Distributed MISO System Capacity over Rayleigh Flat Fading Channels’

Daniel Castanheira
Instituto de Telecomunicações
University of Aveiro
3810-193 Aveiro, Portugal
Email: dcastanheira@av.it.pt

Atílio Gameiro
Instituto de Telecomunicações
University of Aveiro
3810-193 Aveiro, Portugal
Email: amg@det.ua.pt

***Abstract*—For a Distributed Multiple Input Single Output (DMISO) system a terminal can be connected to all system antennas, but this leads, obviously, to a high processing capacity requirement, which is not practical. Since capacity increases only slightly with the terminal connection to more antennas, it is important to evaluate the number of antennas that a terminal must be connected to. Thus in this paper we study the ergodic capacity, for the single-user case, and the respective capacity increase by the user connection to K new antennas, over a Rayleigh flat fading channel. For each case we provide an exact closed-form expression and simple to compute upper/lower bounds. Results show that symmetry in the antenna configuration is good since maintains the capacity increase curve approximately flat. They also show that the maximum capacity increase by the connection to K new antennas is obtained when we have all mean SNR's equal, which happens when all antennas are co-located, and not distributed.**

I. Introduction

The provision of broadband services to everyone is considered one of the key components for enabling the so-called information society. It is more or less consensual that to achieve targets outlined for systems beyond IMT-2000 [1] of providing around 1Gbit/s for pedestrian and 100Mbit/s for high mobility, will require the use of multiple antennas at the transceivers to exploit the scattering properties of the wireless medium. Unfortunately due to the physical limitations in the size of the transceivers, the number of antenna elements cannot be large and the spacing between them is limited, which implies that the degree of channel independence achieved is insufficient in most scenarios to reach the high capacities envisioned. One solution to achieve the fundamental results predicted by the theory is to have the mobiles communicating simultaneously with several antennas with perfect cooperation between them. Conceptually, this allows the antennas to be treated as physically distributed antennas of one composite base station. The key to achieve perfect cooperation is to have the radio signals transparently transmitted / received to / from a central unit (CU) where all the signal processing is performed [2]. Considering the high capacities envisioned optical fibre, due to its low attenuation and enormous bandwidth, is the obvious technology choice to build these transparent interconnections. In this context it is worth to mention the FUTON integrated project [3].

Following the law of diminishing returns it is expected that as the number of antennas increase the complexity increases and the improvement in throughput may not increase in the same way. In this paper we address this problem in terms of the ergodic capacity of the channel in the downlink.

Work on the achievement of a closed-form expression for particular cases of the ergodic DMISO channel capacity was already carried on [4] and [5]. In [4] the authors study the ergodic capacity of a orthogonalized (by orthogonal space time block codes (OSTBC)) DMISO channel. In [5] the authors study the more general case of a DMISO channel but they consider that all channels gains are different. One measure that is of interest, when the system is connected to N antennas, is to check if it is worth to use additional connections. In this context we consider the differential capacity ΔC_{N-1}^{N}, that is the increase in capacity when starting with $N-1$ antennas the terminal is connected to 1 more antenna. While numerically the results can be obtained through the formulas of [4] and [5], no theoretical expression was given in the referred papers. In this paper we derive an expression for this differential capacity and provide upper bounds that are simple to compute and give us information on the maximum capacity increase one can expect by connecting the terminal to additional antennas. Such a result is of interest when managing the radio resources, since assuming that one wants to connect to the antennas providing the best SNR's it gives us indication when for a given network state we should add or drop an antenna.

This paper is organized as follows: In section II we describe the channel and system model. In section III we obtain a closed-form expression for the ergodic capacity and for the differential capacity, for a Rayleigh flat fading channel. We also provide upper and lower bounds for this expressions. Next, in section IV, we analyze the differential capacity (DCAP) expression and respective upper and lower bounds for the specific case of a grid antenna placement.

II. Channel and System Model

We consider geographically separated multiple antenna transmission to a single user, with one antenna, over a Rayleigh flat fading channel. We define a configuration $(K_1, K_2, \ldots, K_M)$ as a "configuration" where the terminal is connected to K_i antennas through a link with average SNR, λ_i^{-1}. Considering that the mobile is at the center of the area and assuming only path loss in our model, which implies an

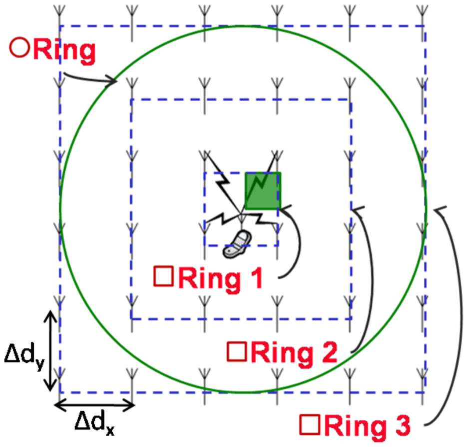


Fig. 1. Geographical antenna placement.

equal received SNR for an equal transmitted power if the transmit antennas are at the same distance from the user, we depict a $(4, 8, 4, 8, 8, 4)$ configuration in Fig. 1 as an example.

Our focus will be in the downlink, where all antennas transmit information to the user. We assume that the channel is ergodic and memoryless, that the transmitters have only Channel Distribution Information (CDI), which we assume Rayleigh, and the receiver has perfect Channel State Information (CSI).

For a MISO channel with N transmit antennas and one receive antenna the input-output relationship can be written mathematically as follows [6], if the channel is flat:

$$y = [h_1 h_2 \dots h_N] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} + n \tag{1}$$

In a more compact format we have:

$$y_{1\times1} = \mathbf{H}_{1\times N}\, \mathbf{x}_{N\times1} + n_{1\times1}$$

where $y \in \mathbb{C}$ is the received signal, h_i is the antenna i complex channel gain, $x_i \in \mathbb{C}$ is the antenna i transmitted signal and $n \in \mathbb{C}$ is the received thermal Gaussian noise with equal variance real and imaginary parts and zero mean.

The channel gains are assumed independent (geographically separated antennas) with zero mean and their real and imaginary parts are assumed independent and equally distributed.

III. Ergodic DMISO System Capacity

Ergodic capacity defines the maximum rate, averaged over all channel realizations, that can be transmitted over the channel for a transmission strategy based only on the distribution of H.

A. General Case

According to [6] and [7] the ergodic capacity can be expressed as:[1]

$$C = E_H \left[log \left(det \left[\mathbf{I}_1 + \frac{\mathbf{H}.\mathbf{R}_x.\mathbf{H}^H}{\sigma^2} \right] \right) \right] \tag{2}$$

where $\mathbf{R}_x$ is the transmit signal covariance matrix and σ^2 is the noise power.

To maximize the capacity, subject to the power constrain $\mathbf{Tr}(\mathbf{R_x}) = \mathbf{P}$, $\mathbf{x}$ must be circularly symmetric complex Gaussian [8] and its correlation matrix, $\mathbf{R}_x$, must be diagonal[2] [9], which is equivalent to independent transmit signals. Let then $\mathbf{R}_x = diag(P_1, P_2, \dots, P_N)$ where P_i is the signal x_i mean transmit power. Thus the ergodic capacity formula simplifies to:

$$C = E_H \left[log \left(1 + \sum_{i=1}^{N} \frac{|h_i|^2 P_i}{\sigma^2} \right) \right] \tag{3}$$

Since the SNR of link i (γ_i) is given by $\frac{|h_i|^2 P_i}{\sigma^2}$ then the capacity is equal to:[3]

$$C = E_{\Gamma_N}[log(1+\gamma)] \tag{4}$$

where Γ_N is a random variable (RV) corresponding to the sum of N exponential distributed RV with mean λ_i^{-1} each, which obey the pdf:

$$f_{\Gamma_N}(\gamma) = \sum_{i=1}^{M} \sum_{n=1}^{K_i} \frac{a_{in}}{(n-1)!}\, \gamma^{n-1} e^{-\lambda_i \gamma} \tag{5}$$

where M is the number of different mean SNR's, λ_i^{-1} is the mean SNR of link i, K_i is the number of antennas with SNR λ_i^{-1} and a_{in} are constants related to the partial fraction expansion [10] of the product of M Erlang distribution characteristic functions [11].

The ergodic capacity is then given by:

$$C = \int_0^{\infty} log(1+\gamma) f_{\Gamma_N}(\gamma) dz \tag{6}$$

Using [4] or [12] for the evaluation of the integral we get:

$$C = \sum_{i=1}^{M} \sum_{n=1}^{K_i} \sum_{k=0}^{n-1} \frac{a_{in}}{\lambda_i^n}\, C_i(\lambda_i, k) \tag{7}$$

where

$$C_i(\lambda_i, k) = \frac{(-\lambda_i)^k}{k!} \left[C_0(\lambda_i) + u(k-1) \sum_{p=1}^{k} \frac{(p-1)!}{(-\lambda_i)^p} \right]$$

$$C_0(\lambda_i) = e^{\lambda_i} E_1(\lambda_i)$$

[1] We consider for now on that the capacity units are nats/s/Hz, when omitted.

[2] The covariance matrix of the channel gains is diagonal, because the channel gains are independent. So their unitary singular value decomposition matrices are equal to the identity matrix.

[3] According to [7] the $|h_i|^2$ random variable is exponential distributed and consequently γ_i.

$C_0(\lambda_i)$ is equal to the capacity of the link associated with a single transmit antenna with SNR λ_i^{-1} and is also equal to $C_i(\lambda_i, 0)$. $E_1(x)$ is the exponential integral function, given by $E_1(x) = \int_x^{\infty} e^t/t\, dt$ and $u(n)$ is the unit step function.

B. Differential Capacity

In this section we derive a recursive expression for the ergodic capacity. With this expression we obtain a closed-form formula for the absolute differential capacity, which we define as the increase in capacity when starting with $N-1$ antennas the terminal is connected to 1 more antenna, and based on that expression upper and lower bounds are derived. We also provide a bound on the DCAP by the connection to K new antennas.

We can see from the moment generating function (MGF) of the Γ_N RV that:

$$-\frac{df_{\Gamma_N}(\gamma)}{d\gamma} = \lambda_N[f_{\Gamma_N}(\gamma) - f_{\Gamma_{N-1}}(\gamma)] \quad (8)$$

We can also prove that:

$$f_{\Gamma_N}(\gamma) = \lambda_N\, e^{-\lambda_N \gamma} \int_0^{\gamma} f_{\Gamma_{N-1}}(x)\, e^{\lambda_N x}\, dx \quad (9)$$

with which we can derive the previous formula and obtain a recursive algorithm for the calculation of the a_{in} coefficients.

So the absolute capacity difference by the user connection to one more antenna is given by:

$$\Delta C_{N-1}^{N} = C_N - C_{N-1} = I_N \lambda_N^{-1} \quad (10)$$

where:

$$\begin{aligned} I_N &= -\int_0^{\infty} \frac{df_{\Gamma_N}(\gamma)}{d\gamma}\, log(1+\gamma)\, d\gamma \\ &= \int_0^{\infty} \frac{f_{\Gamma_N}(\gamma)}{1+\gamma}\, d\gamma = \sum_{i=1}^{M}\sum_{n=1}^{K_i} \frac{a_{in}}{\lambda_i^{n-1}}\, C_i(\lambda_i, n-1) \end{aligned} \quad (11)$$

The differential capacity can also be expressed as:

$$\begin{aligned} \Delta C_{N-1}^{N} &= \frac{\lambda_{N-1}}{\lambda_N}\Delta C_{N-2}^{N-1} \\ &+ \frac{1}{\lambda_N^2}\left[C_0(\lambda_N) f_{\Gamma_N}(0) - \int_0^{\infty} \frac{f_{\Gamma_N}(\gamma)}{(1+\gamma)^2}\, d\gamma\right] \end{aligned} \quad (12)$$

$f_{\Gamma_N}(0)$ is equal to 0 for $N > 1$ and equal to λ_1 for $N = 1$.

C. Differential Capacity Bounds

The $g(\gamma) = 1/(1+\gamma)$ function is always less than one for all γ in $[0, \infty[$. Thus, we can easily get a simple upper bound:

$$\Delta C_{N-1}^{N} = \frac{I_N}{\lambda_N} \le \frac{1}{\lambda_N}\int_0^{\infty} f_{\Gamma_N}(\gamma)\, d\gamma = \lambda_N^{-1} \quad (13)$$

and by consequence:

$$C_N \le \sum_{k=1}^{N} \lambda_k^{-1} = \sum_{i=1}^{M} K_i\, \lambda_i^{-1} \quad (14)$$

Tighter bounds can be derived verifying that the $g(\gamma)$ function is also always greater or equal than $e^{-\gamma}$ (Bernoulli's inequality) for all γ in $[0, \infty[$, with a maximum difference of $(M_d \approx 0.204)$[4] so:

$$e^{-\gamma} \le g(\gamma) \le e^{-\gamma} + M_d \quad (15)$$

and by consequence:

$$\frac{\varphi_{\Gamma_N}(-1)}{\lambda_N} \le \Delta C_{N-1}^{N} \le \frac{\varphi_{\Gamma_N}(-1)}{\lambda_N} + \frac{M_d}{\lambda_N} \quad (16)$$

where $\varphi_{\Gamma_N}(s)$ is the Γ_N RV MGF.

From the previous bounds one can see that in the case of low SNR's the DCAP is approximated by λ_N^{-1}. Thus the capacity expression for low SNR's is given by:

$$C_N \approx \sum_{i=1}^{M} K_i\, \lambda_i^{-1} \quad (17)$$

Assuming that the new mean SNR is the smallest one, it can be shown that the maximum DCAP is achieved when all antennas have the same mean SNR, in other words when they are co-located[5].

$$\begin{aligned} \Delta C_{N-1}^{N}(\lambda_1, \ldots, \lambda_N) &\le \Delta C_{N-1}^{N}(\lambda_N, \ldots, \lambda_N) \\ \lambda_N &\ge \lambda_n, \forall\, n \in [1, 2, \ldots, N-1] \end{aligned} \quad (18)$$

In the limit of high SNR the previous expression is maximum and equal to:

$$\lim_{\lambda_N \to 0} \Delta C_{N-1}^{N}(\lambda_N, \ldots, \lambda_N) = \frac{1}{N-1} \quad (19)$$

Thus in the limit case of high SNR we can see that the DCAP value is independent of the SNR, and only depends on the number of connected antennas to the mobile terminal. This maximum can be approached with a difference of less than 0.1 bps/Hz if all mean SNR's are equal and higher than 17 dB.

The previous expression can be considered as a limit bound, but a tighter bound for the general case maximum achievable DCAP can be obtained:

$$\Delta C_{N-1}^{N} \le \frac{1}{\lambda_N + N - 1} \le \frac{1}{N-1} \quad (20)$$

The DCAP by the connection to K new antennas, if the user is only connected to one antenna and that antenna has the greatest SNR of all of them, is upper bounded by:

$$\Delta C_1^{K+1} \le \sum_{n=2}^{K+1} \frac{1}{n-1} = \sum_{n=1}^{K} \frac{1}{n} \quad (21)$$

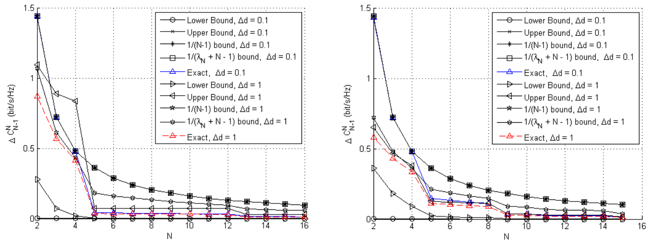
with equality if all SNR's are equal and high.

For a high number of antennas this formula can be approximated by:

$$\Delta C_1^{K+1} \le \sum_{n=1}^{K} \frac{1}{n} \approx \gamma + log(K) \quad (22)$$

[4] Obtained numerically, and knowing that this maximum is global.

[5] $\frac{d\Delta C_{N-1}^{N}}{d\lambda_i} \ge 0$ for $N > 1$ and $i \ne N \Rightarrow \Delta C_{N-1}^{N}(\lambda_1, \ldots, \lambda_N) \le \Delta C_{N-1}^{N}(\lambda_N, \ldots, \lambda_N)$ if $\lambda_N \ge \lambda_n, \forall\, n \in [1, 2, \ldots, N-1]$.



(a) Central Point.

(b) $(0.5\Delta d_x, 0.5\Delta d_y)$ Point.

Fig. 2. Differential capacity by the connection to one more antenna, exact, upper and lower bounds.

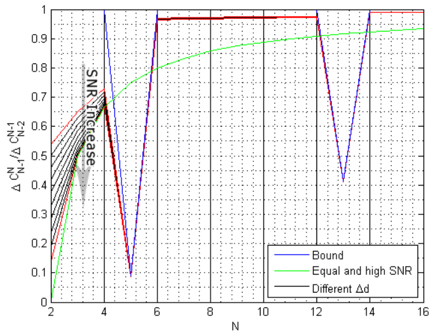


Fig. 3. Differential capacity variation with respect to N, for the central point.

and upper bounded by:

$$\Delta C_1^{K+1} \leq \sum_{n=1}^{K} \frac{1}{n} \leq \gamma + log(1+K) \qquad (23)$$

where γ is the Euler constant (0.577215665). So as the number of connected antennas increase the DCAP is smaller and smaller, because $1/K$ decreases as K increases. If the number of antennas grows to infinity so does the capacity but of course in practice one can only have a limited number of antennas.

IV. Numerical Results and Comparisons

In this section, the numerical evaluation of the bounds presented in the previous section is given. We compare this bounds with the exact value for two points of a grid antenna placement, as shown in Fig. 1. Next we analyze the DCAP variation for a range of mean SNR's with respect to the number of connected antennas. Finally we perform a DCAP analysis for a representative area covered by the antennas.

In all analysis presented in this section we consider that the mean SNR is only dependent on the signal path loss (Simplified Path Loss Model from [7]), that[6] $P_t K d_0^{\gamma}/\sigma_n^2 = 1$ and that $\gamma = 3$. It is also always assumed that the new connected antenna is the one that provides among all that are not already connected the highest SNR.

In Fig. 2 we show, for the central area point and for a point that is in the same place as one of the antennas of the configuration in Fig. 1, a plot of exact, upper and lower bounds of ΔC_{N-1}^{N}. In the second point we do not take into account the closest antenna. In this analysis we evaluate the aforementioned expressions on two distinct cases, one for which the inter antenna distance is equal to $\Delta d = 1$ and another for which $\Delta d = 0.1$, considering that $\Delta d_x = \Delta d_y = \Delta d$.

We can see from this figure that when we begin to connect to the next circular ring antennas the DCAP value decrease a lot and while we stay in a given circular ring the DCAP keeps constant. So for the central point, because of the existing symmetry, we can say that either we connect the user to 4 or 12 or 16 or 24 or 32 or ... antennas, depending on the wanted ΔC_{N-1}^{N}. This approximation of a constant capacity increase in a circular ring is better for rings far apart of the user. This can be explained by the fact that ΔC_{N-1}^{N} is dependent on ΔC_{N-2}^{N-1} by a factor of λ_{N-1}/λ_N, an approximation that due to its importance will be analyzed next, or by the fact that in the upper bound, equation (16), $\varphi_{\Gamma_N}(-1)$ tends to zero as we connect to more and more antennas, and in that way the bound becomes independent off all SNR's minus the new one, which for a ring is constant.

This figure also shows that for high SNR's the best bound is the one from equation (20), but for low SNR's the upper bound provided by (16) is more precise. The upper bound provided by equation (16) for the case of $\Delta d = 0.1$ is not shown in this figure because its higher than the bound provided by equation (20) for all N.

It is also important to stress that the most interesting bounds are the ones that are accurate for a small number of antennas, equation (20), because the user will probably only connect to a small number of antennas due to the diminishing returns that one gets as the number of connected antennas increases.

These two figures, more specifically the red and blue lines, show a convergence of the DCAP to a same value has the terminal connect to more and more antennas, in the case of SNR vectors that are multiple among themselves. It is also easy to prove that as the SNR's increase this convergence occurs at a smaller N and in the case of high SNR's the DCAP cannot be higher than a given value/line, having as the ultimate line $1/(N-1)$ in the case of all equal SNR's. Showing in that way that if we bring all antennas closer to the terminal by a given factor we only obtain an increase in the DCAP in the closest antennas.

Another aspect that it is worthwhile to analyze, as seen previously in the analysis of the bounds, is the variation/sensitivity of the DCAP value with respect to N. In this context we have evaluated the DCAP ratio, $\Delta C_{N-1}^{N}/\Delta C_{N-2}^{N-1}$, and plotted it in figure 3 and 4, for the same two previously used points and for a group of inter antenna distances between 0.1 and 1 with increments of 0.1, which corresponds to SNR's in the range of approximately $-10dB$ to $35dB$. For analysis purposes a plot of the bound given by the first term of equation (12) and a plot of the same ratio but for the case of all

[6] K is a unitless constant which depends on the antenna characteristics and on the average channel attenuation, d_0 is a reference distance for the antenna far-field, γ is the path loss exponent and P_t is the transmitted power at a distance d_0.

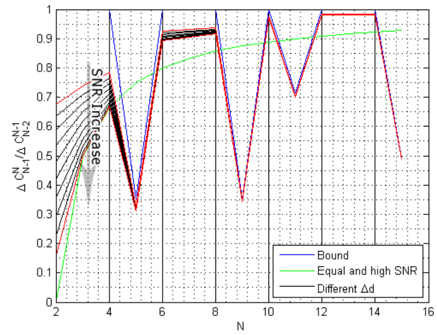


Fig. 4. Differential capacity variation with respect to N, for the $(0.5\Delta d_x, 0.5\Delta d_y)$ point

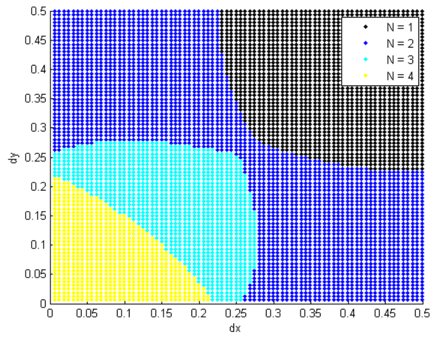


Fig. 5. Number of antennas till where the differential capacity is greater than 0.2 bps/Hz.

equal and high SNR's is also represented in this figures, in blue and green respectively. One can see from this two figures that the ratio variation with respect to the mean SNR decrease as the number of connected antennas increase and that the same happens with the difference between the exact ratio value and the respective bound. It can also be seen that the DCAP ratio tends to be constant in a circular ring, as previously observed.

In the next paragraphs we will perform a DCAP analysis for a representative area covered by the antennas. For this analysis we consider $\Delta d_x = \Delta d_y = 1$. In this context the results presented in Fig. 5 are obtained. The region shown in this figure is the green one in Fig. 1. The best way to explain the type of information contained in this figure is giving an example. For which we consider the point (0.2, 0.4). For this point the corresponding N is equal to 2 which indicates that when the user connects to the first and second ($N = 2$) antennas he obtains a capacity increase greater than 0.2 bps/Hz but departing from that number of antennas, from 2 to 3, 3 to 4, ..., the capacity increase obtained is less than 0.2 bps/Hz. One thing we see from Fig. 5 is that, for this case, only the first ring, the four closest antennas, is used. This figure also shows a circular pattern in the number of antennas which is related to the fact that the received SNR from one antenna is always equal at a distance d from that antenna. The (0, 0) point, in the middle of the area, has the greatest number of antennas, retrieving in that way more system capacity, which is related to the fact that this point is the one with greatest symmetry.

If all SNR's are equal and high we can say that the total capacity obtained by the terminal connection to $K+1$ is equal to a constant plus the capacity of a SISO link constitued by one of the antennas. Then the total capacity, due to the fact that the SISO link is for a high SNR, $C_{SISO_{high}} \approx -\gamma - log(\lambda)$, and making the approximation $\gamma + log(K)$ for ΔC_1^{K+1}, can be approximated by $log(K) - log(\lambda) = log(K\lambda^{-1})$. So as the number of antennas increases the Rayleigh flat fading channel capacity increase and approaches the capacity of a Gaussian channel.

V. Conclusion

We have examined the Shannon capacity, or equivalently, the upper-bound on spectral efficiency of a Rayleigh flat fading channel. In particular we obtained a closed-form expression for the general case configuration capacity. We have also derived a closed form expression for ΔC_{N-1}^{N}, for which upper and lower bounds were given. Results show that symmetry in the antenna configuration is good, since it mantains approximately flat the DCAP curve. If all SNR's are equal and high ΔC_{N-1}^{N} decreases exponentially. However, for this case, the maximum of ΔC_1^{K+1} can be approached closely if the SNR is greater than 17dB. If the number of connected antennas was also high, the capacity of the Rayleigh channel would be close to the one of the AWGN. ΔC_1^{K+1} was shown to be maximum when all mean SNR's are equal, which happens when all antennas are co-located, and not distributed.

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