LINEAR FRACTIONAL DISCRETE-TIME SYSTEMS

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Abstract: Mathematicians have been discussing about the existence (and the meaning) of derivatives and integrals of fractional order since the beginnings of differential calculus. Various concepts of fractional calculus have been developed and some of them were already applied to dynamical systems. In particular, the author already proposed a way to consider systems defined by linear differential equations of fractional order within the so-called behavioral approach.
In this paper, it is shown how to generalize, analogously, discrete-time linear systems by defining a certain type of difference equations of fractional order. Some of the ideas and techniques which will be used belong to the theory of dynamical systems on time scales.

Keywords: Linear dynamical systems, Behavioral approach, Fractional calculus, Time scales

1. INTRODUCTION

At the end of the 17^{th} century, while differential calculus was still a recent discovery, Leibniz, Bernoulli, L'Hôpital and other mathematicians were already talking about its generalizations. They asked themselves which could be the meaning of a derivative of order $\frac{1}{2}$, whence the name *fractional* calculus (which, nowadays, is a misnomer because the order may be a complex number).

Since then, that idea has been developed, leading to a great variety of definitions of not integer order derivatives and integrals. Indeed, as it usually happens in these cases, there is no unique way of extending the ordinary *integer* calculus. In a certain sense, different solutions were given to different problems.

In particular, the goal of this paper consists in finding a suitable framework for studying linear fractional dynamical systems (i.e., defined by linear fractional equations) using the so-called *behavioral* approach — which will be described in Section 2.

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One possible definition of fractional derivative for continuous-time systems, which has been sketched by the author at the MTNS conference in 2008, will be summed up in Section 3.

However, this contribution is mainly concerned with the analogous problem for discrete-time systems. In this case too, many notions of fractional differences have been introduced. Nevertheless, they all have one common property: they give fractional derivatives as the step length tends to zero.

Having this in mind, some basic ideas of time scales theory, which unifies and extends continuous and discrete analysis, will be presented in Section 4.

Using these concepts, a fractional difference operator will be defined in Section 5, which represents the discrete-time version of the aforementioned fractional differential operator.

2. BEHAVIORS

Linear systems will be defined following the behavioral approach, as introduced by Jan C. Willems at the end of the eighties (Willems, 1986–1987; Willems, 1989; Willems, 1991). In this framework, the following is the main definition.

Definition 1. A dynamical system is given by a *time set* $\mathbb{T}$, a *signal space* $\mathbb{W}$, and a *behavior* $\mathcal{B}$, which is a set of functions, called *trajectories*, having domain $\mathbb{T}$ and codomain $\mathbb{W}$.

In other words, once time and signal sets have been fixed, a dynamical system is characterized by its behavior

$$\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}} = \{w : \mathbb{T} \to \mathbb{W}\}.$$

In the linear case, the signal space is a vector space, usually $\mathbb{W} = \mathbb{R}^q$ with $q \in \mathbb{N}$, and the time space is $\mathbb{T} = \mathbb{R}$ for continuous-time or $\mathbb{T} = \mathbb{Z}$ for discrete-time systems.

The classical theories of dynamical systems deal with mathematical models (input/output or state space, for instance) which correspond to equations that exhibit some particular structure, while the behavioral point of view focuses on their solutions. Therefore, a behavior may be given as a set of solutions of a system of equations, but this is just one of many possible representations.

To be more concrete, a behavior may be defined as the set of functions $w : \mathbb{R} \to \mathbb{R}^q$ which satisfy a system of differential equations like $\sum_{i=0}^{n} R_i w^{(i)}(t) = 0$, where $R_i \in \mathbb{R}^{p\times q}$ are matrix coefficients. By writing the i-th derivative of w as $\frac{\mathrm{d}^i}{\mathrm{d}t^i} w$ instead of $w^{(i)}$, we obtain the equation $\sum_{i=0}^{n} R_i \frac{\mathrm{d}^i}{\mathrm{d}t^i} w(t) = 0$ that expresses w as an element of the kernel of the differential operator $R\big(\frac{\mathrm{d}}{\mathrm{d}t}\big) = \sum_{i=0}^{n} R_i \frac{\mathrm{d}^i}{\mathrm{d}t^i}$. Thus, we obtained

$$\mathcal{B} = \ker R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right), \tag{2.1}$$

which is the so-called *kernel representation* of the behavior $\mathcal{B}$. For example, the state space equation $x' = Ax$ is a kernel representation, being equivalent to $\frac{\mathrm{d}}{\mathrm{d}t}x - Ax = (I\frac{\mathrm{d}}{\mathrm{d}t} - A)x = 0$.

This is a rather general representation, but it is not the only one. A very important concept, which has been introduced in this approach, is the image representation $\mathcal{B} = \operatorname{im} M\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)$, meaning that for any $w \in \mathcal{B}$ there exists some function v such that $w = M\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)v$, where $M\big(\frac{\mathrm{d}}{\mathrm{d}t}\big) = \sum_{i=0}^{m} M_i \frac{\mathrm{d}^i}{\mathrm{d}t^i}$.

One of the goals of the behavioral theory consists in the characterization of analytical properties of $\mathcal{B}$. For example, a behavior is controllable (i.e., the *past* of any trajectory can be linked to the *future* of any other one, obtaining again an element of the behavior) if and only if it admits an image representation.

However, the importance of the theory is a consequence of the following idea: to study the polynomial matrix $R(s) = \sum_{i=0}^{n} R_i s^i$ instead of the differential operator $R\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)$. This makes sense (the product $P(s)R(s)$ corresponds with the composition of $P\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)$ and $R\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)$, since $\frac{\mathrm{d}^i}{\mathrm{d}t^i}\frac{\mathrm{d}^j}{\mathrm{d}t^j} = \frac{\mathrm{d}^{i+j}}{\mathrm{d}t^{i+j}}$) and important conditions may be obtained. For example, it can be proved that $B = \ker R\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)$ is controllable if and only the (constant) matrices $R(\lambda)$ have the same rank for any $\lambda \in \mathbb{C}$.

In this paper, we shall just show how it is possible to extend this setup to systems defined by differential or difference equations of fractional order. Before doing this, we recall some definitions for discrete-time behaviors.

When $\mathbb{T} = \mathbb{Z}$, the most general linear (finite) difference equation is

$$\sum_{i=m}^{n} R_i w(t-i) = 0, \tag{2.2}$$

where $m \le n$ are integers, $w(t) \in \mathbb{R}^q$, with $t \in \mathbb{Z}$, is a sequence of vectors, and R_i are suitable matrix coefficients. In order to transform Equation (2.2) as in the continuous case, the following operator is introduced.

Definition 2. The *backward shift operator* is denoted by σ^τ, with $\tau \in \mathbb{R}$. It acts on a real function w as follows:

$$(\sigma^\tau w)(t) = w(t-\tau), \text{ for any } t \in \mathbb{R}.$$

Thus, Equation (2.2) is equal to $\sum_{i=m}^{n} R_i \sigma^i w(t) = 0$, whence w belongs to the kernel of the difference operator $R(\sigma) = \sum_{i=m}^{n} R_i \sigma^i$. More concisely, as in the continuous case (2.1), we have the kernel representation

$$\mathcal{B} = \ker R(\sigma). \tag{2.3}$$

Observe that, in the discrete case, R may be not polynomial. However, the behavior is time invariant, i.e., $w \in \mathcal{B} \Leftrightarrow \sigma^t w \in \mathcal{B}$ for any $t \in \mathbb{Z}$ and, therefore, $\ker R(\sigma) = \ker R(\sigma)\sigma^t$, allowing to consider, without loss of generality, $m = 0$ in the difference equation. Consequently, the matrix $R(s)$ may be polynomial, with no negative powers of the variable.

In this case too, the polynomial matrix $R(s)$ may be studied instead of the difference operator $R(\sigma)$. Indeed, from an algebraic point of view, the product $P(s)R(s)$ corresponds to the composition of $P(\sigma)$ and $R(\sigma)$, being $\sigma^i\sigma^j = \sigma^{i+j}$.

3. CONTINUOUS-TIME FRACTIONAL SYSTEMS

We briefly review a setup for behaviors defined by fractional order differential equations. In what follows, all the missing necessary hypotheses (e.g. differentiability or integrability of functions) will be tacitly assumed.

The linear operator (definite integration)

$$D_c^{-1} : f \mapsto \int_c^t f(\tau)\,\mathrm{d}\tau$$

can be extended to the n-th iterated integration (Cauchy's formula)

$$D_c^{-n} : f \mapsto \int_c^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau)\, \mathrm{d}\tau. \qquad (3.1)$$

Recalling that $\Gamma(n) = (n-1)!$ for any positive integer n, it is easy to generalize the operator (3.1) to a positive non integer order, getting one of the definitions of fractional integral of real order $\alpha > 0$

$$D_c^{-\alpha} : f \mapsto \int_c^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)\, \mathrm{d}\tau. \qquad (3.2)$$

This is usually applied to functions f defined on the positive real axis, thus giving the operator $D_0^{-\alpha}$ which corresponds to a convolution. Indeed, if we let $d_{-\alpha}(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$ for $t > 0$ and 0 elsewhere, then (3.2) may be written

$$D_0^{-\alpha} f(t) = \int_0^t d_{-\alpha}(t-\tau) f(\tau)\, \mathrm{d}\tau = d_{-\alpha} * f(t).$$

As is well-known, the Laplace transform of $d_{-\alpha}$ is $\mathscr{L}[d_{-\alpha}](s) = s^{-\alpha}$ and so, if $F(s)$ is the Laplace transform of f, then

$$\mathscr{L}[D_0^{-\alpha} f] = \mathscr{L}[d_{-\alpha} * f] = \mathscr{L}[d_{-\alpha}]\mathscr{L}[f] = s^{-\alpha} F(s).$$

As concerns fractional derivatives, i.e., the operator D_c^{α} with $\alpha > 0$, different definitions have been given. If $m-1 < \alpha < m$, with $m \in \mathbb{N}$, then we could consider the Riemann-Liouville derivative $\frac{\mathrm{d}^m}{\mathrm{d}t^m} D_c^{\alpha-m}$ (fractional integration followed by an integer order derivative) or the Caputo derivative $D_c^{\alpha-m} \frac{\mathrm{d}^m}{\mathrm{d}t^m}$ (the order is here reversed). See Gorenflo and Mainardi (1997) for more details.

As a first observation, note that the order of derivation matters: the Caputo derivative of a constant is zero, but the Riemann-Liouville derivative is not! Therefore, we loose the connection with the product of powers of the variable s that holds for integer order derivatives.

Then, the Laplace transform of the order α derivative of f is not $s^{\alpha} F(s)$ (with different expressions for the Riemann-Liouville and the Caputo derivatives).

Third, going back to behaviors, trajectories are usually defined also for negative values — while starting at $t = 0$ is perfectly suitable for the state space model.

Therefore, a different definition was proposed by the author. Consider trajectories $f \in \mathscr{D}_+$, where $\mathscr{D}_+$ is the set of smooth functions with compact support on the left. Then $d_{-\alpha}$ may be thought of as a distribution on $\mathscr{D}_+$, i.e., $d_{-\alpha} \in \mathscr{D}'_+$, by defining the linear functional

$$\int_{-\infty}^{\infty} d_{-\alpha}(-t) f(t)\, \mathrm{d}t.$$

According to the properties of $\mathscr{D}'_+$ (Schwartz, 1966), $d_{-\alpha}$ admits a convolutional inverse $d_{\alpha} \in \mathscr{D}'_+$ such that $d_{-\alpha} * d_{\alpha} = \delta$, which is the Dirac delta.

Thus, fractional integrals and derivatives, acting on $\mathscr{D}_+$, may be defined as follows: as before, for any $\alpha > 0$,

$$D^{-\alpha} f(t) = d_{-\alpha} * f(t) = \int_{-\infty}^{t} d_{-\alpha}(t-\tau) f(\tau)\, \mathrm{d}\tau,$$

where the integral converges, being restricted to a finite interval by the supports of $d_{-\alpha}$ and of f, and

$$D^{\alpha} f(t) = d_{\alpha} * f(t), \text{ with } d_{\alpha} * d_{-\alpha} = \delta.$$

Note that the Laplace transform of the distribution d_{α} is $\mathscr{L}[d_{\alpha}] = s^{\alpha}$, whence it follows that the composition of the *fractional* operators D^{α} and D^{β} corresponds with the product $s^{\alpha} s^{\beta} = s^{\alpha+\beta}$.

Resuming, the fractional equation $\sum_{i=0}^{n} R_i D^{\alpha_i} w(t)$, where $\alpha_i \in \mathbb{R}$ and $w \in \mathscr{D}_+^q$, may be studied through the matrix function $R(s) = \sum_{i=0}^{n} R_i s^{\alpha_i}$.

4. TIME SCALES

In this contribution, the theory of time scales will not be used in its full generality. The interested reader may find more information in the books by Bohner and Peterson (2001) and Bohner and Peterson (2003).

Time scales theory aims at unifying continuous and discrete analysis by defining calculus on any closed set $\mathbb{T} \subseteq \mathbb{R}$, called *time scale*. Obviously, $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$ are just two particular cases — in general, $\mathbb{T}$ may contain both intervals and isolated points.

As usual, there are many concepts of derivative which generalize to time scales the continuous time derivative. In this paper we only make use of the so-called *nabla derivative* (Bohner and Peterson, 2003, Ch. 3), which will be shortly defined. Observe that the Laplace transform is an integral transform, thus depending on the chosen derivative. Therefore, in the next section, we shall give a definition of Laplace transform, which is consistent with the following *derivative* for the time scale $\mathbb{T} = \mathbb{Z}$.

Definition 3. The *backward jump operator* is the function $\rho : \mathbb{T} \to \mathbb{T}$, such that $t \mapsto \sup\{\tau \in \mathbb{T} : \tau < t\}$.

Definition 4. The *nabla derivative* of a function $f : \mathbb{T} \to \mathbb{R}$ at $t \in \mathbb{T}$ is the number $f^{\nabla}(t)$, provided it exists, such that for any $\varepsilon > 0$ there is a neighborhood U of t such that for any $\tau \in U$,

$$|f(\tau) - f(\rho(t)) - f^{\nabla}(t)(\tau - \rho(t))| < \varepsilon |\tau - \rho(t)|.$$

When $\mathbb{T} = \mathbb{R}$, $\rho(t) = t$ and $f^{\nabla}(t) = f'(t)$ is the classical derivative of f. On the other hand, if $\mathbb{T} = \mathbb{Z}$ then $\rho(t) = t-1$ and $f^{\nabla}(t) = \nabla f(t) = f(t) - f(t-1)$ is the backward difference. Using the backward shift operator as in Definition 2, we have $f(t) - f(t-1) = f(t) - \sigma f(t) = (1-\sigma) f(t)$, and so the iterated nabla operator is

$$\nabla^n = (1-\sigma)^n. \qquad (4.1)$$

5. DISCRETE-TIME FRACTIONAL SYSTEMS

Consider now discrete-time systems with time set (scale) $\mathbb{T} = \mathbb{Z}$. As we wrote, the (chosen) derivative of a sequence will be $f^\nabla = \nabla f = (1-\sigma)f$ and therefore, as we showed, derivatives of any *integer* order are given by $\nabla^n = (1-\sigma)^n$.

As for negative powers of ∇, a formula can be given, which is equivalent to the continuous-time scale Cauchy formula (3.1). Before showing it, we have to introduce the necessary concepts and notation.

Definition 5. For any $n \in \mathbb{N}$, let

$$t^{\overline{n}} = t(t+1)(t+2)\cdots(t+n-1),$$

which is called rising factorial or also Pochhammer symbol. The latter is usually denoted by the different notation $t^{\overline{n}} = (t)_n$ (Knuth, 1992).

Theorem 6. (Bohner and Peterson, 2003, Th. 3.99) The expression

$$\nabla_{c+1}^{-n} f(t) = \sum_{\tau=c+1}^{t} \frac{(t-\tau+1)^{\overline{n-1}}}{(n-1)!} f(\tau) \qquad (5.1)$$

is the solution of the discrete-time Cauchy problem $\nabla^n y(t) = f(t)$ with initial conditions $\nabla^i y(c) = 0$, for any $i = 0, \ldots, n-1$.

Since $t^{\overline{n}} = \frac{(t+n-1)!}{(t-1)!} = \frac{\Gamma(t+n)}{\Gamma(t)}$, we may generalize Formula (5.1) to any real order $\alpha > 0$, obtaining the discrete equivalent of the operator $D_c^{-\alpha}$ defined in (3.2):

$$\nabla_c^{-\alpha} f(t) = \sum_{\tau=c}^{t} \frac{(t-\tau+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(\tau). \qquad (5.2)$$

Analogously to the continuous-time case, the (nabla) derivative ∇_c^α of order $\alpha > 0$ has been defined (Atıcı and Eloe, 2009) as $\nabla^m \nabla_c^{\alpha-m}$ where $m \in \mathbb{N}$ and $m-1 < \alpha < m$, using formula (5.2) for $\nabla_c^{\alpha-m}$. Here we choose a different definition, which is similar to the one proposed in Section 3 for the continuous-time case.

Definition 7. Let $\mathcal{S}_+$ be the set of sequences with left compact support, i.e., $f \in \mathcal{S}_+$ if and only if there exists $t_f \in \mathbb{Z}$ such that $f(t) = 0$ for $t < t_f$.

Next, convolution of sequences in $\mathcal{S}_+$ has to be defined.

Definition 8. The convolution of $d, f \in \mathcal{S}_+$ is

$$d * f(t) = \sum_{\tau \in \mathbb{Z}} d(t-\tau+1) f(\tau).$$

Note that convolution is well-defined. Indeed, the sum is always finite, since $d(t-\tau+1)f(\tau) = 0$ when $\tau < t_f$ or $\tau > t+1-t_d$. Moreover, it is commutative, i.e., $d*f = f*d$, as can be easily seen by substituting $n = t-\tau+1$ (that is, $\tau = t-n+1$) in its definition.

As for functions in $\mathcal{D}_+$, also the derivative of sequences in $\mathcal{S}_+$ may be defined by a convolution, leading to a formula similar to (5.2).

Definition 9. Let $\delta_{-\alpha} = \frac{1}{\Gamma(\alpha)} t^{\overline{\alpha-1}}$ for $t > 0$ and zero elsewhere. Then the derivative of order $-\alpha < 0$ of $f \in \mathcal{S}_+$ is

$$\nabla^{-\alpha} f(t) = \delta_{-\alpha} * f(t) = \sum_{\tau \le t} \frac{(t-\tau+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(\tau).$$

For any $\alpha, \beta > 0$, we have that $\delta_{-\alpha} * \delta_{-\beta} = \delta_{-(\alpha+\beta)}$ (Atıcı and Eloe, 2009, Lemma 2.3), thus

$$\nabla^{-\alpha}\nabla^{-\beta} f = \nabla^{-(\alpha+\beta)} f = \nabla^{-\beta}\nabla^{-\alpha} f, \ \forall f \in \mathcal{S}_+.$$

Define $\delta_0(t) = \delta_{1t}$,[1] and observe that it is the unit of convolution, i.e., $\delta_0 * f = f * \delta_0 = f$ for any $f \in \mathcal{S}_+$. Note that, since there is a unit, it is possible to define the convolutional inverse of $f \in \mathcal{S}_+$, being that sequence $g \in \mathcal{S}_+$ such that $f * g = \delta_0$. It is easy to show that every sequence in $\mathcal{S}_+$ has an inverse and so, in this way, we obtain a definition of fractional nabla derivative for any real value.

Definition 10. If $\alpha > 0$, let δ_α be the convolutional inverse of $\delta_{-\alpha}$. Then the nabla derivative of order $\alpha \in \mathbb{R}$ of $f \in \mathcal{S}_+$ is

$$\nabla^\alpha f = \delta_\alpha * f.$$

Note that, unlike the continuous-time case where, for $\alpha > 0$, d_α is a distribution, in the discrete-time case δ_α is a sequence in $\mathcal{S}_+$ for any $\alpha \in \mathbb{R}$.

Let us investigate the use of the Laplace transform which, as we said, depends on the definition of the derivative. In this case, the so-called *nabla Laplace transform* has been introduced in Atıcı and Eloe (2009). Until now, no definition has been given in the literature for the bilateral nabla Laplace transform, which we will need in this context. However, using some general definitions (see, for instance, Bohner and Peterson (2003), Davis *et al.* (2007), and Davis *et al.* (2010)), we propose the following transformation.

Definition 11. The nabla Laplace transform of the sequence $f \in \mathcal{S}_+$ is

$$\mathscr{L}^\nabla[f](s) = \sum_{t \in \mathbb{Z}} (1-s)^{t-1} f(t). \qquad (5.3)$$

Transformation (5.3) is the bilateral version of the $\mathcal{N}$-transform presented in Atıcı and Eloe (2009). Since the interest of the transform in this paper lies in its formal properties, we will not discuss here issues related to its region of convergence.

[1] δ_{ij} is the Kronecker's symbol, equal to 1 when $i = j$ and to 0 when $i \neq j$

Notice that, even if it is not equal to the more common z-transform $\mathcal{Z}[f](z) = \sum_{\tau\in\mathbb{Z}} f(\tau)z^{-\tau}$, the discrete Laplace transform (5.3) is related to it by

$$z^{-1}\mathcal{Z}[f](z^{-1}) = \mathscr{L}^{\nabla}[f](1-z).$$

Some properties of $\mathscr{L}^{\nabla}$ follow.

Theorem 12. Let $f, g \in \mathcal{S}_+$. Then

(1) $\mathscr{L}^{\nabla}[\delta_{-\alpha}](s) = s^{-\alpha}$, $\alpha \geq 0$;
(2) $\mathscr{L}^{\nabla}[f * g](s) = \mathscr{L}^{\nabla}[f](s) \cdot \mathscr{L}^{\nabla}[g](s)$;
(3) $\mathscr{L}^{\nabla}[\sigma^{\tau} f](s) = (1-s)^{\tau}\mathscr{L}^{\nabla}[f](s)$, $\tau \in \mathbb{Z}$.

Proof. Property 1) is proved in (Atıcı and Eloe, 2009, Lemma 3.1) for $\alpha > 0$ and, for $\alpha = 0$,

$$\mathscr{L}^{\nabla}[\delta_0](s) = \sum_{t\in\mathbb{Z}}(1-s)^{t-1}\delta_0(t) = (1-s)^0 = 1.$$

Properties 2) and 3) hold true, since

$$\begin{aligned}\mathscr{L}^{\nabla}[f * g](s) &= \sum_{t\in\mathbb{Z}}(1-s)^{t-1}\sum_{\tau\in\mathbb{Z}} f(t-\tau+1)g(\tau)\\ &= \sum_{n\in\mathbb{Z}}(1-s)^{n-1}f(n)\sum_{\tau\in\mathbb{Z}}(1-s)^{\tau-1}g(\tau)\\ &= \mathscr{L}^{\nabla}[f](s)\cdot\mathscr{L}^{\nabla}[g](s),\end{aligned}$$

where $n = t - \tau + 1$, and

$$\begin{aligned}\mathscr{L}^{\nabla}[\sigma^{\tau} f](s) &= \sum_{t\in\mathbb{Z}}(1-s)^{t-1}f(t-\tau)\\ &= \sum_{n\in\mathbb{Z}}(1-s)^{n+\tau-1}f(n)\\ &= (1-s)^{\tau}\mathscr{L}^{\nabla}[f](s),\end{aligned}$$

where $n = t - \tau$. □

Corollary 13. $\mathscr{L}^{\nabla}[\delta_\alpha](s) = s^{\alpha}$ for every $\alpha \in \mathbb{R}$.

Proof. By Theorem 12, the claimed property is true for $\alpha \leq 0$ and $\mathscr{L}^{\nabla}[\delta_\alpha * \delta_{-\alpha}](s) = \mathscr{L}[\delta_0] = 1$. So, $\mathscr{L}^{\nabla}[\delta_\alpha] = \left(\mathscr{L}^{\nabla}[\delta_{-\alpha}]\right)^{-1} = s^{\alpha}$ for $\alpha > 0$. □

Remark 14. Definition 9 could be given for any α which is not a negative integer, i.e., $-\alpha \notin \mathbb{N}_0$ and also the proof of Theorem 12(1), in (Atıcı and Eloe, 2009), is given for these values of α. What is missing, in these cases, is exactly the definition of δ_n with $n \in \mathbb{N}_0$, which gives the integer order derivative $\nabla^n f = \delta_n * f$.

However, besides being the convolutional inverse of δ_{-n} as in Definition 10, we will show that δ_n may be seen as the limit of δ_α as $\alpha \to n$. First of all, note that, by (4.1),

$$\begin{aligned}\nabla^n f(t) &= (1-\sigma)^n f(t) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\sigma^k f(t)\\ &= \sum_{\tau=1}^{n+1}\binom{n}{\tau-1}(-1)^{\tau-1}f(t-\tau+1)\\ &= \sum_{\tau=1}^{n+1}\frac{n!}{(n+1-\tau)!\Gamma(\tau)}(-1)^{\tau-1}f(t-\tau+1).\end{aligned}$$

So, by Definition 8, $\delta_n(t) = \frac{n!}{(n+1-t)!\Gamma(t)}(-1)^{t-1}$ when $t = 1, \dots, n+1$ and zero elsewhere. By Definition 9, also $\delta_\alpha(t)$ is zero for $t \leq 0$. Therefore, we just have to calculate its limit for $t > 0$

Remember (Abramowitz and Stegun, 1984, Formula 6.1.3) that $\Gamma(s)$ has simple poles at $s = -n$, where $n \in \mathbb{N}_0$, with residue

$$\operatorname{Res}\Gamma(s)\Big|_{s=-n} = \frac{(-1)^n}{n!}.$$

So, the limit of $\delta_\alpha(t) = \frac{1}{\Gamma(-\alpha)}t^{\overline{-\alpha-1}} = \frac{\Gamma(t-\alpha-1)}{\Gamma(-\alpha)\Gamma(t)}$ as $\alpha \to n \in \mathbb{N}_0$ is zero when $t > n+1$, because only $\Gamma(-\alpha)$, in the denominator, tends to infinity.

On the contrary, when $t = 1, \dots, n+1$ the limit of $\delta_\alpha(t)$ is finite and, as expected, is equal to

$$\begin{aligned}\delta_\alpha(t) = \frac{\Gamma(t-\alpha-1)}{\Gamma(-\alpha)\Gamma(t)} &\to \frac{(-1)^{t-n-1}}{(n+1-t)!}\cdot\frac{n!}{(-1)^n}\cdot\frac{1}{\Gamma(t)}\\ &= \frac{n!}{(n+1-t)!\Gamma(t)}(-1)^{t-1}\\ &= \delta_n(t),\end{aligned}$$

where we considered both the residues of the numerator and of the denominator.

Finally, the main result is obtained.

Theorem 15. For any $f \in \mathcal{S}_+$ and $\alpha \in \mathbb{R}$,

$$\mathscr{L}^{\nabla}[\nabla^{\alpha} f](s) = s^{\alpha}\mathscr{L}^{\nabla}[f](s).$$

Proof. By Corollary 13,

$$\mathscr{L}^{\nabla}[\nabla^{\alpha} f](s) = \mathscr{L}^{\nabla}[\delta_\alpha * f](s) = s^{\alpha}\mathscr{L}^{\nabla}[f](s),$$

for any $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}_+$. □

So, since the derivative $\nabla^{\alpha} f$, with $\alpha \in \mathbb{R}$, can be represented by a product of the nabla Laplace transform of the sequence $f \in \mathcal{S}_+$ with s^{α}, the power rule

$$\nabla^{\alpha}\nabla^{\beta} f = \nabla^{\alpha+\beta} f = \nabla^{\beta}\nabla^{\alpha} f.$$

holds for any $\alpha, \beta \in \mathbb{R}$.

6. CONCLUSIONS

Summing up, we showed that a behavior containing all the sequences $w \in \mathcal{S}_+^q$ which satisfy a fractional order difference equation $\sum_{i=0}^{n} R_i\nabla^{\alpha_i} w(t) = 0$, with $R_i \in \mathbb{R}^{p\times q}$ and $\alpha_i \in \mathbb{R}$, can be associated, as in the continuous-time case, to the matrix-valued function $R(s) = \sum_{i=0}^{n} R_i s^{\alpha_i}$.

As we saw, $R(s)$ is the nabla Laplace transform of some matrix sequence $R(t) \in \mathcal{S}_+^{p\times q}$. Therefore, the behavior $\mathcal{B}$ can be seen as the solution set of the convolutional equation $R * w(t) = 0$.

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