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A constructive algorithm for determination of immobile indices in convex SIP problems with polyhedral index sets *

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Abstract

We consider convex Semi-Infinite Programming (SIP) problems with polyhedral index sets. For these problems, we generalize the concepts of *immobile indices* and their *immobility orders* (see [5]-[8]) that are objective and important characteristics of the feasible sets permitting to formulate new efficient optimality conditions.

We describe and justify a finite constructive algorithm (DIIPS algorithm) that determines immobile indices and their immobility orders along the feasible directions. This algorithm is based on a representation of the cones of feasible directions of polyhedral index sets in the form of linear combinations of the extremal rays and on the approach described in [5]- [8] for the cases of multidimensional immobile sets of more simple structure. A constructive procedure of determination of the extremal rays is described and an example illustrating the application of the DIIPS algorithm is provided.

Key words. Semi-Infinite Programming (SIP), Convex Programming (CP), immobile index, immobility order, cone of feasible directions, extremal ray.
AMS subject classification. 90C25, 90C30, 90C34

1 Introduction

Semi-Infinite Programming (SIP) deals with extremal problems that involve infinitely many constraints in a finite dimensional space. Due to numerous theoretical and practical applications, today semi-infinite optimization is a topic of a special interest (see [3], [4], and the references therein). The most efficient methods for solving optimization problems are usually based on optimality conditions that permit not only to test the optimality of a given feasible solution, but also to find the better direction to optimality. Usually the optimality conditions are formulated for certain classes of optimization problems that permit to use more efficiently specific structure of problems under consideration.
In the paper, we consider convex Semi-Infinite Programming (SIP) problems with polyhedral index sets. For these problems, we generalize the concepts of *immobile indices* and their *immobility orders* (see [5]-[8]) that are objective and important characteristics of the feasible sets permitting to formulate new efficient optimality conditions.

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The main aim of the paper is to describe and justify a finite constructive algorithm (DIIPS algorithm) that determines immobile indices and their immobility orders along the feasible directions. This algorithm is based on representation of the cones of feasible directions of polyhedral index sets in the form of linear combinations of the extremal rays and on the approach described in [5]- [8] for the cases of multidimensional immobile sets of more simple structure. A constructive procedure of determination of the extremal rays is described and an example illustrating the application of the DIIPS algorithm is provided.

2 Immobile Indices and immobility orders in SIP problems with polyhedral index sets

Consider a convex Semi-Infinite Programming problem in the form

$$(P): \qquad \min_{x\in\mathbb{R}^n} c(x) \tag{1}$$

$$\text{s.t.} \quad f(x,t) \le 0 \ \ \forall\, t \in T = \{t \in \mathbb{R}^s : \ h_k^T t \le \Delta h_k, \ k \in K\}, \tag{2}$$

where K is a finite index set, the constraint function $f(x,t), t \in T$, is linear w.r.t. $x \in \mathbb{R}^n$; vectors $h_k \in \mathbb{R}^s$ and numbers $\Delta h_k, \ k \in K$ are given. Notice that here the index set T is a convex polyhedron.
Denote by X the feasible set of problem (P):

$$X = \{x \in \mathbb{R}^n : \ f(x,t) \le 0, \forall t \in T\}. \tag{3}$$

Given $t \in T$, denote by $K_a(t) \subset K$ the set of active at t indices:

$$K_a(t) := \{k \in K : \ h_k^T t = \Delta h_k\}, \tag{4}$$

and by $L(t)$ the set of *feasible directions* in the set T corresponding to t:

$$L(t) := \{l \in \mathbb{R}^s : \ h_k^T l \le 0, \ k \in K_a(t)\}. \tag{5}$$

Given $x \in X$, let $T_a(x) \subset T$ denote the set of active at x indices:

$$T_a(x) := \{t \in T : \ f(x,t) = 0\}. \tag{6}$$

Definition 1 *Let us say that an index $\bar{t} \in T$ is immobile in problem (P) if*

$$f(x,\bar{t}) = 0 \text{ for all } x \in X. \tag{7}$$

Denote by T^* the *set of all immobile indices* in (P). It is evident that $T^* \subset T_a(x)$ for all $x \in X$.

In the papers [5] and [8], the immobile indices were defined for the convex SIP problems with box-constrained one- and multidimensional index sets respectively, being used later (see also [6]) to obtain new efficient CQ-free optimality conditions. In what follows, we generalize this approach to the more general case of convex SIP problems with polyhedral index sets.
We can characterize each immobile index with the help of a special quantitative characteristic called order of immobility or immobility order. In the multidimensional case, the immobility orders are defined w.r.t. feasible directions in T.

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Definition 2 *Given an immobile index $\bar{t} \in T^*$ and a nontrivial feasible direction $\bar{l} \in L(\bar{t})$, let us say that $\bar{t}$ has the immobility order $q(\bar{t}, \bar{l})$ along $\bar{l}$ if*

1. $\left.\frac{d^i f(x,\bar{t}+\alpha\bar{l})}{d\alpha^i}\right|_{\alpha=+0} = 0, \forall x \in X, i = 0, 1, \ldots, mq(\bar{t}, \bar{l}),$
2. *there exists a feasible* $\bar{x} = x(\bar{t}, \bar{l}) \in X$ *such that* $\left.\frac{d^{(q(\bar{t},\bar{l})+1)} f(\bar{x},\bar{t}+\alpha\bar{l})}{d\alpha^{(q(\bar{t},\bar{l})+1)}}\right|_{\alpha=+0} \neq 0.$

Here and in what follows we consider the set $\{i = s, s+1, ..., k\}$ to be empty if $k < s$, and $\left.\frac{d^0 f(x,\bar{t}+\alpha\bar{l})}{d\alpha^0}\right|_{\alpha=+0} = f(x, \bar{t})$.

3 Alternative representation of the sets of feasible directions in polyhedral index sets

Given the convex SIP problem (P), consider an index $\bar{t} \in T$. Here we will give another description of the set (5) of feasible directions in $\bar{t}$.
Denote $\bar{L} := L(\bar{t})$. Consider the set $\bar{K} := \bar{K}_a(\bar{t})$ defined at (4). In this section, we will present the set $\bar{L}$ as a linear combination of some vector sets in $\mathbb{R}^s$ and show how these vector sets can be obtained.

3.1 Properties of the set $\bar{L}$.

It is easy to verify that $\bar{L}$ is a cone in $\mathbb{R}^s$.
Consider the set $\Delta\bar{L} \subset \mathbb{R}^s$ defined as follows:

$$\Delta\bar{L} = \{l \in \mathbb{R}^s : \; h_k^T l = 0, \; k \in \bar{K}\}. \tag{8}$$

Evidently, $\Delta\bar{L} = \{0\}$ for $m = s$ and $\Delta\bar{L}$ is a subspace of $\mathbb{R}^s$ for $m < s$ where

$$m = rank(h_k, \; k \in \bar{K}). \tag{9}$$

Set $p = s - m$ and denote by

$$\{b_i, \; i = 1, \ldots, p\} \tag{10}$$

a basis of $\Delta\bar{L}$. Consider the set $\Delta\bar{\bar{L}} = \bar{L} \bigcap \Delta\bar{L}^{\perp}$, where $\Delta\bar{L}^{\perp}$ is the orthogonal complement of the subspace $\Delta\bar{L}$ in $\mathbb{R}^s$.
One can easily check that the set $\Delta\bar{\bar{L}}$ is a pointed cone, i.e. it is the cone with the following property:

$$\text{for any } l \neq 0 : \quad l \in \Delta\bar{\bar{L}} \quad \Rightarrow \quad -l \notin \Delta\bar{\bar{L}}.$$

Then there exists a finite set of vectors

$$a_i \in \Delta\bar{\bar{L}}, i \in I, \tag{11}$$

such that the cone $\bar{L}$ can be represented in the form

$$\bar{L} = \{l \in \mathbb{R}^s : \; l = \sum_{i=1}^{p} \beta_i b_i + \sum_{i \in I} \alpha_i a_i, \; \alpha_i \geq 0, \; i \in I\}, \tag{12}$$

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where vectors $b_i,\ i \in \{1,\dots,p\}$ are defined in (10) and $\alpha_i \in \mathbb{R}$, $i \in I$, $\beta_i \in \mathbb{R}, i \in \{1,\dots,p\}$. Therefore we have shown that for any $\bar{t} \in T$ there exist (finite) sets of vectors (10) and (11) such that the set of feasible directions in $\bar{t}$ can be represented in the form (12).
Vectors (10), (11) are usually referred to as *extremal rays*, vectors (10) being *bidirectional* and vectors (11) being *unidirectional* rays.

3.2 The rules for constructing the extremal rays

In the literature, different algorithms for presentation of polyhedral cones can be find (see for example, [1] and [2]). Here we describe one more procedure that can be used to find the sets of vectors (10) and (11) and therefore to describe explicitly the set $\bar{L}$.
Given $k \in \bar{K}$, denote by $h_{ki},\ i \in S := \{1, 2, \dots, s\}$ the components of the vector h_k:

$$h_k^T = (h_{ki},\ i \in S)$$

and by H the $|\bar{K}| \times |S|-$ matrix

$$H = \begin{pmatrix} h_{ki}, & i \in S \\ & k \in \bar{K} \end{pmatrix}.$$

Consider subsets $S_0 \subset S$ and $N_0 \subset \bar{K}$ such that $|S_0| = |N_0| = m$ and the matrix

$$H_0 = H(N_0, S_0) = \begin{pmatrix} h_{ki}, & i \in S_0 \\ & k \in N_0 \end{pmatrix} \tag{13}$$

is not singular: $\det(H_0) \neq 0$. By construction, H_0 is a square sub-matrix of the matrix H of the same rank: $rank\ H = rank\ H_0 = m$.
Construct vectors

$$\bar{b}_i = (\bar{b}_{ij}, j \in S),\ i \in S \backslash S_0, \tag{14}$$

whose components are as follows:

$$\begin{gathered} \bar{b}_{ij} = 0,\ j \in S \backslash (S_0 \bigcup i),\ \bar{b}_{ii} = 1, \\ (\bar{b}_{ij}, j \in S_0)^T = -H_0^{-1} \begin{pmatrix} h_{ki} \\ k \in N_0 \end{pmatrix},\ i \in S \backslash S_0. \end{gathered} \tag{15}$$

It is easy to verify that these vectors form a basis of the subspace $KerH = \Delta\bar{L}$. Therefore we can set

$$\{b_i, i = 1, \dots, p\} := \{\bar{b}_i,\ i \in S \backslash S_0\}. \tag{16}$$

Let $h_0 = \sum\limits_{k \in \bar{K}} h_k$. If $h_0 = 0 \in \mathbb{R}^s$, then the set of vectors (11) is empty.
Suppose that $h_0 \neq 0$. Denote by Ω the set of subsets $N_* \subset \bar{K}$ such that $|N_*| = m - 1$ and $det(D(N_*)) \neq 0$, where $D(N_*) = (h_0,\ h_k,\ k \in N_*;\ b_i, i = 1, \dots, p)^T \in \mathbb{R}^{s \times s}$.
Given $N_* \in \Omega$, let $a(N_*)$ be the first column of the matrix $-D^{-1}(N_*)$, i.e. $a(N_*) = -D^{-1}(N_*)e_1$. Set

$$\Omega_* := \{N_* {\in} \Omega :\ h_k^T a(N_*) \le 0,\ k \in \bar{K} \backslash N_*\}.$$

It can be easily verified that the set

$$\{a_i, i \in I\} := \{a(N_*),\ N_* {\in} \Omega_*\} \tag{17}$$

is a set of vectors defined in (11), (12).

Remark 1 *From the constructions above it follows that in the case* $m = |S_0| = |\bar{K}|$*, we have* $I = \{1, \dots, m\}$*, and it is easy to construct vectors* $a_i = (a_{ij},\ j \in S),\ i \in I$*:*

$$a_{ij} = 0,\ j \in S \backslash S_0; \quad (a_{ij}, j \in S_0)^T = -H_0^{-1} e_i,\ i = 1, \dots, m,$$

where $e_i \in \mathbb{R}^m$ *is the i-th vector of the canonic basis of* $\mathbb{R}^m$*; and the matrix* H_0 *is given in (13).*

Remark 2 *In the case* $m = |S| = s$*, the set of vectors* $b_i, i = 1, \dots, p$ *is empty since* $p = 0$*.*

Remark 3 *As noted above, the set of vectors* $a_i,\ i \in I$*, is empty* ($I = \emptyset$) *when* $h_0 = 0$*. Notice here that* $h_0 \neq 0$ *when the interior of the polyhedral index set* T *is not empty, i.e. the constraints defining* T*, satisfy the Slater condition:* $\exists \tilde{t} \in T :\ h_k^T \tilde{t} < \Delta h_k,\ \forall k \in K$*.*

4 Determination of immobile indices and their immobility orders

4.1 Assumptions and notations

Assumption 1 *Suppose that* $X \neq \emptyset$*, the set* T *is bounded and*

$$q(t, l) \leq 1,\ \forall t \in T^*,\ \forall l \in L(t) \setminus \{0\}. \tag{18}$$

We consider that conditions (18) are trivially fulfilled if $T^* = \emptyset$.

Proposition 1 *Assumption 1 implies that the set of immobile indices* T^* *consists of a finite number of elements:*

$$T^* = \{t_j^*,\ j \in J_*\} \text{ with some finite index set } J_*$$

and there exists $\bar{x} \in X$ *such that* $|T_a(\bar{x})| < \infty$*.*

Proof. The proof of the proposition is similar to proof of Lemma 2.1 from [7].

Remark 4 *The condition of boundeness of the index set* T *was introduced into Assumption 1 with the only purpose to prove Proposition 1. Notice here that instead of this condition we can suppose that the set* T^* *is finite and there exists* $\bar{x} \in X$ *such that* $|T_a(\bar{x})| < \infty$*.*

Here and in what follows, given $t \in T$, the set of feasible directions $L(t)$ is defined as in (5).

Given vector $\bar{x} \in X$ such that $|T_a(\bar{x})| < \infty$, the set of active at $\bar{x}$ indices has the form

$$T_a(\bar{x}) = \{\bar{t}_j,\ j \in \bar{J}\} \text{ with } |\bar{J}| < \infty. \tag{19}$$

For any $\bar{t}_j,\ j \in \bar{J}$, let us find the corresponding extremal rays of the cone of feasible directions defined in (10), (11),

$$b_i(j), i = 1, \dots, p_j; \quad a_i(j), i \in I(j), j \in \bar{J}, \tag{20}$$

according to the rules described in the previous section.
Set

$$\tilde{I}(j) = \{i \in I(j) :\ \frac{\partial^T f(\bar{x}, \bar{t}_j)}{\partial t} a_i(j) = 0\},\ j \in \bar{J}. \tag{21}$$

Notice that $T^* \subset T_a(\bar{x}) = \{\bar{t}_j,\ j \in \bar{J}\}$.
Now we can describe the algorithm that, for problem (P), determines the set of immobile indices and immobility orders along their extremal rays under the assumptions made in this section. We call this algorithm DIIPS since it determines the set of immobile indices in (P) with polyhedral index set.

4.2 Algorithm DIIPS

Suppose that for a given feasible $\bar{x}$ in problem (P), the corresponding index set $\{\bar{t}_j,\ j \in \bar{J}\}$ (see (19)), vectors (20) and the index sets (21) are known.

Initializing. Set $J_*^{(0)} := \emptyset$, $k = 0$.

General iteration. We start the $(k+1)$-th iteration of the algorithm ($k \geq 0$) having the following sets constructed on the previous iteration:

$$J_*^{(k)} \subset \bar{J},\ I_0^{(k)}(j) \subset \tilde{I}(j),\ j \in J_*^{(k)}.$$

Notice that at the first iteration ($k = 0$) we do not use the sets $I_0^{(k)}(j) \subset \tilde{I}(j),\ j \in J_*^{(k)}$, since the set $J_*^{(0)}$ is empty.

Given $j \in J_*^{(k)}$, find

$$L_j^{(k)} := \{l \in \mathbb{R}^s :\ l = B(j)\beta_j + A_0^{(k)}(j)\alpha_j^{(k)},\ \alpha_j^{(k)} \geq 0, l^T \frac{\partial^2 f(\bar{x}, \bar{t}_j)}{\partial t^2} l = 0, ||l|| = 1\}, \tag{22}$$

where

$$B(j) = (b_i(j), i = 1, ..., p_j),\ \beta_j \in \mathbb{R}^{p_j};\ A_0^{(k)}(j) = (a_i(j), i \in I_0^{(k)}(j)),\ \alpha_j^{(k)} \in \mathbb{R}_+^{|I_0^{(k)}(j)|}, \tag{23}$$

and construct the following set:

$$X^{(k+1)} := \{x \in \mathbb{R}^n :\ f(x, \bar{t}_j) \leq 0, j \in \bar{J} \setminus J_*^{(k)};\ f(x, \bar{t}_j) = 0,\ \frac{\partial^T f(x, \bar{t}_j)}{\partial t} b_i(j) = 0,\ i = 1, \dots, p_j;$$

$$\frac{\partial^T f(x, \bar{t}_j)}{\partial t} a_i(j) \begin{cases} = 0, & \text{for } i \in I_0^{(k)}(j) \\ \leq 0, & \text{for } i \in \tilde{I}(j) \backslash I_0^{(k)}(j) \end{cases},\quad l^T \frac{\partial^2 f(x, \bar{t}_j)}{\partial t^2} l \leq 0,\ \forall l \in L_j^{(k)},\ j \in J_*^{(k)}\}.$$

It can be shown that $\bar{x} \in X^{(k+1)}$.

For all $j \in \bar{J} \backslash J_*^{(k)}$, solve the auxiliary problem:

$$\min f(x, \bar{t}_j), \quad \text{s.t. } x \in X^{(k+1)}. \tag{$Aux1$}$$

Set $x^{(j)} := \bar{x}$
if $\bar{x}$ is optimal in this problem; otherwise let $x^{(j)}$ be any vector satisfying the following conditions: $x^{(j)} \in X^{(k+1)},\ f(x^{(j)}, \bar{t}_j) < 0$.
Set $\Delta J_*^{(k+1)} := \{j \in \bar{J} \backslash J_*^{(k)} :\ f(x^{(j)}, \bar{t}_j) = 0\}$.

For all $i \in \tilde{I}(j) \backslash I_0^{(k)}(j),\ j \in J_*^{(k)}$, solve the following auxiliary problem:

$$\min \frac{\partial f^T(x, \bar{t}_j)}{\partial t} a_i(j), \quad \text{s.t. } x \in X^{(k+1)}. \tag{$Aux2$}$$

Set $x^{(ij)} := \bar{x}$ if vector $\bar{x}$ is optimal in problem $(Aux2)$, otherwise choose any vector $x^{(ij)} \in X^{(k+1)}$ such that $(\frac{\partial f(x^{(ij)}, \bar{t}_j)}{\partial t})^T a_i(j) < 0$.

Set $\Delta I_0^{(k+1)}(j) = \{i \in \tilde{I}(j) \backslash I_0^{(k)}(j) : \ \frac{\partial f^T(x^{(ij)}, \bar{t}_j)}{\partial t} a_i(j) = 0\}, \ j \in J_*^{(k)}$.
If $\Delta J_*^{(k+1)} = \emptyset$ and $\Delta I_0^{(k+1)}(j) = \emptyset, \ \forall j \in J_*^{(k)}$, then the algorithm stops with

$$T^* = \{t_j^* := \bar{t}_j, j \in J_* := J_*^{(k)}\}, \tag{24}$$

and

$$q(t_j^*, a_i(j)) = 1, j \in I_0(j) := I_0^{(k)}(j); \ \ q(t_j^*, a_i(j)) = 0, i \in I_*(j) = I(j) \setminus I_0^{(k)}(j), \ j \in J_*. \tag{25}$$

Otherwise (if at least one of the sets $\Delta J_*^{(k+1)}$ and $\Delta I_0^{(k+1)}(j)$ is not empty), we set

$$J_*^{(k+1)} := J_*^{(k)} \bigcup \Delta J_*^{(k+1)},$$

$$I_0^{(k+1)}(j) := I_0^{(k)}(j) \bigcup \Delta I_0^{(k+1)}(j) \text{ for } \ j \in J_*^{(k)}, \ \text{ and}$$

$$I_0^{(k+1)}(j) := \emptyset \text{ for } \ j \in \Delta J_*^{(k+1)},$$

and pass to the next iteration.

The algorithm is described.

4.3 Justification of the algorithm DIIPS

Suppose that we apply the algorithm DIIPS to the convex SIP problem (P) that satisfies Assumption 1.

First of all, notice that it is evident that the algorithm should stop in a finite number of iterations.
Suppose that the algorithm has stopped on the $(k+1)$-th iteration. Then we have the sets $J_*^{(k)} \subset \bar{J}, \ I_0^{(k)}(j) \subset \tilde{I}(j), j \in J_*^{(k)}$, and vectors $x^{(j)} \in X^{(k+1)}, j \in \bar{J} \backslash J_*^{(k)}, \ \ x^{(ij)} \in X^{(k+1)}, \ i \in \tilde{I}(j) \backslash I_0^{(k)}(j), j \in J_*^{(k)}$ such that

$$f(x^{(j)}, \bar{t}_j) < 0, \ j \in \bar{J} \backslash J_*^{(k)}, \ \ \frac{\partial f^T(x^{(ij)}, \bar{t}_j)}{\partial t} a_i(j) < 0, \ i \in \tilde{I}(j) \backslash I_0^{(k)}(j), \ j \in J_*^{(k)}.$$

Since the function $f(x,t)$ is linear w.r.t. x and the set $X^{(k+1)}$ is convex, there exists $\hat{x} \in X^{(k+1)}$ satisfying

$$f(\hat{x}, \bar{t}_j) < 0, j \in \bar{J} \backslash J_*^{(k)}, \ \frac{\partial f^T(\hat{x}, \bar{t}_j)}{\partial t} a_i(j) < 0, i \in \tilde{I}(j) \backslash I_0^{(k)}(j), j \in J_*^{(k)}. \tag{26}$$

It follows from the Algorithm that $\{\bar{t}_j, \ j \in J_*^{(k)}\} \subset T^*$ and $q(\bar{t}_j, l) > 0$ for

$$l = B(j)\beta_j + A_0^{(k)}(j)\alpha_{0j}^{(k)} = (B(j), \ A_0^{(k)}(j)) \begin{pmatrix} \beta_j \\ \alpha_{0j}^{(k)} \end{pmatrix} \neq 0, \ \alpha_{0j}^{(k)} \geq 0, j \in J_*^{(k)},$$

where $A_0^{(k)}(j)$, $B(j)$, $\alpha_{0j}^{(k)}$, β_j are defined in (23).
Hence from Assumption 1, it follows

$$q(l, \bar{t}_j) = 1 \ \text{ for } \ l = (B(j), \ A_0^{(k)}(j)) \begin{pmatrix} \beta_j \\ \alpha_{0j}^{(k)} \end{pmatrix} \neq 0, \ \alpha_{0j}^{(k)} \geq 0, j \in J_*^{(k)}. \tag{27}$$

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Lemma 1 *Let $j \in J_*^{(k)}$. Then there exists $x^{*j} \in X$ such that*

$$(\mathcal{A}\tilde{\beta})^T \frac{\partial^2 f(x^{*j}, \bar{t}_j)}{\partial t^2} \mathcal{A}\tilde{\beta} < 0 \textit{ for all } \tilde{\beta} = \tilde{\beta}(j) = (\beta_j, \alpha_{0j}^{(k)})^T, \alpha_{0j}^{(k)} \geq 0, \ ||\tilde{\beta}|| = 1.$$

where $\mathcal{A} = \mathcal{A}(j) := (B(j), \ A_0^{(k)}(j))$.

Proof. Define the following function:

$$F(x, \tilde{\beta}) := (\mathcal{A}\tilde{\beta})^T \frac{\partial^2 f(x, \bar{t}_j)}{\partial t^2} \mathcal{A}\tilde{\beta} \leq 0, \ \tilde{\beta} \in \mathbf{B}, x \in X, \tag{28}$$

where $\mathbf{B} := \{\tilde{\beta} = (\beta_j, \alpha_{0j}^{(k)})^T, \ \alpha_{0j}^{(k)} \geq 0, \ ||\tilde{\beta}|| = 1\}$.
By construction, $X \subset \mathbb{R}^n$ is a convex set, $\mathbf{B}$ is a compact, and the function $F(x, \tilde{\beta})$ is convex w.r.t. x.
Consider any set of vectors

$$\{\tilde{\beta}^r : \ \tilde{\beta}^r \in \mathbf{B}, \ r = 1, \ldots, n+1\}. \tag{29}$$

Remind that according to (27) we have $q(\bar{t}_j, l) = 1$, for all $l = \mathcal{A}\tilde{\beta} \neq 0$, $\tilde{\beta} \in \mathbf{B}$. Then, by Definition 2, for each $w = 1, \ldots, n+1$, there exists $x^{(w)} \in X$, satisfying inequality $F(x^{(w)}, \tilde{\beta}^w) < 0$.
From the condition $x^{(w)} \in X$, it follows that

$$F(x^{(w)}, \tilde{\beta}^r) \leq 0, \forall r \neq w, r = 1, \ldots, n+1.$$

Set $\bar{\bar{x}} = \frac{1}{n+1} \sum_{i=1}^{n+1} x^{(i)}$. It is easy to check that

$$\bar{\bar{x}} \in X \text{ and } F(\bar{\bar{x}}, \tilde{\beta}^r) < 0, \ \forall r = 1, \ldots, n+1. \tag{30}$$

Therefore we have showed that for any set (29) there exists vector $\bar{\bar{x}}$ satisfying (30).
Hence, according to Proposition 3 from [5], for the given $j \in J_*^{(k)}$, there exists $x^{*j} \in X$ such that $F(x^{*j}, \tilde{\beta}) < 0, \forall \tilde{\beta} \in \mathbf{B}$, i.e.

$$\exists x^{*j} \in X : \ \tilde{\beta}^T \mathcal{A}^T \frac{\partial^2 f(x^{*j}, \bar{t}_j)}{\partial t^2} \mathcal{A}\tilde{\beta} < 0 \text{ for all } \tilde{\beta} = (\beta_j, \alpha_{0j}^{(k)}) \text{ such that } ||\tilde{\beta}|| = 1, \alpha_{0j}^{(k)} \geq 0.$$

∎

Consider vector $\tilde{x}^* = \sum_{j \in J_*^{(k)}} \frac{x^{*j}}{|J_*^{(k)}|}$ where $x^{*j} \in X$, $j \in J_*^{(k)}$, are the vectors considered in Lemma 1. Then $\tilde{x}^*$ satisfies the following conditions:

$$\tilde{x}^* \in X, \quad l^T \frac{\partial^2 f(\tilde{x}^*, \bar{t}_j)}{\partial t^2} l < 0, \ \forall l = B(j)\beta_j + A_0^{(k)}(j)\alpha_{0j}^{(k)} \neq 0, \alpha_{0j}^{(k)} \geq 0, j \in J_*^{(k)}.$$

Moreover, we know that given an immobile index $\bar{t}_j$, $j \in J_* = J_*^{(k)}$, for any $x \in X$, it holds

$$f(x, \bar{t}_j) = 0, \quad \frac{\partial f(x, \bar{t}_j)}{\partial t} b_i(j) = 0, i = 1, \ldots, p_j; \quad \frac{\partial f(x, \bar{t}_j)}{\partial t} a_i(j) = 0, i \in I_0^{(k)}(j), j \in J_*^{(k)}.$$

Then evidently, for the vector $\tilde{x}^*$ constructed above, the following relations take a place:

$$\begin{array}{l} f(\tilde{x}^*,\bar{t}_j)=0,\ \ \dfrac{\partial f(\tilde{x}^*,\bar{t}_j)}{\partial t}b_i(j)=0,\ i=1,\dots,p_j;\\ \dfrac{\partial f(\tilde{x}^*,\bar{t}_j)}{\partial t}a_i(j)=0, i\in I_0^{(k)}(j),\ \ \dfrac{\partial f(\tilde{x}^*,\bar{t}_j)}{\partial t}a_i(j)\le 0, i\in I(j)\setminus I_0^{(k)}(j), j\in J_*^{(k)}.\end{array}$$

Consider vector $z=\frac{1}{2}(\tilde{x}^*+\bar{x})\in X$, where $\bar{x}$ is the vector introduced in section 4.1. Then by construction

$$f(z,\bar{t}_j)\le 0,\ j\in\bar{J}\setminus J_*^{(k)},\ \ f(z,\bar{t}_j)=0,\ j\in J_*^{(k)};$$

$$\frac{\partial f(z,\bar{t}_j)}{\partial t}b_i(j)=0, i=1,\dots,p_j;\ \ \frac{\partial f(z,\bar{t}_j)}{\partial t}a_i(j)\begin{cases}<0, & i\in I(j)\setminus\tilde{I}(j)\\ \le 0, & i\in\tilde{I}(j);\end{cases}$$

$$l^T\frac{\partial^2 f(z,\bar{t}_j)}{\partial t}l<0,\forall l\in L^{0k}(\bar{t}_j);\ \ l^T\frac{\partial^2 f(z,\bar{t}_j)}{\partial t}l\le 0,\forall l\in L(z,\bar{t}_j),\ j\in J_*^{(k)},$$

where

$$\begin{array}{l} L^{0k}(\bar{t}_j):=\{l=B(j)\beta_j+A_0^{(k)}(j)\alpha_{0j}^{(k)},\ \alpha_{0j}^{(k)}\ge 0,\ (\beta_j,\alpha_{0j}^{(k)})\ne 0\};\\ L(z,\bar{t}_j):=\{l=B(j)\beta_j+A(j)\alpha_j,\ \alpha_j\ge 0,\ \dfrac{\partial f(z,\bar{t}_j)}{\partial t}l=0\ \},\ j\in J_*^{(k)}.\end{array}$$

Given $\lambda\in[0,1]$, let us consider now vector $x(\lambda)=(1-\lambda)z+\lambda\hat{x}$. Remind here that vector $\hat{x}\in X^{(k+1)}$ satisfies (26).
Taking into account linearity of $f(x,t)$ w.r.t. x, we have

$$f(x(\lambda),\bar{t}_j)=(1-\lambda)f(z,\bar{t}_j)+\lambda f(\hat{x},\bar{t}_j).$$

Then we can conclude that for $0<\lambda<1$, the following relations take place:

$$\begin{array}{l} f(x(\lambda),\bar{t}_j)<0 \text{ for } j\in\bar{J}\setminus J_*^{(k)};\ f(x(\lambda),\bar{t}_j)=0 \text{ for } j\in J_*^{(k)};\\ \dfrac{\partial f^T(x(\lambda),\bar{t}_j)}{\partial t}b_i(j)=0, i=1,\dots,p_j;\\ \dfrac{\partial f^T(x(\lambda),\bar{t}_j)}{\partial t}a_i(j)=0, i\in I_0^{(k)}(j),\ \dfrac{\partial f^T(x(\lambda),\bar{t}_j)}{\partial t}a_i(j)<0, i\in I(j)\setminus I_0^{(k)}(j);\\ l^T\dfrac{\partial^2 f(x(\lambda),\bar{t}_j)}{\partial t^2}l<0,\ \forall l\in L^{0k}(\bar{t}_j),\ j\in J_*^{(k)}.\end{array}\tag{31}$$

It is evident that for sufficiently small $\lambda>0$ we can guarantee that there exists $\varepsilon(\lambda)\ge 0$ such that $\varepsilon(\lambda)\to 0$ as $\lambda\to 0$ and

$$f(x(\lambda),t)<0,\ t\in T\setminus\bigcup_{j\in J_*^{(k)}}T_{\varepsilon(\lambda)}(\bar{t}_j),\tag{32}$$

where $T_\varepsilon(t)=\{\tau\in T:\ ||t-\tau||\le\varepsilon\}$.

Suppose that $j \in J_*^{(k)}$. Then any $t \in T_{\varepsilon(\lambda)}(\bar{t}_i)$ can be presented in the form $t = \bar{t}_j + \Delta t_j,\ \Delta t_j \in L(\bar{t}_j),\ ||\Delta t_j|| \leq \varepsilon(\lambda)$ and in its neighborhood the following asymptotic expansion holds:

$$
\begin{aligned}
f(x(\lambda), t) &= f(x(\lambda), \bar{t}_j + \Delta t_j) \\
&= f(x(\lambda), \bar{t}_j) + \frac{\partial f^T(x(\lambda), \bar{t}_j)}{\partial t}\Delta t_j + \frac{1}{2}\Delta t_j^T \frac{\partial^2 f(x(\lambda), \bar{t}_j)}{\partial t^2}\Delta t_j + o(||\Delta t_j||^2)
\end{aligned}
$$

$$
= \sum_{i \in I(j)\setminus I_0^{(k)}(j)} \frac{\partial f^T(x(\lambda), \bar{t}_j)}{\partial t} a_i(j)\alpha_i(j) + \frac{1}{2}(\beta_j, \alpha_j)^T (B_j, A_j)^T \frac{\partial^2 f(x(\lambda), \bar{t}_j)}{\partial t^2}(B_j, A_j) \begin{pmatrix} \beta_j \\ \alpha_j \end{pmatrix}
$$

$$
+ o(||(\beta_j, \alpha_j)||^2),
$$

where $\alpha_j = (\alpha_i(j), i \in I(j)) \geq 0$.

Notice here that if $(\alpha_i(j),\ i \in I(j) \setminus I_0^{(k)}(j)) \neq 0$, then the first-order term in this expansion is negative. If $(\alpha_i(j),\ i \in I(j) \setminus I_0^{(k)}(j)) = 0$, then the term mentioned above vanishes and we get

$$
f(x(\lambda), t) = (\beta_j, \alpha_{0j}^{(k)})^T (B(j), A_0^{(k)}(j))^T \frac{\partial^2 f(x(\lambda), \bar{t}_j)}{\partial t^2} (B(j), A_0^{(k)}(j)) \begin{pmatrix} \beta_j \\ \alpha_{0j}^{(k)} \end{pmatrix} + o(||(\beta_j, \alpha_j^{(k)})||^2).
$$

In this case, evidently, $f(x(\lambda), t) < 0$ when $(\beta_j, \alpha_{0j}^{(k)}) \neq 0$ (taking into account (31)); and $f(x(\lambda), t) = f(x(\lambda),\ \bar{t}_j) = 0$ when $(\beta_j, \alpha_{0j}^{(k)}) = 0$. Then for sufficiently small $\lambda > 0$ we have

$$
f(x(\lambda), t) < 0,\ t \in T_{\varepsilon(\lambda)}(\bar{t}_j) \setminus \bar{t}_j,\ j \in J_*^{(k)}. \tag{33}
$$

Therefore we have proved that for sufficiently small $\lambda > 0$, vector $\tilde{x} = x(\lambda)$ has the following properties:

P1. $\tilde{x} \in X$, i.e. $\tilde{x}$ is a feasible solution of problem (P) (it follows from (32), (33));

P2. the following relations are valid:

$$
\begin{aligned}
& f(\tilde{x}, \bar{t}_j) = 0,\ \frac{\partial f^T(\tilde{x}, \bar{t}_j)}{\partial t} b_i(j) = 0, i = 1, \ldots, p_j; \\
& \frac{\partial f^T(\tilde{x}, \bar{t}_j)}{\partial t} a_j(j) = 0, i \in I_0^{(k)}(j),\ \frac{\partial f^T(\tilde{x}, \bar{t}_j)}{\partial t} a_i(j) < 0, i \in I(j) \setminus I_0^{(k)}(j); \\
& l^T \frac{\partial^2 f(\tilde{x}, \bar{t}_j)}{\partial t^2} l < 0,\ \forall l \in L^{0k}(\bar{t}_j),\ j \in J_*^{(k)}; \\
& f(\tilde{x}, \bar{t}_j) < 0, \quad t \in T \setminus \{\bar{t}_j, j \in J_*^{(k)}\}.
\end{aligned}
$$

Recall that by construction, $\{\bar{t}_j, j \in J_*^{(k)}\} \subset T^*,\ I_0^{(k)}(j) \subset I(j),\ j \in J_*^{(k)}$. Then, taking into account Definition 2, we can conclude that relations (24) and(25) take place and thus the algorithm DIIPS is justified.
Notice that from the considerations above it follows that

$$
q(\bar{t}_j, l) = 1,\ l \in L^{0k}(\bar{t}_j);\ q(\bar{t}_j, l) = 0,\ l \in L(\bar{t}_j) \setminus L^{0k}(\bar{t}_j),\ j \in J_*^{(k)}.
$$

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Lemma 2 *In Assumption 1, condition (18) is equivalent to the following statement: for any immobile index $\bar{t} \in T^*$, there exists $\bar{x} = \bar{x}(\bar{t}) \in X$ such that vector $\bar{t}$ satisfies the sufficient conditions of strict local minimum in the problem*

$$\max f(\bar{x}, t), \quad s.t.\ t \in T, \tag{34}$$

that have the form

$$\begin{aligned} &\exists y_k \geq 0,\ k \in \bar{K} = \{k \in K:\ h_k^T \bar{t} = \Delta h_k\} \text{such that } \frac{\partial f(\bar{x}, \bar{t})}{\partial t} = \sum_{k \in \bar{K}} h_k y_k, \\ &l^T \frac{\partial^2 f(\bar{x}, \bar{t})}{\partial t^2} l < 0,\ \forall l \in L(\bar{t}, \bar{x}) := \{l \in \mathbb{R}^s:\ l \neq 0,\ \frac{\partial f^T(\bar{x}, \bar{t})}{\partial t} l = 0,\ h_k^T l \leq 0,\ k \in \bar{K}\}. \end{aligned} \tag{35}$$

Proof.
$\Rightarrow$) Suppose that Assumption 1 is satisfied. It was proved above that there exists vector $\tilde{x}$ that satisfies properties P1 and P2. Hence for any $\bar{t} \in T^*$ we can choose vector $\bar{x} = \bar{x}(\bar{t}) = \tilde{x}$.

$\Leftarrow$) Now let us consider a situation when for some $\bar{t} \in T^*$ there exists vector $\bar{x} \in X$ satisfying (35). If suppose that condition (18) is not satisfied for this index $\bar{t}$, we get that there exists $\bar{l} \in L(\bar{t})$, $\bar{l} \neq 0$ such that $q(\bar{t}, \bar{l}) > 1$. Then from the definition of the immobility order it follows that

$$\frac{\partial f(x, \bar{t})}{\partial t} l = 0,\ l^T \frac{\partial^2 f(x, \bar{t})}{\partial t} l = 0,\ \forall x \in X. \tag{36}$$

But equalities (36) with $x = \bar{x} \in X$ contradict (35). The contradiction proves the lemma. ■

5 Example

We consider here an example of a convex SIP problem with polyhedral index set in the form (P).
Let $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4,\ t = (t_1, t_2)^T \in \mathbb{R}^2$, and

$$\begin{aligned} &f_1(x, t) = -t_1^2 x_1 + t_1 t_2 x_1 + t_1 x_2 + (\sin t_1) x_3 + t_1 x_4 - t_2^2, \\ &f_2(x, t) = t_2 x_1 + (t_2 + 1)^2 x_2 + (1 - t_2) x_3 + x_4 - (t_1 - 3)^2 + (t_1 - 3) t_2; \\ &T_1 = \{t \in \mathbb{R}^2 : -t_1 + t_2 \leq 0,\ t_1 \leq 2,\ -1 \leq t_2\}, \\ &T_2 = \{t \in \mathbb{R}^2 : t_1 - t_2 \leq 3,\ 2 \leq t_1 \leq 4,\ 0 \leq t_2 \leq 2\}. \end{aligned}$$

Consider the following SIP problem:

$$\begin{aligned} &\min(-x_2 + x_3), \\ \text{s.t. } &f_1(x, t) \leq 0,\ \forall t \in T_1, \quad f_2(x, t) \leq 0,\ \forall t \in T_2. \end{aligned} \tag{37}$$

The index set here has the form $T = T_1 \bigcup T_2$ where the sets T_1 and T_2 are polyhedrons defined as follows:

$$T_1 = \{t \in \mathbb{R}^2:\ h_1^T t \leq 0, h_2^T t \leq 2, h_3^T t \leq 1\},$$
$$T_2 = \{t \in \mathbb{R}^2:\ g_1^T t \leq 3, g_2^T t \leq 4, g_3^T t \leq -2, g_4^T t \leq 2, g_5^T t \leq 0\},$$

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if suppose that

$$h_1^T = (-1,1), h_2^T = (1,0), h_3^T = (0,-1),$$
$$g_1^T = (1,-1),\ g_2^T = (1,0), g_3^T = (-1,0), g_4^T = (0,1), g_5^T = (0,-1).$$

Problem (37) admits a feasible solution $x^0 = (x_1^0, x_2^0, x_3^0, x_4^0)^T$ where

$$x_1^0 = \frac{\sin 2 + 2}{2} \approx 1.455, \quad x_2^0 = \frac{(x_1^0)^2 + x_1^0(\sin 2 - 6)}{-2(\sin 2 - 2)} \approx -2.425, \tag{38}$$
$$x_3^0 = x_1^0 + 2x_2^0, \quad x_4^0 = -3x_2^0 - x_1^0.$$

It can be shown that the active index set at x^0 has the form $T_a(x^0) = \{t^{(1)}, t^{(2)}, t^{(3)}\}$, where $t^{(1)} := (0,0)^T \in T_1,\ t^{(2)} := (3,0)^T \in T_2,\ t^{(3)} := (2, x_1^0)^T \in T_1$, and

$$f_1(x^0, t^{(1)}) = f_2(x^0, t^{(2)}) = f_1(x^0, t^{(3)}) = 0.$$

Let us describe the cones of feasible directions $\bar{L}(t^{(i)}), t^{(i)} \in T_a(x^0),\ i = 1,2,3$, defined in (5) using the rules described in section 3.
Consider, first, the active index $t^{(1)} = (0,0)$. Since $K_a(t^{(1)}) = \{1\}$ and $h_1 = (-1,1)^T$, we conclude that $\bar{L}_1 = \bar{L}(t^{(1)}) = \{(l_1,l_2)^T : -l_1 + l_2 \le 0\}$. Then

$$\Delta\bar{L}_1 = \{(l_1,l_2)^T : -l_1 + l_2 = 0\} = \{(l_1,l_2) : l_1 = l_2\} = \{(\alpha,\alpha),\ \alpha \in \mathbb{R}\}.$$

Hence $\Delta\bar{L}_1^{\perp} = \{(l_1,l_2) : l_1 + l_2 = 0\}$ and $\bar{\bar{L}}_1 = \bar{L}_1 \bigcap \Delta\bar{L}_1^{\perp} = \{(l_1,l_2) : l_1 = -l_2,\ l_2 \le 0\} = \{(\beta,-\beta),\ \beta \ge 0\}$.
In this example we have $s = 2$, therefore $S = \{1,2\}$.

Since $K_a(t^{(1)}) = \{1\}$, the corresponding matrix H has the form $H = [h_{11} = -1, h_{12} = 1]$. Having supposed $S_0 = \{1\}$, $N_0 = \{1\}$, we get $H_0 = H(N_0, S_0) = [-1]$ and $H_0^{-1} = [-1]$. Taking into account that $S \setminus S_0 = \{2\}$, we can find the components $\bar{b}_{22} = 1$ and $\bar{b}_{21} = -H_0^{-1}h_{12} = 1$ of the bidirectional extremal ray $b(1)$ corresponding to $t^{(1)}$ and then $b(1) = (1,1)$. Now, let us find the unidirectional rays corresponding to $t^{(1)}$. Consider vector $h_0 = (-1,1)^T \ne 0$. Since $m = 1$, we get that $|N_*| = m - 1 = 0$ and hence $N_* = \emptyset$. Then the matrix $D(N_*)$ has the form $D(N_*) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ and $D(N_*)^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Hence $a(N_*) = -\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, $h_1^T a(N_*) = (-1,1)\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = -1 \le 0$, and vector $(\frac{1}{2}, -\frac{1}{2})^T$is a unidirectional ray of the set $\bar{L}_1$. It is evident that vector $a(1) := 2 \cdot (\frac{1}{2}, -\frac{1}{2})^T = (1,-1)^T$ is a unidimensional ray as well. Therefore we conclude that the set $\bar{L}_1$ has two extremal rays, $b(1) = (1,1)^T$ and $a(1) = (1,-1)^T$.

In analogous way, using the rules described in 3.2 we can find that the extremal rays of the set $\bar{L}_2 = \bar{L}(t^{(2)}) = \{(l_1,l_2) : l_1 \le l_2, l_2 \ge 0\}$, have the form $a_1(2) = (1,1)^T$, $a_2(2) = (-1,0)^T$, and the extremal rays of the set $\bar{L}_3 = \bar{L}(t^{(3)}) = \{(l_1,l_2) : l_1 \ge 0\}$ have the form $a(3) = (-1,0)^T$ and $b(3) = (0,1)^T$.

It is evident that $T^* \subset T_a(x^0) = \{t^{(1)}, t^{(2)}, t^{(3)}\}$.
Now let us apply the algorithm DIIPS and determine the immobile indices and their immobility orders along the corresponding extremal rays.

Notice that

$$\frac{\partial^T f_1(x^0,T^{(1)})}{\partial t}a(1)=0,\ \frac{\partial^T f_2(x^0,t^{(2)})}{\partial t}a_1(2)=0,\ \frac{\partial^T f_2(x^0,t^{(2)})}{\partial t}a_2(2)=0,\ \frac{\partial^T f_1(x^0,t^{(3)})}{\partial t}a(3)\neq 0.$$

Using the same notations as in 4.2, we consider the following sets:

$$\bar{J}=\{1,2,3\},\ \tilde{I}(1)=\{1\},\tilde{I}(2)=\{1,\ 2\},\ \tilde{I}(3)=\emptyset.$$

On the first iteration of the algorithm set $k=0$, $J_*^{(0)}=\emptyset$, and construct the set

$$X^{(1)}=\{x\in\mathbb{R}^n:\ f_1(x,t^{(1)})\leq 0,\ f_2(x,t^{(2)})\leq 0,\ f_1(x,t^{(3)})\leq 0\}$$

$$=\{x\in\mathbb{R}^4: x_2+x_3+x_4\leq 0,\ -4x_1+2x_1^0x_1+2x_2+\sin 2\cdot x_3+2x_4-(x_1^0)^2\leq 0\}.$$

Consider the auxiliary problem $(Aux1)$ for each $j\in\bar{J}=\{1,2,3\}$.
When $j=1$, this problem has the form

$$\min_{x\in X^{(1)}} f_1(x,t^{(1)}).$$

Since for each $x\in X^{(1)}$ it holds $f_1(x,t^{(1)})=0$, we can set $x^{(1)}=x^0$.
Let $j=2$. In that case problem $(Aux1)$ takes the form

$$\min_{x\in X^{(1)}} f_2(x,t^{(2)}).$$

Since the objective function of this problem, $f_2(x,t^{(2)})=x_2+x_3+x_4$, is unbounded from below, then according to the algorithm we can choose the feasible $x^{(2)}=(0,0,0,-1)$ with $f_2(x^{(2)},t^{(2)})=-1<0$.
The same situation occurs for $j=3$: the objective function of the problem

$$\min_{x\in X^{(1)}} f_1(x,t^{(3)}),$$

is unbounded from below: $f_1(x,t^{(3)})=(2x_1^0-4)x_2+2x_2+\sin 2\cdot x_3+2x_4-(x_1^0)^2)$, and we can set $x^{(3)}=(0,0,0,0)$ since $f_1(x^{(3)},t^{(3)})=-(x_1^0)^2<0$.
Find the sets $\Delta J_*^{(1)}:=\{j\in\bar{J}: f(x^{(j)},t^{(j)})=0\}=\{1\}$, $\Delta I_0^{(1)}(1):=\emptyset$.
Since $\Delta J_*^{(1)}=\{1\}\neq\emptyset$, we pass to the next iteration with

$$J_*^{(1)}=J_*^{(0)}\bigcup\Delta J_*^{(1)}=\{1\},\ I_0^{(1)}(1)=\Delta I_0^{(1)}(1)=\emptyset \text{ and } \bar{J}\backslash J_*^{(1)}=\{2,3\}.$$

On the next iteration $(k=1)$ we construct the set

$$X^{(2)}=\{x\in\mathbb{R}^4:\ f_2(x,t^{(2)})\leq 0,\ f_1(x,t^{(3)})\leq 0,\ f_1(x,t^{(1)})=0,\frac{\partial^T f_1(x,t^{(1)})}{\partial t}b(1)=0,$$

$$\frac{\partial^T f_1(x,t^{(1)})}{\partial t}a(1)\leq 0,\ l^T\frac{\partial^2 f_1(x,t^{(1)})}{\partial t^2}l\leq 0,\ l\in L_1^{(1)}\},$$

where the set $L_1^{(1)}$ is defined by formula (22) for $J_*^{(1)}=\{1\}$, and it is empty: $L_1^{(1)}=\emptyset$, as $\frac{\partial^2 f_1(x^0,t^{(1)})}{\partial t^2}\prec 0$.

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Then

$$X^{(2)} = \{x \in \mathbb{R}^4 : x_2 + x_3 + x_4 = 0,\ -4x_1 + 2x_1^0 x_1 + 2x_2 + \sin 2 \cdot x_3 + 2x_4 - (x_1^0)^2 \leq 0\}.$$

For $j = 2$, the auxiliary problem $(Aux1)$ has the form

$$\min_{x \in X^{(2)}} f_2(x, t^{(2)}).$$

Since $f_2(x, t^{(2)})= x_2 + x_3 + x_4 = 0\ \forall x \in X^{(2)}$, we can set $x^{(2)} = x^0$.
For $j = 3$, problem $(Aux1)$ takes the form

$$\min_{x \in X^{(2)}} f_1(x, t^{(3)}),$$

and it easy to conclude that the objective function of this problem, $f_1(x, t^{(3)}) = -4x_1 + 2x_1^0 x_2 + 2x_2 + \sin 2 \cdot x_3 + 2x_4 - (x_1^0)$, is unbounded from below. Then we can choose $x^{(3)} = (0, 0, 0, 0)$, as $f_1(x^{(3)}, t^{(3)}) = -(x_1^0)^2 < 0$.
Construct the set $\Delta J_*^{(2)} = \{j \in \{2, 3\} :\ f(x^{(j)}, t^{(j)}) = 0\} = \{2\}$. Since $\tilde{I}(1) \backslash I_0^{(1)}(1) = \{1\}$, we have to solve the auxiliary problem $(Aux2)$

$$\min_{x \in X^{(2)}} \frac{\partial^T f_1(x, t^{(1)})}{\partial t} a(1).$$

Since $\frac{\partial^T f_1(x,t^{(1)})}{\partial t} a(1) = x_2 + x_3 + x_4$, the objective function of this problem is equal to zero for all feasible solutions and therefore we can choose $x^{(11)} = x^0$.
According to the Algorithm,

$$\Delta I_0^{(2)}(1) = \{j \in \{1\} : \frac{\partial^T f_1(x^{(11)}, t^{(1)})}{\partial t} a(1) = 0\}.$$

Evidently, $\Delta I_0^{(2)}(1) = \{1\} \neq \emptyset$.
Construct the sets

$$J_*^{(2)} = J_*^{(1)} \bigcup \Delta J_*^{(2)} = \{1, 2\},\ I_0^{(2)}(1) = I_0^{(1)}(1) \bigcup \Delta I_0^{(2)}(2) = \{1\},\ I_0^{(2)}(2) = \emptyset$$

and pass to the next iteration.
For $k = 2$, we construct the set

$$\begin{aligned} X^{(3)} = \{x \in \mathbb{R}^4 :\ & f_1(x, t^{(3)}) \leq 0,\ f_1(x, t^{(1)}) = 0,\ f_2(x, t^{(2)}) = 0, \\ & \frac{\partial^T f_1(x, t^{(1)})}{\partial t} b(1) = 0,\ \frac{\partial^T f_1(x, t^{(1)})}{\partial t} a(1) = 0,\ l^T \frac{\partial^2 f_1(x, t^{(1)})}{\partial t^2} l \leq 0,\ l \in L_1^{(2)}, \\ & \frac{\partial^T f_2(x, t^{(2)})}{\partial t} a_1(2) \leq 0,\ \frac{\partial^T f_2(x, t^{(2)})}{\partial t} a_2(2) \leq 0,\ \ l^T \frac{\partial^2 f_2(x, t^{(2)})}{\partial t^2} l \leq 0,\ l \in L_2^{(2)}\} \end{aligned}$$

where $L_1^{(2)}, L_2^{(2)}$ are defined in (22). Since $\frac{\partial^2 f_i(x^0,t^{(i)})}{\partial t^2} \prec 0, i = 1, 2$, we have $L_1^{(2)} = L_2^{(2)} = \emptyset$.
Having substituted the functions and simplifying the expression, we get

$$\begin{aligned} X^{(3)} = \{x \in \mathbb{R}^4 : -4x_1 + 2x_1^0 x_1 + 2x_2 + \sin 2 \cdot x_3 + 2x_4 - (x_1^0)^2 \leq 0, \\ x_2 + x_3 + x_4 = 0,\ x_1 + 2x_2 - x_3 \leq 0\}. \end{aligned}$$

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Then the problem $(Aux1)$, takes the form

$$\min_{x \in X^{(3)}} f_1(x, t^{(3)}),$$

or explicitly

$$\begin{aligned} &\min\ -4x_1 + 2x_1^0 x_1 + 2x_2 + \sin 2 \cdot x_3 + 2x_4 - (x_1^0)^2 \\ \text{s.t. }& x_2 + x_3 + x_4 = 0, x_1 + 2x_2 - x_3 \le 0, \\ &-4x_1 + 2x_1^0 x_1 + 2x_2 + \sin 2 \cdot x_3 + 2x_4 - (x_1^0)^2 \le 0. \end{aligned} \tag{39}$$

The objective function is unbounded from below. Choose $x^{(3)} = (0, -1, 0, 1)$ with $f_1(x^{(3)}, t^{(3)}) = -(x_1^0)^2 < 0$ and construct $\Delta J_*^{(3)} = \{j \in \{3\} : f_1(x^{(j)}), t^{(j)}) = 0\} = \emptyset$.
For $j \in J_*^{(2)} = \{1, 2\}$, consider the sets $\tilde{I}(j) \setminus I_0^{(2)}(j)$:

$$\tilde{I}(1) \setminus I_0^{(2)}(1) = \emptyset,\ \tilde{I}(2) \setminus I_0^{(2)}(2) = \tilde{I}(2) = \{1, 2\}.$$

For $i \in \tilde{I}(2) \setminus I_0^{(2)}(2) = \{1, 2\}$, the corresponding auxiliary problems $(Aux2)$ take the forms

$$\min_{x \in X^{(3)}} \frac{\partial^T f_2(x, t^{(2)})}{\partial t} a_1(2),$$

and

$$\min_{x \in X^{(3)}} \frac{\partial^T f_2(x, t^{(2)})}{\partial t} a_2(2),$$

or equivalently,

$$\min_{x \in X^{(3)}} (x_1 + 2x_2 - x_3), \tag{40}$$

$$\min_{x \in X^{(3)}} 0. \tag{41}$$

The problem (40) can be rewritten in the form

$$\begin{aligned} &\min\ x_1 + 2x_2 - x_3, \\ \text{s.t. }& x_2 + x_3 + x_4 = 0, x_1 + 2x_2 - x_3 \le 0,\ (\sin 2 - 2)x_3 - (x_1^0)^2 \le 0. \end{aligned}$$

Since the objective function of this problems is unbounded, we choose $x^{(12)} = (0, 0, 2, -2)$ such that $f_2(x^{(12)}, t^{(2)}) = -2$.
Now notice that in the auxiliary problem (41), the value of the objective function is constant and equal to zero, therefore we can consider any feasible solution as an optimal and set, for example, $x^{(22)} = (0, 0, 2, -2)$.
Then $\Delta I_0^{(3)}(1) = \emptyset$ and $\Delta I_0^{(3)}(2) = \{i \in \{1, 2\} : \frac{\partial^T f_2(x^{(i2)}, t^{(2)})}{\partial t} a_i(2) = 0\} = \{2\}$. Notice that we have here $\Delta J_*^{(3)} = \emptyset, \Delta I_0^{(3)}(1) = \emptyset$, but $\Delta I_0^{(3)}(2) \neq \emptyset$.
Therefore we pass to the next iteration, with $k = 3$, and the following sets:

$$J_*^{(3)} = J_*^{(2)} \bigcup \Delta J_*^{(3)} = \{1, 2\},\quad I_0^{(3)}(1) = I_0^{(2)}(1) \bigcup \Delta I_0^{(3)}(1) = \{1\},$$

$$I_0^{(3)}(2) = I_0^{(2)}(2) \bigcup \Delta I_0^{(3)}(2) = \{2\},$$

and

$$X^{(4)} = \{x \in \mathbb{R}^4 : \ f_1(x, t^{(3)}) \le 0, \ f_1(x, t^{(1)}) = 0, \ f_2(x, t^{(2)}) = 0,$$
$$\frac{\partial^T f_1(x,t^{(1)})}{\partial t} b(1) = 0, \ \frac{\partial^T f_1(x,t^{(1)})}{\partial t} a(1) = 0, \ l^T \frac{\partial^2 f_1(x,t^{(1)})}{\partial t^2} l \le 0, \ l \in L_1^{(3)}, \tag{42}$$
$$\frac{\partial^T f_2(x,t^{(2)})}{\partial t} a_1(2) \le 0, \ \frac{\partial^T f_2(x,t^{(2)})}{\partial t} a_2(2) = 0, \ l^T \frac{\partial^2 f_2(x,t^{(2)})}{\partial t^2} l \le 0, \ l \in L_2^{(3)}\}.$$

Having substituted explicit presentations of the sets and functions involved in (42) and simplifying the obtained expressions, we get

$$X^{(4)} = \{x \in \mathbb{R}^4 : -4x_1 + 2x_1^0 x_1 + 2x_2 + \sin 2 \cdot x_3 + 2x_4 - (x_1^0)^2 \le 0,$$
$$x_2 + x_3 + x_4 = 0, \ x_1 + 2x_2 - x_3 \le 0\},$$

and therefore for $j = 3$ the auxiliary problem $(Aux1)$ takes the form

$$\min \ -4x_1 + 2x_1^0 x_1 + 2x_2 + \sin 2 \cdot x_3 + 2x_4 - (x_1^0)^2,$$
$$\text{s.t. } x \in X^{(4)},$$

and coincides with problem (39) from the previous iteration. As above, since the objective function is unbounded from below, we can choose $x^{(3)} = (0, -1, 0, 1)$. Then $f_1(x^{(3)}, t^{(3)}) = -(x_1^0)^2 < 0$, and $\Delta J_*^{(4)} = \emptyset$.
For $i \in \tilde{I}(2) \setminus I_0^{(3)}(2) = \{1\}$, the auxiliary problem $(Aux2)$ has the form

$$\min \ x_1 + 2x_2 - x_3, \quad \text{s.t.} \quad x \in X^{(4)}.$$

The objective function of this problem is unbounded.
For $x^{(12)} = (0, -1, 0, 1) \in X^{(4)}$ we have $f_2(x^{(12)}, t^{(2)}) = -2 < 0$. Consequently we get $\Delta I_0^{(4)}(2) = \emptyset$.
Since $\Delta I_0^{(4)}(2) = \emptyset, \ \Delta J_*^{(4)} = \emptyset$, the algorithm stops with $T^* = \{t^{(j)}, \ j \in J_*^{(3)}\} = \{t^{(1)}, t^{(2)}\}$.
For the immobile indices found, the immobility orders along the extremal rays are as follows:

$$q(t^{(j)}, a_i(j)) = 1, \ i \in I_0^{(3)}(j); \quad q(t^{(j)}, a_i(j)) = 0, i \in \tilde{I}(i) \setminus I_0^{(3)}(j), j \in J_*^{(3)}.$$

Hence we conclude that $q(t^{(1)}, a(1)) = 1, \ q(t^{(1)}, b(1)) = 1$, and $q(t^{(2)}, a_1(2)) = 0, \ q(t^{(2)}, a_2(2)) = 1$.

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