* Finally, let $F_p^\alpha = [x_0^{\alpha,p}, u]$ where u is chosen so that $|F_p^\alpha| = m + 2 + \sum_{r<p} |F_r^\alpha|$.

(iv) Let $x^{\alpha,p} = x_0^{\alpha,p}, x_1^{\alpha,p}, \dots, x_m^{\alpha,p}, y_m^{\alpha,p}, y_{m-1}^{\alpha,p}, \dots, y_0^{\alpha,p}$ be the entourage of $x^{\alpha,p}$ at the end of stage s (note that the entourage consists of A-numbers $x_l^{\alpha,p}$ and B-numbers $y_l^{\alpha,p}$ only). Append the least new large number $z_m^{\alpha,p}$ to the entourage of $x^{\alpha,p}$, target $z_m^{\alpha,p}$ for C and let $z_m^{\alpha,p}$ be the C-trace of $x_m^{\alpha,p}$ and $y_m^{\alpha,p}$. Moreover, declare $x^{\alpha,p}$ to be *C-certified.*

(v) Assign the least new large number $x^{\alpha,p}$ as an additional follower to $\mathcal{N}_\alpha$ targeted for B_i where N_n is of the form $B_i \neq \Phi_e^{A_{1-i} B_{1-i} C}$.

(vi) Declare $x^{\alpha,p}$ to be *realized.* Assign the least new large number $y^{\alpha,p}$ as a trace of $x^{\alpha,p}$ targeted for C.

In any case, declare that $\mathcal{N}_\alpha$ *receives attention* or *becomes active* at stage $s+1$ (via $x^{\alpha,p}$ where $p = 0$ for Clause (i)) and initialize all strategies $\mathcal{N}_\beta$ with $\alpha < \beta$, i.e., declare them to be unsatisfied and cancel all followers, entourages and intervals of such strategies. Furthermore, for any $\beta < \alpha$, to any existing entourage whose last element $x_m^{\beta,p}$ is targeted for some A_i, append a new large number $x_{m+1}^{\beta,p}$ as a trace of $x_m^{\beta,p}$ targeted for A_{1-i}. Finally, for any critical strategy $\mathcal{N}_\beta$ and any follower $x = x^{\beta,p}$ of $\mathcal{N}_\beta$ such that x is realized but not admissible at the end of stage s and (2.61) holds, let $\gamma(x^{\beta,p}, s+1) = \max\{\gamma(x^{\beta,p}, s), f(\max F_p^\beta)\} + 1$ and declare that $x^{\beta,p}$ becomes *admissible* at stage $s+1$.

(Any sets, concepts and parameters remain unchanged unless mentioned otherwise above.)

This completes the construction.

Verification.

We begin with a few observations and explanations before we turn to proving that the constructed sets have the required properties.

We let $x^{\alpha,0}[s], x^{\alpha,1}[s], \ldots, x^{\alpha,p}[s]$ denote the followers of $\mathcal{N}_\alpha$ at the end of stage s in order of their appointment (if any). Similarly, if $\mathcal{N}_\alpha$ is critical we denote the entourage of $x^{\alpha,r}[s]$ by $x_0^{\alpha,r}[s], \ldots, x_m^{\alpha,r}[s]$ (if $x^{\alpha,r}[s]$ is not yet realized) respectively by $x_0^{\alpha,r}[s], \ldots, x_m^{\alpha,r}[s], y_m^{\alpha,r}[s], \ldots, y_0^{\alpha,r}[s]$ (if $x^{\alpha,r}[s]$ is realized but not C-certified) respectively $x_0^{\alpha,r}[s], \ldots, x_m^{\alpha,r}[s], y_m^{\alpha,r}[s], \ldots, y_0^{\alpha,r}[s], z_m^{\alpha,r}[s]$ (if $x^{\alpha,r}[s]$ is realized and C-certified) where the x-numbers are targeted for A_0 or A_1, the y-numbers are targeted for B_0 or B_1 and the z-numbers are targeted for C. We drop the parameter $[s]$ if it is obvious from the context. Since followers and traces are appointed in order, $x^{\alpha,0}[s] < x^{\alpha,1}[s] < \ldots < x^{\alpha,p}[s]$ and the members of any entourage are strictly increasing. Moreover, if there is more than one $\mathcal{N}_\alpha$-follower at the same stage then the members of the entourage of the stronger follower are less than all members of the entourage of the weaker follower unless the weaker follower is injured:

$$\text{If } p < p',\ x^{\alpha,p}[s] \text{ and } x^{\alpha,p'}[s] \text{ are defined, } x^{\alpha,p'}[s] \text{ is not injured and } v \text{ and } v' \text{ are in the entourage of } x^{\alpha,p}[s] \text{ and } x^{\alpha,p'}[s], \text{ respectively, then } x^{\alpha,p}[s] \text{ is realized and } v < v'. \tag{2.62}$$

The first part of (2.62) is immediate since a new $\mathcal{N}_\alpha$-follower is appointed only if all stronger followers are realized. For a proof of the second part, it suffices to show that v' is appointed later than v. Fix the stage $t+1 < s$ such that $x^{\alpha,p}[s]$ becomes realized at stage $t+1$. Then $x^{\alpha,p'}[s]$ is appointed after stage $t+1$. So if v is appointed by the end of stage $t+1$ then the claim is trivial. This leaves the case that v is appointed after stage $t+1$. Then v must be a C-trace and there must be stages t' and t'' such that $t < t' < t'' < s$, $\mathcal{N}_\alpha$ gets permitting via $x^{\alpha,p}[s]$ at stage $t'+1$ and acts accordingly and v is appointed as a C-trace at stage $t''+1$ where $t''+1$ is the least stage $> t'+1$ at which α is accessible and $\mathcal{N}_\alpha$ becomes active via $x^{\alpha,p}[s]$ or a weaker follower (since, for any α-stage $\hat{t}$ with $t' < \hat{t} \leq t''$, $\mathcal{N}_\alpha$ requires attention via $x^{\alpha,p}[s]$ and Clause (iv) at stage $\hat{t}+1$). Since all followers weaker than $x^{\alpha,p}[s]$ are canceled or injured at stage $t'+1$, it follows that $x^{\alpha,p'}[s]$ is appointed at a stage $t'''+1 > t'+1$. In fact, since t''' is an α-stage and $\mathcal{N}_\alpha$ acts at stage $t'''+1$ via $x^{\alpha,p'}[s]$, it follows by minimality of t'' that $t'' < t'''$. So v' is appointed later than v in this case, too.

Finally, observe that whenever a number y enters any set, there is a C-trace $z \geq y$ entering C at the same stage.

We prove a series of claims very similar to those in the proof of Theorem 77 to show that the constructed sets have the required properties. Claims 1, 2 and 5 and their proofs are the same as there.

Claim 3. Every strategy $\mathcal{N}_\alpha$ on the true path (i.e., $\alpha \sqsubset f$) is initialized only finitely often and requires attention only finitely often. Moreover, $N_{|\alpha|}$ is met.

Proof. The proof is by induction on $|\alpha|$. Fix $\alpha \sqsubset f$. By Claim 2 and by inductive hypothesis, we may fix the least stage s_0 such that $\alpha \leq \delta_s$ for all $s \geq s_0$ and no strategy $\mathcal{N}_\beta$ with $\beta \sqsubset \alpha$ requires attention after stage s_0. Since no $\gamma <_{\mathrm{L}} \alpha$ is accessible after stage s_0, it follows that there are only finitely many followers which are assigned to strategies $\mathcal{N}_\beta$ with $\beta <_{\mathrm{L}} \alpha$. Since any strategy acts via the same follower only finitely often (and since a follower is assigned to one strategy only), this implies that there is a stage s such that no strategy $\mathcal{N}_\beta$, $\beta <_{\mathrm{L}} \alpha$, requires attention after stage s. So, since there are infinitely many α-stages, we may let $s_1 \geq s_0$ be the least α-stage such that $\alpha \leq \delta_s$ for all $s \geq s_1$ and no strategy $\mathcal{N}_\beta$ with $\beta < \alpha$ requires attention after stage s_1. Then $\mathcal{N}_\alpha$ acts whenever it requires attention and is not initialized after stage s_1. Moreover, by minimality of s_1, $\mathcal{N}_\alpha$ is initialized at the end of stage s_1 (hence is not satisfied at the end of stage s_1) and a follower $x_0^{\alpha,0}$ of $\mathcal{N}_\alpha$ is appointed at stage s_1+1. Since $\mathcal{N}_\alpha$ is not initialized after stage s_1 and since $x_0^{\alpha,0}$ is the strongest $\mathcal{N}_\alpha$-follower existing after stage s_1, $x_0^{\alpha,0}$ is never canceled hence permanent.

Case 1. $\mathcal{N}_\alpha$ is critical.

Fix $e \geq 0$ and $i \leq 1$ such that $|\alpha| = 4e+i$. Now, first assume that $\mathcal{N}_\alpha$ is declared to be satisfied at a stage $s+1 \geq s_1+1$. Then $\mathcal{N}_\alpha$ does not require attention after stage $s+1$. Moreover, $N_{|\alpha|}$ is met. Namely, fix the follower $x^{\alpha,p}$ such that $\mathcal{N}_\alpha$ gets permitting via $x^{\alpha,p}$ and Clause (I) at stage $s+1$. Then $x^{\alpha,p}$ is admissible – hence realized – and not injured at stage s and $x^{\alpha,p}$ is enumerated into A_i at stage $s+1$. Moreover, for the stage $s'+1 < s+1$ at which $x^{\alpha,p}$ becomes realized, $\Phi_{e,s'}^{B_{0,s'}B_{1,s'}C_{s'}}(x^{\alpha,p}) = 0$. So it suffices to show that no number $\leq s'$ enters any of the sets B_0, B_1, C after stage s'. By initialization and by choice of s_1 no strategy $\mathcal{N}_\beta$ with $\beta \neq \alpha$ will enumerate numbers $\leq s'$ after stage s'. Moreover no numbers from entourages of $\mathcal{N}_\alpha$-followers stronger than $x^{\alpha,p}$ are enumerated after stage s'. (Namely, this had to happen by stage s thereby injuring $x^{\alpha,p}$.) On the other hand any weaker $\mathcal{N}_\alpha$-followers acting after stage s' are appointed after this stage hence may enumerate only numbers $> s'$. Finally, the B_0-, B_1- and C-numbers in the entourage of $x^{\alpha,p}$ are appointed at stage $s'+1$ or at later stages hence are greater than s', too.

Next assume that there is a permanent $\mathcal{N}_\alpha$-follower $x^{\alpha,p}$ which is never realized (i.e., such that $\mathcal{N}_\alpha$ never acts via $x^{\alpha,p}$ and Clause (iii)). Then, once $x^{\alpha,p}$ is appointed, no weaker $\mathcal{N}_\alpha$-follower may be appointed, hence there are only finitely many $\mathcal{N}_\alpha$-followers. Since a strategy receives attention via a fixed follower only finitely often, it follows that $\mathcal{N}_\alpha$ receives attention only finitely often, hence, by choice of s_1, requires attention only finitely often. It follows that for all sufficiently large α-stages s, $\Phi_{e,s}^{B_{0,s}B_{1,s}C_s}(x^{\alpha,p}) \neq 0$ hence $\Phi_e^{B_0B_1C}(x^{\alpha,p}) \neq 0$. Moreover, $x^{\alpha,p}$ is not enumerated into A_i. So $N_{|\alpha|}$ is met.

By the preceding observations, it suffices to show that $\mathcal{N}_\alpha$ will be satisfied at some stage $\geq s_1+1$ or will have a permanent follower $x^{\alpha,p}$ which is never realized. For a contradiction, assume that neither is the case. Then $\mathcal{N}_\alpha$ requires (hence receives) attention infinitely often. (Namely, assume not and fix the least α-stage $s \geq s_1+1$ such that $\mathcal{N}_\alpha$ does not require attention after stage s. Then there are $\mathcal{N}_\alpha$-followers at the end of stage s and none of this followers acts later. Since, by assumption, all of these followers are realized and since $\mathcal{N}_\alpha$ is not satisfied at any stage $\geq s$, any of these followers must be injured or C-certified at stage s (since otherwise $\mathcal{N}_\alpha$ will require attention via Clause (iv) at stage $s+1$). But this implies that $\mathcal{N}_\alpha$ requires attention via Clause (ii) at stage $s+1$, a contradiction.) Since a strategy receives attention via a fixed follower only finitely often, it follows that there are infinitely many $\mathcal{N}_\alpha$-followers. In fact, since after stage s_1 an $\mathcal{N}_\alpha$-follower can be canceled only if a stronger $\mathcal{N}_\alpha$-follower acts, there are infinitely many permanent $\mathcal{N}_\alpha$-followers, say $x^{\alpha,0} < x^{\alpha,1} < x^{\alpha,2} < \ldots$. Moreover, these followers are just the $\mathcal{N}_\alpha$-followers which become realized, i.e., which get an interval F assigned to it after stage s_1. (Namely, once realized, an $\mathcal{N}_\alpha$-follower x cannot be canceled by stronger $\mathcal{N}_\alpha$-followers. So any $\mathcal{N}_\alpha$-follower x realized after stage s_1 is permanent. The converse holds by assumption.) It follows that the sequence of the permanent followers is computable and so is the sequence $\mathcal{F} = \{F_p^\alpha\}_{p\geq 0}$ of the intervals associated with these followers. Since, by definition, $\max F_p^\alpha < \min F_{p+1}^\alpha$ and $|F_p^\alpha| < |F_{p+1}^\alpha|$, this implies that $\mathcal{F}$ is a very strong array of intervals. Moreover, the stage t_p such that $x^{\alpha,p}$ becomes realized at stage $t_p + 1$ hence F_p^α becomes assigned to $x^{\alpha,p}$ at stage $t_p + 1$ and the entourage

$$x_0^{\alpha,p} = x^{\alpha,p}, \ldots x_{m_p}^{\alpha,p}, y_0^{\alpha,p}, \ldots, y_{m_p}^{\alpha,p}, z_{m_p}^{\alpha,p}$$

of $x^{\alpha,p}$ defined at the end of this stage can be computed from p. Note that $t_p < t_{p+1}$ and that $x^{\alpha,p}$ is not injured at stage $t_p + 1$ ($p \geq 0$). Moreover, for any stage $t \geq t_p + 1$, the followers $x < x^{\alpha,p}$ of $\mathcal{N}_\alpha$ at stage t are just the followers $x^{\alpha,0}, \ldots, x^{\alpha,p-1}$ and all of them are realized at stage t.

Moreover, by definition, for any p the interval F_p^α is large enough so that we can effectively (uniformly in p) split an appropriate initial segment of F_p^α into intervals $I_{p,r}^\alpha$, $r \leq p$ such that

$$I_{p,p}^\alpha \text{ contains the } m_p + 2 \text{ least elements } x_0^{\alpha,p} < \hat{x}_0^{\alpha,p,p} < \cdots < \hat{x}_{m_p}^{\alpha,p,p} \text{ of } F_p^\alpha$$

(note that $x^{\alpha,p} = x_0^{\alpha,p}$ is the least element of F_p^α) and, for $r < p$,

$$I_{p,r}^\alpha \text{ contains the } m_r + 1 \text{ least elements } \hat{x}_0^{\alpha,p,r} < \cdots < \hat{x}_{m_r}^{\alpha,p,r} \text{ of } F_r^\alpha.$$

Based on these effective partitions define the partial computable function ψ on F_p^α ($p \geq 0$) by letting

$$\psi(x_0^{\alpha,p}) = t_p + 1 \tag{2.63}$$

and

$$\begin{aligned}\psi(\hat{x}_m^{\alpha,p,r}) = \mu\, t > t_p + 1\, [z_m^{\alpha,r}[t]\downarrow \ \text{ and } \ x^{\alpha,r} \text{ is admissible at stage } t \\ \text{and } \ \gamma(x^{\alpha,r}, t) > f(\max F_p^\alpha)]\end{aligned}$$

(for $p \geq 0, r \leq p, m \leq m_r$). Now, since D is $\mathcal{F}$-permitting via f, we may fix p minimal such that

$$\forall x \in F_p^\alpha(\psi(x)\downarrow \Rightarrow \ D \upharpoonright f(x) + 1 \neq D_{\psi(x)} \upharpoonright f(x) + 1), \tag{2.64}$$

holds. In order to get the desired contradiction, distinguish the following two cases.

First assume that $x^{\alpha,p}$ is never injured. (We will show that $\mathcal{N}_\alpha$ becomes satisfied via $x^{\alpha,p}$ contrary to assumption.) Note that, by (2.63) and (2.64), $x^{\alpha,p}$ becomes admissible, say at stage $t' + 1 > t_p + 1$. So, since $\mathcal{N}_\alpha$ does not become satisfied, we may fix $m \leq m_p$ maximal such that $x_m^{\alpha,p}$ is not enumerated into $A_0 \cup A_1$. Now if $m = m_p$ then $t'' = t' + 1$ is the least stage such that $z_{m_p}^{\alpha,p}$ is defined and $x^{\alpha,p}$ is admissible. If $m < m_p$ then, since $x^{\alpha,p}$ is never injured, there is an α-stage $t'' - 1 \geq t' + 1$ such that $\mathcal{N}_\alpha$ acts via $x^{\alpha,p}$ according to Clause

(iv) at stage t'' and the C-trace $z_m^{\alpha,p}$ is appointed as the maximum element of the entourage $x_0^{\alpha,p}, \dots x_m^{\alpha,p}, y_m^{\alpha,p}, \dots, y_0^{\alpha,p}, z_m^{\alpha,p}$ at this stage. In either case this implies that $\psi(\hat{x}_m^{\alpha,p,p}) = t''$. So, by (2.64), there is a stage $t''' + 1 > t''$ such that $\mathcal{N}_\alpha$ gets permitting via $x^{\alpha,p}$ at stage $t''' + 1$. Moreover, since $x^{\alpha,p}$ does not become injured at stage $t''' + 1$, $\mathcal{N}_\alpha$ acts via $x^{\alpha,p}$. So $x_m^{\alpha,p}$ is enumerated into $A_0 \cup A_1$ contrary to choice of m.

Finally assume that $x^{\alpha,p}$ becomes injured. Then there are a number $r < p$ and a stage $t + 1 > t_p + 1$ such that $\mathcal{N}_\alpha$ gets permitting and acts via $x^{\alpha,r}$ at stage $t + 1$. Fix the least such r and fix the least corresponding t. (We will show that $\mathcal{N}_\alpha$ becomes satisfied via $x^{\alpha,r}$ contrary to assumption.) Note that, by minimality of r, $x^{\alpha,r}$ is never injured while, by choice of t,

$$\forall\, s \geq t + 1\, (\gamma(x^{\alpha,r}, s) > f(\max F_p^\alpha)) \tag{2.65}$$

and the entourage of $x^{\alpha,r}$ at the end of stage $t + 1$ has the form $x_0^{\alpha,r}, \dots, x_{m^*}^{\alpha,r}, y_{m^*}^{\alpha,r}, \dots, y_0^{\alpha,r}$ for some $m^* \geq 0$. (Note that the entourage cannot be empty since $\mathcal{N}_\alpha$ is not satisfied.) So, since $\mathcal{N}_\alpha$ is not satisfied after stage t_p, there is a greatest number $m \leq m^*$ such that $x_m^{\alpha,r}$ is not enumerated into $A_0 \cup A_1$. But this is impossible. Namely, as in the first case, we may argue that there is a stage $t'' > t + 1$ at which the C-trace $z_m^{\alpha,r}$ becomes appended to the entourage $x_0^{\alpha,r}, \dots x_m^{\alpha,r}, y_m^{\alpha,r}, \dots, y_0^{\alpha,r}$ of $x^{\alpha,r}$ and that (by (2.65)) $\psi(\hat{x}_m^{\alpha,p,r}) = t''$. So, by (2.64) (and (2.65)), $\mathcal{N}_\alpha$ will get permitting via $x^{\alpha,r}$ after stage t'' and $x_m^{\alpha,r}$ will be enumerated into $A_0 \cup A_1$ contrary to choice of m.

Case 2. $\mathcal{N}_\alpha$ is uncritical.

Fix $e \geq 0$ and $i \leq 1$ such that $|\alpha| = 4e + 2 + i$. If $\mathcal{N}_\alpha$ is declared to be satisfied at a stage $s + 1 \geq s_1 + 1$ or if there is a permanent $\mathcal{N}_\alpha$-follower which is never realized then, by a straightforward variant of the argument given in the first case, we may argue that $\mathcal{N}_\alpha$ requires attention only finitely often and requirement $N_{|\alpha|}$ is met. So it suffices to show that $\mathcal{N}_\alpha$ will be satisfied at some stage $\geq s_1 + 1$ or will have a permanent follower $x^{\alpha,p}$ which is never realized.

For a contradiction, assume that neither is the case. Then $\mathcal{N}_\alpha$ requires (hence receives) attention via Clause (v) infinitely often. So infinitely many followers $x^{\alpha,0} < x^{\alpha,1} < \dots$ are assigned to $\mathcal{N}_\alpha$ after stage s_1, say at stages $t_0 + 1 < t_1 + 1 < \dots$. All of these followers are permanent and eventually realized. (Namely, if $x^{\alpha,p}$ is appointed at stage $t_p + 1$ then the followers $x^{\alpha,0}, \dots, x^{\alpha,p-1}$ are realized at stage t_p and, since $\mathcal{N}_\alpha$ does not become satisfied after stage s_1, $\mathcal{N}_\alpha$ does not get permitting via any of these followers after stage t_p hence does not act via any of these followers after stage t_p. So $x^{\alpha,p}$ is permanent.) So we may fix the stage $t'_p + 1$ at which $x^{\alpha,p}$ becomes realized ($p \geq 0$). Note that the sequences $\{x^{\alpha,p}\}_{p \geq 0}$ and $\{t'_p\}_{p \geq 0}$ are computable and strictly increasing. Moreover, since $\mathcal{N}_\alpha$ does not get permitting via $x^{\alpha,p}$ after stage $t'_p + 1$, it follows (by $x \leq \gamma(x, s)$ for all x, s) that $D \restriction x^{\alpha,p} + 1 = D_{t'_p+1} \restriction x^{\alpha,p} + 1$ for $p \geq 0$. So D is computable, a contradiction. ☐

Claim 4. For $n \geq 0$, requirement M_n is met.

Proof. The proof is similar to the proof of the corresponding claim in the proof of Theorem 77. In particular, assume that M_n is infinitary and define s_0 and $S = \{s_l : l \geq 0\}$ as there (where the existence of s_0 can be established in a similar way as we established the existence of s_1 in the proof of Claim 3). Observe that, by choice of S, at every stage $s_l + 1$ with $s_l \in S$, a strategy $\mathcal{N}_\beta$ with $\alpha 0 \sqsubset \beta$ becomes active and hence all strategies $\mathcal{N}_\gamma$ with $\alpha 0 <_{\mathrm{L}} \gamma$ are initialized at stage $s_l + 1$. So, for all nodes γ and $s_l \in S$, the following holds.

$$\text{If } \mathcal{N}_\gamma \text{ enumerates a number } x \leq s_l \text{ into some set at stage } t+1 \text{ and } t \geq s_l \text{ then } \alpha 0 \sqsubset \gamma. \tag{2.66}$$

We prove one more auxiliary observation before we proceed with the same case distinction as above. We claim that for all $t_0, t, y, z \geq 0$, the following holds.

$$\left.\begin{array}{l} y \text{ is enumerated into any set at stage } t_0 + 1 \\ C_{t_0} \restriction y = C \restriction y \\ z \text{ is a } C\text{-trace active at stage } t_0 + 1 \\ z \text{ is enumerated into } C \text{ at stage } t + 1 \end{array}\right\} \Rightarrow t = t_0. \tag{2.67}$$

For a proof, first observe that, as z is active at stage $t_0 + 1$, $t_0 \leq t$ holds. Since z enters C at stage $t + 1$ and since $C_{t_0} \restriction y = C \restriction y$, it follows that $y \leq z$. Now assume for a contradiction that $t_0 < t$. Fix nodes β and γ such that y is enumerated for the sake of $\mathcal{N}_\beta$ and z enters C for the sake of $\mathcal{N}_\gamma$. Observe that $\mathcal{N}_\beta$ acts at stage $t_0 + 1$ while $\mathcal{N}_\gamma$ is not initialized at any stage s with $t_0 + 1 \leq s \leq t + 1$. Hence $\gamma \leq \beta$. Moreover, if $\gamma < \beta$, then at the stage $t_z + 1$ where z is appointed to $\mathcal{N}_\gamma$, $\mathcal{N}_\beta$ is initialized. Hence, as z is active at stage $t_0 + 1$, z is appointed before y is appointed to $\mathcal{N}_\beta$, hence $z < y$ contradicting $y \leq z$. It follows that $\beta = \gamma$. If $\mathcal{N}_\beta$ is uncritical, then a realized follower of $\mathcal{N}_\beta$ is enumerated and $\mathcal{N}_\beta$ is declared to be satisfied at stage $t_0 + 1$. This implies that no trace of $\mathcal{N}_\beta$ active at stage t_0+1 is enumerated by $\mathcal{N}_\beta$ after t_0+1, contradicting the assumption $t_0 < t$. This leaves the case that $\mathcal{N}_\beta$ is critical. Then there are followers $x^{\beta,p}$ and $x^{\beta,p'}$ such that, at the end of stage t_0, y is in the entourage of $x^{\beta,p}$ and z is in the entourage of $x^{\beta,p'}$. Moreover, none of these followers is injured (or canceled) by the end of stage $t_0 + 1$, since $\mathcal{N}_\beta$ acts via $x^{\beta,p}$ and $x^{\beta,p'}$ at stage t_0+1 and $t+1$, respectively. Since $\mathcal{N}_\beta$ enumerates y at stage t_0+1 hence injures or cancels all weaker $\mathcal{N}_\beta$-followers it follows that $p' \leq p$. So, by $y \leq z$ and (2.62) (applied to stage $s = t_0$), $p' = p$. So z is the unique C-trace in the entourage of $x^{\beta,p}$ at the end of stage t_0. Hence z is enumerated together with y at stage $t_0 + 1$, which implies $t = t_0$. This completes the proof of (2.67). Now distinguish the following cases.

Case 1: $n = 3e + 2$.

For given x, fix $s_k \in S$ minimal such that $l(n, s_k) > x$ and such that the following holds for $i = 0$ and $i = 1$.

$$C_{s_k} \restriction \varphi_{e,s_k}^{A_{i,s_k} B_{i,s_k} C_{s_k}}(x) = C \restriction \varphi_{e,s_k}^{A_{i,s_k} B_{i,s_k} C_{s_k}}(x).$$

Note that s_k exists by the assumption that M_n is infinitary and that s_k is computable in C. As in Case 1 of the above proof, we show that $\Phi_e^{A_0 B_0 C}(x) = \Phi_{e,s_k}^{A_{0,s_k} B_{0,s_k} C_{s_k}}(x)$ by proving that for all $l \geq k$, the following holds.

$$\exists i (\Phi_{e,s_{l+1}}^{A_{i,s_{l+1}} B_{i,s_{l+1}} C_{s_{l+1}}}(x) = \Phi_{e,s_l}^{A_{i,s_l} B_{i,s_l} C_{s_l}}(x)). \tag{2.68}$$

Simultaneously, we show by induction that the following holds for all $l \geq k$.

$$\forall i (C_{s_l} \restriction \varphi_{e,s_l}^{A_{i,s_l} B_{i,s_l} C_{s_l}}(x) = C \restriction \varphi_{e,s_l}^{A_{i,s_l} B_{i,s_l} C_{s_l}}(x)). \tag{2.69}$$

For k in place of l, (2.69) is immediate by choice of k. So assume that (2.69) holds for some $l \geq k$. We prove (2.68) as well as (2.69) for $l+1$ in place of l where we refer to the latter as $(2.69)_{l+1}$.

If, for both $i = 0$ and $i = 1$,

$$A_{i,s_{l+1}} B_{i,s_{l+1}} \upharpoonright \varphi_{e,s_l}^{A_{i,s_l} B_{i,s_l} C_{s_l}}(x) = A_{i,s_l} B_{i,s_l} \upharpoonright \varphi_{e,s_l}^{A_{i,s_l} B_{i,s_l} C_{s_l}}(x), \tag{2.70}$$

then (2.68) and $(2.69)_{l+1}$ are immediate by (2.69). Assume for the rest of the proof that (2.70) fails for some i and let $t_0 + 1 > s_l$ be the least stage witnessing that. Note that $t_0 < s_{l+1}$. By symmetry, w.l.o.g. assume that $i = 0$. By this assumption, fix a number $y_0 < \varphi_{e,s_l}^{A_{0,s_l} B_{0,s_l} C_{s_l}}(x)$ such that y_0 enters $A_0 B_0$ at stage $t_0 + 1$.

We claim that (2.70) holds for $i = 1$. Assume not and fix t_1 with $t_0 \leq t_1 < s_{l+1}$ such that a number $y_1 < \varphi_{e,s_l}^{A_{1,s_l} B_{1,s_l} C_{s_l}}(x)$ is enumerated into $A_1 B_1$ at stage $t_1 + 1$ by some strategy $\mathcal{N}_\gamma$. Then, a C-trace z is enumerated at stage $t_1 + 1$ and, by (2.66), $\alpha 0 \sqsubset \gamma$. As z is appointed to $\mathcal{N}_\gamma$ at a stage $t + 1$ where γ is accessible and by $s_{l+1} > t_1$, it follows that z is appointed at a stage $\leq s_l + 1$, hence active at stage $t_0 + 1$ (note that by construction, z cannot be appointed at stage $t_0 + 1$). As $y_0 < \varphi_{e,s_l}^{A_{0,s_l} B_{0,s_l} C_{s_l}}(x)$, together with (2.69), it follows that the hypotheses of (2.67) hold for $t = t_1$ and $y = y_0$, so $t_0 = t_1$. Hence y_0 enters $A_0 B_0$ and y_1 enters $A_1 B_1$ at the same stage, contradicting the construction. So (2.70) holds for $i = 1$ which implies (2.68).

Finally, assume that $(2.69)_{l+1}$ fails. By this assumption, fix $z < \varphi_{e,s_{l+1}}^{A_{i,s_{l+1}} B_{i,s_{l+1}} C_{s_{l+1}}}(x)$ entering C at stage $t + 1$ where $t \geq s_{l+1}$ for the sake of $\mathcal{N}_\gamma$. Then, $z \leq s_{l+1}$, hence $\alpha 0 \sqsubset \gamma$ by (2.66). It follows that z is appointed at a stage $\leq s_l + 1$, hence, as above, z is active at stage $t_0 + 1$. So, again, the hypotheses of (2.67) hold for $y = y_0$, so $t = t_0$. But $t_0 < s_{l+1} \leq t$, a contradiction. This completes the proof for Case 1.

Case 2. $n = 3e + i$.

Given x, fix s_k minimal such that the following hold.

(a) $l(n, s_k) > x$,

(b) none of the numbers targeted for B_i or C which exist at the end of stage s_k will enter its target after stage s_k.

Similarly as in Case 2 of the proof of Claim 4 in the proof of Theorem 77, we can argue that such a k exists. Since k can be computed from $B_i C$ uniformly in x, it suffices to show

$$\Phi_e^{A_i B_i C}(x) = \Phi_{e,s_k}^{A_{i,s_k} B_{i,s_k} C_{s_k}}(x). \tag{2.71}$$

As above, it suffices to define a strictly increasing function g such that, for all $r \geq 0$, the following hold.

$$\Phi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}} C_{s_{g(r)}}}(x) = \Phi_{e,s_k}^{A_{i,s_k} B_{i,s_k} C_{s_k}}(x), \tag{2.72}$$

$$B_{i,s_{g(r)}} C_{s_{g(r)}} \upharpoonright \varphi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}} C_{s_{g(r)}}}(x) = B_i C \upharpoonright \varphi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}} C_{s_{g(r)}}}(x), \tag{2.73}$$

$$B_{i,s_{g(r)}} C_{s_{g(r)}} \upharpoonright \varphi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}} C_{s_{g(r)}}}(x) = B_i C \upharpoonright \varphi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}} C_{s_{g(r)}}}(x). \tag{2.74}$$

Define the function g correspondingly as in the corresponding case above: let $g(0) = k$ and, for $r \geq 0$, let $g(r+1) = g(r) + 1$ unless the nonordering strategy $\mathcal{N}_\beta$ which acts at stage $s_{g(r)} + 1$ is critical and acts via Clause (iii) or (iv). In this case, let $g(r+1) = q$ where $q > g(r)$ is minimal such that $\mathcal{N}_\beta$ does not act via any of its followers existing at the end of stage $s_{g(r)} + 1$ at any stage $\geq s_q + 1$. Note that here, we cannot argue that $\mathcal{N}_\beta$ acts at stage $s_{g(r+1)-1} + 1$, but by definition of g we may fix the greatest stage t_r (which is not necessarily a β-stage) such that $\mathcal{N}_\beta$ acts at stage $t_r + 1$ via some follower existing at the end of stage $s_{g(r)} + 1$ and such that $s_{g(r+1)-1} + 1 \leq t_r < s_{g(r+1)}$.

For $r = 0$, $g(r) = k$, hence (2.72), (2.73) and (2.74) are immediate by choice of k. Now fix $r \geq 0$ such that (2.72), (2.73) and (2.74) hold. We need to establish (2.72), (2.73) and (2.74) for $r+1$ in place of r, to which we refer as $(2.72)_{r+1}$, $(2.73)_{r+1}$ and $(2.74)_{r+1}$, respectively. For establishing $(2.72)_{r+1}$, by inductive hypotheses and as $l(n, s_{g(r+1)}) > l(n, s_{g(r)}) \geq l(n, s_k) > x$, it is enough to prove the following.

$$\Phi_{e,s_{g(r+1)}}^{A_{i,s_{g(r+1)}} B_{i,s_{g(r+1)}} C_{s_{g(r+1)}}}(x) = \Phi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}} C_{s_{g(r)}}}(x)$$
$$\text{or} \tag{2.75}$$
$$\Phi_{e,s_{g(r+1)}}^{B_{0,s_{g(r+1)}} B_{1,s_{g(r+1)}} C_{s_{g(r+1)}}}(x) = \Phi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}} C_{s_{g(r)}}}(x).$$

Fix β such that $\mathcal{N}_\beta$ acts at stage $s_{g(r)} + 1$. Note that we cannot prove (2.55) anymore, as now strategies $\mathcal{N}_\gamma$ are allowed to act at stages at which γ it not accessible. However, we know that $\mathcal{N}_\beta$ acts at stages $s_{g(r)} + 1$ and $t_r + 1$ (where $s_{g(r+1)-1} \leq t_r < s_{g(r+1)}$) without being initialized in between and that all assignments of B- and C-traces to a strategy $\mathcal{N}_\gamma$ are still performed only at stages where γ is accessible, so the following variant of (2.55) holds.

$$\begin{gathered}\text{No strategy } \mathcal{N}_\gamma \text{ with } \gamma < \beta \text{ gets any } B\text{- or } C\text{-trace assigned} \\ \text{at any stage } s+1 \text{ with } s_{g(r)} + 1 \leq s + 1 \leq s_{g(r+1)}.\end{gathered} \tag{2.76}$$

We begin with showing that at least one of the following holds.

$$A_{i,s_{g(r+1)}} \restriction \varphi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}} C_{s_{g(r)}}}(x) = A_{i,s_{g(r)}} \restriction \varphi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}} C_{s_{g(r)}}}(x), \tag{2.77}$$
$$B_{1-i,s_{g(r+1)}} \restriction \varphi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}} C_{s_{g(r)}}}(x) = B_{1-i,s_{g(r)}} \restriction \varphi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}} C_{s_{g(r)}}}(x). \tag{2.78}$$

For a contradiction assume failure of both (2.77) (witnessed by $y_0 < \varphi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}} C_{s_{g(r)}}}(x)$ entering A_i at stage $t_0 + 1$ where $s_{g(r)} \leq t_0 < s_{g(r+1)}$) and (2.78) (witnessed by $y_1 < \varphi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}} C_{s_{g(r)}}}(x)$ entering B_0 at stage $t_1 + 1$ where $s_{g(r)} \leq t_1 < s_{g(r+1)}$). Similarly as in Case 1, we can use (2.66) and (2.67) to argue that $t_0 = t_1$. So y_0 enters A_i and y_1 enters B_{1-i} at the same stage. By construction, this cannot happen, a contradiction. So, (2.77) or (2.78) holds which together with (2.73) and (2.74) implies (2.75).

If now suffices to show $(2.73)_{r+1}$ and $(2.74)_{r+1}$. In case both (2.77) and (2.78) hold, these are immediate by inductive hypotheses. So from now on assume failure of either (2.77) or (2.78) (witnessed by $z_0 < \max\{\varphi_{e,s_{g(r)}}^{A_{i,s_{g(r)}} B_{i,s_{g(r)}} C_{s_{g(r)}}}(x), \varphi_{e,s_{g(r)}}^{B_{0,s_{g(r)}} B_{1,s_{g(r)}} C_{s_{g(r)}}}(x)\} < s_{g(r)}$ enumerated into either A_i or B_{1-i} at a stage $t_0 + 1 > s_{g(r)}$ where $t_0 + 1 \leq s_{g(r+1)}$) and, for a contradiction, assume failure of $(2.73)_{r+1}$ or $(2.74)_{r+1}$ (witnessed by a number $z_1 <$

$\max\{\varphi_{e,s_{g(r+1)}}^{A_{i,s_{g(r+1)}} B_{i,s_{g(r+1)}} C_{s_{g(r+1)}}}(x), \varphi_{e,s_{g(r+1)}}^{B_{0,s_{g(r+1)}} B_{1,s_{g(r+1)}} C_{s_{g(r+1)}}}(x)\} < s_{g(r+1)}$ entering B_iC at a stage $t_1 + 1 > s_{g(r+1)}$).

Fix β' such that z_0 is enumerated by $\mathcal{N}_{\beta'}$. Let $t_0' + 1$ be the stage at which z_0 is appointed. Then, the following holds by choice of z_0 and t_0.

$$t_0' + 1 < z_0 < s_{g(r)} < t_0 + 1 \leq s_{g(r+1)} \tag{2.79}$$

and z_0 is not canceled at any stage $t + 1$ with $t_0' \leq t \leq t_0$. Moreover, by (2.66) for $l = g(r)$, $\alpha 0 \sqsubset \beta'$.

Similarly, if we fix β'' such that z_1 is enumerated into its target by $\mathcal{N}_{\beta''}$ and the stage $t_1' + 1$ at which z_1 is appointed to $\mathcal{N}_{\beta''}$, the following holds

$$t_1' + 1 < z_1 < s_{g(r+1)} < t_1 + 1, \tag{2.80}$$

z_1 is not canceled at any stage $t + 1$ with $t_1' \leq t \leq t_1$ and by (2.66) for $l = g(r+1)$, $\alpha 0 \sqsubset \beta''$ holds. Furthermore, as z_1 is a B- or a C-number, t_1' is a β''- hence an $\alpha 0$-stage. Note that (2.79) and (2.80) imply that $t_0 + 1 < t_1 + 1$. Moreover, by inductive hypotheses (2.73) and (2.74), $z_0 < z_1$ hence $t_0' + 1 \leq t_1' + 1$.

We claim that $\beta' = \beta'' \leq \beta$. First observe that z_0 is not canceled at stage $s_{g(r)} + 1$ whence $\beta' \leq \beta$ holds.

Now assume $\beta < \beta''$. Then $\mathcal{N}_{\beta''}$ is initialized at stage $t_r + 1$ (recall that $\mathcal{N}_\beta$ acts at stage $t_r + 1$ where $s_{g(r+1)-1} \leq t_r < s_{g(r+1)}$). Hence, by (2.80), $t_1' + 1 > t_r + 1$. As t_1' is an $\alpha 0$-stage, it follows that $t_1' + 1 \geq s_{g(r+1)} + 1$, contradicting (2.80). So $\beta'' \leq \beta$ holds, as well.

Next, assume that $\beta' < \beta''$. Then $\mathcal{N}_{\beta''}$ is initialized at stage $t_0 + 1$, hence, by (2.79) and (2.80), $t_1' + 1 > t_0 + 1$ holds. Moreover, as $\beta'' \leq \beta$, $\mathcal{N}_\beta$ is initialized at stage $t_0 + 1$, as well, hence, by (2.79), $t_0 + 1 > t_r + 1$. As t_1' is an $\alpha 0$-stage, it follows altogether that $t_1' + 1 \geq s_{g(r+1)} + 1$, contradicting (2.80) as in the preceding case. It follows that $\beta'' \leq \beta'$.

If $\beta'' = \beta$, then it follows that $\beta' = \beta'' = \beta$, hence $\beta' = \beta'' \leq \beta$ is immediate. So finally assume that $\beta'' < \beta$. Then, by (2.80) and by (2.76), $t_1' + 1 \leq s_{g(r)}$. So, by $t_0' + 1 \leq t_1' + 1 \leq s_{g(r)}$ and by (2.79), z_0 exists and is not canceled at stage $t_1' + 1$. Hence $\beta' \leq \beta''$ whence $\beta' = \beta'' \leq \beta$ follows in this case, too.

Having established that z_0 and z_1 are both associated with $\mathcal{N}_{\beta'}$, we may use a very similar argument as in Case 2 of the proof of Claim 4 in the proof of Theorem 77 to get to a contradiction. For completeness, we repeat it here with the necessary adjustments.

First assume that $\mathcal{N}_\beta$ acts via permitting or via one of the Clauses (i), (ii), (v) or (vi) at stage $s_{g(r)} + 1$. Then $g(r+1) = g(r) + 1$. Since t_1' is an $\alpha 0$-stage and $t_1' + 1 < s_{g(r+1)}$, the latter implies that $t_1' \leq s_{g(r)}$. In fact, since z_1 becomes appointed as a B_i- or C-trace of $\mathcal{N}_{\beta'}$ at stage $t_1' + 1$ and since no new trace is appointed at stage $s_{g(r)} + 1$, $t_1' < s_{g(r)}$ holds.

Now assume that $\mathcal{N}_\beta$ acts at stage $s_{g(r)} + 1$ via Clause (iii) or (iv). Then, by definition of g, any of the B- or C-numbers associated with a follower of $\mathcal{N}_\beta$ existing at the end of stage

$s_{g(r)}+1$ (where a number z is *associated* with a follower y if $z = y$ or z is a trace of y or z is in the entourage of y) which enter their targets later do so by the end of stage $t_r + 1 \leq s_{g(r+1)}$. Hence, if we assume that $\beta = \beta'$, then (by (2.80) and as t'_1 in an $\alpha 0$-stage),

$$s_{g(r)} + 1 < t'_1 + 1 \leq s_{g(r+1)-1} + 1 \leq t_r + 1 \tag{2.81}$$

holds and z_1 is associated with a follower x^β of $\mathcal{N}_\beta$ that has not yet existed at the end of stage $s_{g(r)} + 1$. As $\mathcal{N}_\beta$ is not initialized at any stage $t+1$ with $s_{g(r)} + 1 \leq t + 1 \leq t_r + 1$, if follows with (2.81) and as followers are assigned in order of magnitude that x^β is weaker than all followers of $\mathcal{N}_\beta$ existing at the end of stage $s_{g(r)} + 1$. In particular, by choice of t_r, x^β is weaker than the follower via which $\mathcal{N}_\beta$ acts at stage $t_r + 1$ whence x^β is canceled or injured at stage $t_r + 1 \leq s_{g(r+1)}$, contradicting the enumeration of z_1 after stage $s_{g(r+1)}$. It follows that $\beta' < \beta$, hence, by (2.80) and (2.76), $t'_1 \leq s_{g(r)}$. As β acts at stage $s_{g(r)} + 1$, it follows that $t'_1 < s_{g(r)}$.

So, in both cases, $t'_1 < s_{g(r)}$ holds. Moreover, $\mathcal{N}_{\beta'}$ becomes active via (iii) or (iv) at stage $t'_1 + 1$. Now, in order to get the desired contradiction, we look at the size of t'_1. If $t'_1 < s_{g(0)}$ then z_1 is associated with $\mathcal{N}_{\beta'}$ at the end of stage $s_{g(0)}$ and enters B_i or C after stage $s_{g(0)}$. Since $g(0) = k$ this contradicts the choice of k.

So w.l.o.g. we may assume that $t'_1 \geq s_{g(0)}$. Fix the unique number $r' < r$ such that $s_{g(r')} \leq t'_1 < s_{g(r'+1)}$ and fix the strategy $\mathcal{N}_{\beta'''}$ which acts at stage $s_{g(r')} + 1$. Now, if $t'_1 = s_{g(r')}$ then $\beta''' = \beta'$, $\mathcal{N}_{\beta'}$ becomes active via (iii) or (iv) at stage $s_{g(r')} + 1$ and z_1 is appointed as a B_i- or C-trace at stage $s_{g(r')} + 1$. So, by definition of g, z_1 cannot be enumerated after stage $t_{r'} + 1$. So, by $r' < r$, $t_1 + 1 \leq t_{r'} + 1 < s_{g(r+1)}$ contrary to choice of t_1.

This leaves the case that $s_{g(r')} < t'_1 < s_{g(r'+1)}$. Since t'_1 is an $\alpha 0$-stage, it follows that $s_{g(r')+1} < s_{g(r'+1)}$ and that $t'_1 \leq s_{g(r'+1)-1}$. By the former, $\mathcal{N}_{\beta'''}$ becomes active via Clause (iii) or (iv) at stage $s_{g(r')} + 1$. Since a B_i- or C-trace is assigned to $\mathcal{N}_{\beta'}$ at stage $t'_1 + 1$, it follows by (2.76) (applied to β''' and r' in place of β and r) that $\beta''' \leq \beta'$. If $\beta' = \beta'''$, using the same argument (considering the strength of the follower $x^{\beta'}$ such that z_1 is associated with $x^{\beta'}$) as in the case above that $\beta = \beta'$ and that $\mathcal{N}_\beta$ acts via Clause (iii) or (iv) at stage $s_{g(r)} + 1$, we get to a contradiction. Hence we may conclude that $\beta''' < \beta'$, so $\mathcal{N}_{\beta'}$ is initialized at stage $t_{r'} + 1 \geq s_{g(r'+1)-1} + 1 \geq t'_1 + 1$ and z_1 is canceled. By the latter, $t_1 + 1 < t_{r'} + 1 \leq s_{g(r+1)}$. But, just as in the preceding case, this contradicts the choice of t_1.

All in all, it follows that $(2.73)_{r+1}$ and $(2.74)_{r+1}$ hold which completes the proof of Claim 4. □

Claim 6. $A_0, A_1, B_0, B_1, C \leq_{\mathrm{T}} D$.

Proof. The marker function $\gamma(x, s)$ is computable and nondecreasing in the second argument. Moreover, if $\gamma(x, s) < \gamma(x, s+1)$ then x is a follower of a critical strategy $\mathcal{N}_\alpha$ and either x becomes admissible at stage $s+1$ or $\mathcal{N}_\alpha$ acts via x at stage $s+1$ and $D_{s+1} \upharpoonright \gamma(x, s) + 1 \neq D_s \upharpoonright \gamma(x, s) + 1$. Since any follower becomes admissible only once and since any strategy acts via the same follower only finitely often, the former implies that $\gamma^*(x) = \lim_{s \to \omega} \gamma(x, s) = \sup_{s \to \omega} \gamma(x, s) < \omega$ exists, while the latter implies that the function $\gamma^*(x)$ is computable in D. Since any number y enters any set under construction at stage $s+1$ only if it is in the entourage of a follower x – hence $x \leq y$ – and $D_{s+1} \upharpoonright \gamma(x,s)+1 \neq D_s \upharpoonright \gamma(x,s)+1$, it follows that $A_0, A_1, B_0, B_1, C \leq D$. □

As Claims 2 to 6 imply that the constructed sets have the required properties, this completes the proof of Theorem 79. □

Chapter 3

Join and Meet Preservation for Bounded Turing Reducibilities

3.1 Introduction

Various notions of reducibilities stronger than Turing reducibility have been studied in computability theory, e.g., the so called classical strong reducibilities: one-one reducibility (1-reducibility), many-one reducibility (m-reducibility), truth-table reducibility (tt-reducibility) and weak truth-table reducibility (wtt-reducibility) (see e.g. Odifreddi [Odi81]). More recently, one has started to look at the so called strongly bounded Turing reducibilities: identity bounded Turing reducibility (ibT-reducibility) and computable Lipschitz reducibility (cl-reducibility) which are defined in terms of Turing functionals where the use is bounded by the identity function and the identity function plus a constant and which were introduced by Soare [Soa04] and Downey, Hirschfeldt and LaForte [DHL01], [DHL04], respectively. cl-reducibility is not only a notion of relative complexity but can also be viewed as a notion of relative randomness and hence is important in the field of algorithmic randomness (see the monograph [DH10] by Downey and Hirschfeldt for more background). The degree structures of the strongly bounded Turing reducibilities on the c.e. sets have been studied intensively. Barmpalias [Bar05] showed that the partial ordering $(\mathbf{R}_{\text{cl}}, \leq)$ of the c.e. cl-degrees has no maximal elements; Barmpalias and Fan and Lu [FL05] showed that there are maximal pairs, hence the partial orderings of the ibT- and cl-degrees are not upper semilattices; and Barmpalias and Lewis [BL06b] and Day [Day10] showed that these partial orderings are not dense. In another article, Barmpalias and Lewis [LB06] prove various decidability results for the global structure of cl-degrees and further results on this reducibility. Ambos-Spies, Bodewig, Kräling and Yu [ASBKY] embedded the nonmodular lattice $\mathcal{N}_5$ into the c.e. ibT- and cl-degrees thereby showing that these partial orderings are not distributive and Ambos-Spies [AS17] proved some global results; e.g., he showed that the first order theories of the partial orderings of the c.e. ibT- and cl-degrees are undecidable. Recently, Ambos-Spies [ASb] introduced a more general class of bounded Turing reducibilities, the uniformly bounded Turing reducibilities. A reducibility r is a (uniformly) bounded Turing reducibility ((u)bT-reducibility) if there is a family $\mathcal{F}$ of (uniformly) computable functions such that, for all sets A and B, A is r-reducible to B if and only if A is

Turing reducible to B with use bounded by some function f in $\mathcal{F}$. We call a (uniformly) bounded Turing reducibility admissible if it is reflexive and transitive and we call it monotone if it is induced by a family of strictly increasing functions. Examples of monotone admissible ubT-reducibilities are the strongly bounded Turing reducibilities ibT and cl as well as the linearly bounded and the primitive recursively bounded Turing reducibilities. An example of a monotone admissible bT-reducibility which is not uniformly bounded is wtt-reducibility. Here, we mainly look at the monotone admissible bT reducibilities.

If a reducibility r is stronger than a reducibility r', of course, every upper r-bound for some sets A_0 and A_1 is also an upper r'-bound for A_0 and A_1 and the same holds for lower bounds. But this does not necessarily imply that least upper r-bounds (joins) have to be least upper r'-bounds, too. Again, the same holds for greatest lower bounds (meets). Here, we ask the question for which reducibilities r and r', joins and meets in the c.e. r-degrees are preserved in the c.e. r'-degrees. We say r-r' join (meet) preservation holds if, for all noncomputable c.e. sets A_0, A_1 and B such that the r-degree of B is the join (meet) of the r-degrees of A_0 and A_1, it holds that the r'-degree of B is the join (meet) of the r'-degrees of A_0 and A_1, too.

For most of the classical strong reducibilities mentioned above, the structure of the c.e. degrees is an upper semilattice where the join of the degrees of two sets A_0 and A_1 is induced by the effective disjoint union $A_0 \oplus A_1$. So, for two such reducibilities where r is stronger than r', of course, r-r' join preservation holds. So, for example, m-tt join preservation, tt-wtt join preservation and wtt-T join preservation hold. For reducibilities r whose degree structures are not upper semilattices with join induced by the effective disjoint union, the question of r-r' join preservation is less obvious. For the classical strong reducibilities, 1-reducibility is an example of such a reducibility, but, as one can easily show (see Lemma 82 below), 1-m join preservation holds. It easily follows that r-r' join preservation holds for all classical strong reducibilities where r is stronger than r'. For the (uniformly) bounded Turing reducibilities, the question of join preservation is less straightforward. Ambos-Spies, Ding, Fan and Merkle [ASDFM13] showed that ibT-cl join preservation holds and Ambos-Spies, Bodewig, Kräling and Yu (see [AS17]) showed that cl-wtt join preservation holds, too. This may lead one to conjecture that – just as in case of the classical strong reducibilities – r-r' join preservation holds for any monotone admissible (u)bT-reducibilities where r is stronger than r', too. As we show here, however, this is not the case. In fact, for r = ibT,cl and for *any* monotone admissible ubT-reducibility r' which is strictly weaker than cl, r-r' join preservation fails (see Theorem 87 below).

We complement the main result of this chapter by considering meet preservation in the monotone admissible bT-reducibilities, too. There we generalize the result in [ASDFM13] that ibT-cl meet preservation holds by showing that indeed, r-r' meet preservation holds for all monotone admissible bT-reducibilities r and r' such that r is stronger than r' (see Lemma 89).

So, for the monotone admissible (uniformly) bounded Turing reducibilities, meet preservation holds in general while, in some instances, join preservation fails. For the classical reducibilities, i.e., the classical strong reducibilities (1-, m-, tt-, and wtt-reducibility) together with Turing reducibility, the converse is true. There join preservation holds in general, whereas, as Downey and Stob [DS86] showed, wtt-T meet preservation fails. We complete the picture by showing that 1-m meet preservation holds while r-r' meet preservation fails for all other pairs of classical reducibilities r and r' such that r is strictly stronger than r'.

The outline of the chapter is as follows. In Section 3.2, we give the definitions and notation needed. In Section 3.3, we show that join preservation holds in general in the classical strong reducibilities together with Turing reducibility. Contrasting this, we see in Section 3.4 that, while join preservation holds in many cases of admissible ubT-reducibilities, there are examples of nonmonotone admissible ubT-reducibilities where join preservation fails. Our main result of this chapter shows that there are also cases of monotone admissible ubT-reducibilities where join preservation fails. On the other hand, meet preservation holds in general for the monotone admissible bounded Turing reducibilities as we show in Section 3.5. In Section 3.6, we give a complete picture of meet preservation in the classical strong reducibilities together with Turing reducibility. Finally, in Section 3.7, we state some open problems.

Large parts of this chapter have been published in two papers by Losert in Lecture Notes in Computer Science [Los15] as well as in Information and Computation [Los17].

3.2 Preliminaries

A reducibility r is *admissible* if it is reflexive and transitive. For two reducibilities r and r', we say that r is *stronger* than r' (denoted by $r \preceq r'$) if, for all sets A and B, from $A \leq_r B$, it follows that $A \leq_{r'} B$ and r is *strictly stronger* than r' ($r \prec r'$) if $r \preceq r'$ and $r \neq r'$.

Definition 80. *For two admissible reducibilities r and r', we say that r-r'* join preservation holds *(in the c.e. degrees) if, for any noncomputable c.e. sets A_0, A_1 and B,*

$$deg_r(A_0) \vee deg_r(A_1) = deg_r(B) \Rightarrow deg_{r'}(A_0) \vee deg_{r'}(A_1) = deg_{r'}(B)$$

holds. Otherwise, we say that r-r' join preservation fails. *Similarly, r-r'* meet preservation holds *(in the c.e. degrees) if, for any noncomputable c.e. sets A_0, A_1 and B,*

$$deg_r(A_0) \wedge deg_r(A_1) = deg_r(B) \Rightarrow deg_{r'}(A_0) \wedge deg_{r'}(A_1) = deg_{r'}(B)$$

holds and r-r' meet preservation fails *otherwise.*

Note that, if r is not stronger than r' on the noncomputable c.e. sets (i.e., if there are noncomputable c.e. sets A and B such that $A \leq_r B$ and $A \not\leq_{r'} B$), then r-r' join preservation fails. This can be seen by considering the degrees of such sets A and B. Namely, $deg_r(A) \vee deg_r(B) = deg_r(B)$, but $deg_{r'}(A) \vee deg_{r'}(B) \neq deg_{r'}(B)$ because B is not an upper r'-bound for A and B. Similarly, in that case, r-r' meet preservation fails. So in the following we discuss join and meet preservation only for reducibilities r and r' such that r is stronger than r'.

From the enumeration $\{\Phi_e^X\}_{e\geq 0}$ of all Turing functionals, we obtain an enumeration $\{\Phi_e^{X,f}\}_{e\geq 0}$ of all f-bounded Turing functionals by bounding the use of each Φ_e^X on input x by $f(x)+1$ (by making the computation divergent in case of greater oracle queries). For any pair of sets A and B, A is f-bounded Turing reducible to B if and only if there is an index e such that $A = \Phi_e^{B,f}$. By letting $f = id$, we obtain an enumeration $\{\hat{\Phi}_e^X\}_{e\geq 0}$ of all identity bounded Turing functionals.

We call a reducibility r a *bounded Turing reducibility* (*bT-reducibility*) if there is a family $\mathcal{F}$ of computable functions such that $A \leq_r B$ if and only if $A \leq_{f-\mathrm{T}} B$ for some function $f \in \mathcal{F}$; in this case we say that r is *induced* by $\mathcal{F}$. If $\mathcal{F}$ is uniformly computable, r is called a *uniformly bounded Turing reducibility* (*ubT-reducibility*). We call a bounded Turing reducibility *monotone* if it is induced by a family $\mathcal{F}$ which only consists of strictly increasing functions. Note that ibT and cl are monotone admissible ubT-reducibilities which are induced by $\mathcal{F}_{\text{ibT}} = \{id\}$ and $\mathcal{F}_{\text{cl}} = \{id + e : e \geq 0\}$, respectively. Another example of a monotone admissible ubT-reducibility we consider here is linearly bounded Turing reducibility. Note that lbT-reducibility is induced by $\mathcal{F}_{\text{lbT}} = \{x \mapsto c_0 x + c_1 : c_0, c_1 \geq 0\}$. The strength of ubT-reducibilities is determined by the growth rates of the corresponding bounds:

Lemma 81 (Ambos-Spies [ASb])**.** *Let r and r' be admissible ubT-reducibilities. Then, $r \preceq r'$ if and only if there are uniformly computable families $\mathcal{F}$ and $\mathcal{F}'$ that induce r and r', respectively, such that $\mathcal{F} \leq^* \mathcal{F}'$, i.e., for every function $f \in \mathcal{F}$, there is a function $f' \in \mathcal{F}'$ such that $f(x) \leq f'(x)$ for almost all $x \geq 0$.*

3.3 Join Preservation in the Classical Strong Reducibilities

It is a straightforward observation that r-r' join preservation holds for reducibilities r and r' such that r is stronger than r' and such that the structures of the c.e. r-degrees and of the c.e. r'-degrees form upper semilattices with join induced by the effective disjoint union. This is the case for all pairs $r, r' \in \{\text{m, tt, wtt, T}\}$ such that r is stronger than r', so in these cases, r-r' join preservation holds. We show that this is still true if we include the strongest classical reducibility, namely 1-reducibility. The c.e. 1-degrees do not form an upper semilattice, but it is possible to generalize the above observation as follows.

Lemma 82. *Let r and r' be admissible reducibilities such that $r \preceq r'$ and, for any c.e. sets A_0 and A_1, the following hold.*

$$A_i \leq_r A_0 \oplus A_1 \ (i \leq 1), \tag{3.1}$$

$$deg_{r'}(A_0) \vee deg_{r'}(A_1) = deg_{r'}(A_0 \oplus A_1). \tag{3.2}$$

Then, r-r' join preservation holds.

Proof. Given c.e. sets A_0, A_1 and B such that

$$deg_r(A_0) \vee deg_r(A_1) = deg_r(B) \tag{3.3}$$

holds, it suffices to show that $A_0 \oplus A_1 =_{r'} B$.

For a proof of $A_0 \oplus A_1 \leq_{r'} B$ it suffices to note that by (3.3) and by $r \preceq r'$, $A_0, A_1 \leq_{r'} B$, hence $A_0 \oplus A_1 \leq_{r'} B$ by (3.2). Finally, for a proof of $B \leq_{r'} A_0 \oplus A_1$, note that $B \leq_r A_0 \oplus A_1$ by (3.1) and (3.3), hence $B \leq_{r'} A_0 \oplus A_1$ by $r \preceq r'$. □

Theorem 83. *Let $r, r' \in \{1, \text{ m, tt, wtt, T}\}$ be given such that $r \preceq r'$. Then, r-r' join preservation holds.*

Proof. This is immediate by the preceding lemma as for all $r \in \{1, \text{ m, tt, wtt, T}\}$, (3.1) holds and for all $r' \in \{\text{m, tt, wtt, T}\}$, (3.2) holds. □

3.4 Join Preservation in the ubT-Reducibilities

For the admissible ubT-reducibilities, we will see that join preservation does not always hold as it is the case for the classical reducibilities. First, we give some positive examples from the literature.

Theorem 84 (Ambos-Spies [ASb])**.** *Let r be an admissible* ubT*-reducibility such that* lbT $\preceq r$*. Then, the following are equivalent.*

(i) r is monotone.

(ii) For any c.e. sets A_0 and A_1, $deg_r(A_0) \vee deg_r(A_1)$ exists.

(iii) For any c.e. sets A_0 and A_1, $deg_r(A_0) \vee deg_r(A_1) = deg_r(A_0 \oplus A_1)$ holds.

This implies that for *monotone* ubT-reducibilities r and r' such that lbT $\preceq r \prec r'$, r-r' join preservation holds. In fact, the following is true.

Corollary 85. *Let r and r' be admissible* ubT*-reducibilities such that* lbT $\preceq r \prec r'$ *and such that r or r' is monotone. Then, r-r' join preservation holds if and only if r' is monotone.*

Proof. If r' is monotone, then r-r' join preservation holds by Lemma 82. Namely, by lbT $\preceq r$, (3.1) holds while, by Theorem 84, (3.2) holds.

On the other hand, if r' is not monotone, then r is monotone. Hence, by Theorem 84, $deg_r(A_0) \vee deg_r(A_1) = deg_r(A_0 \oplus A_1)$ holds for all c.e. sets A_0 and A_1 while, again by Theorem 84, there are c.e. sets A_0 and A_1 such that $deg_{r'}(A_0) \vee deg_{r'}(A_1) = deg_{r'}(A_0 \oplus A_1)$ fails. It follows that r-r' join preservation fails. □

For admissible bounded Turing reducibilities r and r' such that r is strictly stronger than lbT, we cannot apply Lemma 82 to show that r-r' join preservation holds, because here, (3.1) in general fails. Still, some positive results are known.

Lemma 86 (Ambos-Spies, Ding, Fan and Merkle [ASDFM13]; Ambos-Spies [AS17])**.** ibT-cl*,* ibT-wtt *and* cl-wtt *join preservation hold.*

Together with the fact that r-r' join preservation holds for all monotone ubT-reducibilities r and r' such that lbT $\preceq r \prec r'$, this leads to the question if r-r' join preservation holds for all *monotone* admissible (uniformly) bounded Turing reducibilities with $r \preceq r'$. If we consider the *nonmonotone* case, Corollary 85 gives us a counterexample, namely two admissible ubT-reducibilities weaker than lbT where join preservation fails, as follows. For admissible reducibilities r and r' such that lbT $\preceq r \prec r'$ and such that r is monotone and r' is nonmonotone, r-r' join preservation fails by Corollary 85. Such reducibilities exist because lbT-reducibility is monotone and in Ambos-Spies [ASb] it has been shown that, for any admissible ubT-reducibility r, there is a nonmonotone admissible ubT-reducibility r' with $r \preceq r'$. Since nonmonotone ubT-reducibilities are rather artificial, a further question is whether there are also natural examples of (monotone) ubT-reducibilities such that join preservation fails.

In the following, we give such an example. We show that cl-r' join preservation (as well as ibT-r' join preservation) fails for *all* monotone admissible ubT-reducibilities r' with cl $\prec r'$. So, for example, join preservation fails for the naturally defined monotone admissible ubT-reducibilities cl and lbT.

Theorem 87. *Let r' be a monotone admissible ubT-reducibility such that* $\mathrm{cl} \prec r'$*. Then, for* $r = \mathrm{ibT}, \mathrm{cl}$, *$r$-$r'$ join-preservation fails.*

Proof. By Lemma 86, ibT-cl join preservation holds. So, it is enough to prove the theorem for $r = \mathrm{ibT}$. Since, by $\mathrm{cl} \prec r'$, any upper ibT-bound for two sets A_0 and A_1 is also an upper r'-bound for A_0 and A_1, it suffices to construct c.e. sets A_0, A_1, B and C such that $deg_{\mathrm{ibT}}(A_0) \vee deg_{\mathrm{ibT}}(A_1) = deg_{\mathrm{ibT}}(B)$ and such that $A_0, A_1 \leq_{r'} C$ but $B \not\leq_{r'} C$. Let $\mathcal{F}$ be a uniformly computable family of strictly increasing functions such that r' is induced by $\mathcal{F}$. As $\mathcal{F}$ is uniformly computable, we can fix a computable function f such that $f \geq^* h$ for all $h \in \mathcal{F}$. As $\mathrm{cl} \prec r'$, hence $r' \not\preceq \mathrm{cl}$, $\mathcal{F} \not\leq^* \{id + e : e \geq 0\}$ holds, so, there is a function $g \in \mathcal{F}$ such that $\{g\} \not\leq^* \{id + e : e \geq 0\}$, i.e., for any $e \geq 0$, $g(x) > x + e$ for infinitely many x. Since g is strictly increasing, this implies that for all $e \geq 0$, $g(x) > x + e$ for all but finitely many x, so, $id + e \leq^* g$ for all $e \geq 0$. So, in order to complete the proof, it suffices to show that the following lemma holds.

Lemma 88. *Let g be a strictly increasing computable function such that $id + e \leq^* g$ for all e and let f be any computable function (in particular, f can be chosen as above). Then, there are c.e. sets A_0, A_1, B and C such that the following hold.*

$$deg_{\mathrm{ibT}}(A_0) \vee deg_{\mathrm{ibT}}(A_1) = deg_{\mathrm{ibT}}(B), \tag{3.4}$$

$$A_0, A_1 \leq_{g-\mathrm{T}} C, \tag{3.5}$$

$$B \not\leq_{f-\mathrm{T}} C. \tag{3.6}$$

Proof. We enumerate c.e. sets A_0, A_1, B and C such that (3.4) to (3.6) hold using a tree argument. We let $A_{0,s}$, $A_{1,s}$, B_s and C_s be the finite parts of A_0, A_1, B and C enumerated by the end of stage s, respectively. Moreover, for any of these sets X, we write $x \searrow_{s+1} X$ if x enters X at stage s, i.e., if $x \in X_{s+1} \setminus X_s$.

In order to guarantee (3.4), we use the join technique introduced in [ASBKY] where it is shown that the nondistributive lattice $\mathcal{N}_5$ can be embedded into the partial orderings $(\mathbf{R}_{\mathrm{ibT}}, \leq)$ and $(\mathbf{R}_{\mathrm{cl}}, \leq)$. The proof we give is self-contained hence duplicates some of the arguments given there and some parts of our proof follow the corresponding parts in [ASBKY] quite closely.

To guarantee that (3.5) holds and that B is an upper ibT-bound for A_0 and A_1, we meet the following global *permitting* (or *coding*) *requirement* for $i = 0, 1$.

$$(x \searrow_{s+1} A_i \Rightarrow \exists y \leq x (y \searrow_{s+1} B)) \;\&\; (x \searrow_{s+1} A_i \Rightarrow \exists y \leq g(x)(y \searrow_{s+1} C)) \tag{3.7}$$

To guarantee that B is in fact the least upper ibT-bound for A_0 and A_1, i.e., that (3.4) holds, we meet the following *join requirements* for $e \geq 0$ (where, here and in the following, $e = \langle e_0, e_1, e_2 \rangle$).

$$Q_e : A_0 = \hat{\Phi}_{e_1}^{W_{e_0}} \;\&\; A_1 = \hat{\Phi}_{e_2}^{W_{e_0}} \Rightarrow B \leq_{\mathrm{ibT}} W_{e_0}.$$

Finally, we satisfy condition (3.6) by meeting the *nonordering requirements*

$$P_e : B \neq \Phi_e^{C,f}$$

for $e \geq 0$.

Before giving the actual construction, we explain the ideas underlying the strategies for meeting the individual requirements and how to combine them.

As the join requirements Q_e are conditional requirements whose hypotheses are not decidable, we have to guess on the correctness of the hypotheses. We define the length of agreement between A_0 and $\hat{\Phi}_{e_1}^{W_{e_0}}$ and between A_1 and $\hat{\Phi}_{e_2}^{W_{e_0}}$ at stage s by letting

$$l(e,s) = \max\{x : \forall y < x(A_{0,s}(y) = \hat{\Phi}_{e_1,s}^{W_{e_0,s}}(y) \ \&\ A_{1,s}(y) = \hat{\Phi}_{e_2,s}^{W_{e_0,s}}(y))\}.$$

Note that $l(e,s)$ is computable. Moreover, the functionals $\hat{\Phi}_{e_i}$ are bounded, so for all $e \geq 0$, $\lim_{s\to\infty} l(e,s) \leq \infty$ exists and the following holds.

$$(A_0 = \hat{\Phi}_{e_1}^{W_{e_0}} \ \&\ A_1 = \hat{\Phi}_{e_2}^{W_{e_0}}) \Leftrightarrow \lim_{s\to\infty} l(e,s) = \infty \Leftrightarrow \limsup_{s\to\infty} l(e,s) = \infty. \tag{3.8}$$

We call a join requirement Q_e *infinitary* if its hypothesis is true (i.e., if $\lim_{s\to\infty} l(e,s) = \infty$) and we call Q_e *finitary* otherwise. The strategy for meeting the join requirements is the join technique introduced by Ambos-Spies, Bodewig, Kräling and Yu in [ASBKY]. Assuming Q_e is infinitary, we aim to make the conclusion $(B \leq_{\mathrm{ibT}} W_{e_0})$ true by a permitting strategy (up to some computable subset of B). We define a computable set $S = \{s_n : n \geq 0\}$ of Q_e*-expansionary stages*, i.e., stages $s_0 < s_1 < s_2 < \ldots$ such that $l(e,s_0) < l(e,s_1) < l(e,s_2) < \ldots$. Between stages $s_n + 1$ and $s_{n+1} + 1$, only numbers that are greater than $s_n + 1$ are allowed to enter B. The subset of B consisting of the numbers that enter B at a stage that is not expansionary will hence be computable. Furthermore, the subset of B consisting of the numbers that enter B at an expansionary stage but are greater or equal to the length of agreement at that stage is computable, too. So, only numbers x that enter B at a stage $s+1$ where $s \in S$ and $x < l(e,s)$ need permitting by W_{e_0}, i.e., some number $\leq x$ has to be enumerated into W_{e_0} after stage s. We cannot control W_{e_0} directly, but we put a sufficiently small number into A_0 or A_1. Then, W_{e_0} is forced to also enumerate a small number, otherwise the hypothesis of Q_e would become false. As one can easily check, this is achieved by guaranteeing the following for all numbers x.

$$\begin{gathered} x \searrow_{s+1} B \ \&\ x < l(e,s) \Rightarrow \exists y < \min\{x', l(e,s)\}(y \searrow_{s+1} A_0 \text{ or } y \searrow_{s+1} A_1) \\ \text{where } x' = \mu z(z > x \ \&\ z \notin W_{e_0,s}). \end{gathered} \tag{3.9}$$

In case that the hypothesis of Q_e is true, a Turing functional Γ that computes $B(x)$ given $W_{e_0} \upharpoonright x+1$ works as follows. On input x, find a Q_e-expansionary stage s with $l(e,s) > x$ such that $W_{e_0,s} \upharpoonright x+1 = W_{e_0} \upharpoonright x+1$. Then, let $\Gamma^{W_{e_0}}(x) = B_s(x)$. By the assumption that the hypothesis of Q_e is true and by (3.9), $B(x)$ cannot change after stage s, i.e., $\Gamma^{W_{e_0}}(x) = B_s(x) = B(x)$.

For meeting the nonordering requirements P_e, we use the standard Friedberg-Muchnik strategy. For some fixed new number x (i.e., a number that has not yet been enumerated into any of the sets we construct), we wait for a stage s such that $\Phi_{e,s}^{C_s,f}(x) = 0$. If such a stage does not exist, we never enumerate x into B, so, by the use principle, P_e is met. Otherwise, at stage $s+1$, we put x into B and, in order to preserve the computation $\Phi_{e,s}^{C_s,f}(x)$, we impose a restraint of length $f(x)+1$ on C, thereby ensuring

$$B(x) = 1 \neq 0 = B_s(x) = \Phi_{e,s}^{C_s,f}(x) = \Phi_e^{C,f}(x). \tag{3.10}$$

In the presence of the join requirements and the global permitting requirement, this strategy needs some amendments. To describe the potential conflicts, consider the situation in which we wish to meet requirement P_e and simultaneously meet the global permitting requirement (3.7) and follow the join technique (3.9) for a single infinitary join requirement $Q_{e'}$ ($e' = \langle e'_0, e'_1, e'_2 \rangle$) of higher priority.

When we put a number x into B at stage $s+1$ in order to guarantee (3.10), then, according to (3.9), we have to put a number $y < x'$ into A_0 or A_1 at stage $s+1$ where

$$x' = \mu z (z > x \ \& \ z \notin W_{e'_0,s}).$$

(In the actual construction, we choose to put y into A_1.) If we do so, then, as long as $x \leq y$, this is consistent with the first part of condition (3.7). But, for the second part of this condition, we have to put a number $z \leq g(y)$ into C. In case that $z \leq f(x)$, however, this injures the restraint imposed on C in order to preserve the computation $\Phi_{e,s}^{C_s,f}(x)$. In order to overcome this problem, we make sure that we can find a number y such that $f(x) < y < x'$ where y is not yet in A_1 and the interval $[y, g(y)]$ is not yet completely enumerated into C. (Then putting y into A_1 and some new number z with $y \leq z \leq g(y)$ into C makes the enumeration of x into B compatible with (3.7) and (3.9).)

For that matter, we assign a sufficiently long interval I_n of unused numbers to P_e. I_n contains finitely many candidates $x_{n,k}$ for a possible attack on P_e where these numbers are chosen so that $x_{n,k+1} > f(x_{n,k})$ and $g(x_{n,k}) \geq x_{n,k}+k+2$ for all k. (Note that the latter can be achieved since, by choice of g, $g(y) > y+k+2$ for all sufficiently large y; also note that $g(x_{n,k}) \geq x_{n,k}+k+2$ implies $g(y) \geq y+k+2$ for all $y \geq x_{n,k}$.) We arrange that, for some k (and some stage s), $(x_{n,k}, x_{n,k+1}] \subseteq W_{e'_0,s}$ where $x_{n,k}$ is not in B_s, $x_{n,k+1}$ is not in $A_{1,s}$ and the interval $[x_{n,k+1}, g(x_{n,k+1})]$ is not completely contained in C_s. (Hence, for $x = x_{n,k}$ and $y = x_{n,k+1}$, $y < x'$ whence we can ensure (3.10) and simultaneously obey (3.7) and (3.9) by putting $x_{n,k}$ into B, $x_{n,k+1}$ into A_1 and some unused number from the interval $[x_{n,k+1}, g(x_{n,k+1})]$ into C at stage $s+1$.)

In order to ensure $(x_{n,k}, x_{n,k+1}] \subseteq W_{e'_0}$ for some k, we will successively and in decreasing order put numbers w from I_n into A_0 at stages $s+1$ where $l(e', s)$ is greater than the endpoint of I_n. This forces $W_{e'_0}$ to respond by enumerating more and more numbers from I_n (or smaller ones). As we will argue, this implies that, at some point s, there is a subinterval $(x_{n,k}, \dots x_{n,k+1}] \subset I_n$ such that the enumeration of the numbers $\geq x_{n,k}+1$ from I_n into A_0 has forced all the numbers $x_{n,k}+1, \dots, x_{n,k+1}$ into $W_{e'_0}$. (In the actual construction, all the numbers have to be forced simultaneously into all sets $W_{e'_0}$ attached to the infinitary higher priority join requirements, but we will show that this can be achieved by the above strategy.) So we can use $x_{n,k}$ for an attack on P_e – provided that $x_{n,k} \notin B_s$, $x_{n,k+1} \notin A_{1,s}$ and $[x_{n,k+1}, g(x_{n,k+1})] \not\subseteq C_s$.

The latter, however, is not trivially true, since to make the enumeration of w into A_0 compatible with (3.7), simultaneously, we have to put a trace $w_B \leq w$ into B and a trace $w_C \leq g(w)$ into C. So whenever we put w into A_0, then, we simultaneously put w into B (which is compatible with (3.9) since w also goes into A_0) and a number from the interval $[w, g(w)]$ into C. Since we put only numbers $w > x_{n,k}$ into A_0 this procedure also puts only numbers $> x_{n,k}$ into B and no numbers into A_1 hence guarantees $x_{n,k} \notin B_s$ and $x_{n,k+1} \notin A_{1,s}$. To ensure that $[x_{n,k+1}, g(x_{n,k+1})] \not\subseteq C_s$, however, we have to carefully choose the trace $w_C \in [w, g(w)]$ to be put into C. Here we let $w_C = w + k' + 1$ for the unique k' such that $w \in (x_{n,k'}, x_{n,k'+1}]$. Note that, by choice of the numbers $x_{n,k'}$, this ensures that $w_C \leq g(w)$. On the other hand, this ensures that $x_{n,k+1}+k+2$ is not enumerated into C since, if $w \in I_n$, for $w \leq x_{n,k+1}$, then $w_C \leq w+k+1 < x_{n,k+1}+k+2$ while, for $x_{n,k+1} < w$, then $w_C \geq w+(k+1)+1 > x_{n,k+1}+k+2$ and if $w \in I_{n'}$ for $n' \neq n$, then the definition of the intervals will ensure that either $w_C < x_{n,0} < x_{n,k+1}+k+2$ or $w_C > x_{n,k+1}+k+2$.

This completes the discussion of the basic conflicts among the different goals of the construction and how these conflicts can be resolved.

Before giving the construction of the sets, we need the following notions and notation. We implement the guesses about which of the join requirements are infinitary on the full binary tree $T = \{0,1\}^{<\omega}$. A node α of length n codes a guess about the hypotheses of the first n join requirements $Q_0, \dots, Q_{n-1}$ where, for $e < n$, $\alpha(e) = 0$ codes the guess that Q_e is infinitary and $\alpha(e) = 1$ codes the guess that Q_e is finitary. The *true path* $f : \omega \to \{0,1\}$ of the construction is defined by

$$f(e) = \begin{cases} 0 & \text{if } A_0 = \hat{\Phi}_{e_1}^{W_{e_0}} \ \& \ A_1 = \hat{\Phi}_{e_2}^{W_{e_0}} \\ 1 & \text{otherwise.} \end{cases}$$

To each node α of length e, we assign a strategy $\mathcal{P}_\alpha$ for meeting requirement P_e which is based on the guess α. We show that the strategy $\mathcal{P}_{f\restriction e}$ on the true path succeeds in meeting P_e.

At any stage s of the construction we have an approximation δ_s of $f \restriction s$, i.e., a guess on which of the first s join requirements are infinitary. For the definition of δ_s, we inductively define *α-stages* for each node α as follows. Each stage $s \geq 0$ is a λ-stage. If s is an α-stage, then we call s *α-expansionary* if $l(|\alpha|, s) > l(|\alpha|, t)$ for all α-stages $t < s$ and we call s an $\alpha 0$-stage if s is α-expansionary and an $\alpha 1$-stage if s is an α-stage but not α-expansionary. Now, for each $s \geq 0$, let $\delta_s \in T$ be the unique α of length s such that s is an α-stage. It easily follows from (3.8) that the true path is the leftmost path visited infinitely often in the construction.

The intervals I_n which might be assigned to the strategies for meeting the nonordering requirements are inductively defined as follows, where the nth interval I_n consists of $n(x_{n,0}+1)$ many subintervals $I_{n,k} = (x_{n,k}, x_{n,k+1}]$.

$$\begin{aligned}
x_{0,0} &= \mu x(g(x) \geq x+2), \\
x_{n,k} &= \mu x(x > x_{n,k-1}, f(x_{n,k-1}) \ \& \ g(x) \geq x+k+2) \\
&\qquad \text{for } n \geq 0 \text{ and } 1 \leq k \leq n(x_{n,0}+1), \\
x_{n+1,0} &= \mu x(x > x_{n,n(x_{n,0}+1)} + n(x_{n,0}+1) + 2 \ \& \ g(x) \geq x+2) \text{ for } n \geq 0, \\
I_{n,k} &= (x_{n,k}, x_{n,k+1}] \ \text{ for } n \geq 0 \text{ and } 0 \leq k \leq n(x_{n,0}+1)-1, \\
I_n &= \bigcup_{k=0}^{n(x_{n,0}+1)-1} I_{n,k}.
\end{aligned}$$

Note that this definition ensures that $x_{n,k+1} > f(x_{n,k})$, $g(w) \geq w+k+2$ and $w+k+2 < x_{n+1,0}$ for $w \in I_{n,k}$.

For a node α of length e, we call a number $x \in I_n \cup \{x_{n,0}\}$ *α-safe* at stage s if for $e' = \langle e'_0, e'_1, e'_2 \rangle$, the following hold.

$$x = x_{n,k} \text{ for some } k \text{ with } 0 \leq k \leq n(x_{n,0}+1)-1, \tag{3.11}$$

$$x \notin B_s, x_{n,k+1} \notin A_{1,s} \text{ and } x_{n,k+1}+k+2 \notin C_s, \tag{3.12}$$

$$\forall e'([e' < e \ \& \ \alpha(e') = 0] \Rightarrow I_{n,k} \subseteq W_{e'_0,s}). \tag{3.13}$$

Now we are ready to give the actual construction of the sets A_0, A_1, B and C.

Construction.

Stage 0 is vacuous (i.e., $A_{0,0} = A_{1,0} = B_0 = C_0 = \emptyset$).

Stage $s+1$. A strategy $\mathcal{P}_\alpha$ with $|\alpha| = e$ *requires attention* at stage $s+1$ if $\alpha \sqsubseteq \delta_s$, $\mathcal{P}_\alpha$ is not satisfied at the end of stage s and one of the following holds.

(i) No interval is assigned to $\mathcal{P}_\alpha$ at the end of stage s.

(ii) Interval $I_n = \big(x_{n,0}, x_{n,n(x_{n,0}+1)}\big]$ is assigned to $\mathcal{P}_\alpha$ at the end of stage s,

$$\forall e'([e' < e \;\&\; \alpha(e') = 0] \Rightarrow l(e', s) > x_{n,n(x_{n,0}+1)}) \tag{3.14}$$

holds, no number $x \in I_n \cup \{x_{n,0}\}$ is α-safe at stage s and $I_n \not\subseteq A_{0,s}$.

(iii) Interval I_n is assigned to $\mathcal{P}_\alpha$ at the end of stage s, (3.14) holds and there is a number $x \in I_n \cup \{x_{n,0}\}$ such that x is α-safe at stage s and $B_s(x) = \Phi_{e,s}^{C_s, f}(x) = 0$.

Fix the least α such that $\mathcal{P}_\alpha$ requires attention at stage $s+1$ (as $\mathcal{P}_{\delta_s}$ requires attention at stage $s+1$, there is such an α). Declare that $\mathcal{P}_\alpha$ *receives attention* or *becomes active* at stage $s+1$, initialize all strategies $\mathcal{P}_\beta$ with $\alpha < \beta$ (i.e., if an interval is assigned to $\mathcal{P}_\beta$ at the end of stage s, then cancel this assignment and if $\mathcal{P}_\beta$ is satisfied at the end of stage s, then declare $\mathcal{P}_\beta$ to be unsatisfied) and perform the following action according to the clause above via which $\mathcal{P}_\alpha$ requires attention.

(i) Assign I_{s+1} to $\mathcal{P}_\alpha$. (Note that $e < s+1$.)

(ii) Let y be the greatest number in $I_n \setminus A_{0,s}$. Put y into A_0 and B and, for the unique k such that $y \in I_{n,k}$, put $y + k + 1$ into C.

(iii) Let x be the greatest α-safe number in $I_n \cup \{x_{n,0}\}$ such that $B_s(x) = \Phi_{e,s}^{C_s, f}(x) = 0$. Let k be the unique number such that $x = x_{n,k}$. Put x into B, $x_{n,k+1}$ into A_1 and $x_{n,k+1} + k + 2$ into C. Then, declare $\mathcal{P}_\alpha$ to be *satisfied*.

This completes the construction.

Verification.

Note that all constructed sets are c.e. We prove a series of claims to show that the construction meets all of our requirements.

Claim 1 (True Path Lemma). $f = \liminf_{s\to\infty} \delta_s$, i.e., for any α, $\alpha \sqsubset f$ if and only if $\alpha \sqsubset \delta_s$ for infinitely many s and there are only finitely many s such that $\delta_s <_L \alpha$.

Proof. This is immediate by (3.8) and by definition of δ_s and f. □

Claim 2. Every strategy $\mathcal{P}_\alpha$ on the true path (i.e., $\alpha \sqsubset f$) is initialized only finitely often and requires attention only finitely often. Moreover, for any such strategy, there is an interval I_n which is permanently assigned to it, i.e., there is a stage s such that I_n is assigned to $\mathcal{P}_\alpha$ and this assignment is never canceled after stage s.

Proof. The proof is by induction on $|\alpha|$. Given α and $e = |\alpha|$, by Claim 1 and by inductive hypothesis fix s_0 minimal such that $\alpha \leq \delta_s$ for all $s \geq s_0$ and such that no strategy $\mathcal{P}_\beta$ with $\beta \sqsubset \alpha$ requires attention after stage s_0. Then, $\mathcal{P}_\alpha$ is not initialized after stage s_0. So $\mathcal{P}_\alpha$ receives attention whenever it requires attention after stage s_0. Moreover, by minimality of s_0, an interval I_n is permanently assigned to $\mathcal{P}_\alpha$ at stage $s_1 + 1$ where s_1 is the least α-stage $\geq s_0$ (note that s_1 exists because $\mathcal{P}_\alpha$ is on the true path). Then, after stage $s_1 + 1$, P_α receives attention at most $|I_n|$ many times via Clause (ii) and at most once via Clause (iii), hence P_α requires attention only finitely often. ☐

Claim 3. The global permitting requirement (3.7) is met.

Proof. It is crucial to note that numbers x with $x_{n,0} \leq x < x_{n+1,0}$ can be enumerated into any of the sets under construction at stage $s+1$ only by the strategy $\mathcal{P}_\alpha$ to which I_n is assigned at this stage. So, it follows by a straightforward induction that if a strategy $\mathcal{P}_\alpha$ acts via (ii) at stage $s+1$ then, for the number y being enumerated there, neither y is in B_s nor $y + k + 1$ is in C_s. And, similarly, if a strategy $\mathcal{P}_\alpha$ acts via (iii) and some $x_{n,k}$ at stage $s+1$ then neither $x_{n,k}$ is in B_s nor $x_{n,k+1} + k + 2$ is in C_s by α-safeness of $x_{n,k}$. This easily implies the claim, since a number x is enumerated into A_0 at some stage $s+1$ only if some strategy $\mathcal{P}_\alpha$ acts at stage $s+1$ via (ii), hence $x \in I_{n,k}$ for some k and, at stage $s+1$, x is enumerated into B and $x + k + 1$ is enumerated into C where $x + k + 1 \leq g(x)$ by choice of $I_{n,k}$; and since a number x is enumerated into A_1 at some stage $s+1$ only if some strategy $\mathcal{P}_\alpha$ acts at stage $s+1$ via (iii), hence $x = x_{n,k+1}$ for some n, k and, at stage $s+1$, $x_{n,k} < x_{n,k+1}$ is enumerated into B and $x_{n,k+1} + k + 2$ is enumerated into C where by choice of $x_{n,k+1}$, $x_{n,k+1} + k + 2 \leq g(x)$. ☐

Claim 4. The join requirements Q_e are met.

Proof. Fix e such that Q_e is infinitary, so, $\alpha 0 \sqsubset f$ for $\alpha = f \upharpoonright e$. (Otherwise, the requirement is trivially met.) Hence there are infinitely many $\alpha 0$-stages. By Claims 1 and 2, we can fix an $\alpha 0$-stage $s_0 > e$ such that no strategy $\mathcal{P}_\beta$ with $\beta \leq \alpha 0$ becomes active after this stage. Let $S = \{s_n : n \geq 0\}$ be the set of the $\alpha 0$-stages $\geq s_0$. Then, S is computable, $s_0 < s_1 < s_2 < \ldots$ and $l(e, s_0) < l(e, s_1) < l(e, s_2) < \ldots$. So, as explained in the discussion of the strategy for meeting the requirements Q_e, it suffices to show that (3.9) holds for $s \in S$. But this is immediate by construction since at a stage $s_m + 1$ only a strategy $\mathcal{P}_\beta$ with $\alpha 0 \sqsubseteq \beta$ may act. Namely, if $\mathcal{P}_\beta$ acts via (ii) then the number x enumerated into B is simultaneously enumerated into A_0 and if $\mathcal{P}_\beta$ acts via (iii) then the claim follows from the corresponding action by β-safeness of the number x put into B. ☐

Claim 5. The nonordering requirements P_e are met.

Proof. The proof closely follows the proof of the corresponding claim in [ASBKY]. For fixed e, assume for a contradiction that P_e is not met. Let $\alpha = f \upharpoonright e$, let I_n be the interval permanently assigned to $\mathcal{P}_\alpha$ and let $s_1 + 1$ ($= n$) be the stage at which I_n is assigned to $\mathcal{P}_\alpha$. Then the assumption that P_e is not met easily implies that $\mathcal{P}_\alpha$ is not satisfied after stage $s_1 + 1$ and that no number in I_n is α-safe after stage $s_1 + 1$. So all numbers in I_n are enumerated into A_0 in decreasing order after stage $s_1 + 1$ according to Clause (ii) in the definition of requiring and receiving attention. For $x \in I_n$, fix the α-stage $t_x > s_1$ such that x is enumerated into A_0 at stage $t_x + 1$ and let $t_{x_{n,0}}$ be the least α-stage greater than $t_{x_{n,0}+1}$. (Note that, for $x \in I_n$, $t_x < t_{x-1}$ since numbers from I_n are enumerated into A_0 in decreasing order.) Since, for any $x \in I_n$, (3.14) holds for $s = t_x$, it follows that

$$W_{e'_0,t_x} \upharpoonright x+1 \neq W_{e'_0,t_{x-1}} \upharpoonright x+1$$

for any infinitary higher priority join requirement $Q_{e'}$. So, for

$$J = \{e'_0 : \exists e'_1, e'_2(\langle e'_0, e'_1, e'_2\rangle < e \;\&\; Q_{\langle e'_0,e'_1,e'_2\rangle} \text{ is infinitary})\},$$

the following holds.

$$\forall j \in J \; \forall x \in I_n(W_{j,t_x} \upharpoonright x+1 \subset W_{j,t_{x-1}} \upharpoonright x+1). \tag{3.15}$$

For $x \in I_n$ and $j \in J$, let

$$w_j(x) = |W_{j,t_x} \upharpoonright x+1| \text{ and } w_J(x) = \sum_{j \in J} w_j(x),$$

and call x *unsaturated* if $x \notin W_{j,t_x}$ for some $j \in J$. By definition, $|J| \leq e$ and $w_j(x) \leq x+1$, hence

$$w_J(x_{n,0}) \leq e(x_{n,0}+1). \tag{3.16}$$

On the other hand,

$$w_J(x_{n,0}) \geq |\{x \in I_n : x \text{ is unsaturated}\}|. \tag{3.17}$$

holds. Namely, it follows by (3.15) that, for $x \in I_n$, $w_j(x-1) \geq w_j(x)$ if $x \in W_{j,t_x}$ and $w_j(x-1) > w_j(x)$ if $x \notin W_{j,t_x}$, whence $w_J(x-1) \geq w_J(x)$ and $w_J(x-1) > w_J(x)$ if x is unsaturated.

Now, by (3.16) and (3.17), in order to get the desired contradiction, it suffices to show that

$$|\{x \in I_n : \; x \text{ is unsaturated}\}| > e(x_{n,0}+1). \tag{3.18}$$

This is done as follows. For a number $x = x_{n,k} \in I_n \cup \{x_{n,0}\}$ $(0 \leq k \leq n(x_{n,0}+1)-1)$, as $x \notin A_{0,t_x}$, as $\mathcal{P}_\alpha$ is not satisfied at stage t_x and by choice of the numbers w_C enumerated into C together with numbers w entering A_0 (see also the discussion preceding the construction), (3.12) holds for $s = t_x$. So, since there are no α-safe numbers in $I_n \cup \{x_{n,0}\}$ after stage s_1+1, (3.13) must fail for $s = t_x$. It follows that, for every k, at least one number in $I_{n,k}$ must be unsaturated. As there are $n(x_{n,0}+1)$ many subintervals $I_{n,k}$ in I_n each of which must contain at least one unsaturated number and as $e < n$ by construction, it follows that there are at least $(e+1)(x_{n,0}+1)$ many unsaturated numbers in I_n. So (3.18) holds. □

Claims 3 to 5 show that the constructed sets have the required properties, hence this completes the proof of Lemma 88. □

As we have seen, this also completes the proof of Theorem 87. □

3.5 Meet Preservation in the ubT-Reducibilities

In contrast to Theorem 87, meet preservation holds for the monotone admissible bounded Turing reducibilities in general. This is immediate by the following theorem which generalizes the observation by Ambos-Spies, Ding, Fan and Merkle in [ASDFM13] that ibT-cl and cl-wtt meet preservation hold.

Theorem 89. *Let r and r' be admissible bounded Turing reducibilities such that r is stronger than r' and such that r' is monotone. Then, r-r' meet preservation holds.*

Proof. The proof is essentially the same as the one for the results in [ASDFM13]. Let $\mathcal{F}'$ be a family of strictly increasing computable functions inducing r'. As shown by Ambos-Spies in [ASb], w.l.o.g., we may assume that $\mathcal{F}'$ is closed under composition. Let A_0, A_1 and B be c.e. sets such that

$$deg_r(A_0) \wedge deg_r(A_1) = deg_r(B) \tag{3.19}$$

holds. As r is stronger than r', B is also a lower r'-bound for A_0 and A_1, so, it suffices to show that for a given c.e. set C such that $C \leq_{r'} A_0, A_1$, $C \leq_{r'} B$ holds. Fix functions $f_i \in \mathcal{F}'$ such that $C \leq_{f_i-\mathrm{T}} A_i$ for $i = 0, 1$. Since $\mathcal{F}'$ is closed under composition, $f_0 \circ f_1 = f \in \mathcal{F}'$ and, since f_0 and f_1 are strictly increasing, $\max\{f_0, f_1\} \leq f$. It follows that $C \leq_{f-\mathrm{T}} A_0, A_1$. Let $C_f = \{f(x) : x \in C\}$ be the f-shift of C. Then, $C_f \leq_{\mathrm{ibT}} A_0, A_1$. As ibT is stronger than r, $C_f \leq_r A_0, A_1$, so, by (3.19), $C_f \leq_r B$, hence $C_f \leq_{r'} B$. We know that $C \leq_{f-\mathrm{T}} C_f$, hence by $f \in \mathcal{F}'$, $C \leq_{r'} C_f$, so, by transitivity of r', $C \leq_{r'} B$. □

3.6 Meet Preservation in the Classical Strong Reducibilities

While we have seen in Section 3.3 that r-r' join preservation holds for all classical strong reducibilities such that r is stronger than r', meet preservation does not always hold in the classical strong reducibilities. Downey and Stob [DS86] use an embedding result into the c.e. T-degrees together with the distributivity of the c.e. wtt-degrees to prove the following.

Lemma 90 ([DS86])**.** *There is a* wtt*-minimal pair that is not a* T*-minimal pair.*

Proof. The proof uses the fact that sufficiently complex nondistributive lattices can be embedded into the c.e. T-degrees while the c.e. wtt-degrees are distributive. Namely, the nondistributive finite 1-4-1 lattice can be embedded into the c.e. T-degrees preserving the least element, i.e., there are noncomputable c.e. sets A_0, A_1, A_2, A_3 and A such that for all $i, j \in \{0, 1, 2, 3\}$ with $i \neq j$, the following hold.

$$\begin{aligned} &A_i \text{ and } A_j \text{ are a T-minimal pair,} \\ &deg_{\mathrm{T}}(A_i) \vee deg_{\mathrm{T}}(A_j) = deg_{\mathrm{T}}(A). \end{aligned} \tag{3.20}$$

Now consider the sets $A_0 \oplus A_1$ and $A_2 \oplus A_3$. By distributivity of the c.e. wtt-degrees (shown by Lachlan; see e.g. Stob [Sto83]), $A_0 \oplus A_1$ and $A_2 \oplus A_3$ form a wtt-minimal pair. Namely, for given c.e. $B \leq_{\mathrm{wtt}} A_0 \oplus A_1, A_2 \oplus A_3$, it suffices to show that B is computable. By distributivity, as $B \leq_{\mathrm{wtt}} A_0 \oplus A_1$, there are c.e. sets $B_0 \leq_{\mathrm{wtt}} A_0$ and $B_1 \leq_{\mathrm{wtt}} A_1$ such that $B_0 \oplus B_1 = B$. Moreover, again by distributivity, as $B_0, B_1 \leq_{\mathrm{wtt}} B \leq_{\mathrm{wtt}} A_2 \oplus A_3$, there are c.e. sets $B_{02} \leq_{\mathrm{wtt}} A_2$ and $B_{03} \leq_{\mathrm{wtt}} A_3$ such that $B_{02} \oplus B_{03} = B_0$ and c.e. sets $B_{12} \leq_{\mathrm{wtt}} A_2$ and $B_{13} \leq_{\mathrm{wtt}} A_3$ such that $B_{12} \oplus B_{13} = B_1$. Altogether, it follows that $B_{ij} \leq_{\mathrm{wtt}} A_i, A_j$ for $i \in \{0, 1\}$ and $j \in \{2, 3\}$. But, as A_i and A_j form a T-minimal pair and hence a wtt-minimal pair, this implies that all B_{ij} are computable. So $B = (B_{02} \oplus B_{03}) \oplus (B_{12} \oplus B_{13})$ is computable, too.

On the other hand, by (3.20) and by noncomputability of A, $A_0 \oplus A_1$ and $A_2 \oplus A_3$ do not form a T-minimal pair. □

Using an embedding result into the c.e. tt-degrees by Fejer and Shore, this idea can be transferred to show that there is an m-minimal pair that is not a tt-minimal pair. Furthermore, we can use a similar method to show that there is a tt-minimal pair that is not a wtt-minimal pair.

Lemma 91. *There is an* m*-minimal pair that is not a* tt*-minimal pair.*

Proof. Fejer and Shore have shown in [FS85] that every finite lattice can be embedded into the c.e. tt-degrees preserving the least element. So, in particular, the 1-4-1 lattice can be embedded into the c.e. tt-degrees preserving zero. Since the c.e. m-degrees are distributive (Lachlan [Lac70]), we can apply the argument used in the proof of Lemma 90. □

Lemma 92. *There is a* tt*-minimal pair that is not a* wtt*-minimal pair.*

Proof. Suppose that every tt-minimal pair is also a wtt-minimal pair. As shown by Jockusch and Mohrherr in [JM85], the diamond lattice can be embedded into the c.e. tt-degrees preserving both least and greatest elements, i.e., there are noncomputable c.e. sets A_0 and A_1 such that

$$A_0 \text{ and } A_1 \text{ form a tt-minimal pair,}$$
$$deg_{\mathrm{tt}}(A_0) \vee deg_{\mathrm{tt}}(A_1) = deg_{\mathrm{tt}}(\emptyset').$$

Since, by Theorem 83, tt-wtt join preservation holds and by assumption, it follows that

$$A_0 \text{ and } A_1 \text{ form a wtt-minimal pair,}$$
$$deg_{\mathrm{wtt}}(A_0) \vee deg_{\mathrm{wtt}}(A_1) = deg_{\mathrm{wtt}}(\emptyset'),$$

which is a contradiction to the nondiamond theorem for the wtt-degrees stated by Ladner and Sasso in [LS75]. □

We give an alternative and direct proof of Lemma 92, showing that there is a pair of noncomputable c.e. sets that form a tt-minimal pair but which are wtt-comparable, hence they have a noncomputable wtt-meet.

Theorem 93. *There are c.e. sets A_0 and A_1 such that the following hold.*

$$A_0 \text{ is not computable,} \tag{3.21}$$
$$A_0 \leq_{\mathrm{wtt}} A_1 \text{ (in fact, } A_0 \leq_{\mathrm{ibT}} A_1\text{),} \tag{3.22}$$
$$deg_{\mathrm{tt}}(A_0) \wedge deg_{\mathrm{tt}}(A_1) = \mathbf{0}. \tag{3.23}$$

Proof. We enumerate sets c.e. A_0 and A_1 in stages such that (3.21), (3.22) and (3.23) hold.

To guarantee that (3.21) holds, we meet the following *noncomputability requirements* for all $e \geq 0$.

$$P_e : \ A_0 \neq \varphi_e.$$

To make sure that (3.22) holds, we have the *global permitting requirement* as follows.

$$x \searrow_{s+1} A_0 \Rightarrow x \searrow_{s+1} A_1.$$

Finally, to guarantee that (3.23) holds, we meet the following *minimal pair requirements* for all $e = \langle e_0, e_1 \rangle$.

$$Q_e : \text{ If } \Phi_{e_0} \text{ and } \Phi_{e_1} \text{ are tt-functionals and } \Phi_{e_0}^{A_0} = \Phi_{e_1}^{A_1} = f, \text{ then } f \text{ is computable.}$$

For a priority ordering of the requirements, we let $R_{2i} = P_i$ and $R_{2i+1} = Q_i$ for $i \geq 0$.

Our strategy is as follows. To meet the noncomputability requirements, we use a standard diagonalization strategy, i.e., we appoint a follower x to P_e and wait for a stage s such that $\varphi_{e,s}(x) = 0$. If such a stage exists, we put x into A_0 at stage $s+1$. For the sake of the global permitting requirement, in that case, we also put x into A_1.

To meet the minimal pair requirements, we aim to destroy their hypothesis. To do so, we wait for a stage s such that it is possible to achieve $\Phi_{e_0,s+1}^{A_{0,s+1}}(x) \neq \Phi_{e_1,s+1}^{A_{1,s+1}}(x)$ by putting finitely many numbers into A_1 at stage $s+1$ without changing A_0. We can then show that if we do not succeed in destroying the hypothesis, then the conclusion holds.

For resolving conflicts between different requirements, we use a standard finite injury strategy, i.e., we force lower priority requirements to respect restraints imposed by higher priority requirements, so each requirement is injured at most finitely often and is eventually met. This is achieved by choosing the stage $s+1$ as the follower that can be appointed to any diagonalization requirement at some stage $s+1$ and by ensuring that a minimal pair requirement acting at some stage $s+1$ can only enumerate numbers greater than s' where $s'+1$ is the last stage before $s+1$ at which a requirement of higher priority has acted. So, by the convention that if a computation is defined at some stage s, the use of this computation is less than s, all computations seen at stages less than or equal to such stages s' are not changed if we enumerate followers appointed at stages greater than s' or if a lower priority minimal pair requirement acts after s'. Now we are ready to give the actual construction of the sets A_0 and A_1 where $A_{0,s}$ and $A_{1,s}$ denote the finite parts of A_0 and A_1 enumerated by the end of stage s, respectively.

Construction.

Stage 0 is vacuous, i.e., $A_{0,0} = A_{1,0} = \emptyset$.

Stage $s+1$. We first define under which circumstances a requirement R_i *requires attention* at a stage $s+1$.

Case 1. $R_i = P_e$ *for some* $e \geq 0$.

P_e requires attention at stage $s+1$ if $e \leq s$, P_e is not satisfied at the end of stage s and one of the following holds.

(i) P_e has no follower at the end of stage s.

(ii) P_e has a follower x at the end of stage s and $\varphi_{e,s}(x) = 0$.

Case 2. $R_i = Q_e$ *for some* $e \geq 0$, $e = \langle e_0, e_1 \rangle$.

Q_e requires attention at stage $s+1$ if $e \leq s$, Q_e is not satisfied at the end of stage s and there is a number $x < s$ such that *all* of the following hold.

- $\Phi_{e_0,s}^{A_{0,s}}(x) \downarrow$,
- $\Phi_{e_1,s+1}^{\sigma}(x) \downarrow$ for all $\sigma \in \{0,1\}^{s+1}$,

- There is a string $\sigma \in \{0,1\}^{s+1}$ such that $A_{1,s} \subseteq \sigma$ (i.e., for all $x' < s$, if $A_{1,s}(x') = 1$ holds, then $\sigma(x') = 1$ holds, too), such that, for the greatest $s' < s$ such that a requirement $R_{i'}$ with $i' < i$ becomes active at stage $s'+1$, $A_{1,s} \upharpoonright s'+1 = \sigma \upharpoonright s'+1$ holds and such that $\Phi_{e_0,s}^{A_{0,s}}(x) \neq \Phi_{e_1,s+1}^{\sigma}(x)$.

If no requirement requires attention at stage $s+1$, let $A_{0,s+1} = A_{0,s}$ and $A_{1,s+1} = A_{1,s}$. Otherwise, fix the least i such that R_i requires attention at stage $s+1$. We say R_i *becomes active* or *receives attention.*

Case 1. $R_i = P_e$ for some $e \geq 0$.

Perform the following action according to the clause via which P_e requires attention.

(i) Appoint $s+1$ as a follower to P_e.

(ii) Let $A_{0,s+1} = A_{0,s} \cup \{x\}$, $A_{1,s+1} = A_{1,s} \cup \{x\}$ and declare P_e to be *satisfied*.

Case 2. $R_i = Q_e$ for some $e \geq 0$.

Let $A_{0,s+1} = A_{0,s}$ and for the least x and σ that make the conditions for Q_e to require attention true, let $A_{1,s+1} = \sigma$ and declare Q_e to be *satisfied*.

In all cases, cancel all followers of requirements of lower priority than R_i (i.e., of requirements $R_{i'}$ such that $i' < i$) and declare these requirements to be unsatisfied.

This completes the construction. For the verification, we prove a series of claims.

Verification.

It is immediate by construction that A_0 and A_1 are c.e., so it remains to show the following claims.

Claim 1. Every requirement requires attention at most finitely often.

Proof. The proof is by a standard induction argument. For fixed R_i, by inductive hypothesis fix the least stage s_0 such that no requirement $R_{i'}$ with $i' < i$ requires attention after stage s_0.

If $R_i = P_e$ for some $e \geq 0$, a follower x is permanently assigned to P_e at some stage $s_1 + 1 > s_0$. Then, P_e becomes active at most once after stage $s_1 + 1$. Since P_e receives attention whenever it requires attention after stage s_0, it follows that P_e requires attention at most finitely often.

If $R_i = Q_e$ for some $e \geq 0$, Q_e becomes active at most once after stage s_0, so, as in the case of P_e, Q_e requires attention at most finitely often. □

Claim 2. The noncomputability requirements P_e are met.

Proof. For fixed e, by Claim 1, fix a stage s_0 such that no requirement of higher priority than P_e requires attention after stage s_0 and a stage $s_1 + 1 > s_0$ such that a follower x $(= s_1 + 1)$ becomes permanently assigned to P_e at stage $s_1 + 1$. If P_e requires attention at some stage $s_2 + 1 > s_1 + 1$, this must be via Clause (ii), so $\varphi_e(x) = \varphi_{e,s_2}(x) = 0$ and we put x into A_0 at stage $s_2 + 1$, so $A_0(x) = 1$, hence P_e is met.

Otherwise, $\varphi_e(x) \neq 0$. Furthermore, x never enters A_0, so $A_0(x) = 0$ and P_e is met. □

Claim 3. The global permitting requirement is met.

Proof. If a number x is enumerated into A_0, this can only happen if some noncomputability requirement P_e becomes active via Clause (ii). But then, x is also enumerated into A_1. $\square$

Claim 4. The minimal pair requirements Q_e are met.

Proof. Fix e such that the hypothesis of Q_e holds, i.e., Φ_{e_0} and Φ_{e_1} are tt-functionals and $\Phi_{e_0}^{A_0} = \Phi_{e_1}^{A_1} = f$. (Otherwise, the claim is trivial.) Now, to compute $f(x)$ for some $x \geq 0$, by Claim 1, fix the least stage s_0 such that neither Q_e nor any higher priority requirement requires attention after stage s_0. Then, wait for a stage $s_1 > s_0$ such that $\Phi_{e_0,s_1}^{A_{0,s_1}}(x) \downarrow = \Phi_{e_1,s_1}^{A_{1,s_1}}(x)$ and such that $\Phi_{e_1,s_1+1}^{\sigma} \downarrow$ for all $\sigma \in \{0,1\}^{s_1+1}$. Note that such a stage must exist by the assumption that Φ_{e_0} and Φ_{e_1} are tt-functionals and that $\Phi_{e_0}^{A_0} = \Phi_{e_1}^{A_1}$. We claim that then, $f(x) = \Phi_{e_1}^{A_1}(x) = \Phi_{e_1,s_1}^{A_{1,s_1}}(x)$.

Suppose not. As $\Phi_{e_1,s_1}^{A_{1,s_1}}(x)$ is defined, by convention, $\varphi_{e_1,s_1}^{A_{1,s_1}}(x) < s_1$. Moreover, by choice of s_0, Q_e does not require attention at stage s_1+1, so for all $\sigma \in \{0,1\}^{s_1+1}$, if $A_{1,s_1} \subseteq \sigma$ and for the greatest $s' < s_1$ such that a requirement of higher priority than Q_e becomes active at stage $s'+1$, $A_{1,s_1} \upharpoonright s'+1 = \sigma \upharpoonright s'+1$ holds, $\Phi_{e_1,s_1}^{A_{1,s_1}}(x) = \Phi_{e_0,s_1}^{A_{0,s_1}}(x) = \Phi_{e_1,s_1+1}^{\sigma}(x)$. Hence the enumeration of numbers y with $s' < y < s_1$ into A_1 after stage s_1 does not change the computation of $\Phi_{e_1,s_1}^{A_{1,s_1}}(x)$.

So $\Phi_{e_1}^{A_1}(x)$ can only after stage s_1 if a number $z \leq s'$ enters A_1 after stage s_1. By definition of s_0, we know that $s'+1 \leq s_0$. Since after stage s_0, only requirements with lower priority than Q_e become active and, by construction, all of them are only allowed to enumerate numbers greater than or equal to s_0 (because, for the lower priority noncomputability requirements, by definition of s_0, all followers get canceled at stage s_0 and, for the lower priority minimal pair requirements, the corresponding stage $s'+1$ in the definition of requiring attention is at least s_0, again by definition of s_0), hence greater than s', this cannot happen. This completes the proof of Claim 4. $\square$

Claims 2 to 4 show that A_0 and A_1 have the required properties which completes the proof of Theorem 93. $\square$

Despite of the above negative results, there is a pair of classical strong reducibilities where meet preservation holds as we show next.

Theorem 94. *1-m meet preservation holds.*

For the proof of Theorem 94, we use some facts about simple sets which were defined by Post [Pos44] and are widely used in computability theory.

Definition 95 ([Pos44])**.** *A set A is* simple *if it is c.e., coinfinite and for every infinite c.e. set B, $A \cap B \neq \emptyset$ holds.*

Before we turn to the proof of Theorem 94, we need the following lemma.

Lemma 96. *Let A_0 and A_1 be 1-incomparable c.e. sets such that $deg_1(A_0) \wedge deg_1(A_1)$ exists. Then, A_0 and A_1 are neither simple nor computable.*

A special case of Lemma 96 – namely that, for 1-incomparable simple sets A_0 and A_1, $deg_1(A_0) \wedge deg_1(A_1)$ does not exist – is proven in Odifreddi [Odi99], Proposition VI.5.1. and attributed to Young [You64]. The proof of Lemma 96 is similar to the proof of this special case.

First, we state some well known facts on the 1-degrees of simple and computable sets which we later need in the proof.

$$\text{If } A_1 \text{ is computable then } A_0 \leq_1 A_1 \text{ if and only if } \\ A_0 \text{ is computable and } |A_0| \leq |A_1| \text{ and } |\overline{A_0}| \leq |\overline{A_1}| \text{ hold.} \tag{3.24}$$

$$\text{If } A_0 \text{ and } A_1 \text{ are c.e., } A_0 \text{ is computable and } A_1 \text{ is not computable} \\ \text{then } A_0 \leq_1 A_1 \text{ if and only if } A_0 \text{ is cofinite or } A_1 \text{ is not simple.} \tag{3.25}$$

$$\text{If } A_0 \text{ is noncomputable, } A_1 \text{ is simple and } A_0 \leq_1 A_1 \text{ then } A_0 \text{ is simple.} \tag{3.26}$$

$$\text{If } A \text{ is simple and } x \in A \text{ then } A <_1 A \setminus \{x\}. \tag{3.27}$$

$$\text{If } A_0 \text{ and } A_1 \text{ are c.e., } A_0 \text{ is infinite, } A_0 \leq_1 A_1 \text{ via } f \text{ and } A_1 \not\leq_1 A_0 \\ \text{then } f(\overline{A_0}) \subset \overline{A_1}. \tag{3.28}$$

Claims (3.24) through (3.26) are straightforward and Claim (3.27) is due to Dekker (see e.g. Rogers [Rog67], Theorem 8.XIV). It remains to show that (3.28) holds. Fix c.e. sets A_0 and A_1 such that A_0 is infinite, $A_0 <_1 A_1$ and a computable one-one function f such that $A_0 \leq_1 A_1$ via f. Then, $f(\overline{A_0}) \subseteq \overline{A_1}$ is immediate. Now assume that $f(\overline{A_0}) = \overline{A_1}$. Then, $\omega = A_1 \cup range(f)$. To get a contradiction, we obtain a computable one-one function g such that $A_1 \leq_1 A_0$ via g as follows. Fix an infinite computable subset B of A_0 and a computable one-one function b enumerating B. Now, given y, simultaneously enumerate A_1 and $range(f)$. If y first occurs in the enumeration of $range(f)$ and if, for the least x such that $f(x) = y$, $x \notin B$ holds, then let $g(y) = x$. Otherwise, let $g(y) = b(y)$.

We now turn to the proof of Lemma 96.

Proof of Lemma 96. First, we show that neither A_0 nor A_1 are computable. By symmetry, it suffices to show that A_0 is not computable. Assume for a contradiction that A_0 is computable. If A_1 is computable, too, by 1-incomparability of A_0 and A_1, it follows from (3.24) that either A_0 is finite and A_1 is cofinite or vice versa. But then, again by (3.24), A_0 and A_1 do not have a common lower bound which contradicts the assumption that $deg_1(A_0) \wedge deg_1(A_1)$ exists.

It remains to consider the case where A_1 is not computable. Then, by 1-incomparability of A_0 and A_1, it follows from (3.25) that A_0 is not cofinite and A_1 is simple. By simplicity of A_1 and again by (3.25), the computable sets that are 1-reducible to A_1 are only the cofinite sets. Since A_1 is computable but not cofinite, A_0 and A_1 do not have a common lower bound which contradicts the assumption as in the case above.

Finally, we prove that A_0 and A_1 are not simple. Again, by symmetry, it suffices to consider A_0. So assume for a contradiction that A_0 is simple. W.l.o.g., we may assume that A_1 is not computable. Let B be a c.e. set such that

$$deg_1(B) = deg_1(A_0) \wedge deg_1(A_1) \tag{3.29}$$

holds. In order to get a contradiction, it suffices to show that there is a c.e. set $\widehat{B}$ such that the following holds.

$$B <_1 \widehat{B} \leq_1 A_0, A_1 \tag{3.30}$$

By assumption, A_0 is simple and $B \leq_1 A_0$, so it follows by (3.26) that B is either simple or computable. If B is simple, fix an element $x_0 \in B$ and let $\widehat{B} = B \setminus \{x_0\}$. Then, by (3.27), $B <_1 \widehat{B}$.

In order to prove that (3.30) holds, it remains to show that $\widehat{B} \leq A_i$ for $i = 0, 1$. For fixed i, by 1-incomparability of A_0 and A_1 and by (3.29), $A_i \not\leq B$. Again by (3.29), fix a computable one-one function f such that $B \leq_1 A_i$ via f. Now, by (3.28), $f(\overline{B}) \subset \overline{A_i}$ holds. Fix $y_0 \in \overline{A_i} \setminus f(\overline{B})$. Then $\widehat{B}$ is 1-reducible to A_i by the computable one-one function $\widehat{f}$ defined as follows.

$$\widehat{f}(x) = \begin{cases} y_0 & \text{if } x = x_0 \\ f(x) & \text{otherwise.} \end{cases}$$

Now consider the case where B is computable. Then, by (3.25) and by simplicity of A_0, B is cofinite. By (3.24), the cofinite 1-degrees form an infinite ascending chain. Furthermore, by (3.25), any cofinite set is 1-reducible to any noncomputable c.e. set. So (3.30) holds for any cofinite set $\widehat{B}$ such that $|\overline{B}| < |\overline{\widehat{B}}|$. $\square$

Now we are ready to prove Theorem 94 using Lemma 96.

Proof of Theorem 94. Given c.e. sets A_0, A_1, B and C such that $deg_1(A_0) \wedge deg_1(A_1) = deg_1(B)$ and $C \leq_{\mathrm{m}} A_0, A_1$ hold, it suffices to show that $C \leq_{\mathrm{m}} B$ holds. If A_0 and A_1 are 1-comparable, then this is trivial. So, for the remainder of the proof, we may assume that A_0 and A_1 are 1-incomparable, hence, by Lemma 96, neither simple nor computable.

Fix computable functions f_0 and f_1 such that C is m-reducible to A_i via f_i for $i = 0, 1$. Let

$$M = \{x : \forall i \leq 1 \; \forall x' < x \; (f_i(x') \neq f_i(x))\}$$

and let $\widehat{C} = C \cap M$. Then,

$$C \leq_{\mathrm{m}} \widehat{C} \tag{3.31}$$

via g where g is inductively defined as follows. Let $g(x) = x$ if $x \in M$. Otherwise, let $g(x) = g(x')$ for the least x' such that $f_i(x') = f_i(x)$ for some $i \leq 1$. Note that by the definition and computability of M, g is well-defined and computable.

On the other hand, for $i \leq 1$,

$$\widehat{C} \leq_1 A_i \tag{3.32}$$

via the functions $\widehat{f_i}$, where $\widehat{f_i}$ is defined as follows. For $i \leq 1$, fix infinite computable sets $D_i \subset \overline{A_i}$ (this is possible because A_0 and A_1 are neither simple nor computable) together with strictly increasing computable functions d_i enumerating D_i in order, respectively. Then, let

$$\widehat{f_i} = \begin{cases} f_i(x) & \text{if } x \in M \text{ and } f_i(x) \notin D \\ d_i(x) & \text{otherwise.} \end{cases}$$

Now, as $deg_1(A_0) \wedge deg_1(A_1) = deg_1(B)$ and by (3.32), it follows that $\widehat{C} \leq_1 B$, hence $\widehat{C} \leq_{\mathrm{m}} B$. Together with (3.31), this gives us $C \leq_{\mathrm{m}} B$, which completes the proof. $\square$

We can summarize the results from this section as follows.

Theorem 97. *Let $r, r' \in \{1, \mathrm{m}, \mathrm{tt}, \mathrm{wtt}, \mathrm{T}\}$ be given such that $r \prec r'$. Then, r-r' meet preservation holds if and only if $r = 1$ and $r' = \mathrm{m}$.*

Proof. By Theorem 94, 1-m meet preservation holds. On the other hand, by Lemma 90, there is a minimal pair in the c.e. wtt-degrees that is not a minimal pair in the c.e. T-degrees; by Lemma 91, there is a minimal pair in the c.e. m-degrees that is not a minimal pair in the c.e. tt-degrees; and by Lemma 92, there is a minimal pair in the c.e. tt-degrees that is not a minimal pair in the c.e. wtt-degrees. Considering the fact that for $r, r' \in \{\mathrm{m}, \mathrm{tt}, \mathrm{wtt}, \mathrm{T}\}$ such that $r \preceq r'$, every r'-minimal pair is also an r-minimal pair, it follows from these results that r-r' meet preservation fails for all $r, r' \in \{\mathrm{m}, \mathrm{tt}, \mathrm{wtt}, \mathrm{T}\}$ such that $r \prec r'$.

For the case of 1-reducibility note that for c.e. sets A_0 and A_1 such that $deg_{\mathrm{m}}(A_0)$ and $deg_{\mathrm{m}}(A_1)$ form a minimal pair in the c.e. m-degrees, $deg_1(2A_0)$ and $deg_1(2A_1)$ form a minimal pair in the c.e. 1-degrees in the sense that $deg_1(2A_0) \wedge deg_1(2A_1) = \mathbf{0}_1$ where $\mathbf{0}_1$ is the 1-degree of the infinite and coinfinite computable sets. Furthermore, $deg_{\mathrm{m}}(A_i) = deg_{\mathrm{m}}(2A_i)$ for $i = 0, 1$. Together with the observations above, it follows that 1-tt meet preservation, 1-wtt meet preservation and 1-T meet preservation all fail. This completes the proof of the theorem. □

3.7 Open Problems

Contrasting previous positive results on join preservation in the bounded Turing degrees (see Lemma 86) we have shown that r-r' join preservation fails for the strongly bounded Turing reducibilities $r = \mathrm{ibT}, \mathrm{cl}$ and any monotone admissible uniformly bounded Turing reducibility r' with $\mathrm{cl} \prec r'$. On the other hand, by Corollary 85, r-r' join preservation holds for all monotone admissible uniformly bounded Turing reducibilities r and r' such that $\mathrm{lbT} \preceq r \prec r'$. This naturally leads to the question of a classification of the monotone admissible (uniformly) bounded Turing reducibilities r and r' for which r-r' join preservation holds. In particular, can our main theorem be extended to show that, for any monotone admissible uniformly bounded Turing reducibilities r and r' such that $\mathrm{lbT} \not\preceq r$, $\mathrm{cl} \prec r'$ and $r \prec r'$, r-r' join preservation fails? Moreover, one may consider nonmonotone reducibilities, too. For the latter, a classification of the (u)bT-reducibilities for which meet preservation holds is open, too.

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