Discrete Direct Methods in the Fractional Calculus of Variations [⋆]

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Abstract: Finite differences, as a subclass of direct methods in the calculus of variations, consist in discretizing the objective functional using appropriate approximations for derivatives that appear in the problem. This article generalizes the same idea for fractional variational problems. We consider a minimization problem with a Lagrangian that depends only on the left Riemann–Liouville fractional derivative. Using Grünwald–Letnikov definition, we approximate the objective functional in an equispaced grid as a multi-variable function of the values of the unknown function on mesh points. The problem is then transformed to an ordinary static optimization problem. The solution to the latter problem gives an approximation to the original fractional problem on mesh points.

Keywords: Fractional calculus, fractional calculus of variations, direct methods.

1. INTRODUCTION

A fractional variational problem consists in finding the extremizer of a functional that depends on fractional derivatives and/or integrals subject to some boundary conditions and possibly some extra constraints. In this work we consider the following minimization problem:

$$\begin{cases} J[x(\cdot)] = \int_a^b L(t, x(t), {}_aD_t^\alpha x(t))dt \to \min \\ x(a) = x_a, \ x(b) = x_b, \end{cases} \quad (1)$$

that depends on the left Riemann–Liouville derivative, ${}_aD_t^\alpha$, which is defined as follows.

Definition 1. (see Kilbas et al. (2006)). Let $x(\cdot)$ be an absolutely continuous function in $[a, b]$ and $0 \le \alpha < 1$. The left Riemann–Liouville fractional derivative of order α, ${}_aD_t^\alpha$, is given by

$$ {}_aD_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} x(\tau)d\tau, \qquad t \in [a, b]. $$

There are several different definitions for fractional derivatives, left and right, that can be found in the literature and could be included. They posses different properties: each one of those definitions has its own advantages and disadvantages. In certain conditions they are equivalent and can be used interchangeably.

There are two major approaches in the classical theory of calculus of variations. In one hand, using Euler–Lagrange necessary conditions, we can reduce a variational problem to the study of a differential equation. Hereafter, one can

use either analytical or numerical methods to solve the differential equation and reach the solution of the original problem (see, e.g. Kirk (1970)). This approach is referred as indirect methods in the literature. On the other hand, we can tackle the functional itself, directly. Direct methods are used to find the extremizer of a functional in two ways: Euler's finite differences and Ritz methods. In the Ritz method, we either restrict admissible functions to all possible linear combinations

$$ x_n = \sum_{i=1}^n \alpha_i \phi_i(t), $$

with constant coefficients α_i and a set of known base functions ϕ_i, or we approximate the admissible functions with such combinations. Using x_n and its derivatives whenever needed, one can transform the functional to a multivariate function of unknown coefficients α_i.

By finite differences, however, we consider the admissible functions not on the class of arbitrary curves, but only on polygonal curves made upon a given grid on the time horizon. Using an appropriate discrete approximation of the Lagrangian, and substituting the integral with a sum, we can transform the main problem to the optimization of a function of several parameters: the values of the unknown function on mesh points (see, e.g. Elsgolts (1973)).

Indirect methods for fractional variational problems have a vast background in the literature and can be considered a well studied subject: see Agrawal (2002); Atanacković et al. (2008); Frederico and Torres (2008); Jelicic and Petrovacki (2009); Klimek (2001); Almeida et al. (2012); Odzijewicz et al. (in press); Riewe (1997) and references therein that study different variants of the problem and discuss a bunch of possibilities in the presence of fractional

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terms, Euler–Lagrange equations and boundary conditions. Direct methods, however, to the best of our knowledge, have got less interest and are not well studied. A brief introduction of using finite differences has been made in Riewe (1996) that can be regarded as a predecessor to what we call Euler-like direct method here. A generalization of Leitmann's direct method can be found in Almeida and Torres (2010) while Lotfi et al. (2011) discusses the Ritz direct method for optimal control problems that can easily be reduced to a problem of the calculus of variations.

2. DIRECT METHODS IN THE CLASSICAL THEORY

The basic idea of direct methods is to consider the variational problem as a limiting case of a certain extremum problem of a multi-variable function. This is then an ordinary static optimization problem. The solution to the latter problem can be regarded as an approximate solution to the original variational problem. There are three major methods: Euler's finite differences, Ritz and Kantorovich's methods. We are going to discuss the Euler's method briefly: see, e.g., Elsgolts (1973). The basic idea of a finite differences method is that instead of considering the values of a functional

$$J[x(\cdot)] = \int_a^b L(t, x(t), \dot{x}(t))dt$$

with boundary conditions $x(a) = x_a$ and $x(b) = x_b$, on arbitrary admissible curves, we only track the values at an $n + 1$ points grid, t_i, $i = 0, \ldots, n$, in the interested time interval. The functional $J[x(\cdot)]$ is then transformed into a function $\Psi(x(t_1), x(t_2), \ldots, x(t_{n-1}))$ of the values of the unknown function on mesh points. Assuming $h = t_i - t_{i-1}$, $x(t_i) = x_i$ and $\dot{x}_i \simeq \frac{x_i - x_{i-1}}{h}$, one has

$$\begin{aligned} J[x(\cdot)] &\simeq \Psi(x_1, x_2, \ldots, x_{n-1}) \\ &= h \sum_{i=1}^n L\left(t_i, x_i, \frac{x_i - x_{i-1}}{h}\right), \\ &x_0 = x_a, \quad x_n = x_b. \end{aligned}$$

The desired values of x_i, $i = 1, \ldots, n-1$, give the extremum to the multi-variable function Ψ and satisfy the system

$$\frac{\partial \Psi}{\partial x_i} = 0, \quad i = 1, \ldots, n-1.$$

The fact that only two terms in the sum, $(i-1)$th and ith, depend on x_i makes it rather easy to find the extremum of Ψ solving a system of algebraic equations. For each n, we obtain a polygonal line which is an approximate solution of the original problem. It has been shown that passing to the limit, as $h \to 0$, the linear system corresponding to finding the extremum of Ψ is equivalent to the Euler–Lagrange equation for the original problem Tuckey (1993).

3. FINITE DIFFERENCES FOR FRACTIONAL DERIVATIVES

In classical theory, given a derivative of certain order $x^{(n)}$, there is a finite difference approximation of the form

$$x^{(n)}(t) = \lim_{h \to 0^+} \frac{1}{h^n} \sum_{k=0}^n (-1)^k \binom{n}{k} x(t - kh)$$

where

$$\binom{n}{k} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!}. \tag{2}$$

This is generalized to derivatives of arbitrary order and gives rise to the Grünwald–Letnikov fractional derivative.

Definition 2. (See Kilbas et al. (2006)). Let $0 < \alpha < 1$. The left Grünwald–Letnikov fractional derivative is defined as

$$^{GL}_a D_t^\alpha x(t) = \lim_{h \to 0^+} \frac{1}{h^\alpha} \sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} x(t - kh). \tag{3}$$

Here $\binom{\alpha}{k}$ is the generalization of binomial coefficients (2) to real numbers.

Similarly, the right Grünwald–Letnikov derivative is given by

$$^{GL}_t D_b^\alpha x(t) = \lim_{h \to 0^+} \frac{1}{h^\alpha} \sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} x(t + kh). \tag{4}$$

The series in (3) and (4) converges absolutely and uniformly if $x(\cdot)$ is bounded. The definition is clearly affected by the non-local property of fractional derivatives. The arbitrary order derivative of a function at a time t depends on all values of that function in $(-\infty, t]$ and $[t, \infty)$ because of the infinite sum, backward and forward difference nature of the left and right derivatives, respectively. Since we are, usually, and specifically in this paper, dealing with closed time intervals, the following remark is made to make the definition clear in closed regions.

Remark 3. For the above definition to be consistent, we need the values of $x(t)$ outside the interval $[a, b]$. To overcome this difficulty, we take

$$x^*(t) = \begin{cases} x(t) & t \in [a, b], \\ 0 & t \notin [a, b]. \end{cases}$$

Thus, one can assume $^{GL}_a D_t^\alpha x(t) = {}^{GL}_a D_t^\alpha x^*(t)$ and $^{GL}_t D_b^\alpha x(t) = {}^{GL}_t D_b^\alpha x^*(t)$ for $t \in [a, b]$.

As we mentioned before, this definition coincides with Riemann–Liouville derivatives. The following proposition establishes the connection between these two definitions and that of Caputo derivative, another type of derivative that is believed to be more applicable in practical fields such as engineering and physics.

Proposition 4. (See Podlubny (1999)). Let $n-1 < \alpha < n$, $n \in \mathbb{N}$ and $x(\cdot) \in C^{n-1}[a, b]$. Suppose also that $x^{(n)}(\cdot)$ is integrable on $[a, b]$. Then the Riemann–Liouville derivative exists and coincides with the Grünwald–Letnikov derivative and the following holds:

$$\begin{aligned} {}_a D_t^\alpha x(t) = {}^{GL}_a D_t^\alpha x(t) &= \sum_{i=0}^{n-1} \frac{x^{(i)}(a)(t-a)^{i-\alpha}}{\Gamma(1+i-\alpha)} \\ &+ \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-1-\alpha} x^{(n)}(\tau) d\tau. \end{aligned}$$

Remark 5. For numerical purposes we need a finite sum in (3) and this goal is achieved by Remark 3. Given a grid on $[a, b]$ as $a = t_0, t_1, \ldots, t_n = b$, where $t_i = t_0 + ih$ for some $h > 0$, we approximate the left Riemann–Liouville derivative as

$$_aD_t^\alpha x(t_i) \simeq \frac{1}{h^\alpha} \sum_{k=0}^{i} (\omega_k^\alpha)\, x(t_i - kh) =: {}_a\widetilde{D}_t^\alpha x(t_i) \qquad (5)$$

where $(\omega_k^\alpha) = (-1)^k \binom{\alpha}{k} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}$.

Similarly, one can approximate the right Riemann–Liouville derivative by

$$_tD_b^\alpha x(t_i) \simeq \frac{1}{h^\alpha} \sum_{k=0}^{n-i} (\omega_k^\alpha)\, x(t_i + kh). \qquad (6)$$

As it is stated in Podlubny (1999), this approximation is of first order, i.e.,

$$_aD_t^\alpha x(t_i) = \frac{1}{h^\alpha} \sum_{k=0}^{i} (\omega_k^\alpha)\, x(t_i - kh) + \mathcal{O}(h).$$

Remark 6. In Meerschaert and Tadjeran (2004), it has been shown that the implicit Euler method is unstable for certain fractional partial differential equations with Grünwald–Letnikov approximations. Therefore, discretizing fractional derivatives, shifted Grünwald–Letnikov derivatives are used. Despite the slight difference with respect to standard Grünwald–Letnikov derivatives, they exhibit, at least for certain cases, a stable performance. The shifted Grünwald–Letnikov derivative is defined by

$$_a^{sGL}D_t^\alpha x(t_i) \simeq \frac{1}{h^\alpha} \sum_{k=0}^{i} (\omega_k^\alpha)\, x(t_i - (k-1)h).$$

Other finite difference approximations can be found in the literature, e.g., the Diethelm's backward finite difference formulas for fractional derivatives Diethelm et al. (2005).

4. EULER-LIKE DIRECT METHOD FOR FRACTIONAL VARIATIONAL PROBLEMS

As mentioned earlier, we consider a simple version of the fractional variational problem where the fractional term has a Riemann–Liouville form on a finite time interval $[a, b]$. The boundary conditions are given and we approximate using Grünwald–Letnikov approximation given by (5). In this context we discretize the functional in (1) using a simple quadrature rule on the mesh points $a = t_0, t_1, \ldots, t_n = b$ with $h = \frac{b-a}{n}$. The goal is to find the values $x_1, x_2, \ldots, x_{n-1}$ of the unknown function $x(\cdot)$ at points t_i, $i = 1, 2, \ldots, n-1$. The values of x_0 and x_n are given. Applying the quadrature rule gives

$$J[x(\cdot)] = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} L(t_i, x_i, {}_aD_t^\alpha x_i)\, dt$$
$$\simeq \sum_{i=1}^{n} hL(t_i, x_i, {}_aD_t^\alpha x_i)$$

and by approximating the fractional derivatives at mesh points using (5) we have

$$J[x(\cdot)] \simeq \sum_{i=1}^{n} hL\left(t_i, x_i, {}_a\widetilde{D}_t^\alpha x_{i-k}\right). \qquad (7)$$

Hereafter the procedure is the same as in classical case. The right hand side of (7) can be regarded as a function Ψ of $n-1$ unknowns $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1})$,

$$\Psi(\mathbf{x}) = \sum_{i=1}^{n} hL\left(t_i, x_i, {}_a\widetilde{D}_t^\alpha x_{i-k}\right). \qquad (8)$$

To find an extremum for Ψ, one has to solve the following system of algebraic equations:

$$\frac{\partial \Psi}{\partial x_i} = 0, \qquad i = 1, \ldots, n-1. \qquad (9)$$

Unlike the classical case, all terms, starting from ith term, in (8) depend on x_i and we have

$$\frac{\partial \Psi}{\partial x_i} = h\frac{\partial L}{\partial x}(t_i, x_i, {}_a\widetilde{D}_t^\alpha x_i)$$
$$+ \sum_{k=0}^{n-i} \frac{h}{h^\alpha} (\omega_k^\alpha) \frac{\partial L}{\partial\, {}_aD_t^\alpha x}(t_{i+k}, x_{i+k}, {}_a\widetilde{D}_t^\alpha x_{i+k}) \quad (10)$$

Equating the right hand side of (10) with zero one has

$$\frac{\partial L}{\partial x}(t_i, x_i, {}_a\widetilde{D}_t^\alpha x_i)$$
$$+ \frac{1}{h^\alpha} \sum_{k=0}^{n-i} (\omega_k^\alpha) \frac{\partial L}{\partial\, {}_aD_t^\alpha x}(t_{i+k}, x_{i+k}, {}_a\widetilde{D}_t^\alpha x_{i+k}) = 0.$$

Passing to the limit and considering the approximation formula for the right Riemann–Liouville derivative (6), it is straightforward to verify that:

Proposition 7. The Euler-like method for a fractional variational problem of the form (1) is equivalent to the fractional Euler–Lagrange equation

$$\frac{\partial L}{\partial x} + {}_tD_b^\alpha \frac{\partial L}{\partial\, {}_aD_t^\alpha x} = 0,$$

as the mesh size, h, tends to zero.

5. EXAMPLES

In this section we try to solve test problems using what has been presented in Section 4. For the sake of simplicity we restrict ourselves to the interval $[0, 1]$. To measure the accuracy of our method we use the maximum norm. Assume that the exact value of the function $x(\cdot)$, at the point t_i, is $x(t_i)$ and it is approximated by $\tilde{x}_i$. The error is defined as

$$E = \max\{|x(t_i) - \tilde{x}_i|,\ i = 1, 2, \cdots, n-1\}. \qquad (11)$$

Example 1. Consider the following minimization problem:

$$\begin{cases} J[x(\cdot)] = \int_0^1 \left({}_0D_t^{0.5}x(t) - \frac{2}{\Gamma(2.5)}t^{1.5} \right)^2 dt \to \min \\ x(0) = 0,\ x(1) = 1. \end{cases}$$
$$(12)$$

Problem (12) has the obvious solution of the form $\hat{x}(t) = t^2$ due to the positivity of the Lagrangian and the zero value of $J[\hat{x}(\cdot)]$.

Using the approximation

$$_0D_t^{0.5}x(t_i) \simeq \frac{1}{h^{0.5}} \sum_{k=0}^{i} (\omega_k^{0.5})\, x(t_i - kh)$$

for a fixed h, and following the routine discussed in Section 4, we approximate problem (12) by

$$\Psi(\mathbf{x}) = \sum_{i=1}^{n} h\left(\frac{1}{h^{0.5}} \sum_{k=0}^{i} (\omega_k^{0.5})\, x_{i-k} - \frac{2}{\Gamma(2.5)}t_i^{1.5} \right)^2.$$
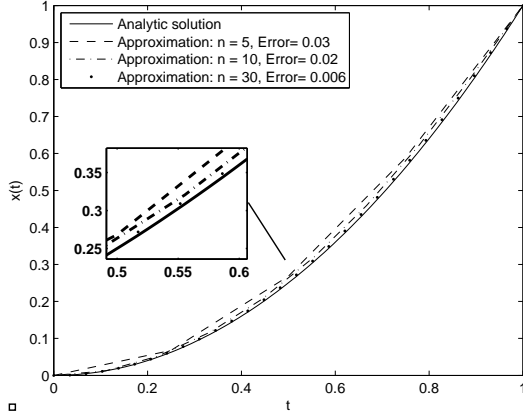
Fig. 1. Analytic and approximate solutions for Example 1.

Since the Lagrangian in this example is quadratic, system (9) is linear and therefore easy to solve. Other problems may end with a system of nonlinear equations. Simple calculations lead to the system

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \qquad (13)$$

in which

$$\mathbf{A} = \begin{bmatrix} \sum_{i=0}^{n-1} A_i^2 & \sum_{i=1}^{n-1} A_i A_{i-1} & \cdots & \sum_{i=n-2}^{n-1} A_i A_{i-n+2} \\ \sum_{i=0}^{n-2} A_i A_{i+1} & \sum_{i=1}^{n-2} A_i^2 & \cdots & \sum_{i=n-3}^{n-2} A_i A_{i-n+3} \\ \sum_{i=0}^{n-3} A_i A_{i+2} & \sum_{i=1}^{n-3} A_i A_{i+1} & \cdots & \sum_{i=n-4}^{n-3} A_i A_{i-n+4} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^{1} A_i A_{i+n-2} & \sum_{i=0}^{1} A_i A_{i+n-3} & \cdots & \sum_{i=0}^{1} A_i^2 \end{bmatrix}$$

where $A_i = (-1)^i h^{1.5} \binom{0.5}{i}$, $\mathbf{x} = (x_1, x_2, \cdots, x_{n-1})^T$ and $\mathbf{b} = (b_1, b_2, \cdots, b_{n-1})^T$ with

$$b_i = \sum_{k=0}^{n-i} \frac{2h^2 A_k}{\Gamma(2.5)} t_{k+i}^{1.5} - A_{n-i} A_0 - \left(\sum_{k=0}^{n-i} A_k A_{k+i} \right).$$

The linear system (13) is easily solved for different values of n. As indicated in Figure 1, by increasing the value of n we get better solutions.

Although we constructed our theory for problems in the form (1), other operators can be included in the Lagrangian. Let us now move to another example that depends also on the first derivative and the solution is obtained via the fractional Euler–Lagrange equation.

Remark 8. Suppose that the objective functional in a fractional variational problem depends on left Riemann–Liouville fractional derivative, $_aD_t^\alpha$, and first derivative, $\dot{x}$. It is easy to verify that if $x(\cdot)$ is an extremizer of such a problem, then it satisfies the fractional Euler–Lagrange equation

$$\frac{\partial L}{\partial x} + {}_tD_b^\alpha \frac{\partial L}{\partial {}_aD_t^\alpha x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) = 0, \qquad (14)$$
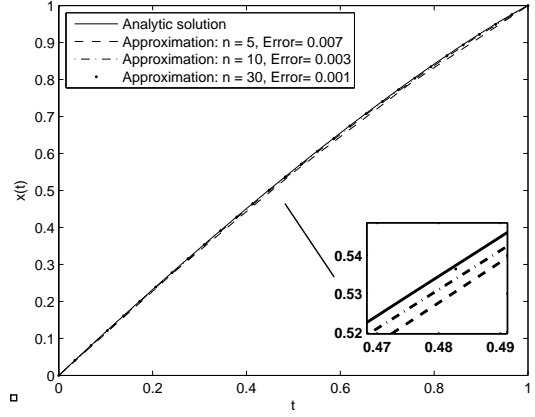
see Odzijewicz et al. (2012).

Fig. 2. Analytic and approximate solutions for Example 2.

Example 2. Consider the following minimization problem:

$$\begin{cases} J[x(\cdot)] = \int_0^1 \left({}_0D_t^{0.5}x(t) - \dot{x}^2(t) \right) dt \to \min \\ x(0) = 0, \ x(1) = 1. \end{cases} \qquad (15)$$

In this case the Euler–Lagrange equation (14) gives

$${}_tD_1^{0.5}1 + 2\ddot{x}(t) = 0.$$

Since $_tD_1^{0.5}1 = \frac{1}{\Gamma(0.5)}(1-t)^{-0.5}$, the fractional Euler–Lagrange equation turns to be an ordinary differential equation

$$\ddot{x}(t) = -\frac{1}{2\Gamma(0.5)}(1-t)^{-0.5},$$

which subject to the given boundary conditions has solution

$$x(t) = -\frac{1}{2\Gamma(2.5)}(1-t)^{1.5} + \left(1 - \frac{1}{2\Gamma(2.5)} \right) t + \frac{1}{2\Gamma(2.5)}.$$

Discretizing problem (15), with the same assumptions of Example 1, ends in a linear system of the form

$$\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{bmatrix}, \quad (16)$$

where

$$b_i = \frac{h}{2} \sum_{k=0}^{n-i-1} (-1)^k h^{0.5} \binom{0.5}{k}, \qquad i = 1, 2, \cdots, n-2$$

and

$$b_{n-1} = \frac{h}{2} \sum_{k=0}^{1} \left((-1)^k h^{0.5} \binom{0.5}{k} \right) + x_n.$$

The linear system (16) can be solved efficiently for any n to reach the desired accuracy. The analytic solution together with some approximated solutions are shown in Figure 2.

Both examples above end with linear systems and their solvability is simply dependent to the matrix of coefficients. Now we try our method on a more complicated problem, yet analytically solvable with an oscillating solution.

Example 3. Consider a minimization problem with the following Lagrangian:

$$L = \left({}_0D_t^{0.5}x(t) - \frac{16\Gamma(6)}{\Gamma(5.5)}t^{4.5} + \frac{20\Gamma(4)}{\Gamma(3.5)}t^{2.5} - \frac{5}{\Gamma(1.5)}t^{0.5} \right)^4$$

and subject to the boundary conditions $x(0) = 0$ and $x(1) = 1$. The functional $\int_0^1 L\, dt$, with positive L, attains its minimum value for

$$x(t) = 16t^5 - 20t^3 + 5t.$$

The appearance of fourth power in the Lagrangian, results in a nonlinear system as we apply the Euler-like direct method to this problem. For $j = 1, 2, \cdots, n-1$ we have

$$\sum_{i=j}^n \left(\omega_{i-j}^{0.5} \right) \left(\frac{1}{h^{0.5}} \sum_{k=0}^i \left(\omega_k^{0.5} \right) x_{i-k} - \phi(t_i) \right)^3 = 0, \quad (17)$$

where

$$\phi(t) = \frac{16\Gamma(6)}{\Gamma(5.5)}t^{4.5} + \frac{20\Gamma(4)}{\Gamma(3.5)}t^{2.5} - \frac{5}{\Gamma(1.5)}t^{0.5}.$$

System (17) is solved for different values of n and the results are given in Figure 3.
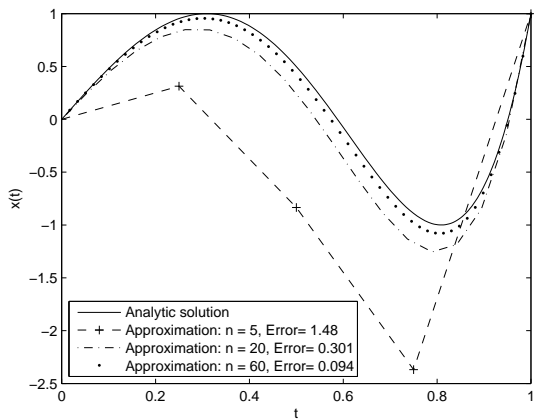

Fig. 3. Analytic and approximate solutions for Example 3.

6. CONCLUSION

Roughly speaking, an Euler-like direct method reduces a variational problem to the solution of a system of algebraic equations. When the system is linear, we can freely increase the number of mesh points, n, and obtain better solutions as long as the resulted matrix of coefficients is invertible. The method is very fast in this case and the execution time is of order 10^{-4} for Example 1 and 2. It is worth, however, to keep in mind that the Grünwald–Letnikov approximation is of first order, $\mathcal{O}(h)$, and even a large n can not result in a high precision. Actually, by increasing n, the solution slowly converges and in Example 2, a grid of 30 points has the same order of error, 10^{-3}, as a 5 points grid.

The situation is completely different when the problem ends in a nonlinear system. In Example 3 a small number of mesh points, $n = 5$, results in a poor solution with the error $E = 1.4787$. The Matlab built in function **fsolve** takes 0.0126 seconds to solve the problem. As one increases the number of mesh points, the solution gets closer to the analytic solution and the required time increases drastically. Finally, by $n = 90$ we have $E = 0.0618$ and

the time is $T = 26.355$ seconds. Table 1 summarizes the results.

	n	T	E
Example 1	5	1.9668×10^{-4}	0.0264
	10	2.8297×10^{-4}	0.0158
	30	9.8318×10^{-4}	0.0065
Example 2	5	2.4053×10^{-4}	0.0070
	10	3.0209×10^{-4}	0.0035
	30	7.3457×10^{-4}	0.0012
Example 3	5	0.0126	1.4787
	20	0.2012	0.3006
	90	26.355	0.0618

Table 1. Number of mesh points, n, with corresponding run time in seconds, T, and error, E (11).

In practice, we have no idea about the solution in advance and the worst case should be taken into account. Comparing the results of the three examples, reveals that for a typical fractional variational problem, Euler-like direct method needs a large number of mesh points and most likely a long running time for the solution procedure.

The Euler-like direct method for fractional variational problems can be improved in some stages. One can try different approximations for the fractional derivative that exhibit higher order precisions. Better quadrature rules can be applied to discretize the functional and, finally, we can apply more sophisticated algorithms for solving the resulting system of algebraic equations. Further works are needed to cover different types of fractional variational problems.

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