Stability of quaternionic systems: a determinantal approach

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Abstract

In this paper we propose a definition of determinant for quaternionic polynomial matrices. This definition is later used in the study of stability of linear quaternionic systems within the behavioral setting.

1 Introduction

The quaternions, introduced by Hamilton in 1843, may be favorably used to describe phenomena occurring in areas such as electromagnetism and quantum physics [11], by means of a compact notation that leads to a higher efficiency in computational terms [5]. In particular, they are a powerful tool in the description of rotations. It is not uncommon to find situations, especially in robotics, where the rotation of a rigid body depends on time, and this dynamics is advantageously written in terms of quaternionic differential or difference equations. The attempt to analyse and to control the corresponding dynamics motivates the study of such equations from a system theoretic point of view.

As is well known, the asymptotical stability (henceforth simply referred as stability) of a discrete-time state space system $x(t+1) = Ax(t)$ with real or complex coefficients, is characterized by the location of the eigenvalues of the system matrix A, i.e., of the zeros of the determinant of the polynomial matrix $sI - A$.

Theorem 1. [6] *The state system* $x(t+1) = Ax(t)$ *is stable if and only if* $\det(\lambda I - A) \neq 0$, $\forall \lambda \in \mathbb{C}$ *such that* $|\lambda| \geq 1$.

In a more general setting, the growth of the solutions (or trajectories) of a linear higher order difference equation with constant real or complex (square) matrix coefficients

$$R_p w(t+p) + \cdots + R_1 w(t+1) + R_0 w(t) = 0, \tag{1}$$

can also be characterized in terms of the zeros of the determinant of the polynomial matrix $R(s) := R_p s^p + \cdots + R_1 s + R_0$.

Theorem 2. [8] *The system described by* (1) *is stable if and only if* $\det R(\lambda) \neq 0$, $\forall \lambda \in \mathbb{C}$ *such that* $|\lambda| \geq 1$.

When trying to generalize these results to systems with quaternionic coefficients, one is confronted with the lack of a notion of determinant for quaternionic polynomial matrices. Indeed, due to the non-commutativity of the field of quaternions, the determinant of quaternionic matrices cannot be defined as in the commutative (eg, real or complex) case. Several definitions have been proposed for matrices over the quaternionic skew field [1], but this work has not been extended to the polynomial ring case.

In this paper we try to fill in this gap by proposing a concept of determinant for quaternionic polynomial matrices. This concept is later used in the study of stability of linear quaternionic systems.

2 Preliminaries

We first introduce some preliminary concepts on quaternions and quaternionic polynomials.

The set

$$\mathbb{H} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : \ a, b, c, d \in \mathbb{R}\},$$

where the *imaginary units* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ and, consequently,

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik},$$

is an associative but noncommutative division algebra over $\mathbb{R}$ called *quaternionic skew field.* The *real* and *imaginary parts* of a quaternion $\eta = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ are defined as $\operatorname{Re}\eta = a$ and $\operatorname{Im}\eta = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, respectively, whereas, similar to the complex case, the *conjugate* $\overline{\eta}$ is given by $\overline{\eta} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ and the *norm* $|\eta|$ is defined as $|\eta| = \sqrt{\eta\overline{\eta}}$.

Two quaternions η and ν are said to be *similar*, $\eta \sim \nu$, if there exists a nonzero $\alpha \in \mathbb{H}$ such that $\eta = \alpha\nu\alpha^{-1}$. Similarity is an equivalence relation and we denote by $[\nu]$ the equivalence class containing ν.

Unlike the real or complex case, there are several possible ways to define quaternionic polynomials since the coefficients can be taken to be on the right, on the left or on both sides of the indeterminate (see, e.g., [9]). In this paper we will adopt the following definition.

The set of *quaternionic polynomials* is defined by

$$\mathbb{H}[s] = \left\{ p(s) = \sum_{l=0}^{n} p_l s^l,\ p_l \in \mathbb{H},\ n \in \mathbb{N} \right\}.$$

The sum and product of polynomials are defined as in the commutative case with the additional rule $(as^n)(bs^m) = abs^{n+m}$.

Conjugacy is extended to quaternionic polynomials by linearity and by the rule $\overline{as^n} = \overline{a}s^n$, $\forall a \in \mathbb{H}$.

3 Determinants of quaternionic and quaternionic polynomial matrices

Before presenting the notion of determinant for quaternionic matrices we note that, due to the noncommutativity of $\mathbb{H}$, the definition adopted for real or complex matrices is not suitable in the quaternionic case.

Indeed, let $A \in \mathbb{R}^{n\times n}$ and denote by A_l, $l = 1, \ldots, n$ the columns of A, i.e., $A = \begin{bmatrix} A_1 | \cdots | A_n \end{bmatrix}$. It is well know that, by definition, the determinant of A satisfies, among others, the following properties [2]

i) $\det\left(\begin{bmatrix} A_1 | \cdots | \alpha A_l | \cdots | A_n \end{bmatrix} \right) = \alpha \det\left(\begin{bmatrix} A_1 | \cdots | A_l | \cdots | A_n \end{bmatrix} \right)$, $\alpha \in \mathbb{R}$;

ii) $\det I = 1$, where I is the identity matrix.

Let, for instance

$$A = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{j} \end{bmatrix},$$

and suppose that the previous properties $i)$ and $ii)$ hold for quaternionic matrices. Then

$$\det A = \det \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{j} \end{bmatrix} = \mathbf{i} \det \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{j} \end{bmatrix} = \mathbf{ij} \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{ij} = \mathbf{k}$$

whereas, on the other hand

$$\det A = \det \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{j} \end{bmatrix} = \mathbf{j} \det \begin{bmatrix} \mathbf{i} & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{ji} \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{ji} = -\mathbf{k},$$

leading to an absurd. Therefore other defining properties must be considered for quaternionic determinants.

After an unsuccessful attempt to define the determinant of a quaternionic matrix made by Arthur Cayley in 1845 [3], only in the twentieth century new developments in this topic were achieved and some different definitions such as the determinants of Dieudonné [4], Study [10] and Moore [1] were given.

Our definition of determinant for quaternionic polynomial matrices is based on the definition of the Dieudonné determinant, with the necessary adjustments to the polynomial case.

In order to introduce the Dieudonné determinant, we will use the following notation, according to [1].

Denote by P_{lm} the matrix that is obtained from the identity by interchanging the l^{th} and m^{th} rows. Denote by $B_{lm}(\alpha)$, where $\alpha \in \mathcal{C}$ and $\mathcal{C}$ is a set, the matrix that is obtained from the identity by adding the m^{th} row multiplied by α to the l^{th} row. Finally denote by $SL(n, \mathcal{C})$ the set of all $n \times n$ matrices that can be decomposed as a product of matrices of the types P_{lm} and $B_{lm}(\alpha)$, $\alpha \in \mathcal{C}$.

Definition 3. [4] Let $A \in \mathbb{H}^{n \times n}$; the *Dieudonné determinant* of A, denoted by $\mathrm{Ddet}(A)$, is defined as follows.

- If A has not full rank, then $\mathrm{Ddet}(A) := 0$.
- Otherwise, let $U \in SL(n, \mathbb{H})$ be such that

$$UA = \mathrm{diag}(1, \ldots, 1, \alpha), \;\; \alpha \in \mathbb{H}. \tag{2}$$

  Then $\mathrm{Ddet}(A) := |\alpha|$.

Note that Ddet can be regarded as a generalized determinant in the sense that it satisfies the following properties:

G_1) It is zero if and only if the corresponding matrix has not full rank.

G_2) It satisfies the product rule.

G_3) It is equal to one for matrices of the type $B_{lm}(\alpha)$.

Definition 3 cannot be directly extended to the polynomial case, since it is in general not possible to bring up a polynomial matrix into the form (2). Therefore, in the definition that we propose, we consider instead a triangular form.

Definition 4. We define the function $\mathrm{Pdet}(\cdot) : \mathbb{H}^{n\times n}[s] \to \mathbb{R}[s]$ as follows.

Let $R \in \mathbb{H}^{n\times n}[s]$. Let further $U \in SL(n, \mathbb{H}[s])$ be such that UR is upper triangular, i.e.,

$$UR = T = \begin{bmatrix} \gamma_1 & * & \cdots & * & * \\ 0 & \gamma_2 & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & * & * \\ \vdots & \vdots & \ddots & \gamma_{n-1} & * \\ 0 & 0 & \cdots & 0 & \gamma_n \end{bmatrix}. \tag{3}$$

Then

$$\mathrm{Pdet}(R) := \prod_{l=1}^{n} \gamma_l \overline{\gamma}_l.$$

Example 5. Let

$$R(s) = \begin{bmatrix} (s+2\mathbf{j})(s+\mathbf{j}) & (s+2\mathbf{j})(2s+\mathbf{k})(s+3\mathbf{i}) + 2s + 3 \\ s+\mathbf{j} & (2s+\mathbf{k})(s+3\mathbf{i}) \end{bmatrix}.$$

Then

$$R = UT = \begin{bmatrix} s+2\mathbf{j} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s+\mathbf{j} & (2s+\mathbf{k})(s+3\mathbf{i}) \\ 0 & 2s+3 \end{bmatrix},$$

and $U \in SL(n, \mathbb{H}[s])$. Therefore

$$\mathrm{Pdet}(R) = (s+\mathbf{j})(\overline{s+\mathbf{j}})(2s+3)(\overline{2s+3}) = (s^2+1)(2s+3)^2.$$

It turns out that the definition of the function Pdet$(\cdot)$ given above is well-posed, i.e., if there exists another triangular matrix $T' \neq T$, where T is the matrix defined in (3), such that $T' = U'R$, with $U' \in SL(n, \mathbb{H}[s])$, then the elements of the main diagonal of T', $\gamma'_1, \ldots, \gamma'_n$, are such that

$$\prod_{l=1}^{n} \gamma'_l \overline{\gamma'}_l = \prod_{l=1}^{n} \gamma_l \overline{\gamma}_l.$$

Moreover, it can be shown that Pdet$(\cdot)$ is a generalized determinant in the sense that it satisfies properties G_1, G_2 and G_3.

As should be expected, the zeros of Pdet$(sI - A)$ can be related with the eigenvalues of the associated matrix A. A quaternion λ is said to be a *right eigenvalue* of $A \in \mathbb{H}^{n\times n}$ if $Av = v\lambda$, for some nonzero quaternionic vector $v \in \mathbb{H}^n$. The vector v is called a *right eigenvector* associated with λ. The set

$$\sigma_r(A) = \{\lambda \in \mathbb{H} : Av = v\lambda, \text{ for some } v \neq 0\}$$

is called the *right spectrum* of A.

Theorem 6. *Let $A \in \mathbb{H}^{n\times n}$. Then*

$$\lambda \in \sigma_r(A) \;\Leftrightarrow\; \lambda \textit{ is a zero of } \mathrm{Pdet}(sI - A).$$

This allows to relate the zeros of Pdet(R) with the right eigenvalues of the companion matrix [7] associated with $R(s)$.

Corollary 7. *Let $R(s) = I_n s^m + R_{m-1}s^{m-1} + \cdots + R_1 s + R_0 \in \mathbb{H}^{n\times n}[s]$ and*

$$A = \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_n \\ -R_0 & -R_1 & -R_2 & \cdots & -R_{m-1} \end{bmatrix} \in \mathbb{H}^{mn\times mn}$$

be the block companion matrix of R. Then

$$\lambda \in \sigma_r(A) \;\Leftrightarrow\; \big(\mathrm{Pdet}\, R\big)(\lambda) = 0.$$

4 Quaternionic system stability

The definition of the polynomial determinant Pdet allows to extend Theorems 1 and 2 on system stability to the quaternionic case.

Theorem 8. *Let $A \in \mathbb{H}^{n\times n}$. Then the following statements are equivalent.*

(*i*) *The quaternionic system described by $x(t+1) = Ax(t)$ is stable.*

(*ii*) $\sigma_r(A) \subset \mathcal{S}_{\mathbb{Z}} := \{q \in \mathbb{H} : |q| < 1\}$

(*iii*) *All the zeros of* $\mathrm{Pdet}(sI - A)$ *lie in* $\mathcal{S}_{\mathbb{Z}}$.

Based on this theorem and Corollary 7, it is not difficult to prove our final result.

Theorem 9. *Consider the system*

$$R(\sigma)\, w = 0, \tag{4}$$

with $R(s) := R_m s^m + \cdots + R_1 s + R_0 \in \mathbb{H}^{n\times n}[s]$. This system is stable if and only if all the zeros of $\mathrm{Pdet}\big(R(s)\big)$ *lie in* $\mathcal{S}_{\mathbb{Z}}$.

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