Universidade de Aveiro Departamento de Matemática,
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Liliana Manuela Gaspar Cerveira da Costa

Politopo de Birkhoff Acíclico

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica de Enide Cascais Silva Andrade Martins, Professora Auxiliar do Departamento de Matemática da Universidade de Aveiro e de Carlos Martins da Fonseca, Professor Auxiliar do Departamento de Matemática da Universidade de Coimbra.

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Palavras-chave

Algoritmos, árvore, diâmetro, grafo, matriz duplamente estocástica, número de arestas, número de faces, número de vértices, politopo de Birkhoff acíclico, politopo de Birkhoff tridiagonal.

Resumo

Neste trabalho estabelece-se uma interpretação geométrica, em termos da teoria dos grafos, para vértices, arestas e faces de uma qualquer dimensão do *politopo de Birkhoff acíclico*, $\mathfrak{T}_n = \Omega_n(T)$, onde T é uma árvore com n vértices. Generaliza-se o resultado obtido por G. Dahl, [18], para o cálculo do diâmetro do grafo $G(\Omega_n^t)$, onde Ω_n^t é o politopo das matrizes tridiagonais duplamente estocásticas. Adicionalmente, para $q = 0, 1, 2, 3$ são obtidas fórmulas explícitas para a contagem do número de $q-$faces do *politopo de Birkhoff tridiagonal*, Ω_n^t, e é feito o estudo da natureza geométrica dessas mesmas faces. São, também, apresentados algoritmos para efectuar contagens do número de faces de dimensão inferior à de uma dada face do *politopo de Birkhoff acíclico*.

Keywords Acyclic Birkhoff polytope, algorithms, diameter, doubly stochastic matrix, graph, number of edges, number of faces, number of vertices, tree, tridiagonal Birkhoff polytope.

Abstract In this work using graph theory, we give a geometrical interpretation of vertices, edges, and faces of any dimension of the *acyclic Birkhoff polytope*, $\mathfrak{T}_n = \Omega_n(T)$, were T is a tree with n vertices. We generalize a proposition from G. Dahl, [18], that allows the calculation of the diameter of the graph $G(\Omega_n^t)$, where Ω_n^t denotes the polytope of tridiagonal doubly stochastic matrices. Furthermore, for $q = 0, 1, 2, 3$ we obtain some explicit formulae for counting the number of $q-$faces of the *tridiagonal Birkhoff polytope*, Ω_n^t, and the study of its geometrical nature is done. For a given p-face of Ω_n^t we determine the number of faces of lower dimension that are contained in it and we discuss its nature. Some algorithms allowing an exhaustive account on the number of edges and faces of the *acyclic Birkhoff polytope* are presented.

Contents

Chapter 1

Introduction

1.1 Background and motivation

General polyhedra are fundamental mathematical objects. They are of great interest in a vast range of fields of pure and applied mathematics, such as probability theory, the optimal assignment problem, linear and integer optimization, just to name few. Though, it is still not known some basic properties of these objects. Counting the number of vertices of a general polyhedra is a major challenging open problem in polyhedral combinatorics. In what follows we adopt the geometric terminology of Grünbaum, [20].

A *polyhedron* can be considered as the intersection of a finite family of closed halfspaces of $\mathbb{R}^n$. As a *halfspace* is the solution of a linear inequality (*i.e.*, an inequality of the form $a^T x \leq \beta$, where $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$), a polyhedron is the solution of a linear system $Ax \leq b$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Furthermore, any polyhedron in $\mathbb{R}^n$ is a closed convex set, [17]. Here, we are interested in a particular family of polyhedra, those who are bounded: namely the polytopes.

A *polytope* $\mathcal{P} \subset \mathbb{R}^n$ is the convex hull of a finite number of points. Thus, for a fixed $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in \mathbb{R}^n$,

$$\mathcal{P} = \left\{ \sum_{i=1}^{m} \lambda_i x_i : \lambda_i \in \mathbb{R}_0^+, \sum \lambda_i = 1 \right\}.$$

Birkhoff studied this issue using matrices, namely *doubly stochastic matrices*, *i.e.*, square matrices with real nonnegative entries and all rows and columns sums equal to one, [1, pg 59], [2]. This denomination is associated to probability distributions and it is amazing the diversity of branches of mathematics in which doubly stochastic matrices arise (geometry, combinatorics, optimization theory, graph theory and statistics). This class of matrices has been studied quite extensively, especially in its relation with the van der Waerden´"conjecture" for the permanent (cf. [6]).

The set of $n \times n$ doubly stochastic matrices, denoted by Ω_n, viewed as a subset of $\mathbb{R}^{n \times n}$ is a closed, bounded convex set. Therefore, Ω_n is a polytope, the so-called *Birkhoff polytope*, whose dimension is $(n-1)^2$. In fact, the dimension of the linear space of $n \times n$ real matrices is n^2. There are $2n$ linear conditions on the row sums and the column sums of $n \times n$ doubly stochastic matrices. Since the sum of all rows sums in these matrices is equal to the sum of all columns sums, only $2n-1$ of those linear conditions are independent at most. Hence, the dimension of Ω_n is equal to $n^2 - (2n-1) \leq (n-1)^2$, [21, pg 70]. The other inequality follows easily.

We present here an example, for the case $n = 3$ we obtain

$$\Omega_3 = \left\{ \begin{bmatrix} 1-p-q & p & q \\ r & 1-r-s & s \\ t & u & 1-t-u \end{bmatrix} : 0 \leq p,q,r,s,t,u \leq 1 \text{ and} \right.$$

$$\left. p+q = r+t, r+s = p+u, t+u = q+s; p+q \leq 1, r+s \leq 1, t+u \leq 1 \right\}$$

and the dimension of Ω_3 is 4. Notice that we have 6 conditions, but only 5 of them are independent.

A point x of a convex set S is an *extreme point* of S if x does not belong to the relative interior of any segment contained in S; the extreme points of a polytope are called *vertices*, [20].

An $n \times n$ permutation matrix is a point of Ω_n, moreover, it is an extreme point of the referred polytope, [1, 4]. According to the Birkhoff-von Neumann theorem, [2, 4], any doubly stochastic matrix can be written as a convex combination of finitely many permutation matrices, and the extreme points, or vertices, of the Birkhoff polytope are exactly the permutation matrices, [1, 4]. In fact, Ω_n is the convex hull of all permutation matrices of order n (cf.[9]). This polytope is also known as *transportation polytope* or *polytope of doubly stochastic matrices.* Observe that, from previous example, if $p = q = r = s = t = u = 0$, we have I_3, the identity matrix of order 3; when either $s = u = 1$ or $q = t = 1$ or $p = r = 1$ and the remaining parameters are zero, we obtain one of the 3×3 transposition matrices; if either $t = p = s = 1$ or $r = u = q = 1$ and the others parameters are zero, then we obtain the remaining permutation matrices of order 3.

R. Brualdi and P. Gibson devoted their attention to the study of affine and combinatorial properties of Ω_n in several papers where they studied the convex polytope of doubly stochastic matrices in association with $(0, 1)-$matrices, see, for instance, [5, 6, 7, 8].

A *face* $\mathcal{F}$ of a polyhedron $\mathcal{P}$ is either $\emptyset$ or $\mathcal{P}$ or there exists a hyperplane $\mathcal{H} = \{x \in \mathbb{R}^n : c^T x = \alpha\}$ such that $\mathcal{F} = \mathcal{H} \cap \mathcal{P}$. Therefore, a face $\mathcal{F}$ of $\mathcal{P}$ is a set of the form $\mathcal{F} = \{x \in \mathcal{P} : c^T x = \alpha\}$, where $c^T x \leq \alpha$, ($c \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$), for all $x \in \mathcal{P}$, [17]. The empty set and $\mathcal{P}$ are the improper faces of $\mathcal{P}$, while all other faces are called proper faces. A vertex of a polyhedron $\mathcal{P}$ is a minimal proper face of $\mathcal{P}$ (*i.e.*, a proper face that does not strictly contain any other proper face of $\mathcal{P}$). A maximal proper face of $\mathcal{P}$ (*i.e.*, a proper face of $\mathcal{P}$ that is not strictly contained in any proper face of $\mathcal{P}$) is called a *facet* of $\mathcal{P}$.

A face $\mathcal{F}$ of a polytope $\mathcal{P}$ is itself a polytope, and the faces of $\mathcal{F}$ are also faces of $\mathcal{P}$.

If $\mathcal{P}$ has dimension n, each face of dimension $n-2$, denoted by $(n-2)$-face, is contained in precisely two facets (faces of dimension $n-1$) of $\mathcal{P}$ and it is equal to their intersection. Furthermore, each p-face, $-1 \leq p < k \leq n-1$, of $\mathcal{P}$ is the intersection of a family of k-faces of $\mathcal{P}$ containing it. Note that when $p = -1$, the correspondent face is $\emptyset$. This family contains at least $k-p+1$ members, [20].

Now, we dedicate our attention to the polytope Ω_n, that is the set of all $n \times n$ real matrices $X = [x_{ij}]$ that satisfy the constraints

$$x_{ij} \geq 0, \; i,j = 1, \ldots, n, \tag{1.1.1}$$

$$\sum_{k=1}^{n} x_{ik} = 1 = \sum_{k=1}^{n} x_{ki}, \; i = 1, \ldots, n, . \tag{1.1.2}$$

We can find the previous concept and next assertions in [6]. The faces of Ω_n can be obtained by replacing some of the inequalities of (1.1.1) by equalities. Note that the constraints of (1.1.1) are not independent (because of the constraints in (1.1.2)). Hence, considering

$$K \subseteq \{(i,j) : i,j = 1, \ldots, n\},$$

a face of Ω_n is determined by replacing (1.1.1) by

$$x_{ij} \geq 0, \; (i,j) \in K$$

$$x_{ij} = 0, \; (i,j) \notin K.$$

If $A = [a_{ij}]$ is an $n \times n$ $(0,1)-$matrix, where $a_{ij} = 1$ if and only if $(i,j) \in K$, the face $\mathcal{F}$ consists of all $n \times n$ doubly stochastic matrices $X = [x_{ij}]$ such that $x_{ij} \leq a_{ij}$, $i,j = 1,\ldots,n$. This face is denoted by $\mathcal{F}_{\mathcal{A}}$. The $n \times n$ permutation matrices P such that $P \leq A$ are precisely the vertices of $\mathcal{F}_{\mathcal{A}}$. If A' is an $n \times n$ $(0,1)-$matrix such that for each permutation matrix P, $P \leq A$ if and only if $P \leq A'$, then $\mathcal{F}_A = \mathcal{F}_{A'}$. Therefore, if there exist $r,s = 1,\ldots,n$ such that $a_{rs} = 1$ for which there is no permutation matrix $P = [p_{ij}]$, with $p_{rs} = 1$ and $P \leq A$, then $\mathcal{F}_A = \mathcal{F}_{A'}$, where A' is obtained from A by replacing a_{rs} by 0. Therefore, to determine the nonempty faces of Ω_n it is only need to consider those $n \times n$ $(0,1)-$matrices $A = [a_{ij}]$ such that $a_{rs} = 1$ implies that there exists a permutation matrix $P = [p_{ij}]$ with $p_{rs} = 1$ and $P \leq A$. These matrices are said to have *total support*, [23].

Another important concept is the concept of *fully indecomposable matrix*. In fact, if A is an $n \times n$ nonnegative matrix, with $n > 1$, A is fully indecomposable provided there do not exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & O \\ A_3 & A_2 \end{bmatrix}, \tag{1.1.3}$$

where A_1 and A_2 are square matrices. If A is an 1×1 matrix, A is fully indecomposable if it is not a zero matrix. Assuming that A is doubly stochastic and (1.1.3) holds, the property that all row and column sums of A equal 1 implies $A_3 = O$. By iterating this argument on A_1 and A_2 it can be said, [4], that there are permutation matrices P' and Q' such that $P'AQ'$ is a direct sum of fully indecomposable matrices. Attending to [22, Theorem 1], previous conclusions can be extended to matrices with the pattern of a doubly stochastic matrix. We mean by the *pattern* or *support* of a matrix, the set of the positions of the nonzero entries of the matrix.

From [6, 22], a $(0,1)-$matrix has *total support* if and only if there exist permutation matrices P' and Q' such that matrix $P'AQ'$ is a direct sum of fully indecomposable matrices. Therefore, from previous assertions the nonempty faces of Ω_n are in one-to-one correspondence with the $n \times n$ $(0,1)-$matrices which can be permuted to direct sums of fully indecomposable matrices, called *fully indecomposable components.* A fully indecomposable component of order 1 is a *trivial component.*

Furthermore, if A is an $n \times n$ $(0,1)-$matrix with total support then we say that the matrix and the face $\mathcal{F}_A$ are in one-to-one correspondence.

Moreover, it is established that the pattern of doubly stochastic matrices of order n are precisely the $(0,1)-$matrices of order n with total support. This result can be found in [4, Theorem 9.2.1], and it is presented below.

Theorem 1.1.1. *[4] Let n be a positive integer. Then the pattern of doubly stochastic matrices of order n are precisely the $(0,1)-$matrices of order n with total support. There is a bijection between the nonempty faces of the polytope Ω_n of doubly stochastic matrices and the $(0,1)-$matrices A of order n with total support. This bijection is given by*

$$A \longleftrightarrow \mathcal{F}_A = \{X \in \Omega_n : X \leq A\};$$

moreover, $\dim \mathcal{F}_A = \sigma_A - 2n + k$ *where k is the number of fully indecomposable components of A and σ_A equals the numbers of 1's in A.*

If A has total support it is a Boolean sum of permutation matrices of order n, [4]. Note that the Boolean arithmetic is based on Boolean operations "and" $(\wedge)$ and "or" $(\vee)$ over the set $\{0,1\}$, in particular the Boolean sum is 0 if both summands are 0, otherwise is equal to 1.

The matrix corresponding to Ω_n is the $n \times n$ matrix whose entries are all equal to 1, [4, pg 385]. In fact, Ω_n is the convex hull of the $n!$ permutation matrices of order n, and it can be represented by the Boolean sum of all these permutation matrices.

From the results already mentioned and cited in [6], the subset of Ω_n whose elements are the $n \times n$ tridiagonal doubly stochastic matrices is a face of Ω_n. Moreover, as a face of a polytope is itself a polytope, [20], then this set is a polytope, called *tridiagonal Birkhoff polytope* and it is denoted by Ω_n^t,

$$\Omega_n^t = \{A \in \Omega_n \ : \ A \text{ is tridiagonal}\}.$$

In [18], G. Dahl discussed and studied the facial structure of this polytope and stated that Ω_n^t is a polytope in $\mathbb{R}^{n \times n}$ of dimension $n-1$. As the vertices of a face of a polytope are also vertices of the polytope, the vertices of Ω_n^t are the $n \times n$ tridiagonal permutation matrices. The author established that Ω_n^t has f_{n+1} vertices, where f_{n+1} denotes the $(n+1)$-th Fibonacci number. Recall that the n-th Fibonacci number is determined by the well-known recurrence relation

$$f_{n+2} = f_n + f_{n+1}, \tag{1.1.4}$$

with initial conditions $f_1 = 1$ and $f_2 = 1$. Furthermore, the edges of Ω_n^t were also described. The author established the next theorem, where it is used the notation $J = [1]$ and $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Theorem 1.1.2. *[18] (i) Ω_n^t is a polytope in $\mathbb{R}^{n \times n}$ of dimension $n-1$ with f_{n+1} vertices (ii) Its vertex set consists of all tridiagonal permutation matrices; these are matrices of order n that can be written as a direct sum*

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_t \tag{1.1.5}$$

where each matrix $A_i (i \leq t)$, called a block, equals either J or K.

(iii) Consider a vertex A as in (1.1.5). Then each adjacent vertex of A is obtained from A by either

(a) *interchanging a sequence of consecutive blocks $J, K, K, \ldots, K$ (with $t \geq 1$ $K s$) and the sequence $K, K, \ldots, K, J$ (with t $K s$), or*

(b) *by interchanging a sequence of consecutive blocks $K, K, \ldots, K$ (with $t \geq 1$ $K s$) and the sequence $J, K, K, \ldots, K, J$ (with $t-1$ $K s$).*

The graph $G(\Omega_n^t)$ denotes the *graph of* Ω_n^t (or skeleton), that is, the graph whose vertices and edges are the vertices and edges of the polytope Ω_n^t. Dahl determine the diameter of $G(\Omega_n^t)$, which is defined as the maximum of $d(u, v)$ taken over all pairs u, v of vertices, where $d(u, v)$ is the smallest number of edges in a path between u and v in $G(\Omega_n^t)$. The result is the next theorem:

Theorem 1.1.3. *[18] The diameter of $G(\Omega_n^t)$ equals $\left\lfloor \frac{n}{2} \right\rfloor$.*

Here, $\lfloor x \rfloor$ represents the largest integer less than or equal to x.

The particular interest of previous theorem, as well of the research involving the study of the diameter of the graph of a polytope, is related to the fact that the diameter gives a bound on the maximum number of iterations needed for solving, using the simplex method, a linear program on a polytope, [4, pg 374].

Recently, C. M. da Fonseca and E. Marques de Sá (cf. [16]), established a closer connection between vertex counting in Ω_n^t and Fibonacci numbers. In particular, the main results on alternating parity sequences - strictly increasing sequences of integers, with a finite numbers of entries, such that any two adjacent entries have opposite parities - are applied to determine the number of vertices of an arbitrarily given face of Ω_n^t. An expression for the number of edges of Ω_n^t is also provided. The next results can be found in [16].

As previously said Ω_n^t is a face of Ω_n and the faces of Ω_n^t are the faces of the latter which are contained in the former. Moreover, the faces of Ω_n^t are in one-to-one correspondence with $n \times n$ tridiagonal $(0,1)-$matrices A that have total support (this means that A is a boolean sum of $n \times n$ tridiagonal permutation matrices), [6]. In their work the authors deal with matrices of this kind. The face of Ω_n^t corresponding to an $n \times n$ tridiagonal $(0,1)-$matrix with total support A is $\mathcal{F}_A = \{X \in \Omega_n^t : a_{ij} = 0 \Rightarrow x_{ij} = 0\}$. It was also established that A is a direct sum of square blocks, $A = A_1 \oplus A_2 \oplus \cdots \oplus A_p$, where each A_i $(i = 1, \ldots, p)$ is one of the following types:

- Type 1 $A_t = [1]$, an 1×1 matrix;
- Type 2 $A_t = K$, where K is the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$;
- Type 3 A_t is not of the previous two types, and all super-diagonal entries of A_t are 1's.

If A_t is of Type 3 then the first and the last diagonal entries are 1, otherwise A would not have total support.

An example of a 4×4 matrix of Type 3 is displayed

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

It is also defined an S-*matrix*, as a symmetric tridiagonal $(0,1)-$matrix, different from [1] and K, with all super-diagonal entries equal to one, and with first and last diagonal entries equal to one. The blocks A_t of Type 3 are called *S-blocks* of A.

Given an S-matrix B, of order m, b_{ii} is an *inner entry* of B, if $b_{ii} = 1$ and $1 < i < m$. In previous example, the entry b_{33} is an inner entry. It was also established that only the multi-set of S-matrices of A is important in the study of a single face $\mathcal{F}_A$ (see [16]) and they prove the following:

Lemma 1.1.4. *[16] Any S-matrix has total support and is fully indecomposable.*

According to [6] and [4, Theorem 9.2.1] previously presented, if B is an $m \times m$ fully indecomposable $(0, 1)-$matrix with total support, dim $\mathcal{F}_B = \sigma_B$ - $2m + 1$, where σ_B is the number of 1's in B. If B is an S-matrix, C. M. da Fonseca and E. Marques de Sá obtained dim $\mathcal{F}_B$=$1 + \omega$, where ω is the number of inner entries of B. Therefore, they stated:

Lemma 1.1.5. *[16] dim $\mathcal{F}_A$= $s_A + \iota_A$, where s_A is the number of $S-$blocks of A, and ι_A is the sum of the numbers of inner entries in the $S-$blocks of A.*

All these results motivated us to the formulation of some problems described in Section 1.3. Before that section we introduce some notation and basic notions underlying the problems formulated during the development of this work.

1.2 Terminology and further definitions

In this section we recall further more-or-less standard definitions on graph theory, which will be used in the sequel. Here, we adopt the same concepts, definitions and terminology used in [3, 19].

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a nonempty finite set of *vertices* (or nodes), and $E(G)$ is a set of unordered pairs of vertices called *edges*. The number of vertices and the number of edges of G are denoted by $|V(G)|$ and $|E(G)|$, respectively. To $|V(G)|$ we also call *order* of G. An edge containing the vertices i and j is represented by $e = ij$ and we say that i and j are *adjacent*, or that j is a *neighbor* of i, and this is denoted by $i \sim j$. We also say that e is *incident* on both i and j. The vertices i and j are the *endpoints* of the edge ij.

The *neighborhood* of a vertex v in G, denoted by $N_G(v)$ (or simply $N(v)$) is the set of the neighbors of v in G. If $S \subset V(G)$, $N(S)$ denotes the vertices in $V(G)\backslash S$ with a neighbor in S.

The *degree* of a vertex v in G, denoted by $d_G(v)$ (or simply $d(v)$) is the number of edges incident in it, and we have $d(v) = |N(v)|$.

A graph G whose vertex set can be partitioned into two subsets X and Y such that each edge of G has a vertex on X and the other vertex in Y, is said a *bipartite graph*, and is denoted by $G = G[X, Y]$.

A graph G whose edges are directed is a *directed graph*, also know as a *digraph*, and its edges are called *arcs*. An arc of a digraph is an ordered pair of distinct vertices.

The *adjacency matrix* of a graph G with $|V(G)| = n$ is the $n \times n$ matrix $A = [a_{ij}]$ whose entries a_{ij} are equal to 1 if the vertices $i \sim j$, otherwise are 0.

Let $A = [a_{ij}]$ be an $m \times n$ matrix. A way to represent the structure of the nonzero entries of A is by a bipartite graph. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$, such that $|U| = m$, $|W| = n$ and $U \cap W = \emptyset$, the *bipartite graph associated* with A is the graph, $BG(A)$, with vertex set $V = U \cup W$ whose edges are all the pairs $u_i w_j$ for which $a_{ij} \neq 0$.

We represent by $diag(a_1, \ldots, a_n)$ the $n \times n$ diagonal matrix whose entry (i, i) is a_i.

If the matrix $A = [a_{ij}]$ is of order n we may represent its nonzero structure by a digraph, $DG(A)$, whose vertex set is a set $V = \{v_1, \ldots, v_n\}$. There is an arc (v_i, v_j) from v_i to v_j if and only if $a_{ij} \neq 0$. Note that a nonzero diagonal entry of A determines an arc from a vertex to itself. If the matrix A is also symmetric, we may represent its nonzero structure by the graph $G(A)$. The vertex set of $G(A)$ is a set $V = \{v_1, \ldots, v_n\}$, and there is an edge joining v_i and v_j if and only if $a_{ij} \neq 0$, [4].

In a graph, to each edge e (or, vertex v) we can associate a numerical positive label $w(e)$ (or, $w(v)$) called the *weight* of the edge e (or, the vertex v). Similarly to the above, we can associate to a weighted graph a matrix whose support corresponds to the weights of edges and/or vertices of the graph.

A *subgraph* of a graph G is a graph G' such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. The subgraph of G *induced* by $V' \neq \emptyset$, denoted by $G[V']$, is the subgraph of G whose vertex set is $V' \subseteq V(G)$ and whose edge set is the set of edges of G that have both endpoints in V'.

Let S be a subset of $E(G)$. We denote the graph with vertex set $V(G)$ and edge set $E(G)\backslash S$ by $G\backslash S$.

A *spanning subgraph* of G is a subgraph of G whose vertex set is $V(G)$. Let G be a graph and G' a subgraph of G. The spanning subgraph of G, denoted by $G\backslash G'$, is the graph such that $V(G\backslash G') = V(G)$ and $E(G\backslash G') = E(G)\backslash E(G')$.

If G is a graph and $I \subseteq V(G)$, $G\backslash I$ denotes the subgraph of G obtained from G deleting all the edges incident on the elements of I and whose vertex set is $V(G\backslash I) = V(G)\backslash I$.

Given two graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, the *union* of G_1 and G_2, here denoted by $G_1 \cup G_2$, is the graph $G = G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. If $V(G_1) \cap V(G_2) = \emptyset$ the union is said to be a *disjoint union* and it is denoted by $G_1 \dot{\cup} G_2$

Given two graphs $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$, the *intersection* of G_1 and G_2, denoted by $G_1 \cap G_2$, is the graph $G = G_1 \cap G_2 = (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$, if $V(G_1) \cap V(G_2) \neq \emptyset$, otherwise $G_1 \cap G_2 = \emptyset$.

A *path* (of length k) in G is a sequence $(v_0, v_1, \ldots, v_k)$ of distinct vertices of G such that $v_{i-1}v_i \in E(G)$, for $i = 1, \ldots, k$. A path from v_0 to v_k in G is denoted by (v_0, v_k)-path, where the vertex v_0 is the *origin*, the vertex v_k is the *terminus* and the vertices $v_1, \ldots, v_{k-1}$ are said *internal* vertices of the path; the edge v_0v_1 is called initial edge. The vertices v_0 and v_k are said to be *joined* by the path. The number of edges is the *length* of the path. A path with n vertices is also denoted by P_n.

A graph is *connected* if any two of its vertices are joined by a path, otherwise the graph is *disconnected*.

The *distance* of two vertices u and v of a connected graph G, $d(u, v)$ is the smallest number of edges in a path between u and v in G.

The *diameter* of a connected graph G, denoted by $diam(G)$ is the maximum of $d(u, v)$ taken over all pairs u, v of vertices in G. For a disconnected graph G, the diameter of G is defined to be the diameter of its largest connected component.

A *cycle* is a connected graph whose vertices are all of degree two. An *acyclic* graph is a graph without cycles.

A *tree* is a connected acyclic graph. Moreover, a tree is a minimal connected graph, in fact deleting any edge it becomes disconnected. Let T_n (or simply T) be a tree with n vertices. We have, $|E(T)| = |V(T)| - 1$. In a tree any two vertices are connected by an unique path. The vertices of T of degree one are called *endpoints*.

A *star* with n vertices, S_n, is a tree in which there is a vertex of degree greater than one, called *central vertex*, and $n-1$ vertices of degree one.

A *generalized star* (or a *starlike tree*) is a tree T having at most one vertex of degree greater than two. This vertex is called the central vertex. Note that this definition also includes the particular case of stars. A *branch* of a generalized star is a path with origin in the central vertex and terminus in an endpoint of the generalized star. A generalized star with $k > 1$ branches of lengths $\ell_1, \ldots, \ell_k$ is denoted by $S_{\ell_1,\ldots,\ell_k}$.

A *matching* on a graph G is a set of edges of G such that no two of them are incident on the same vertex. The largest possible matching on a graph is called a *maximum matching*. The number of edges of a maximum matching on a graph G is called *matching number* of G. A matching that covers every vertex of G is a *perfect matching* of G.

1.3 An overview

In this subsection we present a summary of what is done in the chapters of this thesis. Chapter 2 starts with the definition of the *acyclic Birkhoff polytope*, denoted by $\Omega_n(T)$. Then, we give an interpretation of vertices and edges as well as the faces of any dimension of $\Omega_n(T)$ in terms of graph theory, and we establish a more general result than the one presented in first section – Theorem 1.1.3, due to G. Dahl relating to the diameter of the tridiagonal Birkhoff polytope – for the diameter of the acyclic Birkhoff polytope. As the text develops some illustrative examples are provided. The results of this chapter can be found in [12].

Counting basic objects as the vertices of polyhedra is a demanding problem in general, even for the most basic structured polytopes. In Chapter 3 we determine the number of q-faces, for some $q \geq 1$, of Ω_n^t, the tridiagonal Birkhoff polytope. We present some explicit formulae for counting the number of edges, the number of faces and the number of cells of this polytope. The particular cases of Ω_3^t, Ω_4^t and Ω_5^t are presented, [14].

The summary of Chapter 4 is the following: we start introducing an illustrative example for counting the edges of the polytope associated to a particular tree. Led by this example, we present an algorithm for counting, in the general case, the number of edges of the acyclic Birkhoff polytope. We also give an alternative algorithm to count the number of edges of an acyclic Birkhoff polytope. In Sections 4 and 5 we consider the problem of counting the faces of the acyclic Birkhoff polytope. We also present two algorithms and, in addition, we present some examples. These algorithms allow us to find the number of faces of any acyclic Birkhoff polytope. For stars and starlike trees we present some explicit formulae. Finally, in last sections, we give explicit expressions allowing us to count the number of faces of any dimension of the acyclic Birkhoff polytope in the particular case of a star, and the number of facets of the acyclic Birkhoff polytope. For a general tree it seems harder to present concise formulae. The results of this chapter can be found in [13].

Finally, in Chapter 5, we study the number of vertices of a p-face of Ω_n^t presenting some explicit formulae for particular cases and, an algorithm for the general case. The number of faces of lower dimension that are contained in a face of Ω_n^t is determined and its nature is discussed. In fact, a 2-face of Ω_n^t is a triangle or a rectangle and its cells can only be tetrahedrons, pentahedrons or hexahedrons. This issue is studied in detail, [15].

Chapter 2

The diameter of the acyclic Birkhoff polytope

One of the goals of this chapter is to introduce a new polytope, the acyclic Birkhoff polytope $\Omega_n(T)$, and extend the results established by G. Dahl related with the tridiagonal Birkhoff polytope, Ω_n^t, namely the counting of vertices, the study of its adjacency and the determination of the diameter of its skeleton. These results can be found in [18] and were presented in the previous chapter.

We characterize the acyclic Birkhoff polytope in a matricial way. Also, using as motivation the results presented in [16], already presented in Chapter 1, in a similar way, we establish a result that allow us to determine the dimension of a face of $\Omega_n(T)$. Note that in [16] an expression for the dimension of any face of Ω_n^t is given. Later, we present a way to count the number of vertices of the acyclic Birkhoff polytope and we determine the diameter of its skeleton. The results presented in this chapter can be found in [12].

2.1 Acyclic Birkhoff polytope

Firstly, in this section, we introduce the concept of acyclic Birkhoff polytope:

Definition 2.1.1. The *acyclic Birkhoff polytope*, $\Omega_n(T)$, is the set of nonnegative doubly stochastic matrices whose support correspond to (some subset of) the edges and vertices of a fixed tree T with n vertices.

In fact, $\Omega_n(T)$ is the set of real square matrices $[a_{ij}]$, with nonnegative entries and all rows and columns sums equal to one, such that $a_{ij} = 0$ provided ij is not an edge, for $i \neq j$, in the given tree T.

Since the matrices are doubly stochastic, the diagonal entries are of the form $a_{ii} = 1 - \sum_{j \sim i} a_{ij}$, as we prove in next proposition. Moreover, if $a_{ij} \neq 0$ and $a_{jk} \neq 0$, then $a_{ik} = 0$. To this matrices we call *acyclic* matrices. Note that doubly stochastic tridiagonal matrices are a particular case of acyclic matrices.

Proposition 2.1.2. *Given a tree T with n vertices, each matrix $A = [a_{ij}]$ in $\Omega_n(T)$ is symmetric and*

$$a_{ii} = 1 - \sum_{j \sim i} a_{ij} \ ,$$

for $i = 1, \ldots, n$.

Proof. From definition of doubly stochastic matrix if follows: $a_{ii} = 1 - \sum_{j=1}^{n} a_{ij}$.

As A is nonnegative $\sum_{j=1}^{n} a_{ij} \geq \sum_{j \sim i} a_{ij}$.

As $a_{ij} = 0$ if $i \nsim j$, it results $\sum_{j=1}^{n} a_{ij} \leq \sum_{j \sim i} a_{ij}$.

Therefore, $a_{ii} = 1 - \sum_{j \sim i} a_{ij}$.

The symmetry follows directly from the fact that $A \in \Omega_n(T)$. □

Recalling the structure of $\Omega_n(T)$ and the way to obtain a face of the Birkhoff polytope explained in Chapter 1 and cited in [6], we can say that, as Ω_n^t, the acyclic Birkhoff polytope, $\Omega_n(T)$, is a face of Ω_n. In the present case, $\Omega_n(T)$, the entries equal to zero, resulting from (1.1.1), correspond to the nonexistence of an edge in T. Noting that the vertices of Ω_n are the $n \times n$ permutation matrices, the vertices of $\Omega_n(T)$ are the $n \times n$ permutation matrices whose support correspond to a spanning subgraph of T.

Consider $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$ the set of edges of T ordered lexicographically. For $n \geq 3$, define the polytope

$$T^n = \left\{ x = (x_{e_1}, \ldots, x_{e_{n-1}}) \in \mathbb{R}^{n-1} \ : \ x \geq 0, \text{ and} \sum_{e_j \in E_i} x_{e_j} \leq 1, i = 1, \ldots, n \right\}$$

where each E_i, $i = 1, \ldots, n$, is the set of edges of T incident on the vertex i. From now on, for simplicity, the vector $x \in \mathbb{R}^{n-1}$ with components x_{e_j}, that is, $x = (x_{e_1}, x_{e_2}, \ldots, x_{e_{n-1}})$, will be denoted by $x = (x_{_1}, x_{_2}, \ldots, x_{_{n-1}})$. Moreover, we identify the vertices with their labels. Now we prove that T^n and $\Omega_n(T)$ are affinely isomorphic.

In fact, for each vector $x \in \mathbb{R}^{n-1}$ define the associated $n \times n$ matrix $A_x = [b_{ij}]$ such that,

$$b_{ii} = 1 - \sum_{x_j \in E_i} x_j, \ i = 1, \ldots, n$$

and if $W_i = \{j : j \sim i, j > i\}, b_{i,i+\ell} = x_{i+\ell-1}$ where $\ell \in L_i = \{j - i : \ j \in W_i\}$ and $i = 1, \ldots, n$. Moreover, for each ℓ the entry $b_{i+\ell,i} = x_{i+\ell-1}$. On the other hand, for each $k \in (N(i) \backslash W_i) = W_i'$, we have $b_{i,i-\ell'} = x_{i-\ell'}$, where $\ell' \in L_i' = \{i - k : \ k \in W_i'\}$.

Defined in this way, A_x is symmetric and the entries above its principal diagonal, where for each $k = 1, \ldots, n-1$, x_k lies on the $k+1$ column, are the components of x. If $x \in T^n$, then A_x is doubly stochastic. Therefore $A_x \in \Omega_n(T)$.

Now, we are going to prove that every matrix in $\Omega_n(T)$ has the form A_x for some $x \in T^n$.

Proposition 2.1.3. *The following sets are equal*

$$\{A_x : x \in T^n\} = \Omega_n(T).$$

Proof. The inclusion $\{A_x : x \in T^n\} \subseteq \Omega_n(T)$ follows immediately. For the opposite inclusion, consider a matrix A in $\Omega_n(T)$, for a given tree T with n vertices. Therefore, A has the properties defined in Proposition 2.1.2.

Consider the entries above the principal diagonal and define $x_{i+\ell-1} = a_{i,i+\ell}$ where $\ell \in L_i = \{j - i : \ j \in W_i\}$, $i = 1, \ldots, n-1$ and $W_i = \{j : j \sim i, i > j\}$.

Let $x = (x_1, x_2, \ldots, x_{n-1})$, we will now verify that $A = A_x$:

As A is doubly stochastic $a_{11} = 1 - \sum_{p \sim 1} a_{1p}$.

As $x_\ell = a_{1,\ell+1}$, where $\ell \in L_1 = \{j-1 : \ j \in W_1\}$, $a_{11} = 1 - \sum_{\ell \in L_1} x_\ell$, that is equal to the entry $(1,1)$ of A_x.

For a given $i > 1$, we have $a_{i,i+\ell} = x_{i+\ell-1}$, where $\ell \in L_i = \{j - i : \ j \in W_i\}$, and, for each $k \in W_i'$, $a_{i,k} = a_{i,i-\underbrace{(i-k)}_{\ell'}} = x_{i-\ell'}$.

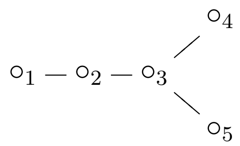
Therefore, as A is doubly stochastic

$$\begin{aligned} a_{ii} = \ & 1 - a_{i,i+\ell_1} - a_{i,i+\ell_2} - \cdots - a_{i,i+\ell_{|W_i|}} - a_{i,i-\ell_1'} - \cdots - a_{i,i-\ell'_{|W_i'|}} \\ = \ & 1 - \sum_{\ell \in L_i} x_{i+\ell-1} - \sum_{\ell' \in L_i'} x_{i-\ell'} \end{aligned}$$

and this entry is equal to the entry (i,i) of A_x, as desired. $\square$

Here, to a bullet circle $\bullet$ and to an open circle, $\circ$, we call, respectively, *closed vertex* and *open vertex* of G. We represent a standard graph with open vertices.

Example 2.1.4. For the tree with five vertices, T_5,



the polytope T^5 is the following set

$$T^5 = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,:\, x \geqslant 0, x_i \leq 1,\ x_1 + x_2 \leqslant 1,\ \text{and } x_2 + x_3 + x_4 \leqslant 1 \right\}.$$

Bearing in mind Proposition 2.1.2, and that T^5 and $\Omega_5(T)$ are affinely isomorphic, we may associate for each vector $x \in T^5$ the following acyclic matrix

$$\begin{bmatrix} 1-x_1 & x_1 & 0 & 0 & 0 \\ x_1 & 1-x_1-x_2 & x_2 & 0 & 0 \\ 0 & x_2 & 1-x_2-x_3-x_4 & x_3 & x_4 \\ 0 & 0 & x_3 & 1-x_3 & 0 \\ 0 & 0 & x_4 & 0 & 1-x_4 \end{bmatrix}.$$

In particular, considering a path P with n vertices, each element of $\Omega_n(P)$ is a tridiagonal matrix and we may state:

Proposition 2.1.5. *[18] Given a path P with n vertices, each matrix of $\Omega_n(P)$ is symmetric, tridiagonal with the form*

$$\begin{bmatrix} 1-x_1 & x_1 & & & & \\ x_1 & 1-x_1-x_2 & x_2 & & & \\ & x_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & x_{n-2} & \\ & & & x_{n-2} & 1-x_{n-2}-x_{n-1} & x_{n-1} \\ & & & & x_{n-1} & 1-x_{n-1} \end{bmatrix}.$$

In the same way, for a star S with n vertices we have:

Proposition 2.1.6. *Given a star S with n vertices, each matrix of $\Omega_n(S)$ is of the form*

$$\begin{bmatrix} 1-x_1-\cdots-x_{n-1} & x_1 & \cdots & x_{n-1} \\ x_1 & 1-x_1 & & 0 \\ \vdots & & \ddots & \vdots \\ x_{n-1} & 0 & \cdots & 1-x_{n-1} \\ \end{bmatrix}.$$

Notice that $\Omega_n(P)$ and $\Omega_n(S)$ are affinely isomorphic, respectively, to

$$P^n = \{x \in \mathbb{R}^{n-1} \,:\, x \geqslant 0 \text{ and } x_i + x_{i+1} \leqslant 1, \text{ for } i = 1, \ldots, n-2\},$$

(cf. [18]), and

$$S^n = \{x \in \mathbb{R}^{n-1} \,:\, x \geqslant 0 \text{ and } x_1 + \cdots + x_{n-1} \leqslant 1\}\ ,$$

respectively.

2.2 Faces of $\Omega_n(T)$

In this section we present more specific notation.

Given an usual graph G we can consider a subgraph of G with the property of assign to each of its vertices two possible colors.

Definition 2.2.1. A *bicolored (vertex) subgraph of G* is a subgraph G' of G such that $G' = (V(G'), E(G'))$ with $E(G') \subseteq E(G)$ and the vertex set is a subset of $V(G)$ where some vertices can be closed, *i.e.*, $V(G')$ can be partitioned into $V_\bullet \dot{\cup} V_\circ$.

In the literature the concept of bicolored graph is also known as 2-stratified graph *i.e.*, a graph where the vertex set is partitioned into two subsets (cf. [11]).

Using the results presented in [16] and described in first chapter as motivation, we present here a different approach for the structure of the faces of the acyclic Birkhoff polytope, $\Omega_n(T)$.

In this section we study the faces of $\Omega_n(T)$. We have already seen in Chapter 1 that $\Omega_n(T)$ is a face of Ω_n, therefore the faces of $\Omega_n(T)$ are the faces of Ω_n which are contained in $\Omega_n(T)$. Also, it was stated in [6] and presented in Chapter 1 that the faces of Ω_n, $\mathcal{F}_{\mathcal{A}}$, are in one-to-one correspondence with $n \times n$ matrices having total support.

That a matrix whose pattern is the same as the pattern of a doubly stochastic matrix has total support also follows from the fact that the extreme points of Ω_n are permutation matrices, [22, Corollary 1]. A doubly stochastic matrix A is then a convex combination of permutation matrices, that is there exist permutation matrices $P_1, P_2, \ldots, P_t$, $(1 \leq t \leq n)$, and positive real numbers $c_1, c_2, \ldots, c_t$ with $c_1+c_2+\cdots+c_t = 1$ such that $A = c_1P_1+c_2P_2+\cdots+c_tP_t$, [4, Theorem 1.7.1], and therefore every nonzero entry of A belongs to a nonzero diagonal, [4, pg 381]. Furthermore, from the above discussion, and apart from row and column permutation, the pattern of a doubly stochastic matrix is a direct sum of fully indecomposable $(0,1)-$matrices, [4, pg 381].

In the case of $\Omega_n(T)$, the faces of $\Omega_n(T)$ are in one-to-one correspondence with $n \times n$ acyclic $(0,1)-$matrices having total support. Let, then, A be an $n \times n$ acyclic $(0,1)-$matrix having total support and consider the face associated to it, denoted by

$$\mathcal{F}_{\mathcal{A}} = \{X \in \Omega_n(T) : a_{ij} = 0 \Rightarrow x_{ij} = 0\}.$$

Apart from permutation of rows and columns, A is a direct sum of blocks

A_t, $A = A_1 \oplus \cdots \oplus A_p$, where each A_t, $t = 1, \ldots, p$ is of the following types:

Type 1 $A_t = [1]$, an 1×1 matrix;

Type 2 $A_t = K$, where K is the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$;

Type 3 This type is not one of the previous two types and at least two of its diagonal entries are equal to one. These matrices are symmetric and in each row there are at least two entries equal to one. If its entries $a_{ij} = 1$ and $a_{ik} = 1$ then $a_{jk} = 0$, for $k \neq i, j$. Moreover, if $a_{ij} = 1$ and for all $k \neq i, j, a_{ik} = 0$, then $a_{ii} = 1$, otherwise A would not have total support.

Example 2.2.2. For example the matrix

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

is a matrix of Type 3.

Definition 2.2.3. A T*-matrix* is a symmetric matrix described in Type 3. The blocks A_t of A of Type 3 are called T*-blocks* of A.

Definition 2.2.4. If B is a T-matrix of order n we say that b_{ii} is an *inner entry* of B if $b_{ii} = 1$ and in row (column) i there exist at least three entries equal to 1.

In the previous example, b_{33} is an inner entry.

Similarly to what was mentioned in Chapter 1, if a matrix $A = [a_{ij}]$ is of order n we may represent its nonzero structure by a bicolored digraph

$DG_b(A)$. The vertex set of $DG_b(A)$ is a set $V = \{1, \dots, n\}$ and there is an arc (i, j) from i to j if and only if $i \neq j$ and $a_{ij} \neq 0$, $(i, j = 1, \dots, n)$, and the vertex i is closed if and only if $a_{ii} \neq 0$. If the matrix $A = [a_{ij}]$ is also symmetric we may represent its nonzero structure by a bicolored graph $G_b(A)$. The vertex set of $G_b(A)$ is a set $V = \{1, \dots, n\}$ and there is an edge ij from i to j if and only if $i \neq j$ and $a_{ij} \neq 0$, $(i, j = 1, \dots, n)$, and the vertex i is closed if and only if $a_{ii} \neq 0$.

As a face of $\Omega_n(T)$ can be represented by an acyclic, total support $(0,1)-$matrix, that (apart permutation of rows and columns) is a direct sum of blocks of Types 1, 2 and 3, we can associate to each block of previous type a bicolored subgraph $G_b(A_j)$, for each $j = 1, \dots, t$. Moreover, as each acyclic total support $(0,1)-$matrix $A = [a_{ij}]$ can be written as a direct sum of blocks A_t, we can associate to a matrix A of this type, the following finite union of bicolored subgraphs $G_b(A_1)\dot{\cup}G_b(A_2)\dot{\cup}\cdots\dot{\cup}G_b(A_t)$ (displayed in this order). Therefore the following correspondence can be established

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_t \longmapsto G_b(A_1)\dot{\cup}G_b(A_2)\dot{\cup}\cdots\dot{\cup}G_b(A_t)$$

Recall that R. Brualdi established the correspondence $A \mapsto \mathcal{F}_A$, for a total support $(0,1)-$matrix, [4, pg 382]. Then, it follows that to a face $\mathcal{F}_A$ of the acyclic Birkhoff polytope we can associate the union of a finite number of bicolored subgraphs $G_b(A_1)\dot{\cup}G_b(A_2)\dot{\cup}\cdots\dot{\cup}G_b(A_t)$, which one can be of the following types:

Type 1 A *closed vertex*, $\bullet$.

Note that from the above definition, to this type of bicolored subgraph we associate an 1×1 matrix $A = [1]$ (the matrix of Type 1 described above).

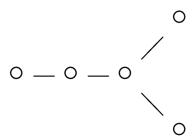
Type 2 An *open edge* $\circ - \circ$.

To an open edge we associate the matrix
$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (the matrix of Type 2 described above).

Type 3 This type is not one of the previous two types, and is a connected bicolored subgraph of T, with all endpoints closed.

Definition 2.2.5. A T-*component* is a bicolored subgraph of T of Type 3. An *inner entry* of a T-component is a closed vertex which is not an endpoint.

Example 2.2.6. Consider the tree with five vertices T_5

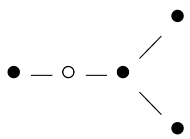


$$\Omega_5(T) = \left\{ X \in \Omega_5 : X = \begin{bmatrix} 1-x_1 & x_1 & 0 & 0 & 0 \\ x_1 & 1-x_1-x_2 & x_2 & 0 & 0 \\ 0 & x_2 & 1-x_2-x_3-x_4 & x_3 & x_4 \\ 0 & 0 & x_3 & 1-x_3 & 0 \\ 0 & 0 & x_4 & 0 & 1-x_4 \end{bmatrix} \right\}.$$

The matrix

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

has total support (for $r, s = 1, \ldots, 5$, $b_{rs} = 1$ implies that there exists a permutation matrix $P = [p_{ij}]$ with $p_{rs} = 1$ and $P \leq B$). We can associate to B the bicolored subgraph



Example 2.2.7. Given the path P_5, the path with 5 vertices,

$$\circ - \circ - \circ - \circ - \circ\ ,$$

one of its bicolored subgraph of Type 3 is, for example,

$$\bullet - \bullet - \circ - \bullet\ .$$

Therefore, its associated matrix is:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The next matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

has total support (it can be directly verified using the definition) and it is the direct sum of a block of Type 1, $A_1 = [1]$, and the T-block presented immediately above. This matrix represents a face of Ω_5^t, and this face is also represented by the union of two bicolored subgraphs:

$$\bullet \quad \bullet - \bullet - \circ - \bullet\ .$$

As we said in Chapter 1, a matrix can be represented by a bipartite graph. Let $H = [h_{ij}]$ be a T-block of order n, $(n > 1)$ and let $BG(H)$ be the bipartite graph associate to H. Then $V(BG(H)) = U \dot{\cup} W$, where $U = \{u_1, \ldots, u_n\}$ and $W = \{w_1, \ldots, w_n\}$ are the sets of vertices corresponding, respectively, to rows and columns of H. As H is a T-block, it has in each row (column) at least two entries equal to one, and it has at least two rows (columns) where there are exactly two entries equal to one. Moreover for $i \neq j$, if $h_{ij} = 1$ then $h_{ji} = 1$. From the support of the matrix H we have:

- $BG(H)$ is connected;
- $\forall\, v \in V(BG(H)), d(v) \geq 2$;
- $\forall\, i = 1, \ldots, n, d(u_i) = d(w_i)$ and it is equal to the number of 1's that are in the respective i-th row (column);
- $u_i \sim w_j \Rightarrow w_i \sim u_j$;
- $i, l \neq j$ and $u_i \sim w_j \wedge u_l \sim w_j \Rightarrow N(u_i) \cap N(u_l) = \{w_j\}$;
- $\forall i, j = 1, \ldots, n$, u_i and u_j have at least a distinct neighbor, that is, $N(u_i) \neq N(u_j)$.

Next lemma is relevant to prove that a T-block H has total support.

Lemma 2.2.8. *Any edge of $BG(H)$ belongs to a perfect matching on it.*

In order to prove this lemma we need the following theorem:

Theorem 2.2.9. *[3, Hall's theorem, Theorem 16.4] A bipartite graph $G = [X, Y]$ has a matching which covers every vertex in X if and only if*

$$|N(S)| \geq |S| \text{ for all } S \subseteq X.$$

Proof. Next we present the proof of previous lemma:

The case $n \leq 3$ is trivial.

Suppose that $n > 3$. Let $u_i w_j$ be any edge in $BG(H)$, and let $G = BG(H)\backslash\{u_i, w_j\}$. Then we have:

- G is bipartite and $V(G) = U\backslash\{u_i\}\dot{\cup}W\backslash\{w_j\}$;

- $E(G) = E(BG(H))\backslash\left(\{u_i w_k : w_k \in N(u_i)\} \cup \{u_l w_j : u_l \in N(w_j)\}\right)$

- $d_G(w_k) = d_{BG(H)}(w_k) - 1$, for all $w_k \in N(u_i)$.

- $d_G(u_l) = d_{BG(H)}(u_l) - 1$, for all $u_l \in N(w_j)$.

- $d_G(v) \geq 1$, for all $v \in G$ and in G, $|N(u_k) \cup N(u_l)| \geq 2$ were u_k and u_l are distinct neighbors in $BG(H)$ of w_j.

Now we use Theorem 2.2.9. Let U' be any subset of $U\backslash\{u_i\}$ and $N(U') = \{w_k \in W\backslash\{w_j\} : \exists u_l \in U' \;\; u_l w_k \in E(G)\}$ we will see that $|U'| \leq |N(U')|$.

In fact if $|U'| = 2$ we have to consider the following situations:

- If for all $u \in U'$, $u \notin N(w_j)$, then $d_G(u) = d_{BG(H)}(u) \geq 2$ and $|N(U')| \geq 3$. Therefore the previous inequality is true.

- If $\exists u \in U' : u \in N_{BG(H)}(w_j)$, as in $BG(H)$, $|N(u_k) \cup N(u_l)| \geq 3$ (note that u_k and u_l can have a common neighbor) and $w_j \in N_{BG(H)}(U')$, then, in G, $N_G(U') = (N(u_k) \cup N(u_l))\backslash\{w_j\}$ and the inequality holds.

  The proof follows by induction on the cardinality of U'. Suppose that the inequality is true for U' such that $|U'| < n$. Join a vertex u_l to U' and consider $U^{\star}$=$U' \cup \{u_l\}$. We can have:

- $u_l \notin N_{BG(H)}(w_j)$, then it has in G at least two neighbors and each of them can not be simultaneously neighbor of two vertices in U' that share a common neighbor (w_j or any other vertex in common) otherwise the matrix H would not be acyclic. Therefore $|N(U^\star)| \geq |N(U')|+1$ then, by induction hypothesis follows, $|U^\star| = |U'|+1 \leq |N(U')| + 1 \leq |N(U^\star)|$.

  If none of elements in U' share a neighbor, $|U'| < |N(U')|$. The vertex u_l, can be neighbor of all neighbors of the elements in U', but $|U'| + 1 \leq |N(U')|$, therefore, $|N(U^\star)| \geq |N(U')| \geq |U'| + 1 = |U^\star|$. The inequality holds.

- Consider now that $u_l \in N_{BG(H)}(w_j)$. By induction hypothesis we have $|U'| \leq |N(U')|$. If we have $|U'| < |N(U')|$, $|U'| + 1 \leq |N(U')|$, furthermore, $|N(U')| \leq |N(U^\star)|$, then $|U^\star| = |U'| + 1 \leq |N(U^\star)|$. If $|U'| = |N(U')|$, this means that w_j was neighbor of some (eventually, all) elements of U' or the elements of U' share neighbors. The added vertex, u_l, cannot be neighbor of a neighbor of an element of U' that was neighbor of w_j, otherwise H is not acyclic. By the same argument, u_l can not be neighbor of any element that was neighbor of a vertex of U' that shares neighbors. Therefore in $|N(u_l)|$ there are at least one element that does not belong to $N(U')$. Therefore $N(U^\star) \geq |N(U')|+1 \geq |U'|+1 = |U^\star|$

Then by, Hall's Theorem, there exists in G a perfect matching, M', with $|M'| = n - 1$.

Thus , $M = \{u_i w_j\} \cup M'$ is a perfect matching in $BG(H)$.

□

Note that the bipartite graph associated to a permutation matrix of order

n, P, has n two by two disjoint edges. If H is an $n \times n$ matrix, such that $P \leq H$, $BG(P)$ is a matching of $BG(H)$. In addition, the first bipartite graph is a perfect matching of the last one.

The following proposition will be useful to establish an expression to calculate the dimension of any face of the acyclic Birkhoff polytope.

Proposition 2.2.10. *Any T-block of order n, $(n > 1)$ has total support and is fully indecomposable.*

Proof. Recalling the definition of a total support matrix for a T-block $H = [h_{ij}]$, for each entry $h_{ij} = 1$ there exists a permutation matrix P such that $p_{ij} = 1$ and $P \leq H$. This means that for each edge $u_i w_j$ of the bipartite graph associate to H, $BG(H)$, there is a perfect matching M in $BG(H)$, such that $u_i w_j \in M$.
Therefore, if H is a T-block, it follows straightforward from previous lemma that H is a matrix with total support.
The matrix H has total support and $BG(H)$ is a connected graph, by [4, Theorem 1.3.7] we conclude that H is fully indecomposable.

□

As for the case of tridiagonal Birkhoff polytope it only matters the multiset of T-block of A in the study of a single face $\mathcal{F}_A$, [16], [5, Corollary 2.5]. In fact, according [6, 7], if A is a $(0,1)-$matrix of order n with total support and considering that $A = A_1 \oplus \cdots \oplus A_p$ has a trivial fully indecomposable component, then A is similar to a matrix A^* such that $A^* = [1] \oplus A'$ where A' is the matrix obtained from A^* by striking out the first row and the first column. Each matrix $X = [x_{ij}]$ in $\mathcal{F}_{A^*}$ has $x_{11} = 1$ and $x_{1j} = x_{i1} = 0,\ i, j = 2, \ldots, n$, then $\mathcal{F}_{A^*}$ and $\mathcal{F}_{A'}$ have the same essential combinatorial and geometric structure, [8, Theorem 2.1]. And the same happens with

$\mathcal{F}_A$. Thus trivial fully indecomposable components can be omitted from the dimension discussion, [4, pg 385].

From [6, Theorem 2.5] and [5, Theorem 2.7], if B, of order m, has total support and is fully indecomposable, the dimension of $\mathcal{F}_B$, $\dim \mathcal{F}_{\mathcal{B}}$, is

$$\dim \mathcal{F}_B = \sigma_B - 2m + 1,$$

where σ_B is the number of entries equal to one in B. So, for a T-component, and considering its T-block B (as it is fully indecomposable and has total support),

$$\dim \mathcal{F}_B = \theta_B - 1 + w,$$

where w and θ_B are, respectively, the number of inner entries and closed endpoints of the T-component. In fact, σ_B is equal to the number of closed endpoints, θ_B plus the number of inner entries, w, and $2(m-1)$ entries equal to one ($m-1$ entries equal to one above and below the principal diagonal of the $T-$block, respectively). Therefore,

$$\dim \mathcal{F}_B = \theta_B + w + (m-1) + (m-1) - 2m + 1 = \theta_B - 1 + w.$$

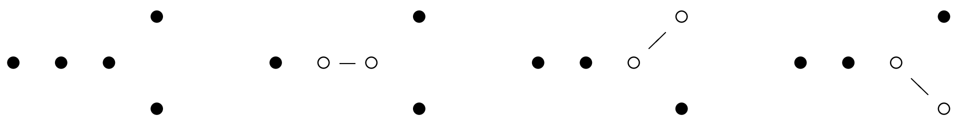
Bearing in mind that $\dim \mathcal{F}_A$ is the sum of $\dim \mathcal{F}_{A_i}$ for the T-blocks A_i of A, [6, Corollary 2.6], we may state the following proposition:

Proposition 2.2.11. *Let t_A be the number of T-components in the union of the bicolored subgraph of T that represents a face $\mathcal{F}_A$. Let θ_A and ι_A be, respectively, the sum of all closed endpoints and the number of inner entries in all T-components. Then*

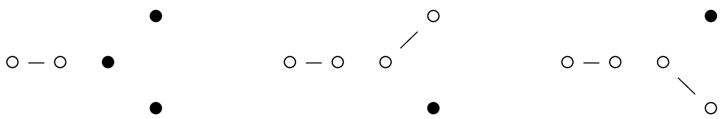
$$\dim \mathcal{F}_A = \theta_A + \iota_A - t_A \, .$$

From now on we only work with bicolored subgraphs. As a consequence of the previous proposition, and as a vertex of a polytope has dimension 0, each vertex of $\Omega_n(T)$ will be identified with a bicolored subgraph of T whose diameter is at most one. In this case we only have the union of bicolored subgraphs of Type 1 and bicolored subgraphs of Type 2.

Example 2.2.12. For the graph T defined in Example 2.1.4, the seven vertices of $\Omega_5(T)$ are:



V_1 V_2 V_3 V_4



V_5 V_6 V_7

These vertices correspond respectively to the following permutation matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Recalling the structure of the faces of $\Omega_n(T)$ previously presented, for example, the vertex V_4 is a 0-face which is the union of three bicolored subgraphs of Type 1 and one bicolored subgraph of Type 2.

2.3 Counting vertices and edges

We are now able to establish a recurrence relation to count the number of vertices of $\Omega_n(T)$, for a given tree T with n vertices. In general, we denote by $f_0(T)$ the number of vertices (0-faces) of the polytope $\Omega_n(T)$ and by $f_{0,ij}(T)$ the number of bicolored subgraphs of T that contains the edge ij and whose diameter is at most one. Note that the number of vertices of $\Omega_n(T)$ is the number of matchings in T.

Let ij be any edge of the tree T. We have

$$f_0(T) = f_0(T \setminus ij) + f_{0,ij}(T), \tag{2.3.1}$$

with initial conditions $f_0(\emptyset) = f_0(v) = 1$, where v is a vertex of T.

Recall that G. Dahl, [18], stated for a path P with n vertices, that

$$f_0(P) = f_{n+1}, \tag{2.3.2}$$

where f_{n+1} is the $(n+1)$-th Fibonacci number.

The previous relation satisfies (2.3.1). In fact, let P_n be a path with n vertices, and consider one of its edges. Without loss of generality we can choose the first edge considered from the left to the right, u_1u_2. Applying (2.3.1), we obtain $f_0(P_n) = f_0(P_n \backslash u_1u_2) + f_{0,u_1u_2}(P_n)$; the first summand is $f_0(P_{n-1}) = f_n$ and the second one is equal to $f_0(P_{n-2}) = f_{n-1}$. Taking into account the recurrence relation (1.1.4), we obtain $f_0(P_n) = f_{n+1}$ and (2.3.1) holds.

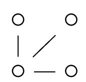
Notice that if

$$G = T_{n_1} \cup T_{n_2} \cup \cdots \cup T_{n_p} \ ,$$

with $n_1, \ldots, n_p$ positive integers, and T_{n_j} are disjoint trees, for $j = 1, \ldots, p$, the number of bicolored subgraphs of G whose diameter is at most one, denoted by $g_0(G)$, follows straightforward from the characterization of the vertices of the acyclic Birkhoff polytope as the union of a finite number of bicolored subgraphs of Types 1 and 2, and is given by

$$g_0(G) = f_0(T_{n_1}) \times f_0(T_{n_2}) \times \cdots \times f_0(T_{n_p}) \tag{2.3.3}$$

Example 2.3.1. Let $S = S_{1,1,1}$ be the star with four vertices presented below:



Let ij be any edge of S. The number of vertices of $\Omega_4(S)$ is $f_0(S) = f_0(S \backslash ij) + f_{0,ij}(S) = f_0(P_3) + f_{0,ij}(S) = 3 + 1 = 4$. The vertices of $\Omega_4(S)$ are represented by the 4×4 permutation matrices:

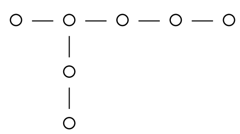
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and recalling the identification of each vertex of the polytope $\Omega_4(S)$ as a bicolored subgraph of S whose diameter is at most one, to each previous permutation matrix corresponds one of the following bicolored subgraphs

The first bicolored subgraph (that results from the union of four bicolored subgraphs of Type 1) and two of the others correspond to the number $f_0(S\backslash ij)$ and the reminder one corresponds to $f_{0,ij}(S)$.

Example 2.3.2. For the graph presented in Example 2.1.4, the number of vertices of $\Omega_5(T)$ is $f_0(T_5) = f_0(T_5 \setminus ij) + f_{0,ij}(T_5) = 4 + 3 = 7$, where ij is taken as the first edge considered from the left to the right. The vertices were presented in Example 2.2.12.

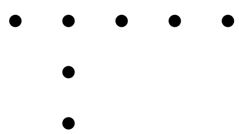
Example 2.3.3. Let $S' = S_{1,2,3}$ be the starlike tree presented below:



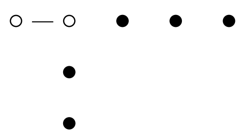
The number of vertices of $\Omega_7(S')$ is $f_0(S') = f_0(S' \backslash ij) + f_{0,ij}(S')$ where ij is any edge. Therefore if ij is the first edge considered from the left to the right we obtain:

$$f_0(S') = f_0(P_6) + f_0(P_3) \times f_0(P_2) = 13 + 3 \times 2 = 19.$$

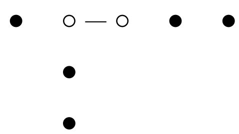
We present the nineteen vertices of S':



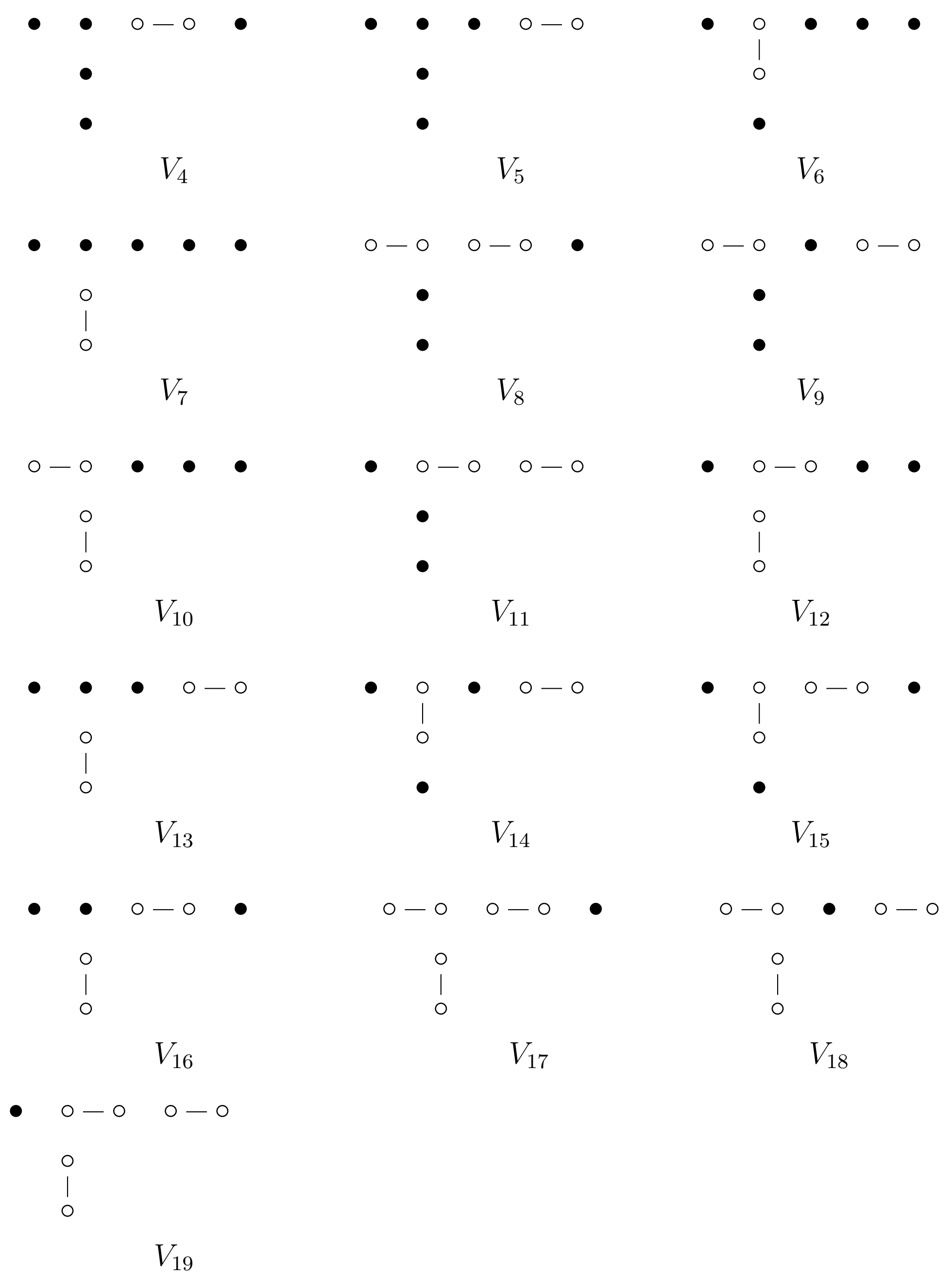
V_1



V_2



V_3



Proposition 2.3.4. *Let $G = S_{q,\dots,q}$ be a generalized star with n branches, with n, q positive integers. Then*

$$f_0(G) = f_{q+1}^{n-1}\left(f_{q+1} + n f_q\right) \ ,$$

where f_q and f_{q+1} are the q-th and the $(q+1)$-th Fibonacci's numbers, respectively.

Proof. The proof follows by induction on n. In fact, for $n = 1$, G is a path with $q + 1$ vertices, and by the recurrence relation presented in (2.3.1) the result follows easily.

If $G = S_{q,\ldots,q}$ is a generalized star with n branches, then by induction hypothesis $f_0(G) = f_{q+1}^{n-1}\,(f_{q+1} + nf_q)$.

Let now G' be a star with $n+1$ branches of length q. We have

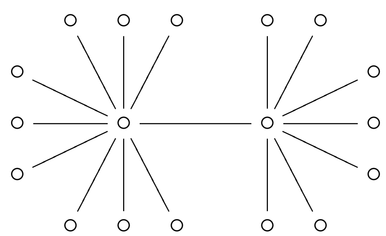
$$\begin{array}{rcl} f_0(G') & = & f_0(G) \times f_0(P_q) + f_0(P_{q-1}) \times (f_0(P_q))^n \\ & = & f_{q+1}^{n-1}\,(f_{q+1} + nf_q)\,f_{q+1} + f_q \times f_{q+1}^n \\ & = & f_{q+1}^n\,(f_{q+1} + (n+1)f_q) \end{array}.$$

□

In the particular case of $q = 1$ we have $f_0(S) = n + 1$.

Given two generalized stars, $S_{p,\ldots,p}$ and $S_{q,\ldots,q}$, with m and n branches, respectively, a *double generalized star*, $G = G(S_{p,\ldots,p}, S_{q,\ldots,q})$, is the tree resulting from $S_{p,\ldots,p}$ and $S_{q,\ldots,q}$ by joining their central vertices by an edge.

Example 2.3.5. Consider the double generalized star:



Using formulae (2.3.1) and (2.3.3), and previous proposition when $q = 1$, we obtain $f_0(G) = 10 \times 8 + 1$.

Proposition 2.3.6. *Let $G = G(S_{p,\ldots,p}, S_{q,\ldots,q})$ be a double generalized star with m and n branches, respectively. Then*

$$f_0(G) = f_{p+1}^{m-1}\left(f_{p+1} + mf_p\right) \cdot f_{q+1}^{n-1}\left(f_{q+1} + nf_q\right) + f_p^m \cdot f_q^n\,.$$

Proof. Consider the edge incident in both central vertices of $S_{p,\ldots,p}$ and $S_{q,\ldots,q}$ and apply (2.3.1),

$$f_0(G) = \quad f_0\left(S_{p,\ldots,p}\dot{\cup}S_{q,\ldots,q}\right) + f_0\left(\cup_{i=1}^{m}(P_{p-1})_i\dot{\cup}\cup_{j=1}^{n}(P_{q-1})_j\right)$$

were for each i and for each j, $(P_{p-1})_i$ and $(P_{q-1})_j$ denote, respectively, a path resulting from each branch of $S_{p,\ldots,p}$ and of $S_{q,\ldots,q}$ obtained by removing the edge incident on the respective center. By (2.3.3) we have

$$= \quad f_0\left(S_{p,\ldots,p}\right) \cdot f_0\left(S_{q,\ldots,q}\right) + \left(\prod_{i=1}^{m} f_0(P_{p-1})_i \cdot \prod_{j=1}^{n} f_0(P_{q-1})_j\right)$$

and applying Proposition 2.3.4, and (2.3.2) we obtain

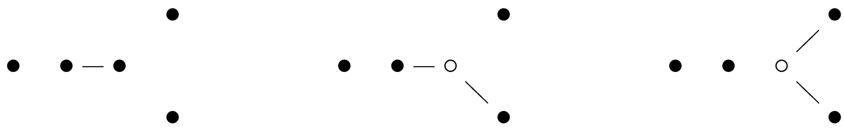
$$= \quad f_{p+1}^{m-1}(f_{p+1} + mf_p) \cdot f_{q+1}^{n-1}(f_{q+1} + mf_q) + (f_m)^p \cdot (f_n)^q$$

□

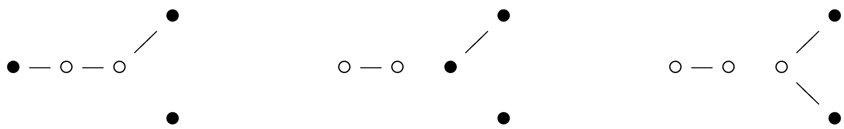
In the particular case of $p = q = 1$ we have $f_0(G) = (m+1)(n+1) + 1$.

Since an edge (1-face) of the acyclic Birkhoff polytope $\Omega_n(T)$, is the union of bicolored subgraphs of Type 1, Type 2, and exactly one bicolored subgraph of Type 3, without inner entries and with two closed endpoints, we can also describe the edges of $\Omega_n(T)$. Next, we provide some examples of the fifteen edges of $\Omega_5(T)$.

Example 2.3.7. For the graph T defined in Example 2.1.4, some of the fifteen edges of $\Omega_5(T)$ are:



V_1V_2 V_2V_4 V_3V_4

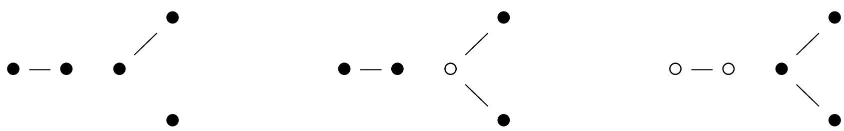


V_2V_6 V_5V_6 V_6V_7

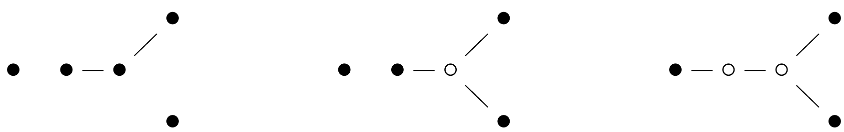
By Proposition 2.2.11, a 2-face of the acyclic Birkhoff polytope $\Omega_n(T)$ is the union of bicolored subgraphs of Type 1, Type 2 and one bicolored subgraph of Type 3 with one inner entry and two closed endpoints; or the union of bicolored subgraphs of Type 1, Type 2 and one bicolored subgraph of Type 3 with three closed endpoints and without inner entries; or the union of bicolored subgraphs of Type 1, Type 2 and two bicolored subgraphs of Type 3 each one with two closed endpoints and without inner entries.

In a 2-face (or simply a *face*) we have at least one bicolored subgraph of Type 3. If the vertices V_i, V_j and V_k belong to a face we denote this face by $V_iV_jV_k$.

Example 2.3.8. Some 2-faces of $\Omega_5(T)$:

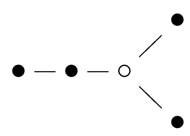


$V_1V_3V_6V_5$ $V_3V_4V_7V_6$ $V_5V_6V_7$



$V_1V_2V_3$ $V_2V_3V_4$ $V_2V_6V_7$

Example 2.3.9. A 3-face (a *cell*) of $\Omega_5(T)$:



2.4 Adjacency of vertices of $\Omega_n(T)$

In this section we will present a result that allow us to say when two vertices of $\Omega_n(T)$ are adjacent. Let $G_1 = (V(G_1), E(G_1)), G_2 = (V(G_2), E(G_2))$ be two bicolored subgraphs of order p, $p \geq 1$ of G, where $V(G_1) = V_\bullet^1 \dot\cup V_\circ^1$ and $V(G_2) = V_\bullet^2 \dot\cup V_\circ^2$. We define *bicolored sum of subgraphs* G_1 *and* G_2 as the bicolored subgraph of G, such that

$$G_1 \boxplus G_2 = G(V_\circ \dot\cup V_\bullet, E(G_1) \cup E(G_2))$$

where $V_\circ = V_\circ^1 \cap V_\circ^2$ and $V_\bullet = V_\bullet^1 \cup V_\bullet^2$, *i.e.*,

$$\begin{array}{rcl} \circ^1 \boxplus \circ^2 & = & \circ, \\ \bullet^1 \boxplus \circ^2 & = & \bullet, \\ \circ^1 \boxplus \bullet^2 & = & \bullet, \\ \bullet^1 \boxplus \bullet^2 & = & \bullet, \end{array}$$

where $\circ^1, \circ^2, \bullet^1, \bullet^2$ denote the open and closed vertices of the bicolored subgraphs G_1 and G_2, respectively.

The cell presented in Example 2.3.9 is obtained from the bicolored sum of faces $V_3V_4V_6$ and $V_2V_3V_4$ or from the faces $V_2V_6V_7$ and $V_2V_3V_4$.

Next, we establish an adjacency criterium for the vertices of the acyclic Birkhoff polytope, $\Omega_n(T)$. Firstly we introduce the concepts of complementary bicolored subgraphs.

Definition 2.4.1. Given the path P_n with n vertices

$$\circ - \circ - \circ - \circ - \circ - \circ \cdots$$

the spanning bicolored subgraphs

$$\bullet \quad \circ - \circ \quad \circ - \circ \cdots$$

and

$$\circ - \circ \quad \circ - \circ \quad \circ \cdots$$

are said *complementary* (in P_n).

Remark 2.4.2. If P_n has odd order then the following two bicolored subgraphs

$$\bullet \quad \circ - \circ \quad \circ - \circ \quad \cdots \quad \circ - \circ$$

and

$$\circ - \circ \quad \circ - \circ \quad \cdots \quad \circ - \circ \quad \bullet$$

are complementary.

If P_n has even order then the two bicolored subgraphs

$$\bullet \quad \circ - \circ \quad \circ - \circ \quad \cdots \quad \circ - \circ \quad \bullet$$

and

$$\circ - \circ \quad \circ - \circ \quad \cdots \quad \circ - \circ \quad \circ - \circ$$

were $\cdots$ represents the union of open edges, are complementary.

Let H_1 and H_2 be the union of a finite number of bicolored subgraphs of Types 1 and 2 obtained from T and P'_k and $let P''_k$, $k \leq n$, be the union of a finite number of bicolored subgraphs of Types 1 and 2 obtained from a path with k vertices. Let V_1, V_2 be two vertices of $\Omega_n(T)$. Suppose that $\dot{\cup}$ denotes the usual union of graphs,

$$V_1 = H_1 \dot{\cup} P'_k \dot{\cup} H_2$$

and

$$V_2 = H_1 \dot{\cup} P_k'' \dot{\cup} H_2.$$

The edge that contains V_1 and V_2, if exists, is of the form:

$$H_1 \quad \bullet - \circ - \circ - \cdots - \circ - \bullet \quad H_2$$

We point out that the bicolored sum of the bicolored subgraphs corresponding to the $0-$faces V_1 and V_2 gives only one T-component with two closed endpoints and no inner entries.

If the sequence of open edges described in the two situations of remark (2.4.2) is empty we have the minimal case of P_2' and P_2'' as follows

$$H_1 \quad \bullet \quad \bullet \quad H_2$$

$$H_1 \quad \circ - \circ \quad H_2$$

The previous observations lead to the main theorem of this section:

Theorem 2.4.3. *Let H_1 and H_2 be the union of a finite number of bicolored subgraphs of Types 1 and 2 obtained from T and P_k' and P_k'' are the union of a finite number of bicolored subgraphs of Types 1 and 2 obtained from a path with k vertices. Let V_1, V_2 be two vertices of $\Omega_n(T)$, then V_1 and V_2 are adjacent if and only if*

$$V_1 = H_1 \dot{\cup} P_k' \dot{\cup} H_2$$

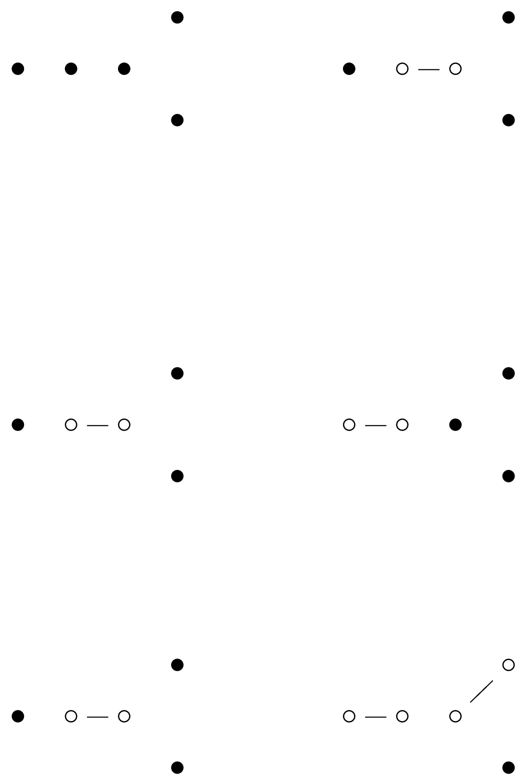
and

$$V_2 = H_1 \dot{\cup} P_k'' \dot{\cup} H_2.$$

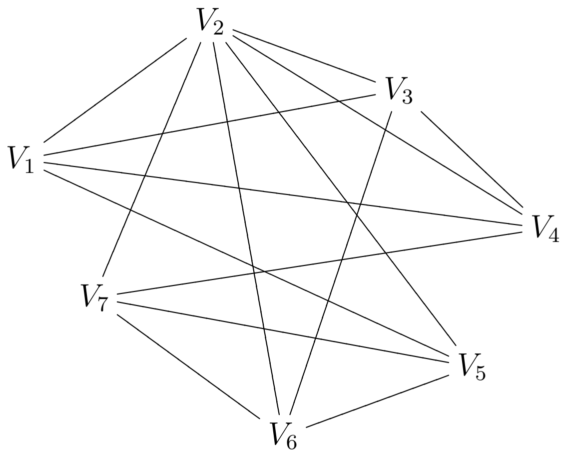
and P_k' and P_k'' are complementary.

The adjacency criterion presented for the tridiagonal Birkhoff polytope by G. Dahl, in a matricial form, in Chapter 1, (also see [18]), follows straightforward from the previous theorem. In fact to the blocks J and K presented in Theorem 1.1.2*(iii)* correspond bicolored subgraphs of Type 1 and Type 2, respectively, and those sequences of blocks that interchanges are here replaced by complementary bicolored subgraphs.

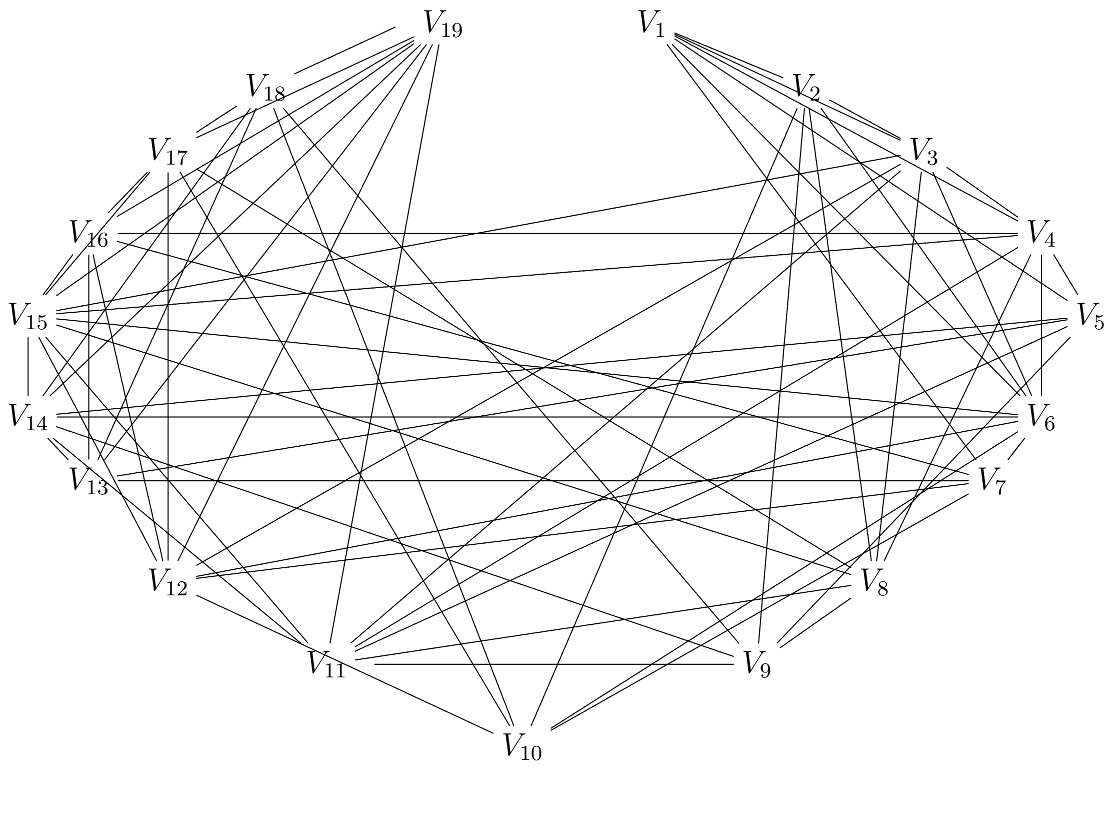
Example 2.4.4. Consider again Example 2.2.12. By Theorem 2.4.3 the following pairs of vertices of the polytope $\Omega_5(T)$ are adjacent:



From the previous criterion and the vertices of Example 2.1.4 we can establish the adjacency relations for all vertices of the polytope $\Omega_5(T)$, and obtain the edges and faces of the respective polytope. See the figure below:



Bearing in mind the 19 vertices of $\Omega_7(S')$, where $S' = S_{1,2,3}$, and using the adjacency criterion, we obtain the skeleton of the respective polytope that is depicted in the next figure:



2.5 The diameter of $G(\Omega_n(T))$

In this section, for a given tree with n vertices we present the diameter of $G(\Omega_n(T))$, the graph of $\Omega_n(T)$, whose vertices and edges correspond to vertices and edges of the polytope $\Omega_n(T)$. Actually this section generalizes an expression for the diameter of the tridiagonal Birkhoff polytope and the result can be seen in Chapter 1, Theorem 1.1.3, see also [18]. In fact the diameter of Ω_n^t is equal to $\lfloor \frac{n}{2} \rfloor$, where n is the number of vertices of a given path.

Recalling the definition of matching in a graph, the *matching number* of G, $\beta(G)$, is the number of edges present in a maximum matching.

We determine in the next theorem the diameter of $G(\Omega_n(T))$ which is defined as the maximum of $d(V, V')$, taken over all pairs of vertices V, V', where $d(V, V')$ is the smallest length of a path between V and V' in the connected graph $G(\Omega_n(T))$.

Theorem 2.5.1. *Given a tree T with n vertices, the diameter of $G(\Omega_n(T))$ is equal to $\beta(T)$.*

Proof. We start by proving that there are two vertices V and V' such that $d(V, V') = \beta(T)$. In fact, let V and V' be two vertices of $\Omega_n(T)$ such that there are no common open edges (at same place) in their representations and the union of the open edges in each bicolored subgraph consists in a maximum matching in T.

For example, V is represented by the bicolored subgraph that is just the union of subgraphs of Type 1, (*i.e.*, closed vertices), and V' is a vertex whose bicolored subgraph is composed by a maximum matching in T (*i.e.*, considered with closed vertices and open edges).

Let V_1 be a vertex of $\Omega_n(T)$ that only differs from V in an open edge, for simplicity, consider that is the first edge considered from left to right, e_1. V_1 is adjacent to V. Compare V_1 with V' by Theorem 2.4.3. If they are adjacent, then $d(V, V') = 2$ and this means that $n = 4$ or $n = 5$ and a maximum matching has 2 edges. If they are not adjacent, let V_2 be the vertex of $\Omega_n(T)$ that differs from V_1 only in an open edge, e_2, different from e_1 (note that V_2 differs from V in two open edges, and it is adjacent to V_1 and $d(V, V_2) = 2$). Again, we use Theorem 2.4.3 to analyze if V_2 is adjacent to V'. If the answer is affirmative then $d(V, V') = 3$, this means that $n = 6$ or $n = 7$ and the matching number is 3; otherwise consider V_3 that differs in its representation from V_2 in one open edge, different from the two previous one, and so on. The process stops after k steps, where k is the number of open edges presents in V'. Therefore,

$$d(V, V') = \beta(T) \ .$$

Next, given two any vertices V and V' of $\Omega_n(T)$, that are not in the previous situation, we always can express V and V' as:

$$V = H_1 \dot{\cup} H_2$$

and

$$V' = H_1' \dot{\cup} H_2',$$

where H_i and H_i' , $i = 1, 2$, represent finite unions of bicolored subgraphs of Types 1 and 2. V and V' are such that:

If $H_1 = H_1' \neq \emptyset$, recalling that we are not in the initial situation, then, in the representations of V and V' either we have at least one open edge in common, or the union of the open edges in their representation is not a maximum matching. As

$$|[E(V) \cup E(V')] \backslash [E(V) \cap E(V')]| < \beta(T),$$

we have

$$d(V, V') \leq \beta(T) - 1 \; .$$

If $H_1 \neq H_1'$, and in this case H_1 and H_1' must be complementary bicolored subgraphs in V and V', in some path in T, see Definition 2.4.1, then let

$$\bar{V} = H_1' \dot{\cup} H_2.$$

By Theorem 2.4.3, V and $\bar{V}$ are adjacent. From previous case, we get $d(\bar{V}, V') \leq \beta(T) - 1$. Therefore

$$d(V, V') \leq \beta(T) \; .$$

$\square$

When the tree is the path with n vertices,

$$\beta(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$$

and, therefore, Theorem 1.1.3 follows from Theorem 2.5.1.

Chapter 3

The number of faces of the tridiagonal Birkhoff polytope

The facial structure of the Birkhoff polytope was extensively studied, see for instance [5, 6, 7, 8]. Recently (in 2008) E. Marques de Sá and C. M. da Fonseca gave expressions that allow to count the vertices and the edges of Ω_n^t, [16]. These authors used the concept of alternating parity subsequences, that are strictly increasing finite sequences of integers such that any two adjacent elements have opposite parities. The results are presented in the form of products of Fibonacci numbers, and in consequence they determine the number of vertices of any face of Ω_n^t, and also they gave the number of edges and the number of faces of Ω_n^t.

Here, considering the representation of a face of Ω_n^t using bicolored subgraphs, we also give the number of vertices and edges of Ω_n^t, however considering a different approach from [14]. Moreover we also count the faces of Ω_n^t and its cells. We present this counting to the particular cases of Ω_3^t, Ω_4^t and Ω_5^t.

3.1 Counting the edges of Ω_n^t

It is referred in previous chapter, that an edge (*i.e.*, 1-face) of the tridiagonal Birkhoff polytope Ω_n^t is the union of bicolored subgraphs of Type 1, Type 2, and exactly one bicolored subgraph of Type 3, without inner entries and with two closed endpoints. As a consequence of Proposition 2.2.11, the number of edges of Ω_n^t is equal to the number of bicolored subgraphs of P_n that have a T-component with two closed endpoints and without inner entries, *i.e.*, a bicolored subgraph of the following type

$$H_i \quad \bullet - \circ - \circ - \cdots - \circ - \bullet \quad H_j$$

where H_i and H_j $(i, j = 1 \ldots, n-2)$ are the union of a finite number of bicolored subgraphs of Types 1 and 2. H_0 represents the empty set and in this case, we convention that $f_0(\emptyset) = 1$.

In this section we present a formula for $f_1(P_n)$, the number of 1-faces of Ω_n^t.

Depending on the number of open vertices that are between the endpoints, the subgraph of Type 3 (T-component) can have one of the following configurations:

I.

$$\bullet - \bullet$$

II.

$$\bullet - \circ - \bullet$$

III.

$$\bullet - \circ - \circ - \bullet$$

and so on.

If the T-component does not have internal open vertices (we are in case I), it can occupy the same position as the edges of P_n, *i.e*, $n-1$ positions. Each of those possibilities gives rise to several distinct bicolored subgraphs. For example, if the T-component occupies the first edge of the path, the remaining $n-2$ vertices give rise to a path P_{n-2}, then

$$f_0(P_{n-2}) = f_{n-2+1}$$

is the number of the possible distinct bicolored subgraphs associated to this case. Doing the same for the remaining possible positions, we obtain the number of bicolored subgraphs associated to this T-component:

$$\sum_{k=0}^{n-2} f_0(P_k)\ f_0(P_{n-2-k}) = \sum_{k=0}^{n-2} f_{k+1} f_{n-2-k+1}.$$

If the T-component has exactly one internal open vertex (case II) it can be in the same position as two consecutive edges of P_n, *i.e.*, $n-2$ possible positions. For example, if the T-component occupies the first two consecutive edges of the given path, we will have P_{n-3} as remaining path, and, in this case, we obtain

$$f_0(P_{n-3}) = f_{n-3+1}$$

distinct bicolored subgraphs. Proceeding in the same way for the remain possible positions, we obtain

$$\sum_{k=0}^{n-3} f_{k+1} f_{n-3-k+1},$$

different bicolored subgraphs.

In general, if the T-component has p internal open vertices, with $p \leq n-2$, the number of bicolored subgraphs associated to this T-component will be given by

$$\sum_{k=0}^{n-2-p} f_{k+1} f_{n-2-p-k+1}.$$

Therefore, we can state the following proposition:

Proposition 3.1.1. *The number of edges of* Ω_n^t, $f_1(P_n)$, *is:*

$$f_1(P_n) = \sum_{p=0}^{n-2} \sum_{k=0}^{n-2-p} f_{k+1} f_{n-p-k-1}. \tag{3.1.1}$$

3.2 Counting the faces of Ω_n^t

We proceed presenting a general formula for the number of faces (2-faces) of the tridiagonal Birkhoff polytope associated to a path with n vertices, ($n \geq 3$), Ω_n^t. Taking into account the formula (2.2.11) this number is equal to a sum whose two summands are the number of all bicolored subgraphs with one T-component with two closed endpoints and one inner entry, and the number of bicolored subgraphs with two T-components with two closed endpoints and without inner entries.

Suppose that the bicolored subgraph has one T-component with two closed endpoints and one inner entry. Taking into account that the open vertices can take several positions, the T-component can have one of the configurations presented below:

- ◇ without open vertex:

$$\bullet - \bullet - \bullet$$

- ◇ one open vertex:

$$\bullet - \circ - \bullet - \bullet \quad , \quad \bullet - \bullet - \circ - \bullet$$

$\diamond$ two open vertices:

$$\bullet - \bullet - \circ - \circ - \bullet \quad , \quad \bullet - \circ - \bullet - \circ - \bullet \quad , \quad \bullet - \circ - \circ - \bullet - \bullet$$

$$\vdots$$

$\diamond$ $n-3$ open vertices:

$$\bullet - \bullet - \circ - \circ \cdots \circ - \bullet \quad , \quad \bullet - \circ - \bullet - \circ - \circ \cdots \bullet \quad , \quad \ldots \quad , \quad \bullet - \circ - \circ \cdots \circ - \bullet - \bullet$$

Let us consider the union of closed vertices, $\bullet$, and open edges, $\circ - \circ$, involving k vertices, denoted by H_k, with the convention that H_0 is empty. If the T-component does not contain internal open vertices, it can occupy $n-2$ different positions on the path. Namely,

$$H_k \quad \bullet - \bullet - \bullet \quad H_{n-3-k} \ , \qquad \text{with } k = 0, \ldots, n-3.$$

Therefore we have

$$\sum_{k=0}^{n-3} f_{k+1} f_{n-k-2}$$

different bicolored subgraphs.

Suppose now that the T-component contains exactly one internal open vertex, $\circ$. The T-component can occupy the $n-3$ positions on the path:

$$H_k \quad \bullet - \bullet - \circ - \bullet \quad H_{n-4-k} \ , \qquad \text{with } k = 0, \ldots, n-4.$$

As the internal open vertex of the T-component can occupy two different positions we obtain, in this case

$$2 \sum_{k=0}^{n-4} f_{k+1} f_{n-k-3}$$

different bicolored subgraphs.

Analogously, we can obtain the number of bicolored subgraphs with one T-component, that is:

$$\sum_{p=1}^{n-2} p \sum_{k=0}^{n-2-p} f_{k+1} f_{n-p-k-1}.$$

When the bicolored subgraph contains two T-components with two closed endpoints and without inner entries, two distinct situations can occur: whether the T-components have internal open vertices or not. If none of the T-components contains internal open vertices, $n-4$, $(n \geq 4)$, of the initial vertices of the path stay free. The T-components can occupy several different positions, which are

$$\begin{array}{ccccc}
H_0 & \bullet - \bullet & H_0 & \bullet - \bullet & H_{n-4} \\
 & & \vdots & & \\
H_0 & \bullet - \bullet & H_{n-4} & \bullet - \bullet & H_0 \\
H_1 & \bullet - \bullet & H_0 & \bullet - \bullet & H_{n-5} \\
 & & \vdots & & \\
H_1 & \bullet - \bullet & H_{n-5} & \bullet - \bullet & H_0 \\
 & & \vdots & & \\
H_{n-4} & \bullet - \bullet & H_0 & \bullet - \bullet & H_0
\end{array}$$

The number of different bicolored subgraphs with these two T-components is:

$$\sum_{j=0}^{n-4} \sum_{k=0}^{n-4-j} f_{k+1} f_{j+1} f_{n-j-k-3}.$$

If one of the T-components has an internal open vertex, $\circ$, and if the other one does not have any internal open vertices, $n-5$ of the initial vertices of the path stay free. The T-components can occupy some different positions:

$$H_0 \quad \bullet - \circ - \bullet \quad H_0 \quad \bullet - \bullet \quad H_{n-5}$$

$$\vdots$$

$$H_0 \quad \bullet - \circ - \bullet \quad H_{n-5} \quad \bullet - \bullet \quad H_0$$

$$\vdots$$

$$H_{n-5} \quad \bullet - \circ - \bullet \quad H_0 \quad \bullet - \bullet \quad H_0.$$

Since the order of the T-components can be switched, the final number of different bicolored subgraphs is:

$$2\sum_{j=0}^{n-5}\sum_{k=0}^{n-5-j} f_{k+1}f_{j+1}f_{n-j-k-4}.$$

If both of the T-components have one internal open vertex, we can have all the following situations considered below:

$$H_0 \quad \bullet - \circ - \bullet \quad H_0 \quad \bullet - \circ - \bullet \quad H_{n-6}$$

$$\vdots$$

$$H_0 \quad \bullet - \circ - \bullet \quad H_{n-6} \quad \bullet - \circ - \bullet \quad H_0$$

$$H_1 \quad \bullet - \circ - \bullet \quad H_0 \quad \bullet - \circ - \bullet \quad H_{n-7}$$

$$\vdots$$

$$H_1 \quad \bullet - \circ - \bullet \quad H_{n-7} \quad \bullet - \circ - \bullet \quad H_0$$

$$\vdots$$

$$H_{n-6} \quad \bullet - \circ - \bullet \quad H_0 \quad \bullet - \circ - \bullet \quad H_0.$$

Note that the number of different bicolored subgraphs obtained in this way is equal to the number of different bicolored subgraphs obtained in the case when one of the T-component has two internal open vertices and the other one does not have any internal open vertices. In this situation, six of the n vertices of the path are used by the T-components. Therefore we have

$$3\sum_{j=0}^{n-6}\sum_{k=0}^{n-6-j} f_{k+1}f_{j+1}f_{n-j-k-5}$$

different bicolored subgraphs.

Similarly, the number of distinct bicolored subgraphs obtained from the configurations of the T-components if one of them has an internal open vertex and the other one has two internal open vertices is the same as if one of the configurations of the T-component has three internal open vertices and the other one does not have internal open vertices. So, when seven of the n vertices of the path are used by the T-components, the final number of different bicolored subgraphs is

$$4\sum_{j=0}^{n-7}\sum_{k=0}^{n-7-j} f_{k+1}f_{j+1}f_{n-j-k-6}.$$

We point out that there exists a relation between the number of sums with two integer summands and the possible configurations that can occur in the T-components.

For example, if the number of vertices involved in the T-components is seven, as the T-components have four closed endpoints, only three of the vertices are internal, and we have four possibilities, as we show in the following decomposition:

$$3 = 3 + 0 = 2 + 1. \tag{3.2.1}$$

This means that the number of bicolored subgraphs having a T-component without internal open vertices and the other T-component with 3 internal open vertices is the same as the number of bicolored subgraphs that have a T-component with an internal open vertex and the other one with 2 internal open vertices.

The number of different sums that appears in (3.2.1) is the number of integer partitions of 3 in two parts. In fact, given an integer i, the number of integer partitions of i in two parts is equal to $\left\lfloor \frac{i}{2}+1 \right\rfloor$, see [10]. If the T-components contain k vertices of the initial path, $k-4$ of these vertices are inner entries. Therefore, we are going to have $\left\lfloor \frac{k-2}{2} \right\rfloor$ integer partitions of $k-4$ in two parts and, attending to addition commutativity, we have $k-3$ different sums. Therefore, the number of different bicolored subgraphs with two T-components that use $p-1$, $(1 \leq p \leq n-3)$internal open vertices is equal to

$$p\sum_{j=0}^{n-3-p}\sum_{k=0}^{n-3-p-j} f_{k+1}f_{j+1}f_{n-p-j-k-2},$$

and the number of different bicolored subgraphs with two T-components is

$$\sum_{p=1}^{n-3} p\sum_{j=0}^{n-3-p}\sum_{k=0}^{n-3-p-j} f_{k+1}f_{j+1}f_{n-p-j-k-2}.$$

Finally, we can write the following proposition:

Proposition 3.2.1. *The number of faces of Ω_n^t, $f_2(P_n)$, is given by:*

$$f_2(P_n) = \sum_{p=1}^{n-2} p \sum_{k=0}^{n-2-p} f_{k+1}f_{n-p-k-1} + \sum_{p=1}^{n-3} p \sum_{j=0}^{n-3-p}\sum_{k=0}^{n-3-p-j} f_{k+1}f_{j+1}f_{n-p-j-k-2}\,. \tag{3.2.2}$$

3.3 Counting the cells of Ω_n^t

In this section we present a formula for counting the number of cells of Ω_n^t using the representation of a face by the union of bicolored subgraphs. This number is equal to the sum of the number of bicolored subgraphs with one T-component with two closed endpoints and two inner entries, with the number of bicolored subgraphs with two T-components with two closed endpoints, and one of them with one inner entry, and with the number of bicolored subgraphs with three T-components and without inner entries.

Suppose that the bicolored subgraph that represents a cell has one T-component with two closed endpoints and two inner entries. Since P_n has n vertices, the T-component, and the respective bicolored subgraph, can have one of the configurations presented below:

$$H_k \quad \bullet - \bullet - \bullet - \bullet \quad H_{n-4-k}\ , \qquad \text{with } k = 0, \ldots, n-4.$$

If we have p open vertices, for $p = 0, \ldots, n-4$ and $k = 0, \ldots, n-4-p$, another configuration can be, for example:

$$H_k \quad \bullet - \bullet - \bullet - \circ - \circ - \cdots - \circ - \circ - \circ - \bullet \quad H_{n-4-k-p}.$$

As we have discussed in the previous section, the internal open vertices can occupy several positions and the T-component can be without internal open vertices, with one internal open vertex and so on. Similarly to previous discussion, we must consider now the number and the position of the internal open vertices varying. Note that if we have p internal open vertices, the T-component have $p+2$ internal vertices and two of them are closed. Choosing, successively, two of the $p+2$ vertices we get the $\mathcal{C}_2^{p+2} = \dfrac{(p+2)(p+1)}{2}$ different T-components of this kind, and we obtain the number of bicolored subgraphs with one T-component. :

$$\sum_{p=0}^{n-4} \frac{(p+2)(p+1)}{2} \sum_{k=0}^{n-4-p} f_{k+1} f_{n-p-k-3}. \tag{3.3.1}$$

Now, suppose that the bicolored subgraph has two T-components with two closed endpoints and with one inner entry, that is:

$$H_k \quad \bullet - \bullet - \bullet \quad H_j \quad \bullet - \bullet \quad H_{n-5-j-k},$$

where $k, j = 0, \dots, n-5$ and $j + k \leq n-5$.

Let us denote the number of internal open vertices present in both T-components by p, $(0 \leq p \leq n-5)$.

If $p = 0$, as the T-components can switch, the number of bicolored subgraphs is two;

If $p = 1$, we can have $2(\mathcal{C}_1^2\,\mathcal{C}_0^0 + \mathcal{C}_1^1\,\mathcal{C}_0^1) = 2(2+1)$ bicolored subgraphs;

If $p = 2$, we can have $2(\mathcal{C}_1^3\;\mathcal{C}_0^0 + \mathcal{C}_1^2\;\mathcal{C}_0^1 + \mathcal{C}_1^1\;\mathcal{C}_0^2) = 2(3+2+1)$ bicolored subgraphs; and so on.

If we have p internal open vertices, the number of bicolored subgraphs is $2(\mathcal{C}_1^{p+1}\,\mathcal{C}_0^0 + \mathcal{C}_1^p\,\mathcal{C}_0^1 + \cdots + \mathcal{C}_1^1\,\mathcal{C}_0^p) = 2[(p+1) + p + \cdots + 2 + 1] = 2\frac{(p+2)(p+1)}{2}$.

Hence the number of different bicolored subgraphs with these two T-components is presented below:

$$\sum_{p=0}^{n-5} (p+2)(p+1) \sum_{j=0}^{n-5-p} \sum_{k=0}^{n-5-p-j} f_{k+1} f_{j+1} f_{n-p-j-k-4}. \tag{3.3.2}$$

The last case occurs when the bicolored subgraph has three T-components with two closed endpoints and without inner entries:

$$H_k \quad \bullet - \bullet \quad H_j \quad \bullet - \bullet \quad H_\ell \quad \bullet - \bullet \quad H_{n-6-\ell-j-k},$$

where $\ell, k, j = 0, \dots, n-6$ and $j + k + \ell \leq n-6$. The number of different bicolored subgraphs with these three T-components is presented below:

$$\sum_{p=0}^{n-6} \frac{(p+2)(p+1)}{2} \sum_{\ell=0}^{n-6-p} \sum_{j=0}^{n-6-p-\ell} \sum_{k=0}^{n-6-p-\ell-j} f_{k+1} f_{j+1} f_{\ell+1} f_{n-p-\ell-j-k-5}. \quad (3.3.3)$$

Proposition 3.3.1. *The total number of cells of Ω_n^t, $f_3(P_n)$, is given by the sum of the expressions (3.3.1), (3.3.2) and (3.3.3).*

3.4 Some particular cases

We conclude this chapter providing a full description of the previous results in the particular cases of the tridiagonal Birkhoff polytopes Ω_3^t, Ω_4^t and Ω_5^t.

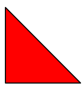
Given the path P_3

$$\circ - \circ - \circ$$

the associated tridiagonal Birkhoff polytope Ω_3^t is affinely isomorphic to the set:

$$\left\{(x,y) \in \mathbb{R}^2 \,:\, x, y \geqslant 0 \text{ and } x+y \leqslant 1\right\},$$

and it can be geometrically represented by



Bearing in mind that dimension of Ω_3^t is two, its proper faces are only vertices and edges.

Setting

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

according to [16], $\mathcal{F}_A$ is Ω_3^t and its vertices are $\mathcal{F}_{A_1}$, $\mathcal{F}_{A_2}$, $\mathcal{F}_{A_3}$, the three permutation matrices:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

Recalling [16] $\mathcal{F}_B$ is an edge if and only if B has only a S-block and no inner entry. Therefore, the edges of Ω_3^t are $\mathcal{F}_{B_1}$, $\mathcal{F}_{B_2}$, $\mathcal{F}_{B_3}$ for

$$B_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Using the graph representation introduced by us, Ω_3^t can be represented by:

$$\bullet - \bullet - \bullet$$

each vertex is represented by the union of bicolored subgraphs of Type 1 and Type 2, and they are:

$$\underset{V_1}{\bullet \quad \bullet \quad \bullet} \qquad\qquad \underset{V_2}{\circ - \circ \quad \bullet} \qquad\qquad \underset{V_3}{\bullet \quad \circ - \circ}$$

and each edge is represented by the union of bicolored subgraphs, as follows:

$$\underset{V_1V_2}{\bullet - \bullet \quad \bullet} \qquad\qquad \underset{V_1V_3}{\bullet \quad \bullet - \bullet} \qquad\qquad \underset{V_2V_3}{\bullet - \circ - \bullet}$$

Finally, it is straightforward from formulae (2.3.1) and (3.1.1) that the number of vertices and the number of edges is three.

Now, given the path P_4,

$$\circ - \circ - \circ - \circ$$

the associated tridiagonal Birkhoff polytope, Ω_4^t, is affinely isomorphic to:

$$\left\{(x,y,z) \in \mathbb{R}^3 \,:\, x,y,z \geqslant 0 \text{ and } x+y \leqslant 1,\ y+z \leqslant 1\right\}.$$

Therefore, to each vector of the previous set we can associate the following matrix, (see Proposition 2.1.5)

$$\begin{bmatrix} 1-x & x & 0 & 0 \\ x & 1-x-y & y & 0 \\ 0 & y & 1-y-z & z \\ 0 & 0 & z & 1-z \end{bmatrix}.$$

Similarly, in a matricial way, and according to [16], we can represent $\mathcal{F}_A$ by the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Its five vertices, $\mathcal{F}_{A_1}$, $\mathcal{F}_{A_2}$, $\mathcal{F}_{A_3}$, $\mathcal{F}_{A_4}$ and $\mathcal{F}_{A_5}$, are represented by the five permutation matrices:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

According to [16, Proposition 5.2], $\mathcal{F}_B$ is an edge of Ω_4^t if and only if, B has only a S-block , without inner entries. Each of the following matrices can be written as a direct sum of J and K matrices and with only one S-block without inner entries. Therefore, the eight edges of Ω_4^t are represented by $\mathcal{F}_{B_1}$, $\mathcal{F}_{B_2}$, $\mathcal{F}_{B_3}$, $\mathcal{F}_{B_4}$, $\mathcal{F}_{B_5}$ $\mathcal{F}_{B_6}$, $\mathcal{F}_{B_7}$, $\mathcal{F}_{B_8}$, where

$$B_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$B_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad B_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$B_7 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad B_8 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Again by [16, pg 1318] the faces of dimension two can be split into two cases: (i) the matrix that represents the face has only a S-block with one inner entry, (ii) the matrix that represents the face has two S-blocks, and no inner entries. As each of the five matrices C_i represented below are in the described situation and satisfies the definition of face of the polytope, and there are no more matrices in this situation, the five faces of Ω_4^t are $\mathcal{F}_{C_1}$, $\mathcal{F}_{C_2}$, $\mathcal{F}_{C_3}$, $\mathcal{F}_{C_4}$, $\mathcal{F}_{C_5}$ where

$$C_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$C_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad C_5 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

As $\dim\Omega_4^t=3$, there are no proper faces of this polytope with dimension greater than 2.

According to graph representation, Ω_4^t is represented by the bicolored graph:

$$\bullet - \bullet - \bullet - \bullet$$

and the five vertices are represented by

$$\underset{V_1}{\bullet \quad \bullet \quad \bullet \quad \bullet} \qquad \underset{V_2}{\circ - \circ \quad \bullet \quad \bullet} \qquad \underset{V_3}{\bullet \quad \circ - \circ \quad \bullet}$$

$$\underset{V_4}{\bullet \quad \bullet \quad \circ - \circ} \qquad \underset{V_5}{\circ - \circ \quad \circ - \circ} \; .$$

Each of the eight edges of Ω_4^t is represented by one of the following bicolored subgraphs:

$$\underset{V_1V_2}{\bullet - \bullet \quad \bullet \quad \bullet} \qquad \underset{V_1V_3}{\bullet \quad \bullet - \bullet \quad \bullet} \qquad \underset{V_1V_4}{\bullet \quad \bullet \quad \bullet - \bullet} \qquad \underset{V_2V_3}{\bullet - \circ - \bullet \quad \bullet}$$

$$\underset{V_2V_5}{\circ - \circ \quad \bullet - \bullet} \qquad \underset{V_3V_4}{\bullet \quad \bullet - \circ - \bullet} \qquad \underset{V_3V_5}{\bullet - \circ - \circ - \bullet} \qquad \underset{V_4V_5}{\bullet - \bullet \quad \circ - \circ}$$

And, finally, its five faces are:

$$\underset{V_1V_2V_3}{\bullet - \bullet - \bullet \quad \bullet} \qquad \underset{V_1V_2V_4}{\bullet - \bullet \quad \bullet - \bullet} \qquad \underset{V_1V_3V_4}{\bullet \quad \bullet - \bullet - \bullet}$$

$$\underset{V_2V_3V_4}{\bullet - \bullet - \circ - \bullet} \qquad \underset{V_2V_3V_5}{\bullet - \circ - \bullet - \bullet}$$

Applying formulae (2.3.1), (3.1.1) and (3.2.1) we can easily get to above results, *i.e.*, five, eight and five, for its number of vertices, edges and faces, respectively.

The tridiagonal Birkhoff polytope, Ω_5^t, is associated to the path P_5

$$\circ - \circ - \circ - \circ - \circ$$

and is affinely isomorphic to:

$$\left\{(x,y,z,t) \in \mathbb{R}^4 \,:\, x,y,z,t \geqslant 0 \text{ and } x+y \leqslant 1,\; y+z \leqslant 1 \text{ and } z+t \leqslant 1\right\}.$$

To each vector (x,y,z,t) of this set we can associate the 5×5 tridiagonal $(0,1)-$matrix (see Proposition 2.1.5)

$$\begin{bmatrix} 1-x & x & 0 & 0 & 0 \\ x & 1-x-y & y & 0 & 0 \\ 0 & y & 1-y-z & z & 0 \\ 0 & 0 & z & 1-z-t & t \\ 0 & 0 & 0 & t & 1-t \end{bmatrix}.$$

The polytope Ω_5^t is represented by the 5×5 tridiagonal matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

From (1.1.2) the number of vertices of Ω_5^t is $f_0(P_5) = f_6 = 8$. This vertices, seen in a matricial way, are the 5×5 permutation matrices P_5 such that $P_5 \leq A$.

We will not make an exhaustive representation of all the faces of Ω_5^t. We will apply the formulae obtained in previous sections to count those faces.

The number of edges of Ω_5^t is, applying (3.1.1):

$$f_1(P_5) = \sum_{p=0}^{3} \sum_{k=0}^{3-p} f_{k+1}\, f_{4-p-k} = 18.$$

From (3.2.1), we obtain for the number of the faces of Ω_5^t:

$$f_2(P_5) = \sum_{p=1}^{3} p \sum_{k=0}^{3-p} f_{k+1}\, f_{4-p-k} + \sum_{p=1}^{2} p \sum_{j=0}^{2-p} \sum_{k=0}^{2-j-p} f_{k+1}\, f_{j+1} f_{3-p-j-k} = 17.$$

Finally, the number of cells of Ω_5^t is given by Proposition 3.3.1. As $n < 6$, we consider only the two first summands, (3.3.1), (3.3.2), and we obtain

$$\begin{aligned} f_3(P_5) = & \sum_{p=0}^{1} \frac{(p+2)(p+1)}{2} \sum_{k=0}^{1-p} f_{k+1}\, f_{2-p-k} + \\ & \sum_{p=0}^{0} (p+2)(p+1) \sum_{j=0}^{-p} \sum_{k=0}^{-j} f_{k+1}\, f_{j+1}\, f_{1-p-j-k} \\ = & \; 7. \end{aligned}$$

Chapter 4

Face counting on an acyclic Birkhoff polytope

In the sequel of the study done in the previous chapter, we are now concerned with several counting problems associated to faces of $\Omega_n(T)$. In this chapter we present algorithms for counting the number of edges of $\Omega_n(T)$, in general, and also we find explicit expressions for this number when T has certain structure such as stars and starlike trees. Moreover, we describe algorithms for counting the number of faces of $\Omega_n(T)$ and we also discuss its complexity restricted only to the number of necessary iterations to obtain results. In this way we compare the algorithms presented. Some examples are provided.

The next proposition follows from Proposition 2.2.11 and it is used to determine the number of faces of dimension m of the polytope associated to two disjoint trees.

Proposition 4.0.1. *Let T and T' be two disjoint trees with n and n' vertices, respectively, if g_m is the number of bicolored subgraphs of an acyclic graph* *corresponding to faces of dimension m, then, for $m < \min\{n, n'\}$,*

$$g_m(T \cup T') = \sum_{k=0}^{m} f_{m-k}(T) f_k(T').$$

Proof. Let $T = (V(T), E(T))$ and $T' = (V(T'), E(T'))$ be two trees with n and n' vertices, respectively, such that $V(T) \cap V(T') = \emptyset$. Consider two vertices, v_T in T and $v_{T'}$ in T'. Let $T_1 = (V(T_1), E(T_1))$ be a tree with $V(T_1) = V(T) \cup V(T')$ and $E(T_1) = E(T) \cup E(T') \cup \{v_T v_{T'}\}$. The number of bicolored subgraphs of T_1 that do not contain the edge $v_T v_{T'}$ and represent a face of dimension $m < min\{n, n'\}$ of $\Omega_{n+n'}(T_1)$ is

$$f_0(T) f_m(T') + f_1(T) f_{m-1}(T') + \cdots + f_m(T) f_0(T')$$

.

For each $k = 0 \ldots, m$ let:

- $f_k(T)$ be the number of bicolored subgraphs that correspond to faces of $\Omega_n(T)$, with dimension k, where each of them has t_0 T-components and, from Proposition 2.2.11, the sum of its closed endpoints and inner entries is equal to $k + t_0$, (*);

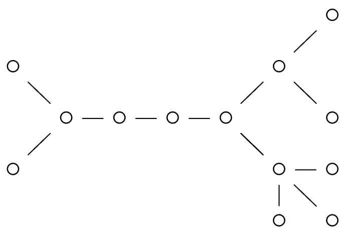
- $f_{m-k}(T')$ be the number of bicolored subgraphs that correspond to faces of $\Omega_{n'}(T')$, with dimension $m - k$, where each of them has t_0' T-components, and from Proposition 2.2.11, the sum of its closed endpoints and inner entries is equal to $m - k + t_0'$, (**).

Let us consider the union of a bicolored subgraph described in (*) with a bicolored subgraph described in (**). It is a bicolored subgraph of T_1 that does not have the edge $v_T v_{T'}$, it has $t_0 + t_0'$ T-components and the sum of its closed endpoints and inner entries is $(k+t_0)+(m-k+t_0')$. The dimension of a face of $\Omega_{n+n'}(T_1)$ corresponding to this bicolored subgraph is, by Proposition

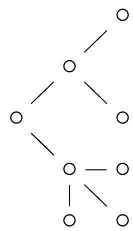
2.2.11, equal to m. Considering the union of each bicolored subgraphs in (*) with each bicolored subgraphs in (**) we obtain $f_k(T)f_{m-k}(T')$ bicolored subgraphs of T_1 and to each of them corresponds a face of dimension m of $\Omega_{n+n'}(T_1)$. For $k = 0 \ldots, m$, sum all the previous values and we obtain the desire number.

□

For better understanding let us present the example below. Consider the next two trees

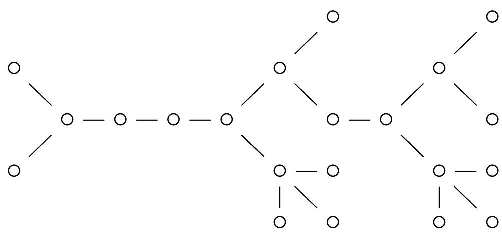


T



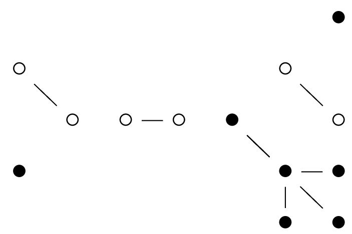
T'

The tree T_1 can be the following

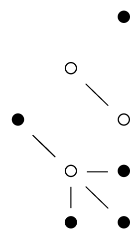


Therefore:

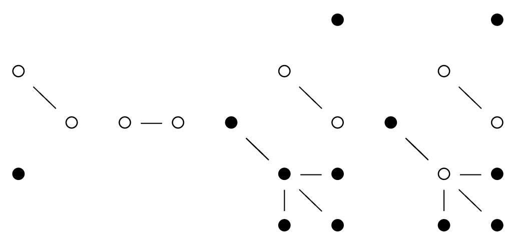
- a bicolored subgraph of T (with one T-component and the sum of its closed endpoints and inner entries equal to five) that corresponds to a face of dimension four of $\Omega_n(T)$ is:



- a bicolored subgraph of T' (with one T-component and the sum of its closed endpoints and inner entries is equal to foour) that corresponds to a face of dimension three of $\Omega_{n'}(T')$ is:



- The union of the two previous bicolored subgraphs is a bicolored subgraph of T_1 (with two T-components, and its sum of closed endpoints and inner entries is equal to nine) that corresponds to a face of dimension seven of $\Omega_{21}(T_1)$.



4.1 Counting the edges of $\Omega_n(T)$

In the previous chapter using bicolored subgraphs we presented a formula to count the number of edges of the tridiagonal Birkhoff polytope, Ω_n^t:

$$f_1(P_n) = \sum_{p=0}^{n-2} \sum_{k=0}^{n-2-p} f_{k+1}\, f_{n-p-k-1},$$

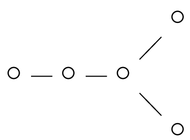
This section is devoted to present algorithms for counting the number of edges of $\Omega_n(T)$ for a given tree with n vertices. Those algorithms are compared.

In Chapter 2 we describe an edge of $\Omega_n(T)$ in terms of bicolored subgraphs. In fact, an edge is the union of bicolored subgraphs of Type 1, Type 2, and exactly one bicolored subgraph of Type 3, without any inner entries and exactly two closed endpoints. Consequently it is represented by the union of bicolored subgraphs in the following form:

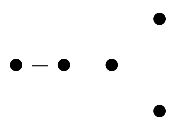
$$H_i \quad \bullet - \circ - \circ - \cdots - \circ - \bullet \quad H_j$$

where H_i and H_j have the same meaning presented in Section 3.1 of Chapter 3. H_0 represents the empty set and, in this case, it is conventioned that $f_0(\emptyset) = 1$ (recall 2.3.1).

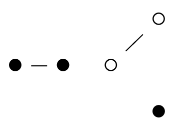
In order to motivate what following we start by presenting the edges of $\Omega_5(T_5)$, where T_5 is the tree:



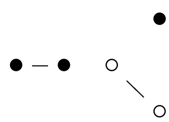
The fifteen edges of $\Omega_5(T)$ are:



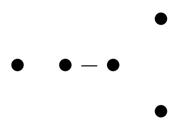
a_1



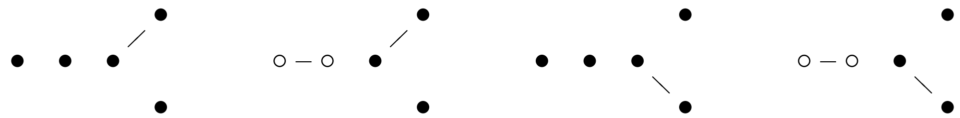
a_2



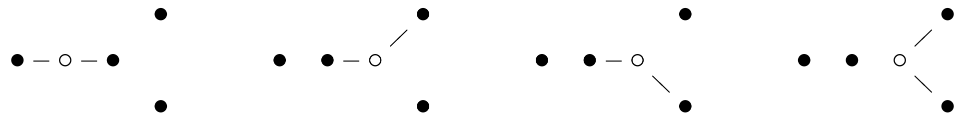
a_3



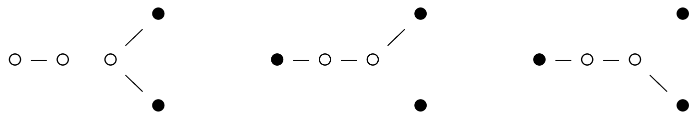
a_4



a_5 a_6 a_7 a_8



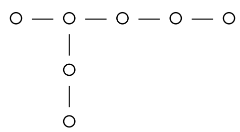
a_9 a_{10} a_{11} a_{12}



a_{13} a_{14} a_{15}

If the number of vertices of the tree increases, even with a small growth, an exhaustive exhibition of all edges of $\Omega_n(T)$ becomes harder.

The next example provides a motivation for an algorithm to calculate the number of edges of $\Omega_n(T)$, for a given tree with n vertices. In this example, we consider the starlike tree $S' = S_{1,2,3}$ with 3 branches of lengths $1, 2, 3$ presented below:



According to Proposition 2.2.11, the number of edges of the polytope $\Omega_7(S')$ is equal to the number of bicolored subgraphs of S' that have one bicolored subgraph of Type 3 with two closed endpoints and without inner entries.

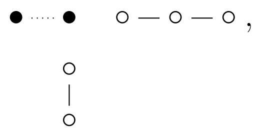
The bicolored subgraph of Type 3 (a T-component) has one of the following configurations:

$$\bullet - \bullet\,,\ \bullet - \circ - \bullet\,, \cdots ,\ \bullet - \circ - \circ - \circ - \circ - \bullet\,.$$

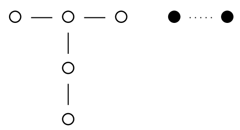
As $diam(S') = 5$, there is no other possibility for the configuration of the T-component. Each of those possibilities gives rise to several distinct bicolored subgraphs that correspond to an edge of $\Omega_7(S')$. For example, if the T-component has the first configuration, it can occupy the same position as the edges of S'. Using the previous procedure, we will distinguish the cases to be discussed.

Suppose that the T-component "occupies" the position of:

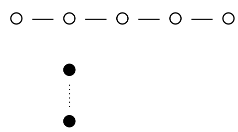
1. the first edge of S',



2. the fourth edge of S',



3. the sixth edge of S',



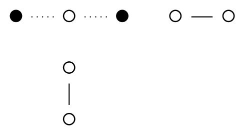
The number of edges of the polytope $\Omega_7(S')$ having the T-component in each of the previous positions is given respectively by: $f_0(P_3)\, f_0(P_2) = 3 \times 2 = 6$, $f_0(T_5) = 7$ and $f_0(P_5) = 8$.

For the remaining cases, that are not presented in this description, the calculation of the number of edges of $\Omega_7(S')$ is determined using similar arguments. Therefore, the total number of edges obtained from this T-component is given by the expression:

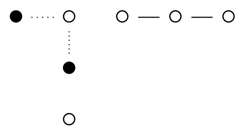
$$\begin{aligned} & f_0(P_3)\, f_0(P_2) + f_0(P_1)\, f_0(P_2)\, f_0(P_2) + f_0(P_4)\, f_0(P_1) + \\ & \qquad + f_0(T_5) + f_0(P_1)\, f_0(P_3)\, f_0(P_1) + f_0(P_5) \quad = 33. \end{aligned}$$

If the T-component has the second configuration, it can "occupy" the same position as two consecutive edges of S', as we can see next. Assume that it "occupies" the position of:

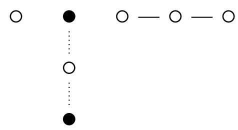
1. the first two edges of S',



2. the first and fifth edges of S',



3. the last two edges of S',



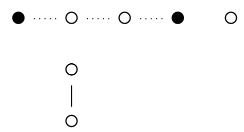
The number of edges of the polytope $\Omega_7(S')$ having the T-component in each of the previous positions is given respectively by: $f_0(P_2)\, f_0(P_2) = 2 \times 2 = 4$, $f_0(P_3)\, f_0(P_1) = 3 \times 1 = 3$ and $f_0(P_1)\, f_0(P_3) = 3$.

Again, for the remaining cases the calculation uses similar arguments. Therefore, the number of edges obtained from this T-component is given by the following expression:

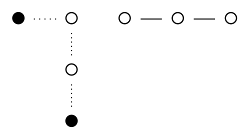
$$\begin{aligned} f_0(P_2)\, f_0(P_2) + f_0(P_1)\, f_0(P_2)\, f_0(P_1) + f_0(P_4) + f_0(P_3)\, f_0(P_1) + \\ + f_0(P_1)\, f_0(P_2)\, f_0(P_1) + f_0(P_1)\, f_0(P_3) \quad = 19. \end{aligned}$$

If the T-component has the third configuration, it can "occupy" the same position as three consecutive edges of S'. Suppose that it "occupies" the position of:

1. the three first consecutive edges of S',



2. the first, fifth and sixth edges of S',

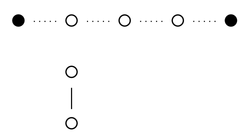


The number of edges of the polytope $\Omega_7(S')$ having the T-component in each of the previous positions is given respectively by: $f_0(P_1)\, f_0(P_2) = 2$ and $f_0(P_3) = 3$. For the remaining cases the calculation is similar. Therefore, the total number of edges that is obtained from this T-component is given by the following expression:

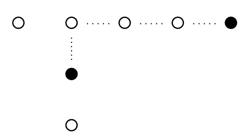
$$\begin{aligned} f_0(P_1)f_0(P_2) + f_0(P_1)f_0(P_2) + f_0(P_1)f_0(P_2) + \\ f_0(P_1)f_0(P_1)f_0(P_1) + f_0(P_3) \quad &= 10. \end{aligned}$$

If the T-component has the fourth configuration, it can "occupy" the same position as four consecutive edges of S'. Consider that it "occupies" the position of :

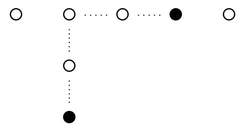
1. the first four edges of S',



2. the second, third, fourth and fifth edges of S',



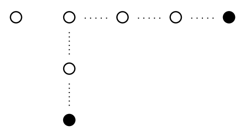
3. the second, third, fifth and sixth edges of S',



Similarly, the number of edges of the polytope $\Omega_7(S')$ having the T-component in each of the previous positions is given respectively by: $f_0(P_2) = 2$, $f_0(P_1)\, f_0(P_1) = 1$ and $f_0(P_1)\, f_0(P_1) = 1$.

If the T-component has the last configuration, it can "occupy" the same position as five consecutive edges of S'. Consider that it "occupies" the position of:

1. the second, third, fourth, fifth and sixth edges of S',



The number of edges of the polytope $\Omega_7(S')$ having the previous T-component is given by $f_0(P_1) = 1$.

Therefore the total number of edges of $\Omega_7(S')$ is the sum of all the previous values, i.e., $f_1(S') = 67$.

The illustration presented above gives rise to the first algorithm that allow us to count the number of edges of $\Omega_n(T)$, for a given tree T.

An edge is a face of dimension 1. From Proposition 2.2.11, as $1 = \theta + \iota - t$, $\theta \geq 2$, $t \geq 1$ and $\iota \geq 0$, we obtain the unique solution $\theta = 2$, $t = 1$ and $\iota = 0$. Therefore, the bicolored subgraphs that represent an edge of $\Omega_n(T)$, have one T-component with two endpoints and without inner entries.

The idea is firstly consider each edge individually, as a path of length one and can be regarded as a T-component with two closed endpoints and without internal open vertices. Each of these paths gives origin to $g_0(T \backslash P_2)$ different bicolored subgraphs.

Then, each pair of consecutive edges, considered individually, is a path of length two and can be regarded as a T-component with two endpoints and without inner entries but with an open vertex. Each path of length two gives origin to $g_0(T \backslash P_3)$ different bicolored subgraphs. We continue in a constructive process until the number of consecutive adjacent edges reaches the diameter of the tree.

We proceed as follows:

Algorithm 1

The input is a tree T such that $diam(T) = p$.

step 1 ♦ For $i \in \{1, \ldots, p\}$ consider each path P of T, with i edges, and calculate the value $g_0(T \setminus P)$;

final step ♦ Sum all the values obtained in the previous steps and exit.

The sum obtained in the final step is the number of edges of $\Omega_n(T)$.

If T is a path P with n vertices, this algorithm leads to Proposition 3.1.1 obtained in Chapter 3. However, if T is a star S with n vertices, the application of this algorithm provides a closed formula for the number of edges of the polytope $\Omega_n(S)$.

Proposition 4.1.1. *Let S be a star with n vertices, then*

$$f_1(S) = \frac{n(n-1)}{2}.$$

Proof. In fact, S has $n-1$ edges and each one of them gives rise to an edge of the acyclic Birkhoff polytope $\Omega_n(S)$; the graph S has C_2^{n-1} pairs of consecutive edges and each pair gives rise again to an edge of $\Omega_n(S)$. Since $diam(S) = 2$, there is only one possibility to count the edges of $\Omega_n(S)$ and therefore,

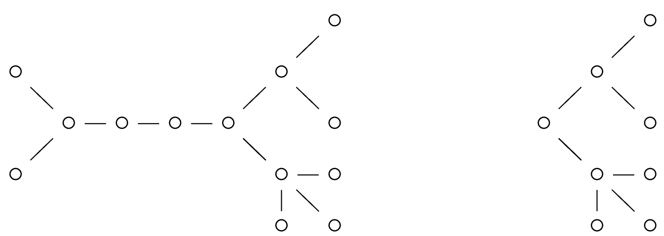
$$f_1(S) = (n-1) + \frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2}.$$

□

4.2 An alternative algorithm for $f_1(T)$

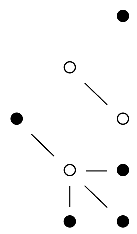
Let T and T' be two trees with n and n' vertices, respectively, such that T' is a subgraph of T. Suppose that T_1' is a bicolored subgraph of T' that represents a p-face, with $0 \leq p \leq min\{n, n'\}$, of the acyclic Birkhoff polytope associate to T'. Then, in T, we can identify $g_0(T \backslash T')$ different bicolored subgraphs, such that each of them represents a p-face of $\Omega_n(T)$ that "contains" the configuration of the referred p-face of $\Omega_{n'}(T')$. Recall that $g_0(T \backslash T')$ denotes the number of bicolored subgraphs of T' that are the union of a finite number of bicolored subgraphs of Type 1 and Type 2 whose diameter is at most one, and it is also equal to the number of matchings in the graph $T \backslash T'$.

In order to illustrate this property let us consider the trees T and T' presented below:

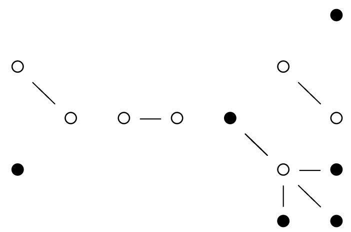


T T'

One 3-face of $\Omega_8(T')$ has the following configuration :



The polytope $\Omega_{13}(T)$ has $g_0(T\backslash T') = g_0(T_5) = 7$ faces of dimension three that “contains” the referred configuration. The next example is one of these seven faces:



Expressions giving the number of edges of the tridiagonal Birkhoff polytope are known, see Chapter 3 and reference [14]. Therefore, the next algorithm for calculating $f_1(T)$ has the underlying idea to consider all different paths obtained using two endpoints of the original tree. We compute the number of the edges of tridiagonal Birkhoff polytopes associated to them. However, in this counting, some of the configurations of the edges of $\Omega_n(T)$ are going to be repeated and must be removed.

Algorithm 2

The input is a tree T with n vertices.

step 1 ♦ for each pair of different endpoints of the tree T consider the path P between them and compute $f_1(P)\, g_0(T\backslash P)$;

step 2 ♦ sum all the numbers obtained in step 1;

step 3 ♦ for each pair of different paths considered at step 1, let P' be the path from their intersection. Compute $f_1(P')\, g_0(T\backslash P')$;

step 4 ♦ sum all the numbers obtained in step 3;

final step ♦ Calculate the difference between the numbers obtained in steps 2 and 4.

Example 4.2.1. For $S' = S_{1,2,3}$ presented in the first part of this section, we count $f_1(S')$. Here attending to the last algorithm we have:

$$\begin{aligned} f_1(S') = \;& f_1(P_5)\, f_0(P_2) + f_1(P_4)\, f_0(P_3) + f_1(P_6)\, f_0(P_1) - \\ & [f_1(P_2)\, f_0(P_2)\, f_0(P_3) + f_1(P_3)\, f_0(P_1)\, f_0(P_3) + \\ & f_1(P_4)\, f_0(P_1)\, f_0(P_2)] \\ = \;& 18 \times 2 + 8 \times 3 + 38 \times 1 - [1 \times 2 \times 3 + 3 \times 1 \times 3 \\ & +8 \times 1 \times 2] \\ = \;& 67. \end{aligned}$$

Applying Algorithm 2 to a starlike tree, next proposition follows:

Proposition 4.2.2. *Let $S' = S_{p_1,p_2,\cdots,p_n}$ be a starlike tree with n branches of lengths $p_1,\ldots,p_n$ and $N = p_1 + p_2 + \cdots + p_n + 1$ vertices. The number of edges of $\Omega_N(S')$ is given by*

$$f_1(S') = \sum_{1\le i<j\le n} \left[f_1(P_{p_i+p_j+1}) \prod_{k\neq i,j} f_0(P_{p_k}) \right] - (n-2) \sum_{i=1}^{n} \left[f_1(P_{p_{i+1}}) \prod_{k\neq i} f_0(P_{p_k}) \right]$$

Proof. For each pair of different branches of S' of lengths p_i and p_j, consider the path $P_{p_i+p_j+1}$. Without loss of generality, for each $i = 1,\ldots,n-1$, j will run over all values from $i+1$ to n. From this, the next sum represents the number of the configurations of all edges of polytopes associated to the mentioned paths

$$\sum_{1\leq i<j\leq n}\left[f_1(P_{p_i+p_j+1})\prod_{k\neq i,j}f_0(P_{p_k})\right]. \tag{4.2.1}$$

The previous number includes edges of $\Omega_N(S')$ that appear repeatedly and must be removed. They correspond to edges of polytopes associated to paths that result from the intersection of two different paths that have a common part and that number is given by the following expression:

$$(n-2)\sum_{i=1}^{n}\left[f_1(P_{p_{i+1}})\prod_{k\neq i}f_0(P_{p_k})\right]. \tag{4.2.2}$$

The final number of the edges of $\Omega_N(S')$ is given by the difference between the expressions (4.2.1) and (4.2.2). □

Recall that the formulae (4.2.1) and (4.2.2) involve only the expressions of $f_0(P_n)$ and $f_1(P_n)$ for any n, that can easily be determined (c.f. [18, 16, 14]).

4.3 Comparing algorithms

Restricting the complexity of the algorithms only to the number of necessary iterations to get results, maintaining the other operational parameters constant, Algorithm 1 needs δ^2 iterations while Algorithm 2 needs $\delta+\delta^2$, where δ is the total number of paths obtained from T considering all possible lengths. However, it depends on the implementations that can be done. The computational implementation of the algorithms is not already done but we present below a different form to write the Algorithms 1 and 2 in such a way that we can study its complexity and compare them.

Consider the following data structure: V is the set of all vertices of the tree T and $Paths$ is the set of all different paths in the tree considering all possible lengths. All paths saved in this data structure have an unique index.

Consider now the following methods:

- $diam(T)$ gives the length of one of the largest path of T;

- LengthPath(P) gives 0 if the path does not exist and gives l_P if l_P is the length of the path P;

- δ =LengthPaths(T) gives the total number of paths existing in the tree and saved in the data structure $Paths$;

- PathTerminal(P) gives $true$ if the path P is formed with terminal vertices of the tree, $false$ other cases;

- IntersectPath(P_a, P_b) gives 0 if the path P_a does not intersect the path P_b and gives the path P_i if $P_a \cap P_b = P_i$.

As we said we can rewrite the previous algorithms in the following form:

Algorithm 1

Let $Sum = 0$

For $i = 1$ to $diam(T)$

  For $p = 1$ to LengthPaths(T)

    If LengthPath(P) $= i$ then let $Sum = Sum + g_0(T \backslash P)$

  Next p

Next i

To get the final result, after running Algorithm 1, the total number of iterations is δ^2. Note that, to determine $diam(T)$ we need to run all the paths of the tree and therefore we need δ iterations.

We present **Algorithm** 2 in two stages. The Stage I finds all the paths P of the tree with terminal vertices and, for each one, calculate $f_1(P)g_0(T\backslash P)$.

The Stage II finds all the intersections P' between two different paths and, for each one, calculate $f_1(P')g_0(T\backslash P')$.

We present the details below:

Algorithm 2

Stage I

Let $Sum1 = 0$

For $p = 1$ to LengthPaths(T)

  If PathTerminal(P) $= true$ then

  let $Sum1 = Sum1 + f_1(P)g_0(T\backslash P)$

Next p

To obtain a result after running this stage, we need δ iterations.

The Stage II can be implemented in two forms:

Stage II

(Implementation 1)

Let $Sum2 = 0$

For $P_a = 1$ to LengthPaths(T)

 For $P_b = 1$ to LengthPaths(T)

  If $P_a \neq P_b$ then

   P' =IntersectPath(P_a, P_b)

   If $P' \neq 0$ then let $Sum2 = Sum2 + f_1(P')g_0(T\backslash P')$

  end If

 Next P_b

Next P_a.

To obtain a result after running Stage II with this implementation, we need δ^2 iterations.

Stage II

(Implementation 2)

Let $Sum2 = 0$

For $P_a = 1$ to LengthPaths$(T) - 1$

For $P_b = P_a + 1$ to LengthPaths(T)

$P' =$IntersectPath$(P_a,\, P_b)$

If $P' \neq 0$ then let $Sum2 = Sum2 + f_1(P')g_0(T\backslash P')$

Next P_b

Next P_a.

We need $1 + \sum\limits_{j=3}^{\delta} (j-1) = \frac{\delta^2-\delta}{2}$ iterations to obtain a result after running Stage II with this implementation. This expression can be obtained by induction.

Now, we can compare the different implementations of Algorithm 2:

a) Algorithm 2 with Stage I and Stage II (implementation 1) needs $\delta + \delta^2$ iterations to obtain the final result.

b) Algorithm 2 with Stage I and Stage II (implementation 2) needs $\delta + 1 + \sum\limits_{j=3}^{\delta} (j-1) = \frac{\delta^2+\delta}{2}$ iterations to obtain the final result.

In conclusion, b) is more efficient than a).

Now, we can compare Algorithms 1 and 2.

Algorithm 1 is more efficient than Algorithm 2 (with implementation 1) but, Algorithm 2 (with implementation 2) is more efficient than Algorithm 1.

4.4 Counting the 2-faces of $\Omega_n(T)$

In Chapter 3, the Proposition 4.4.1 gives a formula to count the number of 2-faces of Ω_n^t:

$$f_2(P_n) = \sum_{p=1}^{n-2} p \sum_{k=0}^{n-2-p} f_{k+1}\, f_{n-p-k-1} + \sum_{p=1}^{n-3} p \sum_{j=0}^{n-3-p} \sum_{k=0}^{n-3-p-j} f_{k+1}\, f_{j+1}\, f_{n-p-j-k-2}\,. \tag{4.4.1}$$

All these problems related with the counting of faces of Ω_n^t motivated us to study the same problem for the acyclic Birkhoff polytope. We start enumerating the faces of $\Omega_n(S)$, where S is a star with n vertices.

Proposition 4.4.1. *Let S be a star with n vertices. The number of 2-faces of $\Omega_n(S)$ is*

$$f_2(S) = \frac{n(n-1)(n-2)}{6}.$$

Proof. According to Proposition 2.2.11, the number of faces of $\Omega_n(S)$ is equal to the sum of the number of all bicolored subgraphs with one T-component, with two closed endpoints and one inner entry, with the number of all bicolored subgraphs with one T-component with three closed endpoints and without inner entries.

The T-component can have one of the configurations presented below:

⊛ two closed endpoints and one inner entry:

$$\bullet - \bullet - \bullet$$

⊛ three closed endpoints without inner entries:

$$\begin{array}{c} \bullet - \circ - \bullet \\ | \\ \bullet \end{array}$$

Recall that S has $n-1$ edges. Therefore, due to the configurations of the T-component it follows that we have C_2^{n-1}, in the first case, and C_3^{n-1}, in the second case, different bicolored subgraphs of S. The sum of these two values gives $f_2(S)$. □

As in Section 2, we use again Proposition 2.2.11, to obtain the composition (in number and structure) of the T-components presented in the configuration of a bicolored subgraph that represents a 2-face.

Let T be a tree with n vertices, and $\mathcal{F}$ be a 2-face of $\Omega_n(T)$. As $\dim \mathcal{F} = 2$, from the relation $2 = \theta + \iota - t$ where $\theta \geq 2$, $\iota \geq 0$ and $t \geq 1$, we only have three possibilities.

1. $\theta = 2, \iota = 1$ and $t = 1$;

2. $\theta = 3, \iota = 0$ and $t = 1$;

3. $\theta = 4, \iota = 0$ and $t = 2$.

Each one leads to a different stage of the next algorithm that will allow an exhaustive account for the number of the 2-faces of $\Omega_n(T)$.

Algorithm 3

The input is a tree, T, with vertex set V and $diam(T) = q$.

Stage I Computation of the number of bicolored subgraphs with a T-component with two endpoints and one inner entry:

step 1 ▲ For $i = 2, \ldots, q$ consider each path P of T with i edges calculate $g_0(T \setminus P)$, compute $(i-1)g_0(T \setminus P)$, and sum all the values obtained;

final step ▲ Sum all the values obtained at step 1 and exit.

Stage II Computation of the number of bicolored subgraphs with a T-component with three endpoints and without inner entries:

step 1 ▲ for each vertex v of T whose degree is greater than 2, we consider each of the triplets of incident edges on v, *i.e.*, stars with three branches and a central vertex v;

step 2 ▲ for each of these stars, S, we consider all starlike trees, S' with central vertex v containing S as a subgraph. For each S' we calculate $g_0(T \setminus S')$;

step 3 ▲ sum all the values obtained in step 2;

step 4 ▲ consider the starlike trees with origin in the same star and with a common vertex $i \neq v$ whose degree in T is greater than 2, for each pair of these starlike trees consider their intersection $S^\star$ and calculate $g_0(T \setminus S^\star)$;

step 5 ▲ sum all the values obtained in step 4;

final step ▲ calculate the difference between the values obtained in steps 3 and 5, respectively.

Stage III Computation of the number of bicolored subgraphs with two T-components each one with two endpoints and without inner entries:

We start fixing a T-component (called first T-component) and we vary the another one in configuration and in position. Then, for each $p = 1, \ldots, diam(T)$ the first T-component can occupy the position of a path in T, $P_{e_i,p}$, with length p, where e_i is its initial edge.

step 1 ▲ If the first T-component occupies the position of a path in T with initial edge e_1 and length p, $P_{e_1,p}$, compute $f_0(\emptyset)g_1(T[V \setminus V(P_{e_1,p})])$;

step 2 ▲ If the first T-component occupies the position of a path in T with initial edge e_2 and length p, $P_{e_2,p}$ compute $f_0(P_1)g_1(T[V \setminus (V(e_1) \cup V(P_{e_2,p}))]$;

step i ▲ If the first T- component occupies the position of a path in T with initial edge e_i and length p, $P_{e_i,p}$ compute $g_0(H)g_1(T[V\backslash(V(H)\cup V(P_{e_i,p}))]$.

Here, H is the subgraph of T with edge set

$E(H)=\{e_1,e_2,\dots,e_{i-1}\}\backslash\{e : e \text{ is incident on some vertex of } P_{e_i,p}\}$

and vertex set:

$$V(H)=V(e_1\cup e_2\cup\cdots\cup e_{i-1})\backslash V(P_{e_i,p}).$$

final step ▲ Repeat the previous step until the diameter of each connected component of the induced subgraph $T[V\backslash(V(H)\cup V(P_{e_i,p}))]$ is equal to 0 and sum all values obtained at this stage.

Stage IV Sum all values computed at Stages I, II and III.

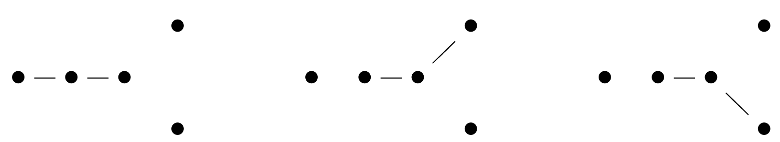
In order to illustrate this algorithm we present a description of the way we can count the 2-faces of $\Omega_5(T)$.

Stage I If we have exactly one T-component, then one of the following configurations may occur:

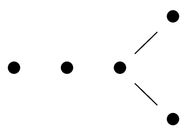
⊛ the T-component has two endpoints and one inner entry:

$$\bullet - \bullet - \bullet$$

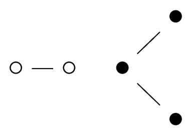
From the first step of the algorithm we have: $g_0(T\backslash\{1,2,3\})=g_0(T\backslash\{2,3,5\})$ $=g_0(T\backslash\{2,3,4\})=f_0(P_1)=1$ and $g_0(T\backslash\{3,4,5\})=f_0(P_2)=2$. We sum all of these numbers. Therefore, we have five 2-faces of $\Omega_5(T)$ and the respective configurations are:



F_1 F_2 F_3



F_4



F_5

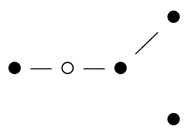
⊛ the T-component has one open vertex and its configuration can be

$$\bullet - \circ - \bullet - \bullet$$

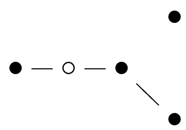
From the application of the step 2 we have:

$$g_0(T\backslash\{1,2,3,4\}) = 1 \text{ and } g_0(T\backslash\{1,2,3,5\}) = 1.$$

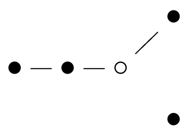
We compute $2g_0(T\backslash\{1,2,3,4\}) + 2g_0(T\backslash\{1,2,3,5\}) = 4$. The corresponding configurations of the four 2-faces of $\Omega_5(T)$ are:



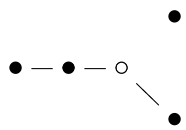
F_6



F_7



F_8



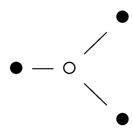
F_9

Note that the two last one are due from the fact that the configuration of the T-component can also be

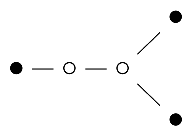
$$\bullet - \bullet - \circ - \bullet$$

As $diam(T_5) = 3$ we do not have another possibility, we finish this stage adding the values obtained: $5 + 4 = 9$.

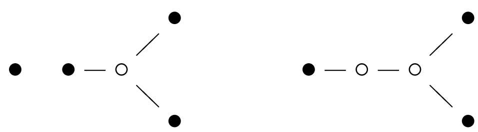
Stage II If the T-component has three endpoints without inner entries it can have the following configuration:



or



As in T_5, we only have one vertex with degree greater than 2 and we only have a triplet of incident edges in this vertex, which correspond to the 2-faces, respectively,



F_{10} F_{11}

Note that from the application of step 2 we only have two starlike trees. As result of this stage we have two faces of $\Omega_5(T)$ and this number of faces results from the sum of $g_0(T\backslash\{2,3,4,5\}) = f_0(v) = 1$ with $g_0(T\backslash\{1,2,3,4,5\}) = f_0(\emptyset) = 1$.

Stage III If the bicolored subgraph has two T-components with two endpoints and without inner entries.

We start fixing the first T-component without open vertices. The second one can have, or not, open vertices. As $diam(T_5) = 3$, the number of open vertices must be necessarily one. Therefore we have two possibilities:

$$\bullet - \bullet \quad \bullet - \bullet$$

and

$$\bullet - \bullet \quad \bullet - \circ - \bullet$$

attending to step1, we calculate

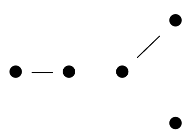
$$f_0(\emptyset)f_1(P_3) = 3.$$

Suppose now that the first T-component will occupy the position of e_2, as

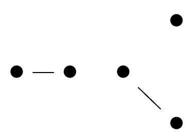
$$diam(T[V \backslash V(e_1 \cup e_2)]) = 0$$

we must stop the process.

So, the configurations of the three faces obtained are:

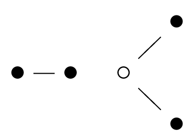


F_{12}



F_{13}

and



F_{14}

Stage IV In this way we obtain the number of all the 2-faces of $\Omega_5(T)$

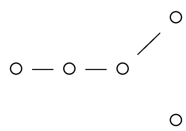
$$5 + 2 \times 2 + 2 + 3 = 14.$$

4.5 Counting the faces of $\Omega_n(T)$ revisited

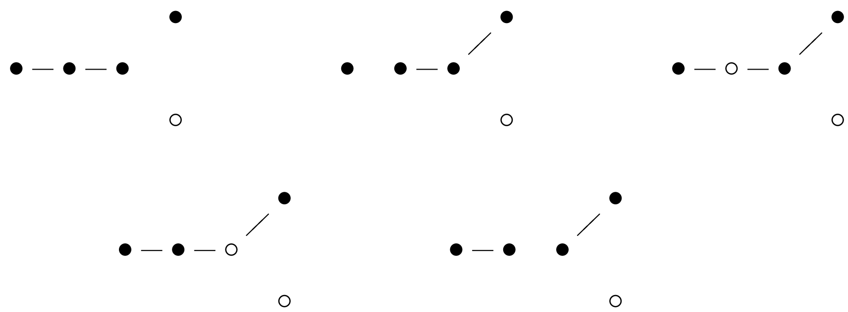
In this section, as we have previously done for the counting of the edges of $\Omega_n(T)$, we will consider paths between terminal vertices of the tree. We count the faces associated to them.

So, in order to count the faces of $\Omega_5(T)$ we start with maximal paths between terminal vertices in T_5, that is, whose diameter is equal to $diam(T_5)$. We have two possibilities:

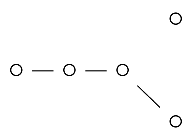
⊛ let P_4 be the path constituted by the vertices $1, 2, 3$ and 4



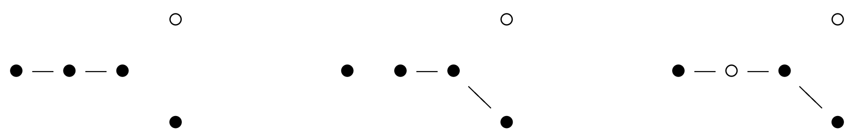
The faces of the tridiagonal Birkhoff polytope associated to the previous path correspond to the following representations:



⊛ let P_4' be the path formed by vertices $1, 2, 3$ and 5



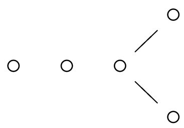
the faces of the respective tridiagonal Birkhoff polytope correspond to:



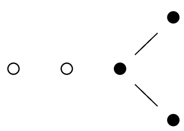
The previous representations will correspond to the 2-faces $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_6$, $\mathcal{F}_7$, $\mathcal{F}_8$, $\mathcal{F}_9$, $\mathcal{F}_{12}$ and $\mathcal{F}_{13}$ of $\Omega_5(T)$. Observe that the representation of $\mathcal{F}_1$ appears in both cases. This is due to the fact that the vertices $1, 2$ and 3 are common to both paths, $P_4 \cap P_4' = P_3$, here is the path with vertices $1, 2$ and 3. So in the end we must remove the faces of $\Omega_5(T)$ that appear repeated. Therefore, so far we have nine different faces.

Now we consider all paths between terminal vertices of T_5 whose diameter is equal to $diam(T_5) - 1$.

⊛ Let P_3' be the path constituted by the vertices $3, 4$ and 5 of T_5



The face of the respective tridiagonal Birkhoff polytope associated to this path is obtained from:



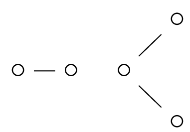
The representation above corresponds to the faces $\mathcal{F}_4$ and $\mathcal{F}_5$ of $\Omega_5(T)$.

Until here we have obtained twelve 2-faces but one of them is repeated. Therefore we have eleven different faces.

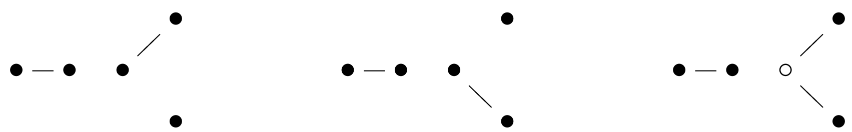
There is no possibility to obtain from the original tree more paths with terminal vertices of T_5.

As in the configuration of a face we can have two T-components, we must consider the situation that involves two disjoint paths. Each T-component belongs to one of the paths, corresponding to an edge of the corresponding tridiagonal Birkhoff polytope.

The faces of $\Omega_5(T)$ obtained from a 1-face of $\Omega_2(P_2)$, where P_2 has vertices 1 and 2 of T_5, and from an 1-face of $\Omega_3(P_3')$, that is,



are



In this case, the first two 2-faces correspond to $\mathcal{F}_{12}$ and $\mathcal{F}_{13}$ and they have previously appeared from the polytopes associated to P_4 and P_4', and the third 2-face corresponds to the face $\mathcal{F}_{14}$ of $\Omega_5(T)$.

Due to the nature of the initial graph, previous bicolored subgraphs are the only possibility.

Finally, we are going to analyze the faces for which bicolored subgraphs have a T-component with three endpoints and without inner entries. As this T-component needs at least 4 vertices, 3 endpoints and one open vertex in its interior, it remains only a free vertex. From this, we obtain two configurations corresponding to the faces $\mathcal{F}_{10}$ and $\mathcal{F}_{11}$ of $\Omega_5(T)$.

Therefore, it is possible to express $f_2(T_5)$ from the number of faces, edges and vertices of polytopes corresponding to paths and from the number of bicolored subgraphs which have a T-component with 3 endpoints and without inner entries.

This illustration leads to the following algorithm for counting the faces of $\Omega_n(T)$.

Algorithm 4

The input is a tree T with vertex set V.

Stage I Computation of the number of configuration of all faces of polytopes associated to different paths.

- step 1 ♦ For each pair of different terminal vertices of the tree T consider the path P between them and compute $f_2(P)g_0(T\backslash P)$;
- step 2 ♦ Sum all the numbers obtained in step 1;
- step 3 ♦ For each pair of different paths of step1 let P' be its intersection. Compute $f_2(P')g_0(T\backslash P')$;
- step 4 ♦ Sum all the numbers obtained in step 3;
- step 5 ♦ Calculate the difference between the numbers obtained in steps 2 and 4;
- step 6 ♦ For each path P considered in step 1 compute $f_1(P)$;
- step 7 ♦ Delete all edges incident on any vertex of P and call to the remaining graph G;
- step 8 ♦ Let $\widetilde{G}$ be any connected component of G. For each pair of terminal vertices of $\widetilde{G}$ consider the path $\widetilde{P}$ between them and compute $f_1(\widetilde{P})$;
- step 9 ♦ Compute $f_1(P) \times f_1(\widetilde{P})$;
- step 10 ♦ Consider all paths of T formed with all terminal vertices of P and $\widetilde{P}$ and distinct from them;
- step 11 ♦ In each paths formed in step 10, delete all edges that join P and $\widetilde{P}$ and let M and M' be the two subgraphs obtained;
- step 12 ♦ Compute $g_1(M) \times g_1(M')$;

step 13 ♦ Sum all the products obtained in step 12;

step 14 ♦ Subtract the value obtained in step 13 to the value obtained in step 9;

step 15 ♦ Calculate $g_0(T\backslash(P \cup \widetilde{G}))$;

step 16 ♦ Multiply the values obtained in steps 14 and 15;

final step ♦ Sum the values obtained in steps 5 and 16.

Stage II Computation of the number of bicolored subgraphs corresponding to the T-component which has 3 endpoints and no inner entries.

step 1 ♦ For each vertex v of T, whose degree is greater than 2, we consider each of the triplets of incident edges on v, i.e, starlike trees with branches with maximum length.

step 2 ♦ For each $S_{p_1,p_2,p_3} = S'$ let $i = 0, 1 \ldots, p_1 - 1$, compute

$f_0(P_{p_1-i})f_0(P_{p_2-j})f_0(P_{p_3-k})g_0(T\backslash S')$,

for all $j = 0, 1, \ldots, p_2 - 1$ and $k = 0, 1, \ldots, p_3 - 1$;

final step ♦ Sum all the values determined in previous step.

Stage III Sum the values obtained in final steps of stages I and II.

Recall that, as we have seen, the maximum number of closed endpoints of a T-component is at most three.

Proposition 4.5.1. *Let $S' = S_{p_1,p_2,\ldots,p_n}$ be a starlike tree with n branches of lengths $p_1, \ldots, p_n$ and $N = p_1 + p_2 + \cdots + p_n + 1$ vertices. The number of faces of $\Omega_N(S')$ is given by*

$$f_2(T) =$$

$$\sum_{1\leq i<j\leq n} \left[f_2(P_{p_i+p_j+1}) \prod_{k\neq i,j} f_0(P_{p_k}) \right] - (n-2) \sum_{i=1}^{n} f_2(P_{p_i+1}) \prod_{k\neq i} f_0(P_{p_k}) +$$

$$\left[\sum_{\substack{1\le i<j\le n\\ j\ne k\ne i}} \left(f_1(P_{p_i+p_j+1}) - f_1(P_{p_i+1}) - f_1(P_{p_j+1})\right) f_1(P_{p_k})\right] \prod_{\ell\ne i,j,k} f_0(P_{p_\ell})$$

$$+\sum_{\substack{j\ne k\ne i\\ i\ne j}} \left(\sum_{\substack{0\le r\le p_i-1\\ 0\le s\le p_j-1\\ 0\le t\le p_k-1}} f_0(P_{p_i-r}) f_0(P_{p_j-s}) f_0(P_{p_k-t}) \prod_{\ell\ne i,j,k} f_0(P_{p_l}) \right).$$

Proof. For each pair of different branches of S' p_i and p_j, consider the path $P_{p_i+p_j+1}$. Without loss of generality, for each i, $i = 1,\ldots,n-1$, j will run all values from $i+1$ to n.

The next sum represents the number of the configurations of all faces of polytopes associated to the referred paths,

$$\sum_{1\le i<j\le n} \left[f_2(P_{p_i+p_{j+1}}) \prod_{i\ne k\ne j} f_0(P_{p_k}) \right].$$

When the faces of the polytopes corresponding to those paths are considered,

$$(n-2)\sum_{i=1}^{n} f_2(P_{p_i+1}) \prod_{k\ne i} f_0(P_{p_k})$$

is the number of faces that are going to appear repeated. Therefore, that number has to be excluded from the previous expression.

Now, we must consider the 2-faces resulting from two T-components where one of them is in a branch and the other T-component has its two endpoints in two different branches. It results the following number

$$\left[\sum_{\substack{1\le i<j\le n\\ j\ne k\ne i}} \left(f_1(P_{p_i+p_j+1}) - f_1(P_{p_i+1}) - f_1(P_{p_j+1})\right) f_1(P_{p_k})\right] \prod_{\ell\ne i,j,k} f_0(P_{p_\ell}).$$

Finally, the number of bicolored subgraphs corresponding to the T-components which have three endpoints and no inner entries is

$$\sum_{\substack{j\neq k\neq i\\ i\neq j}} \left(\sum_{\substack{0\leq r\leq p_i-1\\ 0\leq s\leq p_j-1\\ 0\leq t\leq p_k-1}} f_0(P_{p_i-r}) f_0(P_{p_j-s}) f_0(P_{p_k-t}) \prod_{\ell\neq i,j,k} f_0(P_{p_\ell}) \right).$$

Here, if $k \leq n$, then $f_n(P_k) = 0$.

From the previous considerations we get the desired result.

□

4.6 Counting faces of any dimension of $\Omega_n(S)$

For a given star with n vertices, we have already seen in sections 4.1 and 4.4, Propositions 4.1.1 and 4.4.1, respectively, that

$$f_1(S) = (n-1) + C_2^{n-1} = \frac{n(n-1)}{2}$$

and

$$f_2(S) = C_2^{n-1} + C_3^{n-1} = \frac{n(n-1)(n-2)}{6}.$$

In this section our aim is to obtain an expression for the number of p-faces of the acyclic Birkhoff polytope associated to S, for $3 \leq p \leq n-1$.

This number is equal to the number of bicolored subgraphs with one T-component and whose sum of closed endpoints and inner entries equals to $p+1$.

Bearing in mind that $diam(S) = 2$, the T-components that we can consider to characterize a p-face, with $3 \leq p \leq n-1$, are only of the two different types:

i) with p closed endpoints and one inner entry; and

$ii)$ with $p+1$ closed endpoints and without inner entries.

The number of p-faces of $\Omega_n(S)$, whose bicolored subgraphs have, respectively, the first and second T-components are given by

$$C_p^{n-1} \text{ and } C_{p+1}^{n-1}.$$

Therefore, the number of p-faces of $\Omega_n(S)$ is given by

$$f_p(S) = C_p^{n-1} + C_{p+1}^{n-1} = \frac{n!}{(p+1)!(n-p-1)!} = C_{p+1}^{n}.$$

As this last expression is also valid for the cases $p = 0, 1, 2$, we can establish the following:

Proposition 4.6.1. *Let S be a star with n vertices, $f_p(S) = C_{p+1}^{n}$.*

4.7 Counting facets of $\Omega_n(T)$

Finally, we present an expression for the number of facets of $\Omega_n(T)$. Taking into account Proposition 2.2.11, the next proposition allows us to determine the number of facets of any acyclic Birkhoff polytope $\Omega_n(T)$. Here, an *pendant edge* is an edge of the tree that is terminal, *i.e.*, one of its vertices is an endpoint of the graph, and an *inner-edge* is an edge of the tree that is not a pendant edge, *i.e.*, whose both vertices are not endpoints of the tree.

Proposition 4.7.1. *Let T be a tree with n $(n \geq 2)$ vertices, where p of them are endpoints. The number of facets of the polytope $\Omega_n(T)$ is $2n - p - 1$.*

Proof. Considering Proposition 2.2.11, we are looking for all different bicolored subgraphs of T verifying

$$n - 2 = \theta + \iota - t.$$

Regarding that in any bicolored subgraph $\theta + \iota \leq n$, the bicolored subgraphs that we are searching for have at most two T-components. If the bicolored subgraph has only one T-component, we can have two different cases:

Case 1. All the endpoints of T are endpoints of the T-component. By (2.2.11), the T-component has $n - p - 1$ inner entries. Therefore, the T-component must have one (and only one) vertex of T that is not an inner entry. As this vertex can occupy $n - p$ different positions in T, we have $n - p$ different bicolored subgraphs and each of them represents a facet.

Case 2. The T-component has $p - 1$ endpoints that are also endpoints of the graph T and one endpoint that is not an endpoint of the graph T, *i.e.*, one of the pendant edges of the graph T is not an edge of the T-component.

The number of different pendant edges of the graph T is p. Therefore we have p different possibilities to get the bicolored subgraph and we have p different bicolored subgraphs whose corresponding face has dimension $n - 2$, and each of them represents a facet of $\Omega_n(T)$.

If the bicolored subgraph has two T-components, all the vertices of the bicolored subgraph must be closed. In this particular situation, the T-component must be "apart" by one inner edge. Since the number of edges of the tree T is $n - 1$, we have $n - 1 - p$ inner edges and consequently we have $n - 1 - p$ bicolored subgraphs of the graph T and each of them represents a facet of $\Omega_n(T)$. From the previous calculations we obtain the number of facets of the polytope $\Omega_n(T)$:

$$(n - p) + p + (n - 1 - p) = 2n - p - 1.$$

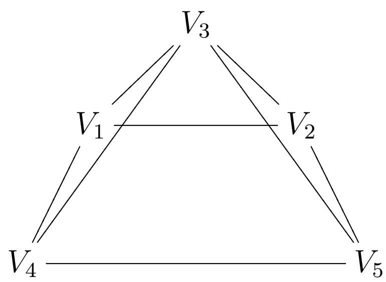
$\square$

Chapter 5

Faces of faces of the tridiagonal Birkhoff polytope

In this chapter using bicolored subgraphs we extend some results of [16], namely we count the faces of lower dimension contained in a given p-face of the tridiagonal Birkhoff polytope, $\Omega_n(P) = \Omega_n^t$, where P is a path with n vertices. Observe that, in this case, the configuration of a face of dimension greater than 0, has always a T-component with the shape of a path.

In the last section of Chapter 3 we made the description of the faces of Ω_4^t. Regarding the previous study, we have the skeleton of this polytope:



As we can see, one face of the previous three dimensional polytope is a rectangle while the others are triangles. Note that the rectangular face is represented by:

$$\bullet - \bullet \quad \bullet - \bullet$$

and it can be determined by two of its four edges:

$$\begin{array}{cccc} \bullet - \bullet \quad \bullet \quad \bullet & \circ - \circ \quad \bullet - \bullet & \bullet \quad \bullet \quad \bullet - \bullet & \bullet - \bullet \quad \circ - \circ \\ V_1V_2 & V_2V_5 & V_1V_4 & V_4V_5 \end{array}$$

It has the four vertices:

$$\begin{array}{cc} \bullet \quad \bullet \quad \bullet \quad \bullet & \circ - \circ \quad \bullet \quad \bullet \\ V_1 & V_2 \end{array}$$

and

$$\begin{array}{cc} \bullet \quad \bullet \quad \circ - \circ & \circ - \circ \quad \circ - \circ \\ V_4 & V_5 \end{array}$$

One of the triangular faces is, for example:

$$\bullet - \bullet - \circ - \bullet$$

whose three edges and three vertices are, respectively:

$$\begin{array}{ccc} \bullet \quad \bullet - \circ - \bullet & \bullet - \circ - \circ - \bullet & \bullet - \bullet \quad \circ - \circ \\ V_3V_4 & V_3V_5 & V_4V_5 \end{array}$$

and

$$\begin{array}{ccc} \bullet \quad \circ - \circ \quad \bullet & \bullet \quad \bullet \quad \circ - \circ & \circ - \circ \quad \circ - \circ \\ V_3 & V_4 & V_5 \end{array}$$

Motivated by these examples we are interested in the study of the nature and number of faces of lower dimension contained in a given p-face of Ω_n^t.

In the next section we determine the maximum number of different possible T-components that can be present in the bicolored subgraph that represents a given p-face of Ω_n^t, with $p \geq 2$. In Section 2 we give the number of vertices of a given face and in Sections 3 and 4 we present the number of faces of lower dimension contained in a given 2-face and a 3-face (a cell) of Ω_n^t, respectively. For the obtained formulae we need to manipulate some relations with Fibonacci numbers, namely, attending to the Fibonacci's numbers property: $f_{n+m} = f_n f_{m-1} + f_{n+1} f_m$, we obtain the following relation $f_n f_{m+1} + f_{n+1} f_m = f_n f_m + f_{n+m}$, with $n, m \in \mathbb{N}$. In Section 5 we present some applications of the results given in the previous sections. The results presented in this chapter can be found in [15].

5.1 Configurations of the T-components in a face of Ω_n^t

In this section we are interested to find what is the maximum number of T-components that can be present in the bicolored subgraph that represents a p-face. According to Chapter 2, Proposition 2.2.3, the configuration of a p-face of $\Omega_n(T)$ is the union of bicolored subgraphs of Type 1, Type 2 and t bicolored subgraphs of Type 3 with ι inner entries and θ closed endpoints such that:

$$p = \theta + \iota - t. \tag{5.1.1}$$

Observe also that only the bicolored subgraphs of Type 3 vary in shape, the remaining bicolored subgraphs are static whereas these ones are dynamic. Therefore, in order to find the number of possibilities that allow us to construct a p-face we only use the number and the shape of the T-components.

In the specific case of a path, the number of T-components, t, is half of the number of its closed endpoints, θ. Therefore, the expression (5.1.1) becomes

$$p = \iota + t, \tag{5.1.2}$$

and the total number of T-components included on the representation of a p-face is given by:

$$t = p - \iota. \tag{5.1.3}$$

In fact, for each $p_0 \geq 1$ the pairs of integers $(\iota, p_0 - \iota)$, $\iota \geq 0$, lye on a straight line, (5.1.3) and give all the possibilities for the number of T-components that can be present in the bicolored subgraph that represents the p-face. The maximum value of t is reached at $\iota = 0$ and this maximum equals to p_0.

From previous considerations follows the proposition that gives the maximum number of T-components that allow us to obtain a p-face:

Proposition 5.1.1. *The maximum number of T-components that can be present in the bicolored subgraph that represents a p-face of Ω_n^t is p.*

We present next example to illustrate Proposition 5.1.1. Here H_i has the same meaning as presented in Chapter 2.

Example 5.1.2. The number of T-components that can be present in the bicolored subgraph that represents a 4-face of Ω_n^t is, from Proposition 5.1.1, equal to 4. As $t = 4 - \iota$, $\iota \geq 0$ and $t \geq 1$, we have to discuss the number of inner entries. In fact, if:

* $\iota = 0$, then $t = 4$ and the representation of the face is:

$$H_1 \quad \bullet - \bullet \quad H_2 \quad \bullet - \bullet \quad H_3 \quad \bullet - \bullet \quad H_4 \quad \bullet - \bullet \quad H_5$$

* $\iota = 1$, then $t = 3$ and the representation of the face can be the following one (or a permutation of the position of the three T-components):

$$H_1 \quad \bullet - \bullet \quad H_2 \quad \bullet - \bullet - \bullet \quad H_3 \quad \bullet - \bullet \quad H_4$$

* $\iota = 2$, then $t = 2$.

  As $2 = 2{+}0 = 1{+}1$, now we can have two situations: one T-component has two inner entries and the other one does not have inner entries, or each of the T-components has one inner entry.

  In the first case, the representation of the face can be the following one (or a permutation of the position of the two T-components):

$$H_1 \quad \bullet - \bullet \quad H_2 \quad \bullet - \bullet - \bullet - \bullet \quad H_3$$

  In the second case, the representation of the face is:

$$H_1 \quad \bullet - \bullet - \bullet \quad H_2 \quad \bullet - \bullet - \bullet \quad H_3$$

* $\iota = 3$, then $t = 1$. The representation of the face is:

$$H_1 \quad \bullet - \bullet - \bullet - \bullet - \bullet \quad H_2$$

We can say that the representation of a p-face can have at least one and at most p T-components. However, when ι and t are both greater than one, to determine the distribution of inner entries in each T-component we must consider the integer partitions of ι in t parts. The number of these partitions can be determined appealing to generating functions, see [10].

5.2 Number of vertices in one p-face of Ω_n^t

From previous section we can say that we have at most p T-components in the configuration of a p-face of Ω_n^t. Due to the dynamic nature of the bicolored subgraphs of Type 3, is the variation of these that gives origin to distinct vertices of a face. Next, we present the study of number of vertices contained in a p-face of Ω_n^t in different situations. Firstly, we present a way of counting the number of vertices contained in a p-face of Ω_n^t whose configuration has only a T-component. We start our study considering a T-component with two sequences of closed vertices and one sequence of open vertices between them.

Definition 5.2.1. A *string* of open (closed) vertices is a sequence of open (closed) vertices. We say that the string has length k if it has exactly k consecutive open (closed) vertices.

5.2.1 A configuration with just a T-component with one string of open vertices

Suppose that the face has the configuration present below. Let $i, k \geq 1$. We have $i + j + k = m$, where j is the length of the string of open vertices that is between two strings of consecutive closed vertices with lengths i and k, respectively:

$$H_1 \quad \bullet - \bullet \cdots \bullet - \circ - \circ \cdots \circ - \circ - \bullet \cdots \bullet - \bullet \quad H_2$$

When $j = 0$ the p-face coincide with the configuration of the polytope Ω_m^t and this study has already been done in Chapter 2. Therefore, from now on, we consider $j > 0$.

Let P_m be the path that has the same shape of the previous T-component, that is, the path with $i + j + k = m$ vertices.

The next proposition gives the number of vertices contained in the previous p-face.

Proposition 5.2.2. *The number of vertices contained in the face that has only a T-component with the previous configuration is:*

- $f_0(P_{i+k})$ *if j is even*
- $f_0(P_{i+k-1}) + f_0(P_{i-1})f_0(P_{k-1})$ *if j is odd.*

Proof. In order to construct the vertices of the p-face and bearing in mind its nature – that is, a vertex is identified as a bicolored subgraph whose diameter is at most one – the strings of the i initial and the k final vertices of the bicolored subgraph can be replaced by closed vertices and/or open edges while the string of j open vertices can only be replaced by open edges.

If j is even, we can have the following two situations. The first one is described below:

$$\underbrace{\cdots}_{i \text{ positions}} \qquad \underset{\frac{j}{2} \text{ open edges}}{\circ - \circ \cdots \circ - \circ} \qquad \underbrace{\cdots}_{k \text{ positions}}$$

Here, we have i positions that can be occupied either by closed vertices or open edges, $\frac{j}{2}$ open edges followed by k positions that can also be occupied either by closed vertices or open edges. This gives origin to $f_0(P_i)f_0(P_k)$ vertices of the p-face. The second situation is:

$$\underbrace{\cdots}_{i-1 \text{ positions}} \qquad \underset{\frac{j+2}{2} \text{ open edges}}{\circ - \circ \cdots \circ - \circ \cdots \circ - \circ} \qquad \underbrace{\cdots}_{k-1 \text{ positions}}$$

In this case, we have $i-1$ positions that can be occupied either by closed vertices or open edges, $\frac{j+2}{2}$ open edges followed by $k-1$ positions that can be occupied either by closed vertices or open edges. This gives origin to $f_0(P_{i-1})f_0(P_{k-1})$ vertices of the p-face.

Therefore, when j is even the total number of vertices of the p-face with the previous configuration is:

$$f_0(P_i)f_0(P_k) + f_0(P_{i-1})f_0(P_{k-1}) = f_{i+1}f_{k+1} + f_i f_k = f_{i+k+1} = f_0(P_{i+k}).$$

Consider now that j is an odd number. We also have two cases. The first one is the following:

$$\underbrace{\cdots}_{i-1 \text{ positions}} \qquad \underset{\frac{j+1}{2} \text{ open edges}}{\circ - \circ \cdots \circ - \circ \cdots \circ - \circ} \qquad \underbrace{\cdots}_{k \text{ positions}}$$

In this case we have $i-1$ positions that can be occupied either by closed vertices or open edges; $\frac{j+1}{2}$ open edges followed by k positions that can be occupied either by closed vertices or open vertices. This will origin $f_0(P_{i-1})f_0(P_k)$ vertices of the p-face. The second case is:

$$\underbrace{\cdots}_{i \text{ positions}} \qquad \underset{\frac{j+1}{2} \text{ open edges}}{\circ - \circ \cdots \circ - \circ \cdots \circ - \circ} \qquad \underbrace{\cdots}_{k-1 \text{ positions}}$$

Here, we have i positions that can be occupied by closed vertices or open edges, $\frac{j+1}{2}$ open edges followed by $k-1$ positions that can be occupied by closed vertices or open edges. This gives origin to $f_0(P_i)f_0(P_{k-1})$ vertices of the p-face.

Therefore, when j is an odd number the total number of vertices of the

p -face with this configuration is:

$$\begin{aligned}
f_0(P_{i-1})f_0(P_k) + f_0(P_i)f_0(P_{k-1}) &= f_i f_{k+1} + f_{i+1} f_k \\
&= f_i f_{k+1} + (f_i + f_{i-1}) f_k \\
&= f_i f_{k+1} + f_{i-1} f_k + f_i f_k \\
&= f_{i+k} + f_i f_k \\
&= f_0(P_{i+k-1}) + f_0(P_{i-1}) f_0(P_{k-1}).
\end{aligned}$$

$\square$

Example 5.2.3. Suppose that we have a 4-face in which configuration has only a T-component as the following:

$$\bullet - \bullet - \bullet - \circ - \circ - \bullet - \bullet$$

From Proposition 5.2.2 the 4-face has $f_0(P_{3+2}) = f_0(P_5) = 8$ vertices. The referred vertices are:

$$\begin{array}{ccccccc}
H_1 & \bullet \quad \bullet \quad \bullet & \circ - \circ & \bullet \quad \bullet & & & H_2 \\
H_1 & \bullet \quad \bullet \quad \bullet & \circ - \circ & \circ - \circ & & & H_2 \\
H_1 & \bullet & \circ - \circ & \circ - \circ & \bullet \quad \bullet & & H_2 \\
H_1 & \bullet & \circ - \circ & \circ - \circ & \circ - \circ & & H_2 \\
H_1 & \circ - \circ & \bullet & \circ - \circ & \bullet \quad \bullet & & H_2 \\
H_1 & \circ - \circ & \bullet & \circ - \circ & \circ - \circ & & H_2 \\
H_1 & \bullet \quad \bullet & \circ - \circ & \circ - \circ & \bullet & & H_2 \\
H_1 & \circ - \circ & \circ - \circ & \circ - \circ & \bullet & & H_2
\end{array}$$

Example 5.2.4. Suppose that we have a 3-face that has in its configuration only a T-component with the following configuration:

$$\bullet - \bullet - \circ - \circ - \circ - \bullet - \bullet$$

From Proposition 5.2.2 the number of vertices of the 3-face is:

$$f_0(P_{2+2-1}) + f_0(P_{2-1})f_0(P_{2-1}) = f_0(P_3) + f_0(P_1)f_0(P_1) = 4.$$

Such vertices are:

$$\begin{array}{ccccccccc}
H_1 & \bullet & \bullet & \circ - \circ & \circ - \circ & \bullet & H_2 \\
H_1 & \circ - \circ & \circ - \circ & \circ - \circ & \bullet & & H_2 \\
H_1 & \bullet & \circ - \circ & \circ - \circ & \bullet & \bullet & H_2 \\
H_1 & \bullet & \circ - \circ & \circ - \circ & \circ - \circ & & H_2
\end{array}$$

Corollary 5.2.5. *The number of vertices of a p-face of a polytope that has in its configuration only a T-component with only one open vertex that occupies the position immediately after (or before) to an endpoint is $f_0(P_{p+1})$.*

Proof. As $j = i = 1$, the configuration of the T-component is

$$\bullet - \circ - \bullet - \bullet \cdots \bullet - \bullet - \bullet$$

Note that the T-component has $p+2$ vertices and $p-1$ of them are inner entries. It results from Proposition 5.2.2 that, as j is odd, the number of vertices of the p-face is:

$$f_0(P_{1+p-1}) + f_0(P_{1-1})f_0(P_{p-1}) = f_{p+1} + f_p = f_{p+2} = f_0(P_{p+1}).$$

□

5.2.2 A configuration with just a T-component with two strings of open vertices

Consider now that the p-face has in its configuration only a T-component where, from the left to the right, we have a string of closed vertices with length i_1, a string of open vertices with length j_1, a string of closed vertices with length k, a string of open vertices with length j_2 and finally a string of closed vertices with length i_2. See the graph depicted in the next figure:

$$\underbrace{\bullet - \bullet \cdots \bullet}_{i_1} - \underbrace{\circ - \circ \cdots \circ}_{j_1} - \underbrace{\circ - \bullet - \bullet \cdots \bullet}_{k} - \underbrace{\circ - \circ \cdots \circ}_{j_2} - \underbrace{\bullet - \bullet \cdots \bullet - \bullet}_{i_2}$$

We can state the next proposition that gives the number of vertices contained in the face that has the configuration described above.

Proposition 5.2.6. *The number of vertices contained in the face that has only a T-component with the previous configuration is*

- *for $k > 1$,*

  - $f_0(P_{i_1+k+i_2})$ *if j_1 and j_2 are even numbers*
  - $f_0(P_{i_1-1})f_0(P_{i_2-1})f_0(P_{k-2}) + f_0(P_{i_2-1})f_0(P_{i_1+k-2}) +$ $f_0(P_{i_1-1})f_0(P_{i_2+k-2}) + f_0(P_{i_1+i_2+k-2})$ *if j_1 and j_2 are odd numbers*
  - $f_0(P_{i_1+k+i_2-1}) + f_0(P_{i_2-1})f_0(P_{i_1+k-1})$ *if j_1 is even and j_2 is odd*
  - $f_0(P_{i_1+k+i_2-1}) + f_0(P_{i_1-1})f_0(P_{i_2+k-1})$ *if j_1 is odd and j_2 is even*

- *for $k = 1$,*

  - $f_0(P_{i_1+i_2+1})$ *if j_1 and j_2 are both even*
  - $f_0(P_{i_1})f_0(P_{i_2+1}) + f_0(P_{i_1-1})f_0(P_{i_2-1})$ *if j_1 is even and j_2 is odd*
  - $f_0(P_{i_1+1})f_0(P_{i_2}) + f_0(P_{i_1-1})f_0(P_{i_2-1})$ *if j_1 is odd and j_2 is even*
  - $f_0(P_{i_1-1})f_0(P_{i_2+1}) + f_0(P_{i_1})f_0(P_{i_2-1})$ *if j_1 and j_2 are odd numbers.*

Proof. When we construct the vertices contained in the p-face the strings of open vertices give origin to open edges. Therefore we have to consider two different hypothesis: j_1 and j_2 have the same parity or j_1 and j_2 have different parities.

Firstly suppose that $k > 1$.

- $*$ If j_1 and j_2 are both even numbers, we can consider the following four situations:

$$\underbrace{\cdots}_{i_1 \text{ positions}} \quad \underset{\frac{j_1}{2} \text{ open edges}}{\circ - \circ \;\cdots\; \circ - \circ} \quad \underbrace{\cdots}_{k \text{ positions}} \quad \underset{\frac{j_2}{2} \text{ open edges}}{\circ - \circ \;\cdots\; \circ - \circ} \quad \underbrace{\cdots}_{i_2 \text{ positions}}$$

$$\underbrace{\cdots}_{i_1 - 1 \text{ p.}} \quad \underset{\frac{j_1+2}{2} \text{ o. e.}}{\circ - \circ \;\cdots\; \circ - \circ} \quad \underbrace{\cdots}_{k-1 \text{ p.}} \quad \underset{\frac{j_2}{2} \text{ o. e.}}{\circ - \circ \;\cdots\; \circ - \circ} \quad \underbrace{\cdots}_{i_2 \text{ p.}}$$

$$\underbrace{\cdots}_{i_1 \text{ p.}} \quad \underset{\frac{j_1}{2} \text{ o. e.}}{\circ - \circ \;\cdots\; \circ - \circ} \quad \underbrace{\cdots}_{k-1 \text{ p.}} \quad \underset{\frac{j_2+2}{2} \text{ o. e.}}{\circ - \circ \;\cdots\; \circ - \circ} \quad \underbrace{\cdots}_{i_2 - 1 \text{ p.}}$$

$$\underbrace{\cdots}_{i_1 - 1 \text{ p.}} \quad \underset{\frac{j_1+2}{2} \text{ o. e.}}{\circ - \circ \;\cdots\; \circ - \circ} \quad \underbrace{\cdots}_{k-2 \text{ p.}} \quad \underset{\frac{j_2+2}{2} \text{ o. e.}}{\circ - \circ \;\cdots\; \circ - \circ} \quad \underbrace{\cdots}_{i_2 - 1 \text{ p.}}$$

The previous cases lead, respectively, to the following numbers of vertices of the p-face:

$f_0(P_{i_1})f_0(P_k)f_0(P_{i_2})$, $f_0(P_{i_1-1})f_0(P_{k-1})f_0(P_{i_2})$, $f_0(P_{i_1})f_0(P_{k-1})f_0(P_{i_2-1})$ and $f_0(P_{i_1-1})f_0(P_{k-2})f_0(P_{i_2-1})$.

Therefore, the number of distinct configurations of the vertices contained in the respective p-face is:

$$\begin{aligned}
f_0(P_{i_1})f_0(P_k)f_0(P_{i_2}) + f_0(P_{i_1-1})f_0(P_{k-1})f_0(P_{i_2}) &\quad + \\
f_0(P_{i_1})f_0(P_{k-1})f_0(P_{i_2-1}) + f_0(P_{i_1-1})f_0(P_{k-2})f_0(P_{i_2-1}) &\quad = \\
(f_{i_1+1}f_{k+1} + f_{i_1}f_k)f_{i_2+1} + (f_{i_1+1}f_k + f_{i_1}f_{k-1})f_{i_2} &\quad = \\
f_{i_1+k+1}f_{i_2+1} + f_{i_1+k}f_{i_2} &\quad = \\
f_{i_1+k+i_2+1} &\quad = \\
f_0(P_{i_1+k+i_2})
\end{aligned}$$

$*$ If j_1 is even and j_2 is odd, we can obtain the following four situations:

$$\underbrace{\cdots}_{i_1 \text{ p.}} \quad \underset{\frac{j_1}{2} \text{ o. e.}}{\circ - \circ \ \cdots \ \circ - \circ} \quad \underbrace{\cdots}_{k-1 \text{ p.}} \quad \underset{\frac{j_2+1}{2} \text{ o. e.}}{\circ - \circ \ \cdots \ \circ - \circ} \quad \underbrace{\cdots}_{i_2 \text{ p.}}$$

$$\underbrace{\cdots}_{i_1 \text{ p.}} \quad \underset{\frac{j_1}{2} \text{ o. e.}}{\circ - \circ \ \cdots \ \circ - \circ} \quad \underbrace{\cdots}_{k \text{ p.}} \quad \underset{\frac{j_2+1}{2} \text{ o. e.}}{\circ - \circ \ \cdots \ \circ - \circ} \quad \underbrace{\cdots}_{i_2-1 \text{ p.}}$$

$$\underbrace{\cdots}_{i_1-1 \text{ p.}} \quad \underset{\frac{j_1+2}{2} \text{ o. e.}}{\circ - \circ \ \cdots \ \circ - \circ} \quad \underbrace{\cdots}_{k-2 \text{ p.}} \quad \underset{\frac{j_2+1}{2} \text{ o. e.}}{\circ - \circ \ \cdots \ \circ - \circ} \quad \underbrace{\cdots}_{i_2 \text{ p.}}$$

$$\underbrace{\cdots}_{i_1-1 \text{ p.}} \quad \underset{\frac{j_1+2}{2} \text{ o. e.}}{\circ - \circ \ \cdots \ \circ - \circ} \quad \underbrace{\cdots}_{k-1 \text{ p.}} \quad \underset{\frac{j_2+1}{2} \text{ o. e.}}{\circ - \circ \ \cdots \ \circ - \circ} \quad \underbrace{\cdots}_{i_2-1 \text{ p.}}$$

Each one of these cases will rise to
$f_0(P_{i_1})f_0(P_{k-1})f_0(P_{i_2})$, $f_0(P_{i_1})f_0(P_k)f_0(P_{i_2-1})$, $f_0(P_{i_1-1})f_0(P_{k-2})f_0(P_{i_2})$ and $f_0(P_{i_1-1})f_0(P_{k-1})f_0(P_{i_2-1})$ vertices of the p-face, respectively.

Therefore, using same reasoning, the number of distinct configurations of the vertices that are contained in the p-face is:

$$\begin{aligned} f_0(P_{i_1})f_0(P_{k-1})f_0(P_{i_2}) + f_0(P_{i_1})f_0(P_k)f_0(P_{i_2-1}) &\quad + \\ f_0(P_{i_1-1})f_0(P_{k-2})f_0(P_{i_2}) + f_0(P_{i_1-1})f_0(P_{k-1})f_0(P_{i_2-1}) &\quad = \\ f_0(P_{i_1+k-1})f_0(P_{i_2-1}) + f_0(P_{i_1+i_2+k-1)} & \end{aligned}$$

* If j_1 is odd and j_2 is even, changing in this last expression i_1 by i_2 and vice versa, the number of the vertices of the p-face is:

$$f_0(P_{i_2+k-1})f_0(P_{i_1-1}) + f_0(P_{i_1+i_2+k-1})$$

* If j_1 and j_2 are both odd numbers, we use similar reasoning to obtain the desired expression.

Now we will study the remaining case.

Let $k = 1$, $i_1, i_2 \geq 1$ and the representation of the T-component is:

$$\bullet - \bullet \cdots \bullet - \circ - \circ \cdots \circ - \circ - \bullet - \circ - \circ \cdots \circ - \bullet \cdots \bullet - \bullet$$

We split the proof into four subcases:

* if j_1 and j_2 are both even, the number of vertices contained in the p-face is:

$$\begin{aligned} f_0(P_{i_1})f_0(P_{i_2}) + f_0(P_{i_1-1})f_0(P_{i_2}) + f_0(P_{i_1})f_0(P_{i_2-1}) &\quad = \\ f_0(P_{i_1+i_2+1}) & \end{aligned}$$

* if j_1 is even and j_2 is odd, the number of vertices contained in the p-face is :

$$\begin{aligned} f_0(P_{i_1})f_0(P_{i_2}) + f_0(P_{i_1})f_0(P_{i_2-1}) + f_0(P_{i_1-1})f_0(P_{i_2-1}) &\quad = \\ f_0(P_{i_1-1})f_0(P_{i_2-1}) & \end{aligned}$$

* if j_1 is odd and j_2 is even, the number of vertices contained in the p-face is:

$$f_0(P_{i_2})f_0(P_{i_1+1}) + f_0(P_{i_1-1})f_0(P_{i_2-1})$$

$*$ if j_1 and j_2 are both odd the number of vertices contained in the p-face is:

$$\begin{aligned} f_0(P_{i_1-1})f_0(P_{i_2}) + f_0(P_{i_1-1})f_0(P_{i_2-1}) + f_0(P_{i_1})f_0(P_{i_2-1}) &= \\ f_0(P_{i_1-1})f_0(P_{i_2+1}) + f_0(P_{i_1})f_0(P_{i_2-1}). & \end{aligned}$$

$\square$

The next corollary follows immediately from the previous proposition.

Corollary 5.2.7. *Let $i_1 = i_2 = 1$ and $j_1, k, j_2 \geq 1$. The representation of the T-component is:*

$$\bullet - \circ - \circ \cdots \circ - \circ - \bullet - \bullet \cdots \bullet - \circ - \circ \cdots \circ - \bullet$$

The number of vertices contained in the face that has only a T-component in the previous conditions is $f_0(P_{k+2})$.

5.2.3 A configuration with just a T-component with s strings of open vertices

Given a p-face of Ω_n^t, it is important to notice what are the differences between a configuration that can be associated to a vertex that belongs to the p-face and a configuration of a vertex of Ω_n^t which is not a vertex of a given p-face.

In fact, all vertices contained in the p-face are vertices of Ω_n^t, but there are vertices of Ω_n^t that are not vertices contained in the p-face namely, those in whose representation it appears as closed, at least one of the vertices of the bicolored subgraph that are open in the configuration of the p-face.

Therefore, a vertex V of Ω_n^t is a vertex of the p-face if and only if all bicolored subgraphs of Type 1 and Type 2 of the configuration of the p-face are the same as in the configuration of V, and if all vertices that are open in the T-components of the representation of the p-face still remain open in the configuration of V.

Let us consider that the T-component has $s+1$ strings of closed vertices with lengths $i_1, \ldots, i_{s+1}$, and s strings of open vertices with lengths $j_1, \ldots, j_s$, placed alternately, where $i_1 + \cdots + i_{s+1} + j_1 + \cdots + j_s = m$. Bearing in mind the previous observations, in order to count the number of vertices of the p-face we proceed as follows:

(1) We count the number of vertices of Ω_m^t, $f_0(P_m) = f_{m+1}$;

(2) we must exclude the vertices of Ω_m^t that have in its configuration at least a closed vertex in a position of an open one of the T-component. The remaining vertices of the polytope Ω_m^t are vertices of the p -face.

Using this procedure we present an algorithm to count the number of vertices of the p-face described in the previous section. This algorithm, when $s = 1$, provides another way to count the number of vertices in the p-face present in the Subsection 5.2.1. In the following algorithm we will represent an edge $e = ij$ by $\{i, j\}$.

Algorithm

The input is a bicolored subgraph with m vertices that corresponds to the T-component with $s = 1$, of the representation of the p-face.

step 1 ♦ Let P_{j_1} be the path corresponding to the string of open vertices of length j_1. Compute the number of vertices of Ω_m^t in which configuration have, at least, a closed vertex in any position of the vertices of P_{j_1} and does not have any of the open edges $\{i_1, i_1+1\}$ and $\{i_1+j_1, i_1+j_1+1\}$, that is

$$f_0(P_{j_1})g_0(P_m \backslash P_{j_1}),\ \text{if } j_1 \text{ is odd} \tag{o1}$$

or

$$(f_0(P_{j_1}) - 1)g_0(P_m \backslash P_{j_1}),\ \text{if } j_1 \text{ is even.} \tag{e1}$$

step 2 ♦ Compute the number of vertices of Ω_m^t that have in its configuration at least a closed vertex in any position of P_{j_1} and that have in its configuration the open edge $\{i_1, i_1 + 1\}$ and does not have the open edge $\{i_1 + j_1, i_1 + j_1 + 1\}$, that is

$$f_0(P_{i_1-1})(f_0(P_{j_1-1}) - 1)f_0(P_{i_2}),\ \text{if } j_1 \text{ is odd} \tag{o2}$$

or

$$f_0(P_{i_1-1})f_0(P_{j_1-1})f_0(P_{i_2}),\ \text{if } j_1 \text{ is even.} \tag{e2}$$

step 3 ♦ Compute the number of vertices of Ω_m^t that have in its configuration at least a closed vertex in any position of P_{j_1} and that have in its configuration the open edge $\{i_1 + j_1, i_1 + j_1 + 1\}$ and does not have the open edge $\{i_1, i_1 + 1\}$, that is

$$f_0(P_{i_1})(f_0(P_{j_1-1}) - 1)f_0(P_{i_2-1}), \text{if } j_1 \text{ is odd} \tag{o3}$$

or

$$f_0(P_{i_1})f_0(P_{j_1-1})f_0(P_{i_2-1}), \text{if } j_1 \text{ is even.} \tag{e3}$$

step 4 ♦ Compute the number of vertices of Ω_m^t that have in its configuration at least a closed vertex of P_{j_1} and that have the two open edges $\{i_1, i_1+1\}$ and $\{i_1 + j_1, i_1 + j_1 + 1\}$, that is

$$f_0(P_{i_1-1})f_0(P_{j_1-2})f_0(P_{i_2-1}),\ \text{if } j_1 \text{ is odd} \tag{o4}$$

or

$$f_0(P_{i_1-1})(f_0(P_{j_1-2}) - 1)f_0(P_{i_2-1}),\ \text{if } j_1 \text{ is even.} \tag{e4}$$

final step ♦ Compute the number of vertices of the p-face, that is, for j_1 odd, the difference between $f_0(P_m)$ and the sum of the expressions (e1), (e2), (e3) and (e4) and, for j_1 even, is the difference between $f_0(P_m)$ and the sum of the expressions (o1),(o2), (o3), (o4).

Recalling the examples studied in the first subsection and applying the previous algorithm we obtain:

Example 5.2.8. The number of vertices contained in the 4-face in which configuration has only the following T-component with one string of open vertices

$$\bullet - \bullet - \bullet - \circ - \circ - \bullet - \bullet$$

is:

$$\begin{aligned} f_0(P_7) - (f_0(P_2)-1)f_0(P_3)f_0(P_2) - f_0(P_2)f_0(P_1)f_0(P_2) & \\ -f_0(P_3)f_0(P_1)f_0(P_1) - f_0(P_2)(f_0(P_0)-1)f_0(P_2) & = 8. \end{aligned}$$

Example 5.2.9. The number of vertices contained in the 3-face in which configuration has only the next T-component

$$\bullet - \bullet - \circ - \circ - \circ - \bullet - \bullet$$

is:

$$\begin{aligned} f_0(P_7) - f_0(P_2)f_0(P_3)f_0(P_2) - f_0(P_1)(f_0(P_2)-1)f_0(P_2) & \\ -f_0(P_2)(f_0(P_2)-1)f_0(P_1) - f_0(P_1)f_0(P_1)f_0(P_1) & = 4. \end{aligned}$$

The next algorithm provides a way to count the number of vertices of the p-face for $s \geq 2$.

For each $k = 1, 2, \ldots, s$ consider the open edges

$$e_{2k-1} = \left\{ \sum_{l=1}^{k} (i_l + j_l) - j_k, \sum_{l=1}^{k} (i_l + j_l) - j_k + 1 \right\}$$

and

$$e_{2k} = \left\{ \sum_{l=1}^{k} (i_l + j_l), \sum_{l=1}^{k} (i_l + j_l) + 1 \right\}.$$

Let us denote the set of these edges by $\mathcal{L} = \{e_c, c = 1, \ldots, 2s\} \subset E(P_m)$. Given an open edge of $\mathcal{L}$ each of its vertices belongs to a different string. One vertex belongs to a string of open vertices and the other one belongs to a string of closed vertices, not necessarily by this order. If a string has only a vertex, and it is neither the first nor the last string, there are two open edges of $\mathcal{L}$ with a common vertex, we denote the number of those open edges by ν.

Algorithm: The input is a bicolored subgraph with m vertices that corresponds to a T-component of the representation of a p-face that has only a T-component in which configuration there are s strings of open vertices with lengths $j_1, \ldots, j_s$ and $s+1$ strings of closed vertices with lengths $i_1, \ldots, i_{s+1}$ placed alternately. We denote by P_a a path corresponding to a string of length a and by $P_a \backslash v$ the path resulting from P_a deleting its origin or its terminus, which is called v; $\mathcal{P}_c = \{P_a : a = i_1, \ldots, i_{s+1}\}$ and $\mathcal{P}_o = \{P_a : a = j_1, \ldots, j_s\}$. Let $m = i_1 + \cdots + i_{s+1} + j_1 + \cdots + j_s$.

stage I . Computation of the number of vertices of Ω_m^t whose configuration has at least a closed vertex in any position of the vertices of one of the $P_{j_1}, \ldots, P_{j_s}$ and does not have any of the open edges that belongs to $\mathcal{L}$:

step 1 ♦ If at least one of the j_k is odd compute:

$$\prod_{P\in\mathcal{P}_o\cup\mathcal{P}_c} f_0(P).$$

If all j_k are even compute:

$$\left[\prod_{P\in\mathcal{P}_o} f_0(P)-1\right]\prod_{P\in\mathcal{P}_c} f_0(P).$$

stage II ▪ Computation of the number of vertices of Ω_m^t whose configuration has at least a closed vertex in any position of the vertices of one of the $P_{j_1},\ldots,P_{j_s}$, and have exactly one of the open edges that belongs to $\mathcal{L}$:

step 1 ♦ For each $c=1,\ldots,2s$ consider $e_c\in\mathcal{L}$, and let $j_k=j_{\frac{c}{2}}$, if c is even or $j_k=j_{\frac{c+1}{2}}$, if c is odd.

⋆ If c is odd and j_k is odd and all j_d, with $d\neq k$, are even compute:

$$f_0(P_{i_k}\backslash v)\cdot\prod_{P\in\mathcal{P}_c\backslash\{P_{i_k}\}} f_0(P)\cdot\left[f_0(P_{j_k}\backslash v)\cdot\prod_{P\in\mathcal{P}_o\backslash\{P_{j_k}\}} f_0(P)-1\right]$$

or, if c is odd and j_k is even or at least one of the j_d, with $d\neq k$, is odd compute:

$$f_0(P_{i_k}\backslash v)\cdot\prod_{P\in\mathcal{P}_c\backslash\{P_{i_k}\}} f_0(P)\cdot f_0(P_{j_k}\backslash v)\cdot\prod_{P\in\mathcal{P}_o\backslash\{P_{j_k}\}} f_0(P)$$

⋆ if c is even and j_k is odd and all j_d, with $d\neq k$, are even compute:

$$f_0(P_{i_{k+1}}\backslash v)\cdot \prod_{P\in\mathcal{P}_c\backslash\{P_{i_{k+1}}\}} f_0(P)\cdot \left[f_0(P_{j_k}\backslash v)\cdot \prod_{P\in\mathcal{P}_o\backslash\{P_{j_k}\}} f_0(P)-1\right]$$

or, if c is even and j_k is even or at least one of the j_d, with $d \neq k$, is odd compute:

$$f_0(P_{i_{k+1}}\backslash v)\cdot \prod_{P\in\mathcal{P}_c\backslash\{P_{i_{k+1}}\}} f_0(P)\cdot f_0(P_{j_k}\backslash v)\cdot \prod_{P\in\mathcal{P}_o\backslash\{P_{j_k}\}} f_0(P)$$

final step ♦ Sum all the values obtained in previous step.

stage III ▪ Computation of the number of vertices of Ω_m^t whose configuration has at least a closed vertex in any position of the vertices of one of the $P_{j_1},\ldots,P_{j_s}$, and has exactly two of the open edges of $\mathcal{L}$. For each pair of edges e_{c_1}, $e_{c_2} \in \mathcal{L}$, Let $P_{c_1}(P_{o_1})$ and $P_{c_2}(P_{o_2})$ be the paths corresponding to the strings of closed (open) vertices such that e_{c_1} has a vertex on P_{c_1} (P_{o_1}) and e_{c_2} has a vertex on P_{c_2} (P_{o_2}).

step1 ♦ If $P_{o_1} \neq P_{o_2}$ and $P_{c_1} \neq P_{c_2}$ and

- ⋆ if the lengths of P_{o_1} and P_{o_2} are both odd and all the remain paths of $\mathcal{P}_o$ have length even compute

$$f_0(P_{c_1}\backslash v)\cdot f_0(P_{c_2}\backslash v)\cdot \prod_{P\in\mathcal{P}_c\backslash\{P_{c_1},P_{c_2}\}} f_0(P)\cdot$$
$$\left[f_0(P_{o_1}\backslash v)\cdot f_0(P_{o_2}\backslash v)\cdot \prod_{P\in\mathcal{P}_o\backslash\{P_{o_1},P_{o_2}\}} f_0(P)-1\right].$$

- ⋆ if at least one of the lengths of P_{o_1}, P_{o_2} is even or at last one of the remain paths of $\mathcal{P}_o$ has length odd compute

$$f_0(P_{c_1}\backslash v)\cdot f_0(P_{c_2}\backslash v)\cdot \prod_{P\in\mathcal{P}_c\backslash\{P_{c_1},P_{c_2}\}} f_0(P)\cdot f_0(P_{o_1}\backslash v)\cdot f_0(P_{o_2}\backslash v)\cdot \prod_{P\in\mathcal{P}_o\backslash\{P_{o_1},P_{o_2}\}} f_0(P).$$

step2 ♦ If $P_{0_1}=P_{0_2}$ and

- ⋆ if the lengths of all paths in $\mathcal{P}_o$ are even compute

$$f_0(P_{c_1}\backslash v)\cdot f_0(P_{c_2}\backslash v)\cdot \prod_{P\in\mathcal{P}_c\backslash\{P_{c_1},P_{c_2}\}} f_0(P)\cdot \left[f_0(P_{o_1}\backslash\{v_1,v_2\})\cdot \prod_{P\in\mathcal{P}_o\backslash\{P_{o_1}\}} f_0(P)-1\right].$$

- ⋆ if at least one of the paths in $\mathcal{P}_o$ has length odd compute:

$$f_0(P_{c_1}\backslash v)\cdot f_0(P_{c_2}\backslash v)\cdot \prod_{P\in\mathcal{P}_c\backslash\{P_{c_1},P_{c_2}\}} f_0(P)\cdot f_0(P_{o_1}\backslash\{v_1,v_2\})\cdot \prod_{P\in\mathcal{P}_o\backslash\{P_{o_1}\}} f_0(P).$$

step3 ♦ If $P_{c_1}=P_{c_2}$ and

- ⋆ if the lengths of P_{o_1} and P_{o_2} are both odd and all the remain paths of $\mathcal{P}_o$ have length even compute

$$f_0(P_{c_1}\backslash\{v_1,v_2\})\cdot \prod_{P\in\mathcal{P}_c\backslash\{P_{c_1}\}} f_0(P)\cdot \left[f_0(P_{o_1}\backslash v)\cdot f_0(P_{o_2}\backslash v)\cdot \prod_{P\in\mathcal{P}_o\backslash\{P_{o_1},P_{o_2}\}} f_0(P)-1\right].$$

- ⋆ if at least one of the lengths of P_{o_1}, P_{o_2} is even or at last one of the remain paths of $\mathcal{P}_o$ has length odd compute

$$f_0(P_{c_1}\backslash\{v_1,v_2\})\cdot \prod_{P\in\mathcal{P}_c\backslash\{P_{c_1}\}} f_0(P)\cdot f_0(P_{o_1}\backslash v)\cdot f_0(P_{o_2}\backslash v)\cdot \prod_{P\in\mathcal{P}_o\backslash\{P_{o_1},P_{o_2}\}} f_0(P).$$

final step ♦ Sum all values obtained in previous steps.

stage IV ▪ For each triplet of edges in $\mathcal{L}$, we proceed with the same reasoning as in the previous stage.

⋮

We continue this procedure until we reach $|\mathcal{L}| - \nu$.

⋮

final stage ▪ Computation of the number of vertices of the face.

step 1 ♦ Sum all the values obtained at final steps of Stages I, II and so on;

step 2 ♦ compute $f_0(P_m)$

step 3 ♦ compute the difference between the values obtained in step 2 and in step 1.

In order to illustrate the previous algorithm we present an example:

Example 5.2.10. Consider a 4-face of Ω_n^t represented by the following bicolored subgraph

$$H_1 \quad \bullet - \bullet - \circ - \circ - \circ - \bullet - \bullet - \circ - \circ - \bullet \quad H_2$$

The T-component has 10 vertices spread over 3 strings of closed vertices and 2 strings of open vertices; $|\mathcal{L}| = 4$ and $\nu = 0$. Next, we present the calculations for each stage.

Stage I (As the lengths of the strings of open vertices are odd, we are in the first situation)

$$f_0(P_2) \cdot f_0(P_3) \cdot f_0(P_2) \cdot f_0(P_2) \cdot f_0(P_1) = 2 \times 3 \times 2 \times 2 \times 1 = \mathbf{24}$$

stage II (As $|\mathcal{L}| = 4$, we have four possibilities in step 1)

step 1 $f_0(P_1) \cdot f_0(P_2) \cdot f_0(P_1) \cdot [f_0(P_2) \cdot f_0(P_2) - 1] = 6$

$f_0(P_1) \cdot f_0(P_2) \cdot f_0(P_1) \cdot [f_0(P_2) \cdot f_0(P_2) - 1] = 6$

$f_0(P_1) \cdot f_0(P_2) \cdot f_0(P_1) \cdot f_0(P_3) \cdot f_0(P_1) = 6$

$f_0(P_0) \cdot f_0(P_2) \cdot f_0(P_2) \cdot f_0(P_3) \cdot f_0(P_1) = 12$

final step $6 + 6 + 6 + 12 = \mathbf{30}$

stage III (As the open edges do not have vertices in common, we have 6 possibilities in this stage)

step 1 $f_0(P_1) \cdot f_0(P_1) \cdot f_0(P_1) \cdot f_0(P_2) \cdot f_0(P_1) = 2$

$f_0(P_1) \cdot f_0(P_0) \cdot f_0(P_2) \cdot f_0(P_2) \cdot f_0(P_1) = 4$

$f_0(P_1) \cdot f_0(P_0) \cdot f_0(P_2) \cdot f_0(P_2) \cdot f_0(P_1) = 4$

step 2 $f_0(P_1) \cdot f_0(P_1) \cdot f_0(P_1) \cdot f_0(P_1) \cdot f_0(P_2) = 2$

$f_0(P_0) \cdot f_0(P_2) \cdot f_0(P_1) \cdot f_0(P_0) \cdot f_0(P_3) = 6$

step 3 $f_0(P_0) \cdot f_0(P_1) \cdot f_0(P_2) \cdot f_0(P_1) \cdot f_0(P_2) = 4$

final step $2 + 4 + 4 + 2 + 6 + 4 = \mathbf{22}$

stage IV (We have 4 triplets of open edges)

step 1 $f_0(P_1) \cdot f_0(P_0) \cdot f_0(P_1) \cdot f_0(P_1) \cdot f_0(P_1) = 1$

$f_0(P_1) \cdot f_0(P_0) \cdot f_0(P_1) \cdot f_0(P_0) \cdot f_0(P_1) = 1$

$f_0(P_1) \cdot f_0(P_0) \cdot f_0(P_1) \cdot [f_0(P_2) \cdot f_0(P_0) - 1] = 1$

$f_0(P_0) \cdot f_0(P_0) \cdot f_0(P_2) \cdot [f_0(P_2) \cdot f_0(P_0) - 1] = 2$

final step $1 + 1 + 1 + 2 = \mathbf{5}$

stage V (The maximum number of open edges is 4, we have one case) step 1 $f_0(P_1) \cdot f_0(P_0) \cdot f_0(P_0) \cdot f_0(P_1) \cdot f_0(P_0) = 1$

final step **1**

final Stage (To complete the precedure)

step 1 $f_{11} = \mathbf{89}$

step 2 $24 + 30 + 22 + 5 + 1 = \mathbf{82}$

final step $89 - 82 = \mathbf{7}$

This 4-face has 7 vertices.

5.2.4 A configuration with ℓ T-components

Consider now a p-face with the next configuration:

$$H_1 \ \dot{\cup} \ T_1 \ \dot{\cup} \ H_2 \ \dot{\cup} \ T_2 \ \dot{\cup} \ \cdots \ \dot{\cup} \ T_\ell \ \dot{\cup} \ H_{\ell+1}$$

For each T-component T_i, $i = 1, \ldots, \ell$, we compute the number of vertices of the face of Ω_n^t whose configuration has only T_i as T-component. We represent this number by h_i.

Proposition 5.2.11. *The number of the vertices of the described p-face is:*

$$\prod_{i=1}^{\ell} h_i \tag{5.2.1}$$

Proof. It follows directly using the same argument as in the proof of Proposition 4.0.1 when $m = 0$.

□

5.3 Faces of lower dimension of a given $2-$face of Ω_n^t

In this section we present the number of vertices and edges that gives origin to the 2-faces of the tridiagonal Birkhoff polytope, Ω_n^t, and we describe their representation in terms of bicolored subgraphs. We denote by ω, $\omega \geq 0$, the number of internal vertices of a T-component.

For the representation of a 2-face we describe the following cases:

Case 1. We have one T-component with one inner entry and two closed endpoints. Between these closed endpoints we have $\omega \geq 1$. One of these internal vertices is the inner entry and it can occupy all the positions of the internal vertices as we can see in the following representation:

$$H_1 \quad \bullet - \circ - \circ \cdots \bullet \cdots \circ - \bullet \quad H_2$$

Case 2. We have two T-components each one with two closed endpoints and without inner entries. Between each pair of endpoints we can have a finite number of internal vertices. The bicolored subgraph represented below describe this situation:

$$H_1 \quad \bullet - \circ \cdots \circ - \bullet \quad H_2 \quad \bullet - \circ \cdots \circ - \bullet \quad H_3$$

5.3.1 Number and representation of vertices of a 2-face of Ω_n^t

In both previously cases, the T-components satisfy the conditions of Proposition 5.2.2 and therefore the number of vertices of a 2-face, $\mathcal{F}, is$:

- In the first case $f_0(\mathcal{F}) = f_0(P_3) = 3$;

- In the second case each T-component gives origin to $f_0(P_2)$ vertices, thus, by Proposition 5.2.10, the number of vertices of the 2-face is $f_0(\mathcal{F}) = f_0(P_2)f_0(P_2) = 4$.

Next we represent the vertices in terms of bicolored subgraphs.

Consider the bicolored subgraph given at Case 1. The number of vertices that gives origin to this configuration is three. Without loss of generality we assume that the inner entry occupies the first internal position, from the left to the right.

If $\omega = 1$, the vertices that gives origin to the T-component have the following representation:

$$H_1 \quad \bullet \quad \bullet \quad \bullet \quad H_2$$

$$H_1 \quad \circ - \circ \quad \bullet \quad H_2$$

and

$$H_1 \quad \bullet \quad \circ - \circ \quad H_2$$

If $\omega > 1$ is odd, from the previous configurations we can obtain the representation of each of the three vertices, adding immediately to the left of the last closed vertex or open edge, the union of $[\frac{\omega-1}{2}]$ open edges. We represent this union by the symbol $\sim$. Therefore, the representation of each of the three vertices that gives origin to the configuration of the 2-face is:

$$H_1 \quad \bullet \quad \bullet \sim \bullet \quad H_2$$

$$H_1 \quad \circ - \circ \sim \bullet \quad H_2$$

and

$$H_1 \quad \bullet \sim \circ - \circ \quad H_2$$

If $\omega = 2$, the configuration of the face results from the bicolored sum of the three vertices represented by:

$$H_1 \quad \bullet \quad \bullet \quad \circ - \circ \quad H_2$$

$$H_1 \quad \bullet \quad \circ - \circ \quad \bullet \quad H_2$$

and

$$H_1 \quad \circ - \circ \quad \circ - \circ \quad H_2$$

If $\omega > 2$ is even, from this last configurations we can obtain the representation of each of the three vertices, adding immediately to the left of the last closed vertex or open edge, the union of $\frac{\omega-2}{2}$ open edges, whose union is represented by $\sim$. Therefore the configuration of the three vertices that gives origin to the face is:

$$H_1 \quad \bullet \quad \bullet \sim \circ - \circ \quad H_2$$

$$H_1 \quad \bullet \quad \circ - \circ \sim \bullet \quad H_2$$

and

$$H_1 \quad \circ - \circ \sim \circ - \circ \quad H_2$$

Consider now that the bicolored subgraph is given as in Case 2. In this case the 2-face results from the bicolored sum of three of the four vertices whose representation is the following:

$$H_1 \quad \bullet \quad \bullet \quad H_2 \quad \bullet \quad \bullet \quad H_3$$

$$H_1 \quad \circ - \circ \quad H_2 \quad \bullet \quad \bullet \quad H_3$$

$$H_1 \quad \bullet \quad \bullet \quad H_2 \quad \circ - \circ \quad H_3$$

$$H_1 \quad \circ - \circ \quad H_2 \quad \circ - \circ \quad H_3$$

Here, if we have a finite number of internal vertices in each T-component, we use a similar procedure as we did in Case 1. The representation of the four vertices are:

* when the numbers of internal vertices are both odd

$$H_1 \quad \circ - \circ \sim \bullet \quad H_2 \quad \bullet \sim \circ - \circ \quad H_3$$

$$H_1 \quad \bullet \sim \circ - \circ \quad H_2 \quad \bullet \sim \circ - \circ \quad H_3$$

$$H_1 \quad \bullet \sim \circ - \circ \quad H_2 \quad \circ - \circ \sim \bullet \quad H_3$$

$$H_1 \quad \circ - \circ \sim \bullet \quad H_2 \quad \circ - \circ \sim \bullet \quad H_3$$

* when the numbers of internal vertices are both even

$$H_1 \quad \bullet \sim \circ - \circ \quad \bullet \quad H_2 \quad \bullet \sim \circ - \circ \quad \bullet \quad H_3$$

$$H_1 \quad \circ - \circ \sim \circ - \circ \quad H_2 \quad \bullet \sim \circ - \circ \quad \bullet \quad H_3$$

$$H_1 \quad \circ - \circ \sim \circ - \circ \quad H_2 \quad \circ - \circ \sim \circ - \circ \quad H_3$$

$$H_1 \quad \bullet \sim \circ - \circ \quad \bullet \quad H_2 \quad \circ - \circ \sim \circ - \circ \quad H_3$$

* when the numbers of internal vertices of the first and second T-components are respectively odd and even

$$H_1 \quad \circ - \circ \sim \bullet \quad H_2 \quad \bullet \sim \circ - \circ \quad \bullet \quad H_3$$

$$H_1 \quad \bullet \sim \circ - \circ \quad H_2 \quad \bullet \sim \circ - \circ \quad \bullet \quad H_3$$

$$H_1 \quad \bullet \sim \circ - \circ \quad H_2 \quad \circ - \circ \sim \circ - \circ \quad H_3$$

$$H_1 \quad \circ - \circ \sim \bullet \quad H_2 \quad \circ - \circ \sim \circ - \circ \quad H_3$$

* when the numbers of internal vertices of the first and second T-components are even and odd, respectively, we have a similar situation as we described above, in the previous representation of the vertices we only change the order of the T-components.

So, in general, the number of vertices of a face of Ω_n^t is three or four and it is equal to the number of closed vertices (closed endpoints plus inner entries) that can be present in the T-components of the representation of the 2-face.

5.3.2 Edges of a 2-face of Ω_n^t

As the number of edges of a 2-face is equal to the number of its vertices, the number of edges of a 2-face of Ω_n^t is three or four and the faces are triangles or rectangles. When the face has in its configuration the T-component described at Case 1 it has the three vertices present at Subsection 5.3.1 and the following three edges:

$$H_1 \quad \bullet - \bullet \quad \bullet \quad H_2$$

$$H_1 \quad \bullet \quad \bullet - \bullet \quad H_2$$

and

$$H_1 \quad \bullet - \circ - \bullet \quad H_2$$

Without loss of generality, here, we considered the simplest situation. Similarly, when $\omega > 1$, we obtain the general configurations of the three edges of the face.

The bicolored subgraphs present at Case 1 results from three vertices and three edges. Therefore, the faces that are associated to them are triangles.

If the representation of the 2-face contains the bicolored subgraph given by Case 2, we have a face with four vertices and consequently with four edges.

These last ones, in the simplest situation, are represented by:

$$H_1 \quad \bullet - \bullet \quad H_2 \quad \bullet \quad \bullet \quad H_3$$

$$H_1 \quad \circ - \circ \quad H_2 \quad \bullet - \bullet \quad H_3$$

$$H_1 \quad \bullet - \bullet \quad H_2 \quad \circ - \circ \quad H_3$$

$$H_1 \quad \bullet \quad \bullet \quad H_2 \quad \bullet - \bullet \quad H_3$$

Similarly, when $\omega > 1$, we obtain the configuration of the four edges of the face. The faces that are associated to these bicolored subgraphs are rectangles.

Therefore, from previous considerations, we can state the following proposition:

Proposition 5.3.1. *A 2-face of Ω_n^t is a triangle or a rectangle.*

5.4 Faces of lower dimension of a given cell of Ω_n^t

Now we will study the elements (vertices, edges and faces) of a 3-face of a tridiagonal Birkhoff polytope. As $p = 3$ we have, from Proposition 5.1.1 that the total number of different configurations, in number of T-components, to obtain a 3 -face of Ω_n^t is 3 and their representations are:

Case 1. one T-component with two closed endpoints and two inner entries;

Case 2. two T-components each of them with two closed endpoints and one of them with one inner entry;

Case 3. three T-components with two closed endpoints and without inner entries.

As in the previous section, we can have in each T-component a finite number of open internal vertices, but first we consider the simplest situation: the T-component does not have open vertices.

Suppose that the T-component has the configuration as in Case 1.

- If all internal vertices are inner entries the configuration of the cell is:

$$H_1 \quad \bullet - \bullet - \bullet - \bullet \quad H_2$$

In this case the cell has the "same elements" as Ω_4^t, *i.e.* $f_0(\mathcal{F}) = f_0(P_4) = 5$, and its configuration is obtained from the bicolored sum of at least three of the following five vertices:

$$H_1 \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad H_2$$

$$H_1 \quad \circ - \circ \quad \bullet \quad \bullet \quad H_2$$

$$H_1 \quad \bullet \quad \bullet \quad \circ - \circ \quad H_2$$

$$H_1 \quad \bullet \quad \circ - \circ \quad \bullet \quad H_2$$

$$H_1 \quad \circ - \circ \quad \circ - \circ \quad H_2$$

Applying the adjacency criterium, (Proposition 2.4.3), to these vertices we find eight edges, and five faces. Attending to previous section, four of these faces are triangles and one of them is a rectangle. This cell is a pentahedron.

Suppose that the T-component presents in the configuration of the cell has internal open vertices. If we have one internal open vertex, it can be, or not, adjacent to an endpoint:

$*$ if the internal open vertex is adjacent to an endpoint, the configuration of the cell is

$$H_1 \quad \bullet - \circ - \bullet - \bullet - \bullet \quad H_2$$

and, attending to Proposition 5.2.2,

$$f_0(\mathcal{F}) = f_0(P_4)$$

and we are in the previous situation: this cell is a pentahedron (a pyramid).

$*$if the internal open vertex is not adjacent to an endpoint the cell has the following configuration:

$$H_1 \quad \bullet - \bullet - \circ - \bullet - \bullet \quad H_2$$

Bearing in mind Proposition 5.2.2,

$$f_0(\mathcal{F}) = f_0(P_3) + f_0(P_1)f_0(P_1) = 4.$$

The previous configuration is obtained from three of the following four vertices:

$$H_1 \quad \bullet \quad \bullet \quad \circ - \circ \quad \bullet \quad H_2$$

$$H_1 \quad \bullet \quad \circ - \circ \quad \bullet \quad \bullet \quad H_2$$

$$H_1 \quad \circ - \circ \quad \circ - \circ \quad \bullet \quad H_2$$

$$H_1 \quad \bullet \quad \circ - \circ \quad \circ - \circ \quad H_2$$

These vertices are pairwise adjacent and we have four $2-$faces, all of them are triangles. This cell is a tetrahedron.

- If $\omega > 1$, bearing in mind Propositions 5.2.2 and 5.2.6, the remain possibilities in this case will be similar to the previous ones.

Suppose now that we are in Case 2. In this case, if the T-components do not have open internal vertices, its configuration is:

$$H_1 \quad \bullet - \bullet - \bullet \quad H_2 \quad \bullet - \bullet \quad H_3$$

and, by Proposition 5.2.10, the number of vertices of the cell is $f_0(\mathcal{F}) = f_0(P_3)f_0(P_2) = 3 \times 2 = 6$. They correspond to five faces: the two first ones are triangles and the three last ones are rectangles, whose configurations are the following:

$$H_1 \quad \bullet - \circ \cdots \circ - \bullet - \circ \cdots \circ \cdots \bullet \quad H_2 \quad \circ - \circ \sim \circ - \circ \quad H_3$$

$$H_1 \quad \bullet - \circ \cdots \circ - \bullet - \circ \cdots \circ \cdots \bullet \quad H_2 \quad \bullet \sim \bullet \quad H_3$$

and, if in the first T-component ω is even

$$H_1 \quad \bullet - \circ \cdots \circ - \bullet \quad \bullet \quad H_2 \quad \bullet - \circ \cdots \circ - \bullet \quad H_3$$

or, if in the first T-component ω is odd

$$H_1 \quad \bullet \sim \bullet - \bullet \quad H_2 \quad \bullet - \circ \cdots \circ - \bullet \quad H_3$$

$$H_1 \quad \circ - \circ \sim \bullet - \bullet \quad H_2 \quad \bullet - \circ \cdots \circ - \bullet \quad H_3$$

In this case, the cell is a pentahedron (a prism).

Suppose, now, that we are in Case 3. If in all T-components, $\omega = 0$, the configuration of the cell is:

$$H_1 \quad \bullet - \bullet \quad H_2 \quad \bullet - \bullet \quad H_3 \quad \bullet - \bullet \quad H_4$$

Independently of the number of the internal open vertices that can be present in each T-component, each of these corresponds to $f_0(P_2) = 2$ configurations. Therefore, by Proposition 5.2.10, we have $2 \times 2 \times 2 = 8$ vertices.

They correspond to six faces, each of them is a rectangle. In this case the cell is a hexahedron.

In conclusion, the cells of the tridiagonal Birkhoff polytope can only be: tetrahedrons; pentahedrons with four triangular faces and one rectangular face, pentahedrons with two triangular faces and three rectangular faces; hexahedrons with six rectangular faces. Therefore we can establish the following:

Proposition 5.4.1. *A 3-face of Ω_n^t, is a tetrahedron, or a pentahedron or an hexahedron.*

The number of 2-faces of a cell depends on the number of closed endpoints and inner entries and, in certain cases, it depends also of the position of the internal open vertices that can exist in its configuration.

5.5 Consequences of previous sections

In this section we will present some results derived from the previous ones. Firstly we illustrate the situation with an example.

Let P_7 be the path

$$\circ - \circ - \circ - \circ - \circ - \circ - \circ$$

One face of Ω_7^t is:

$$\bullet - \bullet \quad \bullet - \bullet \quad \bullet \quad \bullet \quad \bullet$$

This face results from the following vertices

V_1

V_2

V_3

V_4

In this case the face is a rectangle. Note that if we consider the bicolored sum of the configurations of the vertices V_1 and V_2 we obtain the configuration of an edge, the vertices are adjacent. We have the same situation if we consider the vertices V_1 and V_3. However, if we consider the bicolored sum of the configurations of the vertices V_2 and V_3 we obtain the configuration of the face. This means that the previous vertices are not adjacent. In fact, if we consider the polygonal line that join the vertices V_1, V_2 and V_3, it is an open line.

The bicolored sums of V_1 with V_4 and V_2 with V_3 are equal to the configuration of the face where the vertices belong. This means that V_1 is an opposite vertex to V_4 and that V_2 is opposite to V_3, concerning to this face. In fact, if V_i and V_j with $i \neq j$, are two vertices of a 2-face of Ω_n^t, as the 2-faces of Ω_n^t are rectangles or triangles, then V_i and V_j are adjacent or are opposite. Therefore we can state:

Proposition 5.5.1. *Let V_i and V_j with $i \neq j$, be two vertices of a 2-face of Ω_n^t. If the bicolored sum of V_i and V_j is equal to the bicolored subgraph that represents the referred 2-face then V_i and V_j are not adjacent (they are opposite vertices) and the face is a rectangle.*

Example 5.5.2. Consider the following vertices from Ω_7^t:

$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \circ - \circ \qquad \text{and} \qquad \circ - \circ \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$$

If we consider their bicolored sum we obtain the following bicolored subgraph

$$\bullet - \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet - \bullet$$

This bicolored subgraph is the configuration of a 2-face, this means that the two given vertices are opposite and the face is a rectangle.

From this last configuration we can get the two remaining vertices of the face. Namely:

$$\circ - \circ \quad \bullet \quad \bullet \quad \bullet \quad \circ - \circ \qquad \text{and} \qquad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$$

Note that this last ones are opposite vertices too.

In a general way, given two vertices of Ω_n^t, V_i and V_j its bicolored sum is the configuration of a face $\mathcal{F}$ of Ω_n^t whose dimension is p. If $p = 1$ the vertices are adjacent, if $p > 1$ the vertices are opposite in the p-face. Returning to Ω_7^t, if we consider the vertices

$$V_1 : \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$$

and

$$V_8 : \quad \circ - \circ \quad \circ - \circ \quad \circ - \circ \quad \bullet$$

Their bicolored sum is

$$\bullet - \bullet \quad \bullet - \bullet \quad \bullet - \bullet \quad \bullet$$

and this is the bicolored subgraph of P_7 that represents a 3-face, so this means that V_1 and V_8 are opposite vertices of the referred cell.

Attending to the nature and number of the configuration of the T-components that belongs to the configuration of a face we can state the following propositions:

Proposition 5.5.3. *Every tridiagonal Birkhoff polytope whose dimension is greater than one, has at least one triangular face.*

Proof. As the dimension of the tridiagonal Birkhoff polytope is greater than one, it corresponds to a path P_n with at least three vertices. Therefore, P_n has, at least, one bicolored subgraph with a T-component with two closed endpoints and one inner entry. □

Proposition 5.5.4. *Every tridiagonal Birkhoff polytope whose dimension is greater than two, has at least one rectangular face.*

Proof. As the dimension of the tridiagonal Birkhoff polytope is greater than two, it corresponds to a path P_n with at least four vertices. Therefore, P_n has, at least, one bicolored subgraph with two T-components each of them with two closed endpoints and without inner entries. □

Moreover, from the discussion we made in Chapter 3, Proposition 3.2.1 we have that the number of triangles present in Ω_n^t is given by

$$\sum_{p=1}^{n-2} p \sum_{k=0}^{n-2-p} f_{k+1} f_{n-p-k-1}.$$

and the number of quadrangular faces of Ω_n^t is

$$\sum_{p=1}^{n-3} p \sum_{j=0}^{n-3-p} \sum_{k=0}^{n-3-p-j} f_{k+1} f_{j+1} f_{n-p-j-k-2}.$$

The next proposition follows from the discussion made in Chapter 3 for Proposition 3.3.1:

Proposition 5.5.5. *Every tridiagonal Birkhoff polytope whose dimension is greater than three (four or five), has at least a cell that is a pentahedron (also, hexahedron or tetrahedron).*

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