Physical Behavior of Eigenvalues and Singular Values in Matrix Decompositions

Azmol Huda, R. L. Aguiar

Abstract— An apposite as well as realistic treatment of eigenvalue and singular value problems are potentially of interest to a wide variety of people, including among others, design engineers, theoretical physicists, classical applied mathematics and numerical analysis who yearn to carry out research in the matrix field. In real world, it is extensively used but at sometimes scantily understood. This paper focuses on building a solid perception of eigenvalue as well as singular value with their substantial meanings. The main goals of this paper are to present an intuitive experience of both eigenvalue and singular value in matrix decompositions throughout a discussion by largely building on ideas from linear algebra and will be proficiently to gain a better perceptive of their physical meanings from graphical representation.

Index Terms—Eigenvalue, singular value, matrix decomposition, orthonormal basis, linear mapping.

◆

1 Introduction

EIGENVALUE as well as singular value has widespread application in diverse fields of empirical science from mathematics to neuroscience because they are a straightforward, non-parametric value that extracts pertinent information from a large matrix. With nominal effort they make available a roadmap to divulge the hidden, simplified structures in a large matrix that frequently lie beneath it. Eigenvalues take part in a significant role in situations where the matrix is a transformation form one vector space onto itself. The primary paradigms of it are systems of linear ordinary differential equations. The eigenvalues of a matrix be capable of keeping up a correspondence to frequencies of vibration, or critical values of stability factors or energy level of atoms. The most application of eigenvalue probably in the field of dimension reduction.

Singular values also play a vital role where the matrix is a transformation from one vector space to a different vector space, possibly with a dissimilar dimension. Systems of over or undetermined algebraic equations are the most important examples. The term “singular value” relates to the distance between a matrix and the set of singular matrices. From the seminal study of eigen and singular value, there has been a lot of research work related to this field [1], [2], [3], and incredibly common to all mathematicians or engineers although nearby there is very little works straightforwardly in attendance to the relation of eigenvalues and singular values simultaneously with their graphical properties, in an efficient and representative way. In paper [4], the author addresses a number of numerical issues of singular value arising in the study of models of linear systems with its applications. Numerical computation of the characteristic values of a real symmetric matrix and numerically stable, fairly fast technique for achieving the singular values are discussed in paper [5] and [6]. Authors in paper [7], describe the solution of large scale eigenvalue problems. In paper [8] and [9], the authors illustrate that the largest eigenvalue of the sample covariance matrix have a tendency to a limit under certain conditions and the limit of the cumulative distribution function of the eigenvalues is likely to determine by using a technique of moments. In paper [10], C.B. Moler, presents a graphical depiction, but the singular value decomposition as well as the relation with the eigenvalues is not discussed in a broader consideration. In this paper, we put in plain words an instictive feel of the physical meaning of eigenvalues and singular values along with emphasizing more on their graphical consequence with numerical example in a meticulous way.

The rest of this paper is structured as follows. The notations used all the way throughout of this paper are given in section 2. We discuss the eigenvalue and singular value matrix decompositions in section 3. Relation between eigenvalue and singular value and their geometrical elucidation are given in section 4 and 5 respectively. Graphical representations are presented in section 6. Concluding remarks are presented in section 7.

2 Background Notations

Symbol	Meaning
A	: square or rectangular matrix of order *n* over a field *F* ;
λ_i	: eigenvalues of matrix *A* ;
Λ	: n-by-n diagonal matrix with the λ_j on the diagonal ;
X	: denotes n-by-n diagonal matrix whose j-th column is x_i ;
U,V	: orthogonal or unitary matrices;
σ_i	: singular values of *A*;
Σ	: n-by-n diagonal matrix with the σ_r on the diagonal;
A^T	: transpose matrix of *A*;
A^H	: Hermitian matrix of *A*;
$\Re$	: set of real numbers;

International Journal of Scientific & Engineering Research, Volume 2, Issue 6, June-2011
ISSN 2229-5518

3 Eigenvalue and singular value decompositions

3.1 Eigenvalue Decompositions

Let *A* be a square matrix of order *n* over a field *F*. A scalar $\lambda \in F$ is an eigenvalue of a matrix *A* if there exists a non-zero column vector *x* for which

$$Ax = \lambda x \tag{1}$$

The eigenvalue-eigenvector equation for a square matrix can be written as $(A - \lambda I)x = 0,\ x \neq 0$ where *I* denotes the identity matrix of order *n*. This implies that $(A - \lambda I)$ is singular and hence that $\det(A - \lambda I) = 0$. This is christened the characteristic equation or characteristic polynomial of *A*. The degree of polynomial is the order of the matrix and an n-by-n matrix has *n* eigenvalues, counting repeated roots. Suppose $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of a matrix *A* and $x_1, x_2, ..., x_n$ be a set of corresponding eigenvectors. Let Λ denote the n-by-n diagonal matrix with the λ_j on the diagonal, and let *X* denotes the n-by-n matrix whose j-th column is x_j, then we can write

$$AX = X\Lambda \tag{2}$$

From equation (2), it is indispensable to put Λ on the right side so that each column of *X* can be multiplied by its resultant eigenvalues. A noteworthy key conjecture is that, this statement is not factual for all matrices. At this juncture, we assume that the eigenvectors are linearly independent, afterward the inverse of matrix *X* exists and

$$A = X\Lambda X^{-1} \tag{3}$$

with the non singular matrix *X*. This is well-known as the eigenvalue decomposition of matrix *A*. When it exists, it consents us to investigate the properties and characterization of *A* by exploring the diagonal elements of Λ. For illustration, repeated matrix powers can be put across in terms of powers of scalars i.e.

$$A^k = X\Lambda^K X^{-1} \tag{4}$$

But, if the eigenvectors of *A* are not linearly independent, then such a diagonal does not exist and the powers of *A* give us an idea or an evidence of a more complex behavior. If *T* is any non singular matrix, then $A = TQT^{-1}$ is known as similarity transformation and *A* and *Q* are said to be similar. If $Ax = \lambda x$ and $x = Ty$ then $Qy = \lambda y$. So a similarity transformation preserves eigenvalues. Commonly, the eigenvalue decomposition is an attempt to find a similarity transformation to diagonal form.

3.2 Singular value Decompositions

Let A be a rectangular matrix of order m-by-n and the rank of AA^T is r. Therefore, AA^T is a square symmetric matrix of order m-by-m. Let us define some more quantities: let $V = \{v_1, v_2, ... v_r\}$ be the set of orthonormal $m \times 1$ eigenvectors with associated eigenvalues $\{\lambda_1, \lambda_2, ... \lambda_r\}$ for the symmetric matrix AA^T. Therefore, $(AA^T)v_i = \lambda_i v_i$. Also let, $U = \{u_1, u_2, ... u_r\}$ be the set of $n \times 1$ vectors defined by

$$u_i \equiv \frac{1}{\sigma_i} Av_i \text{ where } u_i . u_j = \begin{cases} 1 & if\ i = j \\ 0 & otherwise \end{cases}$$

and σ_i is related with the eigenvalues by the relation $\sigma_i = \sqrt{\lambda_i}$. This makes an unexpected property, $\|Av_i\| = \sigma_i$. A scalar $\sigma \in F$ is called a singular value of a rectangular matrix A, with a pair of singular vectors U and V if it satisfies the relation

$$AV = \sigma U \tag{5}$$

This can be written as in matrix form $AV = U\Sigma$ or $A^H U = V\Sigma^H$ where Σ stands for the n-by-n diagonal matrix with the σ_r on the diagonal. The superscript *H* stands for Hermitian transpose. The mathematical intuition behind the construction of the matrix is that we craving to express all *n* scalar equations in just one equation. The ultimate form of SVD is thick and it is uncomplicated to comprehend this process in graphically. The complete structures of singular value decomposition are described in Figure 1.

Figure 1.(a) is the basic form of singular value decomposition of a matrix, whereas, Figure 1.(b) and 1.(c) are the form of singular value decomposition of the matrix when $m < n$ and $m > n$ respectively. It yields that singular vectors can constantly be chosen to be perpendicular to each other, so the matrices *U* and *V*, whose columns are the normalized singular vectors, satisfy $U^H U = I$ and $V^H V = I$. In other words, *U* and *V* are orthogonal, if they are real and unitary, if they are complex. From equation (4), we can effortlessly derive

$$A = U\Sigma V^H \tag{6}$$

with the diagonal matrix Σ and orthogonal or unitary matrices *U* and *V*. This is well-known as the singular value decomposition or SVD, of matrix *A*.

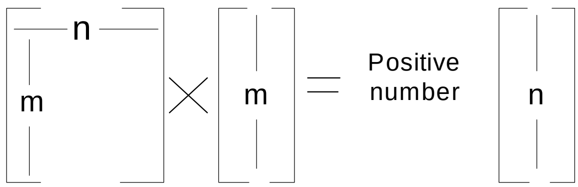

Figure. 1(a) General form of SVD

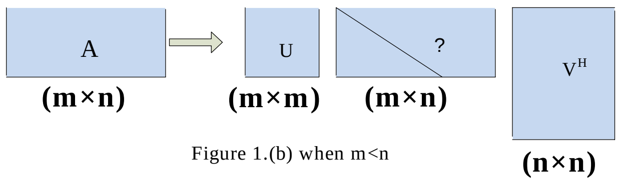

Figure 1.(b) when m<n

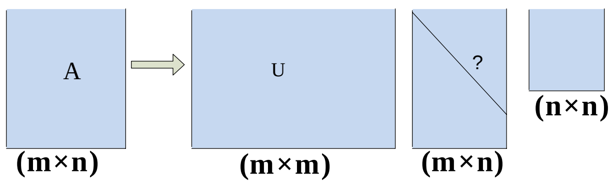

Figure 1.(c) when m>n

Fig.1. Structures of singular value decompositions.

4 RELATION BETWEEN EIGENVALUE AND SINGULAR VALUE DECOMPOSITIONS

In our earlier section, we have established that $A = U\Sigma V^H$. Now we can obtain the relation between eigenvalues and singular values by a trouble-free computation.

$$\begin{aligned} A^H A &= \left(U\Sigma V^H\right)^H \left(U\Sigma V^H\right) = V\Sigma^H U^H U\Sigma V^H = V\left(\Sigma^H \Sigma\right)V^H \\ AA^H &= \left(U\Sigma V^H\right)\left(U\Sigma V^H\right)^H = U\Sigma V^H V\Sigma^H U^H = U\left(\Sigma\Sigma^H\right)U^H \end{aligned} \tag{7}$$

The right hand side of these two equations expresses the eigenvalue decompositions of the left hand sides. It is evidently seen that, the squares of the non-zero singular values of *A* are equal to the non-zero eigenvalues of their $A^H A$ and AA^H. Furthermore, the columns of *U* are eigenvectors of AA^H and the columns of *V* are eigenvectors of $A^H A$.

5 GEOMETRICAL INTERPRETATION OF EIGENVALUE AND SINGULAR VALUE

Let us take a closer look carefully of equation (1), $Ax = \lambda x$ and let us now ask whether there are any vectors which are not changed in direction by the deformation? The answer may be found easily from the equation (1). We could make some quick concluding remarks of equation (1). The square matrix *A* can be thought of as a transformation matrix. The multiplication of the matrix *A*, on the left of a vector *x*, the answer is another vector that is transformed from its original position. The transformed vector *x* does not change its direction, only changes its magnitude.

Moreover, it is effortlessly seen that, even if we scale the vector *x*, by a few amount before we multiply it, we at a standstill get the same multiple of it as a result. This is for the reason that, if we scale by some amount, all we are doing is, building it longer, not change its direction. Perhaps the preeminent way to think of an eigenvector *x* of a matrix *A* is that it represents a direction which remains invariant under multiplication by *A*. The corresponding eigenvalues of *A* are then the representation of *A* in the subspace spanned by the eigenvector *x*. The benefit of having this representation is that multiplication by *A* is reduced to a scalar operation along the eigenvector. For instance, from the equation (4), $A^k = X\Lambda^K X^{-1}$ we can state that the effect of a power of *A* along *x* can be determined by taking the subsequent power of *A*. From the point of linear algebra, eigenvalues are pertinent of a square n-by-n matrix *A* which is reflection of as a mapping of *n* dime sional space onto itself. Here, we attempt to find a basis for the space so that the matrix is converted into diagonal. Even if *A* is real, this basis might be complex. In fact, if the eigenvectors are not linearly independent, such a basis does not subsist.

On the contrary, SVD is relevant to a possibly rectangular m-by-n matrix *A* which is thought of as mapping n-space onto m-space. Eigenvalue decomposition is applicable only for square matrix; in contrast, the singular value decomposition is possible for any rectangular matrix. In SVD, we strive to find one change of basis in the domain and a typically different change of basis in the range with the intention that the matrix becomes diagonal. Such bases always exist and real if *A* is real. Usually, the transforming matrices are orthogonal or unitary so as they preserve lengths and angles and do not magnify errors. The geometrical outlook of the SVD decompositions can be summarized as follows: for every linear map $\Omega : F^n \rightarrow F^m$ of the field *F* one can find orthonormal bases of F^n and F^m such that Ω maps the i-th basis vector of F^n to a non-negative multiple of the i-th basis vector of F^m. With respect to these bases, the map Ω is therefore represented by a diagonal matrix with non-negative real diagonal entries. Finally, let we look a more visual flavor of singular values and singular value decompositions, at least when it works on a real vector space. If *S* is a sphere of radius one in $\Re^n$, after that the linear map $\Omega : F^n \rightarrow F^m$ maps this sphere onto an ellipsoid in $\Re^m$ and usually, nonzero singular values are simply the lengths of the semi-axes of this ellipsoid.

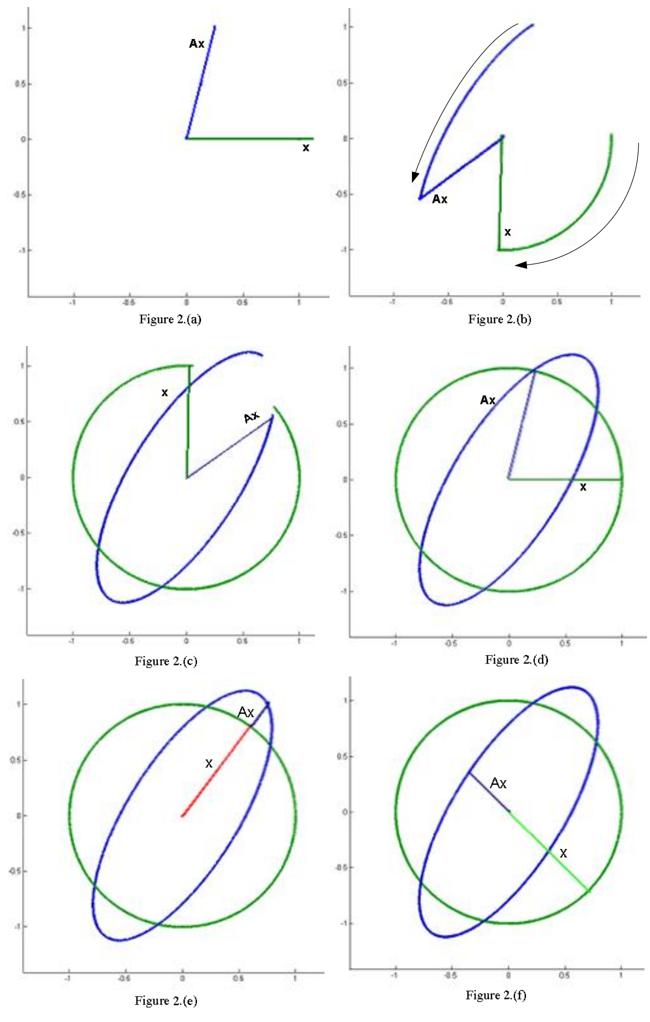

Fig. 2. Plot of eigenvalues.

ISSN 2229-5518

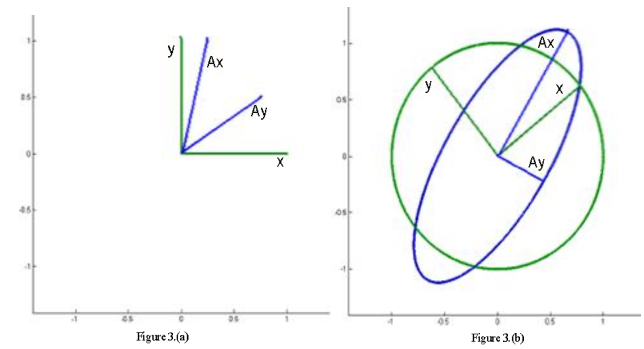

Fig. 3. Plot of singular values.

6 Graphical Presentation of Eigenvalue and Singular Value

Here we demonstrate a graphical view of eigenvalues and singular values by Mat Lab with a numerical example. We use the transformation matrix

$$A = \begin{bmatrix} 1/4 & 3/4 \\ 1 & 1/2 \end{bmatrix}$$

of order 2 and a unit vector $x = [1 \quad 0]'$ to facilitate of drawing a unit circle.

The resulting trajectory of *Ax* is plotted. In Figure-2, the first four subplots, Figure-2.(a), 2.(b), 2.(c) and 2.(d), are the intermediate steps of their traversed orbits. The goal is to find the vectors *x* so that *Ax* is parallel to *x*. Generally, for such a direction *x*, the transformation matrix *A* is simply a stretching or shrinking by a factor of λ. Each such *x* is an eigenvector and the length of *Ax* corresponding eigenvalue. From the last two subplots Figure-2.(e), 2.(d) of Figure-2 , the first eigenvalue is positive, so *Ax* lies on top of the eigenvector *x* and the second eigenvalue is negative and *Ax* is parallel to *x* but points in the opposite direction. We have plotted the SVD orbit in Figure-3 with subplots 3.(a) and 3.(b) respectively. The vectors *x* and *y* are perpendicular each other and the resulted *Ax* and *Ay* are plotted. In SVD mode, the axes of the ellipse do play a key role. The resulting *Ax* and *Ay* traverse on the ellipse, but are not perpendicular to each other though *x* and *y* are perpendicular. When *Ax* and *Ay* are perpendicular, they form the axes of the ellipse. In this case, *x* and *y* are right singular vectors as well as the columns of *U* in the SVD, the vectors *Ax* and *Ay* are multiples of the columns of *V* and the lengths of the axes of ellipse are the singular values.

7 Conclusions

In this paper, we have discussed several aspects of eigenvalues and singular values from the point of view of their underlying relations. To catch a better understanding, a pictorial visualization and elucidation, are presented. Here, we have characterized and discussed the effect of the traverse orbit both of eigenvalues and singular values. It puts on display of a mapping from sphere onto an ellipsoid. When the vector *x* moves on a circle, *Ax* moves on an ellipsoid and the length of *Ax* is the eigenvalue. In singular value mode, non-zero singular values are the simply the lengths of the semi axes of the ellipsoid.

References

[1] G.W. Stewart, "Matrix Algorithms: Basic Decompositions," SIAM, Philadelphia, 1998.

[2] H. Anton, C. Rorres, "Elementary Linear Algebra with Applications," 9-th Edition, John Wiley and Sons, 2005.

[3] V.C. Klema, "The Singular Value Decomposition: Its Computation and Some Applications," IEEE Trans. Automatic Control, Vol. 25, pp.164-176, 1980.

[4] W. Givens, "Numerical Computation of the Characteristic Values of a Real Symmetric Matrix," ORNL Rep. 1574.

[5] G.H. Golub and W. Kahan, "Calculating the Singular Values and Pseudo-Inverse of a Matrix," SIAMJ. Nwner. Anal., vol. 2, pp. 205-224, 1965.

[6] R.B. Lehoucq, D.C. Sorensen and C. Yang, "ARPACK User's Guide: Solution of Large-Scale Eigenvalue Problems with Implicitly Restarted Arnoldi Menhods," SIAM, Philadelphia, 1998.

[7] D. Jonsson, "Some Limit Theorems for the Eigenvalues of a Sample Covariance Matrix," J. Multivariate Analysis 12, pp.1-38, 1982.

[8] Y.Q. Yin, Z.D. Bai and P.R. Krishnajah, "On the Limit of the Largest Eigenvalue of the Large Dimension Sample Covariance Matrix,"Probability Theory and Related Fields, Volume 78, Number 4, 509-521, 1998.

[9] C.B. Moler, "Numerical Computing with MATLAB," February 15, 2008.

[10] Y. Jiao, B. Yang, H. Wang, X. Niu, "SVD Based Robust Image Content Retrieval," International Conference on Intelligent Information Hiding and Multimedia Signal Processing, pp. 351-354, Dec. 2006.

[11] Y.H. Wang, T.N. Tan and Y. Zhu, "Face Verification Based on Singular Value Decomposition and Radial Basis Function Neural Network," (NLPR), Institute of Automation, Chinese Academy of Sciences, 2005.

[12] C.L., Sun, J. Hahn, "Parameter Reduction for Stable Dynamical Systems Based on Hankel Singular Values and Sensitivity Analysis," Chem. Eng.Sci., 61(16), pp. 5393–5403, 2006.

[13] S. Lall, J.E. Marsden, S. Glavaski, "Empirical Model Reduction of Controlled Nonlinear Systems," 14th IFAC World Congress, Beijing, 1999.

[14] C. Sun, J. Hahn, "Reduction of Differential-Algebraic Equation Systems Via Projections and System Identification," Journal of Process Control, vol. 15, pp. 639-650, 2005.

[15] G.H. Golub and C. Reinsch, "Singualr Value Decomposition and Least Squares Solutions," Numer. Math., vol. 14, pp. 403-420, 1970.

[16] M.H.A. Biswas, M.H. Azmol, Munnujahan Ara, M.A. Rahman, "Optimal Control Theory and its Applications in Aerospace Engineering," International Journal on Academic Research, Vol. 3, No. 2, March 2011.