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Complexity results for the gap inequalities for the max-cut problem

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ABSTRACT

We prove several complexity results about the gap inequalities for the max-cut problem, including (i) the gap-1 inequalities do not imply the other gap inequalities, unless $\mathcal{NP} = Co\ \mathcal{NP}$; (ii) there must exist non-redundant gap inequalities with exponentially large coefficients, unless $\mathcal{NP} = Co\ \mathcal{NP}$; (iii) the associated separation problem can be solved in finite (doubly exponential) time.

1. Introduction

Given an edge-weighted undirected graph, the *max-cut* problem calls for a partition of the vertex set into two subsets, such that the total weight of the edges having exactly one end-vertex in each subset is maximized. The max-cut problem is a fundamental and well-known combinatorial optimization problem, proven to be strongly $\mathcal{NP}$-hard in [12]. It has a surprisingly large number of important practical applications, and has received a great deal of attention (see, e.g., the book [9] and the survey [17]).

It is usual in combinatorial optimization to formulate a problem as a zero-one linear program, and then derive strong linear inequalities that must be satisfied by all feasible solutions. Such inequalities can then be exploited algorithmically within a branch-and-cut framework (see, e.g., [6]). A wide array of such inequalities have been discovered for the max-cut problem (see again [9]). In particular, Laurent and Poljak [18] introduced an intriguing class of inequalities, known as *gap* inequalities, which includes several other known classes as special cases.

Unfortunately, computing the right-hand side of a gap inequality is itself an $\mathcal{NP}$-hard problem [18]. Perhaps for this reason, the gap inequalities have received very little attention in the literature. The present paper is concerned with certain complexity aspects of gap inequalities.

We assume throughout the paper that the reader is familiar with the fundamental concepts of computational complexity; in particular, the definition of the classes $\mathcal{NP}$ and Co $\mathcal{NP}$ of decision problems (see, e.g., [11]). We also use the term *extreme* in several places. An inequality in a given class is said to be extreme if it cannot be expressed as a non-negative linear combination of two or more other inequalities in that class.

The structure of the paper is as follows. In Section 2, the relevant literature is reviewed. In Section 3, several results are proved concerned with the complexity of the coefficients that an extreme gap inequality can have. Then, in Section 4, some results are proved concerned with the complexity of the separation problem for gap inequalities and some of their special cases. Some open problems are also mentioned.

2. Literature review

Let $G = (V, E)$ be an undirected graph. For any $S \subseteq V$, the set of edges having exactly one end-vertex in S is called an *edge-cutset* or *cut*, and denoted by $\delta(S)$. A vector $x \in \{0, 1\}^{\binom{n}{2}}$ is the incidence vector of a cut in the complete graph K_n if and only if it satisfies the following *triangle inequalities*:

$$x_{ij} + x_{ik} + x_{jk} \le 2 \quad (1 \le i < j < k \le n) \tag{1}$$

$$x_{ij} - x_{ik} - x_{jk} \le 0 \quad (1 \le i < j \le n;\ k \ne i, j). \tag{2}$$

The *cut polytope*, which we will denote by CUT_n, is the convex hull in $\mathbb{R}^{\binom{n}{2}}$ of such incidence vectors [4].

Many classes of strong valid inequalities have been discovered for CUT_n; see again [9,17]. Here, we are interested in the *gap* inequalities of Laurent and Poljak [18], which take the following form:

$$\sum_{1 \le i < j \le n} b_i b_j x_{ij} \le (\sigma(b)^2 - \gamma(b)^2)/4 \quad (\forall b \in \mathbb{Z}^n). \tag{3}$$

Here, $\sigma(b)$ denotes $\sum_{i \in V} b_i$, and

$$\gamma(b) := \min\{|z^T b| : z \in \{\pm 1\}^n\} \tag{4}$$

is the so-called *gap* of b.

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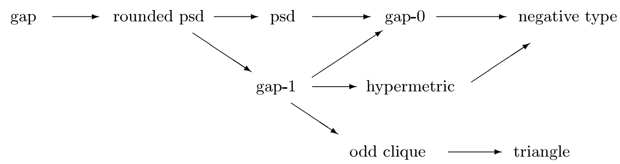

Fig. 1. Inequalities for the cut polytope.

It is shown in [18] that every gap inequality defines a proper face of CUT_n. In the same paper, some sufficient conditions are given for gap inequalities to define facets of CUT_n, and it is conjectured that all facet-defining gap inequalities have $\gamma(b) = 1$.

As mentioned in the introduction, the gap inequalities include several other important classes of inequalities as special cases. They also dominate various other inequalities. A graphical representation of the situation is given in Fig. 1. An arrow from one class to another means that the former is a generalization of, or dominates, the latter.

By a 'gap-0' or 'gap-1' inequality, we simply mean a gap inequality with $\gamma(b)$ equal to 0 or 1, respectively. The other inequalities mentioned in the diagram are as follows:

- The *triangle* inequalities (1), (2).
- *Negative-type* inequalities [21], obtained when $\sigma(b) = \gamma(b) = 0$.
- *Hypermetric* inequalities [7,16], obtained when $\sigma(b) = \gamma(b) = 1$.
- *Odd clique* inequalities [4], obtained when $b \in \{0, \pm 1\}^n$ and $\sigma(b)$ is odd.
- *Positive semidefinite* (psd) inequalities [19], obtained by replacing the right-hand side of (3) with $\sigma(b)^2/4$.
- *Rounded psd* inequalities [1,3,9,13,20], obtained when $\sigma(b)$ is odd, and the right-hand side of (3) is replaced with $\lfloor \sigma(b)^2/4 \rfloor$.

Thus, the gap inequalities are extremely general. Unfortunately, as pointed out in [19], computing $\gamma(b)$ is $\mathcal{NP}$-hard. Indeed, testing if $\gamma(b) = 0$ is equivalent to the *partition problem*, proven to be $\mathcal{NP}$-complete in [15]. This suggests that it might be difficult to use gap inequalities computationally. Perhaps for this reason, they have received little attention from researchers. The only papers we are aware of that concern them are [1], which briefly discusses the complexity of the associated separation problem, and our own paper [10], which adapts the gap inequalities to non-convex Mixed-Integer Quadratic Programs.

For the purposes of what follows, we will need the following three results from the literature.

Theorem 1 (*Barahona & Mahjoub [4]*). *If the inequality $\alpha^T x \le \beta$ is valid for CUT_n, then the 'switched' inequality*

$$\sum_{e \in E \setminus \delta(S)} \alpha_e x_e - \sum_{e \in \delta(S)} \alpha_e x_e \le \beta - \sum_{e \in \delta(S)} \alpha_e$$

is also valid for CUT_n, for any $S \subset V$.

Theorem 2 (*Avis & Grishukhin [2]*). *If a hypermetric inequality is extreme, then the encoding length of the corresponding b-vector is bounded by a polynomial in n.*

Theorem 3 (*Letchford & Sørensen [20]*). *Let $x^* \in \mathbb{R}^{\binom{n}{2}}$ and $b \in \mathbb{Z}^n$ be given, and suppose that $\sigma(b)$ is odd. Construct modified vectors $\tilde{x} \in \mathbb{R}^{\binom{n+1}{2}}$ and $\tilde{b} \in \mathbb{Z}^{n+1}$ as follows. Let $\tilde{x}_{ij} = x^*_{ij}$ for $1 \le i < j \le n$, $\tilde{x}_{i,n+1} = 1 - x^*_{i,n}$ for $i = 1, \dots, n-1$, and $\tilde{x}_{n,n+1} = 1$. Also let $\tilde{b}_i = b_i$ for $i = 1, \dots, n-1$, $\tilde{b}_n = b_n + (1 - \sigma(b))/2$, and $\tilde{b}_{n+1} = (1 - \sigma(b))/2$. Then x^* violates the rounded psd inequality $\sum_{1 \le i < j \le n} b_i b_j x_{ij} \le \lfloor \sigma(b)^2/4 \rfloor$ if and only if $\tilde{x}$ violates the hypermetric inequality $\sum_{1 \le i < j \le n+1} \tilde{b}_i \tilde{b}_j x_{ij} \le 0$.*

3. On extreme gap inequalities

In this section, we present two theorems indicating that there exist extreme gap inequalities with rather complex coefficients. We begin with the following three lemmas.

Lemma 1. *If a rounded psd inequality is extreme, then the encoding length of the corresponding b-vector is bounded by a polynomial in n.*

Proof. Theorem 3 establishes a one-to-one correspondence between rounded psd inequalities for CUT_n and hypermetric inequalities for CUT_{n+1}. One can check that this correspondence preserves the property of being extreme. The result then follows from Theorem 2. □

Lemma 2. *The following decision problem is $\mathcal{NP}$-complete: given positive integers n and k and a vector $b \in \mathbb{Z}^n$, is $\gamma(b) < k$?*

Proof. To show that $\gamma(b) < k$, it suffices to exhibit a set $S \subset V$ such that $\sum_{i \in S} b_i - \sum_{i \in V \setminus S} b_i < k$. Therefore the problem lies in $\mathcal{NP}$.

To show that the problem is $\mathcal{NP}$-hard, we reduce the 'partition' problem to it. The partition problem [15] takes a positive integer p and positive integers $d_1, \dots, d_p$ as input, and asks whether there exists a subset of those integers summing to $\frac{1}{2} \sum_{i=1}^{p} d_i$. The reduction simply sets $n = p$, and $b_i = k d_i$ for $i = 1, \dots, p$. □

Lemma 3. *The following decision problem is Co $\mathcal{NP}$-complete: given positive integers n and k and a vector $b \in \mathbb{Z}^n$, is the inequality*

$$\sum_{1 \le i < j \le n} b_i b_j x_{ij} \le \frac{\sigma(b)^2 - k^2}{4} \tag{5}$$

valid for CUT_n?

Proof. This follows from Lemma 2 and the fact that the inequality (5) is *not* valid for CUT_n if and only if $\gamma(b) < k$. □

We are now ready to prove our first main result.

Theorem 4. *Suppose that every gap inequality is a non-negative linear combination of one or more rounded psd inequalities. Then $\mathcal{NP} = \mathrm{Co}\,\mathcal{NP}$.*

Proof. The inequality (5) is either a gap inequality or implied by a gap inequality. So, if the statement were true, the inequality (5) would be implied by rounded psd inequalities. In particular, by Carathéodory's theorem, there would exist a set of at most $\binom{n}{2}$ extreme rounded psd inequalities that collectively implied the inequality (5). Now, Lemma 1 implies that, for each of those rounded psd inequalities, the corresponding b-vector would be a short certificate of validity. Thus, we would have a short certificate for a *Co* $\mathcal{NP}$-complete problem, and $\mathcal{NP}$ would equal *Co* $\mathcal{NP}$. □

Since the gap-1 inequalities are a special case of the rounded psd inequalities, Theorem 4 has the following corollary.

Corollary 1. *Suppose that every gap inequality is a non-negative linear combination of one or more gap-1 inequalities. Then $\mathcal{NP} = \mathrm{Co}\,\mathcal{NP}$.*

Before presenting our second main result, we need the following lemma.

Lemma 4. *Let $\|b\|_1$ denote $\sum_{i \in V} |b_i|$. One can compute $\gamma(b)$ in $\mathcal{O}(n\|b\|_1)$ time.*

Proof. To compute $\gamma(b)$, it suffices to solve the subset-sum problem

$$\mathrm{SSP} = \max\left\{\sum_{i\in V} |b_i| y_i : \sum_{i\in V} |b_i| y_i \le \|b\|_1/2,\ y \in \{0,1\}^n\right\},$$

and then set $\gamma(b)$ to $\|b\|_1 - 2\,\mathrm{SSP}$. This subset-sum problem can be solved in $\mathcal{O}(n\|b\|_1)$ time with dynamic programming [5]. □

Armed with this lemma, we can prove our second main result, which essentially states that there should exist extreme gap inequalities with 'large' coefficients.

Theorem 5. *Suppose there exists a polynomial $p(n)$ such that every extreme gap inequality satisfies $\|b\|_1 \le p(n)$. Then $\mathcal{NP} = \mathrm{Co}\,\mathcal{NP}$.*

Proof. The proof is similar to that of Theorem 4. The difference is that the inequality (5) would be implied by a set of $\mathcal{O}(n^2)$ extreme gap inequalities, each with $\|b\|_1 \le p(n)$. In light of Lemma 4, the b-vectors associated with these gap inequalities would provide a short certificate of validity. □

4. On the complexity of separation

The *separation problem*, for a given class of valid inequalities, is the problem of detecting when an inequality in that class is violated by some given input vector x^* [14]. The separation problem for triangle inequalities can be solved in $\mathcal{O}(n^3)$ time by mere enumeration, and polynomial-time separation algorithms are known for psd inequalities [14,19] and negative-type inequalities [9]. To our knowledge, the complexity of separation for the remaining inequalities in Fig. 1 is unknown (even if Theorem 3 implies that rounded psd separation can be reduced to hypermetric separation).

The following two lemmas are relevant to the complexity of the separation problem for gap-1 inequalities.

Lemma 5. *If a gap-1 inequality is extreme, then the encoding length of the corresponding b-vector is bounded by a polynomial in n.*

Proof. It is well known and easy to see that the gap-1 inequalities are nothing but the inequalities that can be obtained from hypermetric inequalities by the switching operation, mentioned in Theorem 1. Moreover, if a gap-1 inequality is extreme, it will be a switching of an extreme hypermetric inequality. The result then follows from Theorem 2. □

Lemma 6. *The following problem is in $\mathcal{NP}$: 'Given an integer $n \ge 2$ and a vector $x^* \in [0,1]^{\binom{n}{2}}$, does x^* violate a gap-1 inequality?'.*

Proof. If x^* violates a gap-1 inequality, then it violates an extreme gap-1 inequality. From Lemma 5, the encoding length of the associated b vector is polynomially bounded. This b vector, along with a set S such that $\sum_{i\in S} b_i - \sum_{i\in V\setminus S} b_i = 1$, constitutes a short certificate of validity of the gap-1 inequality, and therefore also of violation. □

In fact, it is possible to formulate the separation problem for gap-1 inequalities as an Integer Quadratic Program (IQP) of 'small' size:

Theorem 6. *The separation problem for gap-1 inequalities can be formulated as an IQP with $\mathcal{O}(n)$ variables and $\mathcal{O}(n)$ constraints.*

Proof. Let $x^* \in \mathbb{Q}^{\binom{n}{2}}$ be the point to be separated, and let U be an upper bound on the value of $\|b\|_1$ implied by Lemma 5. From the form of the gap inequality (3), a violated gap-1 inequality exists if and only if the solution to the following optimization problem has a cost of less than 1:

$$\min\left\{\sum_{i=1}^n b_i^2 + \sum_{1\le i<j\le n} (2 - 4x^*_{ij}) b_i b_j : \gamma(b) = 1,\ b \in [-U,U]^n \cap \mathbb{Z}^n\right\}.$$

To put the constraint $\gamma(b) = 1$ in a more tractable form, we use the fact that $\gamma(b) = 1$ if and only if there exists a set $S \subseteq V$ such that $\sum_{i\in S} b_i - \sum_{i\in V\setminus S} b_i = 1$. Accordingly, for each $i \in V$, we introduce the binary variable s_i, taking the value 1 if and only if $i \in S$. The constraint $\gamma(b) = 1$ can then be replaced with the quadratic constraint

$$\sum_{i=1}^n b_i s_i - \sum_{i=1}^n b_i(1 - s_i) = \sum_{i=1}^n b_i(2s_i - 1) = 1. \tag{6}$$

Finally, to linearize the constraint (6), we introduce for each $i \in V$ the general-integer variable p_i, representing the product $b_i s_i$. The constraint (6) can then be replaced with the constraint

$$\sum_{i=1}^n (2p_i - b_i) = 1,$$

together with the following linking constraints for $i = 1, \ldots, n$:

$$\begin{aligned}
&p_i \le U s_i\\
&p_i \ge -U s_i\\
&b_i - p_i + U s_i \le U\\
&-b_i + p_i + U s_i \le U.
\end{aligned}$$

Now all constraints are linear. □

We remark that the separation problems for odd clique, hypermetric and rounded psd inequalities can also be easily formulated as IQPs. We do not know whether the same is true for gap-0 inequalities. As for general gap inequalities, it is unlikely that the separation problem can be formulated as an IQP of polynomial size. Indeed, if it could, then a feasible solution of that IQP could be used as a short certificate of the validity of a gap inequality, which we have already seen cannot exist unless $\mathcal{NP} = \mathrm{Co}\,\mathcal{NP}$.

On a more positive note, we now show that a *finite* separation algorithm exists for general gap inequalities.

Theorem 7. *The separation problem for general gap inequalities can be solved in finite time.*

Proof. Observe that the definition of $\gamma(b)$ given by (4) can be applied to arbitrary rational vectors as well as integer vectors. Then, the gap inequalities can be written in the following alternative form:

$$\sum_{1\le i<j\le n} b_i b_j x_{ij} \le (\sigma(b)^2 - 1)/4 \quad (\forall b \in \mathbb{Q}^n : \gamma(b) = 1).$$

So, a violated gap inequality exists if and only if the solution to the following optimization problem has a cost of less than 1:

$$\min\left\{\sum_{i=1}^n b_i^2 + \sum_{1\le i<j\le n} (2 - 4x^*_{ij}) b_i b_j : \gamma(b) = 1,\ b \in \mathbb{Q}^n\right\}. \tag{7}$$

Since the separation problem for psd inequalities can be solved in polynomial time, and psd inequalities are weaker than gap inequalities, we can assume w.l.o.g. that all psd inequalities are satisfied. Then, the objective function in (7) will be non-negative for all $b \in \mathbb{R}^n$, and therefore convex. It follows that one can relax the constraint $\gamma(b) = 1$ to $\gamma(b) \ge 1$, without affecting the optimal solution. Now, observe that the condition $\gamma(b) \geq 1$ is equivalent to an exponential number of disjunctions:

$$\left(\sum_{i\in S} b_i - \sum_{i\in V\setminus S} b_i \geq 1\right) \vee \left(\sum_{i\in S} b_i - \sum_{i\in V\setminus S} b_i \leq -1\right) \quad (\forall S \subset V).$$

Accordingly, we let $\mathcal{F} = 2^V$ denote the family of all possible sets $S \subseteq V$. To solve the separation problem, it suffices to solve the following Convex Quadratic Program (CQP) for all subsets $\mathcal{F}' \subseteq \mathcal{F}$:

$$\min \sum_{i=1}^{n} b_i^2 + \sum_{1\leq i<j\leq n} (2 - 4x_{ij}^*) b_i b_j$$

$$\text{s.t.} \sum_{i\in S} b_i - \sum_{i\in V\setminus S} b_i \geq 1 \quad (\forall S \in \mathcal{F}')$$

$$\sum_{i\in S} b_i - \sum_{i\in V\setminus S} b_i \leq -1 \quad (\forall S \in \mathcal{F} \setminus \mathcal{F}')$$

$$b \in \mathbb{Q}^n.$$

Each of these CQP instances can be solved in finite time using, e.g., the simplex method of Wolfe [22]. □

Observe that the running time of this algorithm is doubly exponential. We leave it as an open question whether an algorithm can be devised whose running time is singly exponential. In any case, it is clear that fast heuristics for separation would be essential if one wished to use gap inequalities as cutting planes in an exact algorithm for the max-cut problem and related problems. We hope to address this issue in a future paper.

To close, we mention two other open questions. The first is whether the gap inequalities define a polyhedron. (It is known that the hypermetric and rounded psd inequalities define polyhedra [8, 20], whereas the negative type and psd inequalities do not [9,19].) The second is whether there exists a gap inequality with $\gamma(b) > 1$ that induces a facet of CUT_n. (As mentioned in Section 2, it is conjectured in [18] that no such inequality exists.)

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